Real analysis - Tutorial

School of science
Specialisation class in the bachelor of mathematics
Class ID: CA-MATH-801

Professor: Shahnaz Farhat
sfarhat@constructor.university - Office 129.a Reasearch I

Teaching assitant: Périce Denis
dperice@constructor.university - Office 129.b Reasearch I

Contents

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Metric spaces

Let (X, d) be a metric space.

Problem I.1: Open and closed sets

Let $x \in X$ and $r > 0$.

1. Prove that open balls $B(x, r) \coloneqq \{y \in X | d(x, y) < r\}$ are opened.
2. Prove that closed balls $CB(x, r) \coloneqq \{y \in X | d(x, y) \leqslant r\}$ are closed.
3. Provide an example of a metric space and a ball for which $CB(x, r) \neq \overline{B(x, r)}$.
4. Prove that open sets are stable under countable unions and finite intersections.
5. Provide an example of a non opened countable intersection of open sets.
6. Prove that closed sets are stable under countable intersections and finite unions.
7. Provide an example of a non closed countable union of closed sets.

Problem I.2: Basic properties

1. Prove that (X, d) is first countable, i.e.: $\forall x \in X, \exists (O_n)_{n \in \mathbb{N}}$ a family of open neighbourhoods of x such that if $\mathcal{U}$ is an open neighbourhood of x, then $\exists n \in \mathbb{N} | O_n \subset \mathcal{U}$.
2. Prove that (X, d) is separated (also called Hausdorff), i.e.: $\forall x, y \in X, \exists \mathcal{U}, \mathcal{V}$ respectively open neighbourhoods of x and y such that $\mathcal{U} \cap \mathcal{V} = \varnothing$.

Problem I.3: Cutoff metric

Let $\alpha > 0$ and define $\forall x, y \in X, d'(x, y) = \min(\alpha, d(x, y))$.

1. Prove that d' is a metric on X.
2. Prove that d and d' give rise to the same topology.

Problem I.4: Completeness (*)

Prove that $\mathbb{R}^d$ is complete and that $\mathbb{Q}$ it not.

Problem I.5: Separability (*)

(X, d) is said to be separable if there exists a dense countable subset.

(X, d) is said to be second countable if these exists a countable family $\mathcal{T}$ of open sets such that every open set is a countable union of elements in $\mathcal{T}$.

1. Prove that (X, d) is separable $\iff$ (X, d) is second countable.
2. Prove that the set of bounded real sequences l^∞ is not separable.

Problem I.6: Normal spaces (**)

1. Urysohn lemma: let A, B be two disjoint closed subsets of X, prove that there exists a continuous function $f : X \to [0, 1]$ such that $A = f^{-1}(\{0\})$ and $B = f^{-1}(\{1\})$.
2. Deduce that (X, d) is normal, i.e.: every two disjoint closed sets have disjoint open neighbourhoods.

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Point set topology

Problem II.1: A bit more about metric spaces

Let (X, d) be a metric space.

1. Prove that (X, d) is separated (Hausdorff).
2. Prove that (X, d) is first countable.
3. Prove that (X, d) is separable $\iff$ (X, d) is second countable.

Problem II.2: Topological basis

Let $(X, \mathcal{T})$ be a topological space with a topological basis β. Let $\mathcal{O} \subset X$, prove that

$$\mathcal{O} \in \mathcal{T} \iff \forall x \in \mathcal{O}, \exists \mathcal{U}_x \in \beta | x \in \mathcal{U}_x \subset \mathcal{O}.$$

Problem II.3: G_δ sets

A G_δ set is a countable intersection of open sets. Prove that in a first countable T_1 topological space singletons are G_δ sets.

Problem II.4: Co-finite topology

Let X be an infinite set and $\mathcal{T} \coloneqq \{\mathcal{O} \subset X | \mathcal{O}^c \text{ is finite } \} \cup \{\varnothing\}$.

1. Prove that $(X, \mathcal{T})$ is a topological space.
2. Let $(x_n)_{n \in \mathbb{N}} \subset X$ be a sequence of distinct elements, prove that
$$\forall x \in X, x_n \underset{n \to \infty}{\to} x.$$
3. Is the co-finite topology separated?

We now assume that $X \coloneqq \mathbb{R}$.

4. Prove that closed sets are 0 of polynomials (this is what we call Zariski's topology).
5. Compute the interior and the closure of $[0, 1]$ and $\mathbb{Z}$.

Problem II.5: Co-countable topology (*)

Let X be an infinite set and $\mathcal{T} \coloneqq \{\mathcal{O} \subset X | \mathcal{O}^c \text{ is countable } \} \cup \{\varnothing\}$.

1. Prove that $(X, \mathcal{T})$ is a topological space.
2. Prove that the co-countable topology is stronger than the co-finite topology.
3. Under which condition do we have equality between these two topology?
4. Prove that X uncountable $\implies$ $(X, \mathcal{T})$ is not first-countable.

Problem II.6: Lexicographic order topology (**)

Let $(X, \leqslant)$ be an ordered space. The order topology on X, denoted $\mathcal{T}_\leqslant$, is the topology generated by open intervals.

1. Prove that $\mathcal{T}_\leqslant$ is the set of unions of open intervals.
2. Check that the order topology of the usual order on $\mathbb{R}$ is the usual topology.

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We introduce the lexicographic order on $\mathbb{R}^2$: $(x, y) \leqslant (x_0, y_0)$ if $x < x_0$ or ($x = x_0$ and $y \leqslant y_0$).

3. Prove that $\leqslant$ is total.
4. Draw the open intervals $((0,1),(1,0))$ and $((0,1),(0,2))$.
5. Prove that $(\mathbb{R}^2, \mathcal{T}_\leqslant)$ is not separable.
6. Let $\mathcal{O} \subset \mathbb{R}^2$, prove that

$$\mathcal{O} \in \mathcal{T}_\leqslant \iff \forall x \in \mathbb{R}, \mathcal{O}_x \coloneqq \{y \in \mathbb{R} | (x, y) \in \mathcal{O}\} \text{ is open in the usual topology of } \mathbb{R}.$$

7. Prove that vertical lines are clopen (close and open).

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Continuous functions & compactness I

Problem III.1: Density

Let X, Y be a topological spaces, assume that Y is Hausdorff and $g, f : X \to Y$ continuous.

1. Prove that $A \coloneqq \{x \in X | f(x) = g(x)\}$ is closed.
2. Prove that A dense $\implies f = g$.

Problem III.2: Quotient topology

Let $(X, \mathcal{T})$ be a topological space, $\sim$ an equivalence relation on X and $\pi : x \in X \mapsto [x] \coloneqq \{y \in X | t \sim x\} \in X/\sim$. We set $\mathcal{T}_\sim \coloneqq \{\mathcal{O} \subset X/\sim | \pi^{-1}(\mathcal{O}) \in \mathcal{T}\}$.

1. Prove that $\mathcal{T}_\sim$ is a topology.
2. Let $A \subset X$ define $A_\sim \coloneqq \{x \in X | \exists y \in X | x \sim y\}$. Prove that π is an open map if and only if $\forall A \subset \mathcal{T}, A_\sim \in \mathcal{T}$.
3. Let Z be a topological space. Prove that a map $g : X/\sim \longrightarrow Z$ is continuous if and only if $g \circ \pi : X \longrightarrow Z$ is continuous.
4. Let $f : X \longrightarrow Z$ be a continuous map such that $x \sim y \implies f(x) = f(y)$. Prove that there exists a unique continuous map $\bar{f} : X/\sim \longrightarrow Z$ such that $\bar{f} \circ \pi = f$.
5. Prove that the above property, along with the continuity of π, characterises the quotient topology.
6. Let $A \subset X$. The collapsing of X onto A refers to the space $X/\sim$, where $\sim$ is the equivalence relation generated by $x \sim y$ for every pair $(x, y) \in A^2$. Describe the sphere S^2 as the collapsing of a space onto a suitable subspace.

Problem III.3: Compactness in term of close sets

Let $(K, \mathcal{T})$ be a topological space. Prove that K is compact if and only if for all family of closed sets $(F_i)_{i \in I}$,

$$\left(\forall J \subset I \text{ finite}, \bigcap_{i \in J} F_i \neq \varnothing\right) \implies \bigcap_{i \in I} F_i \neq \varnothing.$$

Problem III.4: A characterisation of continuity (*)

Let X, Y be topological spaces and $f : X \to Y$. Prove that f is continuous if and only if $\forall A \subset X, f(\overline{A}) \subset \overline{f(A)}$.

Problem III.5: Diagonal extraction (**)

We consider the metric space $C \coloneqq [0,1]^{\mathbb{N}}$ with the metric

$$d(x, y) = \sum_{n \in \mathbb{N}} 2^{-n} |x_n - y_n|.$$

Prove that C is sequentially compact.

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Continuous functions & compactness II

Problem IV.1:

Let $f : (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ between two topological spaces.

1. Let $x \in X$, prove that if f is continuous at x then

$$\forall (x_n)_{n\in\mathbb{N}} \subset X, x_n \underset{n\to\infty}{\to} x \implies f(x_n) \underset{n\to\infty}{\to} f(x).$$

2. Prove the converse proposition assuming that X is first countable.
3. We assume that f is continuous, injective and that Y is Hausdorff, prove that X is Hausdorff.

Problem IV.2: Homeomorphism

Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a continuous bijection such that $\lim_{\|x\|\to+\infty} \|f(x)\| = +\infty$.

1. Prove that $\forall K \subset \mathbb{R}^n$ compact, $f^{-1}(K)$ is compact.
2. Let $C \subset \mathbb{R}^n$ closed, prove that $f(C)$ is closed.
3. Conclude that f is a homeomorphism.

Problem IV.3: Application of the Arzelà–Ascoli theorem

Let $f \in C([0,1])$ and $K \in C([0,1] \times [0,1])$, define

$$\forall x \in [0,1], Tf(x) := \int_0^1 K(x,y)f(y)dy.$$

1. Prove that Tf is continuous.
2. Prove that $\left\{Tf, f \in C([0,1]) \middle| \sup_{x\in[0,1]} |f(x)| \leqslant 1\right\}$ is relatively compact (i.e. has compact closure) in $C([0,1])$.

Problem IV.4: First Dini theorem (*)

Let K be a compact metric space and $(f_n)_n$ a sequence of functions in $C^0(K,\mathbb{R})$ that converges pointwise to a function f continuous on K.

1. Suppose that the sequence $(f_n)_n$ is increasing. Prove that the convergence is uniform.
2. Application: Prove that $t \mapsto \sqrt{t}$ is the uniform limit of polynomials P on $[0,1]$. *Hint: Use the sequence* $P_0 = 0, P_{n+1}(t) = P_n(t) + \frac{1}{2}(t - P_n^2(t))$.

Problem IV.5: Compactness in metric spaces (**) Prove that compactness is equivalent to sequential compactness in metric spaces.

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Arzelà–Ascoli theorem & Banach spaces

Problem V.1: Non compactness of closed balls in infinite dimensions

Prove that $\overline{B(0,1)}$ is not compact in $l^\infty(\mathbb{C})$.

Problem V.2: Hölder continuity

Let (X,d) be a compact metric space, $f \in C(X,\mathbb{C})$ is called Hölder continuous of exponent $\alpha > 0$ if

$$N_\alpha(f) \coloneqq \sup_{x,y\in X, x\neq y} \frac{|f(x)-f(y)|}{d(x,y)^\alpha} < \infty.$$

Prove that $K \coloneqq \{f \in C(X,\mathbb{C}) | \, \|f\|_u \leqslant 1 \text{ and } N_\alpha(f) \leqslant 1\}$ is compact in $C(X,\mathbb{C})$.

Problem V.3: Application with integral kernel

Let $f \in C([0,1],\mathbb{R})$ and $K \in C([0,1]\times[0,1],\mathbb{R})$, define $\forall x \in [0,1], Tf(x) \coloneqq \int_0^1 K(x,y)f(y)dy$.

1. Prove that Tf is continuous.
2. Prove that $\{Tf, f \in C([0,1],\mathbb{R}) | \, \|f\|_u \leqslant 1\}$ is relatively compact in $C([0,1],\mathbb{R})$.
3. Prove that if $K \leqslant 1$ then T has a fixed point.

Problem V.4: Complete functional spaces

1. Let X, Y be two Banach space. Prove that $C(X,Y)$ is complete with respect to the uniform norm.
2. Prove that $\mathcal{C} \coloneqq \{f \in C([0,1],\mathbb{R}) | f(0) = 1\}$ is closed in $C([0,1],\mathbb{R})$.

Problem V.5: (*)

Let $f \in C([0,1],\mathbb{R})$ and $K \in C([0,1]\times[0,1],\mathbb{R})$, define $\forall x \in [0,1], Tf(x) \coloneqq \int_0^1 K(x,y)f(y)^2dy$. Prove that T has a fixed point.

Problem V.6: Compactness in Banach spaces (**)

1. Let E be a Banach space and X a subset of E. Prove that X is compact if and only if X is closed, bounded, and for every $\varepsilon > 0$, there exists a finite-dimensional vector subspace F_ε of E such that $d(x, F_\varepsilon) < \varepsilon$ for every $x \in X$.

   Hint: a metric space is compact if and only if it is totally bounded and complete.
2. A subset A of $l^1(\mathbb{N})$ is said to be equisummable if

   $$\forall \varepsilon > 0, \exists N \geqslant 0, \forall (x_n)_{n\in\mathbb{N}} \in A, \quad \sum_{n\geqslant N} |x_n| < \varepsilon.$$

   Prove that a subset A of $l^1(\mathbb{N})$ is compact if and only if it is closed, bounded, and equisummable. Provide an example.

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Banach spaces

Problem VI.1: Evaluation map

Let X, Y be normed vector spaces and $\mathcal{L}(X,Y)$ denote the space of continuous linear maps, prove that

$$\begin{array}{ccccc} \mathcal{L}(X,Y) & \times & X & \to & Y \\ T & , & x & \mapsto & Tx \end{array}$$

is continuous.

Problem VI.2:

Let $L \subset C([0,1],\mathbb{R})$ be the subspace consisting of Lipschitz functions.

1. Is $(L, \|\cdot\|_\infty)$ complete?
2. What about the set of 1-Lipschitz functions?
3. Prove that $N(f) \coloneqq \|f\|_\infty + \sup_{x\neq y \in [0,1]} \left(\left| \frac{f(x)-f(y)}{x-y} \right| \right)$ defines a norm on L.
4. Prove that (L, N) is complete.
5. Prove that $C^1([0,1],\mathbb{R})$ is complete with respect to the norm $\|f\| \coloneqq \|f\|_\infty + \|f'\|_\infty$.

Problem VI.3:

1. Find a continuous linear map $f : \mathbb{R} \to \mathbb{R}$ that is not open.
2. Find a continuous surjective map $f : \mathbb{R} \to \mathbb{R}$ that is not open.
3. Let $\mathbb{R}^{(\mathbb{N})}$ be the space of real sequences that are null except for a finite number of terms, equipped with the norm $\|\cdot\|_\infty$. Verify that this normed vector space is not complete, and find a bijective, continuous map $T \in \mathcal{L}(\mathbb{R}^{(\mathbb{N})})$ whose inverse is not continuous.

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Banach spaces II: Hahn-Banach Theorem

Problem VII.1: Dual norm

Let E be a Banach space.

1. Let $x \in E$, prove that

$$\|x\|_E = \sup\{\varphi(x), \varphi \in E^* |\ \|\varphi\|_{E^*} \leqslant 1\}.$$

2. We assume that E is reflexive. Let $\varphi \in E^*$, deduce that $\exists x \in E$ such that $\|\varphi\|_{E^*} = \varphi(x)$.

Problem VII.2: l^1 is not reflexive

Prove that $l^1(\mathbb{R}) \subsetneq l^\infty(\mathbb{R})^*$, meaning that there exists $T \in l^\infty(\mathbb{R})^*$ which is not of the following form

$$\begin{array}{ccc} l^\infty(\mathbb{R}) & \to & \mathbb{R} \\ (y_n)_{n\in\mathbb{N}} & \mapsto & \sum x_n y_n \end{array}$$

with $(x_n)_{n\in\mathbb{N}} \in l^1(\mathbb{R})$.

Problem VII.3: Functional vanishing on a subspace

Let $X := \{f \in C([0,1],\mathbb{R}) | f(0) = 0\}$ and $\mathcal{C}$ be the subset of $C([0,1],\mathbb{R})$ made up of constant functions.

1. Prove that $C([0,1],\mathbb{R}) = X \oplus \mathcal{C}$.
2. Prove that $\exists F \in C([0,1],\mathbb{R})^*$ such that $F_{|X} = 0$ and $\forall c \in \mathcal{C}, c \neq 0 \implies F(c) \neq 0$.

   Hint: consider the evaluation at 0.

Problem VII.4: Density and duality (*)

Let U be a dense subspace of a normed vector space X. Prove that U^* and X^* are isometrically isomorphic.

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Banach spaces III: uniform boundedness & open mapping theorem

Problem IX.1: Continuity via duality

Let E and F be two Banach spaces, and let $T : E \to F$ be a linear map. We assume that $\forall \varphi \in F^*, \varphi \circ T \in E^*$. Prove that T is continuous.

Hint: use the fact that $\forall x \in E$

$$\|Tx\|_F = \sup_{\varphi \in F^*, \|\varphi\|_{F^*} \leqslant 1} \varphi(Tx).$$

Problem IX.2: Open mapping theorem

Let E and F be two Banach spaces, and let $T \in \mathcal{L}_c(E, F)$. Prove that the following two statements are equivalent:

1. There exists $\alpha > 0$ such that for all $x \in E$, $\|Tx\| \geqslant \alpha\|x\|$.
2. T is injective and has closed range.

Problem IX.3: Dual of l^p

Let E and F be two Banach spaces. Let (T_n) be a sequence of continuous linear maps from E to F such that, $\forall x \in E$, $(T_n x)_{n\in\mathbb{N}}$ converges to a limit denoted Tx.

1. Prove that the mapping $x \mapsto Tx$ is linear.
2. Prove that $\sup\limits_{n\in\mathbb{N}} ||T_n|| < +\infty$. Deduce that T is continuous.
3. Prove that
$$||T|| \leqslant \liminf_{n\to+\infty} ||T_n||.$$

Let $1 < p, q < +\infty$ be real numbers such that $1/p + 1/q = 1$. Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of complex numbers such that for every sequence $(b_n)_{n\in\mathbb{N}}$ in $\ell^p(\mathbb{C})$, the series $\sum |a_n b_n|$ converges.

4. Viewing
$$T : \begin{array}{ccc} \ell^p(\mathbb{C}) & \to & \mathbb{C} \\ (b_n)_{n\in\mathbb{N}} & \mapsto & \sum \overline{a_n} b_n \end{array}$$
as a pointwise limit, deduce that T is continuous.
5. Prove that $(a_n)_{n\in\mathbb{N}} \in \ell^q(\mathbb{C})$.

   Hint: consider $b_n \coloneqq |a_n|^{q-1} s(a_n)\mathbb{1}_{n\leqslant N}$ *where s is the sign function defined as* $s(z) \coloneqq \frac{z}{|z|}\mathbb{1}_{z\neq 0}$.

Problem IX.4: (*)

Let E be a Banach space and F and G two closed subspaces of E. We assume that F and G are algebraic complements, *i.e.*

$$F + G = E, \quad F \cap G = \{0\}.$$

Prove that F and G are topological complements, *i.e.* that the projections associated with the sum are continuous.

Problem IX.5: (**)

Let K be a compact subset of $\mathbb{R}^n$. Consider a norm N on the space $\mathcal{C}^0(K, \mathbb{R})$ that makes it complete and satisfies the condition that any sequence of functions (f_n) in $\mathcal{C}^0(K, \mathbb{R})$ that converges in the norm N also converges pointwise to the same limit. Prove that the norm N is then equivalent to the supremum norm, *i.e.*

$$\exists c_1, c_2 > 0, \forall f \in \mathcal{C}^0(K, \mathbb{R}), \quad c_1 N(f) \leqslant \|f\|_\infty \leqslant c_2 N(f).$$

*: Bonus problem

**: Bonus and harder problem

Hilbert spaces

Problem X.1: Projections

Let H be a Hilbert space and C a closed convex subset of H. We denote by p_C the orthogonal projection onto C, and let u and v be elements of H.

1. Prove that $v = p_C(u)$ if and only if $v \in C$ and $\langle u - v, c - v\rangle \leqslant 0$ for all c in C.
2. Prove that p_C is 1-Lipschitz continuous.
3. Prove that $\langle p_C(u) - p_C(v), u - v\rangle \geqslant 0$.

Problem X.2: Countable basis

1. Prove that a Hilbert space is separable if and only if there exists a countable Hilbert basis.
2. (*) Prove that a vector space of infinite dimension with a countable algebraic basis cannot be equipped with a norm that makes it complete.

Problem X.3: Geometric forms of the Hahn Banach theorem (**)

Let A and B be two disjoint convex subsets of a Hilbert space H.

1. Assume that A is closed and let $x \notin A$. Prove that there exists $\phi \in H^\star$ such that $\sup_A \phi < \phi(x)$.
2. Prove that if A is compact and B is closed, there exists $\phi \in H^\star$ such that $\sup_A \phi < \inf_B \phi$.
3. Prove that if A is open, there exists $\phi \in H^\star\backslash\{0\}$ such that $\sup_A \phi \leqslant \inf_B \phi$.

*: Bonus problem
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Lebesgue spaces

Problem XI.1: Shift operator

1. Prove that the shift operator:

$$S : \begin{array}{ccc} l^1 & \to & l^2 \\ (u_n)_{n\in\mathbb{N}} & \to (0, u_0, u_1, \dots) & \end{array}$$

   is a bounded linear map and compute Compute $\|S\|$.
2. Compute the adjoint S^* as a linear map $l^2 \to l^\infty$ and compute $\|S^*\|$.

Problem XI.2:

Let $p_1 \leqslant p \leqslant p_2 \leqslant +\infty$, prove that $L^p \subset L^{p_1} + L^{p_2}$.

Hint: Let $\Lambda > 0, f \in L^p$, *and consider* $f_\Lambda \coloneqq f\mathbb{1}_{|f|\leqslant\Lambda}$.

Problem XI.3: Separability (**)

Let $1 \leqslant p < +\infty$, let $a < b \in \mathbb{R}$, prove that $L^p([a, b])$ is separable.

Weak convergence

Problem XII.1:

Let $(E, \|\cdot\|)$ be a Banach space and $(x_n)_{n\in\mathbb{N}} \subset E$ such that $x_n \rightharpoonup x \in E$.

1. Prove that $(x_n)_{n\in\mathbb{N}}$ is bounded with respect to $\|\cdot\|$.

   Hint: consider the canonical injection of E into E^{**} and use uniform boundedness.

2. Deduce that a weakly compact subset $A \subset E$ is bounded with respect to $\|\cdot\|$.

3. Prove that

$$\frac{1}{n}\sum_{k=1}^{n} x_k \underset{n\to\infty}{\rightharpoonup} x.$$

Problem XII.2:

1. Let $\varphi \in C_c^\infty(\mathbb{R})$ be smooth and compactly supported function. Prove that

$$\int_n^{n+1} \varphi(x)dx \underset{n\to\infty}{\to} 0.$$

2. Deduce that $\mathbb{1}_{[n,n+1]} \rightharpoonup 0$ in $L^2(\mathbb{R})$.

3. Does this convergence holds in $\|.\|_2$?

4. Admitting the fact that $(e_k : x \mapsto e^{2i\pi kx})_{k\in\mathbb{Z}}$ is a Hilbert basis of $L^2([0,1],\mathbb{C})$, prove that $\sin(2\pi n\cdot) \rightharpoonup 0$ in $L^2([0,1],\mathbb{C})$.

   Hint: Use Parseval's identity.

Thomas-Fermi functional

Let $V \in L^2(\mathbb{R}^2, \mathbb{R}), w \in L^1(\mathbb{R}^2, \mathbb{R})$ such that $\|w\|_1 < 1$ and

$$\hat{w}(\nu) := \int_{\mathbb{R}^2} w(x)e^{-2i\pi\nu\cdot x}dx \geqslant 0.$$

We aim to minimise to following functional:

$$E(\rho) := \int_{\mathbb{R}^2} \rho^2(x)dx + \int_{\mathbb{R}^2} V(x)\rho(x)dx + \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} w(x-y)\rho(x)\rho(y)dxdy.$$

with

$$\rho \in \mathcal{D} := \left\{ \rho \in L^2(\mathbb{R}^2, \mathbb{R}_+) \text{ such that } \int_{\mathbb{R}^2} \rho(x)dx = 1 \right\}.$$

Denote

$$E_0 = \inf_{\rho\in\mathcal{D}} E(\rho).$$

1. Prove that $\|(w \star \rho)\rho\|_1 \leqslant \|w\|_1 \|\rho\|_2^2$.
2. Deduce that E is well defined and that $E_0 < +\infty$.
3. Prove that $\forall a, b \in \mathbb{R}, \forall \epsilon > 0, ab \leqslant \frac{a^2}{2\epsilon} + \epsilon\frac{b^2}{2}$.
4. Prove that

$$\|\rho\|_2^2 \leqslant \frac{1}{1-\|w\|_1}\left(2E(\rho) + \frac{\|V\|_2^2}{1-\|w\|_1}\right).$$

Let $(\rho_n)_{n\in\mathbb{N}}$ be a minimising sequence of E in $\mathcal{D}$, meaning that

$$E(\rho_n) \underset{n\to\infty}{\to} E_0.$$

5. Prove that, up to an extraction, $\exists\mu \in L^2(\mathbb{R}^2, \mathbb{R})$ such that

$$\rho_n \underset{n\to\infty}{\rightharpoonup} \mu.$$

6. Prove that

$$\int_{\mathbb{R}^2} \mu^2(x)dx + \int_{\mathbb{R}^2} V(x)\mu(x)dx \leqslant \liminf_{n\to\infty}\left(\int_{\mathbb{R}^2} \rho_n^2(x)dx + \int_{\mathbb{R}^2} V(x)\rho_n(x)dx\right).$$

7. Prove that

$$\int_{\mathbb{R}^2}(\mu \star w)(x)\mu(x)dx \leqslant \liminf_{n\to\infty}\int_{\mathbb{R}^2}(\rho_n \star w)(x)\rho_n(x)dx$$

What we previously obtained still remains valid if we only assume that the negative part of V satisfy $V_- \in L^2(\mathbb{R}^2, \mathbb{R})$. Now, we additionally assume that

$$V(x) \underset{|x|\to\infty}{\to} \infty.$$

8. (*) Prove that

$$\int_{\mathbb{R}} V(x)\rho(x)dx \leqslant E(\rho)$$

and deduce that for $r > 0$,

$$\int_{\mathbb{R}} \mathbb{1}_{x>r}\rho(x)dx \leqslant \frac{2E(\rho)}{\inf_{|x|>r} V(x)}.$$

9. (*) Prove that $\|\mu\|_1 = 1$
10. (*) Conclude that $\mu \in \mathcal{D}$ and that μ is a minimizer of E.

*: Bonus problem
**: Bonus and harder problem

Topological, metric, Banach, Hilbert & Lebesgue spaces

Problem 1: Hausdorff spaces

1. Provide an example of a non-Hausdorff topological space.
2. Prove that metric spaces are Hausdorff.

Let X be a Hausdorff topological space.

3. We assume that X is compact. Prove that we can separate a closed set from a point, meaning: $\forall F \subset X$ closed, $\forall x \in F^c, \exists \mathcal{U}, \mathcal{V}$ open such that $x \in \mathcal{U}, F \subset \mathcal{V}$ and $\mathcal{U} \cap \mathcal{V} = \varnothing$.
4. Same question with X metrizable.

Problem 2: Distance to a set

Let (X, d) be a metric space. We define the distance from $x \in X$ to $A \subset X$ as

$$d(x, A) \coloneqq \inf_{a \in A} d(x, a).$$

1. Prove that $d(\cdot, A)$ is 1-Lipschitz.
2. Prove that A is closed if and only if $\forall x \in A, d(x, A) = 0 \implies x \in A$.

Problem 3: Bounded linear maps

Decide for which topological spaces $X, Y \in \{(C([-1,1]), \|\cdot\|_u), (L^2([-1,1]), \|\cdot\|_2)\}$ the map

$$T : \begin{array}{ccc} X & \to & Y \\ f & \mapsto & \begin{pmatrix} [-1,1] & \to & \mathbb{R} \\ x & \mapsto & f(x^2) \end{pmatrix} \end{array}$$

is a bounded linear map and compute $\|T\|_{\mathcal{L}(X,Y)}$ when it is the case.

Problem 4: A characterization of the uniform topology

Let K be a compact subset of $\mathbb{R}^n$ and $\mathcal{N}$ a complete norm on $C(K, \mathbb{R})$ such that any sequence of functions $(f_n)_{n \in \mathbb{N}}$ in $C(K, \mathbb{R})$ that converges in the norm $\mathcal{N}$ also converges pointwise to the same limit. We aim to prove that $\mathcal{N}$ is then equivalent to the uniform norm, *i.e.*

$$\exists c_1, c_2 > 0 \text{ such that } \forall f \in C(K, \mathbb{R}),\ c_1 \mathcal{N}(f) \leqslant \|f\|_u \leqslant c_2 \mathcal{N}(f).$$

Let $x \in K$, we consider the evaluation map

$$T_x : \begin{array}{ccc} (C(K, \mathbb{R}), \mathcal{N}) & \to & \mathbb{R} \\ f & \mapsto & f(x) \end{array}.$$

1. Prove that T_x is continuous.
2. Prove that $\sup_{x \in K} \|T_x\| < +\infty$.
3. Prove that $\|\cdot\|_u \leqslant \sup_{x \in K} \|T_x\| \mathcal{N}$
4. Deduce that

$$i : \begin{array}{ccc} (C(K, \mathbb{R}), \mathcal{N}) & \to & (C(K, \mathbb{R}), \|\cdot\|_u) \\ f & \mapsto & f \end{array}$$

is a continuous bijection.

5. Conclude that $\mathcal{N}$ and $\|\cdot\|_u$ are equivalent.

Problem 5: Polarization identity

Let H be a complex pre-Hilbert space with $\langle\cdot,\cdot\rangle_H$ its inner product linear in the second variable end skew linear in the first variable. Denote $\forall x \in H, \|x\|_H \coloneqq \sqrt{\langle x, x\rangle_H}$.

1. Prove the polarization identity:

$$\forall x, y \in H, \langle x, y\rangle_H = \frac{1}{4}\left(\|x+y\|_H^2 - \|x-y\|_H^2 - i\,\|x+iy\|_H^2 + i\,\|x-iy\|_H^2\right)$$

2. Assume that $(H, \|\cdot\|)$ is a normed vector space with $\|\cdot\|$ satisfying the parallelogram law, prove that there exists a Hermitian inner product $\langle\cdot,\cdot\rangle$ on H such that $\forall x \in H, \|x\| = \sqrt{\langle x, x\rangle}$.

Problem 6: Lebesgue spaces

1. Let $a < b$, prove that $L^\infty([a,b])$ is not separable.
2. Let $x \coloneqq (x_n)_{n\in\mathbb{N}} \in l^\infty(\mathbb{R})$, prove that

$$\Phi(x) : \begin{array}{ccc} l^1(\mathbb{R}) & \to & \mathbb{R} \\ y \coloneqq (y_n)_{n\in\mathbb{N}} & \mapsto & \langle x, y\rangle \coloneqq \sum_{n\in\mathbb{N}} x_n y_n \end{array} \in l^1(\mathbb{R})^*$$

3. Prove that $\Phi : l^\infty(\mathbb{R}) \to l^1(\mathbb{R})^*$ is an isometry.