Multiple Landau level filling for a large magnetic field limit of 2D fermions

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Abstract :

Motivated by the quantum hall effect, we study N two dimensional interacting fermions in a large magnetic field limit. We work in a bounded domain, ensuring finite degeneracy of the Landau levels. In our regime, several levels are fully filled and inert: the density in these levels is constant. We derive a limiting mean-field and semi classical description of the physics in the last, partially filled Landau level.

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I Context and result

I.1 Model

We consider a system of N interacting fermionic particles in two dimensions. They are placed under a homogeneous magnetic field perpendicular to the domain. In this context the kinetic energy of the particles is quantized into discrete energy levels called Landau levels, separated by a finite energy gap. This problem has initially been studied by Lieb Solovej and Yngvason in [12], [14], [15], [16], [17] and more recently by Fournais, Lewin and Madsen in [7], [8].

Our goal is to study the mean field and semi-classical limit under high magnetic field so the Landau level quantization plays an important role. This setup is related to that of [17] where three regimes are studied. In the first one, the energy gap is small with respect to the potential contributions in the energy so particles occupy all Landau levels and a standard Thomas–Fermi model is obtained in the limit. In the second one, the energy gap is comparable to the potential energy terms, particles optimise both their Landau level and their position in the potentials and the limit is a magnetic Tomas-Fermi model. For the last scaling, the gap is large compared to the potential energies so all particles occupy the lowest Landau level and the limit is described with a classical continuum electrostatic theory in this level. We want to deal with the intermediate situation where only a finite number of Landau levels are completely filled. Precisely, our result is a limit where the q first Landau Level are fully filled, the next Landau level is partially filled with filling ratio $r < 1$ and all higher Landau levels are empty. We also provide a model for the physics in the partially filled Landau level. This setup is physically motivated by the quantum hall effect which mostly takes place in a partially filled Landau level while lower Landau levels are filled and inert, and higher levels are empty, see [9].

In this perspective we want to fix the limit ratio of the number of particles to the degeneracy of Landau levels. On the whole space $\mathbb{R}^2$ this degeneracy is infinite. To ensure finiteness of the Landau levels' degeneracy (see Proposition II.9), we work on a bounded domain. For simplicity, we would like to consider a torus with periodic boundary conditions. But, in the presence of a magnetic field the periodic boundary conditions must be modified. This is a well known issue, for example see [9: Section 3.9]. As explained in Subsection II.1, we define magnetic translation operators to ensure commutation with the magnetic momentum. These magnetic translations operators define the so called magnetic periodic boundary conditions.

Notation I.1: *Model*

We work on the domain $\Omega \coloneqq [0, L]^2$ of fixed size $L > 0$. The one body kinetic energy operator, also called magnetic Laplacian, is

$$\mathscr{L}_{\hbar,b} \coloneqq (i\hbar\nabla + bA)^2 \tag{I.1}$$

We work in the Coulomb gauge:

$$\nabla \cdot A = 0$$

where $A \in C^\infty(\mathbb{R}^2, \mathbb{R}^2)$ is the vector potential generating the constant magnetic field

$$\nabla \wedge A = (0, 0, 1) \tag{I.2}$$

b is the magnetic field amplitude with associated magnetic length

$$l_b \coloneqq \sqrt{\frac{\hbar}{b}}$$

We identify $\mathbb{R}^2$ with $\mathbb{C}$ and use complex notation for the variables $(x,y) \in \mathbb{R}^2$ namely

$$(x,y) = x + iy \in \mathbb{C}$$

Let $z_0 \in \mathbb{C}$, by (I.2)

$$\nabla \wedge (A - A(\bullet - z_0)) = 0$$

so we can choose $\varphi_{z_0} \in C^\infty(\mathbb{R}^2, \mathbb{R})$ such that

$$A - A(\bullet - z_0) \eqqcolon l_b^2 \nabla \varphi_{z_0} \tag{I.3}$$

For some usual expressions see (II.3) and (II.4). As detailed in Subsection II.1, for a wave-function $\psi \in L^2(\Omega)$ the magnetic periodic boundary conditions are

$$\forall t \in [0,L], \begin{cases} \psi(L+it) = e^{i\varphi_L(L+it)}\psi(it) \\ \psi(t+iL) = e^{i\varphi_{iL}(t+iL)}\psi(t) \end{cases} \tag{I.4}$$

and the domain of the magnetic Laplacian is

$$\mathrm{Dom}\left(\mathscr{L}_{\hbar,bA}\right) \coloneqq \left\{\psi \in H^2(\Omega) \text{ such that (I.4) holds}\right\} \tag{I.5}$$

Now, the N-body Hamiltonian is

$$\mathscr{H}_N \coloneqq \sum_{j=1}^{N} \left(\left(i\hbar\nabla_{x_j} + bA(x_j)\right)^2 + V(x_j)\right) + \frac{2}{N-1} \sum_{1 \leqslant j < k \leqslant N} w(x_j - x_k) \tag{I.6}$$

acting on the space of N-body fermionic wave-functions

$$L^2_-\left(\Omega^N\right) \coloneqq \bigwedge^N L^2(\Omega).$$

We denote $\mathbb{T} \coloneqq \mathbb{R}^2/L\mathbb{Z}^2$. $V \in L^2(\mathbb{T})$ is the external potential and $w \in L^2(\mathbb{T})$ the interaction potential assumed to be radial for the metric on the torus:

$$w(x-y) \eqqcolon \widetilde{w}(d(x,y)) \text{ with } d(x,y) \coloneqq \min_{r \in L\mathbb{Z}^2} |x - y + r|$$

The domain of the N-body Hamiltonian (I.6) is

$$\mathrm{Dom}\left(\mathscr{H}_N\right) \coloneqq \bigwedge^N \mathrm{Dom}\left(\mathscr{L}_{\hbar,b}\right)$$

We define the N-body ground state energy

$$E_N^0 \coloneqq \inf\left\{\left\langle \psi_N | \mathscr{H}_N \psi_N \right\rangle, \psi_N \in \mathrm{Dom}(\mathscr{H}_N) \text{ such that } \|\psi_N\|_{L^2} = 1\right\} \tag{I.7}$$

There are $N(N-1)/2$ interacting pairs of fermions. Thus, we divide the interactions term by $(N-1)/2$ so that the order of the contribution coming from interactions is $\mathcal{O}(N)$ and comparable to the contribution coming from the external potential.

As we will see in Subsection II.2, the self adjointness of the magnetic Laplacian and the existence of its eigenvectors require the magnetic field flux bL^2 going through the domain to be quantized in multiples of $2\pi\hbar$:

$$\exists d \in \mathbb{N} \text{ such that } 2\pi d = \frac{b}{\hbar}L^2 = \frac{L^2}{l_b^2}$$

We will prove in Proposition II.9 that d is the degeneracy of Landau levels. Now, we can fix the number of filled Landau levels by choosing a scaling for which the ratio N/d is fixed.

Notation I.2: *Scaling*

We take Planck's constant to be $\hbar \coloneqq (\hbar_N)_N$ such that

$$N^{-\frac{1}{2}} \ll \hbar \ll N^{-\frac{1}{4}} \tag{I.8}$$

Let $q \in \mathbb{N}, r \in [0,1), b \coloneqq (b_N)_{N\in\mathbb{N}}$ be such that

$$d \coloneqq \frac{L^2}{2\pi l_b^2} \subset \mathbb{N}^* \tag{I.9}$$

and

$$\frac{N}{d} \underset{N\to\infty}{=} q + r + o\left(\frac{1}{\hbar b}\right) \tag{I.10}$$

where $E^* \coloneqq E \backslash \{0\}$ for $E \subset \mathbb{R}$.

q will give the number of fully filled Landau levels and r the filling ratio of the q^{th} Landau level. Note that the lowest Landau level index is 0 in our convention. With this notation,

$$\frac{N}{d} = \frac{2\pi l_b^2 N}{L^2} \underset{N\to\infty}{\to} q + r \quad \text{and} \quad \frac{1}{l_b^2} = \frac{b}{\hbar} \underset{N\to\infty}{\sim} \frac{2\pi N}{(q+r)L^2} \tag{I.11}$$

With this scaling, we find that the order of the magnetic field is $b = \mathcal{O}(\hbar N)$. It is known (II.14), that the order or the kinetic energy is

$$\hbar b = \mathcal{O}(\hbar^2 N) \gg 1 \tag{I.12}$$

The kinetic energy contribution needs to be of leading order compared to the potential terms if we want to impose the number of filled Landau level and this is true if and only if

$$\hbar^2 N \gg 1$$

hence the upper bound in (I.8). The condition $\hbar \gg N^{-\frac{1}{2}}$ is necessary in our approach to control some error terms coming from the kinetic energy. This is also the reason why we impose the convergence rate in (I.10). This scaling is a semi-classical limit because Planck's constant goes to 0.

To satisfy (I.10), one can take for example

$$d := \left\lfloor \frac{N}{q+r} \right\rfloor$$

so

$$\frac{N}{q+r} - 1 \leqslant d \leqslant \frac{N}{q+r} \implies q+r \leqslant \frac{N}{d} \leqslant \frac{N}{\frac{N}{q+r} - 1} = \frac{q+r}{1 - \frac{q+r}{N}} \underset{N\to\infty}{=} q+r+\mathcal{O}\left(\frac{1}{N}\right)$$

Note that if r is rational one can take sequences such that there is no error in (I.10), and if r is irrational (see [6: Proposition 1.4]) it is always possible to have

$$\frac{N}{d} \underset{N\to\infty}{=} q+r+\mathcal{O}\left(\frac{1}{N^2}\right)$$

I.2 Semi-classical limit model

In the limit, we obtain a semi-classic model where the energy no longer depend on the wave-function but on the density in phase space. This comes with a non linearity in the interaction term. The phase space is $\mathbb{N} \times \Omega$. This means that particles have two degrees of freedom: the first one is $n \in \mathbb{N}$ the quantum number representing the Landau Level index and $R \in \Omega$ representing the position of particles in space. In classical mechanics, one can think of R as the center of the cyclotron orbit of the particles and n as the index of the quantized angular velocity of the cyclotron orbit. This model is semi-classical in the sense that the Pauli principle still holds as a bound on the density.

Notation I.3: *Semi-classical functional*

We consider the measure on phase space

$$\eta := \left(\sum_{n\in\mathbb{N}} \delta_n\right) \otimes \lambda_\Omega$$

where λ_Ω is the Lebesgue measure on Ω. For a phase space density $m \in L^1(\mathbb{N} \times \Omega, \mathbb{R}_+)$, the semi-classical energy is

$$\mathcal{E}_{sc,\hbar b}[m] := \int_{\mathbb{N}\times\Omega} E_n m(n,R) d\eta(n,R) + \int_{\mathbb{N}\times\Omega} V m d\eta + \int_{(\mathbb{N}\times\Omega)^2} w m^{\otimes 2} d\eta^{\otimes 2} \tag{I.13}$$

where, as we will see in Section II,

$$E_n := 2\hbar b\left(n + \frac{1}{2}\right) \tag{I.14}$$

is the energy of the n^{th} Landau level. Define the semi-classical domain

$$\mathcal{D}_{sc} := \left\{ m \in L^1(\mathbb{N}\times\Omega) \text{ such that } \int_{\mathbb{N}\times\Omega} m d\eta = 1 \text{ and } 0 \leqslant m \leqslant \frac{1}{(q+r)L^2} \right\} \tag{I.15}$$

and the semi-classical ground state energy

$$E^0_{sc,\hbar b} \coloneqq \inf_{m \in \mathcal{D}_{sc}} \mathcal{E}_{sc,\hbar b}\,[m]$$

We also define the electrostatic model for the partially filled Landau level that only depends on the density.

Notation I.4: *Electrostatic model for the partially filled level*

Define

$$\mathcal{E}_{qLL}[\rho] \coloneqq \int_{\Omega} V\rho + \iint_{\Omega^2} w(x-y)\rho(x)\rho(y)dxdy \tag{I.16}$$

with domain

$$\mathcal{D}_{qLL} \coloneqq \left\{ \rho \in L^1(\Omega) \text{ such that } \int_{\Omega} \rho = \frac{r}{q+r} \text{ and } 0 \leqslant \rho \leqslant \frac{1}{(q+r)L^2} \right\} \tag{I.17}$$

The associated ground state energy is

$$E^0_{qLL} \coloneqq \inf_{\mathcal{D}_{qLL}} \mathcal{E}_{qLL}$$

We define the following energies:

$$E^{q,r} \coloneqq \frac{q^2+2qr+r}{q+r} \tag{I.18}$$

$$E^{q,r}_V \coloneqq \frac{q}{q+r} \fint_{\Omega} V \tag{I.19}$$

$$E^{q,r}_w \coloneqq \frac{q^2+2qr}{(q+r)^2} \fint_{\Omega^2} w \tag{I.20}$$

Let $\rho \in \mathcal{D}_{qLL}$, define

$$m_\rho(n,x) \coloneqq \mathbb{1}_{n<q} \frac{1}{L^2(q+r)} + \mathbb{1}_{n=q}\rho(x) \tag{I.21}$$

m_ρ is a phase space density constructed with the qLL lowest Landau levels saturating the Pauli principle in (I.15) and (I.17) and with the density ρ in the partially filled Landau level. The ratio of particles in the partially filled Landau level is

$$\frac{r}{q+r}$$

This corresponds to the normalization constraint in (I.17). With this we see that the Pauli principle is indeed the correct bound on the densities to have

$$\int\limits_{\mathbb{N}\times\Omega} m_\rho d\eta = 1$$

We will see in Proposition VI.1 by a direct computation that

$$\mathcal{E}_{sc,\hbar b}\left[m_\rho\right] = \hbar b E^{q,r} + E_V^{q,r} + E_w^{q,r} + \mathcal{E}_{qLL}\left[\rho\right]$$

where

- $\hbar b E^{q,r}$ is the kinetic energy contribution from the $q+1$ lowest Landau levels
- $E_V^{q,r}$ is the external potential energy contribution from the q lowest Landau levels
- $E_w^{q,r}$ is the energy contribution from interactions between the q lowest Landau levels and the interactions between the q lowest Landau levels and the $(q+1)^{th}$ Landau level. in other words, it contains all the interactions except the ones inside the partially filled level.

The particles in the partially filled Landau level try to optimise their localisation with respect to the self consistent potential $V + w * \rho$:

$$\mathcal{E}_{qLL}\left[\rho\right] = \int\limits_{\Omega} \left(V + w * \rho\right)\rho$$

I.3 Main results

We can now state our main result:

Theorem I.5: *Mean field limit with magnetic periodic conditions*

$$\frac{E_N^0}{N} \underset{N\to\infty}{=} \hbar b E^{q,r} + E_V^{q,r} + E_w^{q,r} + E_{qll}^0 + o(1)$$

This means that in the limit, the first order in the quantum many body energy per particle is the trivial energy $\hbar b E^{q,r}$. Then for terms of order 1, the only non trivial contribution to the energy are the external potential term and the interaction term inside the partially filled Landau level. The lower Landau levels are totally filled and therefore their contribution to the energy is constant. The interaction of the partially filled level with all other level will also be a constant. For higher Landau levels, their contribution to the energy is null because they are totally empty.

The regularity assumptions on the potentials are not minimal, we expect this result to hold true if potentials have a L^1 positive part and a L^2 negative part. Under these assumptions, one needs to prove that the particles will not concentrate in the L^1 positive singularities of the potentials. This has been done in [17] for the repulsive $1/|x|$ Coulomb potential. We will not deal with this issue in this paper.

The number of variables of the densities is going to infinity in our limit. As usual for a large number of particles, obtaining a convergence of densities requires to work in a space with a finite number of variables and therefore look at reduced densities.

Notation I.6: *Reduced densities*

We denote $\mathcal{L}^p$ the set of p-Schatten class operators along with $\|\bullet\|_{\mathcal{L}^p}$ the p-Schatten norm. Let $\gamma_N \in \mathcal{L}^1\left(L^2_-\left(\Omega^N\right)\right)$ a positive operator (thus self adjoint) of trace 1. We call such an operator a N-body density matrix. By the spectral theorem, γ_N is diagonalizable in a Hilbert basis of $L^2_-\left(\Omega^N\right)$:

$$\gamma_N = \sum_{i\in\mathbb{N}} \lambda_i \left|u_i\right\rangle \left\langle u_i\right| \text{ with } 0 \leqslant \lambda_i \leqslant 1 \text{ and } \sum_{i\in\mathbb{N}} \lambda_i = 1$$

We will denote in the same way operators and their integral kernel. We introduce compact notation for lists:

$$\begin{aligned} 1:n &\coloneqq (1,\dots,n) \\ x_{1:n} &\coloneqq (x_1,\dots,x_n) \end{aligned}$$

The density associated to γ_N is

$$\rho_{\gamma_N}(x_{1:N}) \coloneqq \gamma_N(x_{1:N}, x_{1:N})$$

Let Tr_I be the partial trace that traces out coordinates in $I \subset [\![1,N]\!]$ of $L^2(\Omega)^{\otimes N}$, it is defined by

$$\forall A_{1:N} \in \mathcal{L}^1\left(L^2(\Omega)\right)^N, \mathrm{Tr}_I\left[\bigotimes_{i=1}^N A_i\right] \coloneqq \mathrm{Tr}\left[\bigotimes_{i\in I} A_i\right] \bigotimes_{i\notin I} A_i$$

Let $1 \leqslant k < N$, we define the k^{th} reduced density matrix associated to γ_N by

$$\gamma_N^{(k)} = \mathrm{Tr}_{k+1:N}\left[\gamma_N\right] \tag{I.22}$$

with the convention that $\gamma_N^{(N)} \coloneqq \gamma_N$. For a N variables symmetric density ρ_N we denote $\rho_N^{(k)}$ its k^{th} marginal. If one starts from a wave-function $\psi_N \in L^2_-\left(\Omega^N\right)$ we use the notation

$$\begin{aligned} \gamma_{\psi_N} &\coloneqq \left|\psi_N\right\rangle\left\langle\psi_N\right| \\ \rho_{\psi_N} &\coloneqq \rho_{\gamma_{\psi_N}} = \left|\psi_N\right|^2 \end{aligned} \tag{I.23}$$

Note that with this notation

$$\rho_{\gamma_N^{(k)}} = \rho_{\gamma_N}^{(k)} \tag{I.24}$$

In (I.22), we have integrated the last $N-k$ variables but the result does not depend of the choice of these variables. Indeed, a permutation of coordinates brings a sign $\pm$ in front of of each $\left|u_i\right\rangle$ and this keeps γ_N invariant.

Ω is a compact metric space, the set of Radon measures on it is the dual of continuous functions

$$\mathcal{M}(\Omega) = C^0(\Omega)^*$$

We denote $\mathcal{M}_+(\Omega)$ the set of positive Radon measures. Let $\mathcal{P}(\Omega)$ be the set of probabilities on

Ω:

$$\mathcal{P}(\Omega) \coloneqq \{\mu \in \mathcal{M}_+(\Omega) \text{ such that } \mu(\Omega) = 1\}$$

On this space the weak star topology is metrizable using a Wasserstein metric. Moreover Ω is compact so $\mathcal{P}(\Omega)$ is also compact, thus it is possible to iterate and define the space of probability measures on $\mathcal{P}(\Omega)$ namely $\mathcal{P}(\mathcal{P}(\Omega))$.

Now, we have the following theorem for the convergence of reduced densities:

Theorem I.7: *Densities convergence with magnetic periodic conditions*

Let (ψ_N) be a sequence of minimizers of (I.7), then $\exists \mu \in \mathcal{P}(\mathcal{D}_{qLL})$ such that

- μ only charges minimizers of the limit energy functional (I.16)
- $\forall k \in \mathbb{N}^*$, in the sense of Radon measures

$$\rho^{(k)}_{\psi_N} \underset{N\to\infty}{\overset{*}{\rightharpoonup}} \int_{\mathcal{D}_{qLL}} \left(\frac{q}{L^2(q+r)} + \rho\right)^{\otimes k} d\mu(\rho) \tag{I.25}$$

The density of particles converge to a convex combination of densities of the form

$$\frac{q}{L^2(q+r)} + \rho$$

From the Pauli principle in (I.17) we see that the constant term in this expression corresponds to particles in the q lowest and fully filled Landau levels. Then the density of particles in the partially filled Landau level is given by a minimizer ρ of the limit functional (I.16).

I.4 Scaling

Another way to obtain the scaling in Notation I.2 is to observe that we have two characteristic length-scales:

- $\dfrac{L}{\sqrt{N}}$, measuring the mean distance between particles
- l_b, the magnetic length, which, in classical mechanics corresponds to the radius of a cyclotron orbit. Due to the Pauli principle, l_b will be the order of the minimum distance between particles inside a Landau level. More precisely the Pauli principle takes the form of an upper bound on the density in phase space.

The square ratio of these length is

$$\frac{L^2}{N l_b^2} = \frac{bL^2}{\hbar N} \tag{I.26}$$

If this ratio goes to zero, the mean distance between particles is very small compared to the minimal length-scale between two particles in a fixed Landau level. This implies that the particles must fill many Landau levels and this corresponds to the scaling in [17] where the energy gap between Landau level is small compared to the potential terms.

If this ratio goes to infinity, the mean distance between particles is very large compared to the minimal length-scale between two particles in a fixed Landau Level. As a consequence, all particles can be placed in the lowest Landau level and this corresponds to the regime in [17] where particles only occupy the lowest Landau level and do not feel the Pauli principle.

In the limit we study, we see from (I.11) that the ratio (I.26) has been fixed to be

$$\frac{L^2}{N l_b^2} \underset{N\to\infty}{\longrightarrow} \frac{2\pi}{q+r} \tag{I.27}$$

in order to fill a finite number of Landau levels. In our limit we fixed L and took l_b going to zero, but one can also ensure (I.27) by fixing a magnetic length $\widetilde{l}_b > 0$ and taking a domain length $\widetilde{L}$ going to infinity as

$$\widetilde{L} \coloneqq \frac{\widetilde{l}_b}{l_b} L \tag{I.28}$$

In this limit the density of particles in the domain Ω is fixed:

$$\frac{\widetilde{L}^2}{N} \underset{N\to\infty}{\longrightarrow} \widetilde{l}_b^{\,2} \frac{2\pi}{q+r} \tag{I.29}$$

Those limits are equivalent in the sense that the N-body Hamiltonian (I.6) is unitarily equivalent to

$$\frac{1}{\tau}\mathscr{H}_{N,\tau} \coloneqq \frac{1}{\tau}\left(\sum_{j=1}^{N}\left(\left(i\hbar_\tau\nabla_j + b_\tau A_\tau(x_j)\right)^2 + V_\tau(x_j)\right) + \frac{2}{N-1}\sum_{1\leqslant j<k\leqslant N} w_\tau(x_j - x_k)\right) \tag{I.30}$$

where

$$\hbar_\tau \coloneqq \frac{\hbar}{\sqrt{\tau}} \qquad A_\tau \coloneqq \frac{1}{\tau}A\left(\tau\bullet\right) \qquad b_\tau \coloneqq \tau^{\frac{3}{2}} b \qquad V_\tau \coloneqq \tau V\left(\tau\bullet\right) \qquad w_\tau \coloneqq \tau w\left(\tau\bullet\right)$$

Taking, $\tau \coloneqq L/\widetilde{L}$ and using (I.30), we confirm that the new magnetic length is

$$\sqrt{\frac{\hbar_\tau}{b_\tau}} = \frac{\widetilde{L} l_b}{L} = \widetilde{l}_b$$

Moreover if one chooses

$$A = A_{Lan} \qquad V(x) = \varrho * \frac{1}{|x|} \qquad w(x) = \frac{1}{|x|}$$

then the vector potential and the interaction potential are not rescaled :

$$A_\tau = A \qquad w_\tau = w$$

If we assume that the external potential is generated by a background charge density $\varrho \in L^1(\Omega)$ it transforms as

$$V_\tau(x) = \int_\Omega \tau \varrho(\tau x - y) \frac{1}{|y|} dy = \int_{\left[0,\widetilde{L}\right]^2} \tau^2 \varrho(\tau (y - x)) \frac{1}{|y|} dy =: \varrho_\tau * \frac{1}{|x|}$$

The re-scaling preserves the total charge

$$\int_{\left[0,\widetilde{L}\right]^2} \varrho_\tau dx = \int_\Omega \varrho dx$$

and

$$\mathscr{H}_{N,\tau} = \sum_{j=1}^N \left((i\hbar_\tau \nabla_j + b_\tau A(x_j))^2 + \rho_\tau * \frac{1}{|x|} \right) + \frac{2}{N-1} \sum_{1 \leqslant j < k \leqslant N} w(x_j - x_k)$$

We conclude that our initial scaling is equivalent to a thermodynamic limit.

I.5 Organisation of the paper

The next two sections contain preparations and necessary tools. Section II is about the diagonalisation of the magnetic Laplacian (I.1). In Section III we define the orthogonal projection on Landau levels and localise it in space, this will be the central object in the definition of the semi-classical densities. Then we prove a Lieb-Thirring inequality in Section IV to deal with L^2 potentials. The last two sections contain the proof of Theorem I.5 and Theorem I.7. In Section V we justify the semi-classical approximation and express the energy in terms of semi-classical densities. Finally, in Section VI we prove the mean-field approximation giving an upper and a lower energy bound.

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II Quantization

In this Section, we recall the diagonalization of the magnetic Laplacian (I.1). We construct an orthonormal basis of $L^2(\Omega)$ adapted to the Landau levels in terms of magnetic periodic eigenstates of $\mathscr{L}_{\hbar,b}$. This result is stated in Proposition II.14. This fact is already well known in the literature, see [1], [2] or in [9: section 3.9]. Thus the reader may go directly to Proposition II.14 and accept its statement.

To prove Proposition II.14 we will see that on a finite domain, the degeneracy of Landau levels is finite in Proposition II.9. We use the fact that the Landau levels are isomorphic and we study the lowest Landau level for which we prove the following properties:

- the wave-functions have a finite number of zeros inside the domain (Proposition II.8)
- the degeneracy is the number of zeros of the wave-functions (Proposition II.9)

Then, we will prove another expression for the eigenfunctions in Proposition II.16 using the Poisson summation formula.

II.1 Magnetic translation operators

In this subsection we explain the definition of the boundary conditions (I.4). Let $z_0 \in \mathbb{C}$, define $T_{z_0} u \coloneqq u(\bullet - z_0)$ the translation by z_0. If we try to commute the magnetic momentum with a translation we get

$$[i\hbar\nabla + bA, T_{z_0}] = b\,[A, T_{z_0}] = b(A - A(\bullet - z_0))T_{z_0}$$

Thus we cannot impose periodic boundary conditions, which would mean finding joint eigenfunctions of $\mathscr{L}_{\hbar,b}$ and of the translation operators with eigenvalue 1. The remedy is to compose the translation operator with a change of phase chosen to ensure commutation with $\mathscr{P}_{\hbar,b}$. Thus, $\mathscr{L}_{\hbar,b}$ and the magnetic translations can be diagonalized jointly. This means that we can

Notation II.1

We define the magnetic translation operators as

$$\tau_{z_0} \coloneqq e^{i\varphi_{z_0}} T_{z_0} \tag{II.1}$$

They define the conditions (I.4) on $\partial\Omega$. Let $k \geqslant 1$, we define the magnetic periodic Sobolev spaces as

$$H^k_{mp}(\Omega) \coloneqq \left\{\psi \in H^k(\Omega) \text{ such that (I.4) hold}\right\}$$

We will use similar notation for other usual functional spaces where the subscript mp stands for magnetic periodic and p for periodic. The domain of the magnetic momentum

$$\mathscr{P}_{\hbar,b} \coloneqq i\hbar\nabla + bA$$

is

$$\mathrm{Dom}(\mathscr{P}_{\hbar,b}) \coloneqq H^1_{mp}(\Omega)$$

On Coulomb gauge, there exists $\phi \in C^\infty(\mathbb{R}^2, \mathbb{R})$ such that the vector potential satisfies

$$A = \nabla^\perp \phi := \begin{pmatrix} -\partial_y \phi \\ \partial_x \phi \end{pmatrix} \tag{II.2}$$

For $k > 1, H^k(\Omega) \hookrightarrow C^0(\Omega)$, so the conditions (I.4) are well defined. For $k = 1$ they are defined with the trace operator T and $\psi_{|\Omega} := T\psi$.

For some examples of Coulomb gauges, one can take the symmetric gauge:

$$\phi_{sym} := \frac{|z|^2}{4} \qquad A_{Sym} := \frac{1}{2}(x,y)^\perp := \frac{1}{2}(-y,x) \qquad \varphi_{z_0,sym} := \frac{x_0 y - y_0 x}{2l_b^2} \tag{II.3}$$

or the Landau gauge:

$$\phi_{Lan} := \frac{y^2}{2} \qquad A_{Lan} = (-y,0) \qquad \varphi_{z_0,Lan} := -\frac{y_0 x}{l_b^2} \tag{II.4}$$

If we insert the Landau gauge (II.4) in (I.4) we get the boundary conditions in Landau gauge:

$$\forall t \in [0,L], \begin{cases} \psi(L+it) = \psi(it) \\ \psi(t+iL) = e^{-i\frac{Lt}{l_b^2}}\psi(t) \end{cases} \tag{II.5}$$

In complex notation, the vector potential Definition (II.2) becomes

$$2i\partial_{\bar{z}}\phi = A \tag{II.6}$$

and with (I.2),

$$\Delta\phi = 1 \tag{II.7}$$

In the next proposition we also emphasize the importance of the flux quantization. The magnetic translations in the two directions L and iL defining the lattice commute if and only if the flux is quantized (I.9). Therefore when the flux is quantized, we are able to impose magnetic periodic boundary conditions in both directions L and iL.

Proposition II.2: *Commutation between magnetic Laplacian and magnetic translations*

The magnetic Laplacian (I.1) commutes with the magnetic translations defined in (I.3) and (II.1):

$$[\mathscr{P}_{\hbar,b}, \tau_{z_0}] = 0 \text{ and } [\mathscr{L}_{\hbar,b}, \tau_{z_0}] = 0$$

Moreover,

$$[\tau_L, \tau_{iL}] = 0 \iff \frac{L^2}{l_b^2} \in 2\pi\mathbb{Z}$$

Proof:

We compute

$$\mathscr{P}_{\hbar,b}e^{i\varphi_{z_0}} = e^{i\varphi_{z_0}}(i\hbar\nabla + bA - \hbar\nabla\varphi_{z_0})$$

so

$$e^{i\varphi_{z_0}}T_{z_0}\mathscr{P}_{\hbar,b} = (i\hbar\nabla + bA(\bullet - z_0) + \hbar\nabla\varphi_{z_0})e^{i\varphi_{z_0}}T_{z_0}$$

With the definitions (I.3), (II.1) this ensure that $[\mathscr{P}_{\hbar,b}, \tau_{z_0}] = 0$. Next, we compute

$$[\tau_L, \tau_{iL}] = e^{i\varphi_L}T_L e^{i\varphi_{iL}}T_{iL} - e^{i\varphi_{iL}}T_{iL}e^{i\varphi_L}T_L = \left(e^{i(\varphi_L + T_L\varphi_{iL})} - e^{i(\varphi_{iL}+T_{iL}\varphi_L)}\right)T_L T_{iL} \tag{II.8}$$

So

$$[\tau_L, \tau_{iL}] = 0 \iff \exists d \in \mathbb{Z} \text{ such that } \varphi_L + T_L\varphi_{iL} - \varphi_{iL} - T_{iL}\varphi_L = 2\pi d$$

and it is sufficient to prove

$$\varphi_L + T_L\varphi_{iL} - \varphi_{iL} - T_{iL}\varphi_L = \frac{L^2}{l_b^2} \tag{II.9}$$

With the Stokes theorem:

$$\int_{\partial\Omega} A.dl = \int_{\Omega} dS = L^2 \tag{II.10}$$

Using (I.3) we get another computation for this integral

$$\begin{aligned}
\int_{\partial\Omega} A.dl &= \int_0^L (A(u) - A(u+iL))\cdot(1,0)du + i\int_0^L (A(L+iu) - A(iu))\cdot(0,1)du \\
&= l_b^2\int_0^L \left(-\partial_x\varphi_{iL}(u+iL) + i\partial_y\varphi_L(L+iu)\right)du = l_b^2\left[-\varphi_{iL}(u+iL) + \varphi_L(L+iu)\right]_0^L \\
&= l_b^2\left(\varphi_L + T_L\varphi_{iL} - \varphi_{iL} - T_{iL}\varphi_L\right)(L+iL)
\end{aligned}$$

but because of (II.10) this quantity is constant. Dividing by l_b^2 gives (II.9).

II.2 Landau Level quantization

In this subsection, we set up the usual formalism for the description of the magnetic Laplacian in term of annihilation and creation operators. More details about these operators and the properties of Landau levels can be found in [20].

Notation II.3

We denote by π_x, π_y the coordinates of the magnetic momentum:

$$\mathscr{P}_{\hbar,b} =: \begin{pmatrix} i\hbar\partial_x + bA_x \\ i\hbar\partial_y + bA_y \end{pmatrix} =: \begin{pmatrix} \pi_x \\ \pi_y \end{pmatrix}$$

and define the annihilation and creation operators respectively as

$$a := \frac{\pi_y - i\pi_x}{\sqrt{2\hbar b}} \qquad a^\dagger := \frac{\pi_y + i\pi_x}{\sqrt{2\hbar b}} \tag{II.11}$$

and the number of excitation operator $\mathscr{N} := a^\dagger a$.

The quantization of the magnetic Laplacian comes from the following commutation relations:

$$[\pi_x, \pi_y] = i\hbar b \tag{II.12}$$

$$\left[a, a^\dagger\right] = \mathrm{Id} \text{ (canonical commutation relation)}$$

$$[\tau_{z_0}, a] = \left[\tau_{z_0}, a^\dagger\right] = 0 \tag{II.13}$$

and the magnetic Laplacian is diagonal in terms of creation and annihilation operators:

$$\mathscr{L}_{\hbar,b} = 2\hbar b \left(\mathscr{N} + \frac{\mathrm{Id}}{2}\right) \tag{II.14}$$

In the next lemma we prove that the magnetic Laplacian $\mathscr{L}_{\hbar,b}$ defines a Sobolev space whose norm turns out to be equivalent to the $H^1(\Omega)$ norm.

Lemma II.4

$\mathscr{L}_{\hbar,b}$ defines the Sobolev space $(\mathrm{Dom}\,(\mathscr{L}_{\hbar,b}), \langle\bullet\rangle_{\mathscr{L}})$ with

$$\langle\chi|\psi\rangle_{\mathscr{L}} := \langle\mathscr{L}_{\hbar,b}\chi|\psi\rangle$$

which is equivalent to $(H^2_{mp}(\Omega), \langle\bullet\rangle_{H^1})$. The quadratic form defined by $\langle\bullet\rangle_{\mathscr{L}}$ is continuous, Hermitian and coercive on $\mathrm{Dom}\,(\mathscr{L}_{\hbar,b})$.

Proof:

First, we prove a Green formula for the magnetic momentum. Let $\chi \in H^1(\Omega), \vec{\psi} \in H^1(\Omega, \mathbb{C}^2)$, we use the Stokes theorem:

$$\int_{\partial\Omega} \chi\vec{\psi}^\perp \cdot \vec{n}(x)dx = \int_\Omega \nabla\cdot\left(\chi\vec{\psi}\right) = \int_\Omega \nabla\chi\cdot\vec{\psi} + \int_\Omega \chi\nabla\cdot\vec{\psi}$$

where $\vec{n}(x)$ is the outer normal vector of Ω when $x \in \partial\Omega$. So

$$\begin{aligned}\langle\chi|\mathscr{P}_{\hbar,b}\cdot\vec{\psi}\rangle &= \int_\Omega \overline{\chi(x)}\,(i\hbar\nabla + bA)\cdot\vec{\psi}(x)dx \\ &= \int_\Omega \vec{\psi}(x)\cdot\overline{(i\hbar\nabla + bA)\,\chi(x)}dx + i\hbar\int_{\partial\Omega}\overline{\chi(x)}\vec{\psi}^\perp(x)\cdot\vec{n}(x)dx\end{aligned}$$

$$=\overline{\langle\vec{\psi}|\mathscr{P}_{\hbar,b}\chi\rangle}+i\hbar\int_{\partial\Omega}\overline{\chi}\vec{\psi}^{\perp}\cdot\vec{n}$$

Further assume $\chi,\vec{\psi}$ are magnetic periodic, then $\overline{\chi}\vec{\psi}$ is periodic so the boundary term vanishes. Thus $\mathscr{P}_{\hbar,b}$ is symmetric. The symmetry of $\mathscr{L}_{\hbar,b}$ follows from

$$\mathscr{P}_{\hbar,b}H^2_{mp}(\Omega)\subset H^1_{mp}(\Omega,\mathbb{C}^2)$$

Indeed, if $\psi\in H^2_{mp}(\Omega)$, $\mathscr{P}_{\hbar,b}\psi\in H^1(\Omega,\mathbb{C}^2)$ and $\mathscr{P}_{\hbar,b}\psi$ is magnetic periodic since the magnetic translations commute with $\mathscr{P}_{\hbar,b}$. We deduce that π_x and π_y are symmetric on $H^2_{mp}(\Omega)$, so a and $a^\dagger$ are adjoint of one another and

$$\langle\psi|\mathcal{N}\psi\rangle=\langle a\psi|a\psi\rangle\geqslant 0 \tag{II.15}$$

Let $\psi\in H^2_{mp}(\Omega)$, now we prove that the norm

$$\|\psi\|_{\mathscr{L}}\coloneqq\sqrt{\langle\psi|\psi\rangle_{\mathscr{L}}}=\|\mathscr{P}_{\hbar,b}\psi\|_{L^2}$$

is equivalent to the H^1 norm. A and its gradient are bounded so $\left(H^2_{mp}(\Omega),\langle\bullet\rangle_{H^1}\right)$ is continuously embedded in $(H^2_{mp}(\Omega),\|\bullet\|_{\mathscr{L}})$. Moreover

$$\|\mathscr{P}_{\hbar,b}\psi\|_{L^2}\geqslant\hbar\|\nabla\psi\|_{L^2}-\|bA\psi\|_{L^2}\geqslant\hbar\|\nabla\psi\|_{L^2}-b\|A\|_{L^\infty}\|\psi\|_{L^2}$$

And $\|\psi\|_{L^2}$ can be controlled with (II.14) and (II.15)

$$\|\psi\|^2_{\mathscr{L}}=\langle\psi|\left(2\hbar b\mathcal{N}+\hbar b\right)\psi\rangle\geqslant\hbar b\|\psi\|^2_{L^2} \tag{II.16}$$

so

$$\|\hbar\nabla\psi\|_{L^2}\leqslant\|\psi\|_{\mathscr{L}}+\frac{b}{\sqrt{\hbar b}}\|A\|_{L^\infty}\|\psi\|_{\mathscr{L}}$$

Therefore we have the desired continuous embedding:

$$\|\psi\|_{H^1}\leqslant C(b,\hbar)\|\psi\|_{\mathscr{L}} \tag{II.17}$$

Finally to prove that $(\mathrm{Dom}\left(\mathscr{L}_{\hbar,b}\right),\langle\bullet\rangle_{\mathscr{L}})$ is a Hilbert space we need to prove that it is closed in $H^2(\Omega)$. Let $\psi_n\in\mathrm{Dom}\left(\mathscr{L}_{\hbar,b}\right)$ such that $\psi_n\to\psi$ in $H^2(\Omega)$, the limit also satisfies magnetic periodic boundary conditions because

$$\begin{aligned}\tau_{z_0}\psi_n=\psi_n\implies&\|\tau_{z_0}\psi-\psi\|_{L^2}\leqslant\|\tau_{z_0}\psi-\tau_{z_0}\psi_n\|_{L^2}+\|\psi_n-\psi\|_{L^2}=2\|\psi_n-\psi\|_{L^2}\underset{n\to\infty}{\to}0\\\implies&\tau_{z_0}\psi=\psi\end{aligned}$$

Continuity and coercivity are trivial, if $\chi,\psi\in\mathrm{Dom}\left(\mathscr{L}_{\hbar,b}\right)$,

$$\begin{aligned}&|\langle\chi|\psi\rangle|_{\mathscr{L}}=|\langle\mathscr{P}_{\hbar,b}\chi|\mathscr{P}_{\hbar,b}\psi\rangle|\leqslant\|\mathscr{P}_{\hbar,b}\chi\|_{L^2}\|\mathscr{P}_{\hbar,b}\psi\|_{L^2}=\|\chi\|_{\mathscr{L}}\|\psi\|_{\mathscr{L}}\\&\langle\psi|\psi\rangle_{\mathscr{L}}=\|\psi\|^2_{\mathscr{L}}\geqslant\|\psi\|^2_{\mathscr{L}}\end{aligned} \tag{II.18}$$

The proposition implies spectral properties of $\mathscr{L}_{\hbar,b}$.

Corollary II.5: *Spectral analysis of the magnetic Laplacian*

$\mathscr{L}_{\hbar,b}$ is a closed positive self-adjoint operator and the embedding $\mathrm{Dom}\left(\mathscr{L}_{\hbar,b}\right) \hookrightarrow L^2(\Omega)$ is continuous and compact.

Proof:

The positivity of $\mathscr{L}_{\hbar,b}$ follows from that of $\mathcal{N}$. Using the Lax-Milgram theorem, see results of [5: Section 2.5], and Lemma II.4 the operator $\mathcal{L}$ of domain

$$\mathrm{Dom}(\mathcal{L}) := \left\{u \in \mathrm{Dom}\left(\mathscr{L}_{\hbar,b}\right), \text{ such that } \forall v \in \mathrm{Dom}\left(\mathscr{L}_{\hbar,b}\right) |\langle u|v\rangle_{\mathscr{L}}| \leqslant C(u) \left\|v\right\|_{L^2}\right\}$$

defined by

$$\forall v \in \mathrm{Dom}(\mathcal{L}), u \in \mathrm{Dom}\left(\mathscr{L}_{\hbar,b}\right), \langle u|v\rangle_{\mathscr{L}} =: \langle \mathcal{L}u|v\rangle$$

is closed and self adjoint. But this operator is equal to $(\mathscr{L}_{\hbar,b}, \mathrm{Dom}\left(\mathscr{L}_{\hbar,b}\right))$ because it coincides with $\mathscr{L}_{\hbar,b}$ on $\mathrm{Dom}\left(\mathscr{L}_{\hbar,b}\right)$ and the required inequality in the definition of $\mathrm{Dom}(\mathcal{L})$ is satisfied taking $C(u) := \left\|\mathcal{L}_{\hbar,bA}u\right\|_{L^2}$, thus $\mathrm{Dom}\left(\mathscr{L}_{\hbar,b}\right) = \mathrm{Dom}(L)$.
The continuity of $\mathrm{Dom}\left(\mathscr{L}_{\hbar,b}\right) \hookrightarrow L^2(\Omega)$ has been proved in (II.16). Then, we have the canonical embeddings

$$\left(\mathrm{Dom}\left(\mathscr{L}_{\hbar,b}\right) \hookrightarrow L^2(\Omega)\right) = \left(H^1(\Omega) \hookrightarrow L^2(\Omega)\right) \circ \left(\mathrm{Dom}\left(\mathscr{L}_{\hbar,b}\right) \hookrightarrow H^1(\Omega)\right)$$

The boundary of Ω is Lipschitz so the left embedding is compact due to the Rellich–Kondrachov theorem and the right one is continuous from Lemma II.4. Thus, the composition is compact.

$H^2_{mp}(\Omega)$ contains the smooth and compactly supported functions, so it is dense in $L^2(\Omega)$. We can conclude using the Lax-Milgram theorem [5: Corollary 4.26] that the resolvent of $\mathscr{L}_{\hbar,b}$ is well defined and compact. Applying the spectral theorem to the resolvent of $\mathscr{L}_{\hbar,b}$ proves that its spectrum is punctual and $L^2(\Omega)$ is a Hilbertian direct sum of eigenspaces of $\mathscr{L}_{\hbar,b}$. The same conclusions also holds for the N-body Hamiltonian (I.6) since the magnetic Laplacian is of dominant order in it.

$\mathcal{N}$ inherits the properties of $\mathscr{L}_{\hbar,b}$ in Corollary II.5 and it is well known that

$$\mathrm{sp}(\mathcal{N}) = \mathbb{N}$$

Notation II.6: *Landau levels*

We define the n^{th} Landau level as the eigenspace associated to $n \in \mathbb{N}$:

$$\mathrm{nLL} := \left\{\psi \in \mathrm{Dom}\left(\mathscr{L}_{\hbar,b}\right) \text{ such that } \mathcal{N}\psi = n\psi\right\}$$

The ground level, denoted LLL for *Lowest Landau Level* has energy $E_0 = \hbar b$.

It is well known that the Landau levels are isomorphic, and that the operator $a^\dagger/\sqrt{n+1}$ is a unitary mapping from nLL to (n+1)LL of inverse $a/\sqrt{n+1}$. Using the creation operator, if we find a basis of the lowest Landau level we will be able to generate a basis of any Landau level. This is why, in the next session we start with a study of the lowest level.

II.3 Lowest Landau level

We start with the following characterisation:

Proposition II.7: *Lowest Landau level*

Denote by $\mathcal{O}(\Omega)$ the set of holomorphic functions, then

$$\mathrm{LLL} \subset \ker(a) \subset \mathcal{O}(\Omega) e^{-\frac{\phi}{l_b^2}}$$

where ϕ is defined in (II.2).

Proof:

Take $\psi \in \mathrm{LLL}$, then $a\psi = 0$, using (II.6)

$$(\pi_y - i\pi_x)\psi = (i\hbar\partial_y + \hbar\partial_x + bA_y - ibA_x)\psi = (2\hbar\partial_{\bar{z}} - ibA)\psi = 2(\hbar\partial_{\bar{z}} + b\partial_{\bar{z}}\phi)\psi = 0$$

So $\exists f \in \mathcal{O}(\Omega)$ such that $\psi = fe^{-\frac{\phi}{l_b^2}}$.

This proves that the zeros of a wave-function of LLL are given by the zeros of an holomorphic function. Since zeros of an holomorphic function must be isolated, the compactness of the domain implies that wave-functions have a finite number of zeros. Actually, the next proposition says that this number of zeros is d defined in (I.9), and therefore independent of the choice of wave-function. One can see [1: section 1] as a reference.

Proposition II.8: *Zeros of LLL wave-functions*

If $\psi \in \mathrm{LLL}$, then ψ has exactly d zeros inside Ω.

Proof:

Let

$$\psi := fe^{-\frac{\phi}{l_b^2}} \in \mathrm{LLL}$$

and n_0 be the number of zeros of ψ which can be computed with the logarithmic derivative through

$$n_0 = \frac{1}{2i\pi}\int_{\partial\Omega} \partial_z ln(f)dz = \frac{1}{2i\pi}\int_{\partial\Omega} \frac{\partial_z f}{f}dz \tag{II.19}$$

With Definitions (II.2) and (I.3)

$$A - T_{z_0}A = \begin{pmatrix} -\partial_y\phi + T_{z_0}\partial_y\phi \\ \partial_x\phi - T_{z_0}\partial_x\phi \end{pmatrix} = l_b^2\nabla\varphi_{z_0} \tag{II.20}$$

If $\tau_{z_0}\psi = \psi$, the boundary condition on f is

$$f = e^{\frac{\phi}{l_b^2}}\psi = e^{\frac{\phi}{l_b^2}+i\varphi_{z_0}} T_{z_0}\psi = e^{\frac{\phi - T_{z_0}\phi}{l_b^2}+i\varphi_{z_0}} T_{z_0} f \tag{II.21}$$

Using equation (II.20), we get

$$l_b^2 \partial_z \varphi_{z_0} = l_b^2 \frac{\partial_x \varphi_{z_0} - i\partial_y \varphi_{z_0}}{2} = \frac{-i\partial_x - \partial_y}{2}\phi + T_{z_0}\frac{i\partial_x + \partial_y}{2}\phi = -i\partial_z\phi + iT_{z_0}\partial_z\phi \tag{II.22}$$

With equations (II.21) and (II.22), the boundary condition on $\partial_z f$ takes the form

$$l_b^2 \partial_z f = e^{\frac{\phi - T_{z_0}\phi}{l_b^2}+i\varphi_{z_0}} \left(T_{z_0} l_b^2 \partial_z f + 2\left(\partial_z\phi - T_{z_0}\partial_z\phi\right) T_{z_0} f\right)$$

As for $l_b^2 \dfrac{\partial_z f}{f}$, we have

$$l_b^2 \frac{\partial_z f}{f} = T_{z_0} l_b^2 \frac{\partial_z f}{f} + 2\partial_z(\phi - T_{z_0}\phi)$$

Finally, we can compute the integral in (II.19):

$$\begin{aligned}
2i\pi l_b^2 n_0 &= l_b^2 \int_0^L \left(\frac{\partial_z f}{f}(t) + i\frac{\partial_z f}{f}(L+it) - \frac{\partial_z f}{f}(t+iL) - i\frac{\partial_z f}{f}(it)\right) dt \\
&= -\int_0^L 2\partial_z(\phi - T_{iL}\phi)(t+iL)dt + i\int_0^L 2\partial_z(\phi - T_L\phi)(L+it)dt \\
&= \int_0^L \big(2\partial_z\phi(t) - 2\partial_z\phi(t+iL) + 2i\partial_z\phi(L+it) - 2i\partial_z\phi(it)\big)dt \\
&= \int_{\partial\Omega} 2\partial_z\phi dz = i\int_{\partial\Omega} \overline{A(z)}dz
\end{aligned} \tag{II.23}$$

where the last equality comes from Equation (II.6) which implies $2\partial_z\varphi = i\overline{A}$. But the integral of $\overline{A}$ over a complex loop can be related to the integral of A over a loop in $\mathbb{R}^2$. Let $\gamma \coloneqq \gamma_x + i\gamma_y : [0,1] \to \partial\Omega$ be a parametrization of $\partial\Omega$:

$$\begin{aligned}
\int_{\partial\Omega} \overline{A(z)}dz &= \int_0^1 \overline{A(\gamma(u))}\gamma'(u)du \\
&= \int_0^1 A(\gamma(u))\cdot\gamma'(u)du + i\int_0^1 \left(A_x(\gamma(u))\gamma_y'(u) - A_y(\gamma(u))\gamma_x'(u)\right) du \\
&= \int_{\partial\Omega} A\cdot dl + i\int_0^1 \left(A_x(\gamma(u))\gamma_y'(u) - A_y(\gamma(u))\gamma_x'(u)\right) du = \int_{\partial\Omega} A\cdot dl + i\int_{\partial\Omega} A^\perp \cdot dl
\end{aligned}$$

$$= \int_{\partial\Omega} A \cdot dl + i \int_{\Omega} \nabla \cdot A = \int_{\partial\Omega} A \cdot dl$$

Combining this with (II.23) and (II.10) we get

$$2\pi n_0 = \frac{l_b^2}{L^2}$$

so with (I.9), we conclude that $n_0 = d$.

An elliptic function can be expressed as a rational function in terms of the Weierstrass elliptic function and its derivative. In the case of magnetic periodic boundary conditions we will see that we have a decomposition in terms of theta functions from which we construct our basis of LLL. A similar proof of the following proposition can be found in [2] or [4: Chapter V Theorem 8].

Proposition II.9: *Degeneracy of Landau levels*

Landau levels have a finite degeneracy and

$$\forall n \in \mathbb{N}, \mathrm{Dim}(\mathrm{nLL}) = d$$

Proof:

Since all Landau levels are isomorphic, a proof for the lowest Landau level is sufficient. The Landau level dimension is independent of the gauge, for simplicity we use the Landau gauge in this proof.

In Landau gauge (II.4), the boundary condition on f in Equation (II.21) becomes

$$f(z) = e^{\frac{y^2-(y-y_0)^2}{2l_b^2} - i\frac{y_0 x}{l_b^2}} f(z - z_0) = e^{-\frac{y_0^2}{2l_b^2} - i\frac{y_0 z}{l_b^2}} f(z - z_0)$$

using equation (I.9), this can be rewritten as

$$\begin{aligned} f(z-L) &= f(z) \\ f(z-iL) &= f(z) e^{\frac{L^2}{2l_b^2} + i\frac{Lz}{l_b^2}} = f(z) e^{\pi d + 2i\pi d\frac{z}{L}} \end{aligned} \tag{II.24}$$

The periodicity along the real axis allows us to expand in Fourier series:

$$f(z) = \sum_{k\in\mathbb{Z}} c_k(y) e^{2i\pi k\frac{x}{L}} \tag{II.25}$$

The holomorphy of f implies that

$$2\partial_{\bar{z}} f = 0 = \sum_k \left(\frac{2i\pi k}{L} c_k(y) + i c_k'(y) \right) e^{2i\pi k\frac{x}{L}}$$

Solving the EDO in y that we obtain by identifying the Fourier coefficients gives

$$c_k(y) = c_k(0) e^{-2\pi k\frac{y}{L}}$$

Plugging this in (II.25) leads to

$$f(z) = \sum_k c_k(0) e^{2i\pi k \frac{z}{L}} \tag{II.26}$$

Finally we impose the pseudo-periodicity along the imaginary axis (II.24):

$$\sum_k c_k(0) e^{2i\pi k \frac{z}{L} + 2\pi k} = \sum_k c_k(0) e^{\pi d + 2i\pi \frac{z}{L}(k+d)}$$

Identifying the Fourier coefficients

$$c_k e^{2\pi k} = c_{k-d} e^{\pi d} \tag{II.27}$$

implies that they are only d independent Fourier coefficients.

We will prove in the next Section that the above relation between Fourier coefficients gives a decomposition of f in terms of theta functions.

II.4 Magnetic periodic eigenfunctions

This Section contains computations of the eigenfunctions of $\mathscr{L}_{\hbar,b}$ with magnetic periodic boundary conditions.

Notation II.10: *Theta functions*

Let τ be a complex parameter in the upper half plane, we define

$$\theta(z,\tau) \coloneqq \sum_{k\in\mathbb{Z}} e^{i\pi\tau k^2 + 2i\pi k z}$$

Theta functions are pseudo-periodic:

$$\begin{aligned} \theta(z+1,\tau) &= \theta(z,\tau) \\ \theta(z+\tau,\tau) &= \theta(z,\tau) e^{-i\pi(\tau+2z)} \end{aligned}$$

We complete the computation of Proposition II.9 and express the wave-functions of the magnetic Laplacian (I.1) in term of theta functions.

Proposition II.11: *LLL wave-functions*

The following family, indexed by $l \in [\![0, d-1]\!]$, is an orthonormal basis of the lowest Landau level in Landau gauge:

$$\psi_{0l}(z) \coloneqq \frac{\pi^{-\frac{1}{4}}}{\sqrt{L l_b}} e^{2i\pi l \frac{x}{L}} \sum_{k\in\mathbb{Z}} e^{2i\pi k d \frac{x}{L} - \frac{1}{2 l_b^2}\left(y + kL + l\frac{L}{d}\right)^2} \tag{II.28}$$

$$= \frac{\pi^{-\frac{1}{4}}}{\sqrt{L l_b}} e^{-\frac{\pi l^2}{d} - \frac{y^2}{2 l_b^2} + 2i\pi l \frac{z}{L}} \theta\left(d\frac{z}{L} + il, id\right) \tag{II.29}$$

Proof:

With the same notation as in the proof of Proposition II.9, we prove by induction that

$$c_{l+kd} = c_l e^{-2\pi kl - \pi dk^2} \tag{II.30}$$

This is satisfied for $k = 0$. Using (II.27) and assuming the relation (II.30) for $k \in \mathbb{N}$,

$$c_{l+(k+1)d} = c_{l+kd}e^{\pi d - 2\pi(l+[k+1]d)} = c_l e^{-2\pi kl - \pi dk^2 - \pi d - 2\pi(l+kd)} = c_l e^{-2\pi(k+1)l - \pi d(k+1)^2}$$

at this point we are done for $k \geqslant 0$, but $c_l = c_{l+kd}e^{2\pi kl + \pi dk^2}$ so

$$c_{l-kd} = c_l e^{2\pi k(l-kd)+\pi dk^2} = c_l e^{2\pi kl - \pi dk^2} = c_l e^{-2\pi(-k)l - \pi d(-k)^2}$$

and we obtain (II.30) for $k \leqslant 0$. Inserting (II.30) in (II.26) gives

$$\begin{aligned} f(z) &= \sum_{l=0}^{d-1} e^{2i\pi l\frac{z}{L}} \sum_{k\in\mathbb{Z}} c_{l+kd}e^{2i\pi kd\frac{z}{L}} = \sum_{l=0}^{d-1} c_l e^{2i\pi l\frac{z}{L}} \sum_{k\in\mathbb{Z}} e^{-2\pi kl - \pi dk^2 + 2i\pi kd\frac{z}{L}} \\ &= \sum_{l=0}^{d-1} c_l e^{2i\pi l\frac{z}{L}} \sum_{k\in\mathbb{Z}} e^{i\pi(id)k^2 + 2i\pi k\left(d\frac{z}{L}+il\right)} = \sum_{l=0}^{d-1} c_l e^{2i\pi l\frac{z}{L}} \theta\left(d\frac{z}{L} + il, id\right) \end{aligned}$$

We found a family of the lowest Landau level indexed by $l \in [\![0, d-1]\!]$ with expression in Landau gauge. We need to normalise this family and to verify that the wave-functions are orthogonal. We start by proving that

$$e^{-\frac{y^2}{2l_b^2}+2i\pi l\frac{z}{L}}\theta\left(d\frac{z}{L}+il, id\right) = e^{\frac{\pi l^2}{d}+2i\pi l\frac{x}{L}} \sum_{k\in\mathbb{Z}} e^{2i\pi kd\frac{x}{L} - \frac{1}{2l_b^2}\left(y+kL+l\frac{L}{d}\right)^2}$$

Using (I.9),

$$\begin{aligned} e^{-\frac{y^2}{2l_b^2}+2i\pi l\frac{z}{L}}\theta\left(d\frac{z}{L}+il, id\right) &= e^{-\frac{y^2}{2l_b^2}+2i\pi l\frac{z}{L}} \sum_{k\in\mathbb{Z}} e^{-\pi dk^2 - 2\pi kl + 2i\pi kd\frac{z}{L}} \\ &= e^{2i\pi l\frac{x}{L}} \sum_{k\in\mathbb{Z}} e^{-\frac{y^2}{2l_b^2} - 2\pi l\frac{y}{L} - \pi dk^2 - 2\pi kl - 2\pi kd\frac{y}{L} + 2i\pi kd\frac{x}{L}} \\ &= e^{2i\pi l\frac{x}{L}} \sum_{k\in\mathbb{Z}} e^{2i\pi kd\frac{x}{L} - \frac{1}{2l_b^2}\left(y^2 + 2ly\frac{L}{d} + L^2k^2 + 2kl\frac{L^2}{d} + 2kLy\right)} \\ &= e^{\frac{\pi l^2}{d}+2i\pi l\frac{x}{L}} \sum_{k\in\mathbb{Z}} e^{2i\pi kd\frac{x}{L} - \frac{1}{2l_b^2}\left(y+kL+l\frac{L}{d}\right)^2} \end{aligned}$$

Finally we check the orthonormality. Let $0 \leqslant l \leqslant q < d$,

$$\begin{aligned} \langle\psi_{0l}|\psi_{0q}\rangle &= \frac{1}{\sqrt{\pi}Ll_b} \sum_{k,p\in\mathbb{Z}} \int_\Omega e^{2i\pi(q-l)\frac{x}{L} + 2i\pi d(p-k)\frac{x}{L} - \frac{1}{2l_b^2}\left(y+kL+l\frac{L}{d}\right)^2 - \frac{1}{2l_b^2}\left(y+pL+q\frac{L}{d}\right)^2} dxdy \\ &= \frac{1}{\sqrt{\pi}Ll_b} \sum_{k,p\in\mathbb{Z}} \int_0^L e^{2i\pi(q-l+d[p-k])\frac{x}{L}} dx \int_0^L e^{-\frac{1}{2l_b^2}\left(y+kL+l\frac{L}{d}\right)^2 - \frac{1}{2l_b^2}\left(y+pL+q\frac{L}{d}\right)^2} dy \end{aligned}$$

Since $0 \leqslant q-l<d$ we have a simplification:

$$\int_0^L e^{2i\pi(q-l+d[p-k])\frac{x}{L}}dx = L\delta_{lq}\delta_{kp}$$

Therefore

$$\langle\psi_{0l}|\psi_{0q}\rangle = \delta_{lq}\frac{1}{\sqrt{\pi}l_b}\sum_{k\in\mathbb{Z}}\int_0^L e^{-\frac{1}{l_b^2}\left(y+kL+l\frac{L}{d}\right)^2}dy = \delta_{lq}$$

One can check that the above wave-functions satisfy the boundary conditions (II.5). Using (II.28) we observe the L-periodicity along the real axis. Along the imaginary axis we increment the index k by 1:

$$\psi_{0l}(z+iL) = \frac{\pi^{-\frac{1}{4}}}{\sqrt{Ll_b}}e^{2i\pi l\frac{x}{L}}\sum_{k\in\mathbb{Z}}e^{2i\pi kd\frac{x}{L}-\frac{1}{2l_b^2}\left(y+(k+1)L+l\frac{L}{d}\right)^2} = e^{-2i\pi d\frac{x}{L}}\psi_{0l}(z)$$

and obtain the magnetic periodic boundary conditions in Landau gauge (II.4). The lowest Landau level is generated by successive magnetic translations:

Corollary II.12: *Generation of nLL with magnetic translations*

If $l \in [\![0, d-1]\!]$,

$$\psi_{0l} = \left(\tau_{-i\frac{L}{d}}\right)^l\psi_{00} = \tau_{-il\frac{L}{d}}\psi_{00} \tag{II.31}$$

Proof:

$\varphi_{,Lan} = -\dfrac{y_0 x}{l_b^2}$ defined in (II.4) is linear in z_0 and independent of y, thus with (II.1),

$$\tau_{-i\frac{L}{d}} = e^{i\frac{Lx}{dl_b^2}}T_{-i\frac{L}{d}} = e^{2i\pi\frac{x}{L}}T_{-i\frac{L}{d}}$$

and

$$\tau_{-il\frac{L}{d}} = \left(\tau_{-i\frac{L}{d}}\right)^l = e^{2i\pi l\frac{x}{L}}\left(T_{-i\frac{L}{d}}\right)^l$$

With this, (II.28) can be written as (II.31).

In order to obtain a full basis of L^2, we only need to apply successively $a^\dagger$ to generate the Landau levels and $\tau_{-i\frac{L}{d}}$ to generate the wave-functions inside a Landau level. The successive applications of $a^\dagger$ bring out Hermite polynomials.

Notation II.13: *Hermite polynomials*

For $n \in \mathbb{N}$, we define the n^{th} Hermite polynomial by

$$H_n := (-1)^n e^{x^2} \left(\frac{d}{dx}\right)^n e^{-x^2} \tag{II.32}$$

We recall some basis properties of Hermite polynomials that will be useful: $H_0 = 1$ and for all $n \in \mathbb{N}$ we have the relations

$$H_{n+1} = 2xH_n - H_n' \tag{II.33}$$
$$H_n(-x) = (-1)^n H_n(x) \tag{II.34}$$
$$H_n' = 2nH_{n-1} \tag{II.35}$$

Using this, we can give expressions for the full basis.

Proposition II.14: *nLL wave-functions*

The following family indexed by $(n, l) \in \mathbb{N} \times [\![0, d-1]\!]$ is a Hilbert basis of eigenfunctions of $\mathscr{L}_{\hbar,b}$ in Landau gauge:

$$\begin{aligned}\psi_{nl} &:= \frac{a^{\dagger n}}{\sqrt{n!}} \left(\tau_{-i\frac{L}{d}}\right)^l \psi_{00} \\ &= \frac{c_n}{\sqrt{Ll_b}} e^{2i\pi l \frac{x}{L}} \sum_{k\in\mathbb{Z}} H_n\left(\frac{1}{l_b}\left[y + kL + l\frac{L}{d}\right]\right) e^{2i\pi kd\frac{x}{L} - \frac{1}{2l_b^2}\left(y+kL+l\frac{L}{d}\right)^2}\end{aligned} \tag{II.36}$$

with the normalization factor

$$c_n := \frac{1}{\pi^{\frac{1}{4}}\sqrt{n!}} \left(\frac{-i}{\sqrt{2}}\right)^n$$

Proof:

Due to (II.13) the order of application of the magnetic translations and the creation operators does not matter. Also, due to Corollary II.12, it is enough to deal with $l = 0$. In order to lighten the computations we define the dimensionless variable

$$y_k = \frac{y + kL}{l_b}$$

We proceed by induction in n. The initialisation is given by $H_0 = 1$ and (II.28):

$$\psi_{00}(z) = \frac{c_0}{\sqrt{Ll_b}} \sum_{k\in\mathbb{Z}} H_0(y_k) e^{2i\pi kd\frac{x}{L} - \frac{y_k^2}{2}}$$

In complex notation $a^\dagger$ (II.11) becomes in Landau gauge

$$a^\dagger = \frac{-2\hbar\partial_z - iby}{\sqrt{2\hbar b}} = -\sqrt{2}l_b\partial_z - \frac{iy}{\sqrt{2}l_b} = \frac{-i}{\sqrt{2}}\left(-l_b(i\partial_x + \partial_y) + \frac{y}{l_b}\right)$$

so

$$\frac{a^{\dagger n+1}}{\sqrt{(n+1)!}}\psi_{00}(z) = \frac{a^\dagger}{\sqrt{n+1}}\psi_{n,0} = \frac{c_{n+1}}{\sqrt{Ll_b}}\sum_{k\in\mathbb{Z}}\left(-l_b(i\partial_x+\partial_y)+\frac{y}{l_b}\right)H_n(y_k)e^{2i\pi kd\frac{x}{L}-\frac{y_k^2}{2}}$$
$$= \frac{c_{n+1}}{\sqrt{Ll_b}}\sum_{k\in\mathbb{Z}}\left(\left[2\pi k\frac{dl_b}{L}+y_k+\frac{y}{l_b}\right]H_n(y_k)-H_n'(y_k)\right)e^{2i\pi kd\frac{x}{L}-\frac{y_k^2}{2}}$$
$$= \frac{c_{n+1}}{\sqrt{Ll_b}}\sum_{k\in\mathbb{Z}}\left[2y_kH_n(y_k)-H_n'(y_k)\right]e^{2i\pi kd\frac{x}{L}-\frac{y_k^2}{2}}$$
$$= \frac{c_{n+1}}{\sqrt{Ll_b}}\sum_{k\in\mathbb{Z}}H_{n+1}(y_k)e^{2i\pi kd\frac{x}{L}-\frac{y_k^2}{2}}$$

where the last equality uses (II.33).

As expected with our boundary conditions the modulus of the wave-functions:

$$|\psi_{nl}| = \left|\frac{c_n}{\sqrt{Ll_b}}\sum_{k\in\mathbb{Z}}H_n\left(\frac{1}{l_b}\left[y+kL+l\frac{L}{d}\right]\right)e^{2i\pi kd\frac{x}{L}-\frac{1}{2l_b^2}\left(y+kL+l\frac{L}{d}\right)^2}\right| \tag{II.37}$$

is periodic on the lattice $L\mathbb{Z}^2$, but the periodicity along the real axis is even shorter. Indeed we see in (II.37) that $|\psi_{nl}|$ is L/d-periodic in x.

We can write another useful form of equation (II.36) using the Poisson summation formula. The advantage of the expression in Proposition II.16 is the fact that the index l is decoupled from the polynomials and the Gaussian factors which is not the case in (II.36). This will simplify the computation of the Landau level's projector when we will sum over l in (III.5).

Notation II.15: *Fourier transform*

We use the convention

$$\mathcal{F}g(\nu) := \hat{g}(\nu) := \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} g(x)e^{-i\nu x}dx$$

for which $\mathcal{F}$ is unitary on $L^2(\mathbb{R})$. And denote the Hermite function

$$h_n(x) := H_n(x)e^{-\frac{x^2}{2}} \tag{II.38}$$

In this convention the Poisson summation formula is

$$\sum_{k\in\mathbb{Z}} g(k) = \sqrt{2\pi}\sum_{k\in\mathbb{Z}}\hat{g}\left(2\pi k\right) \tag{II.39}$$

h_n are the eigenfunctions of the one dimensional harmonic oscillator and of the Fourier transform:

$$\mathcal{F}h_n = (-i)^n h_n \tag{II.40}$$

with the following normalization

$$\|h_n\|_{L^2}^2 = \sqrt{\pi}2^n n! \tag{II.41}$$

With this we are ready for the next computation:

Proposition II.16: *Poisson summation of eigenfunctions*

$$\psi_{nl}(z) = \widetilde{c}_n \frac{\sqrt{l_b}}{L^{\frac{3}{2}}} e^{-i\frac{xy}{l_b^2}} \sum_{k\in\mathbb{Z}} H_n\left(\frac{1}{l_b}\left[x + k\frac{L}{d}\right]\right) e^{-2i\pi k\left(\frac{y}{L}+\frac{l}{d}\right)-\frac{1}{2l_b^2}\left(x+k\frac{L}{d}\right)^2}$$

with the normalization factor

$$\widetilde{c}_n \coloneqq \frac{\pi^{\frac{1}{4}}(-1)^n 2^{\frac{1-n}{2}}}{\sqrt{n!}}$$

Proof:

We start from (II.36) expressed in terms of h_n:

$$\psi_{nl}(z) = \frac{c_n}{\sqrt{Ll_b}} e^{2i\pi l\frac{x}{L}} \sum_{k\in\mathbb{Z}} h_n\left(\frac{1}{l_b}\left[y + kL + l\frac{L}{d}\right]\right) e^{2i\pi kd\frac{x}{L}}$$

Define

$$g(u) \coloneqq h_n\left(\frac{1}{l_b}\left[y + uL + l\frac{L}{d}\right]\right) e^{2i\pi du\frac{x}{L}}$$

so we have

$$\psi_{nl}(z) = \frac{c_n}{\sqrt{Ll_b}} e^{2i\pi l\frac{x}{L}} \sum_{k\in\mathbb{Z}} g(k) \tag{II.42}$$

in order to apply the Poisson summation formula to g. To do so, we compute $\hat{g}$ with a change of variable and equations (I.9) and (II.40):

$$\begin{aligned}
\hat{g}(\nu) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} h_n\left(\frac{1}{l_b}\left[y + uL + l\frac{L}{d}\right]\right) e^{-iu\left(\nu - 2\pi d\frac{x}{L}\right)} du \\
&= \frac{l_b}{L\sqrt{2\pi}} e^{i\left(\frac{y}{L}+\frac{l}{d}\right)\left(\nu-2\pi d\frac{x}{L}\right)} \int_{\mathbb{R}} h_n(u) e^{-i\frac{ul_b}{L}\left(\nu-2\pi d\frac{x}{L}\right)} du \\
&= \frac{l_b}{L\sqrt{2\pi}} e^{i\left(\frac{y}{L}+\frac{l}{d}\right)\left(\nu-2\pi d\frac{x}{L}\right)} \widehat{h_n}\left(\frac{l_b}{L}\nu - \frac{x}{l_b}\right) = \frac{(-i)^n l_b}{L} e^{i\left(\frac{y}{L}+\frac{l}{d}\right)\left(\nu-2\pi d\frac{x}{L}\right)} h_n\left(\frac{l_b}{L}\nu - \frac{x}{l_b}\right)
\end{aligned}$$

so by using (I.9) again:

$$\begin{aligned}
\hat{g}(2\pi k) &= \frac{(-i)^n l_b}{L} e^{i\left(\frac{y}{L}+\frac{l}{d}\right)\left(2\pi k-2\pi d\frac{x}{L}\right)} h_n\left(2\pi k\frac{l_b}{L} - \frac{x}{l_b}\right) \\
&= \frac{(-i)^n l_b}{L} e^{-i\frac{xy}{l_b^2}-2i\pi l\frac{x}{L}} e^{2i\pi k\left(\frac{y}{L}+\frac{l}{d}\right)} h_n\left(\frac{1}{l_b}\left[k\frac{L}{d} - x\right]\right)
\end{aligned}$$

To conclude the computation we insert this after applying the Poisson summation formula (II.39) to (II.42):

$$\psi_{nl}(z) = \frac{c_n}{\sqrt{Ll_b}} e^{2i\pi l\frac{x}{L}} \sqrt{2\pi} \sum_{k\in\mathbb{Z}} \hat{g}\left(2\pi k\right)$$

$$= \frac{c_n}{\sqrt{Ll_b}} \cdot \frac{\sqrt{2\pi}(-i)^n l_b}{L} e^{-i\frac{xy}{l_b^2}} \sum_{k\in\mathbb{Z}} H_n\left(\frac{1}{l_b}\left[k\frac{L}{d} - x\right]\right) e^{2i\pi k\left(\frac{y}{L}+\frac{l}{d}\right)-\frac{1}{2l_b^2}\left(k\frac{L}{d}-x\right)^2}$$

$$= \tilde{c}_n \frac{\sqrt{l_b}}{L^{\frac{3}{2}}} e^{-i\frac{xy}{l_b^2}} \sum_{k\in\mathbb{Z}} H_n\left(\frac{1}{l_b}\left[x + k\frac{L}{d}\right]\right) e^{-2i\pi k\left(\frac{y}{L}+\frac{l}{d}\right)-\frac{1}{2l_b^2}\left(x+k\frac{L}{d}\right)^2}$$

by changing the sum index k to $-k$, using the parity of Hermite polynomials and the relation

$$c_n\sqrt{2\pi}(-i)^n = \tilde{c}_n$$

III Projectors on Landau levels

From the construction of a $L^2(\Omega)$ basis adapted to Landau levels, we define the projectors on Landau levels in Notation III.1. Since the phase space is $\mathbb{N} \times \Omega$ we also want to localise the projectors in space. Then we prove some properties of the projector that will be needed for the semi-classical analysis. In Proposition III.4 we give an equivalent for the diagonal of the projector's integral kernel, and in Corollary III.6 an equivalent for its trace.

III.1 nLL projectors

Notation III.1: *Projectors*

The orthogonal projector on nLL is

$$\Pi_n \coloneqq \sum_{l=0}^{d-1} |\psi_{nl}\rangle \langle \psi_{nl}|$$

Let $g \in C^\infty(\mathbb{R}^2, \mathbb{R}_+)$ radial with support included in the ball $B(0, L/2)$ such that $\|g\|_{L^2} = 1$. Let $\lambda \geqslant 1$, define the localizer $g_\lambda \in C^\infty(\mathbb{T})$ defined by

$$g_\lambda(x) \coloneqq \begin{cases} \lambda g(\lambda x) & \text{if } x \in B\left(0, \frac{L}{2\lambda}\right) \\ 0 & \text{else} \end{cases}$$

Note that

$$\|g_\lambda\|_{L^2} = 1$$

Then define the localised projector

$$\Pi_{n,R} \coloneqq g_\lambda(\bullet - R)\Pi_n g_\lambda(\bullet - R) \tag{III.1}$$

We assume the following scaling for $\lambda \coloneqq (\lambda_N)_N$:

$$1 \ll \lambda \ll \frac{N^{-\frac{1}{2}}}{\hbar^2} \tag{III.2}$$

This localised projector was introduced by Lieb and Yngvason in [13] and [22] where it has been called coherent operator. We take the bounds (III.2) in order to have $g_\lambda^2 \xrightharpoonup{*} \delta$ so the projector is well localised and

$$\frac{\hbar^2}{l_b}\lambda = \hbar b \lambda l_b \ll 1$$

This is necessary because $\hbar b \lambda l_b$ is the order of some error terms coming from the kinetic energy (for example in Proposition VI.7).

Property III.2

Π_n and $\Pi_{n,R}$ are positive and satisfy the following resolution of identity:

$$\sum_{n\in\mathbb{N}} \Pi_n = \mathrm{Id} \qquad \int_{\mathbb{N}\times\Omega} \Pi_X d\eta(X) = \mathrm{Id} \tag{III.3}$$

Proof:

Let $\psi \in L^2(\Omega)$

$$\langle\psi|\Pi_{n,R}\psi\rangle = \langle g(\bullet - R)\psi|\Pi_n g(\bullet - R)\psi\rangle \geqslant 0$$

because Π_n is a projector. The first resolution of the identity is a consequence of the completeness of the basis and implies the second one:

$$\int_{\mathbb{N}\times\Omega} \Pi_X d\eta(X) = \int_\Omega g_\lambda(\cdot - R)\left(\sum_{n\in\mathbb{N}} \Pi_n\right) g_\lambda(\cdot - R) dR = \int_\Omega g_\lambda(\cdot - R)^2 dR = \mathrm{Id}$$

III.2 Integral kernels of the projectors

The computations of Section II lead to the following expression of the nLL-projector's kernel.

Proposition III.3: *Kernel of the nLL-projector*

With notation (II.38) and $x \coloneqq x_1 + ix_2$, $y = y_1 + iy_2$,

$$\Pi_n(x,y) = \frac{1}{\|h_n\|_{L^2}^2 L l_b} e^{i\frac{y_1y_2 - x_1x_2}{l_b^2}} \sum_{k,q\in\mathbb{Z}} H_n\left(\frac{1}{l_b}\left[x_1 + k\frac{L}{d}\right]\right) H_n\left(\frac{1}{l_b}\left[y_1 + qL + k\frac{L}{d}\right]\right)$$
$$\cdot e^{2i\pi k\frac{y_2 - x_2}{L} + 2i\pi dq\frac{y_2}{L} - \frac{1}{2l_b^2}\left(x_1 + k\frac{L}{d}\right)^2 - \frac{1}{2l_b^2}\left(y_1 + qL + k\frac{L}{d}\right)^2} \tag{III.4}$$

Proof:

From Proposition II.16:

$$\Pi_n(x,y) = \tilde{c}_n^2 \frac{l_b}{L^3} e^{i\frac{y_1y_2 - x_1x_2}{l_b^2}} \sum_{l=0}^{d-1} \sum_{k,p\in\mathbb{Z}} H_n\left(\frac{1}{l_b}\left[x_1 + k\frac{L}{d}\right]\right) H_n\left(\frac{1}{l_b}\left[y_1 + p\frac{L}{d}\right]\right)$$
$$\cdot e^{2i\pi l\frac{p-k}{d} + 2i\pi\frac{py_2 - kx_2}{L} - \frac{1}{2l_b^2}\left(x_1 + k\frac{L}{d}\right)^2 - \frac{1}{2l_b^2}\left(y_1 + p\frac{L}{d}\right)^2}$$

Then, we use

$$\sum_{l=0}^{d-1} e^{2i\pi l\frac{p-k}{d}} = d\mathbb{1}_{p=k\,(\mathrm{mod}\,d)} \tag{III.5}$$

to conclude. The computation of the normalization factor can be performed with (II.41):

$$\tilde{c}_n^2 \frac{l_b}{L^3} \cdot d = \frac{\tilde{c}_n^2}{2\pi} \cdot \frac{1}{Ll_b} = \frac{1}{\sqrt{\pi} 2^n n!} \cdot \frac{1}{Ll_b} = \frac{1}{\|h_n\|_{L^2}^2 Ll_b}$$

The above simplification for the sum in l is the reason why we used the Poisson formula on wave-functions. The argument does not work on the expression in (II.36) since the Gaussian terms depend on l.

If we consider the same setup on the whole space $\mathbb{R}^2$ instead of Ω, the expression of the projector in Landau gauge becomes (see [9: Section 3.2]):

$$\Pi_0^\infty(x,y) \coloneqq \frac{1}{2\pi l_b^2} e^{-\frac{|x-y|}{2l_b^2} + i\frac{\mathrm{Im}[x\overline{y}]}{4l_b^2} + i\frac{y_1 y_2 - x_2 x_1}{2l_b^2}}$$

The next proposition states that the diagonal of the projector's kernel on Ω converges to that of the projector on the whole space. This is expected since the limit is equivalent to a scaling where the size of the domain goes to infinity. This result will be important to estimate the trace of $\Pi_{n,R}$.

Proposition III.4: *Convergence of the integral kernel*

The kernel (III.4) satisfies

$$\Pi_n(z,z) \underset{b\to\infty}{\sim} \frac{1}{2\pi l_b^2}$$

uniformly in z with the convergence rate

$$\left\| \Pi_n(z,z) - \frac{1}{2\pi l_b^2} \right\|_{L^\infty} \leqslant \frac{C(n)}{Ll_b} \tag{III.6}$$

Moreover with notation (II.38),

$$\left\| (\mathscr{P}_{\hbar,b}\Pi_n)(z,z) - \frac{b}{l_b} \cdot \frac{1}{2\pi \|h_n\|_{L^2}^2} \int_{\mathbb{R}} \begin{pmatrix} ih_n'(u) \\ uh_n(u) \end{pmatrix} h_n(u) e^{-u^2} du \right\|_{L^\infty} \leqslant C(n) b \tag{III.7}$$

The proof needs the following technical lemma.

Lemma III.5

Let $m \in \mathbb{N}, c > 0$, the following series are uniformly bounded in α, a, b:

$$\forall \alpha \in \mathbb{R}_+, a, b \in [-1,1], \alpha \sum_{q\in\mathbb{Z}^*} |a+b+\alpha q|^m e^{-c(a+\alpha q)^2} \leqslant C(c,m) \tag{III.8}$$

$$\forall \alpha \in [0,1], a \in \mathbb{R}, b \in [-1,1], \alpha \sum_{k\in\mathbb{Z}} |a+b+\alpha k|^m e^{-c(a+\alpha k)^2} \leqslant C(c,m) \tag{III.9}$$

Moreover, if P_n, Q_n are complex polynomials of degree n, the function

$$\Xi(z) \coloneqq \sum_{k,q\in\mathbb{Z}} P_n\left(\frac{1}{l_b}\left[x+k\frac{L}{d}\right]\right) Q_n\left(\frac{1}{l_b}\left[x+qL+k\frac{L}{d}\right]\right) e^{2i\pi qd\frac{y}{L}-\frac{1}{2l_b^2}\left(x+k\frac{L}{d}\right)^2-\frac{1}{2l_b^2}\left(x+qL+k\frac{L}{d}\right)^2}$$

is of order $\frac{1}{l_b}$ and can be uniformly approximated as

$$\left\|\Xi(z)-\frac{L}{2\pi l_b}\int_{\mathbb{R}} P_n(u)Q_n(u)e^{-u^2}du\right\|_{L^\infty} \leqslant C(n) \tag{III.10}$$

Proof of Proposition III.4:

We start from (III.4):

$$\Pi_n(z,z)=\frac{1}{\|h_n\|_{L^2}^2 Ll_b}\sum_{k,q\in\mathbb{Z}} h_n\left(\frac{1}{l_b}\left[x+k\frac{L}{d}\right]\right) h_n\left(\frac{1}{l_b}\left[x+qL+k\frac{L}{d}\right]\right) e^{2i\pi qd\frac{y}{L}}$$

We apply Lemma III.5 and thus compute

$$\frac{1}{\|h_n\|_{L^2}^2 Ll_b}\int_{\mathbb{R}} h_n\left(\frac{x}{l_b}+2\pi u\frac{l_b}{L}\right)^2 du=\frac{1}{Ll_b}\cdot\frac{L}{2\pi l_b}=\frac{1}{2\pi l_b^2}$$

and obtain (III.6). Starting again from (III.4) and using notation (III.11), we compute in Landau gauge

$$\begin{aligned}
&(\mathscr{P}_{\hbar,b}\Pi_n)(x,y)\\
&=\begin{pmatrix} i\hbar\partial_{x_1}-bx_2\\ i\hbar\partial_{x_2}\end{pmatrix}\frac{1}{\|h_n\|_{L^2}^2 Ll_b}e^{i\frac{y_1y_2-x_1x_2}{l_b^2}}\sum_{k,q\in\mathbb{Z}} h_n\left(k_{b,x_1}\right)h_n\left(k_{b,y_1+qL}\right)\cdot e^{2i\pi k\frac{y_2-x_2}{L}+2i\pi dq\frac{y_2}{L}}\\
&=\frac{1}{\|h_n\|_{L^2}^2 Ll_b}e^{i\frac{y_1y_2-x_1x_2}{l_b^2}}\sum_{k,q\in\mathbb{Z}}\frac{\hbar}{l_b}\begin{pmatrix} ih_n'\left(k_{b,x_1}\right)\\ k_{b,x_1}h_n\left(k_{b,x_1}\right)\end{pmatrix}h_n\left(k_{b,y_1+qL}\right)\cdot e^{2i\pi k\frac{y_2-x_2}{L}+2i\pi dq\frac{y_2}{L}}
\end{aligned}$$

So

$$(\mathscr{P}_{\hbar,b}\Pi_n)(z,z)=\frac{b}{\|h_n\|_{L^2}^2 L}\sum_{k,q\in\mathbb{Z}}\begin{pmatrix} ih_n'\left(k_{b,x}\right)\\ k_{b,x}h_n\left(k_{b,x}\right)\end{pmatrix}h_n\left(k_{b,x+qL}\right)e^{2i\pi dq\frac{y}{L}}$$

and with Lemma III.5,

$$\left\|(\mathscr{P}_{\hbar,b}\Pi_n)(z,z)-\frac{L}{2\pi l_b}\cdot\frac{b}{\|h_n\|_{L^2}^2 L}\int_{\mathbb{R}}\begin{pmatrix} ih_n'(u)\\ uh_n(u)\end{pmatrix}h_n(u)e^{-u^2}du\right\|_{L^\infty}\leqslant C(n)b$$

Finally, we compare the trace of $\Pi_{n,R}$ to the trace of the projector on the whole space.

Corollary III.6: *Approximation of the projector's trace*

$$\left|\mathrm{Tr}\left[\Pi_{n,R}\right] - \frac{1}{2\pi l_b^2}\right| \leqslant \frac{C(n)}{l_b}$$

Proof:

This is a direct consequence of Proposition III.4 after integrating on $z \in \Omega$:

$$\mathrm{Tr}\left[\Pi_{n,R}\right] = \int_\Omega \Pi_{n,R}(z,z)dz = \frac{1}{2\pi l_b^2}\int_\Omega g_\lambda(z-R)^2 dz + \mathcal{O}\left(\frac{1}{l_b}\right) = \frac{1}{2\pi l_b^2} + \mathcal{O}\left(\frac{1}{l_b}\right)$$

We end this section with the proof of the technical Lemma.

Proof of Lemma III.5:

Let $\alpha \in \mathbb{R}_+, a, b \in [-1,1]$. If $q \geqslant 2$ then $q \leqslant 2(q-1)$ so

$$\forall u \in [q-1,q), \left|a+b+\alpha q\right|^m e^{-c(a+\alpha q)^2} \leqslant (2+2\alpha u)^m e^{-c(a+\alpha u)^2}$$

and

$$\alpha \sum_{q\geqslant 2} \left|a+b+\alpha q\right|^m e^{-c(a+\alpha q)^2} \leqslant \int_1^\infty (2+2\alpha u)^m e^{-c(a+\alpha u)^2} \leqslant \int_\mathbb{R} (2+2u)^m e^{-c(a+u)^2} du \leqslant C(c,m)$$

the term for $q = 1$ is

$$\alpha\left|a+b+\alpha\right|^m e^{-c(a+\alpha)^2} \leqslant C$$

for the negative q, we see that

$$\alpha \sum_{q\leqslant -1} \left|a+b+\alpha q\right|^m e^{-c(a+\alpha q)^2} = \alpha \sum_{q\geqslant 1} \left|-a-b+\alpha q\right|^m e^{-c(-a+\alpha q)^2} \leqslant C(c,m)$$

because $-a, -b \in [-1,1]$.
For (III.9), let $\alpha \in [0,1], a \in \mathbb{R}, b \in [-1,1]$. We see that the series is α-periodic in a so we can assume $0 \leqslant a \leqslant 1$ and use (III.8) and for $k = 0$:

$$\alpha\left|a+b\right|^m e^{-ca^2} \leqslant 2^m$$

Now we use this result to prove the approximation of Ξ. Due to the Gaussian factor, all terms for which $q \neq 0$ have a fair chance to vanish when $l_b \to 0$. Thus, we focus first on the term indexed by $q = 0$. To simplify notation we introduce

$$u_{b,x} := \frac{1}{l_b}\left(x + u\frac{L}{d}\right) = \frac{x}{l_b} + 2\pi u \frac{l_b}{L} \tag{III.11}$$

$$\xi(u) := P_n\left(u_{b,x}\right) Q_n\left(u_{b,x}\right) e^{-{u_{b,x}}^2}$$

$$\Xi_{|q\neq 0}(z) := \Xi(z) - \sum_{k\in\mathbb{Z}} \xi(k)$$

so

$$\sum_{k\in\mathbb{Z}}\xi(k)=\sum_{k\in\mathbb{Z}}P_n\left(\frac{1}{l_b}\left[x+k\frac{L}{d}\right]\right)Q_n\left(\frac{1}{l_b}\left[x+k\frac{L}{d}\right]\right)e^{-\frac{1}{l_b^2}\left(x+k\frac{L}{d}\right)^2}$$

is the term for $q=0$ and $\Xi_{|q\neq 0}(z)$ contains the other terms.
Note that Ξ is L/d-periodic in x so we can choose $x\in[0,L/d]$ and

$$\frac{x}{l_b}\leqslant 2\pi\frac{l_b}{L}\underset{N\to\infty}{\longrightarrow}0 \tag{III.12}$$

For $q=0$, if we replace the sum in k by the associated integral we obtain:

$$\int_{\mathbb{R}}\xi(u)du=\frac{L}{2\pi l_b}\int_{\mathbb{R}}P_n(u)Q_n(u)e^{-u^2}du$$

which is the approximation in (III.10). For the convergence of the Riemann sum, we compute the derivative of the integrand. There exists R_n a polynomial of degree $2n+1$ such that

$$\xi'(u)=2\pi\frac{l_b}{L}R_n\left(u_{b,x}\right)e^{-u_{b,x}{}^2}$$

Now, use the mean value theorem:

$$\left|\sum_{k\in\mathbb{Z}}\xi(k)-\int_{\mathbb{R}}\xi(u)du\right|\leqslant\sum_{k\in\mathbb{Z}}\int_k^{k+1}|\xi(k)-\xi(u)|\,du\leqslant 2\pi\frac{l_b}{L}\sum_{k\in\mathbb{Z}}\sup_{k\leqslant u\leqslant k+1}\left|R_n\left(u_{b,x}\right)\right|e^{-u_{b,x}{}^2} \tag{III.13}$$

To control this we only need to control monomials. If $k\leqslant u\leqslant k+1$,

$$\begin{aligned}&\left|u_{b,x}\right|^m e^{-u_{b,x}{}^2}\\ \leqslant&\left|k_{b,x}\right|^m e^{-k_{b,x}{}^2}+\left|(k+1)_{b,x}\right|^m e^{-(k+1)_{b,x}{}^2}+\left|k_{b,x}\right|^m e^{-(k+1)_{b,x}{}^2}+\left|(k+1)_{b,x}\right|^m e^{-k_{b,x}{}^2}\\ =&\left|k_{b,x}\right|^m e^{-k_{b,x}{}^2}+\left|(k+1)_{b,x}\right|^m e^{-(k+1)_{b,x}{}^2}+\left|(k+1)_{b,x-\frac{L}{d}}\right|^m e^{-(k+1)_{b,x}{}^2}+\left|k_{b,x+\frac{L}{d}}\right|^m e^{-k_{b,x}{}^2}\end{aligned}$$

Thus after some change of indices,

$$2\pi\frac{l_b}{L}\sum_{k\in\mathbb{Z}}\sup_{k\leqslant u\leqslant k+1}\left|u_{b,x}\right|^m e^{-u_{b,x}{}^2}\leqslant 2\pi\frac{l_b}{L}\sum_{k\in\mathbb{Z}}\left(2\left|k_{b,x}\right|^m+\left|k_{b,x-\frac{L}{d}}\right|^m+\left|k_{b,x+\frac{L}{d}}\right|^m\right)e^{-k_{b,x}{}^2}$$

Using (III.9) with $\alpha=2\pi\frac{l_b}{L}\to 0, a=\frac{x}{l_b}, b\in\left\{0,2\pi\frac{l_b}{L},-2\pi\frac{l_b}{L}\right\}, c=1$:

$$\left|\sum_{k\in\mathbb{Z}}\xi(k)-\int_{\mathbb{R}}\xi(u)du\right|\leqslant C(n)$$

We next control $\Xi_{|q\neq 0}$. Let $\epsilon>0$, with Young's inequality:

$$-\left(\frac{x}{l_b}+2\pi k\frac{l_b}{L}\right)\cdot q\frac{L}{l_b}\leqslant\frac{\epsilon}{2}\left(\frac{x}{l_b}+2\pi k\frac{l_b}{L}\right)^2+\frac{1}{2\epsilon}\left(q\frac{L}{l_b}\right)^2$$

so

$$e^{-\frac{1}{2}\left(\frac{x}{l_b}+2\pi k\frac{l_b}{L}\right)^2-\frac{1}{2}\left(\frac{x}{l_b}+2\pi k\frac{l_b}{L}+q\frac{L}{l_b}\right)^2} \leqslant e^{-\left(1-\frac{\epsilon}{2}\right)\left(\frac{x}{l_b}+2\pi k\frac{l_b}{L}\right)^2-\left(\frac{1}{2}-\frac{1}{2\epsilon}\right)\left(q\frac{L}{l_b}\right)^2}$$

We take $\epsilon = 3/2$. As in (III.13), we need to deal with monomial terms of the form

$$\sum_{q\neq 0,k}\left|\frac{x}{l_b}+2\pi k\frac{l_b}{L}\right|^m\left|q\frac{L}{l_b}\right|^{\tilde{m}} e^{-\frac{1}{4}\left(\frac{x}{l_b}+2\pi k\frac{l_b}{L}\right)^2-\frac{1}{6}\left(q\frac{L}{l_b}\right)^2}$$

by using

- (III.8) for the sum in q with $\alpha = \dfrac{L}{l_b}, a = 0, b = 0, c = \dfrac{1}{6}$
- (III.9) for the sum in k with $\alpha = 2\pi\dfrac{l_b}{L} \to 0, a = \dfrac{x}{l_b} \to 0, b = 0, c = \dfrac{1}{4}$

We conclude that

$$\left|\Xi_{|q\neq 0}\right| \leqslant C(n)\frac{L}{l_b}\cdot\frac{l_b}{L} = C(n)$$

IV A Lieb-Thirring inequality

In this section we prove a Lieb-Thirring inequality for the magnetic Laplacian with magnetic periodic boundary conditions:

Theorem IV.1: *Kinetic energy inequality*

Let $\gamma \in \mathcal{L}^1(L^2(\Omega))$ a positive operator, then

$$\int_\Omega \rho_\gamma^2 \leqslant \frac{C \|\gamma\|_{\mathcal{L}^\infty}}{\hbar^2} \operatorname{Tr}\left[\mathscr{L}_{\hbar,b}\gamma\right] \tag{IV.1}$$

Moreover if $\psi_N \in L^2_-(\Omega^N)$ with $\|\psi_N\|_{L^2} = 1$,

$$\left\|\rho_{\psi_N}^{(1)}\right\|_{L^2}^2 \leqslant \frac{C}{\hbar b} \operatorname{Tr}\left[\mathscr{L}_{\hbar,b}(x_i)\gamma_{\psi_N}^{(1)}\right] \text{ and } \left|\int_\Omega V\rho_{\psi_N}^{(1)}\right| \leqslant \frac{C}{\hbar b} \operatorname{Tr}\left[\mathscr{L}_{\hbar,b}(x_i)\gamma_{\psi_N}^{(1)}\right] \|V\|_{L^2} \tag{IV.2}$$

$$\int_{\Omega^2} w\rho_{\psi_N}^{(2)} \leqslant \frac{C}{\hbar b} \operatorname{Tr}\left[\mathscr{L}_{\hbar,b}(x_i)\gamma_{\psi_N}^{(1)}\right] \|w\|_{L^2} \tag{IV.3}$$

We follow the proof of [11: Chapter 4]. To achieve this goal we prove the following sequence of inequalities: a Kato inequality (Lemma IV.2), a diamagnetic inequality for Green functions (Proposition IV.5), a Lieb-Thirring inequality (Theorem IV.6) from which we deduce the inequality on the kinetic energy (Theorem IV.1).

IV.1 Reduced densities

We give some usual properties of the reduced density matrices, see Notation I.6. Let γ_N be a N-body density matrix, since the Hamiltonian only contains one-body and two-body terms, the quantum N-body energy in the state γ_N only depends on the two first reduced densities:

$$\frac{\operatorname{Tr}\left[\mathscr{H}_N\gamma_N\right]}{N} = \operatorname{Tr}\left[\left(\mathscr{L}_{\hbar,b} + V\right)\gamma_N^{(1)}\right] + \operatorname{Tr}\left[w\gamma_N^{(2)}\right] \tag{IV.4}$$

moreover, reduced densities inherit trace and Pauli principle from γ_N:

$$\operatorname{Tr}\left[\gamma_N^{(k)}\right] = 1, \quad 0 \leqslant \gamma_N^{(k)} \leqslant \frac{k!\,(N-k)!}{N!} \tag{IV.5}$$

We can also express the reduced density matrices in term of integral kernels:

$$\gamma_N^{(k)}(x_{1:k}, y_{1:k}) \coloneqq \int_{\Omega^{N-k}} \gamma_N(x_{1:k}, x_{k+1:N}; y_{1:k}, x_{k+1:N})dx_{k+1:N} \tag{IV.6}$$

The reduced density matrices are symmetric under permutation of coordinates:

$$\forall \sigma \in S_k, \gamma_N^{(k)}\left(x_{\sigma(1:k)}, y_{\sigma(1:k)}\right) = \gamma_N^{(k)}\left(x_{1:k}, y_{1:k}\right)$$

and consistent:

$$\forall q \in [\![1:k]\!]\,, \gamma_N^{(q)}(x_{1:q}, y_{1:q}) = \int_{\Omega^{k-q}} \gamma_N(x_{1:q}, x_{q+1:k}; y_{1:q}, x_{q+1:k}) dx_{q+1:k}$$

Note that the reduced densities $\rho_{\gamma_N}^{(k)}$ inherit the symmetry and the consistency from the reduced density matrices.

IV.2 A Kato inequality with periodic boundary conditions

One can look at [21: Theorem X.27] for a proof of the Kato inequality in the non magnetic case.

Lemma IV.2: *Periodic Kato inequality*

Define the complex sign

$$s(u) := \begin{cases} \dfrac{\overline{u}}{|u|} & \text{if} \quad u \neq 0 \\ 0 & \text{if} \quad u = 0 \end{cases}$$

Let $u \in C^\infty(\Omega)$ then $|u| \in H^1(\Omega)$ and

$$|\hbar\nabla |u|| \leqslant |\mathscr{P}_{\hbar,b} u| \tag{IV.7}$$

Moreover if $|u|$ is periodic, then

$$-\hbar^2 \Delta |u| \leqslant \operatorname{Re}\left[s(u)\mathscr{L}_{\hbar,b} u\right] \tag{IV.8}$$

in the weak sense on $C_p^\infty(\Omega)$, or equivalently, $\forall \varphi \in C_p^\infty(\Omega, \mathbb{R}_+)$,

$$-\hbar^2 \int_\Omega |u| \Delta\varphi \leqslant \int_\Omega \operatorname{Re}\left[s(u)\mathscr{L}_{\hbar,b} u\right] \varphi$$

Proof:

$$\frac{1}{2}\hbar\nabla |u|^2 = \frac{1}{2}\hbar\nabla(\overline{u}u) = \operatorname{Re}\left[\overline{u}\hbar\nabla u\right] = \operatorname{Re}\left[\overline{u}\left(\hbar\nabla - ibA\right)u\right]$$

so taking absolute values

$$\left|\hbar\frac{\nabla |u|^2}{2}\right| \leqslant |u|\,|\mathscr{P}_{\hbar,b} u| \tag{IV.9}$$

Define

$$u_\epsilon = \sqrt{|u|^2 + \epsilon^2} \in C_p^\infty(\Omega, \mathbb{R}_+^*)$$

Using (IV.9),

$$|\hbar\nabla u_\epsilon| = \frac{|\hbar\nabla |u|^2|}{2u\epsilon} \leqslant \frac{|u|}{u_\epsilon}|\mathscr{P}_{\hbar,b} u| \leqslant |\mathscr{P}_{\hbar,b} u| \tag{IV.10}$$

So ∇u_ϵ is bounded in $L^2(\Omega,\mathbb{R}^2)$ and converges weakly to $v \in L^2(\Omega,\mathbb{R}^2)$ up to sequence of ϵ. Let $\varphi \in C_c^\infty(\mathrm{int}(\Omega),\mathbb{R}^2)$, since $\varphi \in L^2(\Omega,\mathbb{R}^2)$ and $0 \leqslant u_\epsilon - |u| \leqslant \epsilon$

$$\int\limits_\Omega v\cdot\varphi = \lim\int\limits_\Omega \nabla u_\epsilon\cdot\varphi = -\lim\int\limits_\Omega u_\epsilon\nabla\cdot\varphi = -\int\limits_\Omega |u|\,\nabla\cdot\varphi$$

so $v = \nabla|u|$ and the bound (IV.10) passes to the limit and we obtain (IV.7).
To prove (IV.8) we use polar coordinates

$$u =: |u|\,e^{i\theta}$$

Let $x\in\Omega$, if $|u|(x)\neq 0$, $|u|$ is smooth on a neighbourhood V_x of x where $|u|>0$ and thus

$$\nabla u_\epsilon = \frac{|u|}{u_\epsilon}\nabla|u| \to \nabla|u| \text{ pointwise on } V_x$$

$e^{i\theta} = u/|u|$ is also smooth and up to another restriction of V_x we can invert the complex exponential so θ is smooth. Under these conditions, we can do a direct computation and use Cauchy-Schwarz:

$$\begin{aligned}
\mathrm{Re}\left[s(u)\mathscr{L}_{\hbar,b}u\right] =&\mathrm{Re}\left[e^{-i\theta}\left(-\hbar^2\Delta + 2i\hbar bA\cdot\nabla + i\hbar b(\nabla.A) + |bA|^2\right)|u|\,e^{i\theta}\right]\\
=&-\hbar^2\Delta|u| + \mathrm{Re}\left[-|u|\,e^{-i\theta}\hbar^2\Delta e^{i\theta} - 2i\hbar^2\nabla|u|\cdot\nabla\theta + 2i\hbar bA\cdot\nabla|u|\right]\\
&-2\hbar b\,|u|\,A\cdot\nabla\theta + |u|\,|bA|^2\\
=&-\hbar^2\Delta|u| + \mathrm{Re}\left[-i\hbar^2|u|\,\Delta\theta + |u|\,\hbar^2|\nabla\theta|^2\right] - 2\hbar b\,|u|\,A\cdot\nabla\theta + |u|\,|bA|^2\\
=&-\hbar^2\Delta|u| + \hbar^2|u|\,|\nabla\theta|^2 - 2\hbar b\,|u|\,A\cdot\nabla\theta + |u|\,|bA|^2 \geqslant -\hbar^2\Delta|u|
\end{aligned}$$

Note that if $u(x)=0$ then x is a local minimum of u_ϵ so

$$\Delta u_\epsilon(x) \geqslant 0$$

Let $\varphi\in C_p^\infty(\Omega,\mathbb{R}_+)$, since u_ϵ and φ are periodic, the boundary terms vanish in the following integration by parts:

$$\int\limits_\Omega u_\epsilon\Delta\varphi = \int\limits_\Omega \varphi\Delta u_\epsilon \geqslant \int\limits_{\Omega\backslash u^{-1}(\{0\})}\varphi\Delta u_\epsilon \tag{IV.11}$$

Now we take $\epsilon\to 0$, u_ϵ converges uniformly to $|u|$ so

$$\int\limits_\Omega u_\epsilon\Delta\varphi \underset{\epsilon\to 0}{\rightarrow} \int\limits_\Omega |u|\,\Delta\varphi \tag{IV.12}$$

Using $|u|\leqslant u_\epsilon$, when $u(x)\neq 0$,

$$\Delta u_\epsilon = \nabla\cdot\frac{|u|\,\nabla|u|}{u_\epsilon} = \frac{|\nabla|u||^2 + |u|\,\Delta|u|}{u_\epsilon} - \frac{|u|^2\,|\nabla|u||^2}{u_\epsilon^3} \geqslant \frac{|u|}{u_\epsilon}\Delta|u| \tag{IV.13}$$

so (IV.11) implies that

$$\int_{\Omega} u_\epsilon \Delta\varphi \geqslant \int_{\Omega\setminus u^{-1}(\{0\})} \varphi \frac{|u|}{u_\epsilon} \Delta |u| \tag{IV.14}$$

With dominated convergence using inequality (IV.13),

$$\int_{\Omega\setminus u^{-1}(\{0\})} \varphi \frac{|u|}{u_\epsilon} \Delta |u| \underset{\epsilon\to 0}{\to} \int_{\Omega\setminus u^{-1}(\{0\})} \varphi \Delta |u| \tag{IV.15}$$

With (IV.14), (IV.12) and (IV.15) we have

$$\int_{\Omega} |u| \Delta\varphi \geqslant \int_{\Omega\setminus u^{-1}(\{0\})} \varphi\Delta |u|$$

we can conclude that

$$-\hbar^2 \int_{\Omega} |u| \Delta\varphi \leqslant -\hbar^2 \int_{\Omega\setminus u^{-1}(\{0\})} \varphi\Delta |u| \leqslant \int_{\Omega\setminus u^{-1}(\{0\})} \mathrm{Re}\left[s(u)\mathscr{L}_{\hbar,b} u\right]\varphi = \int_{\Omega} \mathrm{Re}\left[s(u)\mathscr{L}_{\hbar,b} u\right]\varphi$$

IV.3 Diamagnetic inequality

The main lemma for the Lieb-Thirring inequality in the magnetic case is the diamagnetic inequality in term of Green functions because it allows to restrict ourselves to the non magnetic case.

Notation IV.3: *Green functions*

The resolvents of $-\hbar^2\Delta$ with periodic boundary conditions and $\mathscr{L}_{\hbar,b}$ are well defined for $\lambda > 0$:

$$G_{bA,\lambda} \coloneqq (\mathscr{L}_{\hbar,b} + \lambda)^{-1} \quad G_\lambda \coloneqq (-\hbar^2\Delta + \lambda)^{-1}$$

Their integral kernels define the corresponding Green functions.

They have the following properties:

Property IV.4

Let $x \in \Omega$, then $G_{bA,\lambda}(x,\bullet), G_\lambda(x,\bullet) \in L^2(\Omega)$ and

$$G_\lambda(x,y) = G_\lambda(x-y) = G_\lambda(y-x) = \frac{1}{L^2}\sum_{k\in\frac{2\pi\hbar}{L}\mathbb{Z}^2} \frac{1}{k^2+\lambda} e^{ik\cdot(x-y)} \geqslant 0$$

Proof:

The periodic Laplacian is diagonalizable in the plane wave basis indexed by $k \in \frac{2\pi}{L}\mathbb{Z}^2$:

$$e_k(x) \coloneqq \frac{1}{L} e^{ik\cdot x}$$

Indeed

$$-\hbar^2\Delta + \lambda = \sum_{k\in\frac{2\pi}{L}\mathbb{Z}^2} \left(\hbar^2 k^2 + \lambda\right) |e_k\rangle \langle e_k|$$

so

$$G_\lambda(x,y) = \frac{1}{L^2} \sum_{k\in\frac{2\pi}{L}\mathbb{Z}^2} \frac{1}{\hbar^2 k^2 + \lambda} e^{ik\cdot(x-y)}$$

A change of index $k \coloneqq -k$ gives $G_\lambda(x,y) \in \mathbb{R}$. Let $f \in L^2(\Omega)$, since $G_\lambda f$ solves

$$(-\hbar^2\Delta + \lambda)u = f, u \in H^2_p(\Omega)$$

by the Lax-Milgram theorem, $G_\lambda f$ is the unique minimizer of the following functional

$$\mathcal{J}(u) \coloneqq \int_\Omega \left(\hbar^2 |\nabla u|^2 + \lambda |u|^2 - fu\right) dx$$

Assuming $f \geqslant 0$, we see that $\mathcal{J}(u) \geqslant \mathcal{J}(|u|)$ and conclude that $G_\lambda f \geqslant 0$. This implies $G_\lambda(x,y) \geqslant 0$. Finally,

$$-\hbar^2\Delta + \lambda \geqslant \lambda \text{ and } \mathscr{L}_{\hbar,b} + \lambda \geqslant \lambda$$

so

$$\|G_\lambda\|_{\mathcal{L}^\infty} \leqslant \frac{1}{\lambda} \text{ and } \|G_{bA,\lambda}\|_{\mathcal{L}^\infty} \leqslant \frac{1}{\lambda}$$

and $G_{bA,\lambda}(x,\bullet), G_\lambda(\bullet,y) \in L^2(\Omega)$.

Now we prove a diamagnetic inequality:

Proposition IV.5: *Diamagnetic inequality for Green functions*

For all $x \in \Omega$ and for almost every $y \in \Omega$,

$$|G_{bA,\lambda}(x,y)| \leqslant G_\lambda(x,y)$$

Proof:

Let $\varphi \in C^\infty(\Omega)$, by definition

$$\int_\Omega G_{bA,\lambda}(x,\bullet)\left(\mathscr{L}_{\hbar,b}+\lambda\right)\varphi = G_{bA,\lambda}\left(\mathscr{L}_{\hbar,b}+\lambda\right)\varphi = \varphi$$

so, in the distributional sense

$$\left(\mathscr{L}_{\hbar,b}+\lambda\right)G_{bA,\lambda}(x,\bullet) = \delta_x \tag{IV.16}$$

Let $\rho \in C^\infty(\mathbb{R}^2,\mathbb{R}_+)$ radial with support included in the ball $B(0,L/2)$ such that $\|\rho\|_{L^1} = 1$.
Let $n \in \mathbb{N}^*$, define the localizer $\rho_n \in C^\infty(\mathbb{T})$ defined by

$$\rho_n(x) \coloneqq \begin{cases} n^2\rho(nx) & \text{if } x \in B\left(0,\frac{L}{2n}\right) \\ 0 & \text{else} \end{cases}$$

Since ρ_n is periodic, the regularisation

$$u_x \coloneqq G_{bA,\lambda}(x,\bullet) * \rho_n \in C_p^\infty(\Omega)$$

Thus, equation (IV.16) becomes

$$\left(\mathscr{L}_{\hbar,b}+\lambda\right)u_x = \delta_x * \rho_n = \rho_n(x-\bullet) \tag{IV.17}$$

We estimate

$$\operatorname{Re}\left[s(u_x)\left(\mathscr{L}_{\hbar,b}+\lambda\right)u_x\right] = \operatorname{Re}\left[s(u_x)\rho_n(x-\bullet)\right] \leqslant \rho_n(x-\bullet)$$

Then we apply Kato's inequality (IV.7) to u_x, use $s(u_x)u_x = |u_x|$ and obtain

$$\left(-\hbar^2\Delta+\lambda\right)|u_x| \leqslant \operatorname{Re}\left[s(u_x)\mathscr{L}_{\hbar,b}u_x\right] + \lambda|u_x| \leqslant \rho_n(x-\bullet) \tag{IV.18}$$

in a weak sense on $C_p^\infty(\Omega)^*$.
Similarly as (IV.17),

$$(-\hbar^2\Delta+\lambda)\rho_n * G_\lambda(\bullet,y) = \rho_n(y-\bullet)$$

Thus testing inequality (IV.18) on $\rho_n * G_\lambda(\bullet,y) \in C_p^\infty(\Omega,\mathbb{R}_+)$ we get

$$\int_\Omega |u_x(z)|\,\rho_n(y-z)dz \leqslant \int_\Omega \rho_n(x-z)\rho_n * G_\lambda(\bullet,y)(z)dz$$

With the changes of variables $t \coloneqq t+x-y$, $z \coloneqq z+x-y$ and Property IV.4,

$$\begin{aligned}|G_{bA,\lambda}(x,\bullet)*\rho_n| * \rho_n(y) &\leqslant \iint_{\Omega^2} \rho_n(x-z)\rho_n(z-t)G_\lambda(t-y)dzdt \\ &= \iint_{\Omega^2} \rho_n(2x-y-z)\rho_n(z-t)G_\lambda(t-x)dzdt\end{aligned}$$

$$=\rho_n * \rho_n * G_\lambda(x,\bullet)(2x-y) \tag{IV.19}$$

If $\varphi_n \to \varphi$ in $L^2(\Omega)$, by Young's inequality

$$\begin{aligned}\|\rho_n * \varphi_n - \varphi\|_{L^2} &\leqslant \|\rho_n * (\varphi_n - \varphi)\|_{L^2} + \|\rho_n * \varphi - \varphi\|_{L^2} \\ &\leqslant \|\rho_n\|_{L^1} \|\varphi_n - \varphi\|_{L^2} + \|\rho_n * \varphi - \varphi\|_{L^2} \underset{n\to\infty}{\longrightarrow} 0\end{aligned}$$

Fix $x \in \Omega$, with Property IV.4 up to a subsequence, $|\rho_n * G_{bA,\lambda}(x,\bullet)| \to |G_{bA,\lambda}(x,\bullet)|$ in $L^2(\Omega)$ so

$$|G_{bA,\lambda}(x,\bullet) * \rho_n| * \rho_n \underset{n\to\infty}{\longrightarrow} |G_{bA,\lambda}(x,\bullet)|$$

in $L^2(\Omega)$ and up to another subsequence almost everywhere. Similarly, almost everywhere

$$\rho_n * \rho_n * G_\lambda(x,\bullet) \underset{n\to\infty}{\longrightarrow} G_\lambda(x,\bullet)$$

So passing to the limit in (IV.19), for almost every $y \in \Omega$,

$$|G_{bA,\lambda}(x,y)| \leqslant G_\lambda(x,2x-y) = G_\lambda(y-x) = G_\lambda(x,y)$$

IV.4 Lieb-Thirring inequality

We would like to prove a kinetic inequality of the form

$$\mathrm{Tr}\left[\gamma \mathscr{L}_{\hbar,b}\right] \geqslant \frac{C}{\|\gamma\|_{\mathcal{L}^\infty}} \int_\Omega \rho_\gamma^2$$

with $\gamma \in \mathcal{L}^1(L^2(\Omega))$ a positive operator. We will deal with the magnetic field with the diamagnetic inequality and use Lieb-Thirring inequality for the Laplacian. But the previous inequality for the Laplacian is false if we take $\gamma := |e_0\rangle \langle e_0|$,

$$\mathrm{Tr}\left[-\Delta\gamma\right] = 0 < \frac{C}{L^2} = \frac{C}{\|\gamma\|_{\mathcal{L}^\infty}} \int_\Omega \rho_\gamma^2$$

To avoid this, we add 1 to the Laplacian so that the constant mode has a non-zero energy.

Theorem IV.6: *Lieb-Thirring Inequality*

Let $\mathcal{V} \in L^2(\Omega, \mathbb{R}_+)$,

$$-\mathrm{Tr}\left[\mathbb{1}_{(\mathscr{L}_{\hbar,b}+1-\mathcal{V})\leqslant 0}\left(\mathscr{L}_{\hbar,b}+1-\mathcal{V}\right)\right] \leqslant \frac{C_{LT}}{\hbar^2}\int_\Omega \mathcal{V}(x)^2 dx \tag{IV.20}$$

Proof:

We denote N_λ the number of eigenvalues of $\mathscr{L}_{\hbar,b}+1$ less than or equal to λ. From [11: Section 4.3],

$$-\mathrm{Tr}\left[\mathbb{1}_{(\mathscr{L}_{\hbar,b}+1-\mathcal{V})\leqslant 0}\left(\mathscr{L}_{\hbar,b}+1-\mathcal{V}\right)\right] = \int\limits_{\mathbb{R}_+} N_\lambda d\lambda$$

Define the Birman-Schwinger operator

$$K_\lambda \coloneqq \sqrt{\mathcal{V}} G_{bA,\lambda} \sqrt{\mathcal{V}}$$

and let B_λ be the number of eigenvalues of K_λ greater or equal to 1. We use the diamagnetic inequality to restrict to the non magnetic case. Since $G_{bA,\lambda}$ is positive, we can define its square root. Using the arguments of [11: Theorem 4.4] we can deduce from Proposition IV.5 the diamagnetic inequality for $\sqrt{G_{bA,\lambda}}$:

$$\left|\sqrt{G_{bA,\lambda}}(x,y)\right| \leqslant \sqrt{G_\lambda}(x,y)$$

Hence with Proposition IV.5,

$$\left|G^{\frac{3}{2}}_{bA,\lambda}(x,y)\right| = \left|\int\limits_\Omega G_{bA,\lambda}(x,z)\sqrt{G_{bA,\lambda}}(z,y)dz\right| \leqslant \int\limits_\Omega G_\lambda(x,z)\sqrt{G_\lambda}(z,y)dz = G^{\frac{3}{2}}_\lambda(x,y)$$

So taking $m \coloneqq 3/2$ and using an inequality on the traces of powers (see [11: Theorem 4.5]),

$$\begin{aligned} N_\lambda = B_\lambda \leqslant \mathrm{Tr}\left[K_\lambda^m\right] \leqslant \mathrm{Tr}\left[\mathcal{V}^{\frac{m}{2}} K_\lambda^m \mathcal{V}^{\frac{m}{2}}\right] \leqslant \int\limits_\Omega \mathcal{V}(x)^m \left|G^m_{A,\lambda+1}(x,x)\right| dx \\ \leqslant \int\limits_\Omega \mathcal{V}(x)^m G^m_{\lambda+1}(x,x)dx \end{aligned}$$

So we obtain

$$-\mathrm{Tr}\left[\mathbb{1}_{(\mathscr{L}_{\hbar,b}+1-\mathcal{V})\leqslant 0}\left(\mathscr{L}_{\hbar,b}+1-\mathcal{V}\right)\right] \leqslant \int\limits_\Omega \int\limits_1^\infty \mathcal{V}(x)^m G^m_\lambda(x,x)dxd\lambda \tag{IV.21}$$

The kernel of G^m_λ is

$$G^m_\lambda(x) = \frac{1}{L^2} \sum_{k\in\frac{2\pi\hbar}{L}\mathbb{Z}^2} \frac{1}{(k^2+\lambda)^m} e^{i\frac{k\cdot x}{\hbar}}$$

We use the integral bound for the sum

$$\sum_{k\in\mathbb{Z}} \frac{1}{(k^2+\lambda)^m} \leqslant \lambda^{-m} + \int\limits_{\mathbb{R}} \frac{1}{(u^2+\lambda)^m} du$$

so

$$G_\lambda^m(0) = \frac{1}{L^2}\sum_{k,q\in\mathbb{Z}} \frac{1}{\left(\left(\frac{2\pi\hbar}{L}\right)^2(k^2+q^2)+\lambda\right)^m} = \frac{1}{L^2}\left(\frac{L}{2\pi\hbar}\right)^{2m}\sum_{k,q\in\mathbb{Z}}\frac{1}{\left(k^2+q^2+\left(\frac{L}{2\pi\hbar}\right)^2\lambda\right)^m}$$
$$\leqslant \frac{1}{L^2}\left(\frac{L}{2\pi\hbar}\right)^{2m}\sum_{k\in\mathbb{Z}}\left(\frac{1}{\left(k^2+\left(\frac{L}{2\pi\hbar}\right)^2\lambda\right)^m}+\int_{\mathbb{R}}\frac{1}{\left(k^2+u^2+\left(\frac{L}{2\pi\hbar}\right)^2\lambda\right)^m}du\right)$$

We estimate the integral

$$\int_{\mathbb{R}}\frac{1}{\left(k^2+u^2+\left(\frac{L}{2\pi\hbar}\right)^2\lambda\right)^m}du = \left(k^2+\left(\frac{L}{2\pi\hbar}\right)^2\lambda\right)^{-m}\int_{\mathbb{R}}\frac{1}{\left(\frac{u^2}{k^2+\left(\frac{L}{2\pi\hbar}\right)^2\lambda}+1\right)^m}du$$
$$= \frac{I(m)}{\left(k^2+\left(\frac{L}{2\pi\hbar}\right)^2\lambda\right)^{m-\frac{1}{2}}}$$

with

$$m>\frac{1}{2} \implies I(m) \coloneqq \int_{\mathbb{R}}\frac{1}{(1+u^2)^m}du<\infty$$

Similarly

$$G_\lambda^m(0) \leqslant \frac{1}{L^2}\left(\frac{L}{2\pi\hbar}\right)^{2m}\sum_{k\in\mathbb{Z}}\left(\frac{1}{\left(k^2+\left(\frac{L}{2\pi\hbar}\right)^2\lambda\right)^m}+\frac{I(m)}{\left(k^2+\left(\frac{L}{2\pi\hbar}\right)^2\lambda\right)^{m-\frac{1}{2}}}\right)$$
$$\leqslant \frac{\lambda^{-m}}{L^2}+\frac{I(m)}{2\pi\hbar L}\lambda^{-m+\frac{1}{2}}$$
$$+\frac{1}{L^2}\left(\frac{L}{2\pi\hbar}\right)^{2m}\left(\int_{\mathbb{R}}\frac{1}{\left(u^2+\left(\frac{L}{2\pi\hbar}\right)^2\lambda\right)^m}du+\int_{\mathbb{R}}\frac{I(m)}{\left(u^2+\left(\frac{L}{2\pi\hbar}\right)^2\lambda\right)^{m-\frac{1}{2}}}du\right)$$
$$\leqslant \frac{\lambda^{-m}}{L^2}+\frac{I(m)}{\pi\hbar L}\lambda^{-m+\frac{1}{2}}+\frac{1}{(2\pi\hbar)^2}I(m)I\left(m-\frac{1}{2}\right)\lambda^{-m+1}$$
$$\leqslant \frac{C(m)}{\hbar^2}\lambda^{-m+1}$$

since $\lambda \geqslant 1$. We need $m>1$ for the integrals to converge. We use the same trick as [11] changing the potential to

$$\mathcal{W}_\lambda(x) \coloneqq \max\left(\mathcal{V}-\frac{\lambda}{2},0\right)$$

Combining this with (IV.21) and the change of variable

$$\mu \coloneqq \frac{2V(x)}{\lambda}, d\lambda = -\frac{2V(x)}{\mu^2} d\mu$$

we obtain

$$\begin{aligned}
-\mathrm{Tr}\left[\mathbb{1}_{(\mathscr{L}_{\hbar,b}+1-\mathcal{V})\leqslant 0}\left(\mathscr{L}_{\hbar,b}+1-\mathcal{V}\right)\right] &\leqslant \frac{C(m)}{\hbar^2}\int_1^\infty\int_\Omega \lambda^{-m+1}\max\left(V(x)-\frac{\lambda}{2},0\right)^m d\lambda dx \\
&\leqslant \frac{C(m)}{\hbar^2}\int_\Omega\left(\int_0^{2V(x)}\lambda\left(\frac{2V(x)}{\lambda}-1\right)^m d\lambda\right)dx \\
&= \frac{C(m)}{\hbar^2}\int_\Omega\left(\int_1^\infty \frac{2V(x)}{\mu}(\mu-1)^m\cdot\frac{2V(x)}{\mu^2}d\mu\right)dx \\
&= \frac{C(m)}{\hbar^2}\int_\Omega V(x)^2\left(\int_1^\infty\frac{(\mu-1)^m}{\mu^3}d\mu\right)dx
\end{aligned}$$

The integral in μ converges if $3-m>1$. To conclude we need $1<m<2$ hence the choice $m=3/2$ is convenient.

This leads to proof of the Fundamental inequality of kinetic energy:

Proof of Theorem IV.1:

With the variational principle and the Lieb-Thirring inequality (IV.20),

$$\begin{aligned}
\mathrm{Tr}\left[(\mathscr{L}_{\hbar,b}+1)\gamma\right] =&\mathrm{Tr}\left[(\mathscr{L}_{\hbar,b}+1-\mathcal{V})\gamma\right]+\mathrm{Tr}\left[\mathcal{V}\gamma\right] \\
\geqslant&\left\|\gamma\right\|_{\mathcal{L}^\infty}\mathrm{Tr}\left[\mathbb{1}_{(\mathscr{L}_{\hbar,b}+1-\mathcal{V})\leqslant 0}\left(\mathscr{L}_{\hbar,b}+1-\mathcal{V}\right)\right]+\mathrm{Tr}\left[\mathcal{V}\gamma\right] \\
\geqslant&-\frac{C_{LT}\left\|\gamma\right\|_{\mathcal{L}^\infty}}{\hbar^2}\int_\Omega\mathcal{V}^2+\int_\Omega\mathcal{V}\rho_\gamma
\end{aligned}$$

Then choose $\mathcal{V}\coloneqq C_N\mathbb{1}_{\rho_\gamma\leqslant c}\rho_\gamma$:

$$\mathrm{Tr}\left[(\mathscr{L}_{\hbar,b}+1)\gamma\right]\geqslant C_N\left(1-C_N\frac{C_{LT}\left\|\gamma\right\|_{\mathcal{L}^\infty}}{\hbar^2}\right)\int_{\rho_\gamma\leqslant c}{\rho_\gamma}^2$$

The constant preceding the integral is maximal when

$$C_N\coloneqq\frac{\hbar^2}{2C_{LT}\left\|\gamma\right\|_{\mathcal{L}^\infty}}$$

and we get

$$\mathrm{Tr}\left[(\mathscr{L}_{\hbar,b}+1)\gamma\right]\geqslant\frac{\hbar^2}{4C_{LT}\left\|\gamma\right\|_{\mathcal{L}^\infty}}\int_{\rho_\gamma\leqslant c}{\rho_\gamma}^2 \tag{IV.22}$$

Since $\mathscr{L}_{\hbar,b} \geqslant \hbar b$,

$$\mathrm{Tr}\left[\mathscr{L}_{\hbar,b}\gamma\right] \geqslant \hbar b \mathrm{Tr}\left[\gamma\right]$$

so because $\hbar b \to \infty$,

$$\mathrm{Tr}\left[\left(\mathscr{L}_{\hbar,b}+1\right)\gamma\right] \leqslant \left(1+\frac{1}{\hbar b}\right)\mathrm{Tr}\left[\mathscr{L}_{\hbar,b}\gamma\right] \leqslant C\mathrm{Tr}\left[\mathscr{L}_{\hbar,b}\gamma\right]$$

With this and monotone convergence we take the limit $c \to \infty$ in inequality (IV.22) and obtain (IV.1). Applying this to (IV.5), we have

$$\frac{1}{\hbar b}\mathrm{Tr}\left[\mathscr{L}_{\hbar,b}\gamma_{\psi_N}^{(1)}\right] \geqslant C\frac{l_b^2}{\left\|\gamma_{\psi_N}^{(1)}\right\|_{\mathcal{L}^\infty}}\left\|\rho_{\psi_N}^{(1)}\right\|_{L^2}^2 \geqslant CNl_b^2\left\|\rho_{\psi_N}^{(1)}\right\|_{L^2}^2 \geqslant C\left\|\rho_{\psi_N}^{(1)}\right\|_{L^2}^2$$

For the second reduced density, by symmetry

$$N\left(\mathrm{Tr}\left[\mathscr{L}_{\hbar,b}\gamma_{\psi_N}^{(1)}\right]-\mathrm{Tr}\left[w\gamma_{\psi_N}^{(2)}\right]\right) = \left\langle \psi_N \middle| \left(\sum_{i=1}^N \mathscr{L}_{\hbar,b}(x_i) - \frac{2}{N-1}\sum_{i<j} w(x_i-w_j)\right)\psi_N\right\rangle$$
$$= \left\langle \psi_N \middle| \left(\sum_{i=1}^N \mathscr{L}_{\hbar,b}(x_i) - \frac{N}{N-1}\sum_{j=2}^N w(x_1-w_j)\right)\psi_N\right\rangle$$
$$\geqslant \left\langle \psi_N \middle| \sum_{j=2}^N \left(\mathscr{L}_{\hbar,b}(x_i) - \frac{N}{N-1}w(x_1-x_j)\right)\psi_N\right\rangle$$
$$= \int_\Omega \left\langle \psi_N(x,\bullet) \middle| \sum_{j=2}^N \left(\mathscr{L}_{\hbar,b}(x_i) - \frac{N}{N-1}w(x-x_j)\right)\psi_N(x,\bullet)\right\rangle dx$$
$$= \int_\Omega \left(\left\langle \psi_N(x,\bullet) \middle| \sum_{j=2}^N \mathscr{L}_{\hbar,b}(x_i)\psi_N(x,\bullet)\right\rangle - N\int_\Omega w(x-y)\rho_{\psi_N(x,\bullet)}^{(1)}dy\right)dx$$

Then using (IV.2) and then Young's inequality,

$$\mathrm{Tr}\left[\mathscr{L}_{\hbar,b}\gamma_{\psi_N}^{(1)}\right]-\mathrm{Tr}\left[w\gamma_{\psi_N}^{(2)}\right]$$
$$\geqslant \frac{1}{N}\int_\Omega\left(C\hbar b(N-1)\left\|\rho_{\psi_N(x,\bullet)}^{(1)}\right\|_{L^2}^2 - N\int_\Omega w(x-y)\rho_{\psi_N(x,\bullet)}^{(1)}dy\right)dx$$
$$\geqslant \int_\Omega\left(C\hbar b\left\|\rho_{\psi_N(x,\bullet)}^{(1)}\right\|_{L^2}^2 - \int_\Omega w(x-y)\rho_{\psi_N(x,\bullet)}^{(1)}dy\right)dx$$
$$\geqslant \int_\Omega\left(C\hbar b\left\|\rho_{\psi_N(x,\bullet)}^{(1)}\right\|_{L^2}^2 - \frac{1}{2}\left(\frac{1}{2C\hbar b}\left\|w\right\|_{L^2}^2 + 2C\hbar b\left\|\rho_{\psi_N(x,\bullet)}^{(1)}\right\|_{L^2}^2\right)\right)dx \geqslant -\frac{C}{\hbar b}\left\|w\right\|_{L^2}^2$$

Changing w to ϵw, dividing by ϵ and using (IV.4) gives

$$\int_{\Omega^2} w\rho^{(2)}_{\psi_N} \leqslant \frac{1}{\epsilon}\mathrm{Tr}\left[\mathscr{L}_{\hbar,b}\gamma^{(1)}_{\psi_N}\right] + \frac{C}{\hbar b}\epsilon \left\|w\right\|^2_{L^2}$$

To optimise in ϵ, we choose $\epsilon \coloneqq \dfrac{\hbar b}{\left\|w\right\|_{L^2}}$, we get

$$\int_{\Omega^2} w\rho^{(2)}_{\psi_N} \leqslant \left(\frac{1}{\hbar b}\mathrm{Tr}\left[\mathscr{L}_{\hbar,b}\gamma^{(1)}_{\psi_N}\right] + C\right)\left\|w\right\|_{L^2} \leqslant \frac{C}{\hbar b}\mathrm{Tr}\left[\mathscr{L}_{\hbar,b}\gamma^{(1)}_{\psi_N}\right]\left\|w\right\|_{L^2}$$

because $\mathscr{L}_{\hbar,b} \geqslant \hbar b$. Similarly with Young's inequality and (IV.2),

$$\left|\int_{\Omega} V\rho^{(1)}_{\psi_N}\right| \leqslant \frac{C}{\hbar b}\mathrm{Tr}\left[\mathscr{L}_{\hbar,b}\gamma^{(1)}_{\psi_N}\right]\left\|V\right\|_{L^2}$$

V Semi-classical limit

In this section we introduce the Husimi functions representing densities in the phase space. The fundamentals properties of these functions can be found in Property V.3. Then we prove that the N-body quantum energy can be approximated by a semi-classical functional depending only on Husimi functions in Proposition V.4.

V.1 Husimi functions

Notation V.1

Let $k \in \mathbb{N}^*, \gamma_k \in \mathcal{L}^1(L^2(\Omega^k))$, recalling Notation III.1 and (I.23) we define the associated Husimi functions or lower symbol as

$$m_{\gamma_k}(X_{1:k}) \coloneqq \operatorname{Tr}\left[\gamma_k \bigotimes_{i=1}^{k} \Pi_{X_i}\right] \text{ with } X_{1:k} \in (\mathbb{N} \times \Omega)^k$$

Conversely, if $m_k \in L^1\left((\mathbb{N} \times \Omega)^k\right)$, define the associated density matrix

$$\gamma_{m_k} \coloneqq (2\pi l_b^2)^k \int_{(\mathbb{N}\times\Omega)^k} m_k(X_{1:k}) \bigotimes_{i=1}^{k} \Pi_{X_i} d\eta^{\otimes k}(X_{1:k})$$

We call m_k the upper symbol of γ_{m_k}. We also associate a density to m_k:

$$\rho_{m_k} \coloneqq \sum_{n_{1:k} \in \mathbb{N}^k} m_k(n_{1:k}; \bullet)$$

we extend the definition (I.13) to Husimi functions, if $k \geqslant 2$:

$$\mathcal{E}_{sc,\hbar b}\left[m_k\right] \coloneqq \int_{\mathbb{N}\times\Omega} E_n m_k^{(1)}(n,R) d\eta(n,R) + \int_{\mathbb{N}\times\Omega} V m_k^{(1)} d\eta + \int_{(\mathbb{N}\times\Omega)^2} w m_k^{(2)} d\eta^{\otimes 2} \tag{V.1}$$

and we also extend (I.16) to $\rho_k \in L^1\left(\Omega^k\right)$:

$$\mathcal{E}_{qLL}\left[\rho_k\right] = \int_{\Omega} V \rho_k^{(1)} + \int_{\Omega^2} w \rho_k^{(2)} \tag{V.2}$$

If one starts from a wave-function $\psi_N \in L^2_-\left(\Omega^N\right)$ we use the notation

$$m_{\psi_N} \coloneqq m_{\gamma_{\psi_N}}$$

For another discussion and further references about lower and upper symbols one can look at [19: Definition 3.13].

The k-body Husimi function is the joint probability distribution for k particles in phase space. Similarly as for (I.24), we have

$$m^{(k)}_{\gamma_N} = m_{\gamma^{(k)}_N} \text{ and } \rho^{(k)}_{m_{\gamma_N}} = \rho_{m^{(k)}_{\gamma_N}}$$

The next lemma provides a translation between Husimi functions and density matrices.

Lemma V.2: *Relations between Husimi functions and reduced densities*

Let $\gamma_k \in \mathcal{L}^1\left(L^2\left(\Omega^k\right)\right)$ be a positive operator, then $m_{\gamma_k} \in L^1\left((\mathbb{N}\times\Omega)^k\right)$ and

$$0 \leqslant m_{\gamma_k} \leqslant \frac{\|\gamma_k\|_{\mathcal{L}^\infty}}{(2\pi l_b^2)^k}(1+\mathcal{O}(l_b)) \qquad \int\limits_{(\mathbb{N}\times\Omega)^k} m_{\gamma_k} d\eta^{\otimes k} = \mathrm{Tr}\left[\gamma_k\right]$$

Conversely if $m_k \in L^1\left((\mathbb{N}\times\Omega)^k\right)$ is positive, then $\gamma_{m_k} \in \mathcal{L}^1(L^2(\Omega^k))$ and

$$0 \leqslant \gamma_{m_k} \leqslant \left(2\pi l_b^2\right)^k \|m_k\|_{L^\infty} \qquad \mathrm{Tr}\left[\gamma_{m_k}\right] = \|m_k\|_{L^1} + \mathcal{O}(l_b)$$

Moreover if $\gamma_N \in \mathcal{L}^1\left(L^2_-\left(\Omega^k\right)\right)$ and $1 \leqslant k \leqslant N$, then

$$m^{(k)}_{\gamma_N} \leqslant \frac{(N-k)!}{\left(2\pi l_b^2\right)^k N!}\mathrm{Tr}\left[\gamma_N\right]\left(1+\mathcal{O}\left(l_b\right)\right)$$

Proof:

m_{γ_k} is positive because $\forall X \in \mathbb{N}\times\Omega$, Π_X and γ_k are positive. With Corollary III.6,

$$m_{\gamma_k}(X_{1:k}) \leqslant \|\gamma_k\|_{\mathcal{L}^\infty} \prod_{i=1}^k \mathrm{Tr}\left[\Pi_{X_i}\right] = \|\gamma_k\|_{\mathcal{L}^\infty}\left(\frac{1}{2\pi l_b^2} + \mathcal{O}\left(\frac{1}{l_b}\right)\right)^k = \frac{\|\gamma_k\|_{\mathcal{L}^\infty}}{(2\pi l_b^2)^k}(1+\mathcal{O}(l_b))$$

Then, with the resolution of identity (III.3) we have

$$\int\limits_{(\mathbb{N}\times\Omega)^k} m_{\gamma_k} d\eta^{\otimes k} = \mathrm{Tr}\left[\gamma_k\right]$$

Since $\forall X \in \mathbb{N}\times\Omega$, Π_X and m_k are positive γ_{m_k} is also positive. (III.3) also implies

$$\gamma_{m_k} \leqslant \left(2\pi l_b^2\right)^k \|m_k\|_{L^\infty} \int\limits_{(\mathbb{N}\times\Omega)^k} \bigotimes_{i=1}^k \Pi_{X_i} d\eta^{\otimes k}(X_{1:k}) = \left(2\pi l_b^2\right)^k \|m_k\|_{L^\infty}$$

Finally, using Corollary III.6,

$$\begin{aligned}\mathrm{Tr}\left[\gamma_{m_k}\right] &= (2\pi l_b^2)^k \int\limits_{(\mathbb{N}\times\Omega)^k} m_k(X_{1:k}) \mathrm{Tr}\left[\bigotimes_{i=1}^k \Pi_{X_i}\right] d\eta^{\otimes k}(X_{1:k}) = \int\limits_{(\mathbb{N}\times\Omega)^k} m_k d\eta^{\otimes k} + \mathcal{O}(l_b) \\ &= \|m_k\|_{L^1} + \mathcal{O}(l_b)\end{aligned}$$

$\Pi_X \in \mathcal{L}^1(L^2(\Omega))$ is positive, thus it can be diagonalized:

$$\Pi_X = \sum_{i\in\mathbb{N}} \lambda_{i,X} \left|\psi_{i,X}\right\rangle\left\langle\psi_{i,X}\right| \text{ with } \lambda_{i,X} \geqslant 0 \text{ and } \sum_{i\in\mathbb{N}} \lambda_{i,X} = \mathrm{Tr}\left[\Pi_X\right]$$

We have

$$
\begin{aligned}
m_{\gamma_N^{(k)}}(X_{1:k}) &= \sum_{i_{1:k}\in\mathbb{N}^k}\left(\prod_{j=1}^k \lambda_{i_j,X_j}\right)\mathrm{Tr}\left[\gamma_N^{(k)}\left|\bigotimes_{j=1}^k \psi_{i_j,X_j}\right\rangle\left\langle\bigotimes_{j=1}^k \psi_{i_j,X_j}\right|\right] \\
&= \sum_{i_{1:k}\in\mathbb{N}^k}\left(\prod_{j=1}^k \lambda_{i_j,X_j}\right)\mathrm{Tr}\left[\gamma_N\left|\bigotimes_{j=1}^k \psi_{i_j,X_j}\right\rangle\left\langle\bigotimes_{j=1}^k \psi_{i_j,X_j}\right|\otimes \mathrm{Id}_{L^2(\Omega^{N-k})}\right] \qquad \text{(V.3)}
\end{aligned}
$$

Let $\psi_{1:N}\in L^2(\Omega)$ be an orthonormal family, we claim that

$$
\left|\bigotimes_{i=1}^k \psi_i\right\rangle\left\langle\bigotimes_{i=1}^k \psi_i\right|\otimes \mathrm{Id}_{L^2(\Omega^{N-k})} \leqslant \frac{(N-k)!}{N!} \text{ on } \mathcal{L}^1\left(L^2_-\left(\Omega^N\right)\right) \qquad \text{(V.4)}
$$

Indeed, if we consider the Slater state

$$
\chi_N \coloneqq \frac{1}{\sqrt{N!}}\sum_{\sigma\in S_N}\epsilon(\sigma)\bigotimes_{j=1}^N \psi_{\sigma(j)}
$$

then

$$
\begin{aligned}
&\left\langle \chi_N\left|\left(\left|\bigotimes_{i=1}^k \psi_i\right\rangle\left\langle\bigotimes_{i=1}^k \psi_i\right|\otimes \mathrm{Id}_{L^2(\Omega^{N-k})}\right)\chi_N\right.\right\rangle \\
=&\frac{1}{N!}\sum_{\sigma,\tau\in S_N}\epsilon(\sigma\tau)\left\langle\bigotimes_{i=1}^k\psi_i\left|\bigotimes_{i=1}^k\psi_{\tau(i)}\right.\right\rangle\left\langle\bigotimes_{i=1}^N\psi_{\sigma(i)}\left|\left(\bigotimes_{i=1}^k\psi_i\right)\otimes\bigotimes_{i=k+1}^N\psi_{\tau(i)}\right.\right\rangle \\
=&\frac{1}{N!}\sum_{\sigma,\tau\in S_N}\epsilon(\sigma\tau)\left(\prod_{i=1}^k\delta_{\sigma(i),i}\delta_{\tau(i),i}\right)\prod_{i=k+1}^N\delta_{\sigma(i),\tau(i)} = \frac{1}{N!}\sum_{\sigma\in S_N}\prod_{i=1}^k\delta_{\sigma(i),i} = \frac{(N-k)!}{N!}
\end{aligned}
$$

If the Slater determinant does not contain the $\psi_{1:k}$ then the result of this computation is 0, thus we obtain (V.4). Then with (V.3) and Corollary III.6,

$$
\begin{aligned}
m_{\gamma_N^{(k)}}(X_{1:k}) &\leqslant \frac{(N-k)!}{N!}\sum_{i_{1:k}\in\mathbb{N}^k}\left(\prod_{j=1}^k\lambda_{i_j,X_j}\right)\mathrm{Tr}\left[\gamma_N\right] = \frac{(N-k)!}{N!}\mathrm{Tr}\left[\gamma_N\right]\prod_{j=1}^k\mathrm{Tr}\left[\Pi_{X_j}\right] \\
&= \frac{(N-k)!}{\left(2\pi l_b^2\right)^k N!}\mathrm{Tr}\left[\gamma_N\right]\left(1+\mathcal{O}\left(l_b\right)\right)
\end{aligned}
$$

We have the following properties for the Husimi functions coming from reduced density matrices.

Property V.3: *Husimi functions*

Let γ_N be a N-body density matrix, then $m_{\gamma_N}^{(k)}$ are symmetric, consistent and satisfy

$$
0 \leqslant m_{\gamma_N}^{(k)} \leqslant \frac{(N-k)!}{(2\pi l_b^2)^k N!} + \mathcal{O}(l_b) \qquad \text{(V.5)}
$$

$$\int_{(\mathbb{N}\times\Omega)^k} m_{\gamma_N}^{(k)} d\eta^{\otimes k} = \left\| m_{\gamma_N}^{(k)} \right\|_{L^1} = 1 \tag{V.6}$$

$$\rho_{m_{\gamma_N}}^{(k)} = (g_\lambda^2)^{\otimes k} * \rho_{\gamma_N}^{(k)} \tag{V.7}$$

Proof:

Let $k > q \geqslant 1$ and $X_{1:q} \in (\mathbb{N} \times \Omega)^q$. Recalling the results of Subsection IV.1, we prove that the N-body Husimi functions are consistent marginals using (III.3):

$$\begin{aligned}\int_{(\mathbb{N}\times\Omega)^{k-q}} m_{\gamma_N}^{(k)}(X_{1:k}) d\eta(X_{q+1:k}) &= \mathrm{Tr}\left[\int_{(\mathbb{N}\times\Omega)^{k-q}} \gamma_N^{(k)} \bigotimes_{i=1}^{k} \Pi_{X_i} d\eta(X_{q+1:k})\right] \\ &= \mathrm{Tr}\left[\gamma_N^{(k)} \bigotimes_{i=1}^{q} \Pi_{X_i} \otimes \mathrm{Id}_{\mathbb{N}\times\Omega}^{\otimes(k-q)}\right] = \mathrm{Tr}\left[\gamma_N^{(q)} \bigotimes_{i=1}^{q} \Pi_{X_i}\right] \\ &= m_{\gamma_N}^{(q)}(X_{1:q}) \end{aligned} \tag{V.8}$$

Let $\sigma \in S_k$, the symmetry of the Husimi measures follows from the symmetry of the reduced density matrices:

$$\begin{aligned}\sigma^{-1} \cdot m_{\gamma_N}^{(k)}(X_{1:k}) &= \mathrm{Tr}\left[\gamma_N^{(k)} \bigotimes_{i=1}^{q} \Pi_{X_{\sigma(i)}}\right] = \mathrm{Tr}\left[\left(\sigma \cdot \gamma_N^{(k)}\right) \bigotimes_{i=1}^{q} \Pi_{X_i}\right] = \mathrm{Tr}\left[\gamma_N^{(k)} \bigotimes_{i=1}^{q} \Pi_{X_i}\right] \\ &= m_{\gamma_N}^{(k)}(X_{1:k})\end{aligned}$$

(V.5) and (V.6) follow from

$$\mathrm{Tr}\left[\gamma_N^{(k)}\right] = 1$$

and Lemma V.2.
For the last point we perform a straightforward computation:

$$\begin{aligned}\sum_{n_{1:k}\geqslant 0} m_{\gamma_N}^{(k)}(n_{1:k}; R_{1:k}) &= \mathrm{Tr}\left[\gamma_N^{(k)} \sum_{n_{1:k}\geqslant 0} \bigotimes_{i=1}^{k} \Pi_{n_i,R_i}\right] = \int_{\Omega^k} \gamma_N^{(k)}(x_{1:k}, x_{1:k}) \prod_{i=1}^{k} g_\lambda(x_i - R_i)^2 dx_{1:k} \\ &= (g_\lambda^2)^{\otimes k} * \rho_{\gamma_N}^{(k)}(R_{1:k})\end{aligned}$$

Equation (V.7) tells us that summing the densities inside every Landau level approximate the total density.

V.2 Semi-classical energy

We now prove that the quantum energy can be approximated by the following semi-classical energy, only depending on the one body and two body Husimi functions.

Proposition V.4: *Semi-classical approximation*

Let $\psi_N \in L^2_-(\Omega^N)$, $\|\psi_N\|_{L^2} = 1$, the quantum energy can be approximated with the semi-classical energy (V.1)

$$\frac{\langle \psi_N | \mathcal{H}_N \psi_N \rangle}{N} \underset{N\to\infty}{=} \mathcal{E}_{sc,\hbar b}\left[m_{\psi_N}\right] + \mathcal{O}\left(\frac{f(\lambda)}{\hbar b}\mathrm{Tr}\left[\mathscr{L}_{\hbar,b}\gamma_N^{(1)}\right]\right) + \mathcal{O}\left((\hbar\lambda)^2\right) \tag{V.9}$$

where

$$f(\lambda) := \max\left(\left\|g_\lambda^2 * V - V\right\|_{L^2}, \left\|(g_\lambda^2)^{\otimes 2} * w - w\right\|_{L^2}\right) \underset{\lambda\to\infty}{\to} 0 \tag{V.10}$$

The term kinetic energy

$$\frac{1}{\hbar b}\mathrm{Tr}\left[\mathscr{L}_{\hbar,b}\gamma_{\psi_N}^{(1)}\right]$$

will be bounded when we will take a sequence of minimizers of the N-body quantum energy. Recalling (I.8) and (I.11),

$$bl_b = \mathcal{O}\left(\hbar N l_b\right) = \mathcal{O}\left(\hbar N^{\frac{1}{2}}\right) \gg 1$$

so with (III.2)

$$(\hbar\lambda)^2 \ll \hbar\lambda \ll \hbar b \lambda l_b \ll 1 \tag{V.11}$$

Moreover, $\lambda \to \infty$ so the error terms in (V.9) will be small.

Proof of Proposition V.4:

We start with the kinetic term. Inserting the resolution of identity (III.3) we have

$$\mathrm{Tr}\left[\mathscr{L}_{\hbar,b}\gamma_{\psi_N}^{(1)}\right] = \int_{\mathbb{N}\times\Omega} \mathrm{Tr}\left[\mathscr{L}_{\hbar,b} g_\lambda(\bullet - R)\Pi_n g_\lambda(\bullet - R)\gamma_{\psi_N}^{(1)}\right] d\eta(n,R)$$

Now, we use the diagonalization of the magnetic Laplacian $\mathscr{L}_{\hbar,b}\Pi_n = E_n\Pi_n$ by commuting $\mathscr{L}_{\hbar,b}$ with $g_\lambda(\bullet - R)$ to obtain

$$\begin{aligned}
\mathrm{Tr}\left[\mathscr{L}_{\hbar,b}\gamma_{\psi_N}^{(1)}\right] =& \mathrm{Tr}\left[\int_{\mathbb{N}\times\Omega} E_n \Pi_{n,R}\gamma_{\psi_N}^{(1)} d\eta(n,R)\right] \\
&+ \mathrm{Tr}\left[\gamma_{\psi_N}^{(1)} \int_{\mathbb{N}\times\Omega} \left[\mathscr{L}_{\hbar,b}, g_\lambda(\bullet - R)\right]\Pi_n g_\lambda(\bullet - R) dR\right] \\
=& \int_{\mathbb{N}\times\Omega} E_n m_{\psi_N}^{(1)}(n,R) d\eta(n,R) + \mathrm{Tr}\left[\gamma_{\psi_N}^{(1)} \int_{\Omega} \left[\mathscr{L}_{\hbar,b}, g_\lambda(\bullet - R)\right] g_\lambda(\bullet - R) dR\right]
\end{aligned} \tag{V.12}$$

We compute

$$[\mathscr{P}_{\hbar,b}, g_\lambda(\bullet - R)] = i\hbar\nabla g_\lambda(\bullet - R)$$

and

$$\begin{aligned}
[\mathscr{L}_{\hbar,b}, g_\lambda(\bullet - R)] &= [\mathscr{P}_{\hbar,b}, g_\lambda(\bullet - R)] \cdot \mathscr{P}_{\hbar,b} + \mathscr{P}_{\hbar,b} \cdot [\mathscr{P}_{\hbar,b}, g_\lambda(\bullet - R)] \\
&= 2i\hbar\nabla g_\lambda(\bullet - R) \cdot \mathscr{P}_{\hbar,b} - \hbar^2 \Delta g_\lambda(\bullet - R) \\
&= \mathscr{P}_{\hbar,b} \cdot 2i\hbar\nabla g_\lambda(\bullet - R) + \hbar^2 \Delta g_\lambda(\bullet - R)
\end{aligned} \tag{V.13}$$

inserting this in (V.12), we find

$$\begin{aligned}
\mathrm{Tr}\left[\mathscr{L}_{\hbar,b}\gamma^{(1)}_{\psi_N}\right] = &\int\limits_{\mathbb{N}\times\Omega} E_n m^{(1)}_{\psi_N}(n,R) d\eta(n,R) \\
&+ 2i\hbar\mathrm{Tr}\left[\gamma^{(1)}_{\psi_N}\mathscr{P}_{\hbar,b} \cdot \int\limits_{\Omega} \nabla g_\lambda(\bullet - R) g_\lambda(\bullet - R) dR\right] \\
&+ \hbar^2\mathrm{Tr}\left[\gamma^{(1)}_{\psi_N}\int\limits_{\Omega} \Delta g_\lambda(\bullet - R) g_\lambda(\bullet - R) dR\right]
\end{aligned}$$

But because g has a fixed L^2 norm and is periodic

$$\nabla \int\limits_{\Omega} g_\lambda(\bullet - R)^2 dR = 0 = 2\int\limits_{\Omega} \nabla g_\lambda(\bullet - R) g_\lambda(\bullet - R) dR$$

Moreover

$$\int\limits_{\Omega} \Delta g_\lambda(\bullet - R) g_\lambda(\bullet - R) dR = -\int\limits_{\Omega} (\nabla g_\lambda)^2 = -\lambda^4 \int\limits_{\Omega} (\nabla g(\lambda x))^2 dx = -\lambda^2 \|\nabla g\|^2_{L^2} \tag{V.14}$$

Therefore

$$\mathrm{Tr}\left[\mathscr{L}_{\hbar,b}\gamma^{(1)}_{\psi_N}\right] = \int\limits_{\mathbb{N}\times\Omega} E_n m^{(1)}_{\psi_N}(n,R) d\eta(n,R) - (\hbar\lambda)^2 \|\nabla g\|^2_{L^2} \tag{V.15}$$

If we take a k variable potential $V_k \in L^1\left(\Omega^k\right)$,

$$\mathrm{Tr}\left[V_k\gamma^{(k)}_{\psi_N}\right] = \int\limits_{\Omega^k} \gamma^{(k)}_{\psi_N}(x_{1:k}; x_{1:k}) V_k(x_{1:k}) dx_{1:k} = \int\limits_{\Omega^k} \rho^{(k)}_{\psi_N} V_k$$

To express this in terms of Husimi functions we use (V.7):

$$\mathrm{Tr}\left[V_k\gamma^{(k)}_{\psi_N}\right] = \int\limits_{\Omega^k} \rho^{(k)}_{m_{\gamma_N}} V_k + \int\limits_{\Omega^k} \left(\rho^{(k)}_{\psi_N} - (g^2_\lambda)^{\otimes k} * \rho^{(k)}_{\psi_N}\right) V_k$$

$$= \int_{\Omega^k} \rho^{(k)}_{m_{\gamma_N}} V_k + \int_{\Omega^k} \rho^{(k)}_{\psi_N} \left(V_k - (g_\lambda^2)^{\otimes k} * V_k\right)$$

Thus applying (IV.4) and using (V.15),

$$\begin{aligned}
\frac{\langle \psi_N | \mathcal{H}_N \psi_N \rangle}{N} &= \operatorname{Tr}\left[(\mathscr{L}_{\hbar,b} + V)\gamma^{(1)}_{\psi_N}\right] + \operatorname{Tr}\left[w\gamma^{(2)}_{\psi_N}\right] \\
&= \int_{\mathbb{N}\times\Omega} E_n m^{(1)}_{\psi_N}(n,R) d\eta(n,R) + \int_{\Omega} \rho^{(1)}_{\psi_N} V + \int_{\Omega^2} \rho^{(2)}_{\psi_N} w - (\hbar\lambda)^2 \|\nabla g\|^2_{L^2} \\
&= \mathcal{E}_{sc,\hbar b}\left[m_{\psi_N}\right] + \int_{\Omega} \rho^{(1)}_{\psi_N} \left[V - g_\lambda^2 * V\right] + \int_{\Omega^2} \rho^{(2)}_{\psi_N} \left[w - (g_\lambda^2)^{\otimes 2} * w\right] - (\hbar\lambda)^2 \|\nabla g\|^2_{L^2}
\end{aligned}$$

Using $V, w \in L^2(\Omega)$ and the fact that w and thus

$$(g_\lambda^2)^{\otimes 2} * w(x,y) = \iint_{\Omega^2} g_\lambda^2(z) g_\lambda^2(t) w(x-y+t-z) dz dt$$

only depends on $x-y$ we can use the kinetic energy inequalities (IV.2) and (IV.3) to control the errors terms:

$$\begin{aligned}
\left| \frac{\langle \psi_N | \mathcal{H}_N \psi_N \rangle}{N} - \mathcal{E}_{sc,\hbar b}\left[m_{\psi_N}\right] \right| \leqslant & \left| \int_{\omega} \rho^{(1)}_{\psi_N} \left[V - g_\lambda^2 * V\right] \right| + \left| \int_{\Omega^2} \rho^{(2)}_{\psi_N} \left[w - (g_\lambda^2)^{\otimes 2} * w\right] \right| \\
& + (\hbar\lambda)^2 \|\nabla g\|^2_{L^2} \\
\leqslant & \frac{C}{\hbar b} \operatorname{Tr}\left[\mathscr{L}_{\hbar,b} \gamma^{(1)}_N\right] f(\lambda) + (\hbar\lambda)^2 \|\nabla g\|^2_{L^2}
\end{aligned}$$

and we have

$$f(\lambda) \underset{\lambda\to\infty}{\longrightarrow} 0$$

VI Mean field limit

In Section V we went from the quantum N-body energy to the semi-classical energy (V.1) (Proposition V.4). The last step needed to obtain the limit models (I.13) and (I.16) out of (V.1) and (V.2) is to remove correlations. Indeed we see that for $m \in L^1(\mathbb{N} \times \Omega)$ and $\rho \in L^1(\Omega)$

$$\mathcal{E}_{sc,\hbar b}\left[m^{\otimes 2}\right] = \mathcal{E}_{sc,\hbar b}\left[m\right]$$
$$\mathcal{E}_{qLL}\left[\rho^{\otimes 2}\right] = \mathcal{E}_{qLL}\left[\rho\right]$$

For fermionic states there are always some correlations due to anti-symmetry. Therefore the objective of this section is to prove that all other correlations are negligible. Neglecting correlations except those coming from the anti-symmetry is called the mean field approximation. We prove that this approximation holds in the mean field limit using Lieb's variational principle (Theorem VI.5) for the energy upper bound in Subsection VI.1 and the De Finetti Theorem VI.11 for the lower bound in Subsection VI.2.

The next proposition is a computation of the semi-classical energy when the low Landau levels are saturated. In this case the semi-classical energy is a sum of constant energies and of the semi-classical functional (I.16) for particles in qLL.

Proposition VI.1: *Saturated low Landau level energy*

Let $\rho \in \mathcal{D}_{qLL}$, using Notation I.4 and Notation V.1

$$\mathcal{E}_{sc,\hbar b}\left[m_\rho\right] = \hbar b E^{q,r} + E_V^{q,r} + E_w^{q,r} + \mathcal{E}_{qLL}[\rho] = \hbar b E^{q,r} + \mathcal{E}_{qLL}[\rho_{m_\rho}]$$

Proof:

With a straightforward computation:

$$\begin{aligned}
\mathcal{E}_{sc,\hbar b}\left[m_\rho\right] &= \sum_{n\in\mathbb{N}} E_n \int_\Omega m_\rho(n,x)dx + \sum_{n\in\mathbb{N}} \int_\Omega V(x) m_\rho(n,x)dx \\
&+ \sum_{n,\tilde{n}\in\mathbb{N}} \int_{\Omega^2} w(x-y) m_\rho(n,x) m_\rho(\tilde{n},y)dxdy \\
&= \frac{1}{q+r}\sum_{n=0}^{q-1} E_n + \frac{r}{q+r}E_q + \frac{q}{(q+r)} \fint_\Omega V + \int_\Omega V\rho + \frac{q^2}{(q+r)^2}\fint_{\Omega^2} w \\
&+ \frac{2q}{L^2(q+r)} \iint_{\Omega^2} w(x-y)\rho(x)dxdy + \int_{\Omega^2} w(x-y)\rho(x)\rho(y)dxdy \\
&= \mathcal{E}_{qLL}[\rho] + \frac{2\hbar b}{q+r}\cdot\frac{q}{2}\cdot\left(q-1+\frac{1}{2}+\frac{1}{2}\right) + \frac{r2\hbar b}{q+r}\cdot\left(q+\frac{1}{2}\right) \\
&+ \frac{q}{q+r}\fint_\Omega V + \frac{q^2}{(q+r)^2}\fint_\Omega w + \frac{2qr}{(q+r)^2}\fint_\Omega w
\end{aligned}$$

$$
\begin{aligned}
&= \mathcal{E}_{qLL}[\rho] + \frac{q^2+2qr+r}{q+r}\hbar b + \frac{q}{q+r}\fint_{\Omega} V + \frac{q^2+2qr}{(q+r)^2}\fint_{\Omega} w \\
&= \mathcal{E}_{qLL}[\rho] + \hbar b E^{q,r} + E_V^{q,r} + E_w^{q,r}
\end{aligned}
$$

We obtain the second equality with

$$
\begin{aligned}
\mathcal{E}_{qLL}\left[\rho_{m_\rho}\right] &= \sum_{n\in\mathbb{N}} \int_{\Omega} V(x) m_\rho(n,x)dx + \sum_{n,\tilde{n}\in\mathbb{N}} \int_{\Omega^2} w(x-y) m_\rho(n,x) m_\rho(\tilde{n},y)dxdy \\
&= \mathcal{E}_{qLL}[\rho] + E_V^{q,r} + E_w^{q,r}
\end{aligned}
$$

VI.1 Energy upper bound

In this part we prove the energy upper bound:

Proposition VI.2: *Upper energy bound*

$$
\frac{E_N^0}{N} \leqslant \hbar b E^{q,r} + E_V^{q,r} + E_w^{q,r} + \mathcal{E}_{qLL}[\rho] + \hbar b \mathcal{O}\left(1 - \frac{d(q+r)}{N}\right) + \mathcal{O}\left(f(\lambda)\right) + \mathcal{O}\left(\hbar b \lambda l_b\right)
$$

The main tool for this proof is the Hartree-Fock theory obtained when one only consider Slater trial states. For Slater states, many energy computation are simplified (see Wick's Theorem VI.4): the second reduced density matrix can be reconstructed from the first reduced density matrix. The second reduced density matrix is given in term of a perfectly uncorrelated term and an exchange term that will reduce the energy in the case of repulsive interactions. The exchange term contains the correlations due to anti-symmetry, these are the minimal correlations fermionic states can have. Thus Hartree Fock theory is a way to assume that all other correlations are negligible. The Hartree-Fock energy gives a canonical upper bound for the N-body quantum energy since the variational ensemble is restricted to Slater sates. Hartree-Fock theory can be extended to one body operator (see Notation VI.3), and using Lieb's variational principle (Theorem VI.5) one can show that the theory still provides an approximate upper bound for the N-body quantum energy. Then we show that the Hartree-Fock energy is an approximation for the semi-classical energy (Proposition VI.7).

Notation VI.3: *Hartree Fock theory*

Let $s,t,u,v \in L^2(\Omega)$, if one define the exchange operator on $\mathcal{L}^1\left(L^2\left(\Omega^2\right)\right)$ as

$$
\mathrm{Ex}\, |s\otimes t\rangle\langle u\otimes v| := |s\otimes t\rangle\langle v\otimes u| \tag{VI.1}
$$

Let $\gamma \in \mathcal{L}\left(L^2(\Omega)\right)$, define

$$
\gamma_2 := \frac{N}{N-1}\left(1-\mathrm{Ex}\right)\gamma^{\otimes 2} \tag{VI.2}
$$

Define the Hartree-Fock energy

$$
\mathcal{E}_{HF}[\gamma] := \mathrm{Tr}\left[\left(\mathscr{L}_{\hbar,b} + V\right)\gamma\right] + \mathrm{Tr}\left[w\gamma_2\right] \tag{VI.3}
$$

With Wick's theorem definitions (VI.2) and (VI.3) are actual statements for Slater states.

Theorem VI.4: *Wick's theorem*

Let $\psi_N = \frac{1}{\sqrt{N!}} \bigwedge_{j=1}^N \phi_j \in L^2_-(\Omega^N)$ with $(\phi_j)_j$ an orthonormal family, then

$$\gamma^{(1)}_{\psi_N} = \frac{1}{N} \sum_{i=1}^N |\phi_i\rangle \langle \phi_i|$$

and

$$\begin{aligned}\gamma^{(2)}_{\psi_N} &= \frac{N}{N-1} (1 - \mathrm{Ex}) \left(\gamma^{(1)}_N\right)^{\otimes 2} \\ &= \frac{1}{N(N-1)} \sum_{i,j=1}^N |\phi_i \otimes \phi_j\rangle \langle \phi_i \otimes \phi_j - \phi_j \otimes \phi_i|\end{aligned}$$

Thus for a Slater state γ_{ψ_N}

$$\left(\gamma^{(1)}_{\psi_N}\right)_2 = \gamma^{(2)}_{\psi_N}$$

and the Hartree-Fock energy is exactly what we obtain for the quantum N-body energy:

$$\mathcal{E}_{HF}\left[\gamma^{(1)}_{\psi_N}\right] = \mathrm{Tr}\left[(\mathscr{L}_{\hbar,b} + V)\, \gamma^{(1)}_{\psi_N}\right] + \mathrm{Tr}\left[w \gamma^{(2)}_{\psi_N}\right]$$

Lieb's theorem [10] extends the usual variational principle for operators of the form (VI.2).

Theorem VI.5: *Lieb's variational principle*

Let $\gamma \in \mathcal{L}\left(L^2(\Omega)\right)$ satisfying

$$\mathrm{Tr}\left[\gamma\right] = 1 \qquad\qquad 0 \leqslant \gamma \leqslant \frac{1}{N}$$

there exits a N-body density matrix γ_N and a positive operator L_2 such that

$$\begin{aligned}\gamma^{(1)}_N &= \gamma \\ \gamma^{(2)}_N &= \gamma_2 - L_2\end{aligned}$$

We start with Lieb's variational principle to get an energy upper bound in term of the operator γ_2. An important remark here is that we don't assume that the interaction potential is repulsive to get the upper bound as it is usually done when dealing with Lieb's variational principle. The reason why we where able to relax the assumption $w \geqslant 0$ is the computation (VI.6). Lieb's variational principle has also been recently generalised in [3].

Proposition VI.6

Let $\gamma \in \mathcal{L}\left(L^2(\Omega)\right)$ such that $\mathrm{Tr}\left[\gamma\right] = 1$ and $0 \leqslant \gamma \leqslant \frac{1}{N}$, then

$$\frac{E_N^0}{N} \leqslant \mathcal{E}_{HF}\left[\gamma\right] + \frac{\mathrm{Tr}\left[\mathscr{L}_{\hbar,b}\gamma\right]}{\hbar b}\mathcal{O}\left(l_b\right)$$

Proof:

First we prove a lower bound for the interaction term. Using The Gagliardo-Nirenberg inequality for $\psi \in L^2(\Omega)$,

$$\left\|\psi\right\|_{L^4}^2 \leqslant C_{GN}\left(\sqrt{\left\|\psi\right\|_{L^2}\left\|\nabla\psi\right\|}_{L^2} + \left\|\psi\right\|_{L^2}\right)$$

along with Hölder's, Young's and Kato's (IV.7) inequalities,

$$\begin{aligned}
\left|\langle\psi|\mathcal{V}\psi\rangle\right| &\leqslant \left\|\psi\right\|_{L^4}\left\|\mathcal{V}\psi\right\|_{L^{\frac{4}{3}}} \leqslant \left\|\mathcal{V}\right\|_{L^2}\left\|\psi\right\|_{L^4}^2 \leqslant C_{GN}\left\|\mathcal{V}\right\|_{L^2}\left(\left\|\psi\right\|_{L^2}\left\|\nabla\left|\psi\right|\right\|_{L^2} + \left\|\psi\right\|_{L^2}^2\right)\\
&\leqslant C_{GN}\left\|\mathcal{V}\right\|_{L^2}\left(\frac{1}{\hbar}\left\|\psi\right\|_{L^2}\left\|\mathscr{P}_{\hbar,b}\psi\right\|_{L^2} + \left\|\psi\right\|_{L^2}^2\right)\\
&\leqslant C_{GN}\left\|\mathcal{V}\right\|_{L^2}\left(\epsilon\left\|\mathscr{P}_{\hbar,b}\psi\right\|_{L^2}^2 + \left(1 + \frac{1}{4\epsilon\hbar^2}\right)\left\|\psi\right\|_{L^2}^2\right)
\end{aligned}$$

So for $\psi_2 \in L^2(\Omega) \otimes \mathrm{Dom}\left(\mathscr{L}_{\hbar,b}\right)$,

$$\begin{aligned}
\left|\langle\psi_2|w\psi_2\rangle\right| &\leqslant \int_{\Omega^2}\left|w(x-y)\right|\left|\psi_2(x,y)\right|^2 dxdy \leqslant \left\|w\right\|_{L^2}\int_{\Omega}\left\|\psi_2(x,\bullet)\right\|_{L^4}^2\\
&\leqslant C_{GN}\left\|w\right\|_{L^2}\int_{\Omega}\left(\epsilon\left\|\mathscr{P}_{\hbar,b}\psi_2(x,\bullet)\right\|_{L^2}^2 + \left(1+\frac{1}{4\epsilon\hbar^2}\right)\left\|\psi_2(x,\bullet)\right\|_{L^2}^2\right)dx \qquad \text{(VI.4)}\\
&= C_{GN}\left\|w\right\|_{L^2}\left(\epsilon\left\|1\otimes\mathscr{P}_{\hbar,b}\psi_2\right\|_{L^2}^2 + \left(1+\frac{1}{4\epsilon\hbar^2}\right)\left\|\psi_2\right\|_{L^2}^2\right)
\end{aligned}$$

Thus

$$\begin{aligned}
\left\langle\psi_2\middle|\left(C_{GN}\left\|w\right\|_{L^2}\epsilon\left(\mathrm{Id}_{L^2(\Omega)}\otimes\mathscr{L}_{\hbar,b}\right) + w\right)\psi_2\right\rangle &= C_{GN}\left\|w\right\|_{L^2}\epsilon\left\|1\otimes\mathscr{P}_{\hbar,b}\psi_2\right\|_{L^2}^2 + \langle\psi_2|w\psi_2\rangle\\
&\geqslant -C_{GN}\left\|w\right\|_{L^2}\left(1+\frac{1}{4\epsilon\hbar^2}\right)\left\|\psi_2\right\|_{L^2}^2
\end{aligned}$$

and

$$\epsilon C_{GN}\left\|w\right\|_{L^2}\left(\mathrm{Id}_{L^2(\Omega)}\otimes\mathscr{L}_{\hbar,b}\right) + w \geqslant -C_{GN}\left\|w\right\|_{L^2}\left(1+\frac{1}{4\epsilon\hbar^2}\right) \qquad \text{(VI.5)}$$

Let γ_N and L_2 be the operators in Theorem VI.5. Now we use (IV.4), and (VI.5):

$$\frac{E_N^0}{N} \leqslant \frac{\mathrm{Tr}\left[\mathscr{H}_N\gamma_N\right]}{N} = \mathrm{Tr}\left[\left(\mathscr{L}_{\hbar,b} + V\right)\gamma_N^{(1)}\right] + \mathrm{Tr}\left[w\gamma_N^{(2)}\right]$$

$$
\begin{aligned}
&= \mathrm{Tr}\left[\left(\mathscr{L}_{\hbar,b}+V\right)\gamma\right]+\mathrm{Tr}\left[w\left(\gamma_2-L_2\right)\right]=\mathcal{E}_{HF}\left[\gamma\right]-\mathrm{Tr}\left[wL_2\right]\\
&\leqslant \mathcal{E}_{HF}\left[\gamma\right]+C_{GN}\left\|w\right\|_{L^2}\left(\left(1+\frac{1}{4\epsilon\hbar^2}\right)\mathrm{Tr}\left[L_2\right]+\epsilon\mathrm{Tr}\left[\left(\mathrm{Id}_{L^2(\Omega)}\otimes\mathscr{L}_{\hbar,b}\right)L_2\right]\right)
\end{aligned}
\tag{VI.6}
$$

To conclude we need to estimate the error terms. If A is an operator on $L^2(\Omega)$ it follows from (VI.1) that

$$
\mathrm{Tr}\left[\left(\mathrm{Id}_{L^2(\Omega)}\otimes A\right)\mathrm{Ex}\gamma^{\otimes 2}\right]=\mathrm{Tr}\left[A\gamma^2\right]
\tag{VI.7}
$$

Indeed, if we decompose γ in an orthonormal family:

$$
\gamma=:\sum_{i\in\mathbb{N}}\lambda_i\left|u_i\right\rangle\left\langle u_i\right|
$$

then

$$
\begin{aligned}
\mathrm{Tr}\left[\left(\mathrm{Id}_{L^2(\Omega)}\otimes A\right)\mathrm{Ex}\gamma^{\otimes 2}\right]&=\sum_{i,j\in\mathbb{N}}\lambda_i\lambda_j\mathrm{Tr}\left[\mathrm{Id}_{L^2(\Omega)}\otimes A\left|u_i\otimes u_j\right\rangle\left\langle u_j\otimes u_i\right|\right]\\
&=\sum_{i,j\in\mathbb{N}}\lambda_i\lambda_j\mathrm{Tr}\left[\left(\left|u_i\right\rangle\left\langle u_j\right|\right)\otimes\left(A\left|u_j\right\rangle\left\langle u_i\right|\right)\right]\\
&=\sum_{i,j\in\mathbb{N}}\lambda_i\lambda_j\mathrm{Tr}\left[\left|u_i\right\rangle\left\langle u_j\right|\right]\mathrm{Tr}\left[A\left|u_j\right\rangle\left\langle u_i\right|\right]=\sum_{i\in\mathbb{N}}\lambda_i^2\mathrm{Tr}\left[A\left|u_i\right\rangle\left\langle u_i\right|\right]\\
&=\mathrm{Tr}\left[A\gamma^2\right]
\end{aligned}
$$

Taking $A:=\mathrm{Id}_{L^2(\Omega)}$, we obtain

$$
\mathrm{Tr}\left[\mathrm{Ex}\gamma^{\otimes 2}\right]=\mathrm{Tr}\left[\gamma^2\right]
$$

and since γ is positive, with (VI.2) we can estimate

$$
\begin{aligned}
\mathrm{Tr}\left[L_2\right]=\mathrm{Tr}\left[\gamma_2\right]-\mathrm{Tr}\left[\gamma_N^{(2)}\right]&=\frac{N}{N-1}\mathrm{Tr}\left[\gamma^{\otimes 2}-\mathrm{Ex}\gamma^{\otimes 2}\right]-1=\frac{N}{N-1}-\frac{N}{N-1}\mathrm{Tr}\left[\gamma^2\right]-1\\
&\leqslant\frac{1}{N-1}
\end{aligned}
$$

If $\epsilon\to 0$, we can control the first error term in (VI.6) with

$$
0\leqslant\left(1+\frac{1}{4\epsilon\hbar^2}\right)\mathrm{Tr}\left[L_2\right]\leqslant\frac{C}{N\epsilon\hbar^2}
\tag{VI.8}
$$

For the second error term, using Theorem VI.5, (VI.2) and (VI.7) for $A:=\mathscr{L}_{\hbar,b}$,

$$
\begin{aligned}
0\leqslant&\mathrm{Tr}\left[\left(\mathrm{Id}_{L^2(\Omega)}\otimes\mathscr{L}_{\hbar,b}\right)L_2\right]=\mathrm{Tr}\left[\left(\mathrm{Id}_{L^2(\Omega)}\otimes\mathscr{L}_{\hbar,b}\right)\left(\gamma_2-\gamma_N^{(2)}\right)\right]\\
&=\frac{N}{N-1}\mathrm{Tr}\left[\left(\mathrm{Id}_{L^2(\Omega)}\otimes\mathscr{L}_{\hbar,b}\right)\left(1-\mathrm{Ex}\right)\gamma^{\otimes 2}\right]-\mathrm{Tr}\left[\left(\mathrm{Id}_{L^2(\Omega)}\otimes\mathscr{L}_{\hbar,b}\right)\gamma_N^{(2)}\right]\\
&=\frac{N}{N-1}\mathrm{Tr}\left[\mathscr{L}_{\hbar,b}\gamma\right]-\frac{N}{N-1}\mathrm{Tr}\left[\left(\mathrm{Id}_{L^2(\Omega)}\otimes\mathscr{L}_{\hbar,b}\right)\mathrm{Ex}\gamma^{\otimes 2}\right]-\mathrm{Tr}\left[\mathscr{L}_{\hbar,b}\gamma\right]
\end{aligned}
$$

$$=\frac{1}{N-1}\mathrm{Tr}\left[\mathscr{L}_{\hbar,b}\gamma\right]-\frac{N}{N-1}\mathrm{Tr}\left[\mathscr{L}_{\hbar,b}\gamma^2\right]\leqslant\frac{1}{N-1}\mathrm{Tr}\left[\mathscr{L}_{\hbar,b}\gamma\right]$$

When the kinetic energy is minimised $\mathrm{Tr}\left[\mathscr{L}_{\hbar,b}\gamma\right]$ is of order $\hbar b$ so the second error term in (VI.6) will be of order:

$$0\leqslant\epsilon\mathrm{Tr}\left[\left(\mathrm{Id}_{L^2(\Omega)}\otimes\mathscr{L}_{\hbar,b}\right)L_2\right]\leqslant C\frac{\epsilon\hbar b}{N}\cdot\frac{\mathrm{Tr}\left[\mathscr{L}_{\hbar,b}\gamma\right]}{\hbar b}\tag{VI.9}$$

We optimise in ϵ so the bounds in (VI.8) and (VI.9) are of the same order:

$$\frac{1}{N\epsilon\hbar^2}=\frac{\epsilon\hbar b}{N}\implies\epsilon=\frac{1}{\sqrt{\hbar^3 b}}=N^{2\delta-\frac{1}{2}}=o(1)\implies\frac{1}{N\epsilon\hbar^2}=\frac{\epsilon\hbar b}{N}=\frac{1}{l_bN}=\mathcal{O}(l_b)\tag{VI.10}$$

so (VI.6) becomes

$$\frac{E_N^0}{N}\leqslant\mathcal{E}_{HF}\left[\gamma\right]+\left(1+\frac{\mathrm{Tr}\left[\mathscr{L}_{\hbar,b}\gamma\right]}{\hbar b}\right)\mathcal{O}\left(l_b\right)=\mathcal{E}_{HF}\left[\gamma\right]+\frac{\mathrm{Tr}\left[\mathscr{L}_{\hbar,b}\gamma\right]}{\hbar b}\mathcal{O}\left(l_b\right)$$

Now we go from Hartree-Fock energy to the semi-classical energy.

Proposition VI.7: *Semi-classical approximation of Hartree-Fock energy*

Let $n_0\in\mathbb{N}$, $m\in L^1(\mathbb{N}\times\Omega)$ such that $\forall n>n_0, m(n,\bullet)=0$ and

$$0\leqslant m\leqslant\frac{1}{2\pi l_b^2N}\tag{VI.11}$$

then

$$\mathcal{E}_{HF}\left[\gamma_m\right]=\mathcal{E}_{sc,\hbar b}\left[m\right]+\mathcal{O}\left(f(\lambda)\right)+\mathcal{O}\left(\hbar b\lambda l_b\right)$$

Proof:

We start by proving that we recover the semi-classical functional from the direct terms. We compute the kinetic term using the commutation relation (V.13) and Corollary III.6:

$$\begin{aligned}\mathrm{Tr}\left[\mathscr{L}_{\hbar,b}\gamma_m\right]=&2\pi l_b^2\int_{\mathbb{N}\times\Omega}m(X)\mathrm{Tr}\left[\mathscr{L}_{\hbar,b}\Pi_X\right]d\eta(X)\\=&2\pi l_b^2\int_{\mathbb{N}\times\Omega}m(X)E_n\mathrm{Tr}\left[\Pi_X\right]d\eta(X)\\&+2\pi l_b^2\int_{\mathbb{N}\times\Omega}m(n,R)\mathrm{Tr}\left[\left[\mathscr{L}_{\hbar,b},g_\lambda(\bullet-R)\right]\Pi_ng_\lambda(\bullet-R)\right]d\eta(n,R)\\=&\int_{\mathbb{N}\times\Omega}E_nm(X)d\eta(X)+\mathcal{O}(\hbar bl_b)\end{aligned}$$

$$
\begin{aligned}
&+ 2\pi l_b^2 \int\limits_{\mathbb{N}\times\Omega} m(n,R)\mathrm{Tr}\left[2i\hbar\nabla g_\lambda(\bullet - R)\mathscr{P}_{\hbar,b}\Pi_n g_\lambda(\bullet - R)\right] d\eta(n,R) \\
&- 2\pi l_b^2 \int\limits_{\mathbb{N}\times\Omega} m(n,R)\mathrm{Tr}\left[\hbar^2\Delta g_\lambda(\bullet - R)\Pi_n g_\lambda(\bullet - R)\right] d\eta(n,R)
\end{aligned} \tag{VI.12}
$$

Using (III.6), $\exists \mathcal{E} : \mathbb{N}\times\Omega \to \mathbb{R}$ such that

$$
\begin{aligned}
&2\pi l_b^2 \Pi_n(x,x) = 1 + l_b\mathcal{E}(n,x) \\
&|\mathcal{E}(n,x)| \leqslant C(n)
\end{aligned}
$$

With (V.14),

$$
\begin{aligned}
&- 2\pi l_b^2 \int\limits_{\mathbb{N}\times\Omega} m(n,R)\mathrm{Tr}\left[\hbar^2\Delta g_\lambda(\bullet - R)\Pi_n g_\lambda(\bullet - R)\right] d\eta(n,R) \\
&= - \hbar^2 \int\limits_{\mathbb{N}\times\Omega} m(n,R)\left(\int\limits_{\Omega} \Delta g_\lambda(x-R)\left(1 + l_b\mathcal{E}(n,x)\right) g_\lambda(x-R)dx\right) d\eta(n,R) \\
&= (\hbar\lambda)^2 \left\|\nabla g\right\|_{L^2}^2 \left\|m\right\|_{L^1} - \hbar^2 l_b \int\limits_{\mathbb{N}\times\Omega} m(n,R)\left(\int\limits_{\Omega} \lambda^3\Delta g(\lambda x)\mathcal{E}(n,x+R)\lambda g(\lambda x)dx\right) d\eta(n,R) \\
&= (\hbar\lambda)^2 \left\|\nabla g\right\|_{L^2}^2 \left\|m\right\|_{L^1} + (\hbar\lambda)^2\mathcal{O}\left(l_b\right) = \mathcal{O}\left((\hbar\lambda)^2\right)
\end{aligned} \tag{VI.13}
$$

And by (III.7), $\exists \widetilde{\mathcal{E}} : \mathbb{N}\times\Omega \to \mathbb{R}$ such that

$$
\begin{aligned}
&\mathscr{P}_{\hbar,b}\Pi_n(x,x) = \frac{b}{l_b}C(n) + b\widetilde{\mathcal{E}}(n,x) \\
&|\mathcal{E}(n,x)| \leqslant \widetilde{C}(n)
\end{aligned}
$$

so

$$
\begin{aligned}
&2\pi l_b^2 \int\limits_{\mathbb{N}\times\Omega} m(n,R)\mathrm{Tr}\left[2i\hbar\nabla g_\lambda(\bullet - R)\mathscr{P}_{\hbar,b}\Pi_n g_\lambda(\bullet - R)\right] d\eta(n,R) \\
&=4i\pi l_b^2\hbar \int\limits_{\mathbb{N}\times\Omega} m(n,R)\left(\int\limits_{\Omega} \nabla g_\lambda(x-R)\left(C(n)\frac{b}{l_b} + b\widetilde{\mathcal{E}}(n,R)\right) g_\lambda(x-R)dx\right) d\eta(n,R) \\
&=\mathcal{O}\left(\hbar b\lambda l_b\right)
\end{aligned} \tag{VI.14}
$$

Inserting (VI.13) and (VI.14) in (VI.12), we obtain

$$
\mathrm{Tr}\left[\mathscr{L}_{\hbar,b}\gamma_m\right] = \int\limits_{\mathbb{N}\times\Omega} E_n m(X)d\eta(X) + \mathcal{O}\left(\hbar b\lambda l_b\right) + \mathcal{O}\left((\hbar\lambda)^2\right) \tag{VI.15}
$$

Let $k \in \mathbb{N}^*$ and $W_k \in L^2\left(\Omega^k\right)$, with the Fubini theorem

$$\begin{aligned}
\operatorname{Tr}\left[W_k \gamma_m^{\otimes k}\right] =& (2\pi l_b^2)^k \int_{(\mathbb{N}\times\Omega)^k} m^{\otimes k}(X_{1:k}) \operatorname{Tr}\left[W_k \bigotimes_{i=1}^{k} \Pi_{X_i}\right] d\eta^{\otimes k}\left(X_{1:k}\right) \\
&= \left(2\pi l_b^2\right)^k \int_{(\mathbb{N}\times\Omega)^k} m^{\otimes k}(X_{1:k}) \int_{\Omega^k} W_k(x_{1:k}) \left(\bigotimes_{i=1}^{k} \Pi_{X_i}\right)(x_{1:k}, x_{1:k}) dx_{1:k} d\eta^{\otimes k}\left(X_{1:k}\right) \\
&= \int_{\Omega^k} W_k(x_{1:k}) \left(\prod_{i=1}^{k} 2\pi l_b^2 \int_{(\mathbb{N}\times\Omega)} m(X) \Pi_X(x_i, x_i) d\eta\left(X\right)\right) dx_{1:k} \\
&= \int_{\Omega^k} W_k(x_{1:k}) \left(\prod_{i=1}^{k} \int_{(\mathbb{N}\times\Omega)} m(n,R) g_\lambda^2(x_i - R)\left(1 + l_b \mathcal{E}(n, x_i)\right) d\eta\left(n, R\right)\right) dx_{1:k} \\
&= \int_{\Omega^k} W_k \left(\rho_m^{\otimes k} * (g_\lambda^2)^{\otimes k}\right) dx \\
&\quad + l_b \int_{\Omega^k} W_k(x_{1:k}) \left(\prod_{i=1}^{k} \int_{\Omega} g_\lambda^2(x_i - R) \sum_{n=0}^{n_0} m(n,R) \mathcal{E}(n, x_i) dR\right) dx_{1:k}
\end{aligned}$$

But m has finitely many filled Landau level so with the Pauli principle (VI.11), $\rho_m \in L^\infty(\Omega)$ and

$$\operatorname{Tr}\left[W_k \gamma_m^{\otimes k}\right] = \int_{\Omega^k} W_k \rho_m^{\otimes k} + \mathcal{O}\left(\left\|W_k - W_k * (g_\lambda^2)^{\otimes k}\right\|_{L^1}\right) + \mathcal{O}\left(l_b\right) \tag{VI.16}$$

Now we need to control the exchange term. It follows from (VI.1) that

$$\operatorname{Ex}\gamma_m^{\otimes 2}(x, y; z, t) = \gamma_m(x, t)\gamma_m(y, z)$$

so with (VI.4) for $\gamma_m \in L^2(\Omega) \otimes \operatorname{Dom}\left(\mathscr{L}_{\hbar,b}\right)$ as an integral kernel,

$$\begin{aligned}
\left|\operatorname{Tr}\left[w \operatorname{Ex}\gamma_m^{\otimes 2}\right]\right| =& \left|\int_{\Omega^2} w(x-y)\left|\gamma_m(x,y)\right|^2 dx dy\right| \\
\leqslant& C_{GN} \left\|w\right\|_{L^2} \int_{\Omega} \left(\epsilon \left\|\mathscr{P}_{\hbar,b}\gamma_m(x, \bullet)\right\|_{L^2}^2 + \left(1 + \frac{1}{4\epsilon\hbar^2}\right) \left\|\gamma_m(x, \bullet)\right\|_{L^2}^2\right) dx
\end{aligned} \tag{VI.17}$$

With an integration by part,

$$\int_{\Omega} \left\|\mathscr{P}_{\hbar,b}\gamma_m(x, \bullet)\right\|_{L^2}^2 dx = \int_{\Omega^2} \mathscr{P}_{\hbar,b}\gamma_m(x, \bullet)(y) \cdot \overline{\mathscr{P}_{\hbar,b}\gamma_m(x, \bullet)(y)} dx dy$$

$$= \int_{\Omega^2} \mathscr{L}_{\hbar,b}\gamma_m(x,\bullet)(y)\overline{\gamma_m(x,y)}dxdy = \int_{\Omega^2} \overline{\gamma_m}(x,y)\mathscr{L}_{\hbar,b}\overline{\gamma_m}(\bullet,x)(y)dxdy$$
$$= \int_{\Omega^2} \overline{\gamma_m}(x,y)\left(\mathscr{L}_{\hbar,b}\overline{\gamma_m}\right)(y,x)dxdy = \mathrm{Tr}\left[\overline{\gamma_m}\mathscr{L}_{\hbar,b}\overline{\gamma_m}\right]$$

Inserting this in (VI.17), using the cyclicity of the trace we get

$$\begin{aligned}\left|\mathrm{Tr}\left[w\mathrm{Ex}\gamma_m^{\otimes 2}\right]\right| = \left|\mathrm{Tr}\left[w\mathrm{Ex}\overline{\gamma_m}^{\otimes 2}\right]\right| \leqslant & C_{GN}\left\|w\right\|_{L^2}\left(\epsilon\mathrm{Tr}\left[\mathscr{L}_{\hbar,b}\gamma_m^2\right] + \left(1+\frac{1}{4\epsilon\hbar^2}\right)\mathrm{Tr}\left[\gamma_m^2\right]\right)\\ \leqslant & \frac{C_{GN}\left\|w\right\|_{L^2}}{N}\left(\epsilon\mathrm{Tr}\left[\mathscr{L}_{\hbar,b}\gamma_m\right] + \left(1+\frac{1}{4\epsilon\hbar^2}\right)\mathrm{Tr}\left[\gamma_m\right]\right)\end{aligned}$$

With (VI.15), $\mathrm{Tr}\left[\mathscr{L}_{\hbar,b}\gamma_m\right] = \mathcal{O}\left(\hbar b\right)$ and using Lemma V.2, $\mathrm{Tr}\left[\gamma_m\right] = \left\|m\right\|_{L^1} + \mathcal{O}(l_b)$ so the choice of ϵ is the same as in (VI.10) thus

$$\mathrm{Tr}\left[w\mathrm{Ex}\gamma_m^{\otimes 2}\right] = \mathcal{O}(l_b)$$

To conclude, with (VI.3) and (VI.2) then (VI.15) and (VI.16) applied to V and w

$$\begin{aligned}\mathcal{E}_{HF}\left[\gamma_m\right] =& \mathrm{Tr}\left[\mathscr{L}_{\hbar,b}\gamma_m\right] + \mathrm{Tr}\left[V\gamma_m\right] + \frac{N}{N-1}\mathrm{Tr}\left[w\gamma_m^{\otimes 2}\right] + \frac{N}{N-1}\mathrm{Tr}\left[w\mathrm{Ex}\gamma_m^{\otimes 2}\right]\\ =& \int_{\mathbb{N}\times\Omega} E_n m(X)d\eta(X) + \int_{\Omega} V\rho_m + \frac{N}{N-1}\int_{\Omega^2} w\rho_m^{\otimes 2} + \mathcal{O}\left(\left\|V - V*(g_\lambda^2)\right\|_{L^1}\right)\\ &+ \frac{N}{N-1}\mathcal{O}\left(\left\|w - w*(g_\lambda^2)^{\otimes 2}\right\|_{L^1}\right) + \mathcal{O}\left(l_b\right) + \mathcal{O}\left(\hbar b\lambda l_b\right) + \mathcal{O}\left((\hbar\lambda)^2\right)\end{aligned}$$

Recalling (V.10), the semi-classical energy expression (I.13), (V.11) and $\hbar b\lambda \gg 1$,

$$\mathcal{E}_{HF}\left[\gamma_m\right] = \mathcal{E}_{sc,\hbar}\left[m\right] + f(\lambda) + \mathcal{O}\left(\hbar b\lambda l_b\right)$$

With the notation of equation (I.21), we would like to define a one body operator with saturated low Landau levels:

$$\gamma_\rho \coloneqq \frac{L^2(q+r)}{N}\int_{\Omega\times\mathbb{N}} m_\rho(X)\Pi_X d\eta(X)$$

We need to prove that the direct term gives the limit model for qLL and to control the exchange terms. But we cannot apply directly Lieb's principle because with Lemma V.2 we have an error on the trace

$$\mathrm{Tr}\left[\gamma_\rho\right] = 1 + o(1) \text{ and } 0 \leqslant \gamma_\rho \leqslant \frac{1}{N}$$

To cure this we modify m_ρ slightly.

Proposition VI.8: *Corrected Husimi function*

Let $n_0 \in \mathbb{N}$, $m \in L^1(\mathbb{N} \times \Omega)$ such that $\forall n > n_0, m(n, \bullet) = 0, \|m\|_{L^1} = 1 + o(1)$ and

$$0 \leqslant m \leqslant \frac{1}{2\pi l_b^2 N}$$

there exist $\tilde{m} \in L^1(\mathbb{N} \times \Omega), n_1 \in \mathbb{N}$ such that $\forall n > n_1, m(n, \bullet) = 0$ such that

$$\mathrm{Tr}\left[\gamma_m\right] = 1 \qquad\qquad 0 \leqslant \gamma_m \leqslant \frac{1}{N}$$

and

$$\mathcal{E}_{sc,\hbar b}\left[\tilde{m}\right] = \mathcal{E}_{sc,\hbar b}\left[m\right] + \mathcal{O}\left(\hbar b l_b\right) + \mathcal{O}\left(\hbar b\left(1 - \|m\|_{L^1}\right)\right) \tag{VI.18}$$

Proof:

First, by Corollary III.6, $\exists \mathcal{E} : \mathbb{N} \times \Omega \to \mathbb{R}$ such that

$$2\pi l_b^2 \mathrm{Tr}\left[\Pi_X\right] = 1 + l_b \mathcal{E}(X)$$
$$|\mathcal{E}(n, R)| \leqslant C(n)$$

So

$$\mathrm{Tr}\left[\gamma_m\right] = \int_{\mathbb{N}\times\Omega} m(X)\left(1 + l_b\mathcal{E}(X)\right) d\eta(X)$$

If $\mathrm{Tr}\left[\gamma_m\right] = 1$ then m has the desired properties. If $\mathrm{Tr}\left[\gamma_m\right] < 1$ we add some mass to m where it is possible without breaking the Pauli principle. Let $n_1 \in \mathbb{N}$ and

$$0 \leqslant \tau \leqslant \frac{1}{2\pi l_b^2 N}$$

we define

$$\tilde{m}(\tau, n_1) \coloneqq m + \min\left(\tau, \frac{1}{2\pi l_b^2 N} - m\right) \mathbb{1}_{n \leqslant n_1}$$

By construction

$$0 \leqslant m \leqslant \tilde{m}(\tau, n_1) \leqslant \frac{1}{2\pi l_b^2 N} \text{ and } \tau \mathbb{1}_{n \leqslant n_1} \leqslant \tilde{m}(\tau, n_1) \tag{VI.19}$$

We choose $n_1 > n_0$ and remark that

$$\mathrm{Tr}\left[\gamma_{\tilde{m}}\right]\left(\frac{1}{2\pi l_B^2 N}, n_1\right) = \frac{1}{2\pi l_B^2 N}\int_{\mathbb{N}\times\Omega} \mathbb{1}_{n \leqslant n_1}\left(1 + l_b\mathcal{E}(X)\right) d\eta(X) \geqslant \frac{L^2}{2\pi l_b^2 N} n_1 - l_b C(n_1)$$

Since $\exists n_1 \in \mathbb{N}$ such that

$$\frac{L^2}{2\pi l_b^2 N} n_1 > 1$$

for large enough N,

$$\mathrm{Tr}\left[\gamma_{\tilde{m}}\right]\left(\frac{1}{2\pi l_B^2 N}, n_1\right) > 1$$

and

$$\mathrm{Tr}\left[\gamma_{\tilde{m}}\right](0, n_1) = \mathrm{Tr}\left[\gamma_m\right] < 1$$

and $\mathrm{Tr}\left[\gamma_{\tilde{m}}\right]$ is Lipschitz in τ, so by the intermediate value theorem we can conclude $\exists \tau \geqslant 0$ such that if we define

$$\tilde{m} \coloneqq \tilde{m}(\tau, n_1)$$

then

$$\mathrm{Tr}\left[\gamma_{\tilde{m}}\right] = \int\limits_{\mathbb{N}\times\Omega} \tilde{m}(X)\left(1 + l_b \mathcal{E}(X)\right) d\eta(X) = 1$$

Thus we can estimate

$$\begin{aligned}\sum_{n \leqslant n_1} \int\limits_{\Omega} \min\left(\tau, \frac{1}{2\pi l_b^2 N} - m(n, x)\right) dx &= \int\limits_{\mathbb{N}\times\Omega} \left(\tilde{m} - m\right) d\eta = 1 - l_b \int\limits_{\mathbb{N}\times\Omega} \tilde{m}\mathcal{E} d\eta - \int\limits_{\mathbb{N}\times\Omega} m d\eta \\ &= \mathcal{O}(l_b) + \mathcal{O}\left(1 - \|m\|_{L^1}\right)\end{aligned}$$

so

$$\tau = \frac{1}{L^2}\int\limits_{\Omega} \min\left(\tau, \frac{1}{2\pi l_b^2 N} - m(n_1, x)\right) dx \leqslant \mathcal{O}(l_b) + \mathcal{O}\left(1 - \|m\|_{L^1}\right) \tag{VI.20}$$

Now if $\mathrm{Tr}\left[\gamma_m\right] > 1$ we remove some mass to m:

$$\tilde{m}(\tau) \coloneqq \max\left(0, m - \tau\right) = m - \min(m, \tau)$$

by construction

$$0 \leqslant \tilde{m} \leqslant m \leqslant \frac{1}{2\pi l_b^2 N} \tag{VI.21}$$

We see that

$$\mathrm{Tr}\left[\gamma_{\tilde{m}}\right](0) = \mathrm{Tr}\left[\gamma_m\right] > 1 \text{ and } \mathrm{Tr}\left[\gamma_{\tilde{m}}\right]\left(\frac{1}{2\pi l_b^2 N}\right) = 0$$

so $\exists \tau \geqslant 0$ such that if one defines

$$\tilde{m} \coloneqq \tilde{m}(\tau)$$

we find that

$$\mathrm{Tr}\left[\gamma_{\widetilde{m}}\right] = \int\limits_{\mathbb{N}\times\Omega} \widetilde{m}(X)\left(1 + l_b\mathcal{E}(X)\right) d\eta(X) = 1$$

and like before,

$$\begin{aligned}\int\limits_{\mathbb{N}\times\Omega} \min(m,\tau) d\eta = \int\limits_{\mathbb{N}\times\Omega} \left(m - \widetilde{m}\right) d\eta = \|m\|_{L^1} - 1 + l_b \int\limits_{\mathbb{N}\times\Omega} \widetilde{m}\mathcal{E} d\eta = \mathcal{O}(l_b) + \mathcal{O}\left(1 - \|m\|_{L^1}\right)\\ = \int\limits_{m<\tau} m d\eta + \int\limits_{\tau\leqslant m} \tau d\eta = \|m\|_{L^1} + \int\limits_{\tau\leqslant m} (\tau - m) d\eta\end{aligned}$$

So

$$\|m\|_{L^1} + \mathcal{O}(l_b) + \mathcal{O}\left(1 - \|m\|_{L^1}\right) = \int\limits_{\tau\leqslant m} (m-\tau) d\eta \leqslant \frac{1}{\pi l_b^2 N} \left|\mathbb{1}_{\tau\leqslant m}\right|$$

and

$$\begin{aligned}\tau \leqslant& \frac{1}{\left|\mathbb{1}_{\tau\leqslant m}\right|} \int\limits_{\mathbb{N}\times\Omega} \min(m,\tau) d\eta = \frac{1}{\left|\mathbb{1}_{\tau\leqslant m}\right|} \left(\mathcal{O}(l_b) + \mathcal{O}\left(1 - \|m\|_{L^1}\right)\right)\\ \leqslant& \frac{1}{\pi l_b^2 N} \cdot \frac{\mathcal{O}(l_b) + \mathcal{O}\left(1 - \|m\|_{L^1}\right)}{\|m\|_{L^1} + \mathcal{O}(l_b) + \mathcal{O}\left(1 - \|m\|_{L^1}\right)} = \mathcal{O}(l_b) + \mathcal{O}\left(1 - \|m\|_{L^1}\right)\end{aligned} \tag{VI.22}$$

With inequalities (VI.20) and (VI.22) we know that

$$\|m - \widetilde{m}\|_{L^\infty} = \mathcal{O}(l_b) + \mathcal{O}\left(1 - \|m\|_{L^1}\right) \tag{VI.23}$$

Finally we prove the estimate on semi-classical energies (VI.18):

$$\begin{aligned}\left|\mathcal{E}_{sc,\hbar b}\left[\widetilde{m}\right] - \mathcal{E}_{sc,\hbar b}\left[m\right]\right| \leqslant& \sum_{n=0}^{n_1} E_n \int\limits_{\Omega} \left|\widetilde{m}(n,\bullet) - m(n,\bullet)\right| + \sum_{n=0}^{n_1} \int\limits_{\Omega} |V| \left|\widetilde{m}(n,\bullet) - m(n,\bullet)\right|\\ &+ \sum_{n,\tilde{n}=0}^{n_1} \int\limits_{\Omega^2} |w(x-y)| \left|\widetilde{m}(n,x)\widetilde{m}(\tilde{n},y) - m(n,x)m(\tilde{n},y)\right| dxdy\\ \leqslant& L^2 \sum_{n=0}^{n_1} E_n \|m - \widetilde{m}\|_{L^\infty} + (n_1+1)\|V\|_{L^1} \|m - \widetilde{m}\|_{L^\infty}\\ &+ L^2 \|w\|_{L^1} \sum_{n,\tilde{n}=0}^{n_1} \left\|\widetilde{m}(n,\bullet)\widetilde{m}(\tilde{n},\bullet) - m(n,\bullet)m(\tilde{n},\bullet)\right\|_{L^\infty}\end{aligned}$$

Moreover

$$\begin{aligned}\left\|\widetilde{m}(n,\bullet)\widetilde{m}(\tilde{n},\bullet) - m(n,\bullet)m(\tilde{n},\bullet)\right\|_{L^\infty} \leqslant& \|\widetilde{m}(n,\bullet)\|_{L^\infty} \|\widetilde{m}(\tilde{n},\bullet) - m(\tilde{n},\bullet)\|_{L^\infty}\\ &+ \|m(\tilde{n},\bullet)\|_{L^\infty} \|\widetilde{m}(n,\bullet) - m(n,\bullet)\|_{L^\infty}\end{aligned}$$

$$\leqslant \|\widetilde{m}\|_{L^\infty} \|\widetilde{m}-m\|_{L^\infty} + \|m\|_{L^\infty} \|\widetilde{m}-m\|_{L^\infty}$$

so with (VI.19) and (VI.21)

$$\left|\mathcal{E}_{sc,\hbar b}\left[\widetilde{m}\right] - \mathcal{E}_{sc,\hbar b}\left[m\right]\right| \leqslant \left(L^2 \sum_{n=0}^{n_1} E_n + (n_1+1)\|V\|_{L^1} + \frac{L^2}{\pi l_b^2 N}\|w\|_{L^1}(n_1+1)^2\right) \cdot \|m-\widetilde{m}\|_{L^\infty}$$

We conclude with (VI.23).

Putting all of this together we obtain the upper bound.

Proof of Proposition VI.2:

Recalling (I.21), let $\rho \in \mathcal{D}_{qLL}$ and define

$$m_{\rho,N} := \frac{d(q+r)}{N} m_\rho \tag{VI.24}$$

then

$$0 \leqslant m_{\rho,N} \leqslant \frac{d}{L^2 N} = \frac{1}{2\pi l_b^2 N}$$
$$\int_{\mathbb{N}\times\Omega} m_{\rho,N} d\eta = \frac{d(q+r)}{N} = 1 + o(1)$$

We consider $\widetilde{m}_{\rho,N}$ the corrected Husimi function in Proposition VI.8 associated with $m_{\rho,N}$, it satisfies

$$\mathcal{E}_{sc,\hbar b}\left[\widetilde{m}_{\rho,N}\right] = \mathcal{E}_{sc,\hbar b}\left[m_{\rho,N}\right] + \mathcal{O}\left(\hbar b l_b\right) + \hbar b \mathcal{O}\left(1 - \frac{d(q+r)}{N}\right) \tag{VI.25}$$

and $\mathrm{Tr}\left[\gamma_{m_{\rho,N}}\right] = 1, 0 \leqslant \gamma_{m_{\rho,N}} \leqslant \frac{1}{N}$. Moreover by (VI.15),

$$\mathrm{Tr}\left[\mathscr{L}_{\hbar,b}\gamma_{m_{\rho,N}}\right] = \mathcal{O}\left(\hbar b\right)$$

Thus, we can apply Propositions Proposition VI.6, Proposition VI.7 and (VI.25):

$$\begin{aligned}
\frac{E_N^0}{N} \leqslant & \mathcal{E}_{HF}\left[\gamma_{m_{\rho,N}}\right] + \mathcal{O}\left(l_b\right) = \mathcal{E}_{sc,\hbar b}\left[\widetilde{m}_{\rho,N}\right] + \mathcal{O}\left(f(\lambda)\right) + \mathcal{O}\left(\hbar b \lambda l_b\right) \\
= & \mathcal{E}_{sc,\hbar b}\left[m_{\rho,N}\right] + \hbar b \mathcal{O}\left(1 - \frac{d(q+r)}{N}\right) + \mathcal{O}\left(f(\lambda)\right) + \mathcal{O}\left(\hbar b \lambda l_b\right) \\
= & \hbar b E^{q,r} + E_V^{q,r} + E_w^{q,r} + \mathcal{E}_{qll}\left[\frac{d(q+r)}{N}\rho\right] + \hbar b \mathcal{O}\left(1 - \frac{d(q+r)}{N}\right) + \mathcal{O}\left(f(\lambda)\right) + \mathcal{O}\left(\hbar b \lambda l_b\right) \\
= & \hbar b E^{q,r} + E_V^{q,r} + E_w^{q,r} + \mathcal{E}_{qLL}\left[\rho\right] + \hbar b \mathcal{O}\left(1 - \frac{d(q+r)}{N}\right) + \mathcal{O}\left(f(\lambda)\right) + \mathcal{O}\left(\hbar b \lambda l_b\right)
\end{aligned}$$

For the last equality we use the estimate

$$\left|\mathcal{E}_{qll}\left[\frac{d(q+r)}{N}\rho\right]-\mathcal{E}_{qll}\left[\rho\right]\right| \leqslant \left|1-\frac{d(q+r)}{N}\right| \|V\|_{L^2}\|\rho\|_{L^2}+\left(1-\left(\frac{d(q+r)}{N}\right)^2\right)\|w\|_{L^2}\|\rho\|_{L^2}^2$$

and

$$\left|\left(1-\left(\frac{d(q+r)}{N}\right)^2\right)\right| \leqslant C\left|1-\frac{d(q+r)}{N}\right|$$

VI.2 Energy lower bound

In this part we prove the Energy lower bound :

Proposition VI.9: *Lower bound*

Let $(\psi_N)_N$ be a sequence of minimizers of (I.7),

$$\mathcal{E}_{sc,\hbar b}\left[m_{\psi_N}\right] \geqslant \hbar b E^{q,r} + E_V^{q,r} + E_w^{q,r} + \mathcal{E}_{qLL}^0 + o(1)$$

The main tool here is the De Finetti Theorem VI.11. Husimi functions are symmetric and consistent measures. The De Finetti theorem states that such measures are indeed reduced to trivial measure of this kind, namely tensorized products of one body measures and their convex combinations. This result plays an important role in the justification of the decorrelation of densities for the lower bound.

We start by extracting some limit Husimi functions and give their fundamental properties. Similar arguments can be found in [7: Section 2]. With Notation V.1,

Proposition VI.10

Let $(\psi_N)_N$ be a sequence of minimizers of (I.7), up to a subsequence

a) there exists limit Husimi functions $M^{(k)} \in L^\infty\left((\mathbb{N}\times\Omega)^k\right)$ such that

$$m_{\psi_N}^{(k)} \underset{N\to\infty}{\overset{*}{\rightharpoonup}} M^{(k)} \text{ in the weak star topology on } L^\infty\left((\mathbb{N}\times\Omega)^k\right) \tag{VI.26}$$

$$0 \leqslant M^{(k)} \leqslant \frac{1}{\left(L^2(q+r)\right)^k} \tag{VI.27}$$

b) $M^{(1)}(q,\bullet) \in \mathcal{D}_{qLL}$ and

$$M^{(1)}(n,\bullet) = \mathbb{1}_{n<q}\frac{1}{L^2(q+r)} + \mathbb{1}_{n=q}M^{(1)}(q,\bullet)` \tag{VI.28}$$

c) $M^{(k)}$ are the reduced densities of a symmetric measure M on $(\mathbb{N}\times\Omega)^{\mathbb{N}}$ and $\left\|M^{(k)}\right\|_{L^1} = 1$

d) in the sense of Radon measures

$$\rho_{m_{\psi_N}}^{(k)} \underset{N\to\infty}{\overset{*}{\rightharpoonup}} \rho_{M^{(k)}} \tag{VI.29}$$

e) we have convergence of the potential terms:

$$\mathcal{E}_{qLL}\left[\rho_{m_{\psi_N}}\right] \underset{N\to\infty}{\longrightarrow} \mathcal{E}_{qLL}\left[\rho_M\right] \tag{VI.30}$$

Proof:

a) From inequality (V.5) the Husimi functions are uniformly bounded, with a diagonal extraction we obtain (VI.26) and the bound (V.5) with (I.11) induce (VI.27) in the limit.

b) Now since we took a minimizer of the energy, with the upper bound Proposition VI.2 and the Kinetic energy inequalities (IV.2) and (IV.3),

$$\begin{aligned}\frac{E_N^0}{N} =&\operatorname{Tr}\left[\mathscr{L}_{\hbar,b}\gamma_{\psi_N}^{(1)}\right] + \int_{\Omega} V\rho_{\psi_N}^{(1)} + \int_{\Omega^2} w\rho_{\psi_N}^{(2)} = \operatorname{Tr}\left[\mathscr{L}_{\hbar,b}\gamma_{\psi_N}^{(1)}\right]\left(1+\mathcal{O}\left(\frac{1}{\hbar b}\right)\right)\\ \leqslant&\mathcal{E}_{sc,\hbar b}\left[m_\rho\right] + \hbar b\mathcal{O}\left(1-\frac{d(q+r)}{N}\right) + \mathcal{O}\left(f(\lambda)\right) + \mathcal{O}\left(\hbar b\lambda l_b\right)\end{aligned}$$

so by Proposition VI.1 we know that

$$\operatorname{Tr}\left[\mathscr{L}_{\hbar,b}\gamma_{\psi_N}^{(1)}\right] = \mathcal{O}\left(\hbar b\right) \tag{VI.31}$$

Since the contribution of the potential are bounded, the only thing we have to look at are the kinetic terms. Let m_ρ be the Husimi function with saturated low Landau levels defined here (I.21). We denote

$$c_{N,n} \coloneqq \int_{\Omega}\left(m_{\psi_N}^{(1)}(n,.) - m_\rho(n,.)\right)$$

By definition of m_ρ and Lemma V.2 we have

$$\begin{aligned}&\sum_{n\in\mathbb{N}} c_{N,n} = \int_{\mathbb{N}\times\Omega} m_{\psi_N}^{(1)} - \int_{\mathbb{N}\times\Omega} m_\rho = 1-1=0\\ &n<q \implies c_{N,n} \leqslant \frac{L^2}{2\pi l_b^2 N} + \mathcal{O}\left(l_b\right) - \frac{1}{q+r} = \mathcal{O}\left(l_b\right) + \mathcal{O}\left(1-\frac{d(q+r)}{N}\right)\\ &n>q \implies c_{N,n} = \left\|m_{\psi_N}^{(1)}(n,\bullet)\right\|_{L^1} \geqslant 0\end{aligned}$$

Since $(E_n)_n$ is increasing

$$\begin{aligned}\sum_{n\in\mathbb{N}} E_n c_{N,n} \geqslant \sum_{n=0}^{q} E_n c_{N,n} + E_q\sum_{n>q} c_{N,n} =& -\sum_{n=0}^{q-1}(E_q - E_n)c_{N,n}\\ \geqslant&\mathcal{O}\left(\hbar b l_b\right) + \hbar b\mathcal{O}\left(1-\frac{d(q+r)}{N}\right)\end{aligned} \tag{VI.32}$$

Now we compute

$$\mathcal{E}_{sc,\hbar b}\left[m_{\psi_N}\right] - \mathcal{E}_{sc,\hbar b}\left[m_\rho\right] = \sum_{n\in\mathbb{N}} E_n c_{N,n} + \int\limits_{\mathbb{N}\times\Omega} V\left(m^{(1)}_{\psi_N} - m_\rho\right) d\eta$$
$$+ \int\limits_{(\mathbb{N}\times\Omega)^2} w\left(m^{(2)}_{\psi_N} - m^{\otimes 2}_\rho\right) d\eta^{\otimes 2} \tag{VI.33}$$

From the semi-classical approximation (Proposition V.4), (VI.31) and the upper bound (Proposition VI.2),

$$\begin{aligned}\frac{E^0_N}{N} =& \frac{\langle\psi_N|\mathcal{H}_N\psi_N\rangle}{N} = \mathcal{E}_{sc,\hbar b}\left[m_{\psi_N}\right] + \mathcal{O}\left(f(\lambda)\right) + \mathcal{O}\left((\hbar\lambda)^2\right) \\ \leqslant& \mathcal{E}_{sc,\hbar b}\left[m_\rho\right] + \hbar b\mathcal{O}\left(1 - \frac{d(q+r)}{N}\right) + \mathcal{O}\left(f(\lambda)\right) + \mathcal{O}\left(\hbar b\lambda l_b\right)\end{aligned}$$

so with (V.11),

$$\mathcal{E}_{sc,\hbar b}\left[m_{\psi_N}\right] - \mathcal{E}_{sc,\hbar b}\left[m_\rho\right] \leqslant \hbar b\mathcal{O}\left(1 - \frac{d(q+r)}{N}\right) + \mathcal{O}\left(f(\lambda)\right) + \mathcal{O}\left(\hbar b\lambda l_b\right) \tag{VI.34}$$

All the potential terms in (VI.33) are of order 1, therefore the sum in (VI.32) is bounded and we have

$$\mathcal{O}\left(\hbar b l_b\right) + \hbar b\mathcal{O}\left(1 - \frac{d(q+r)}{N}\right) \leqslant -\sum_{n=0}^{q-1}(E_q - E_n)c_{N,n} \leqslant \sum_{n\in\mathbb{N}} E_n c_{N,n} \leqslant C$$

So

$$\sum_{n=0}^{q-1} \frac{E_n - E_q}{\hbar b} c_{N,n} = \mathcal{O}\left(\frac{1}{\hbar b}\right) \tag{VI.35}$$

With a similar inequality as (VI.32) but with E_{q+1} instead of E_q we deduce

$$\begin{aligned}C \geqslant& \sum_{n\in\mathbb{N}} E_n c_{N,n} \geqslant \sum_{n=0}^{q} E_n c_{N,n} + E_{q+1}\sum_{n>q} c_{N,n} = \sum_{n=0}^{q}(E_n - E_{q+1})c_{N,n} \\ \geqslant& \sum_{n=0}^{q} E_n c_{N,n} + E_q\sum_{n>q} c_{N,n} \geqslant \mathcal{O}\left(\hbar b l_b\right) + \hbar b\mathcal{O}\left(1 - \frac{d(q+r)}{N}\right)\end{aligned} \tag{VI.36}$$

and therefore (VI.35) implies

$$c_{N,q} = \mathcal{O}\left(\frac{1}{\hbar b}\right)$$

Then

$$\sum_{n>q} \frac{E_n}{\hbar b} c_{N,n} = \sum_{n\in\mathbb{N}} \frac{E_n}{\hbar b} c_{N,n} - \sum_{n=0}^{q} \frac{E_n}{\hbar b} c_{N,n} = \mathcal{O}\left(\frac{1}{\hbar b}\right) \geqslant \sum_{n>q} \int_\Omega m^{(1)}_{\psi_N}(n, \bullet)$$

and

$$c_{N,q} = \int_\Omega m_N^{(1)}(q,R)dR - \int_\Omega \rho(R)dR = \left\| m_N^{(1)}(q,\bullet) \right\|_{L^1} - \frac{r}{q+r} = \mathcal{O}\left(\frac{1}{\hbar b}\right) \tag{VI.37}$$

From the consistency of $m_{\psi_N}^{(k)}$ in Property V.3,

$$\begin{aligned} \left\| m_{\psi_N}^{(1)}(n_1,\bullet) \right\|_{L^1} &= \int_\Omega \left(\int_{(\mathbb{N}\times\Omega)^{k-1}} m_{\psi_N}^{(k)}(n_1,x_1;X_{2:k}) d\eta^{\otimes(k-1)}(X_{2:k}) \right) dx_1 \\ &= \sum_{n_{2:k}\in\mathbb{N}^{k-1}} \left\| m_{\psi_N}^{(k)}(n_{1:k},\bullet) \right\|_{L^1} \end{aligned} \tag{VI.38}$$

Since

$$\mathbb{N}^k \backslash [\![0:q]\!]^k = \bigsqcup_{i=1}^k \mathbb{N}^{i-1} \times (\mathbb{N}\backslash [\![0:q]\!]) \times \mathbb{N}^{k-i}$$

by the symmetry of $m_{\psi_N}^{(k)}$, (VI.38) and (VI.35),

$$\sum_{n_{1:k}\in\mathbb{N}^k\backslash[\![0:q]\!]^k} \left\| m_{\psi_N}^{(k)}(n_{1:k},\bullet) \right\|_{L^1} = k \sum_{n_1>q} \left\| m_{\psi_N}^{(1)}(n_1,\bullet) \right\|_{L^1} = \mathcal{O}\left(\frac{1}{\hbar b}\right) \tag{VI.39}$$

Ω is bounded, thus testing (VI.26) against $\mathbb{1}_{\{n_{1:k}\}\times\Omega} \in L^1\left((\mathbb{N}\times\Omega)^k\right)$,

$$\left\| m_{\psi_N}^{(k)}(n_{1:k};\bullet) \right\|_{L^1} \underset{N\to\infty}{\to} \left\| M^{(k)}(n_{1:k};\bullet) \right\|_{L^1}$$

So (VI.37) gives

$$\left\| M^{(1)}(q,\bullet) \right\|_{L^1} = \frac{r}{q+r}$$

and with (VI.39), if $n_{1:k} \in \mathbb{N}^k \backslash [\![0:q]\!]^k$, then $M^{(k)}(n_{1:k},\bullet) = 0$ and we see that the norm (V.6) passes to the limit:

$$\begin{aligned} \left\| M^{(k)} \right\|_{L^1} &= \sum_{n_{1:k}\in[\![0:q]\!]^k} \left\| M^{(k)}(n_{1:k},\bullet) \right\|_{L^1} = \lim_{N\to\infty} \sum_{n_{1:k}\in[\![0:q]\!]^k} \left\| m_{\psi_N}^{(k)}(n_{1:k},\bullet) \right\|_{L^1} \\ &= \lim_{N\to\infty} \left(\sum_{n_{1:k}\in[\![0:q]\!]^k} \left\| m_{\psi_N}^{(k)}(n_{1:k},\bullet) \right\|_{L^1} + \sum_{n_{1:k}\in\mathbb{N}^k\backslash[\![0:q]\!]^k} \left\| m_{\psi_N}^{(k)}(n_{1:k},\bullet) \right\|_{L^1} \right) \\ &= \lim_{N\to\infty} \left\| m_{\psi_N}^{(k)} \right\|_{L^1} = 1 \end{aligned}$$

If $n < 0$, by (VI.35),

$$\left\| m_{\psi_N}^{(1)}(n,\bullet) - \frac{1}{L^2(q+r)} \right\|_{L^1} \leqslant \left\| m_{\psi_N}^{(1)}(n,\bullet) - \frac{1}{2\pi l_b^2 N} \right\|_{L^1} + \mathcal{O}\left(1 - \frac{d(q+r)}{N}\right)$$

$$
\begin{aligned}
&= \int\limits_{\Omega} \left(\frac{1}{2\pi l_b^2 N} - m_{\psi_N}^{(1)}(n,\bullet) \right) + \mathcal{O}\left(1 - \frac{d(q+r)}{N}\right) \\
&= \int\limits_{\Omega} \left(\frac{1}{L^2(q+r)} - m_{\psi_N}^{(1)}(n,\bullet) \right) + \mathcal{O}\left(1 - \frac{d(q+r)}{N}\right) \\
&= -C_{N,n} + \mathcal{O}\left(1 - \frac{d(q+r)}{N}\right) = \mathcal{O}\left(\frac{1}{\hbar b}\right) + \mathcal{O}\left(1 - \frac{d(q+r)}{N}\right)
\end{aligned}
$$

so $M^{(1)}(n,\bullet) = \dfrac{1}{L^2(q+r)}$.

c) Testing (V.8) against $\varphi_q \in C_c^0\left((\mathbb{N}\times\Omega)^q\right)$, we have

$$
\int\limits_{(\mathbb{N}\times\Omega)^q} \varphi_q m_{\psi_N}^{(q)} d\eta^{\otimes q} = \int\limits_{(\mathbb{N}\times\Omega)^k} \varphi_q(X_{1:q}) m_{\psi_N}^{(k)}(X_{1:k}) d\eta^{\otimes k}(X_{1:k}) \tag{VI.40}
$$

Since $\varphi_q \in L^1\left((\mathbb{N}\times\Omega)^k\right)$, with (VI.26),

$$
\int\limits_{(\mathbb{N}\times\Omega)^q} \varphi_q m_{\psi_N}^{(q)} d\eta^{\otimes q} \underset{N\to\infty}{\longrightarrow} \int\limits_{(\mathbb{N}\times\Omega)^q} \varphi_q M^{(q)} d\eta^{\otimes q} \tag{VI.41}
$$

In order to pass to the limit in the right term of (VI.40), for the low Landau levels we use (VI.26) on

$$
\mathbb{1}_{([\![0:q]\!]\times\mathbb{N})^k} \left(\varphi_q \otimes \mathrm{Id}_{(\mathbb{N}\times\Omega)^{k-q}}\right) \in L^1\left((\mathbb{N}\times\Omega)^k\right)
$$

and for the high Landau levels we use (VI.39) and $\varphi_q \in L^\infty\left((\mathbb{N}\times\Omega)^k\right)$:

$$
\begin{aligned}
\int\limits_{(\mathbb{N}\times\Omega)^k} \varphi_q(X_{1:q}) m_{\psi_N}^{(k)}(X_{1:k}) d\eta^{\otimes k}(X_{1:k}) &= \int\limits_{\Omega^k} \mathbb{1}_{([\![0:q]\!]\times\mathbb{N})^k} \left(\varphi_q \otimes \mathrm{Id}_{(\mathbb{N}\times\Omega)^{k-q}}\right) m_{\psi_N}^{(k)} d\eta^{\otimes k} \\
&\quad + \sum_{n_{1:k}\in\mathbb{N}^k\setminus[\![0:q]\!]^k} \int\limits_{\Omega^k} \varphi_q(n_{1:q}, x_{1:q}) m_{\psi_N}^{(k)}(n_{1:k}, x_{1:k}) dx_{1:k} \\
&\underset{N\to\infty}{\longrightarrow} \int\limits_{\Omega^k} \mathbb{1}_{([\![0:q]\!]\times\mathbb{N})^k} \left(\varphi_q \otimes \mathrm{Id}_{(\mathbb{N}\times\Omega)^{k-q}}\right) M^{(k)} d\eta^{\otimes k} \\
&= \int\limits_{(\mathbb{N}\times\Omega)^k} \varphi_q(X_{1:q}) M^{(k)}(X_{1:k}) d\eta^{\otimes k}(X_{1:k})
\end{aligned} \tag{VI.42}
$$

Thus passing to the limit in (VI.40) and inserting (VI.41) and(VI.42) we obtain

$$
\forall \varphi_q \in C_c^0\left((\mathbb{N}\times\Omega)^q\right), \int\limits_{(\mathbb{N}\times\Omega)^q} \varphi_q M^{(q)} d\eta^{\otimes q} = \int\limits_{(\mathbb{N}\times\Omega)^k} \varphi_q(X_{1:q}) M^{(k)}(X_{1:k}) d\eta^{\otimes k}(X_{1:k})
$$

and this proves that the limit Husimi functions are also consistent. Testing against φ_q, we also obtain that the symmetry of Husimi functions passes to the limit. Then we can conclude with the Kolmogorov extension theorem that there exists M a symmetric measure on $(\mathbb{N}\times\Omega)^{\mathbb{N}}$ whose marginals are $(M^{(k)})_k$.

d) Let $\varphi_k \in C^0(\Omega^k)$, φ_k is bounded and

$$\mathbb{1}_{[\![0:q]\!]^k} \otimes \varphi_k \in L^1\left((\mathbb{N}\times\Omega)^k\right)$$

so using (VI.26) and (VI.39)

$$\begin{aligned}\int_{\Omega^k} \varphi_k \rho^{(k)}_{m_{\psi_N}} &= \int_{(\mathbb{N}\times\Omega)^k} \left(\mathbb{1}_{[\![0:q]\!]^k} \otimes \varphi_k\right) m^{(k)}_{\psi_N} d\eta^{\otimes k} + \sum_{n_{1:k}\in\mathbb{N}^k\setminus[\![0:q]\!]^k} \int_{\Omega^k} \varphi_k m^{(k)}_{\psi_N} \\ &\underset{N\to\infty}{\rightarrow} \int_{(\mathbb{N}\times\Omega)^k} \left(\mathbb{1}_{[\![0:q]\!]^k} \otimes \varphi_k\right) M^{(k)} d\eta^{\otimes k} = \int_{\Omega^k} \varphi_k \rho^{(k)}_M\end{aligned}$$

e) Let $V_k \in L^2(\Omega^k)$, and $(V_{k,n})_n \subset C^\infty(\Omega^k)$ a sequence regularised with a convolution to a regular function so that

$$\|V_k - V_{k,n}\|_{L^2} \underset{n\to\infty}{\rightarrow} 0$$

we have

$$\int_{\Omega^k} V_k \left(\rho^{(k)}_{m_{\psi_N}} - \rho^{(k)}_M\right) = \int_{\Omega^k} V_{k,n} \left(\rho^{(k)}_{m_{\psi_N}} - \rho^{(k)}_M\right) + \int_{\Omega^k} \rho^{(k)}_{m_{\psi_N}} (V_k - V_{k,n}) + \int_{\Omega^k} \rho^{(k)}_M (V_{k,n} - V_k)$$

For a fixed n, since $V_{k,n} \in C^0(\Omega^k)$ by (VI.29) the first term goes to 0 when $N\to\infty$. For the second term we use (IV.2) if $V_1 = V$, (IV.3) if $V_2 = w$ and (V.7)

$$\begin{aligned}\left|\int_{\Omega^k} \rho^{(k)}_{m_{\psi_N}} (V_k - V_{k,n})\right| &= \left|\int_{\Omega^2} \left(\left(g^k_\lambda\right)^{\otimes k} * \rho^{(k)}_N\right) (V_k - V_{k,n})\right| \leqslant C \left\|(V_k - V_{k,n}) * \left(g^2_\lambda\right)^{\otimes k}\right\|_{L^2} \\ &\leqslant C \left\|(V_k - V_{k,n})\right\|_{L^2}\end{aligned}$$

For the third term we use Hölder's inequality since $\rho^{(k)}_M \in L^\infty\left(\Omega^k\right)$ so we have

$$\lim_{N\to\infty} \left|\int_{\Omega^k} V_k \left(\rho^{(k)}_{m_{\psi_N}} - \rho^{(k)}_M\right)\right| \leqslant C \|V_k - V_{n,k}\|_{L^2} \underset{n\to\infty}{\rightarrow} 0$$

Now we want to apply the De Finetti theorem to M:

Theorem VI.11: *De Finetti or Hewitt-Savage*

Let X be a metric space, $\mu \in \mathcal{P}_s(X^{\mathbb{N}})$ be a symmetric probability measure with marginals $\left(\mu^{(n)}\right)_{n\geqslant 1}$.
$\exists P_\mu \in \mathcal{P}(\mathcal{P}(X))$ such that:

$$\forall n \in \mathbb{N}^*, \mu^{(n)} = \int_{\mathcal{P}(\Omega)} \rho^{\otimes n} dP_\mu(\rho) \tag{VI.43}$$

For a proof of this via the the Diaconis-Freedman theorem see [18: Section 2.1.] and for some further context one can look at [19: Section 2.2.].

Recalling the definition of the semi-classical domain (I.15), we obtain:

Proposition VI.12: *Low Landau level filling of the limit factorised densities*

There exists $\mathcal{P}_M \in \mathcal{P}\left(\mathcal{D}_{sc}\right)$ such that

$$\forall k \in \mathbb{N}^*, M^{(k)} = \int_{\mathcal{D}_{sc}} m^{\otimes k} d\mathcal{P}_M(m) \tag{VI.44}$$

Let μ be the push-forward measure of $\mathcal{P}_M$ by the application

$$\begin{array}{ccc} L^1\left(\mathbb{N}\times\Omega\right) & \to & L^1(\Omega) \\ m & \mapsto & m(q,\bullet) \end{array}$$

then $\mu \in \mathcal{P}\left(\mathcal{D}_{qLL}\right)$ and

$$\rho_M^{(k)} = \int_{\mathcal{D}_{qLL}} \left(\frac{q}{L^2(q+r)} + \rho\right)^{\otimes k} d\mu(\rho) \tag{VI.45}$$

$$\mathcal{E}_{qLL}\left[\rho_M\right] = \int_{\mathcal{D}_{qLL}} \mathcal{E}_{qLL}\left[\frac{q}{L^2(q+r)} + \rho\right] d\mu(\rho) = E_V^{q,r} + E_w^{q,r} + \int_{\mathcal{D}_{qLL}} \mathcal{E}_{qLL}\left[\rho\right] d\mu(\rho) \tag{VI.46}$$

Proof:

Applying Theorem VI.11 to M obtained in Proposition VI.10 gives the existence of $\mathcal{P}_M \in \mathcal{P}\left(\mathcal{P}\left(\mathbb{N}\times\Omega\right)\right)$ such that

$$\forall k \in \mathbb{N}^*, M^{(k)} = \int_{\mathcal{P}(\mathbb{N}\times\Omega)} m^{\otimes k} d\mathcal{P}_M(m) \tag{VI.47}$$

Let $\varphi \in C_c^0(\mathbb{N}\times\Omega, \mathbb{R}_+), \varphi \neq 0, \epsilon > 0, k \in \mathbb{N}^*$, and

$$A_\epsilon(\varphi) \coloneqq \left\{ m \in \mathcal{P}\left(\mathbb{N}\times\Omega\right) \mid \int_{\mathbb{N}\times\Omega} \varphi dm \geqslant \frac{1+\epsilon}{L^2(q+r)} \int_{\mathbb{N}\times\Omega} \varphi \right\}$$

If $m \in A_\epsilon(\varphi)$, then

$$1 \leqslant \frac{L^2(q+r)}{(1+\epsilon)\left\|\varphi\right\|_{L^1(\eta)}} \int_{\mathbb{N}\times\Omega} \varphi dm \leqslant \left(\frac{L^2(q+r)}{(1+\epsilon)\left\|\varphi\right\|_{L^1(\eta)}} \int_{\mathbb{N}\times\Omega} \varphi dm\right)^k$$

so with (VI.27),

$$\begin{aligned}
\mathcal{P}_M\left(A_\epsilon(\varphi)\right) &= \int_{\mathcal{P}(\mathbb{N}\times\Omega)} \mathbb{1}_{A_\epsilon(\varphi)} d\mathcal{P}_M \leqslant \int_{\mathcal{P}(\mathbb{N}\times\Omega)} \left(\frac{L^2(q+r)}{(1+\epsilon)\left\|\varphi\right\|_{L^1(\eta)}} \int_{\mathbb{N}\times\Omega} \varphi dm\right)^k d\mathcal{P}_M(m) \\
&= \left(\frac{L^2(q+r)}{(1+\epsilon)\left\|\varphi\right\|_{L^1(\eta)}}\right)^k \int_{\mathcal{P}(\mathbb{N}\times\Omega)} \left(\int_{(\mathbb{N}\times\Omega)^k} \varphi^{\otimes k} dm^{\otimes k}\right) d\mathcal{P}_M(m) \\
&= \left(\frac{L^2(q+r)}{(1+\epsilon)\left\|\varphi\right\|_{L^1(\eta)}}\right)^k \int_{(\mathbb{N}\times\Omega)^k} \varphi^{\otimes k} dM^{(k)} \leqslant \left(\frac{1}{1+\epsilon}\right)^k \underset{k\to\infty}{\to} 0
\end{aligned}$$

we proved that $\mathcal{P}_M\left(A_\epsilon(\varphi)\right) = 0$ and therefore

$$\mathcal{P}_M\left(\bigcap_{\substack{\varphi\in C_c^0(\mathbb{N}\times\Omega,\mathbb{R}_+)\\ \epsilon>0}} \mathcal{P}\left(\mathbb{N}\times\Omega\right) \backslash A_\epsilon(\varphi)\right) = 1 - \mathcal{P}_M\left(\bigcup_{\substack{\varphi\in C_c^0(\mathbb{N}\times\Omega,\mathbb{R}_+)\\ \epsilon>0}} A_\epsilon(\varphi)\right) = 1$$

therefore for $\mathcal{P}_m$ almost every $m \in \mathcal{P}\left(\mathbb{N}\times\Omega\right)$,

$$\forall \varphi \in C_c^0\left(\mathbb{N}\times\Omega, \mathbb{R}_+\right), \epsilon > 0, \int_{\mathbb{N}\times\Omega} \varphi dm < \frac{1+\epsilon}{L^2(q+r)} \int_{\mathbb{N}\times\Omega} \varphi \tag{VI.48}$$

So for $\mathcal{P}_m$ almost every $m \in \mathcal{P}\left(\mathbb{N}\times\Omega\right)$, m is the density of a probability measure thus a positive function such that $\left\|m\right\|_{L^1} = 1$ and by (VI.48), $m \in L^\infty(\mathbb{N}\times\Omega)$ and

$$m \leqslant \frac{1}{L^2(q+r)} \tag{VI.49}$$

We have shown $\mathcal{P}_M \in \mathcal{P}\left(\mathcal{D}_{sc}\right)$, therefore (VI.47) implies (VI.44).
Moreover if $n < q$ by (VI.28),

$$\int_\Omega \frac{1}{L^2(q+r)} dx = \int_{\mathbb{N}\times\Omega} \mathbb{1}_{\{n\}\times\Omega} dM^{(1)} = \int_{\mathcal{P}(\mathbb{N}\times\Omega)} \left(\int_\Omega m(n,x)dx\right) d\mathcal{P}_M(m)$$

so

$$\int_{\mathcal{P}(\mathbb{N}\times\Omega)} \left(\int_\Omega \left(\frac{1}{L^2(q+r)} - m(n,x)\right) dx\right) d\mathcal{P}_M(m) = 0$$

By (VI.49) the integrand is positive thus null $\mathcal{P}_M$ almost everywhere, we conclude that for $\mathcal{P}_M$ almost every m

$$n < q \implies m(n, \bullet) = \frac{1}{L^2(q+r)} \tag{VI.50}$$

If $n > q$ by (VI.28),

$$0 = \int\limits_{\mathbb{N}\times\Omega} \mathbb{1}_{\{n\}\times\Omega} dM^{(1)} = \int\limits_{\mathcal{P}(\mathbb{N}\times\Omega)} \left(\int\limits_{\Omega} m(n,x)dx \right) d\mathcal{P}_M(m)$$

Once again by (VI.49) the right integrand is positive and thus null so for $\mathcal{P}_M$ almost every m

$$n > q \implies m(n, \bullet) = 0 \tag{VI.51}$$

Finally if $n = q$, since $m \in \mathcal{P}(\mathbb{N} \times \Omega)$ we conclude using (VI.51) and (VI.50): for $\mathcal{P}_M$ almost everywhere m

$$\int\limits_{\Omega} m(q, \bullet) = \int\limits_{\mathbb{N}\times\Omega} m - \sum_{n<q} \int\limits_{\Omega} m(n, \bullet) - \sum_{n>q} \int\limits_{\Omega} m(n, \bullet) = 1 - \frac{q}{q+r} = \frac{r}{q+r} \tag{VI.52}$$

Gathering (VI.49), (VI.50), (VI.51) and (VI.52), we now know that for $\mathcal{P}_M$ almost every m we have $m(q, \bullet) \in \mathcal{D}_{qLL}$. This means that $\mu \in \mathcal{P}(\mathcal{D}_{qLL})$.
Finally we compute

$$\begin{aligned}
\rho_M^{(k)} &= \sum_{n_{1:k}} M^{(k)}(n_{1:k}; \bullet) = \int\limits_{\mathcal{D}_{sc}} \sum_{n_{1:k}} m^{\otimes k}(n_{1:k}; \bullet) d\mathcal{P}_M(m) = \int\limits_{\mathcal{D}_{sc}} \left(\sum_{n\in\mathbb{N}} m(n; \bullet) \right)^{\otimes k} d\mathcal{P}_M(m) \\
&= \int\limits_{\mathcal{D}_{sc}} \left(\frac{q}{L^2(q+r)} + m(q; \bullet) \right)^{\otimes k} d\mathcal{P}_M(m) = \int\limits_{\mathcal{D}_{qLL}} \left(\frac{q}{L^2(q+r)} + \rho \right)^{\otimes k} d\mu(\rho) \\
&= \int\limits_{\mathcal{D}_{qLL}} \left(\frac{q}{L^2(q+r)} + \rho \right)^{\otimes k} d\mu(\rho)
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{E}_{qLL}[\rho_M] = \int\limits_{\mathcal{D}_{qLL}} \mathcal{E}_{qLL}\left[\frac{q}{L^2(q+r)} + \rho \right] d\mu(\rho) = & \int\limits_{\mathcal{D}_{qLL}} \left(E_V^{q,r} + E_w^{q,r} + \mathcal{E}_{qLL}[\rho] \right) d\mu(\rho) \\
= & E_V^{q,r} + E_w^{q,r} + \int\limits_{\mathcal{D}_{qLL}} \mathcal{E}_{qLL}[\rho] \, d\mu(\rho)
\end{aligned}$$

Now we are ready for the proof of the lower bound.

Proof of Proposition VI.9:

Let $\rho \in \mathcal{D}_{qLL}$, starting from (VI.33), using inequality (VI.36) and Proposition VI.1 we have

$$\begin{aligned}\mathcal{E}_{sc,\hbar b}\left[m_{\psi_N}\right] \geqslant \mathcal{E}_{sc,\hbar b}\left[m_\rho\right] + \mathcal{E}_{qLL}\left[\rho_{m_{\psi_N}}\right] - \mathcal{E}_{qLL}\left[\rho_{m_\rho}\right] + \mathcal{O}\left(\hbar b l_b\right) + \hbar b \mathcal{O}\left(1 - \frac{d(q+r)}{N}\right) \\ = \hbar b E_{q,r} + \mathcal{E}_{qLL}\left[\rho_{m_{\psi_N}}\right] + \mathcal{O}\left(\hbar b l_b\right) + \hbar b \mathcal{O}\left(1 - \frac{d(q+r)}{N}\right)\end{aligned}$$

We conclude with (VI.30) and (VI.46) and that

$$\begin{aligned}\mathcal{E}_{sc,\hbar b}\left[m_{\psi_N}\right] \geqslant & \hbar b E_{q,r} + \mathcal{E}_{qLL}\left[\rho_{m_{\psi_N}}\right] + \mathcal{O}\left(\hbar b l_b\right) + \hbar b \mathcal{O}\left(1 - \frac{d(q+r)}{N}\right) \\ = & \hbar b E_{q,r} + \mathcal{E}_{qLL}\left[\rho_M\right] + o(1) = \hbar b E^{q,r} + E_V^{q,r} + E_w^{q,r} + \int_{\mathcal{D}_{qLL}} \mathcal{E}_{qLL}\left[\rho\right] d\mu(\rho) + o(1) \\ \geqslant & \hbar b E^{q,r} + E_V^{q,r} + E_w^{q,r} + \mathcal{E}_{qLL}^0 + o(1)\end{aligned} \tag{VI.53}$$

VI.3 Conclusion

Proof of Theorem I.5:

Let $(\psi_N)_N$ be a sequence of minimizers of (I.7), by (V.9)

$$\frac{E(N)}{N} = \frac{\langle \psi_N | \mathscr{H}_N \psi_N \rangle}{N} = \mathcal{E}_{sc,\hbar b}\left[m_{\psi_N}\right] + o(1)$$

Since the lower bound is true up to a subsequence for which the have Proposition VI.10, for every adherence value of $E(N)/N$ we conclude by gathering Proposition VI.2 and Proposition VI.9.

Proof of Theorem I.7:

With (VI.45) and (VI.29) we get

$$\rho^{(k)}_{m_{\psi_N}} \underset{N\to\infty}{\overset{*}{\rightharpoonup}} \int_{\mathcal{D}_{qLL}} \left(\frac{q}{L^2(q+r)} + \rho\right)^{\otimes k} d\mu(\rho)$$

Let $\varphi \in C^\infty\left(\Omega^k\right)$ with (V.7),

$$\int_{\Omega^k} \varphi\left(\rho^{(k)}_{m_{\psi_N}} - \rho^{(k)}_{\psi_N}\right) = \int_{\Omega^k} \varphi\left((g_\lambda^2)^{\otimes k} * \rho^{(k)}_{\psi_N} - \rho^{(k)}_{\psi_N}\right) = \int_{\Omega^k} \rho^{(k)}_{\psi_N}\left((g_\lambda^2)^{\otimes k} * \varphi - \varphi\right) \underset{N\to\infty}{\to} 0 \tag{VI.54}$$

by Hölder's inequality since

$$\left\|\rho^{(k)}_{\psi_N}\right\|_{L^1} = 1$$

and φ is Lipschitz. Up to a subsequence $\rho^{(k)}_{\psi_N}$ converges $\forall k \in \mathbb{N}^*$ in the sense of Radon measures. But with (VI.54) this limit coincides with the one of $\rho^{(k)}_{m_{\psi_N}}$ so we obtain (I.25). Moreover by (VI.53) and Proposition VI.2

$$\mathcal{E}^0_{qLL} \geqslant \int\limits_{\mathcal{D}_{qLL}} \mathcal{E}_{qLL}\left[\rho\right] d\mu(\rho) + o(1)$$

thus μ only gives mass to minimizers of $\mathcal{E}_{qLL}$.

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