Real Analysis: Measure Theory - Tutorials

School of science
Bachelor of mathematics
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Contents

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Set theory and topology

Problem I.1: Sets and functions

Let $f : E \to F$ a function. Let also A, $B \subseteq E$ and P, $Q \subseteq F$. Determine whether the following statements are true. If the statement is false provide a **necessary and sufficient condition** on f for the statement to be true. (i.e whether f needs to be injective, surjective etc.). A^c is the complement of the set A in E. $f^{-1}(A)$ is the pre-image of the set A.

1. $f(A \cup B) = f(A) \cup f(B)$
2. $f(A \cap B) = f(A) \cap f(B)$
3. $f(A^c) = f(A)^c$
4. $f^{-1}(P \cup Q) = f^{-1}(P) \cup f^{-1}(Q)$
5. $f^{-1}(P \cap Q) = f^{-1}(Q) \cap f^{-1}(Q)$
6. $f^{-1}(P^c) = f^{-1}(P)^c$

Problem I.2: De Morgan's laws

Let I be a set and $(A_i)_{i\in I}$ a family of sets.

1. Prove that
$$\left(\bigcap_{i\in I} A_i\right)^c = \bigcup_{i\in I} A_i^c.$$
2. Prove that
$$\left(\bigcup_{i\in I} A_i\right)^c = \bigcap_{i\in I} A_i^c.$$

Problem I.3: Open sets on $\mathbb{R}^n$

For any $x \in \mathbb{R}^n$ and $r > 0$, the open ball of radius r around x is defined as $B_r(x) = \{y \in \mathbb{R}^n : d(x, y) < r\}$, where $d(x, y) = \sqrt{\sum_{i=1}^n |x_i - y_i|^2}$. Let $\tau_1 = \left\{\bigcup_{(x,r)\in F} B_r(x) : F \subseteq \mathbb{R}^n \times (0, \infty)\right\}$ and $\tau_2 = \{\mathcal{O} \subseteq \mathbb{R}^n : \forall y \in \mathcal{O}\ \exists r > 0 \text{ such that } B_r(y) \subset \mathcal{O}\}$.

1. Prove that $\tau_1 = \tau_2$.
2. Let $N \in \mathbb{N}$ and let $(\mathcal{O}_k)_{k=1}^N$ be a finite collection of sets in τ_1. Prove that $\bigcap_{k=1}^N \mathcal{O}_k \in \tau_1$.
3. Let I be an arbitrary index set and let $(\mathcal{O}_k)_{k\in I}$ be a collection of elements in τ_1. Prove that $\bigcup_{k\in I} \mathcal{O}_k \in \tau_1$.

Problem I.4: Open sets on $\mathbb{R}$ (*)

Prove that an open set in $\mathbb{R}$ (in the standard topology) is a **countable** union of open intervals.

Hint: Use that fact that for any $x \in \mathbb{R}$ and $r > 0$, $\mathbb{Q} \cap B_r(x) \neq \varnothing$.

*: Bonus problem
**: Bonus and harder problem

Sigma-Algebras

Problem II.1: (2p)

Provide an explicit counter-example to show that, in general, the union of two σ-algebras $\mathcal{A}_1$ and $\mathcal{A}_2$ is not a σ-algebra.

Problem II.2: (6p)

In the lecture, the Borel σ-algebra $\mathcal{B}(\mathbb{R})$ was introduced as the σ-algebra generated by the open sets $\tau \subset \mathcal{P}(\mathbb{R})$. Let $\mathcal{E} = \{(a, b) : a, b \in \mathbb{Q} \ , \ a < b\}$ be the set of open intervals with rational endpoints. Prove that $\mathcal{B}(\mathbb{R}) = \sigma(\mathcal{E})$. This shows that $\mathcal{B}(\mathbb{R})$ can be generated by a countable set.

Problem II.3: (4p)

Let M be an uncountably infinite set and let $\mathcal{A} = \sigma(\{\{m\} \subset M \ \mathbf{:} \ m \in M\})$. Prove that $\mathcal{A} = \{A \subset M : A \text{ is countable or } A^c \text{ is countable}\}$

Problem II.4: (4p)

Let $f : E \to F$ a function.

1. Let $\mathcal{A} = \{f^{-1}(B) \mid B \in \mathcal{B}\}$ where $\mathcal{B}$ is a σ-algebra on F.
   Is $\mathcal{A}$ a σ-algebra on E?
2. Let $\mathcal{B} = \{f(A) \mid A \in \mathcal{A}\}$ where $\mathcal{A}$ is a σ-algebra on E.
   Is $\mathcal{B}$ a σ-algebra on F?

Problem II.5: (4p)

Show whether the following are valid σ-algebras on the set X.

1. The σ-algebra formed by the sets $A \subset X$ such that either A or A^c is finite.
2. The σ-algebra formed by the sets $A \subset X$ such that either A or A^c is countable.

*: Bonus problem
**: Bonus and harder problem

Measures

Problem III.1: (2p)

Let $(X, \mathcal{A})$ be a measurable space and $a \in X$. Define $\delta_a : \mathcal{A} \to \mathbb{R}_+$ by $\delta_a(A) = 1$ if $a \in A$ and $\delta_a(A) = 0$ else. Show that δ_a is a measure on $(X, \mathcal{A})$.

Problem III.2: (4p+4p)

Let (X, $\mathcal{A}$, μ) be a measure space. The purpose of this exercise is to prove the so-called *inclusion-exclusion formulas*: Let $n \in \mathbb{N}$ and $A_1, A_2, \ldots, A_n \in \mathcal{A}$ such that $\mu(\bigcup_{k=1}^n A_k) < \infty$. Prove that:

1.
$$\mu\left(\bigcup_{k=1}^{n} A_k\right) = \sum_{k=1}^{n} (-1)^{k-1} \sum_{\{i_1,\ldots,i_k\} \subset \{1,\ldots,n\}} \mu\left(\bigcap_{j=1}^{k} A_{i_j}\right).$$

2.
$$\mu\left(\bigcap_{k=1}^{n} A_k\right) = \sum_{k=1}^{n} (-1)^{k-1} \sum_{\{i_1,\ldots,i_k\} \subset \{1,\ldots,n\}} \mu\left(\bigcup_{j=1}^{k} A_{i_j}\right).$$

Here, the summation is over all subsets of $\{1, \ldots, n\}$ with k elements.

Hint: Use mathematical induction over n.

Problem III.3: (2p+4p+4p)

Let $(X, \mathcal{A}, \mu)$ be a measure space and let $(A_n)_{n\in\mathbb{N}}$ be a collection of elements in $\mathcal{A}$. Put

$$A^* = \limsup_{n\to\infty} A_n = \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} A_m.$$

1. Suppose that $\sum_{n=1}^{\infty} \mu(A_n) < \infty$. Prove that $\mu(A^*) = 0$.
2. Now suppose that μ is a probability measure, i.e. $\mu(X) = 1$. A collection $(B_n)_{n\in\mathbb{N}}$ of elements in $\mathcal{A}$ is called *independent* if
$$\mu\left(\bigcap_{j\in J} B_j\right) = \prod_{j\in J} \mu(B_j),$$
for any finite $J \subset \mathbb{N}$. Prove that independence of $(B_n)_{n\in\mathbb{N}}$ implies independence of $(B_n^c)_{n\in\mathbb{N}}$.
Hint: Use the inclusion-exclusion formula.
3. Suppose that $(A_n)_{n\in\mathbb{N}}$ are independent and $\sum_{n=1}^{\infty} \mu(A_n) = \infty$. Prove that $\mu(A^*) = 1$.
Hint: Compute $\mu((A^*)^c)$. You can use that $\ln(1-x) \leqslant -x$ for $x \in [0, 1]$.

*: Bonus problem
**: Bonus and harder problem

Problem III.4: (*)(2p+3p)

Recall that in the lecture, the outer Lebesgue measure $l^* : \mathcal{P}(\mathbb{R}) \to \overline{\mathbb{R}}$ was defined as

$$l^*(M) := \inf_{(a_n,b_n)_{n\in\mathbb{N}} \in I(M)} \sum_{n\in\mathbb{N}} (b_n - a_n),$$

where $I(M) = \{(a_n, b_n)_{n\in\mathbb{N}} : a_n \leqslant b_n \text{ and } M \subset \bigcup_{n\in\mathbb{N}} (a_n, b_n)\}$ is the set of coverings of $M \subset \mathbb{R}$ through countably many open intervals.

1. Let B be the set of all rational numbers in the interval $[0, 1]$, and let $\{I_k\}_{k=1}^n$ be a finite collection of open intervals that covers B. Prove that $\sum_{k=1}^n l^*(I_k) \geqslant 1$.
2. Let A be the set of all irrational numbers in the interval $[0, 1]$. Prove that $l^*(A) = 1$.
Hint: Begin by showing that the rational numbers in the interval $[0, 1]$ *have outer measure 0.*

*: Bonus problem

**: Bonus and harder problem

The Lebesgue Measure

Problem IV.1: (1p+1p+2p+2p+2p)

The aim of this exercise is to construct Vitali sets, which are an example of non Lebesgue measurable sets. The construction depends on the axiom of choice. Define a relation $\sim$ on $\mathbb{R}$ by

$$x \sim y :\iff x - y \in \mathbb{Q}.$$

1. Show that $\sim$ is an equivalence relation.
2. The equivalence classes of $\mathbb{R}$ under $\sim$ are called cosets of $\mathbb{Q}$ in $\mathbb{R}$ and are denoted by $\mathbb{R}/\mathbb{Q}$. Show that $[0,1] \cap A \neq \varnothing$ for all $A \in \mathbb{R}/\mathbb{Q}$.
3. A set $V \subset [0,1]$ is a Vitali set if contains exactly a single point from each coset of $\mathbb{Q}$ in $\mathbb{R}$. The axiom of choice guarantees the existence of such sets. Show that the sets $q + V$ where $q \in \mathbb{Q}$ are disjoint. Here $q + V$ refers to a translation of V by q.
4. Let $V \subset [0,1]$ be a Vitali set and $C = \mathbb{Q} \cap [-1,1]$.
   Now consider
   $$U = \bigcup_{q \in C} (V + q).$$
   Show that $[0,1] \subset U \subset [-1,2]$.
5. Conclude that V is not measurable.
   Hint: A countable sum of some constant is either 0 *or infinite.*

Problem IV.2: (3p+3p)

Let μ be a measure on $(\mathbb{R}, \mathcal{L}(\mathbb{R}))$ with $\mu([0,1]) = 1$ and let l be the Lebesgue measure.

1. Suppose that $\mu(a + M) = \mu(M)$ for all $M \in \mathcal{L}(\mathbb{R})$ and $a \in \mathbb{R}$. Show that $\mu = l$.
2. Suppose that $\mu(\lambda M) = |\lambda|\, \mu(M)$ for all $M \in \mathcal{L}(\mathbb{R})$ and $\lambda \in \mathbb{R}$. Show that $\mu = l$.

Problem IV.3: (3p+3p)

1. Prove that the Lebesgue measure is *outer regular*, that is, show that for any $A \in \mathcal{L}(\mathbb{R})$
   $$l(A) = \inf\{l(U) : A \subset U \text{ and } U \text{ open}\}.$$
2. Prove that the Lebesgue measure is *inner regular*, that is, show that for any $A \in \mathcal{L}(\mathbb{R})$
   $$l(A) = \sup\{l(K) : K \subset A \text{ and } K \text{ compact}\}.$$

Non-Borel measurable set

The purpose of this exercise is to construct a set that is Lebesgue - but not Borel - measurable. To do so, we make use of the *Cantor function* $c : [0,1] \to [0,1]$ defined as follows: Put $c_0(x) = x$ and for every integer $n \in \mathbb{N}$ let

$$c_n(x) = \begin{cases} \frac{1}{2}c_{n-1}(3x) & \text{if } 0 \leqslant x < \frac{1}{3} \\ \frac{1}{2} & \text{if } \frac{1}{3} \leqslant x < \frac{2}{3} \\ \frac{1}{2}(1 + c_{n-1}(3x-2)) & \text{if } \frac{2}{3} \leqslant x \leqslant 1 \end{cases}$$

1. Show that the sequence c_n converges pointwise as $n \to \infty$. The limiting function is called the Cantor function c.
2. Show that the convergence is uniform and conclude that c is continuous.
3. Show that c is constant on intervals of the form $(\frac{3k+1}{3^n}, \frac{3k+2}{3^n})$ where $n \in \mathbb{N}$ and $k = 0, 1, \ldots, 3^{n-1}-1$. These are the intervals removed from $[0,1]$ in the construction of the Cantor set C. Conclude that $c' = 0$ up to a set of Lebesgue measure zero.

Now define $f : [0,1] \to [0,2]$ as $f(x) = x + c(x)$.

4. Show that f is strictly increasing.
5. Prove that f is a homeomorphism, that is, show that f is bijective and that both f and f^{-1} are continuous.
6. Prove that f maps Borel sets to Borel sets.
7. Let C be the Cantor set. Prove that $l(f(C)) > 0$.

We can now use the fact that every measurable set $M \in \mathcal{L}(\mathbb{R})$ with $l(M) > 0$ contains a set that is not Lebesgue measurable to conclude that there is a $N \subset f(C)$ that is not Lebesgue measurable.

8. Argue that $f^{-1}(N)$ is Lebesgue measurable.
9. Argue that $f^{-1}(N)$ cannot be Borel measurable.

*: Bonus problem
**: Bonus and harder problem

Measurable functions and Lebesgue integral

Problem VI.1: (2p)

Let $f : (\mathbb{R}, \mathcal{B}(\mathbb{R})) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ be a function such that $f^{-1}(\{c\})$ is measurable for every $c \in \mathbb{R}$. Is f necessarily measurable? Prove or disprove.

Problem VI.2: (4p)

Let $(X, \mathcal{A})$ be a measurable space and $(f_n)_{n\in\mathbb{N}}$ a sequence of measurable functions $f_n : (X, \mathcal{A}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$. Let $E_0 = \{x \in X : \lim_{n\to\infty} f_n(x) \text{ exists}\}$. Is E_0 measurable? Prove or disprove.

Problem VI.3: (6p)

Let $(X, \mathcal{A}, \mu)$ be a measure space with μ a non-zero measure and $f : (X, \mathcal{A}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ a measurable function. Show that for any $\epsilon > 0$ there exists a measurable $A \in \mathcal{A}$ with $\mu(A) > 0$ such that $|f(x) - f(y)| < \epsilon$ for all $x, y \in A$.

Problem VI.4: (3p+1p+4p)

Let $(X, \mathcal{A}, \mu)$ be a measure space and $f : (X, \mathcal{A}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ a measurable function.

1. Suppose f is non-negative and that $\int_X f d\mu = 0$. Show that f is the zero function almost everywhere, that is, show that $\mu(\{x \in X : f(x) \neq 0\}) = 0$. Also argue why $\{x \in X : f(x) \neq 0\}$ is measurable.
2. Suppose now that the condition that f be non-negative is dropped. Is f still zero almost everywhere? Provide a proof or counter example.
3. Suppose that $\int_M f d\mu = 0$ where M is any measurable subset of X. Is f zero almost everywhere? Prove or disprove.

Monotone and dominated convergence

Problem VII.1: (3p+3p)

1. Let $f_n : (E, \mathcal{A}, \mu) \mapsto \overline{\mathbb{R}}_+$ be a sequence of measurable functions. Prove that

$$\sum_{n\in\mathbb{N}} \int_E f_n d\mu = \int_E \left(\sum_{n\in\mathbb{N}} f_n \right) d\mu$$

2. Let $f : (E, \mathcal{A}, \mu) \mapsto \overline{\mathbb{R}}_+$. For $A \in \mathcal{A}$ define $\mu_f(A) = \int_A f d\mu$. Prove that $\mu_f : \mathcal{A} \to \overline{\mathbb{R}}_+$ is a measure on $(E, \mathcal{A})$.

Problem VII.2: (6p)

Let $(w_n)_{n\in\mathbb{N}}$ be given by

$$w_n = \int_{\mathbb{R}_+} \frac{\sin(\pi x)}{1 + x^{n+2}} dx,$$

where dx denotes the Lebesgue measure. Evaluate $\lim_{n\to\infty} w_n$.

Problem VII.3: (4p+4p)

We will deduce a limit formula for the Gamma function

$$\Gamma(s) = \int_0^{+\infty} e^{-t} t^{s-1} \mathrm{d}t.$$

The Beta function $\mathcal{B}(x, y)$ is defined as

$$\mathcal{B}(x, y) = \int_0^1 (1-t)^{x-1} t^{y-1} \mathrm{d}t, \qquad \text{where } x,\, y > 0.$$

For this exercise you can use the following relation:

$$\mathcal{B}(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

1. Show that for all $s > 0$:

$$\lim_{n\to\infty} \int_0^n \left(1 - \frac{t}{n}\right)^n t^{s-1} \mathrm{d}t = \Gamma(s)\,.$$

2. Deduce the Gauss formula, which states that for all $s > 0$:

$$\Gamma(s) = \lim_{n\to\infty} \frac{n! n^s}{s(s+1)\cdots(s+n)}\,.$$

Integration

Problem VIII.1: (5p)

Compute the following integral:

$$\int_0^1 \frac{x^2-1}{\log(x)}dx$$

Proceed by defining

$$G(t) := \int_0^1 \frac{x^t-1}{\log(x)}dx$$

Then differentiate $G(t)$. Justify why you can move the differentiation under the integral sign. Afterwards integrate the simplified $G'(t)$ on an appropriate interval.

Problem VIII.2: (3p+4p)

Let $f : \mathbb{R} \to \mathbb{C}$ be an integrable function. Define the Fourier transform $\widehat{f} : \mathbb{R} \to \mathbb{C}$ by

$$\widehat{f}(\xi) = \int_{\mathbb{R}} e^{-i\xi x} f(x)dx.$$

1. Show that $\widehat{f}$ is well-defined and continuous on $\mathbb{R}$.
2. Suppose that $\int_{\mathbb{R}} |xf(x)|dx < \infty$. Show that $\widehat{f}$ is continuously differentiable and that

$$\widehat{f}\,'(\xi) = -i\widehat{xf(x)}.$$

The notation $\widehat{xf(x)}$ is to be understood as the Fourier transform of the mapping $x \mapsto xf(x)$.

Problem VIII.3: (5p+3p)

Let $(E, \mathcal{T}, \mu)$ be a finite measure space with $\mu \neq 0$, and $f : E \to \mathbb{C}$ a μ-integrable function. Suppose that there exists a closed set $F \subset \mathbb{C}$ such that for all $A \in \mathcal{T}$ with $\mu(A) > 0$

$$\frac{1}{\mu(A)} \int_A f(x)\, d\mu(x) \in F.$$

1. Prove that $f(x) \in F$ for μ-a.e. $x \in E$.
   Hint : Prove that for every open ball $B_r(z) \subset F^c$, we have $\mu(f^{-1}(B_r(z))) = 0$.
2. Generalize the result to σ-finite measures.

Product measures and Fubini's Theorem

Problem IX.1: (5p)

Let l denote the Lebesgue measure on $(\mathbb{R}, \mathcal{L}(\mathbb{R}))$ and put $l_2 = l \otimes l$. Argue that $\mathcal{B}(\mathbb{R}^2) \subset \mathcal{L}(\mathbb{R}) \otimes \mathcal{L}(\mathbb{R})$ and show that, up to normalisation, l_2 is the unique translation invariant measure on $(\mathbb{R}^2, \mathcal{B}(\mathbb{R}^2))$.

Problem IX.2: (5p)

Define $\mathcal{L}(\mathbb{R}^2) \coloneqq \overline{\mathcal{B}(\mathbb{R}^2)}^{l \otimes l}$. Prove that

$$\mathcal{L}(\mathbb{R}) \otimes \mathcal{L}(\mathbb{R}) \subsetneq \mathcal{L}(\mathbb{R}^2).$$

Problem IX.3: (5p)

Let $f : \mathbb{R}_+^2 \to \mathbb{R}$ be defined as $f(x, y) = \left[(1 + y)(1 + x^2 y)\right]^{-1}$. Apply Fubini's Theorem to f to calculate

$$\int_0^{+\infty} \frac{\log(x)}{x^2 - 1} dx.$$

Problem IX.4: (5p)

Let $(E, \mathcal{A}, \mu)$ be a measure space and $f : E \to \mathbb{R}_+$ a measurable function. Prove that

$$\int_E f(x) d\mu(x) = \int_0^{\infty} \mu(\{x \in E : f(x) > t\}) dt,$$

where dt is the Lebesgue measure.

Problem IX.5: (*)(5p)

Let μ be a probability measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. For $z \in \mathbb{C} \backslash \mathbb{R}$, define

$$F_\mu(z) = \int_{\mathbb{R}} \frac{1}{\lambda - z} d\mu(\lambda).$$

Prove that

$$\mu((a, b)) + \mu([a, b]) = \frac{2}{\pi} \lim_{\delta \to 0^+} \int_a^b \mathrm{Im}(F_\mu(t + i\delta)) dt,$$

for all $a < b$.

*: Bonus problem

**: Bonus and harder problem

Change of variables

Problem X.1: (5p)

Compute the Gaussian integral:

$$\int_{\mathbb{R}} e^{-x^2} dx.$$

To do so, consider the square of the integral above and change to polar coordinates, that is, use the parametrisation $\varphi : (0, \infty) \times (-\pi, \pi) \to \mathbb{R}^2 \backslash ((-\infty, 0] \times \{0\})$ defined by $\varphi(r, \theta) = (r \cos(\theta), r \sin(\theta))$.

Problem X.2: (4p+3p+3p)

Fix $n \in \mathbb{N}$ and let $S^n := \{x \in \mathbb{R}^{n+1} : ||x|| = 1\}$ be the surface of the unit ball of dimension $n + 1$. Define $\Phi : (0, \infty) \times S^n \to \mathbb{R}^{n+1} \backslash \{0\}$ by $\Phi(r, d) = rd$.

1. Show that there is a unique measure Ω on $\mathcal{B}(S^n)$ with the property that for all $B \in \mathcal{B}((0, \infty))$ and $E \in \mathcal{B}(S^n)$
$$l_{n+1}(\Phi(B \times E)) = \left(\int_B r^n dr \right) \Omega(E).$$
Hint: $\Omega(E) = (n + 1) l_{n+1}(\{re \in \mathbb{R}^{n+1} : r \in (0, 1], e \in E\})$.
2. Conclude that for any measurable $f : \mathbb{R}^{n+1} \to \mathbb{R}_+$
$$\int_{\mathbb{R}^{n+1}} f(x) dx = \int_{S^n} \int_0^\infty f(\Phi(r, d)) r^n dr d\Omega.$$
3. Compute $\Omega(S^n)$. Proceed by evaluating the integral
$$I(n) = \left(\int_{\mathbb{R}} e^{-x^2} dx \right)^{n+1}$$
in two different ways.
Hint: Recall the definition of the Gamma function: $\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt$, $x > 0$.

Problem X.3: (5p)

Consider $\mathbb{R}^n$ equipped with the Euclidean norm. For which α are the following functions integrable?

1. $f(x) = ||x||^\alpha \mathbb{1}_{B_1(0)}$.
2. $f(x) = ||x||^\alpha \mathbb{1}_{\mathbb{R}^d \backslash B_1(0)}$.

*: Bonus problem
**: Bonus and harder problem

Lebesgue spaces

Problem XI.1: (3p)

Let f be bounded and in L^{p_0} for some finite $p_0 \geqslant 1$. Prove that

$$\lim_{p\to\infty} ||f||_p = ||f||_\infty.$$

Problem XI.2: (4p+4p)

1. Let $1 \leqslant p, q, r \leqslant \infty$ with $p^{-1} + q^{-1} = r^{-1}$. Prove that for any $f \in L^p, g \in L^q$

$$||fg||_r \leqslant ||f||_p ||g||_q.$$

2. Let $n \in \mathbb{N}$ and $1 \leqslant p_1, p_2, \ldots, p_n \leqslant \infty$ with $1 = \sum_{i=1}^n p_i^{-1}$. Prove that for any $f_i \in L^{p_i}$ $(i = 1, 2, \ldots, n)$:

$$||f_1 f_2 \cdots f_n||_1 \leqslant ||f_1||_{p_1} ||f_2||_{p_2} \cdots ||f_n||_{p_n}.$$

Problem XI.3: (3p)

Prove that l^p spaces are equivalently defined as $L^p(\mathbb{N}, \mathcal{P}(\mathbb{N}), \#)$ where $\#$ is the counting measure. *Hint: For f $\#$-integrable show that*

$$\int_{\mathbb{N}} f d\# = \sum_{n\in\mathbb{N}} f(n)$$

Problem XI.4: (3p+3p)

Let $E = \{a, b\}$ and consider the measure defined by $\mu(\{a\}) = 1$ and $\mu(\{b\}) = \infty$ on $\mathcal{P}(E)$.

1. Characterise L^1 and L^∞ and their dual spaces as euclidean spaces. What are their dimensions?
2. What do you conclude regarding the Riesz representation theorem? Why does the Riesz representation theorem not apply in this case for L^1?

Problem XI.5: (5p)(*) :

Let A be a positive definite symmetric matrix. In particular this means A can be diagonalized as $A = ODO^t$ where D is a diagonal matrix, O is an orthogonal matrix ($O^{-1} = O^t$). Calculate

$$\int_{\mathbb{R}^d} e^{-<x,Ax>} dl_d(x).$$

*: Bonus problem
**: Bonus and harder problem

Fourier series

Problem XII.1: (2p+3p)

Let $f \in \mathrm{L}^2([0, 2\pi])$ and let $c_k = (2\pi)^{-1/2} \int_0^{2\pi} e^{-ikx} f(x)dx$, with $k \in \mathbb{Z}$ be its Fourier coefficients. For $n \in \mathbb{N}$, let $a_n = \pi^{-1/2} \int_0^{2\pi} \sin(nx)f(x)dx$ and $b_n = \pi^{-1/2} \int_0^{2\pi} \cos(nx)f(x)dx$.

1. What is the relationship between the coefficients (a_n, b_n) and c_n? Conclude that

$$f(x) = \lim_{N\to\infty} \frac{1}{\sqrt{\pi}} \left[\frac{b_0}{2} + \sum_{n=1}^{N} (a_n \sin(nx) + b_n \cos(nx)) \right],$$

   where the limit is taken in the L^2-sense.

2. Compute the coefficients a_n and b_n for the following functions: Id, $\mathbb{1}_{[0,\pi]}$ and the triangle function

$$t(x) = x\mathbb{1}_{[0,\pi]}(x) + (2\pi - x)\mathbb{1}_{]\pi,2\pi]}(x)$$

   on $[0, 2\pi]$.

Problem XII.2: (4p+4p+3p+4p)

View $[0, 2\pi]$ as a group with addition mod 2π. Let $f \in \mathrm{L}^2([0, 2\pi])$ and denote by

$$c_n = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} e^{-inx} f(x)dx$$

its Fourier coefficients.

1. For $N \in \mathbb{N}_0$, let $[S_N(f)](x) = (2\pi)^{-1/2} \sum_{n=-N}^{N} c_n e^{inx}$. Prove that

$$[S_N(f)](x) = \frac{1}{2\pi} \int_0^{2\pi} f(y+x) \frac{\sin[(N+1/2)y]}{\sin(y/2)} dy.$$

2. Let $[C_N(f)](x) = (N+1)^{-1} \sum_{n=0}^{N} [S_n(f)](x)$ be the Cesaro average of the Fourier sum of f. Prove that

$$[C_N(f)](x) = \frac{1}{2\pi(N+1)} \int_0^{2\pi} f(y+x) \frac{\sin^2[(N+1)y/2]}{\sin^2(y/2)} dy.$$

   Hint: You can use the trigonometric relations $2\sin(a)\sin(b) = \cos(a-b) - \cos(a+b)$ and $2\sin^2(a) = 1 - \cos(2a)$.

3. Denote

$$F_N(y) = \frac{\sin^2[(N+1)y/2]}{2\pi(N+1)\sin^2(y/2)}.$$

   Prove that for any $0 < \delta < \pi$, $F_N(y) \to 0$ as $N \to \infty$ uniformly in $[\delta, 2\pi - \delta]$.

4. Prove that $C_N(f)$ converges to f pointwise if f continuous and periodic, that is, if $f(0) = f(2\pi)$.

The Stone-Weierstrass theorem

Problem XIII.1: (2p+2p+2p+2p+2p+2p+2p+2p+2p+2p)

Let (X, d) be a compact metric space and let $C^0(X, \mathbb{R})$ (respectively $C^0(X, \mathbb{C})$) be the space of continuous functions from X to $\mathbb{R}$ (respectively $\mathbb{C}$) equipped with the supremum norm. Our goal is to prove the following theorems:

Theorem XIII.1: *Real Stone-Weierstrass theorem*

If $S \subseteq C^0(X, \mathbb{R})$ is a (unital) sub-algebra such that S separates the points of X:

$$\forall x \neq y \in X, \exists f \in S \text{ such that } f(x) \neq f(y).$$

Then S is dense in $C^0(X, \mathbb{C})$.

Theorem XIII.2: *Complex Stone-Weierstrass theorem*

Let (X, d) be a compact metric space. If $S \subseteq C^0(X, \mathbb{C})$ is a (unital) sub-algebra such that S separates the points of X and is closed under complex conjugation, then S is dense in $C^0(X, \mathbb{C})$.

Let $S \subseteq C^0(X, \mathbb{R})$ satisfying the assumptions of the real Stone-Weierstrass theorem.

Define a sequence of polynomials $P_n : [-1, 1] \to \mathbb{R}$ ($n \in \mathbb{N}_0$), recursively by

$$\begin{aligned} &P_0 := 0 \\ &P_{n+1}(t) := P_n(t) + \frac{1}{2}(t^2 - P_n(t)^2), \quad n \in \mathbb{N}. \end{aligned}$$

1. Prove that for all $t \in [-1, 1]$, $0 \leqslant P_n(t) \leqslant |t|$.
2. Prove that for all $t \in [-1, 1]$, $P_n(t)$ is non-decreasing.
3. Conclude that P_n converges uniformly to $|\cdot|$ on $[-1, 1]$.
4. Let $f, g \in S$, prove that $|f|, \max(f, g), \min(f, g) \in \overline{S}$. Here $\overline{S}$ denotes the closure of S.
5. Deduce that $\forall x \neq y \in X, \forall a, b \in \mathbb{R}, \exists g \in S$ such that $g(x) = a$ and $g(y) = b$.

Now fix $f \in C^0(X, \mathbb{R})$, $x \in \mathbb{R}$ and $\epsilon > 0$. From 5, we have

$$\forall y \in X \setminus \{x\}, \exists g_y \in S \text{ such that } g_y(x) = f(x) \text{ and } g_y(y) = f(y).$$

6. Prove that the sets $U_y := \{z \in X \mid g_y(z) < f(z) + \epsilon\}$ are an open covering of X.
7. Construct a function $g_x \in \overline{S}$ such that $g_x(x) = f(x)$ and $g_x < f + \epsilon$.
   Hint: X is compact so we can extract a finite covering from every open covering.
8. Prove that the sets $V_x := \{z \in X \mid f(z) < g_x(z) + \epsilon\}$ are an open covering of X.
9. Construct a function $h \in \overline{S}$ such that $\forall x \in X, f(x) - \epsilon < h(x) < f(x) + \epsilon$. This proves the real Stone-Weierstrass theorem.
10. Deduce the complex Stone-Weierstrass theorem.

*: Bonus problem
**: Bonus and harder problem

Problem XIII.2: (*)(1p+1p+1p+1p+1p)

Let $a > 0$, and define $g_a : \mathbb{R} \to \mathbb{R}$ by

$$g_a(x) = \frac{\pi^{-\frac{1}{4}}}{\sqrt{a}} e^{-\frac{1}{2}\left(\frac{x}{a}\right)^2}.$$

Recall the definition of the Fourier transform:

$$\mathcal{F}(f)(\nu) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x) e^{-i\nu x} dx.$$

1. Verify that g_a is normalized in L^2, that is, show that $||g_a||_2 = 1$.
2. Prove that for all $\nu \in \mathbb{R}$,

$$\mathcal{F}(g_a)'(\nu) = -\nu a^2 \mathcal{F}(g_a)(\nu).$$

   Hint: Use the results from Problem VIII.2.

3. Deduce that there is a $c \in \mathbb{R}$ such that for all $\nu \in \mathbb{R}$,

$$\mathcal{F}(g_a)(\nu) = c e^{-\frac{1}{2}(\nu a)^2}.$$

4. Compute $\mathcal{F}(g_a)(0)$ and deduce the value of c.
5. Show that $\mathcal{F}(\mathcal{F}(g_a)) = g_a$.

*: Bonus problem

**: Bonus and harder problem

Fourier transform (Bonus)

Problem XIV.1:

Define the Hermite functions $\phi_n : \mathbb{R} \to \mathbb{R}$ by the equation

$$\phi_n(\lambda) = \frac{(-1)^n}{\sqrt{2^n n!}} e^{\lambda^2/2} \left(\frac{d}{d\lambda}\right)^n e^{-\lambda^2}.$$

Also define the Hermite polynomials by $H_n(\lambda) = \sqrt{2^n n!}\phi_n(\lambda) e^{\lambda^2/2}$.

1. Prove that for all $a \in \mathbb{R}$
$$\sum_{n=0}^{\infty} H_n(\lambda)\frac{a^n}{n!} = e^{-a^2+2a\lambda}.$$
2. If $f \in \mathrm{L}^2(\mathbb{R})$ and $\langle f, \phi_n \rangle = 0$ for all $n \in \mathbb{N}_0$, prove that
$$\int_{\mathbb{R}} f(x) e^{-(x-a)^2/2} dx = 0,$$
for all $a \in \mathbb{R}$.
3. Use the Fourier transform to show that if $\int_{\mathbb{R}} f(x)e^{-(x-a)^2/2}dx = 0$ for all $a \in \mathbb{R}$, then $f = 0$.
4. Conclude that $(\phi_n)_{n\in\mathbb{N}_0}$ is a basis for $\mathrm{L}^2(\mathbb{R})$.

Problem XIV.2: Heat equation

Assume that $f : \begin{array}{ccc} \mathbb{R}_+ \times \mathbb{R} & \to & \mathbb{R} \\ (t,x) & \mapsto & f(t,x) \end{array}$ is a smooth function such that $\forall t \in \mathbb{R}$,

$$x \mapsto (1+x^2) f(t,x) \in L^1(\mathbb{R})$$

that solves the 1-dimensional heat equation

$$\partial_t f = \partial_x^2 f$$

We denote $\widehat{f}$ the Fourier transform of f with respect to the space variable:

$$\forall \nu \in \mathbb{R}, \forall t \in \mathbb{R}_+, \widehat{f}(t,\nu) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(t,x) e^{-i\nu x} dx$$

1. Prove that $\forall \nu \in \mathbb{R}, \forall t \in \mathbb{R}_+$
$$\partial_t \widehat{f}(t,\nu) = -\nu^2 \widehat{f}(t,\nu)$$
2. Deduce that $\forall \nu \in \mathbb{R}, \forall t \in \mathbb{R}_+$,
$$\widehat{f}(t,\nu) = g_t(\nu)\widehat{f}(0,\nu)$$
with $g_t(\nu) := e^{-\nu^2 t}$
3. Deduce that $\forall t \in \mathbb{R}_+$
$$f(t,\cdot) = \frac{1}{\sqrt{2\pi}} \mathcal{F}(g_t)^{-1} * f(0,\cdot)$$

4. Prove that $\forall x \in \mathbb{R}, \forall t \in \mathbb{R}_+$,

$$f(t,x) = \frac{1}{2\sqrt{\pi t}} \int_{\mathbb{R}} f(0,y) e^{-\frac{(x-y)^2}{4t}} dy$$

5. Assuming that $f(0,\cdot) \geqslant 0$, prove that the mass is conserved, i.e. that $\forall t \in \mathbb{R}_+$,

$$\|f(t,\cdot)\|_{L^1} = \|f(0,\cdot)\|_{L^1}$$

*: Bonus problem
**: Bonus and harder problem