

Mean-Field limit of the Bose-Hubbard model in high dimension

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Motivation

Study: large system of quantum bosons

Usually [3]: many-body $N \rightarrow \infty$ mean field:

$$H_N := \sum_{i=1}^N (-\Delta_i) + \frac{1}{N} \sum_{1 \leq i < j \leq N} w(X_i - X_j) \quad \text{acting on } L^2(\mathbb{R}^d, \mathbb{C})^{\otimes N}$$

Statistical description of the interaction for a mean particle $\varphi \in L^2(\mathbb{R}^d)$:

$$h_{\text{Hartree}}^\varphi = -\Delta + |\varphi|^2 \star w$$

Bose-Hubbard model: interacting bosons on a lattice

- Great success in physics:
Mott-insulator \ Superfluid phase transition, experimental observation [2] & theoretical description of the mean field theory [1]
- Mean field justified when $d \rightarrow \infty$ and effective in $d = 3$
- Simple mathematical description

Goals:

- Mean field limit as $d \rightarrow \infty$ of the dynamics and the ground state energy
- Describe a phase transition
- Strong and local particle interactions

Bose-Hubbard model

Lattice: $\Lambda := (\mathbb{Z}/L\mathbb{Z})^d$ with $d, L \in \mathbb{N}$ such that $d, L \geq 2$ of volume $|\Lambda| = L^d$

One-lattice-site Hilbert space: $\ell^2(\mathbb{C})$ of canonical basis

$$|n\rangle := (0, \dots, 0, \underbrace{1}_{n^{\text{th index}}}, 0, \dots), \quad n \in \mathbb{N}$$

2nd quantization: creation and annihilation operators:

$$\begin{aligned} a|0\rangle &:= 0, \quad \forall n \in \mathbb{N}^*, \quad a|n\rangle := \sqrt{n} |n-1\rangle, \\ \forall n \in \mathbb{N}, \quad a^\dagger|n\rangle &:= \sqrt{n+1} |n+1\rangle \\ [a, a^\dagger] &= \mathbf{1}_{\ell^2} \end{aligned}$$

(CCR)

Particle number: $\mathcal{N} := a^\dagger a$

Fock space: $\ell^2(\mathbb{C})^{\otimes |\Lambda|} \cong \mathcal{F}_+(L^2(\Lambda, \mathbb{C})) := \bigoplus_{n \in \mathbb{N}} L^2(\Lambda, \mathbb{C})^{\otimes n}$

Results

Theorem: *S.Farhat D.P S.Petrat 2025*

Dynamics [4]: Assume

- γ_d solves (B-H) with $\gamma_d(0) \in S^1(\ell^2(\mathbb{C})^{\otimes |\Lambda|})$ such that $\text{Tr}(\gamma_d(0)) = 1$
- φ solves (mf) with $\varphi(0) \in \ell^2(\mathbb{C})$ such that $\|\varphi\|_{\ell^2} = 1$
- $\exists c_1, c_2 > 0$ such that $\forall n \in \mathbb{N}$,

$$\text{Tr}(p_\varphi(0)\mathbf{1}_{\mathcal{N}=n}) \leq c_1 e^{-\frac{n}{c_2}}, \quad \text{Tr}\left(\gamma_d^{(1)}(0)\mathbf{1}_{\mathcal{N}=n}\right) \leq c_1 e^{-\frac{n}{c_2}}.$$

Then $\exists C := C(J, c_1, c_2, \text{Tr}(p_\varphi(0)\mathcal{N})) > 0$ such that $\forall t \in \mathbb{R}_+$,

$$\left\| \gamma_d^{(1)}(t) - p_\varphi(t) \right\|_{S^1} \leq C \left(\left\| \gamma_d^{(1)}(0) - p_\varphi(0) \right\|_{S^1} + \frac{1}{d\sqrt{\ln(d)}} \right) e^{Cte^{Ct}\sqrt{\ln(d)}}$$

If $\left\| \gamma_d^{(1)}(0) - p_\varphi(0) \right\|_{S^1} = \mathcal{O}\left(\frac{1}{d}\right)$, then $\forall t \in \mathbb{R}_+$, $\left\| \gamma_d^{(1)}(t) - p_\varphi(t) \right\|_{S^1} \lesssim e^{Cte^{Ct}\sqrt{\ln(d)-\ln(d)}} \xrightarrow{d \rightarrow \infty} 0$

Convergence of the order parameter: Use $a \leq \mathcal{N} + 1$ and insert a \mathcal{N} -cut-off:

$$\begin{aligned} \left| \text{Tr}\left(\gamma_d^{(1)}a\right) - \text{Tr}(p_\varphi a) \right| &\leq \left\| \left(\gamma_d^{(1)} - p_\varphi\right)a \right\|_{S^1} = \left\| \left(\gamma_d^{(1)} - p_\varphi\right)a(\mathcal{N}+1)^{-1}(\mathcal{N}+1) \right\|_{S^1} \\ &\leq \left\| \underbrace{\left(\gamma_d^{(1)} - p_\varphi\right)a(\mathcal{N}+1)^{-1}(\mathcal{N}+1)\mathbf{1}_{\mathcal{N} < M}}_{\leq M} \right\|_{S^1} + \left\| \underbrace{\left(\gamma_d^{(1)} - p_\varphi\right)a(\mathcal{N}+1)^{-1}(\mathcal{N}+1)\mathbf{1}_{\mathcal{N} \geq M}}_{\leq 1} \right\|_{S^1} \\ &\leq M \left\| \gamma_d^{(1)} - p_\varphi \right\|_{S^1} + \underbrace{\text{Tr}\left(\gamma_d^{(1)}(\mathcal{N}+1)\mathbf{1}_{\mathcal{N} \geq M}\right) + \text{Tr}(p_\varphi(\mathcal{N}+1)\mathbf{1}_{\mathcal{N} \geq M})}_{\rightarrow 0 \text{ when } M \rightarrow \infty \text{ since the particle numbers are conserved}} \end{aligned}$$

Any choice of $M \gg 1$ such that $M \left\| \gamma_d^{(1)} - p_\varphi \right\|_{S^1} \ll 1$ as $d \rightarrow \infty$ is sufficient to prove that $\left\| \left(\gamma_d^{(1)} - p_\varphi\right)a \right\|_{S^1} \xrightarrow{d \rightarrow \infty} 0$

References

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Bose-Hubbard hamiltonian of parameters $J, \mu, U \in \mathbb{R}$:

$$H_d := -\frac{J}{2d} \sum_{\substack{x,y \in \Lambda \\ x \sim y}}^{\mathcal{O}(2d|\Lambda|)} a_x^\dagger a_y + (J - \mu) \sum_{x \in \Lambda} \mathcal{N}_x + \frac{U}{2} \sum_{x \in \Lambda} \mathcal{N}_x(\mathcal{N}_x - 1)$$

Dynamics for $\gamma_d \in L^\infty(\mathbb{R}_+, S^1(\ell^2(\mathbb{C})^{\otimes |\Lambda|}))$:

$$i\partial_t \gamma_d(t) = [H_d, \gamma_d(t)] \quad (\text{B-H})$$

First reduced one-lattice-site density matrix:

$$\gamma_d^{(1)} := \frac{1}{|\Lambda|} \sum_{x \in \Lambda} \text{Tr}_{\Lambda \setminus \{x\}}(\gamma_d)$$

Mean field theory

Mean field hamiltonian for $\varphi \in \ell^2(\mathbb{C})$:

$$h^\varphi := -J(\overline{\alpha_\varphi}a + \alpha_\varphi a^\dagger - |\alpha_\varphi|^2) + (J - \mu)\mathcal{N} + \frac{U}{2}\mathcal{N}(\mathcal{N} - 1)$$

with the order parameter

$$\alpha_\varphi := \langle \varphi | a \varphi \rangle$$

Phase transition: decompose

$$\begin{aligned} \varphi &:= \sum_{n \in \mathbb{N}} \lambda_n |n\rangle \\ \implies \alpha_\varphi &= \sum_{n \in \mathbb{N}} \sqrt{n+1} \overline{\lambda_n} \lambda_{n+1} \end{aligned}$$

- Mott Insulator (MI): $\alpha_\varphi = 0$
- Superfluid (SF): $\alpha_\varphi > 0$

Dynamics:

For $\varphi \in L^\infty(\mathbb{R}_+, \ell^2(\mathbb{C}))$,

$$i\partial_t \varphi(t) = h^{\varphi(t)} \varphi(t) \quad (\text{mf})$$

Corresponding projection

$$p_\varphi := |\varphi\rangle \langle \varphi|$$

Phase diagram obtained [1] by minimizing $\varphi \mapsto \langle \varphi | h^\varphi \varphi \rangle$

Ground state (WIP): If $J, \mu, U \geq 0$, then for d large enough,

$$-\frac{\ln(d)^3}{d} \lesssim \inf_{\substack{\psi_d \in \ell^2(\mathbb{C})^{\otimes |\Lambda|} \\ \|\psi_d\|=1}} \frac{\langle \psi_d | H_d \psi_d \rangle}{|\Lambda|} - \inf_{\substack{\varphi \in \ell^2(\mathbb{C}) \\ \|\varphi\|=1}} \langle \varphi | h^\varphi \varphi \rangle \leq 0$$