Semi-classical limit of the 2D Hartree equation in a large magnetic field

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Abstract:

We study the dynamic of two dimensional fermionic particles submitted to a magnetic field, assumed to be transverse to the domain and homogeneous. A large magnetic field regime where the gap between Landau levels is of the same order as the other energy contributions is considered. We start from the Hartree equation for the first reduced density matrix, describing the mean field behaviour of a large fermionic system, and derive a gyrokinetic transport equation for the first reduced density. We define a semi-classical density for which the dynamic is computed and compared to the limit gyrokinetic equation. It is shown that the first reduced density almost satisfies the gyrokinetic equation and converges in the sense of measures to a weak solution of this equation.

Contents

I Context and results

I.1 Model

We consider a large system of interacting fermionic particles in two dimensions. They are placed in a homogeneous magnetic field perpendicular to the domain. In this context the kinetic energy of the particles is quantized into discrete energy levels called Landau levels, separated by a finite energy gap. Our goal is to study the semi-classical limit of the dynamics under high magnetic field. This setup is physically motivated by the quantum hall effect, see [23] for some physical context. We will start from the Hartree equation appropriate for a large system of fermions, and obtain a gyrokinetic transport equation for the density.

Notation I.1: *Model*
We work on $\mathbb{R}^2$. The one body kinetic energy operator, also called magnetic Laplacian, is

$$\mathscr{L}_b := (i\hbar\nabla + bA)^2$$

With

$$Dom(\mathscr{L}_b) := \left\{\psi \in L^2\left(\mathbb{R}^2\right) | \mathscr{L}_b\psi \in L^2\left(\mathbb{R}^2\right)\right\}$$

We work in symmetric gauge, namely the vector potential is

$$A = \frac{1}{2}X^{\perp} \tag{1}$$

where X is the position operator in $\mathbb{R}^2$. b is the magnetic field amplitude, we associate to it the magnetic length

$$l_b := \sqrt{\frac{\hbar}{b}}$$

Let V be the external potential and w the interaction potential assumed to be radial:

$$w(x-y) =: \widetilde{w}(|x-y|)$$

We denote $\mathcal{L}^p$ the p^{th} Schatten space. Let $\gamma \in L^\infty\left(\mathbb{R}_+, \mathcal{L}^1\left(L^2\left(\mathbb{R}^2\right)\right)\right)$, and ρ_γ be the associated reduced density

$$\rho_\gamma(t,x) := \gamma(t)(x,x) \tag{2}$$

Here we identified γ with its integral kernel and we will use this convention for the rest of the text. Let $W : \mathbb{R}^2 \to \mathbb{R}$ be a potential, we introduce a notation for the gyrokinetic differential operator:

$$GYRO_W(\mu)(t,z) := \partial_t\mu(t,z) + \nabla^\perp W(z)\cdot\nabla\mu(t,z)$$
$$GYRO_\rho(\mu)(t,z) := GYRO_{V+w\star\rho(t)}(\mu)(t,z) = \partial_t\mu(t,z) + \nabla^\perp\left(V + w\star\rho(t)\right)(z)\cdot\nabla\mu(t,z)$$

Finally denote

$$H_b(t) := \mathscr{L}_b + V + \frac{1}{2}w\star\rho_{\gamma_b(t)} \tag{3}$$

Our goal is to obtain from the Hartree equation

$$i\hbar\partial_t\gamma = [\mathscr{L}_b + V + w\star\rho_\gamma, \gamma] \tag{4}$$

the following gyrokinetic transport equation for a density $\rho : \mathbb{R}_+ \times \mathbb{R}^2 \to \mathbb{R}_+$,

$$\partial_t\rho + \nabla^\perp(V + w\star\rho))\cdot\nabla\rho = 0 \tag{5}$$

The Hamiltonian $\mathscr{L}_b + V + w\star\rho_\gamma$ is time dependent hence the total energy is not preserved through the evolution (4). However, see Proposition III.2, the quantity $\mathrm{Tr}\left[H_b(t)\gamma(t)\right]$ is constant due to the 1/2 factor in front of the interaction term.

I.2 Scaling

In classical mechanics, the movement of a fermion in a transverse magnetic field is decomposed in a cyclotron orbit and the motion of the orbit center. As exposed in Subsection I.4 the latter takes place on a time scale of order b. As recalled in Subsection II.1, the order of magnitude of the kinetic energy is $\hbar b$. Our plan is to look at a scaling where all the terms in the Hamiltonian are of order 1 and the time scale is of order b.

Notation I.2: *scaling*
We take a high magnetic field limit

$$b \to +\infty$$

coupled with a semi-classical scaling

$$\hbar \underset{b\to\infty}{\to} 0$$

such that the magnetic kinetic energy is of order 1*:*

$$\hbar b \underset{b\to\infty}{\to} 1 \tag{6}$$

Let $\gamma \in L^{\infty}\left(\mathbb{R}_+, \mathcal{L}^1\left(L^2(\mathbb{R}^2)\right)\right)$*, such that*

$$Tr\left[\gamma(0)\right] = 1 \quad \text{ and } \quad 0 \leqslant \gamma(0) \leqslant 2\pi l_b^2 \tag{7}$$

define the time rescaled density matrix

$$\forall t \in \mathbb{R}_+, \gamma_b(t) := \gamma(bt) \tag{8}$$

If γ satisfies (4), the equation for the time-rescaled density matrix is

$$\partial_t \gamma_b = \frac{b}{i\hbar}\left[\mathscr{L}_b + V + w \star \rho_{\gamma_b}, \gamma_b\right] = \frac{1}{il_b^2}\left[\mathscr{L}_b + V + w \star \rho_{\gamma_b}, \gamma_b\right] \tag{9}$$

Due to the constraint (7) known to propagate in time (see Proposition III.1), from (2) we see that

$$\int_{\mathbb{R}^2} \rho_{\gamma_b}(t) = \mathrm{Tr}\left[\gamma_b(t)\right]$$

Moreover the Pauli principle $\gamma_b \leqslant 2\pi l_b^2$ guarantees that the system occupies a volume of order 1 in the limit

$$l_b \underset{b\to\infty}{\to} 0$$

Indeed it is known, [23] or [1, subsection I.4], that the degeneracy per area inside a Landau level is of order l_b^{-2}. A typical fermionic state satisfying (7) is a projection onto a N-body Slater determinant of N orthonormal one body wave-functions with

$$N := \mathcal{O}\left(\frac{1}{2\pi l_b^2}\right)$$

Such a N-particles state occupies a volume of order

$$\frac{N}{l_b^{-2}} = \mathcal{O}(1)$$

Hence with (6) this confirms that all the terms in the Hamiltonian $\mathscr{L}_b + V + w \star \rho_\gamma$ are of order 1. If one starts from the N-body Schrödinger dynamics, the fermionic characteristic of the system is imposed by the anti-symmetry of the wave-functions. Starting from the mean field dynamics, the fermionic characteristic is imposed directly via the Pauli principle $\gamma_b \leqslant 2\pi l_b^2$.

As a remark, we give an equivalent formulation of this scaling. If one takes exactly $\hbar = 1/b$, then (9) is equivalent to

$$i\partial_t \gamma = \left[\left(i\nabla + b^2 A\right) + b^2(V + w \star \rho_{\gamma_b}), \gamma_b\right]$$

In other words with the new scaling

$$\widetilde{b} \coloneqq b^2$$
$$\widetilde{\gamma_b} \coloneqq \frac{b^2}{2\pi}\gamma$$

we have

$$\mathrm{Tr}\left[\widetilde{\gamma_b}\right] = \frac{\widetilde{b}}{2\pi}, \quad \widetilde{\gamma_b} \leqslant 1$$
$$i\partial_t \widetilde{\gamma_b} = \left[\left(i\nabla + \widetilde{b}A\right)^2 + \widetilde{b}V + w \star \rho_{\widetilde{\gamma_b}}, \widetilde{\gamma_b}\right]$$

where all the terms in the Hamiltonian $\left(i\nabla + \widetilde{b}A\right)^2 + \widetilde{b}V + w \star \rho_{\widetilde{\gamma_b}}$ are of order $\widetilde{b}$.

I.3 Results

The classical counterpart of this work, starting from the Vlasov equation, has been well studied [29] [26] [28] [20] [25] [22] [15] [7]. Some results start from Newton's dynamics [8]. In the quantum literature it is known that the Hartree equation can be obtained by a mean field limit from the N-body Schrödinger dynamics [19] [14] [16]. It is also known that the Vlasov equation can be derived by a semi-classical limit from the Hartree equation [18] [10]. The mean field and semi-classical limits can be coupled to obtain directly the Vlasov equation from the N-body Schrödinger dynamics [5] [6] [2]. More recent results have been dealing with singular potentials [13] [17] [4]. We also refer to a semi-classical work [3] obtaining Euler's vorticity equation from the N-body Schrödinger dynamics in a regime where the gap between Landau levels is small compared to the interactions. Note that for large magnetic fields, the limit of the fundamental energy of the N-body Hamiltonian and the associated densities have been well studied [30], [31], [32], [33], [34] [11], [12] [1].

Now, with Notation I.1 and Notation I.2, we can state our main results. The first reduced density approximately satisfies a gyrokinetic dynamics:

Theorem I.3: *Gyrokinetic limit of the Hartree solution*
Let $\gamma_b \in L^\infty\left(\mathbb{R}_+, \mathcal{L}^1\left(L^2\left(\mathbb{R}^2\right)\right)\right)$ be a solution of (9) and assume

$$Tr\left[\gamma_b(0)\right] = 1, 0 \leqslant \gamma_b(0) \leqslant 2\pi l_b^2$$
$$Tr\left[\gamma_b(0)H_b(0)\right] < \infty$$

If $V, w \in W^{4,\infty}\left(\mathbb{R}^2\right)$, then $\forall \varphi \in C_c^\infty\left(\mathbb{R}_+ \times \mathbb{R}^2\right)$,

$$\left| \int\limits_{\mathbb{R}_+ \times \mathbb{R}^2} \rho_{\gamma_b}(t,z)\, GYRO_{\rho_{\gamma_b}}(\varphi)(t,z)\,dt\,dz - \int\limits_{\mathbb{R}^2} \varphi(0,z)\rho_{\gamma_b}(0,z)\,dz \right| \leqslant C(\varphi, V, w)\frac{1}{\sqrt{\ln\left(l_b^{-1}\right)}}$$

Moreover the reduced density converges to a weak solution of the gyrokinetic transport equation.

Theorem I.4: *Convergence of densities*
Under the same assumptions as Theorem I.3, up to a subsequence, ρ_{γ_b} converges in the sense of measures:

$$\rho_{\gamma_b} \underset{b\to\infty}{\overset{*}{\rightharpoonup}} \rho \in \mathcal{M}(\mathbb{R}_+ \times \mathbb{R}^2)$$
$$\rho_{\gamma_b}(0) \underset{b\to\infty}{\overset{*}{\rightharpoonup}} \rho_0 \in \mathcal{M}(\mathbb{R}^2)$$

to a weak solution of (5), meaning that $\forall \varphi \in C_c^\infty\left(\mathbb{R}_+ \times \mathbb{R}^2\right)$,

$$\int_{\mathbb{R}_+\times\mathbb{R}^2} \rho\left(\partial_t\varphi + \nabla^\perp (V + w \star \rho) \cdot \nabla_z\varphi\right) - \int_{\mathbb{R}^2} \varphi(0)\rho_0 = 0$$

I.4 Classical orbits

In this part we study a classical particle of charge -1 in a transverse magnetic field of amplitude b in a force field F. Our goal is to extract some useful heuristics from the dynamic of this particle. Newton's fundamental equation of dynamics gives

$$Z''(t) = F(t, Z(t)) + bZ'(t)^\perp \tag{10}$$

For a constant and homogeneous force field,

$$Z(t) = \underbrace{\frac{|Z_c'(0)|}{b}\begin{pmatrix}\cos(bt)\\ \sin(bt)\end{pmatrix}}_{=:Z_c(t)} + \underbrace{\frac{F^\perp}{b}t}_{=:Z_d(t)} \tag{11}$$

is the trajectory satisfying

$$Z_d(0) = (0,0) \quad Z_c(0) = (1,0)$$

The characteristic time for the drift Z_d is of order b. This analysis gives us important information about the motion of the orbit center of particles. First, this motion has to be observed over a time scale of order b, hence the time rescaling of the density matrix (8). Secondly, the shifting motion takes place in the direction perpendicular to the force field. For instance, if one takes

$$F := V + w \star \rho$$

this explains the origin of the $\nabla^\perp(V + w \star \rho)$ in the Gyrokinetic transport equation (5).

For the cyclotron motion Z_c, the characteristic time is $1/b$. We see from (11) that the radius of the cyclotron orbit

$$r_c := \frac{|Z_c'(0)|}{b}$$

is small in the large magnetic field limit. This means that this part of the dynamic, even though it consists on fast oscillations, should disappear in the limit equation.

I.5 Organisation of the paper

Section II and Section III are preliminaries respectively about the magnetic Laplacian and the conserved properties of the dynamics. In Section IV we introduce the semi-classical densities and prove that they approximate the physical density. Then we study the dynamics of the semi-classical densities in Section V. Section VI contains the conclusion of the proofs of the main theorems. Our method is similar to [5] [6] [2] the main difference being about the phase space. Our phase space is $\mathbb{N} \times \mathbb{R}^2$ because the gap between Landau level is of the same order as the other energy contributions, and thus doesn't disappears in the limit. For comparison, when one does semi-classical approximations with a small gap between Landau levels, the phase space is the usual position momentum phase space $\mathbb{R}^2 \times \mathbb{R}^2$.

II Quantization

II.1 Landau quantization

In this subsection, we set up the usual formalism for the description of the magnetic Laplacian in terms of annihilation and creation operators. More details about these operators and the properties of Landau levels can be found in [9]. This gives us a basis of eigenstates indexed by two quantum numbers n and m (12), n being the index describing the Landau level. With the aim of obtaining a projector on a point in phase space, coherent states are used for a fixed n. In dimension two, the complex parameter in the definition of coherent states can be identified with a position. We thus construct (Definition II.4) a one-particle state in the Landau level n located at a point in space. Then, somes properties of the associated projector are given.

Notation II.1
We denote by p_x, p_y the coordinates of the magnetic momentum:

$$\mathscr{P}_{\hbar,b} =: \begin{pmatrix} p_x \\ p_y \end{pmatrix} =: -\begin{pmatrix} i\hbar\partial_x + bA_x \\ i\hbar\partial_y + bA_y \end{pmatrix}$$

and define the annihilation and creation operators respectively as

$$a := \frac{p_x + ip_y}{\sqrt{2\hbar b}} \quad a^\dagger := \frac{p_x - ip_y}{\sqrt{2\hbar b}}$$

and the number of excitation operator $\mathcal{N} := a^\dagger a$.

We have the commutation relations:

$$\begin{aligned} &[p_x, p_y] = i\hbar b \\ &\left[a, a^\dagger\right] = \mathbb{1} \text{ (canonical commutation relation)} \end{aligned}$$

and

$$\mathscr{L}_b = 2\hbar b\left(\mathcal{N} + \frac{\mathrm{Id}}{2}\right)$$

Moreover

$$\mathrm{sp}(\mathcal{N}) = \mathbb{N}$$

Notation II.2: *Landau levels*
We define the n^{th} Landau level as the eigenspace associated to $n \in \mathbb{N}$:

$$nLL := \{\psi \in Dom\,(\mathscr{L}_{\hbar,b}) \ \text{ such that } \mathcal{N}\psi = n\psi\}$$

The ground level, denoted LLL for Lowest Landau Level has energy $E_0 = \hbar b$.

The Landau levels are isomorphic, and the operator $a^\dagger/\sqrt{n+1}$ is a unitary mapping from nLL to (n+1)LL of inverse $a/\sqrt{n+1}$. Therefore we can extend a basis of LLL using $a^\dagger$ to higher Landau levels. It is know [9] that the Lowest Landau level consists of holomorphic functions pondered by a Gaussian factor.

II.2 Landau levels

In classical mechanics, a charged particle in a 2D plane subjected to a transverse magnetic field follows a cyclotron orbit. In the previous section, we quantized the kinetic energy of the cyclotron orbit. To finish the quantization, one can try to quantize the wave functions as a function of the position of the cyclotron orbit, called the guiding center, in the 2D plane. This approach is justified by the fact that in what follows, we will need to reintroduce the potentials V and w, which depend on the position in space.

Notation II.3
For the rest of the text, we will identify

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2$$

with the complex notation $\boldsymbol{x} \coloneqq x_1 + ix_2$. *We introduce the following position operators*

$$\begin{aligned} r &\coloneqq \begin{pmatrix} r_x \\ r_y \end{pmatrix} \coloneqq \frac{\mathscr{P}^{\perp}_{\hbar,b}}{b} = \frac{1}{b}\begin{pmatrix} -p_y \\ p_x \end{pmatrix} \\ R &\coloneqq X - r \end{aligned}$$

We can therefore define creation and annihilation operators by:

$$\widehat{b} = \frac{R_x - iR_y}{\sqrt{2}l_b} \quad \widehat{b}^{\dagger} = \frac{R_x + iR_y}{\sqrt{2}l_b}$$

r represents the position of particles in the center of orbit frame. The classical physics meaning of this definition is that, since particles trajectories' are orbits, their momentum is perpendicular to their relative position with respect to the center to the orbit. Electrons are describing clock orbits thus the momentum rotated of $\pi/2$ gives us r. Moreover, r is related to the quantization of the cyclotron pulsation of the orbit because

$$\begin{aligned} a &= \frac{p_x + ip_y}{\sqrt{2\hbar b}} = \frac{r_y - ir_x}{\sqrt{2}l_b} = \frac{-i\mathbf{r}}{\sqrt{2}l_b} \\ a^{\dagger} &= \frac{i\overline{\mathbf{r}}}{\sqrt{2}l_b} \end{aligned}$$

From the definition of r, the position R of the orbit center is indeed

$$X = R + r$$

and related to the second harmonic oscillator

$$\begin{aligned} \widehat{b} &= \frac{\overline{\mathbf{R}}}{\sqrt{2}l_b} \\ \widehat{b}^{\dagger} &= \frac{\mathbf{R}}{\sqrt{2}l_b} \end{aligned}$$

r, R and X commutes with one another. Moreover

$$\begin{aligned} [r_x, r_y] &= il_b^2 \\ [R_x, R_y] &= -il_b^2 \\ \left[\widehat{b}, \widehat{b}^{\dagger}\right] &= \mathrm{Id} \end{aligned}$$

$$[a,b]=\left[a,\widehat{b}^{\dagger}\right]=\left[a^{\dagger},\widehat{b}\right]=\left[a^{\dagger},\widehat{b}^{\dagger}\right]=0$$

We therefore have two independent harmonic oscillators. By successively applying the creation operators $a^{\dagger}$ and $b^{\dagger}$ we obtain the desired basis. In symmetric gauge (1), the family defined by

$$\varphi_{n,m} \coloneqq \frac{\left(a^{\dagger}\right)^{n}\left(\widehat{b}^{\dagger}\right)^{m}}{\sqrt{n!m!}}\varphi_{0,0} \tag{12}$$

with

$$\varphi_{0,0}=\frac{1}{\sqrt{2\pi}l_b}e^{\frac{-|z|^2}{4l_b^2}}$$

is an orthonormal Hilbert basis of $L^2\left(\mathbb{R}^2\right)$. The full expression, see [27] [24] [21] [9], is

$$\varphi_{n,m}(x)=\frac{\left(\left(-2il_b^2\partial_{\mathbf{x}}+i\overline{\mathbf{x}}\right)^{n}\mathbf{x}^{m}\right)}{\sqrt{\pi n!m!}\left(\sqrt{2}l_b\right)^{n+m+1}}e^{\frac{-|x|^2}{4l_b^2}} \tag{13}$$

II.3 Coherent states

Definition II.4
Define the coherent state

$$\psi_{n,z} \coloneqq e^{\frac{\overline{z}b^{\dagger}-zb}{\sqrt{2}l_b}}\varphi_{n,0}$$

and the associated projector

$$\Pi_{n,z} \coloneqq |\psi_{n,z}\rangle\langle\psi_{n,z}|$$

the Landau level projector

$$\Pi_n \coloneqq \sum_{m\in\mathbb{N}}|\varphi_{n,m}\rangle\langle\varphi_{n,m}|$$

the localised projector

$$\Pi_z=\sum_{n\in\mathbb{N}}\Pi_{n,z}$$

Let $N_1\leqslant N_2$, define the truncated projector

$$\Pi_{N_1:N_2} \coloneqq \sum_{n=N_1}^{N_2}\Pi_n$$
$$\Pi_{\leqslant N} \coloneqq \Pi_{0:N}$$

with similar definitions for $\Pi_{>N}$ and $\Pi_{\leqslant N,z},\Pi_{>N,z}$.

We also have the following resolutions of identity [27], **Lemma II.5**

$$\frac{1}{2\pi l_b^2}\int_{\mathbb{R}^2}\Pi_{n,z}dz=\Pi_n \tag{14}$$

$$\frac{1}{2\pi l_b^2}\sum_{n\in\mathbb{N}}\int_{\mathbb{R}^2}\Pi_{n,z}dz = Id \tag{15}$$

$$\int_{\mathbb{R}^2}\Pi_z dz = Id$$

Note that $\psi_{n,z}$ is localised around z since

$$\overline{\mathbf{R}}\psi_{n,z} = \overline{\mathbf{z}}\psi_{n,z}$$

We have the following expressions for the coherent states (see [27]):

Proposition II.6

$$\psi_{n,z}(x) = \frac{i^n}{\sqrt{2\pi n!}l_b}\left(\frac{\boldsymbol{x}\text{ - }\boldsymbol{z}}{\sqrt{2}l_b}\right)^n e^{-\frac{|x-z|^2-2iz^\perp\cdot x}{4l_b^2}} \tag{16}$$

$$\Pi_{n,z}(x,y) = \frac{1}{2\pi n! l_b^2}\left(\frac{(\overline{\boldsymbol{x}\text{ - }\boldsymbol{z}})(\boldsymbol{y}\text{ - }\boldsymbol{z})}{2l_b^2}\right)^n e^{-\frac{|x-z|^2+|y-z|^2-2iz^\perp\cdot(x-y)}{4l_b^2}}$$

$$\Pi_z(x,y) = \frac{1}{2\pi l_b^2}e^{-\frac{|x-y|^2-2i\left(x^\perp\cdot y+2z^\perp\cdot(x-y)\right)}{4l_b^2}} \tag{17}$$

Proof: By the properties of the coherent states

$$\psi_{n,z} = e^{-\frac{|z|^2}{4l_b^2}}\sum_{m\in\mathbb{N}}\frac{\overline{\mathbf{z}}^m}{\left(\sqrt{2}l_b\right)^m\sqrt{m!}}\varphi_{n,m}$$

so with (13),

$$\begin{aligned}
\psi_{n,z}(x) =& e^{-\frac{|z|^2}{4l_b^2}}\sum_{m\in\mathbb{N}}\frac{\overline{\mathbf{z}}^m}{\left(\sqrt{2}l_b\right)^m\sqrt{m!}}\cdot\frac{\left(\left(-2il_b^2\partial_{\mathbf{x}}+i\overline{\mathbf{x}}\right)^n\mathbf{x}^m\right)}{\sqrt{\pi n! m!}\left(\sqrt{2}l_b\right)^{n+m+1}}e^{\frac{-|x|^2}{4l_b^2}}\\
=& e^{-\frac{|z|^2+|x|^2}{4l_b^2}}\frac{\left(-2il_b^2\partial_{\mathbf{x}}+i\overline{\mathbf{x}}\right)^n e^{\frac{\overline{\mathbf{z}}\mathbf{x}}{2l_b^2}}}{\sqrt{\pi n!}\left(\sqrt{2}l_b\right)^{n+1}} = \frac{i^n}{\sqrt{2\pi n!}l_b}\left(\frac{\overline{\mathbf{x}\text{ - }\mathbf{z}}}{\sqrt{2}l_b}\right)^n e^{-\frac{|z|^2+|x|^2-2\overline{\mathbf{z}}\mathbf{x}}{4l_b^2}}\\
=& \frac{i^n}{\sqrt{2\pi n!}l_b}\left(\frac{\overline{\mathbf{x}\text{ - }\mathbf{z}}}{\sqrt{2}l_b}\right)^n e^{-\frac{|x-z|^2-2i\mathrm{Im}[\overline{\mathbf{z}}\mathbf{x}]}{4l_b^2}}
\end{aligned}$$

note that

$$\mathrm{Im}\left[\overline{\mathbf{z}}\mathbf{x}\right] = z_1x_2 - z_2x_1 = z^\perp\cdot x$$

so

$$\psi_{n,z}(x) = \frac{i^n}{\sqrt{2\pi n!}l_b}\left(\frac{\overline{\mathbf{x}\text{ - }\mathbf{z}}}{\sqrt{2}l_b}\right)^n e^{-\frac{|x-z|^2-2iz^\perp\cdot x}{4l_b^2}}$$

We also obtain the kernel of the projector

$$\Pi_{n,z}(x,y) = \psi_{n,z}(x)\overline{\psi_{n,z}(y)} = \frac{1}{2\pi n! l_b^2}\left(\frac{(\overline{\mathbf{x}\text{ - }\mathbf{z}})(\mathbf{y}\text{ - }\mathbf{z})}{2l_b^2}\right)^n e^{-\frac{|x-z|^2+|y-z|^2-2iz^\perp\cdot(x-y)}{4l_b^2}}$$

$$=\frac{1}{2\pi n! l_b^2}\left(\frac{(\overline{\mathbf{x} - \mathbf{z}})(\mathbf{y} - \mathbf{z})}{2l_b^2}\right)^n e^{-\frac{2|z|^2+|x|^2+|y|^2-2(\overline{\mathbf{z}}\mathbf{x}+\mathbf{z}\overline{\mathbf{y}})}{4l_b^2}} \tag{18}$$

Then compute

$$\begin{aligned}\Pi_z(x,y) &= \sum_{n\in\mathbb{N}} \frac{1}{2\pi n! l_b^2}\left(\frac{(\overline{\mathbf{x} - \mathbf{z}})(\mathbf{y} - \mathbf{z})}{2l_b^2}\right)^n e^{-\frac{2|z|^2+|x|^2+|y|^2-2(\overline{\mathbf{z}}\mathbf{x}+\mathbf{z}\overline{\mathbf{y}})}{4l_b^2}} \\ &= \frac{1}{2\pi l_b^2} e^{-\frac{2|z|^2+|x|^2+|y|^2-2(\overline{\mathbf{z}}\mathbf{x}+\mathbf{z}\overline{\mathbf{y}})-2(\overline{\mathbf{x} - \mathbf{z}})(\mathbf{y} - \mathbf{z})}{4l_b^2}}\end{aligned}$$

and

$$\begin{aligned}&2|z|^2+|x|^2+|y|^2-2(\overline{\mathbf{z}}\mathbf{x}+\mathbf{z}\overline{\mathbf{y}})-2(\overline{\mathbf{x} - \mathbf{z}})(\mathbf{y} - \mathbf{z}) \\ =& |x|^2+|y|^2-2(\overline{\mathbf{z}}\mathbf{x}+\mathbf{z}\overline{\mathbf{y}}+\overline{\mathbf{x}}\mathbf{y}-\overline{\mathbf{x}}\mathbf{z}-\overline{\mathbf{z}}\mathbf{y}) = |x-y|^2 - 2i\mathrm{Im}\left[\overline{\mathbf{x}}\mathbf{y}+2\overline{\mathbf{z}}(\mathbf{x} - \mathbf{y})\right]\end{aligned}$$

so

$$\Pi_z(x,y) = \frac{1}{2\pi l_b^2} e^{-\frac{|x-y|^2-2i\left(x^\perp\cdot y+2z^\perp\cdot(x-y)\right)}{4l_b^2}}$$

□

From (17) we immediately see that

$$\nabla_z^\perp \Pi_z(x,y) = \frac{y-x}{il_b^2}$$

or as operator identity

$$\nabla_z^\perp \Pi_z = \frac{1}{il_b^2}\left[\Pi_z, X\right] \tag{19}$$

This formula will play a key role in the computation of the spacial derivative of the density in Section V. The following lemma is an approximation of (19) for the truncated projector.

Lemma II.7

$$\nabla_z^\perp \Pi_{\leqslant N,z} = \frac{1}{il_b^2}\left[\Pi_{\leqslant N,z}, X\right] - \frac{\sqrt{N+1}}{\sqrt{2}l_b}\begin{pmatrix}1 & 1\\ -i & i\end{pmatrix}\begin{pmatrix}|\psi_{N,z}\rangle\langle\psi_{N+1,z}| \\ |\psi_{N+1,z}\rangle\langle\psi_{N,z}|\end{pmatrix} \tag{20}$$

$$\begin{aligned}\nabla_z^\perp \Pi_{n,z} &= \frac{1}{il_b^2}\left[\Pi_{n,z}, X\right] - \frac{\sqrt{n+1}}{\sqrt{2}l_b}\begin{pmatrix}1 & 1\\ -i & i\end{pmatrix}\begin{pmatrix}|\psi_{n,z}\rangle\langle\psi_{n+1,z}| \\ |\psi_{n+1,z}\rangle\langle\psi_{n,z}|\end{pmatrix} \\ &+ \frac{\sqrt{n}}{\sqrt{2}l_b}\begin{pmatrix}1 & 1\\ -i & i\end{pmatrix}\begin{pmatrix}|\psi_{n-1,z}\rangle\langle\psi_{n,z}| \\ |\psi_{n,z}\rangle\langle\psi_{n-1,z}|\end{pmatrix}\end{aligned} \tag{21}$$

and

$$\begin{aligned}\nabla_z^\perp\otimes\nabla_z^\perp \Pi_{\leqslant N,z}(x,y) &= \frac{-1}{l_b^4}(x-y)^{\otimes 2}\Pi_{\leqslant N,z}(x,y) + \frac{Id_2}{l_b^2}\Pi_{N,z}(x,y) - \frac{\sqrt{N+1}}{\sqrt{2}l_b}\cdot \\ &\left(\nabla_z^\perp\otimes\begin{pmatrix}\psi_{N,z}(x)\overline{\psi_{N+1,z}(y)} \\ \psi_{N+1,z}(x)\overline{\psi_{N,z}(y)}\end{pmatrix}\begin{pmatrix}1 & -i\\ 1 & i\end{pmatrix} + \left(\nabla_z^\perp\otimes\begin{pmatrix}\psi_{N,z}(x)\overline{\psi_{N+1,z}(y)} \\ \psi_{N+1,z}(x)\overline{\psi_{N,z}(y)}\end{pmatrix}\begin{pmatrix}1 & -i\\ 1 & i\end{pmatrix}\right)^T\right)\end{aligned}$$

$$
-\frac{N+1}{2l_b^2}\begin{pmatrix}1 & 1\\ -i & i\end{pmatrix}\begin{pmatrix}\sqrt{\frac{N+2}{N+1}}\psi_{N,z}(x)\overline{\psi_{N+2,z}(y)} & \psi_{N+1,z}(x)\overline{\psi_{N+1,z}(y)}\\ \psi_{N+1,z}(x)\overline{\psi_{N+1,z}(y)} & \sqrt{\frac{N+2}{N+1}}\psi_{N+2,z}(x)\overline{\psi_{N,z}(y)}\end{pmatrix}\begin{pmatrix}1 & -i\\ 1 & i\end{pmatrix} \tag{22}
$$

Proof: **Proof of (20):**

Using

$$
\begin{aligned}
\psi_{n,z}(x) &= \frac{-i}{\sqrt{n}}\frac{\overline{\mathbf{z} - \mathbf{x}}}{\sqrt{2}l_b}\psi_{n-1,z}(x)\\
\overline{\psi_{n,z}(y)} &= \frac{i}{\sqrt{n}}\frac{\mathbf{z} - \mathbf{y}}{\sqrt{2}l_b}\overline{\psi_{n-1,z}(y)}
\end{aligned}
$$

With (18) compute

$$
\begin{aligned}
\partial_{\mathbf{z}}\Pi_{n,z}(x,y) =&\frac{\overline{\mathbf{y} - \mathbf{z}}}{2l_b^2}\Pi_{n,z}(x,y) + \frac{\overline{\mathbf{z} - \mathbf{x}}}{2l_b^2}\Pi_{n-1,z}(x,y)\\
=&\frac{\overline{\mathbf{y} - \mathbf{x}}}{2l_b^2}\psi_{n,z}(x)\overline{\psi_{n,z}(y)} + \frac{\overline{\mathbf{x} - \mathbf{z}}}{2l_b^2}\psi_{n,z}(x)\overline{\psi_{n,z}(y)} + \frac{\overline{\mathbf{z} - \mathbf{x}}}{2l_b^2}\psi_{n-1,z}(x)\overline{\psi_{n-1,z}(y)}\\
=&\frac{\overline{\mathbf{y} - \mathbf{x}}}{2l_b^2}\psi_{n,z}(x)\overline{\psi_{n,z}(y)} - i\frac{\sqrt{n+1}}{\sqrt{2}l_b}\psi_{n+1,z}(x)\overline{\psi_{n,z}(y)} + i\frac{\sqrt{n}}{\sqrt{2}l_b}\psi_{n,z}(x)\overline{\psi_{n-1,z}(y)}
\end{aligned}
$$

and

$$
\begin{aligned}
\partial_{\overline{\mathbf{z}}}\Pi_{n,z}(x,y) =&\frac{\mathbf{x} - \mathbf{z}}{2l_b^2}\Pi_{n,z}(x,y) + \frac{\mathbf{z} - \mathbf{y}}{2l_b^2}\Pi_{n-1,z}(x,y)\\
=&\frac{\mathbf{x} - \mathbf{y}}{2l_b^2}\psi_{n,z}(x)\overline{\psi_{n,z}(y)} + \frac{\mathbf{y} - \mathbf{z}}{2l_b^2}\psi_{n,z}(x)\overline{\psi_{n,z}(y)} + \frac{\mathbf{z} - \mathbf{y}}{2l_b^2}\psi_{n-1,z}(x)\overline{\psi_{n-1,z}(y)}\\
=&\frac{\mathbf{x} - \mathbf{y}}{2l_b^2}\psi_{n,z}(x)\overline{\psi_{n,z}(y)} + i\frac{\sqrt{n+1}}{\sqrt{2}l_b}\psi_{n,z}(x)\overline{\psi_{n+1,z}(y)} - i\frac{\sqrt{n}}{\sqrt{2}l_b}\psi_{n-1,z}(x)\overline{\psi_{n,z}(y)}
\end{aligned}
$$

So

$$
\begin{aligned}
\partial_{z_1}\Pi_{n,z}(x,y) =&\left(\partial_{\mathbf{z}} + \partial_{\overline{\mathbf{z}}}\right)\Pi_{n,z}(x,y)\\
=&i\frac{x_2 - y_2}{l_b^2}\Pi_{n,z}(x,y) + i\frac{\sqrt{n+1}}{\sqrt{2}l_b}\left(\psi_{n,z}(x)\overline{\psi_{n+1,z}(y)} - \psi_{n+1,z}(x)\overline{\psi_{n,z}(y)}\right)\\
&- i\frac{\sqrt{n}}{\sqrt{2}l_b}\left(\psi_{n-1,z}(x)\overline{\psi_{n,z}(y)} - \psi_{n,z}(x)\overline{\psi_{n-1,z}(y)}\right)
\end{aligned}
$$

and

$$
\begin{aligned}
\partial_{z_2}\Pi_{n,z}(x,y) =&i\left(\partial_{\mathbf{z}} - \partial_{\overline{\mathbf{z}}}\right)\Pi_{n,z}(x,y)\\
=&i\frac{y_1 - x_1}{l_b^2}\Pi_{n,z}(x,y) + \frac{\sqrt{n+1}}{\sqrt{2}l_b}\left(\psi_{n,z}(x)\overline{\psi_{n+1,z}(y)} + \psi_{n+1,z}(x)\overline{\psi_{n,z}(y)}\right)\\
&- \frac{\sqrt{n}}{\sqrt{2}l_b}\left(\psi_{n-1,z}(x)\overline{\psi_{n,z}(y)} + \psi_{n,z}(x)\overline{\psi_{n-1,z}(y)}\right)
\end{aligned}
$$

We deduce that

$$
\nabla_z^{\perp}\Pi_{n,z}(x,y) =i\frac{x-y}{l_b^2}\Pi_{n,z}(x,y) - \frac{\sqrt{n+1}}{\sqrt{2}l_b}\begin{pmatrix}1 & 1\\ -i & i\end{pmatrix}\begin{pmatrix}\psi_{n,z}(x)\overline{\psi_{n+1,z}(y)}\\ \psi_{n+1,z}(x)\overline{\psi_{n,z}(y)}\end{pmatrix}
$$

$$+\frac{\sqrt{n}}{\sqrt{2}l_b}\begin{pmatrix}1 & 1\\ -i & i\end{pmatrix}\begin{pmatrix}\psi_{n-1,z}(x)\overline{\psi_{n,z}(y)}\\ \psi_{n,z}(x)\overline{\psi_{n-1,z}(y)}\end{pmatrix}$$

After summation over n, we conclude that

$$\nabla_z^{\perp}\Pi_{\leqslant N,z}(x,y)=i\frac{x-y}{l_b^2}\Pi_{\leqslant N,z}(x,y)-\frac{\sqrt{N+1}}{\sqrt{2}l_b}\begin{pmatrix}1 & 1\\ -i & i\end{pmatrix}\begin{pmatrix}\psi_{N,z}(x)\overline{\psi_{N+1,z}(y)}\\ \psi_{N+1,z}(x)\overline{\psi_{N,z}(y)}\end{pmatrix}$$

Proof of (22):

Noticing that

$$\begin{aligned}\psi_{N+1,z}(x)&=\frac{-i}{\sqrt{N+1}}\frac{\overline{\mathbf{z - x}}}{\sqrt{2}l_b}\psi_{N,z}(x)\\ \overline{\psi_{N+1,z}(y)}&=\frac{i}{\sqrt{N+1}}\frac{\mathbf{z - y}}{\sqrt{2}l_b}\overline{\psi_{N,z}(y)}\end{aligned}$$

we get

$$\begin{pmatrix}\psi_{N,z}(x)\overline{\psi_{N+1,z}(y)}\\ \psi_{N+1,z}(x)\overline{\psi_{N,z}(y)}\end{pmatrix}=\frac{i}{\sqrt{N+1}\sqrt{2}l_b}\Pi_{N,z}(x,y)\begin{pmatrix}\mathbf{z\text{-}y}\\ \overline{\mathbf{x\text{-}z}}\end{pmatrix}$$

Thus inserting (20),

$$\begin{aligned}&\nabla_z^{\perp}\otimes\begin{pmatrix}\psi_{N,z}(x)\overline{\psi_{N+1,z}(y)}\\ \psi_{N+1,z}(x)\overline{\psi_{N,z}(y)}\end{pmatrix}\\
=&\frac{i}{\sqrt{N+1}\sqrt{2}l_b}\left(\nabla_z^{\perp}\Pi_{N,z}(x,y)\right)\otimes\begin{pmatrix}\mathbf{z\text{-}y}\\ \overline{\mathbf{x\text{-}z}}\end{pmatrix}+\frac{i}{\sqrt{N+1}\sqrt{2}l_b}\Pi_{N,z}(x,y)\nabla_z^{\perp}\otimes\begin{pmatrix}\mathbf{z\text{-}y}\\ \overline{\mathbf{x\text{-}z}}\end{pmatrix}\\
=&i\frac{x-y}{l_b^2}\Pi_{\leqslant N,z}(x,y)\otimes\frac{i}{\sqrt{N+1}\sqrt{2}l_b}\begin{pmatrix}\mathbf{z\text{-}y}\\ \overline{\mathbf{x\text{-}z}}\end{pmatrix}\\
&-\frac{\sqrt{N+1}}{\sqrt{2}l_b}\begin{pmatrix}1 & 1\\ -i & i\end{pmatrix}\begin{pmatrix}\psi_{N,z}(x)\overline{\psi_{N+1,z}(y)}\\ \psi_{N+1,z}(x)\overline{\psi_{N,z}(y)}\end{pmatrix}\otimes\frac{i}{\sqrt{N+1}\sqrt{2}l_b}\begin{pmatrix}\mathbf{z\text{-}y}\\ \overline{\mathbf{x\text{-}z}}\end{pmatrix}\\
&+\frac{i}{\sqrt{N+1}\sqrt{2}l_b}\Pi_{N,z}(x,y)\begin{pmatrix}-i & -i\\ 1 & -1\end{pmatrix}\\
=&i\frac{x-y}{l_b^2}\otimes\begin{pmatrix}\psi_{N,z}(x)\overline{\psi_{N+1,z}(y)}\\ \psi_{N+1,z}(x)\overline{\psi_{N,z}(y)}\end{pmatrix}\\
&-\frac{\sqrt{N+1}}{\sqrt{2}l_b}\begin{pmatrix}1 & 1\\ -i & i\end{pmatrix}\begin{pmatrix}\sqrt{\frac{N+2}{N+1}}\psi_{N,z}(x)\overline{\psi_{N+2,z}(y)} & \psi_{N+1,z}(x)\overline{\psi_{N+1,z}(y)}\\ \psi_{N+1,z}(x)\overline{\psi_{N+1,z}(y)} & \sqrt{\frac{N+2}{N+1}}\psi_{N+2,z}(x)\overline{\psi_{N,z}(y)}\end{pmatrix}\\
&+\frac{1}{\sqrt{N+1}\sqrt{2}l_b}\Pi_{N,z}(x,y)\begin{pmatrix}1 & 1\\ i & -i\end{pmatrix}\end{aligned}\tag{23}$$

so

$$\begin{aligned}&\nabla_z^{\perp}\otimes\nabla_z^{\perp}\Pi_{\leqslant N,z}(x,y)\\
=&\nabla_z^{\perp}\Pi_{\leqslant N,z}(x,y)\otimes i\frac{x-y}{l_b^2}-\nabla_z^{\perp}\otimes\frac{\sqrt{N+1}}{\sqrt{2}l_b}\begin{pmatrix}1 & 1\\ -i & i\end{pmatrix}\begin{pmatrix}\psi_{N,z}(x)\overline{\psi_{N+1,z}(y)}\\ \psi_{N+1,z}(x)\overline{\psi_{N,z}(y)}\end{pmatrix}\\
=&\frac{-1}{l_b^4}(x-y)^{\otimes 2}\Pi_{\leqslant N,z}(x,y)-\frac{\sqrt{N+1}}{\sqrt{2}l_b}\begin{pmatrix}1 & 1\\ -i & i\end{pmatrix}\begin{pmatrix}\psi_{N,z}(x)\overline{\psi_{N+1,z}(y)}\\ \psi_{N+1,z}(x)\overline{\psi_{N,z}(y)}\end{pmatrix}\otimes i\frac{x-y}{l_b^2}\end{aligned}$$

$$-\frac{\sqrt{N+1}}{\sqrt{2}l_b}\nabla_z^\perp\otimes\begin{pmatrix}\psi_{N,z}(x)\overline{\psi_{N+1,z}(y)}\\ \psi_{N+1,z}(x)\overline{\psi_{N,z}(y)}\end{pmatrix}\begin{pmatrix}1 & -i\\ 1 & i\end{pmatrix} \tag{24}$$

From (23) we can isolate

$$\begin{aligned}
&\begin{pmatrix}\psi_{N,z}(x)\overline{\psi_{N+1,z}(y)}\\ \psi_{N+1,z}(x)\overline{\psi_{N,z}(y)}\end{pmatrix}\otimes i\frac{x-y}{l_b^2}\\
=&\left(\nabla_z^\perp\otimes\begin{pmatrix}\psi_{N,z}(x)\overline{\psi_{N+1,z}(y)}\\ \psi_{N+1,z}(x)\overline{\psi_{N,z}(y)}\end{pmatrix}\right)^T-\frac{1}{\sqrt{N+1}\sqrt{2}l_b}\Pi_{N,z}(x,y)\begin{pmatrix}1 & i\\ 1 & -i\end{pmatrix}\\
&+\frac{\sqrt{N+1}}{\sqrt{2}l_b}\begin{pmatrix}\sqrt{\frac{N+2}{N+1}}\psi_{N,z}(x)\overline{\psi_{N+2,z}(y)} & \psi_{N+1,z}(x)\overline{\psi_{N+1,z}(y)}\\ \psi_{N+1,z}(x)\overline{\psi_{N+1,z}(y)} & \sqrt{\frac{N+2}{N+1}}\psi_{N+2,z}(x)\overline{\psi_{N,z}(y)}\end{pmatrix}\begin{pmatrix}1 & -i\\ 1 & i\end{pmatrix}
\end{aligned}$$

thus inserting this in (24),

$$\begin{aligned}
&\nabla_z^\perp\otimes\nabla_z^\perp\Pi_{\leqslant N,z}(x,y)\\
=&\frac{-1}{l_b^4}(x-y)^{\otimes 2}\Pi_{\leqslant N,z}(x,y)-\frac{\sqrt{N+1}}{\sqrt{2}l_b}\nabla_z^\perp\otimes\begin{pmatrix}\psi_{N,z}(x)\overline{\psi_{N+1,z}(y)}\\ \psi_{N+1,z}(x)\overline{\psi_{N,z}(y)}\end{pmatrix}\begin{pmatrix}1 & -i\\ 1 & i\end{pmatrix}\\
&-\frac{\sqrt{N+1}}{\sqrt{2}l_b}\begin{pmatrix}1 & 1\\ -i & i\end{pmatrix}\left(\nabla_z^\perp\otimes\begin{pmatrix}\psi_{N,z}(x)\overline{\psi_{N+1,z}(y)}\\ \psi_{N+1,z}(x)\overline{\psi_{N,z}(y)}\end{pmatrix}\right)^T+\frac{1}{2l_b^2}\begin{pmatrix}1 & 1\\ -i & i\end{pmatrix}\Pi_{N,z}(x,y)\begin{pmatrix}1 & i\\ 1 & -i\end{pmatrix}\\
&-\frac{N+1}{2l_b^2}\begin{pmatrix}1 & 1\\ -i & i\end{pmatrix}\begin{pmatrix}\sqrt{\frac{N+2}{N+1}}\psi_{N,z}(x)\overline{\psi_{N+2,z}(y)} & \psi_{N+1,z}(x)\overline{\psi_{N+1,z}(y)}\\ \psi_{N+1,z}(x)\overline{\psi_{N+1,z}(y)} & \sqrt{\frac{N+2}{N+1}}\psi_{N+2,z}(x)\overline{\psi_{N,z}(y)}\end{pmatrix}\begin{pmatrix}1 & -i\\ 1 & i\end{pmatrix}\\
=&\frac{-1}{l_b^4}(x-y)^{\otimes 2}\Pi_{\leqslant N,z}(x,y)+\frac{\mathrm{Id}_2}{l_b^2}\Pi_{N,z}(x,y)-\frac{\sqrt{N+1}}{\sqrt{2}l_b}\cdot\\
&\left(\nabla_z^\perp\otimes\begin{pmatrix}\psi_{N,z}(x)\overline{\psi_{N+1,z}(y)}\\ \psi_{N+1,z}(x)\overline{\psi_{N,z}(y)}\end{pmatrix}\begin{pmatrix}1 & -i\\ 1 & i\end{pmatrix}+\left(\nabla_z^\perp\otimes\begin{pmatrix}\psi_{N,z}(x)\overline{\psi_{N+1,z}(y)}\\ \psi_{N+1,z}(x)\overline{\psi_{N,z}(y)}\end{pmatrix}\begin{pmatrix}1 & -i\\ 1 & i\end{pmatrix}\right)^T\right)\\
&-\frac{N+1}{2l_b^2}\begin{pmatrix}1 & 1\\ -i & i\end{pmatrix}\begin{pmatrix}\sqrt{\frac{N+2}{N+1}}\psi_{N,z}(x)\overline{\psi_{N+2,z}(y)} & \psi_{N+1,z}(x)\overline{\psi_{N+1,z}(y)}\\ \psi_{N+1,z}(x)\overline{\psi_{N+1,z}(y)} & \sqrt{\frac{N+2}{N+1}}\psi_{N+2,z}(x)\overline{\psi_{N,z}(y)}\end{pmatrix}\begin{pmatrix}1 & -i\\ 1 & i\end{pmatrix}
\end{aligned}$$

□

III Properties of the time evolution

The goal of this section is to expose some basic properties of the dynamics: conservation fermionic states (Proposition III.1), conservation of the energy (Proposition III.2) and of the kinetic energy (Proposition III.3).

III.1 Conservation of fermionic states

Proposition III.1
Assume $\gamma_b \in L^\infty\left(\mathbb{R}_+, \mathcal{L}^1\left(L^2\left(\mathbb{R}^2\right)\right)\right)$ *solves*

$$\partial_t \gamma_b(t) = \frac{1}{il_b^2}\left[\mathscr{L}_b + V + w \star \rho_{\gamma_b(t)}, \gamma_b\right]$$

and satisfies

$$Tr\left[\gamma_b(0)\right] = 1, \quad 0 \leqslant \gamma_b(0) \leqslant 2\pi l_b^2$$

then $\forall t \in \mathbb{R}_+$,

$$Tr\left[\gamma_b(t)\right] = 1, \quad 0 \leqslant \gamma_b(t) \leqslant 2\pi l_b^2$$

Proof: First, the trace is preserved by the evolution because

$$\frac{d}{dt}\mathrm{Tr}\left[\gamma_b(t)\right] = \frac{1}{il_b^2}\mathrm{Tr}\left[\left[\mathscr{L}_b + V + W \star \rho_{\gamma_b(t)}, \gamma_b(t)\right]\right] = 0$$

Moreover

$$\gamma_b(t) = \mathcal{U}(t,0)\gamma_b(0)\mathcal{U}(0,t)$$

with

$$\mathcal{U}(t_2,t_1) := e^{\frac{1}{il_b^2}\int_{t_1}^{t_2}\left(\mathscr{L}_b + V + w\star\rho_{\gamma_b(\tau)}\right)d\tau}$$

so

$$\begin{aligned}
&\gamma(0) \geqslant 0 \implies \gamma_b(t) = \mathcal{U}(t,0)\gamma_b(0)\mathcal{U}(t,0)^* \geqslant 0 \\
&\gamma(0) \leqslant 2\pi l_b^2 \implies 2\pi l_b^2 - \gamma_b(t) = \mathcal{U}(t,0)\left(2\pi l_b^2 - \gamma_b(0)\right)\mathcal{U}(t,0)^* \geqslant 0 \implies \gamma_b(t) \leqslant 2\pi l_b^2
\end{aligned}$$

□

III.2 Energy conservation

Recalling the notation (3), we state that the quantity $\mathrm{Tr}\left[\gamma_b H_b\right]$ is preserved through the dynamics.

Proposition III.2: *Energy conservation*
Assume $\gamma_b \in L^\infty\left(\mathbb{R}_+, \mathcal{L}^1\left(L^2\left(\mathbb{R}^2\right)\right)\right)$ *solves*

$$\partial_t \gamma_b(t) = \frac{1}{il_b^2}\left[\mathscr{L}_b + V + w \star \rho_{\gamma_b(t)}, \gamma_b\right]$$

then

$$\frac{d}{dt}Tr\left[\gamma_b(t)H_b(t)\right] = 0$$

Proof: First compute

$$\begin{aligned}\partial_t H_b(t)(z) =& \frac{1}{2}\frac{d}{dt}(w \star \rho_{\gamma_b(t)})(z) = \frac{1}{2}\frac{d}{dt}\mathrm{Tr}\left[\gamma_b(t) w(\bullet - z)\right]\\ =& \frac{1}{2il_b^2}\mathrm{Tr}\left[\left[\mathscr{L}_b + V + w \star \rho_{\gamma_b(t)}, \gamma_b(t)\right] w(\bullet - z)\right]\\ =& \frac{1}{2il_b^2}\mathrm{Tr}\left[\left[w(\bullet - z), \mathscr{L}_b + V + w \star \rho_{\gamma_b(t)}\right]\gamma_b(t)\right] = \frac{1}{2il_b^2}\mathrm{Tr}\left[\left[w(\bullet - z), \mathscr{L}_b\right]\gamma_b(t)\right]\end{aligned}$$

so

$$\begin{aligned}\mathrm{Tr}\left[\gamma_b(t)\partial_t H_b(t)\right] =& \int_{\mathbb{R}^2} \rho_{\gamma_b(t)}(z)\partial_t H_b(t)(z) dz = \frac{1}{2il_b^2}\int_{\mathbb{R}^2}\mathrm{Tr}\left[\left[\rho_{\gamma_b(t)}(z) w(\bullet - z), \mathscr{L}_b\right]\gamma_b(t)\right] dz\\ =& \frac{1}{2il_b^2}\mathrm{Tr}\left[\left[w \star \rho_{\gamma_b(t)}, \mathscr{L}_b\right]\gamma_b(t)\right]\end{aligned} \tag{25}$$

we can compute

$$\begin{aligned}\mathrm{Tr}\left[\partial_t\gamma_b(t) H_b(t)\right] =& \frac{1}{il_b^2}\mathrm{Tr}\left[\left[\mathscr{L}_b + V + w \star \rho_{\gamma_b(t)}, \gamma_b(t)\right] H_b(t)\right]\\ =& \frac{1}{il_b^2}\mathrm{Tr}\left[\left[H_b(t), H_b(t) + \frac{1}{2}w \star \rho_{\gamma_b(t)}\right]\gamma_b(t)\right]\\ =& \frac{1}{2il_b^2}\mathrm{Tr}\left[\left[H_b(t), w \star \rho_{\gamma_b(t)}\right]\gamma_b(t)\right] = \frac{1}{2il_b^2}\mathrm{Tr}\left[\left[\mathscr{L}_b, w \star \rho_{\gamma_b(t)}\right]\gamma_b(t)\right]\end{aligned} \tag{26}$$

Then with (25) and (26) we conclude that

$$\frac{d}{dt}\mathrm{Tr}\left[\gamma_b(t) H_b(t)\right] = \mathrm{Tr}\left[\partial_t\gamma_b(t) H_b(t)\right] + \mathrm{Tr}\left[\gamma_b(t)\partial_t H_b(t)\right] = 0$$

□

Next we use the energy conservation to control the kinetic energy. In Section IV and Section V we will estimate error terms using this control.

Proposition III.3: *Kinetic energy conservation*
Let $\gamma_b \in \mathcal{L}^1\left(L^2\left(\mathbb{R}^2\right)\right), W \in L^\infty(\mathbb{R}^2)$ *and assume*

$$Tr\left[\gamma_b\right] = 1$$

then

$$Tr\left[\gamma_b \mathscr{L}_b\right] \leqslant \left|Tr\left[\gamma_b\left(\mathscr{L}_b + W\right)\right]\right| + \|W\|_{L^\infty}$$

Proof: The kinetic energy is bounded by

$$\mathrm{Tr}\left[\gamma_b \mathscr{L}_b\right] = \mathrm{Tr}\left[\gamma_b\left(\mathscr{L}_b + W\right)\right] - \mathrm{Tr}\left[\gamma_b W\right] \leqslant \left|\mathrm{Tr}\left[\gamma_b\left(\mathscr{L}_b + W\right)\right]\right| + \|W\|_{L^\infty}$$

□

In Proposition V.4 we manage to control the dynamics of a semi-classical density using only the kinetic energy. But we also present some estimates in Proposition V.1 and Proposition V.3 that only work with higher moments of the kinetic energy. Conservation of higher moments of the kinetic energy is a physical assumption. The next proposition is an attempt at controlling the 2^{nd} moment, however conservation could not be obtained yet due to the presence of $1/l_b$ factor in the derivative.

Proposition III.4: *2^{nd} moment of the Kinetic energy bound*
Let $t \in \mathbb{R}_+, \gamma_b(t) \in \mathcal{L}^1\left(L^2\left(\mathbb{R}^2\right)\right), W \in W^{2,\infty}(\mathbb{R}^2)$ and assume

$$
\begin{aligned}
&Tr\left[\gamma_b(t)\right] = 1, \quad 0 \leqslant \gamma_b(t) \\
&\partial_t \gamma_b(t) = \frac{1}{il_b^2}\left[\mathscr{L}_b + W, \gamma_b(t)\right]
\end{aligned}
$$

then

$$
\left|\frac{d}{dt} Tr\left[\gamma_b(t)\mathscr{L}_b^2\right]\right| \leqslant C\left(\|\Delta W\|_{L^\infty} + \frac{\|\nabla W\|_{L^\infty}}{l_b}\right) Tr\left[\gamma_b(t)\mathscr{L}_b^2\right]
$$

Proof: First we compute

$$
\frac{d}{dt}\mathrm{Tr}\left[\gamma_b \mathscr{L}_b^2\right] = \frac{1}{il_b^2}\mathrm{Tr}\left[\left[\mathscr{L}_b + W, \gamma_b\right]\mathscr{L}_b^2\right] = \frac{1}{il_b^2}\mathrm{Tr}\left[\left[\mathscr{L}_b^2, W\right]\gamma_b\right] \tag{27}
$$

With a direct computation

$$
\left[\mathscr{P}_b, W\right] = \left[i\hbar\nabla + bA, W\right] = i\hbar\nabla W
$$

so

$$
\begin{aligned}
\left[\mathscr{L}_b, W\right] =& \left[\mathscr{P}_b^2, W\right] = \mathscr{P}_b \cdot \left[\mathscr{P}_b, W\right] + \left[\mathscr{P}_b, W\right]\cdot \mathscr{P}_b = \mathscr{P}_b \cdot (i\hbar\nabla W) + i\hbar\nabla W \cdot \mathscr{P}_b \\
=& i\hbar\left[\mathscr{P}_b, \nabla W\right] + 2i\hbar\nabla W \cdot \mathscr{P}_b = -\hbar^2\Delta W + 2i\hbar\nabla W\cdot\mathscr{P}_b
\end{aligned}
$$

and

$$
\begin{aligned}
\mathrm{Tr}\left[\left[\mathscr{L}_b^2, W\right]\gamma_b\right] =& \mathrm{Tr}\left[\mathscr{L}_b\left[\mathscr{L}_b, W\right]\gamma_b\right] + \mathrm{Tr}\left[\left[\mathscr{L}_b, W\right]\mathscr{L}_b\gamma_b\right] \\
=& -\hbar^2\left(\mathrm{Tr}\left[\mathscr{L}_b\Delta W\gamma_b\right] + \mathrm{Tr}\left[\Delta W\mathscr{L}_b\gamma_b\right]\right) \\
&+ 2i\hbar\left(\mathrm{Tr}\left[\mathscr{L}_b\nabla W\cdot\mathscr{P}_b\gamma_b\right] + \mathrm{Tr}\left[\nabla W\cdot\mathscr{P}_b\mathscr{L}_b\gamma_b\right]\right)
\end{aligned}
$$

But

$$
\left[\mathscr{P}_b, \nabla W\right] \coloneqq \mathscr{P}_b\cdot\nabla W - \nabla W\cdot\mathscr{P}_b = i\hbar\Delta W
$$

so

$$
\begin{aligned}
\mathrm{Tr}\left[\left[\mathscr{L}_b^2, W\right]\gamma_b\right] =& \hbar^2\left(\mathrm{Tr}\left[\Delta W\mathscr{L}_b\gamma_b\right] - \mathrm{Tr}\left[\mathscr{L}_b\Delta W\gamma_b\right]\right) \\
&+ 2i\hbar\left(\mathrm{Tr}\left[\mathscr{L}_b\nabla W\cdot\mathscr{P}_b\gamma_b\right] + \mathrm{Tr}\left[\mathscr{P}_b\cdot\nabla W\mathscr{L}_b\gamma_b\right]\right)
\end{aligned} \tag{28}
$$

As operators

$$
\Delta W\mathscr{L}_b - \mathscr{L}_b\Delta W \leqslant \epsilon\left|\Delta W\right|^2 + \frac{1}{\epsilon}\mathscr{L}_b^2 \leqslant \epsilon\left\|\Delta W\right\|_{L^\infty}^2 + \frac{1}{\epsilon}\mathscr{L}_b^2 \tag{29}
$$

and with Cauchy-schwarz inequality

$$
\left|\mathrm{Tr}\left[\mathscr{L}_b\nabla W\cdot\mathscr{P}_b\gamma_b\right] + \mathrm{Tr}\left[\mathscr{P}_b\cdot\nabla W\mathscr{L}_b\gamma_b\right]\right|
$$

$$
\begin{aligned}
&= \left| \mathrm{Tr}\left[\gamma_b^{\frac{1}{2}} \mathscr{L}_b \nabla W \cdot \mathscr{P}_b \gamma_b^{\frac{1}{2}}\right] + \mathrm{Tr}\left[\gamma_b^{\frac{1}{2}} \mathscr{P}_b \cdot \nabla W \mathscr{L}_b \gamma_b^{\frac{1}{2}}\right]\right| \\
&\leqslant \sqrt{\mathrm{Tr}\left[\gamma_b^{\frac{1}{2}} \mathscr{L}_b \nabla W \cdot \left(\gamma_b^{\frac{1}{2}} \mathscr{L}_b \nabla W\right)^*\right] \mathrm{Tr}\left[\mathscr{P}_b \gamma_b^{\frac{1}{2}} \cdot \left(\mathscr{P}_b \gamma_b^{\frac{1}{2}}\right)^*\right]} \\
&\quad + \sqrt{\mathrm{Tr}\left[\gamma_b^{\frac{1}{2}} \mathscr{P}_b \cdot \left(\gamma_b^{\frac{1}{2}} \mathscr{P}_b\right)^*\right] \mathrm{Tr}\left[\nabla W \mathscr{L}_b \gamma_b^{\frac{1}{2}} \cdot \left(\nabla W \mathscr{L}_b \gamma_b^{\frac{1}{2}}\right)^*\right]} \\
&\leqslant 2\sqrt{\mathrm{Tr}\left[\gamma_b \mathscr{L}_b \left|\nabla W\right|^2 \mathscr{L}_b\right] \mathrm{Tr}\left[\gamma_b \mathscr{L}_b\right]} \leqslant \left\|\nabla W\right\|_{L^\infty} \sqrt{\mathrm{Tr}\left[\gamma_b \mathscr{L}_b^2\right] \mathrm{Tr}\left[\gamma_b \mathscr{L}_b\right]}
\end{aligned}
\tag{30}
$$

Inserting (29)with $\epsilon \coloneqq \dfrac{\hbar b}{\left\|\Delta W\right\|_{L^\infty}}$ and (30) in (28) we obtain

$$
\left|\mathrm{Tr}\left[\left[\mathscr{L}_b^2, W\right]\gamma_b\right]\right| \leqslant \hbar^2 \left\|\Delta W\right\|_{L^\infty} \left(\hbar b + \frac{1}{\hbar b} \mathrm{Tr}\left[\gamma_b \mathscr{L}_b^2\right]\right) + 2\hbar \left\|\nabla W\right\|_{L^\infty} \sqrt{\mathrm{Tr}\left[\gamma_b \mathscr{L}_b^2\right] \mathrm{Tr}\left[\gamma_b \mathscr{L}_b\right]}
$$

Recalling that $\mathscr{L}_b \geqslant \hbar b = o(1)$ we get

$$
\begin{aligned}
\left|\mathrm{Tr}\left[\left[\mathscr{L}_b^2, W\right]\gamma_b\right]\right| &\leqslant \left(\frac{\hbar^2}{\hbar b} \left\|\Delta W\right\|_{L^\infty} + \frac{2\hbar}{\sqrt{\hbar b}} \left\|\nabla W\right\|_{L^\infty}\right) \mathrm{Tr}\left[\gamma_b \mathscr{L}_b^2\right] \\
&= \left(l_b^2 \left\|\Delta W\right\|_{L^\infty} + 2 l_b \left\|\nabla W\right\|_{L^\infty}\right) \mathrm{Tr}\left[\gamma_b \mathscr{L}_b^2\right]
\end{aligned}
$$

We conclude with (27). □

IV Semi-classical approximations

In this section, we introduce the semi-classical density (32) and the truncated semi-classical density (33) that only takes into account Landau level under a certain threshold. One main difficulty in our method is that the coherent states (Definition II.4) are not well localized on high Landau levels, hence the introduction of the truncated semi-classical density. The Kinetic energy is used to control the number of particles in high Landau levels. The truncated semi-classical density turns out to be a good approximation of the physical density (Proposition IV.3). It is also possible to show that the semi-classical (untruncated) density is a good approximation of the physical density (Proposition IV.2) but this requires higher moments of the kinetic energy.

Definition IV.1
Let $\gamma \in \mathcal{L}^1\left(L^2\left(\mathbb{R}^2\right)\right)$. We define the semi-classical phase space density associated to γ, so-called Husimi function by

$$m_\gamma(n,z) := \frac{1}{2\pi l_b^2}\left\langle \psi_{n,z} | \gamma \psi_{n,z}\right\rangle = \frac{1}{2\pi l_b^2} Tr\left[\Pi_{n,z}\gamma\right] \tag{31}$$

and the semi-classical density

$$\rho_\gamma^{sc}(z) := \sum_{n\in\mathbb{N}} m_\gamma(n,z) = \frac{1}{2\pi l_b^2} Tr\left[\Pi_z \gamma\right] \tag{32}$$

Let N be a sequence such that

$$N \underset{b\to\infty}{\to} \infty$$

and the truncated semi-classical density be

$$\rho_\gamma^{sc,\leqslant N}(z) := \frac{1}{2\pi l_b^2} Tr\left[\gamma \Pi_{\leqslant N,z}\right] \tag{33}$$

The parameter N will represent the number of Landau levels we take into account for the semi-classical approximations in Section IV and Section V.

IV.1 Semi-classical density

Proposition IV.2: *Convergence of the semi-classical density*
Let $k>1, \gamma_b \in \mathcal{L}^1\left(L^2\left(\mathbb{R}^2\right)\right)$ and assume

$$Tr\left[\gamma_b\right] = 1, 0 \leqslant \gamma_b \leqslant 2\pi l_b^2$$

then $\forall \varphi \in C_c^\infty\left(\mathbb{R}^2\right)$,

$$\left|\int_{\mathbb{R}^2} \varphi\left(\rho_{\gamma_b} - \rho_{\gamma_b}^{sc}\right)\right| \leqslant C\sqrt{|supp(\varphi)|}\,\|\varphi\|_{W^{1,\infty}}\sqrt{Tr\left[\gamma_b \mathscr{L}_b^k\right]}\cdot \begin{cases} l_b^{2k-3} & \text{if } k<2 \\ l_b\sqrt{\ln\left(\frac{1}{l_b^2}\right)} & \text{if } k=2 \\ l_b & \text{if } k>2 \end{cases}$$

Notice that this estimate requires $k > \frac{3}{2}$ for the error on the right side to be be small.

Proof of Proposition IV.2: Let $\varphi \in C_c^\infty\left(\mathbb{R}^2\right)$,

$$\int_{\mathbb{R}^2} \varphi\left(\rho_{\gamma_b} - \rho_{\gamma_b}^{sc}\right) = \mathrm{Tr}\left[\varphi\gamma_b\right] - \frac{1}{2\pi l_b^2}\int_{\mathbb{R}^2} \varphi(z)\mathrm{Tr}\left[\Pi_z\gamma_b\right]dz = \frac{1}{2\pi l_b^2}\int_{\mathbb{R}^2}\mathrm{Tr}\left[\left(\varphi - \varphi(z)\right)\Pi_z\gamma_b\right]dz$$

$$=\frac{1}{2\pi l_b^2}\int_{\mathbb{R}^2}\sum_{n\in\mathbb{N}}\mathrm{Tr}\left[\left(\varphi-\varphi(z)\right)\Pi_{n,z}^2\gamma_b\right]dz \tag{34}$$

so with Young's inequality

$$\begin{aligned}
\left|\int_{\mathbb{R}^2}\varphi\left(\rho_{\gamma_b}-\rho^{sc}_{\gamma_b}\right)\right| \leqslant & \frac{1}{4\pi l_b^2}\int_{\mathbb{R}^2}\sum_{n\in\mathbb{N}}\left(\frac{1}{\epsilon_n}\mathrm{Tr}\left[\left|\varphi-\varphi(z)\right|^2\Pi_{n,z}\right]+\epsilon_n\mathrm{Tr}\left[\Pi_{n,z}\gamma_b^2\right]\right)dz\\
=&\sum_{n\in\mathbb{N}}\frac{1}{4\pi\epsilon_n l_b^2}\int_{\mathbb{R}^2}\mathrm{Tr}\left[\left|\varphi-\varphi(z)\right|^2\Pi_{n,z}\right]dz+\sum_{n\in\mathbb{N}}\frac{\epsilon_n}{2}\mathrm{Tr}\left[\Pi_n\gamma_b^2\right]\\
\leqslant&\sum_{n\in\mathbb{N}}\frac{1}{4\pi\epsilon_n l_b^2}\int_{\mathbb{R}^2}\mathrm{Tr}\left[\left|\varphi-\varphi(z)\right|^2\Pi_{n,z}\right]dz+\sum_{n\in\mathbb{N}}\pi\epsilon_n l_b^2\mathrm{Tr}\left[\Pi_n\gamma_b\right] \qquad (35)
\end{aligned}$$

and with the change of variable $x \coloneqq \dfrac{x-z}{\sqrt{2}l_b}$

$$\begin{aligned}
\mathrm{Tr}\left[\left|\varphi-\varphi(z)\right|^2\Pi_{n,z}\right]=&\int_{\mathbb{R}^2}\left|\varphi(x)-\varphi(z)\right|^2\Pi_{n,z}(x,x)dx\\
=&\frac{1}{2\pi n!l_b^2}\int_{\mathbb{R}^2}\left|\varphi(x)-\varphi(z)\right|^2\left|\frac{x-z}{\sqrt{2}l_b}\right|^{2n}e^{-\frac{|x-z|^2}{2l_b^2}}dx\\
=&\frac{1}{\pi n!}\int_{\mathbb{R}^2}\left|\varphi\left(z+\sqrt{2}l_b x\right)-\varphi(z)\right|^2|x|^{2n}e^{-|x|^2}dx\\
\leqslant&\frac{1}{\pi n!}2l_b^2\left\|\nabla\varphi\right\|_\infty^2\int_{\mathbb{R}^2}|x|^{2(n+1)}e^{-|x|^2}dx=2(n+1)l_b^2\left\|\nabla\varphi\right\|_\infty^2 \qquad (36)
\end{aligned}$$

We can also write

$$\mathrm{Tr}\left[\left|\varphi-\varphi(z)\right|^2\Pi_{n,z}\right]\leqslant 4\left\|\varphi\right\|_{L^\infty}^2\mathrm{Tr}\left[\Pi_{n,z}\right]=4\left\|\varphi\right\|_{L^\infty}^2 \tag{37}$$

Since

$$\Pi_{n,z}(x,x)=\Pi_{n,x}(z,z)$$

with (36) and (37), by seeing that the integrand is symmetric in x and z we get

$$\begin{aligned}
\int_{\mathbb{R}^2}\mathrm{Tr}\left[\left|\varphi-\varphi(z)\right|^2\Pi_{n,z}\right]dz=&\int_{(\mathrm{supp}(\varphi)\times\mathbb{R}^2)\cup(\mathbb{R}^2\times\mathrm{supp}(\varphi))}\left|\varphi(x)-\varphi(z)\right|^2\Pi_{n,z}(x,x)dxdz\\
\leqslant&2\int_{\mathbb{R}^2\times\mathrm{supp}(\varphi)}\left|\varphi(x)-\varphi(z)\right|^2\Pi_{n,z}(x,x)dxdz\\
=&2\int_{\mathrm{supp}(\varphi)}\mathrm{Tr}\left[\left|\varphi-\varphi(z)\right|^2\Pi_{n,z}\right]dz\\
\leqslant&C\left|\mathrm{supp}(\varphi)\right|\left\|\varphi\right\|_{W^{1,\infty}}^2\min\left((n+1)l_b^2,1\right) \qquad (38)
\end{aligned}$$

Next we identify the k^{nd} moment of the kinetic energy in the sum:

$$\sum_{n\in\mathbb{N}}(n+1)^k\mathrm{Tr}\left[\gamma_b\Pi_n\right]=\sum_{n\in\mathbb{N}}\left(\frac{n+1}{2\hbar b\left(n+\dfrac{1}{2}\right)}\right)^k\mathrm{Tr}\left[\gamma_b\Pi_n\mathscr{L}_b^k\right]\leqslant\sum_{n\in\mathbb{N}}\mathrm{Tr}\left[\gamma_b\Pi_n\mathscr{L}_b^k\right]=\mathrm{Tr}\left[\gamma_b\mathscr{L}_b^k\right] \tag{39}$$

Inserting (38) in (35), taking $\epsilon_n \coloneqq \sqrt{|\mathrm{supp}(\varphi)|}\, \|\varphi\|_{W^{1,\infty}}\, \epsilon(n+1)^k$ and using (39) we get

$$\begin{aligned}
&\left|\int_{\mathbb{R}^2} \varphi\left(\rho_{\gamma_b} - \rho^{sc}_{\gamma_b}\right)\right| \\
\leqslant & C\left(|\mathrm{supp}(\varphi)|\, \|\varphi\|^2_{W^{1,\infty}} \sum_{n\in\mathbb{N}} \frac{1}{\epsilon_n} \min\left(n+1, \frac{1}{l_b^2}\right) + l_b^2 \sum_{n\in\mathbb{N}} \epsilon_n \mathrm{Tr}\left[\Pi_n \gamma_b\right]\right) \\
\leqslant & C\sqrt{|\mathrm{supp}(\varphi)|}\, \|\varphi\|_{W^{1,\infty}} \left(\sum_{n\in\mathbb{N}} \frac{1}{\epsilon(n+1)^k} \min\left(n+1, \frac{1}{l_b^2}\right) + \epsilon l_b^2 \mathrm{Tr}\left[\gamma_b \mathscr{L}_b^k\right]\right) \\
\leqslant & C\sqrt{|\mathrm{supp}(\varphi)|}\, \|\varphi\|_{W^{1,\infty}} \left(\sum_{(n+1)l_b^2 \leqslant 1} \frac{1}{\epsilon(n+1)^{k-1}} + \frac{1}{\epsilon l_b^2} \sum_{(n+1)l_b^2>1} \frac{1}{(n+1)^k} + \epsilon l_b^2 \mathrm{Tr}\left[\gamma_b \mathscr{L}_b^k\right]\right)
\end{aligned} \tag{40}$$

Introducing the notation

$$p_\lambda(x) \coloneqq x^{-\lambda}\mathbb{1}_{\lambda<0} + \ln(x)\mathbb{1}_{\lambda=0} + \mathbb{1}_{\lambda>0} \tag{41}$$

we have the asymptotics

$$\sum_{(n+1)l_b^2>1} \frac{1}{(n+1)^k} = \sum_{n>\frac{1}{l_b^2}}^{\infty} \frac{1}{n^k} = \mathcal{O}\left(\frac{1}{\left(\frac{1}{l_b^2}\right)^{k-1}}\right) = \mathcal{O}\left(l_b^{2(k-1)}\right) \tag{42}$$

$$\sum_{(n+1)l_b^2\leqslant 1} \frac{1}{(n+1)^{k-1}} = \sum_{1\leqslant n\leqslant \frac{1}{l_b^2}} \frac{1}{n^{k-1}} = \begin{cases} \mathcal{O}\left(l_b^{2(k-2)}\right) & \text{if } k<2 \\ \mathcal{O}\left(\ln\left(l_b^{-2}\right)\right) & \text{if } k=2 \\ \mathcal{O}\left(1\right) & \text{if } k>2 \end{cases} = \mathcal{O}\left(p_{k-2}\left(l_b^{-2}\right)\right) \tag{43}$$

We notice that

$$l_b^{2(k-2)} \leqslant p_{k-2}\left(l_b^{-2}\right) \tag{44}$$

Inserting (42), (43), (44) in (40) and taking $\epsilon \coloneqq \frac{1}{l_b}\sqrt{\frac{p_{k-2}\left(l_b^{-2}\right)}{\mathrm{Tr}\left[\gamma_b \mathscr{L}_b^k\right]}}$ we obtain

$$\begin{aligned}
\left|\int_{\mathbb{R}^2} \varphi\left(\rho_{\gamma_b} - \rho^{sc}_{\gamma_b}\right)\right| \leqslant & C\sqrt{|\mathrm{supp}(\varphi)|}\, \|\varphi\|_{W^{1,\infty}} \left(\frac{p_{k-2}\left(l_b^{-2}\right)}{\epsilon} + \frac{l_b^{2(k-2)}}{\epsilon} + \epsilon l_b^2 \mathrm{Tr}\left[\gamma_b \mathscr{L}_b^k\right]\right) \\
\leqslant & C\sqrt{|\mathrm{supp}(\varphi)|}\, \|\varphi\|_{W^{1,\infty}} \left(\frac{p_{k-2}\left(l_b^{-2}\right)}{\epsilon} + \epsilon l_b^2 \mathrm{Tr}\left[\gamma_b \mathscr{L}_b^k\right]\right) \\
= & C\sqrt{|\mathrm{supp}(\varphi)|}\, \|\varphi\|_{W^{1,\infty}} \sqrt{\mathrm{Tr}\left[\gamma_b \mathscr{L}_b^k\right]}\, l_b \sqrt{p_{k-2}\left(l_b^{-2}\right)}
\end{aligned}$$

Note that this choice of ϵ is possible because

$$\sum_{n\in\mathbb{N}} (n+1)^k \mathrm{Tr}\left[\Pi_n \gamma_b\right] \geqslant \sum_{n\in\mathbb{N}} \mathrm{Tr}\left[\Pi_n \gamma_b\right] = \mathrm{Tr}\left[\gamma_b\right] = 1 \tag{45}$$

□

The second term in (40) requires $k>1$ for the convergence of the series. If we only assume that the kinetic energy is bounded ($k=1$), in order to deal with the divergent series in (40) we need to consider the truncated semi-classical density (33) instead of (32). This is the goal of the next section.

IV.2 Truncated semi-classical density

Proposition IV.3: *Convergence of the truncated semi-classical density*
Let $k > 1, \gamma_b \in \mathcal{L}^1\left(L^2\left(\mathbb{R}^2\right)\right)$ *and assume*

$$Tr\left[\gamma_b\right] = 1, 0 \leqslant \gamma_b \leqslant 2\pi l_b^2$$

then $\forall \varphi \in C_c^\infty\left(\mathbb{R}^2\right)$,

$$\left|\int_{\mathbb{R}^2} \varphi\left(\rho_{\gamma_b} - \rho_{\gamma_b}^{sc,\leqslant N}\right)\right| \leqslant \|\varphi\|_{L^\infty} N^{-\frac{k}{2}} \sqrt{Tr\left[\gamma_b \Pi_{>N} \mathscr{L}_b^k\right]}$$
$$+ C\sqrt{|supp(\varphi)|}\, \|\varphi\|_{W^{1,\infty}} \sqrt{Tr\left[\gamma_b \mathscr{L}_b^k\right]} \cdot \begin{cases} N^{1-\frac{k}{2}} l_b & \text{if } k < 2 \\ \sqrt{\ln(N)} l_b & \text{if } k = 2 \\ l_b & \text{if } k > 2 \end{cases} \tag{46}$$

This time, the right-hand side term is small for $k = 1$ if we choose N such that $\sqrt{N} l_b \ll 1$. This constraint has a physical meaning. Indeed from the expression of the coherent state (16) we can infer that the characteristic length for particles in nLL is $\sqrt{n} l_b$. Hence $\rho_{\gamma_b}^{sc,\leqslant N}$ taken with $\sqrt{N} l_b \ll 1$ is the semi-classical density of well localised particles, and good localisation of the coherent states is key in semi-classical approximations. The first term in the right-hand side of 46 corresponds to high Landau levels. The estimates will use the information about the moments of the kinetic energy to control the number of particles inside high Landau levels.

Proof of Proposition IV.3: Similarly as in the proof of Proposition IV.2, instead of (34) we have

$$\int_{\mathbb{R}^2} \varphi\left(\rho_{\gamma_b} - \rho_{\gamma_b}^{sc,\leqslant N}\right) = \frac{1}{2\pi l_b^2} \sum_{n=0}^{N} \int_{\mathbb{R}^2} \mathrm{Tr}\left[\left(\varphi - \varphi(z)\right) \Pi_{n,z} \gamma_b\right] dz + \sum_{n>N} \mathrm{Tr}\left[\varphi \Pi_n \gamma_b\right] \tag{47}$$

Using

$$\sum_{n\leqslant N} (n+1)^k \mathrm{Tr}\left[\gamma_b \Pi_n\right] \leqslant \sum_{n\in\mathbb{N}} (n+1)^k \mathrm{Tr}\left[\gamma_b \Pi_n\right]$$

we obtain instead of (40):

$$\frac{1}{2\pi l_b^2} \sum_{n=0}^{N} \int_{\mathbb{R}^2} \left|\mathrm{Tr}\left[\left(\varphi - \varphi(z)\right) \Pi_{n,z} \gamma_b\right]\right| dz$$
$$\leqslant C\sqrt{|\mathrm{supp}(\varphi)|}\, \|\varphi\|_{W^{1,\infty}} \left(\sum_{n=0}^{N} \frac{1}{\epsilon (n+1)^{k-1}} + \epsilon l_b^2 \mathrm{Tr}\left[\gamma_b \mathscr{L}_b^k\right]\right)$$

Recalling the notation in (41),

$$\sum_{n=0}^{N} \frac{1}{(n+1)^{k-1}} = \begin{cases} \mathcal{O}\left(N^{2-k}\right) & \text{if } k < 2 \\ \mathcal{O}\left(\ln(N)\right) & \text{if } k = 2 \\ \mathcal{O}(1) & \text{if } k > 2 \end{cases} = \mathcal{O}\left(p_{k-2}(N)\right)$$

With $\epsilon := \frac{1}{l_b}\sqrt{\frac{p_{k-2}(N)}{\mathrm{Tr}\left[\gamma_b \mathscr{L}_b^k\right]}}$ we have

$$\frac{1}{2\pi l_b^2} \sum_{n=0}^{N} \int_{\mathbb{R}^2} \left|\mathrm{Tr}\left[\left(\varphi - \varphi(z)\right) \Pi_{n,z} \gamma_b\right]\right| dz \leqslant C\sqrt{|\mathrm{supp}(\varphi)|}\, \|\varphi\|_{W^{1,\infty}} \sqrt{\mathrm{Tr}\left[\gamma_b \mathscr{L}_b^k\right]} l_b \sqrt{p_{k-2}(N)} \tag{48}$$

Moreover

$$\left|\sum_{n>\mathbb{N}} \operatorname{Tr}\left[\varphi \Pi_n \gamma_b\right]\right| = \left|\operatorname{Tr}\left[\varphi \Pi_{>N} \gamma_b\right]\right| \leqslant \sqrt{\operatorname{Tr}\left[\gamma_b \left|\varphi\right|^2\right] \operatorname{Tr}\left[\gamma_b \Pi_{>N}\right]} \leqslant \|\varphi\|_{L^\infty} \sqrt{\operatorname{Tr}\left[\gamma_b \Pi_{>N}\right]}$$

and

$$\begin{aligned}\operatorname{Tr}\left[\gamma_b \Pi_{>N}\right] &= \sum_{n>N} \operatorname{Tr}\left[\gamma_b \Pi_n\right] \leqslant \sum_{n>N} \frac{n^k}{N^k} \operatorname{Tr}\left[\gamma_b \Pi_n\right] \leqslant \frac{1}{N^k} \sum_{n>N} \operatorname{Tr}\left[\gamma_b \Pi_n \mathscr{L}_b^k\right] \\ &= \frac{1}{N^k} \operatorname{Tr}\left[\gamma_b \Pi_{>N} \mathscr{L}_b^k\right]\end{aligned}$$

so

$$\left|\sum_{n>N} \operatorname{Tr}\left[\varphi \Pi_n \gamma_b\right]\right| \leqslant \|\varphi\|_{L^\infty} N^{-\frac{k}{2}} \sqrt{\operatorname{Tr}\left[\gamma_b \Pi_{>N} \mathscr{L}_b^k\right]} \tag{49}$$

We conclude by assembling (47), (48) and (47). □

V Gyrokinetic dynamics for semi-classical densities

The goal of this section if to prove that the semi-classical densities almost satisfies (5).

V.1 Dynamics of the semi-classical density

Proposition V.1: *Gyrokinetic equation for the semi-classical density*
Let $t \in \mathbb{R}_+, k > 3, \gamma_b(t) \in \mathcal{L}^1\left(L^2\left(\mathbb{R}^2\right)\right), W \in W^{3,\infty}(\mathbb{R}^2)$ *and assume*

$$Tr[\gamma_b(t)] = 1, 0 \leqslant \gamma_b(t) \leqslant 2\pi l_b^2 \tag{50}$$

$$\partial_t \gamma_b(t) = \frac{1}{il_b^2}\left[\mathscr{L}_b + W, \gamma_b(t)\right] \tag{51}$$

then $\forall \varphi \in C_c^\infty\left(\mathbb{R}^2\right)$,

$$\left|\int_{\mathbb{R}^2} \varphi(z)\left(\partial_t \rho^{sc}_{\gamma_b}(t,z) + \nabla^\perp W(z)\cdot \nabla \rho^{sc}_{\gamma_b}(t,z)\right)dz\right|$$
$$\leqslant C\sqrt{|supp(\varphi)|}\,\|\varphi\|_{W^{1,\infty}}\|W\|_{W^{3,\infty}}\sqrt{Tr\left[\gamma_b(t)\mathscr{L}_b^k\right]}\cdot\begin{cases} l_b^{2k-7} & \text{if } k<4 \\ l_b\sqrt{\ln\left(\frac{1}{l_b^2}\right)} & \text{if } k=4 \\ l_b & \text{if } k>4\end{cases} \tag{52}$$

This estimate requires $k > 7/2$ for the right-hand side to be small. We will later in the text improve this constraint, but we start by Proposition V.1 in order to introduce our method of proof. We begin the proof with a technical Lemma.

Lemma V.2
Let $\alpha \in \mathbb{R}_+$, *then*

$$I_n(\alpha) := \frac{1}{\pi n!}\int_{\mathbb{R}^2} |x|^{\alpha+2n} e^{-|x|^2}dx \underset{n\to\infty}{\sim} n^{\frac{\alpha}{2}}$$

and $\exists C > 0$ *such that*

$$\forall n \in \mathbb{N}, I_n(\alpha) \leqslant C(n+1)^{\frac{\alpha}{2}}$$

Proof: With polar coordinates and the change of variable $x := x^2$

$$\frac{1}{\pi}\int_{\mathbb{R}^2}|x|^\alpha e^{-|x|^2}dx = 2\int_{\mathbb{R}_+} x^{\alpha+1}e^{-x^2}dx = \int_{\mathbb{R}_+} x^{\frac{\alpha}{2}}e^{-u}du = \Gamma\left(\frac{\alpha}{2}+1\right)$$

where Γ is the Euler Gamma function,

$$\Gamma(z) := \int_{\mathbb{R}_+} t^{z-1}e^{-t}dt$$

We have the following equivalent for the Euler Gamma function (as a direct consequence of the Stirling formula)

$$\frac{\Gamma(n+x)}{\Gamma(n)} \underset{x\to\infty}{\sim} n^x \tag{53}$$

so

$$I_n(\alpha) = \frac{1}{n!}\Gamma\left(n+\frac{\alpha}{2}+1\right) = \frac{\Gamma\left(n+\frac{\alpha}{2}+1\right)}{\Gamma(n+1)}$$

and by (53),

$$I_n(\alpha) \underset{n\to\infty}{\sim} (n+1)^{\frac{\alpha}{2}} \underset{n\to\infty}{\sim} n^{\frac{\alpha}{2}}$$

$\square$

Proof of Proposition V.1:

a direct computation:

We start from (32) and (51):

$$\begin{aligned}\partial_t \rho^{sc}_{\gamma_b}(z) =& \frac{1}{2\pi l_b^2}\mathrm{Tr}\left[\partial_t \gamma_b \Pi_z\right] = \frac{1}{2i\pi l_b^4}\mathrm{Tr}\left[\left[\mathscr{L}_b + W, \gamma_b\right]\Pi_z\right] = \frac{1}{2i\pi l_b^4}\mathrm{Tr}\left[\gamma_b\left[\Pi_z, \mathscr{L}_b + W\right]\right] \\ =& \frac{1}{2i\pi l_b^4}\mathrm{Tr}\left[\gamma_b\left[\Pi_z, W\right]\right] \end{aligned} \tag{54}$$

On the other hand, by (19) and (32),

$$\begin{aligned}\nabla^\perp W(z)\cdot\nabla\rho^{sc}_{\gamma_b}(z) =& \frac{1}{2\pi l_b^2}\nabla^\perp W(z)\cdot\mathrm{Tr}\left[\gamma_b\nabla_z\Pi_z\right] = -\frac{1}{2\pi l_b^2}\nabla W(z)\cdot\mathrm{Tr}\left[\gamma_b\nabla_z^\perp\Pi_z\right] \\ =& -\frac{1}{2i\pi l_b^4}\mathrm{Tr}\left[\gamma_b\left[\Pi_z, X\cdot\nabla W(z)\right]\right] \end{aligned} \tag{55}$$

Putting together (54) and (55),

$$\partial_t \rho^{sc}_{\gamma_b}(z) + \nabla^\perp W(z)\cdot\nabla\rho^{sc}_{\gamma_b}(z) = \frac{1}{2i\pi l_b^4}\mathrm{Tr}\left[\gamma_b\left[\Pi_z, W - X\cdot\nabla W(z)\right]\right] \tag{56}$$

Taylor expansion of the potential:

We expand W to the second order. Define

$$\mathcal{V}_z(x) \coloneqq W(x) - dW(z)(x-z) - \frac{1}{2}d^2W(z)(x-z,x-z) \tag{57}$$

so that

$$\begin{aligned}W(y) - W(x) - (y-x)\cdot\nabla W(z) =& \mathcal{V}_z(y) - \mathcal{V}_z(x) \\ &+ \frac{1}{2}d^2W(z)(y-z,y-z) - \frac{1}{2}d^2W(z)(x-z,x-z)\end{aligned}$$

Where the notation d^nW stand for the n^{th} differential of W, meaning that $d^nW(z)$ is a n-linear form. Notice that

$$\begin{aligned}d^2W(z)(y-z,y-z) =& d^2W(z)(y-x,y-z) + d^2W(z)(x-z,y-z) \\ =& d^2W(z)(y-x,y-z) + d^2W(z)(x-z,y-x)\end{aligned}$$

$$+ d^2W(z)(x-z,x-z)$$

so

$$\begin{aligned}
&\left[\Pi_z, W - X\cdot\nabla W(z)\right](x,y) = \Pi_z(x,y)\left(W(y)-W(x)-(y-x)\cdot\nabla W(z)\right)\\
=&\Pi_z(x,y)\left(\mathcal{V}_z(y)-\mathcal{V}_z(x)+\frac{1}{2}d^2W(z)(y-x,y-z)+\frac{1}{2}d^2W(z)(x-z,y-x)\right)\\
=&\left[\Pi_z,\mathcal{V}_z\right](x,y)+\frac{1}{2}d^2W(z)(\Pi_z(x,y)(y-x),y-z)+\frac{1}{2}d^2W(z)(x-z,\Pi_z(x,y)(y-x))
\end{aligned}\tag{58}$$

Thus with (19), we have the operator identity

$$\begin{aligned}
\left[\Pi_z, W - X\cdot\nabla W(z)\right] &= \left[\Pi_z,\mathcal{V}_z\right]+\frac{1}{2}d^2W(z)(\left[\Pi_z,X\right],X-z)+\frac{1}{2}d^2W(z)(X-z,\left[\Pi_z,X\right])\\
&= \left[\Pi_z,\mathcal{V}_z\right]+\frac{il_b^2}{2}d^2W(z)(\nabla_z^\perp\Pi_z,X-z)+\frac{il_b^2}{2}d^2W(z)(X-z,\nabla_z^\perp\Pi_z)
\end{aligned}\tag{59}$$

Defining the error terms

$$\mathcal{E}_{1,b}(z) \coloneqq \frac{1}{2i\pi l_b^4}\operatorname{Tr}\left[\gamma_b\left[\Pi_z,\mathcal{V}_z\right]\right]\tag{60}$$

$$\mathcal{E}_{2,b}(z) \coloneqq \frac{1}{4\pi l_b^2}\operatorname{Tr}\left[\gamma_b d^2W(z)(\nabla_z^\perp\Pi_z,X-z)\right]\tag{61}$$

$$\widetilde{\mathcal{E}}_{2,b}(z) \coloneqq \frac{1}{4\pi l_b^2}\operatorname{Tr}\left[\gamma_b d^2W(z)(X-z,\nabla_z^\perp\Pi_z)\right]\tag{62}$$

and inserting (59) (60) (61) (62) in (56) we obtain

$$\partial_t\rho_{\gamma_b}^{sc}(z)+\nabla^\perp W(z)\cdot\nabla\rho_{\gamma_b}^{sc}(z)=\mathcal{E}_{1,b}(z)+\mathcal{E}_{2,b}(z)+\widetilde{\mathcal{E}}_{2,b}(z)\tag{63}$$

Estimate of $\mathcal{E}_{1,b}(z)$:

From (60),

$$\mathcal{E}_{1,b}(z)=\frac{1}{2i\pi l_b^4}\sum_{n\in\mathbb{N}}\operatorname{Tr}\left[\gamma_b\left[\Pi_{n,z},\mathcal{V}_z\right]\right]\tag{64}$$

We introduce another projector:

$$\operatorname{Tr}\left[\gamma_b\left[\Pi_{n,z},\mathcal{V}_z\right]\right]=\operatorname{Tr}\left[\gamma_b\Pi_{n,z}\left[\Pi_{n,z},\mathcal{V}_z\right]\right]+\operatorname{Tr}\left[\Pi_{n,z}\gamma_b\left[\Pi_{n,z},\mathcal{V}_z\right]\right]$$

Since $\Pi_{n,z},\mathcal{V}_z,\gamma_b$ are self-adjoint, with Young's inequality

$$\begin{aligned}
\left|\operatorname{Tr}\left[\gamma_b\left[\Pi_{n,z},\mathcal{V}_z\right]\right]\right| \leqslant& 2\sqrt{\operatorname{Tr}\left[\left[\Pi_{n,z},\mathcal{V}_z\right]\left[\Pi_{n,z},\mathcal{V}_z\right]^*\right]\operatorname{Tr}\left[\gamma_b\Pi_{n,z}\left(\gamma_b\Pi_{n,z}\right)^*\right]}\\
=&2\left\|\left[\Pi_{n,z},\mathcal{V}_z\right]\right\|_{\mathcal{L}^2}\sqrt{\operatorname{Tr}\left[\gamma_b^2\Pi_{n,z}\right]}\leqslant\epsilon_n\left\|\left[\Pi_{n,z},\mathcal{V}_z\right]\right\|_{\mathcal{L}^2}^2+\frac{1}{\epsilon_n}\operatorname{Tr}\left[\gamma_b^2\Pi_{n,z}\right]\\
\leqslant&\epsilon_n\left\|\left[\Pi_{n,z},\mathcal{V}_z\right]\right\|_{\mathcal{L}^2}^2+\frac{2\pi l_b^2}{\epsilon_n}\operatorname{Tr}\left[\gamma_b\Pi_{n,z}\right]
\end{aligned}\tag{65}$$

We estimate the first term with the changes of variables $x\coloneqq\frac{x-z}{\sqrt{2}l_b}$, $y\coloneqq\frac{y-z}{\sqrt{2}l_b}$

$$\left\|\left[\Pi_{n,z},\mathcal{V}_z\right]\right\|_{\mathcal{L}^2}^2=\iint_{\mathbb{R}^2\times\mathbb{R}^2}\left|\left[\Pi_{n,z},\mathcal{V}_z\right](x,y)\right|^2dxdy=\iint_{\mathbb{R}^2\times\mathbb{R}^2}\left(\mathcal{V}_z(x)-\mathcal{V}_z(y)\right)^2\left|\Pi_{n,z}(x,y)\right|^2dxdy$$

$$
\begin{aligned}
=&\frac{1}{(2\pi n! l_b^2)^2} \int\limits_{\mathbb{R}^2\times\mathbb{R}^2} (\mathcal{V}_z(x)-\mathcal{V}_z(y))^2 \left|\frac{(x-z)\,(y-z)}{2l_b^2}\right|^{2n} e^{-\frac{|x-z|^2+|y-z|^2}{2l_b^2}} dxdy \\
=&\frac{1}{(\pi n!)^2} \int\limits_{\mathbb{R}^2\times\mathbb{R}^2} \left(\mathcal{V}_z(z+\sqrt{2}l_b x)-\mathcal{V}_z(z+\sqrt{2}l_b y)\right)^2 |xy|^{2n}\, e^{-|x|^2-|y|^2} dxdy
\end{aligned} \tag{66}
$$

Let $2 \leqslant \alpha \leqslant 3$, using the expansion (57),

$$
\left|\mathcal{V}_z(z+\sqrt{2}l_b x)\right| = \left|W(z+\sqrt{2}l_b x)-\sqrt{2}l_b dW_z(x)-l_b^2 d^2W_z(x,x)\right| \leqslant C\,\|W\|_{W^{\alpha,\infty}}\,|l_b x|^{\alpha}
$$

and similarly with y instead of x. With Lemma V.2,

$$
\begin{aligned}
\|[\Pi_{n,z},\mathcal{V}_z]\|_{\mathcal{L}^2}^2 \leqslant& C\frac{\|W\|_{W^{\alpha,\infty}}^2}{n!^2} \int\limits_{\mathbb{R}^2\times\mathbb{R}^2} l_b^{2\alpha}\left(|x|^{2\alpha}+|y|^{2\alpha}\right)|xy|^{2n}\, e^{-|x|^2-|y|^2} dxdy \\
\leqslant& C\,\|W\|_{W^{3,\infty}}^2 \left((n+1)l_b^2\right)^{\alpha}
\end{aligned}
$$

Inserting this along with (65) in (64) we get

$$
|\mathcal{E}_{1,b}(z)| \leqslant \frac{C}{l_b^4}\left(\|W\|_{W^{3,\infty}}^2 \sum_{n\in\mathbb{N}} \epsilon_n \min_{2\leqslant\alpha\leqslant 3}\left(\left((n+1)l_b^2\right)^{\alpha}\right) + \sum_{n\in\mathbb{N}} \frac{l_b^2}{\epsilon_n}\mathrm{Tr}\left[\gamma_b \Pi_{n,z}\right]\right) \tag{67}
$$

Integrating against $\varphi \in C_c^{\infty}\left(\mathbb{R}^2\right)$, choosing

$$
\epsilon_n \coloneqq \frac{\epsilon}{\sqrt{|\mathrm{supp}(\varphi)|}\,\|W\|_{W^{3,\infty}}\,(n+1)^k}
$$

and using (39) (14) (50), we obtain

$$
\begin{aligned}
&\left|\int\limits_{\mathbb{R}^2} \varphi(z)\mathcal{E}_{1,b}(z)dz\right| \\
\leqslant&\frac{C}{l_b^4}\left(\|W\|_{W^{3,\infty}}^2\|\varphi\|_{L^1} \sum_{n\in\mathbb{N}} \epsilon_n \min_{2\leqslant\alpha\leqslant 3}\left(\left((n+1)l_b^2\right)^{\alpha}\right) + \|\varphi\|_{L^\infty}\sum_{n\in\mathbb{N}} \frac{2\pi l_b^4}{\epsilon_n}\mathrm{Tr}\left[\gamma_b\Pi_n\right]\right) \\
\leqslant& C\,\|\varphi\|_{L^\infty}\left(|\mathrm{supp}(\varphi)|\,\|W\|_{W^{3,\infty}}^2 \sum_{n\in\mathbb{N}} \frac{\epsilon_n}{l_b^4} \min_{2\leqslant\alpha\leqslant 3}\left(\left((n+1)l_b^2\right)^{\alpha}\right) + \sum_{n\in\mathbb{N}}\frac{1}{\epsilon_n}\mathrm{Tr}\left[\gamma_b\Pi_n\right]\right) \\
\leqslant& C\sqrt{|\mathrm{supp}(\varphi)|}\,\|\varphi\|_{L^\infty}\,\|W\|_{W^{3,\infty}}\left(\epsilon\sum_{n\in\mathbb{N}}\frac{\min_{2\leqslant\alpha\leqslant 3}\left(\left((n+1)l_b^2\right)^{\alpha}\right)}{l_b^4(n+1)^k} + \frac{1}{\epsilon}\sum_{n\in\mathbb{N}}(n+1)^k\mathrm{Tr}\left[\gamma_b\Pi_n\right]\right) \\
\leqslant& C\sqrt{|\mathrm{supp}(\varphi)|}\,\|\varphi\|_{L^\infty}\,\|W\|_{W^{3,\infty}}\left(\epsilon\sum_{n\in\mathbb{N}}\frac{\min_{2\leqslant\alpha\leqslant 3}\left(\left((n+1)l_b^2\right)^{\alpha}\right)}{l_b^4(n+1)^k} + \frac{1}{\epsilon}\mathrm{Tr}\left[\gamma_b\mathscr{L}_b^k\right]\right)
\end{aligned}
$$

We split the sum in two part depending on weather $(n+1)l_b^2 \leqslant 1$ or not, for the first part take $\alpha = 3$, and for the second part $\alpha = 2$:

$$
\begin{aligned}
\left|\int\limits_{\mathbb{R}^2} \varphi(z)\mathcal{E}_{1,b}(z)dz\right| \leqslant& C\sqrt{|\mathrm{supp}(\varphi)|}\,\|\varphi\|_{L^\infty}\,\|W\|_{W^{3,\infty}}\left(\epsilon l_b^2\sum_{(n+1)l_b^2\leqslant 1}\frac{1}{(n+1)^{k-3}}\right. \\
&\left.+\epsilon\sum_{(n+1)l_b^2>1}\frac{1}{(n+1)^{k-2}}+\frac{1}{\epsilon}\mathrm{Tr}\left[\gamma_b\mathscr{L}_b^k\right]\right)
\end{aligned} \tag{68}
$$

With the asymptotics (42) and (43),

$$\left| \int_{\mathbb{R}^2} \varphi(z)\mathcal{E}_{1,b}(z)dz \right| \leqslant C\sqrt{|\mathrm{supp}(\varphi)|}\, \|\varphi\|_{L^\infty} \|W\|_{W^{3,\infty}} \left(\epsilon l_b^2 p_{k-4}\left(l_b^{-2}\right) + \epsilon l_b^{2(k-3)} + \frac{1}{\epsilon}\mathrm{Tr}\left[\gamma_b \mathscr{L}_b^k\right]\right)$$

Similarly as in (44),

$$l_b^{2(k-4)} \leqslant p_{k-4}$$

so

$$l_b^{2(k-3)} \leqslant l_b^2 p_{k-4}$$

Hence choosing $\epsilon \coloneqq \frac{1}{l_b}\sqrt{\frac{\mathrm{Tr}\left[\gamma_b \mathscr{L}_b^k\right]}{p_{k-4}\left(l_b^{-2}\right)}}$ we conclude that

$$\begin{aligned}\left| \int_{\mathbb{R}^2} \varphi(z)\mathcal{E}_{1,b}(z)dz \right| \leqslant & C\sqrt{|\mathrm{supp}(\varphi)|}\, \|\varphi\|_{L^\infty} \|W\|_{W^{3,\infty}} \left(\epsilon l_b^2 p_{k-4}\left(l_b^{-2}\right) + \frac{1}{\epsilon}\mathrm{Tr}\left[\gamma_b \mathscr{L}_b^k\right]\right) \\ = & C\sqrt{|\mathrm{supp}(\varphi)|}\, \|\varphi\|_{L^\infty} \|W\|_{W^{3,\infty}} \sqrt{\mathrm{Tr}\left[\gamma_b \mathscr{L}_b^k\right]}\, l_b \sqrt{p_{k-4}\left(l_b^{-2}\right)} \end{aligned} \tag{69}$$

Estimate of $\mathcal{E}_{2,b}(z)$:

We start from (61):

$$\mathcal{E}_{2,b}(z) \coloneqq \frac{1}{4\pi l_b^2} \iint_{\mathbb{R}^2\times\mathbb{R}^2} \gamma_b(x,y)\nabla_z^\perp \Pi_z(y,x)\cdot d^2W(z)(x-z)dxdy$$

Let $\odot$ denote the tensor contraction defined for $n, m \geqslant k$ by

$$u_1\otimes\cdots\otimes u_n \odot^k v_1\otimes\cdots\otimes v_m \coloneqq \langle u_n|v_1\rangle \dots \langle u_{n-k+1}|v_k\rangle\, u_1\otimes\cdots\otimes u_{n-k}\otimes v_{k+1}\otimes\cdots\otimes v_m$$

Identifying $d^nW(z)$ with the associated rank n tensor, we notice that

$$\begin{aligned}\nabla_z^\perp \cdot d^2W(z)(x-z) =& \left(\nabla\odot\nabla\otimes\nabla W(z)\right)(x-z) + \nabla\otimes\nabla W(z)\odot^2 \nabla_z^\perp \otimes (x-z) \\ =& d^2W(z)\odot^2 \begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix} = 0\end{aligned}$$

because $d^2W(z)$ is symmetric and $\begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix}$ anti-symmetric. An integration by parts yields

$$\begin{aligned}\int_{\mathbb{R}^2}\varphi(z)\mathcal{E}_{2,b}(z)dz =& \frac{-1}{4\pi l_b^2}\int_{\mathbb{R}^2} \nabla^\perp\varphi(z)\cdot d^2W(z)\left(\iint_{\mathbb{R}^2\times\mathbb{R}^2}\gamma_b(x,y)\Pi_z(y,x)(x-z)dxdy\right)dz \\ =& \frac{-1}{4\pi l_b^2}\int_{\mathbb{R}^2} \nabla^\perp\varphi(z)\cdot d^2W(z)\mathrm{Tr}\left[\gamma_b\Pi_z(X-z)\right]dz\end{aligned} \tag{70}$$

With Young's inequality and an estimate similar to (36),

$$\left|\mathrm{Tr}\left[\gamma_b\Pi_z(X-z)\right]\right| \leqslant \sum_{n\in\mathbb{N}} \left|\mathrm{Tr}\left[\gamma_b\Pi_{n,z}(X-z)\right]\right|$$

$$\begin{aligned}
&\leqslant \sum_{n\in\mathbb{N}} \epsilon_n \mathrm{Tr}\left[\Pi_{n,z}\left|X-z\right|^2\right] + \sum_{n\in\mathbb{N}} \frac{1}{\epsilon_n}\mathrm{Tr}\left[\gamma_b^2\Pi_{n,z}\right]\\
&\leqslant 2l_b^2\sum_{n\in\mathbb{N}}\epsilon_n(n+1) + 2\pi l_b^2 \sum_{n\in\mathbb{N}}\frac{1}{\epsilon_n}\mathrm{Tr}\left[\gamma_b\Pi_{n,z}\right] \qquad (71)
\end{aligned}$$

Inserting this in (70), taking $\epsilon_n \coloneqq \dfrac{\epsilon}{\sqrt{|\mathrm{supp}(\varphi)|}(n+1)^k}$ and using (39) we obtain

$$\begin{aligned}
&\left|\int_{\mathbb{R}^2}\varphi(z)\mathcal{E}_{2,b}(z)dz\right|\\
\leqslant& C\left(\left\|d^2W\nabla^\perp\varphi\right\|_{L^1}\sum_{n\in\mathbb{N}}\epsilon_n(n+1) + \left\|d^2W\nabla^\perp\varphi\right\|_{L^\infty}\sum_{n\in\mathbb{N}}\frac{l_b^2}{\epsilon_n}\mathrm{Tr}\left[\gamma_b\Pi_n\right]\right)\\
\leqslant& C\sqrt{|\mathrm{supp}(\varphi)|}\left\|d^2W\nabla^\perp\varphi\right\|_{L^\infty}\left(\epsilon\sum_{n\in\mathbb{N}}\frac{1}{(n+1)^{k-1}} + \frac{l_b^2}{\epsilon}\sum_{n\in\mathbb{N}}(n+1)^k\mathrm{Tr}\left[\gamma_b\Pi_n\right]\right)\\
\leqslant& C\sqrt{|\mathrm{supp}(\varphi)|}\left\|\nabla\varphi\right\|_{L^\infty}\left\|d^2W\right\|_{L^\infty}\left(\epsilon+\frac{l_b^2}{\epsilon}\mathrm{Tr}\left[\gamma_b\mathscr{L}_b^k\right]\right) \qquad (72)
\end{aligned}$$

Choosing $\epsilon \coloneqq \dfrac{l_b}{\sqrt{\mathrm{Tr}\left[\gamma_b\mathscr{L}_b^k\right]}}$ we conclude that

$$\left|\int_{\mathbb{R}^2}\varphi(z)\mathcal{E}_{2,b}(z)dz\right| \leqslant C\sqrt{|\mathrm{supp}(\varphi)|}\left\|\nabla\varphi\right\|_{L^\infty}\left\|d^2W\right\|_{L^\infty}\sqrt{\mathrm{Tr}\left[\gamma_b\mathscr{L}_b^k\right]}\,l_b \qquad (73)$$

Conclusion:

We can control (62) with an estimate similar to (61) by exchanging x and y, obtaining

$$\left|\int_{\mathbb{R}^2}\varphi(z)\widetilde{\mathcal{E}}_{2,b}(z)dz\right| \leqslant C\sqrt{|\mathrm{supp}(\varphi)|}\left\|\nabla\varphi\right\|_{L^\infty}\left\|d^2W\right\|_{L^\infty}\sqrt{\mathrm{Tr}\left[\gamma_b\mathscr{L}_b^k\right]}\,l_b \qquad (74)$$

Finally, using the notation (41),

$$1 \leqslant p_{k-4}\left(l_b^{-2}\right) = \begin{cases} l_b^{2(k-4)} & \text{if } k<4\\ \ln\left(l_b^{-2}\right) & \text{if } k=4\\ 1 & \text{if } k>4\end{cases}$$

so with (69) (73) (74) and (63) we obtain (52). □

This proof can be adapted for the estimate to work with $k>2$ if we expand W to the second order in (57) for low Landau levels only, precisely when $(n+1)l_b^2 \leqslant 1$. But this idea works with $k=1$ when we consider the dynamics of the truncated semi-classical density (33) instead of (32). This is the goal of Proposition IV.2 and Proposition IV.3.

V.2 Dynamics of the truncated semi-classical density

Proposition V.3: *Gyrokinetic equation for the truncated semi-classical density*
Let $t \in \mathbb{R}_+, k \geqslant 0, \gamma_b(t) \in \mathcal{L}^1\left(L^2\left(\mathbb{R}^2\right)\right), W \in W^{3,\infty}(\mathbb{R}^2)$ *and assume*

$$\begin{aligned}
&Tr\left[\gamma_b(t)\right] = 1, 0 \leqslant \gamma_b(t) \leqslant 2\pi l_b^2 \\
&\partial_t \gamma_b(t) = \frac{1}{il_b^2}\left[\mathscr{L}_b + W, \gamma_b(t)\right]
\end{aligned}$$

then $\forall \varphi \in C_c^\infty\left(\mathbb{R}^2\right)$,

$$\begin{aligned}
&\left|\int_{\mathbb{R}^2} \varphi(z)\left(\partial_t \rho_{\gamma_b}^{\leqslant N}(t,z) + \nabla^\perp W(z)\cdot\nabla\rho_{\gamma_b}^{\leqslant N}(t,z)\right)dz\right| \\
\leqslant &C\left\|\varphi\right\|_{L^\infty}\left\|\nabla W\right\|_{L^\infty}\frac{1}{l_b N^{k-\frac{1}{2}}}Tr\left[\gamma_b\mathscr{L}_b^k\left(\Pi_N + \Pi_{N+1}\right)\right] + C\sqrt{\left|supp(\varphi)\right|}\left\|\varphi\right\|_{W^{1,\infty}}\left\|W\right\|_{W^{3,\infty}}\Bigg(\\
&N^{1-\frac{k}{2}}\sqrt{Tr\left[\gamma_b\mathscr{L}_b^k\left(\Pi_N+\Pi_{N+1}\right)\right]} + \sqrt{Tr\left[\gamma_b(t)\mathscr{L}_b^k\right]}\cdot\begin{cases} l_b N^{2-\frac{k}{2}} & \text{if } k<4 \\ l_b\sqrt{\ln(N)} & \text{if } k=4 \\ l_b & \text{if } k>4\end{cases}\Bigg)
\end{aligned} \tag{75}$$

Proof: **a direct computation:**

Using (20) in (55),

$$\begin{aligned}
&\nabla^\perp W(z)\cdot\nabla_z\rho_{\gamma_b}^{\leqslant N}(z) = \frac{-1}{2\pi l_b^2}\nabla W(z)\cdot\mathrm{Tr}\left[\gamma_b\nabla_z^\perp\Pi_{\leqslant N,z}\right] = \frac{-1}{2i\pi l_b^4}\mathrm{Tr}\left[\gamma_b\left[\Pi_{\leqslant N,z}, X\cdot\nabla W(z)\right]\right] \\
&+\frac{\sqrt{N+1}}{2\sqrt{2}\pi l_b^3}\nabla W(z)\cdot\begin{pmatrix}1 & 1\\ -i & i\end{pmatrix}\begin{pmatrix}\mathrm{Tr}\left[\gamma_b\left|\psi_{N,z}\right\rangle\left\langle\psi_{N+1,z}\right|\right]\\ \mathrm{Tr}\left[\gamma_b\left|\psi_{N+1,z}\right\rangle\left\langle\psi_{N,z}\right|\right]\end{pmatrix}
\end{aligned}$$

so with the same computation as for (56),

$$\begin{aligned}
&\partial_t\rho_{\gamma_b}^{\leqslant N}(z) + \nabla^\perp W(z)\cdot\nabla_z\rho_{\gamma_b}^{\leqslant N}(z) = \frac{1}{2i\pi l_b^4}\mathrm{Tr}\left[\gamma_b\left[\Pi_{\leqslant N,z}, W - X\cdot\nabla W(z)\right]\right] \\
&+\frac{\sqrt{N+1}}{2\sqrt{2}\pi l_b^3}\nabla W(z)\cdot\begin{pmatrix}1 & 1\\ -i & i\end{pmatrix}\begin{pmatrix}\mathrm{Tr}\left[\gamma_b\left|\psi_{N,z}\right\rangle\left\langle\psi_{N+1,z}\right|\right]\\ \mathrm{Tr}\left[\gamma_b\left|\psi_{N+1,z}\right\rangle\left\langle\psi_{N,z}\right|\right]\end{pmatrix}
\end{aligned} \tag{76}$$

Using (59) for $\Pi_{\leqslant N,z}$ instead of Π_z and then inserting (20),

$$\begin{aligned}
&\left[\Pi_{\leqslant N,z}, W - X\cdot\nabla W(z)\right] \\
&= \left[\Pi_{\leqslant N,z}, \mathcal{V}_z\right] + \frac{1}{2}d^2W(z)(\left[\Pi_{\leqslant N,z}, X\right], X-z) + \frac{1}{2}d^2W(z)(X-z, \left[\Pi_{\leqslant N,z}, X\right]) \\
&= \left[\Pi_{\leqslant N,z}, \mathcal{V}_z\right] + \frac{il_b^2}{2}d^2W(z)(\nabla_z^\perp\Pi_{\leqslant N,z}, X-z) + \frac{il_b^2}{2}d^2W(z)(X-z, \nabla_z^\perp\Pi_{\leqslant N,z}) \\
&+\frac{i\sqrt{N+1}l_b}{2\sqrt{2}}d^2W(z)\left(\begin{pmatrix}1 & 1\\ -i & i\end{pmatrix}\begin{pmatrix}\left|\psi_{N,z}\right\rangle\left\langle\psi_{N+1,z}\right|\\ \left|\psi_{N+1,z}\right\rangle\left\langle\psi_{N,z}\right|\end{pmatrix}, X-z\right) \\
&+\frac{i\sqrt{N+1}l_b}{2\sqrt{2}}d^2W(z)\left(X-z, \begin{pmatrix}1 & 1\\ -i & i\end{pmatrix}\begin{pmatrix}\left|\psi_{N,z}\right\rangle\left\langle\psi_{N+1,z}\right|\\ \left|\psi_{N+1,z}\right\rangle\left\langle\psi_{N,z}\right|\end{pmatrix}\right)
\end{aligned} \tag{77}$$

Defining the error terms

$$\mathcal{E}_{0,b}(z) \coloneqq \frac{\sqrt{N+1}}{2\sqrt{2}\pi l_b^3}\nabla W(z)\cdot\begin{pmatrix}1 & 1\\ -i & i\end{pmatrix}\begin{pmatrix}\mathrm{Tr}\left[\gamma_b\left|\psi_{N,z}\right\rangle\left\langle\psi_{N+1,z}\right|\right]\\ \mathrm{Tr}\left[\gamma_b\left|\psi_{N+1,z}\right\rangle\left\langle\psi_{N,z}\right|\right]\end{pmatrix} \tag{78}$$

$$\mathcal{E}_{1,b}(z) \coloneqq \frac{1}{2i\pi l_b^4}\mathrm{Tr}\left[\gamma_b\left[\Pi_{\leqslant N,z}, \mathcal{V}_z\right]\right] \tag{79}$$

$$\mathcal{E}_{2,b}(z) \coloneqq \frac{1}{4\pi l_b^2}\mathrm{Tr}\left[\gamma_b d^2W(z)\left(\nabla_z^\perp \Pi_{\leqslant N,z}, X-z\right)\right] \tag{80}$$

$$\widetilde{\mathcal{E}}_{2,b}(z) \coloneqq \frac{1}{4\pi l_b^2}\mathrm{Tr}\left[\gamma_b d^2W(z)\left(X-z, \nabla_z^\perp \Pi_{\leqslant N,z}\right)\right] \tag{81}$$

$$\mathcal{E}_{3,b}(z) \coloneqq \frac{\sqrt{N+1}}{4\sqrt{2}\pi l_b^3}\mathrm{Tr}\left[\gamma_b d^2W(z)\left(\begin{pmatrix}1 & 1\\ -i & i\end{pmatrix}\begin{pmatrix}|\psi_{N,z}\rangle\langle\psi_{N+1,z}|\\ |\psi_{N+1,z}\rangle\langle\psi_{N,z}|\end{pmatrix}, X-z\right)\right] \tag{82}$$

$$\widetilde{\mathcal{E}}_{3,b}(z) \coloneqq \frac{\sqrt{N+1}}{2\sqrt{4}\pi l_b^3}\mathrm{Tr}\left[\gamma_b d^2W(z)\left(X-z, \begin{pmatrix}1 & 1\\ -i & i\end{pmatrix}\begin{pmatrix}|\psi_{N,z}\rangle\langle\psi_{N+1,z}|\\ |\psi_{N+1,z}\rangle\langle\psi_{N,z}|\end{pmatrix}\right)\right] \tag{83}$$

and inserting (77) in (76) we have

$$\partial_t \rho_{\gamma_b}^{\leqslant N} + \nabla^\perp W\cdot\nabla\rho_{\gamma_b}^{\leqslant N} = \mathcal{E}_{0,b} + \mathcal{E}_{1,b} + \mathcal{E}_{2,b} + \widetilde{\mathcal{E}}_{2,b} + \mathcal{E}_{3,b} + \widetilde{\mathcal{E}}_{3,b} \tag{84}$$

Estimate of $\mathcal{E}_{0,b}$:

As operators, with Young's inequality

$$\begin{aligned}2\left||\psi_{N+1,z}\rangle\langle\psi_{N,z}|\right| &\leqslant \Pi_{N,z} + \Pi_{N+1,z}\\ 2\left||\psi_{N,z}\rangle\langle\psi_{N+1,z}|\right| &\leqslant \Pi_{N,z} + \Pi_{N+1,z}\end{aligned}$$

so

$$|\mathcal{E}_{0,b}(z)| \leqslant C\left\|\nabla W\right\|_{L^\infty}\frac{\sqrt{N}}{l_b^3}\left(\mathrm{Tr}\left[\gamma_b\Pi_{N+1,z}\right] + \mathrm{Tr}\left[\gamma_b\Pi_{N,z}\right]\right)$$

and

$$\left|\int_{\mathbb{R}^2}\varphi(z)\mathcal{E}_{0,b}(z)dz\right| \leqslant C\left\|\varphi\right\|_{L^\infty}\left\|\nabla W\right\|_{L^\infty}\frac{\sqrt{N}}{l_b}\left(\mathrm{Tr}\left[\gamma_b\Pi_N\right] + \mathrm{Tr}\left[\gamma_b\Pi_{N+1}\right]\right) \tag{85}$$

Note that

$$\begin{aligned}&\mathrm{Tr}\left[\gamma_b\mathscr{L}_b^k\Pi_N\right] = \left(2\hbar b\left(N+\frac{1}{2}\right)\right)^k\mathrm{Tr}\left[\gamma_b\Pi_N\right] \underset{N\to\infty}{\sim} 2^kN^k\mathrm{Tr}\left[\gamma_b\Pi_N\right]\\ &\mathrm{Tr}\left[\gamma_b\mathscr{L}_b^2\Pi_{N+1}\right] \underset{N\to\infty}{\sim} 2^kN^k\mathrm{Tr}\left[\gamma_b\Pi_{N+1}\right]\end{aligned}$$

Inserting this in (85) we have

$$\left|\int_{\mathbb{R}^2}\varphi(z)\mathcal{E}_{0,b}(z)dz\right| \leqslant C\left\|\varphi\right\|_{L^\infty}\left\|\nabla W\right\|_{L^\infty}\frac{1}{l_bN^{k-\frac{1}{2}}}\mathrm{Tr}\left[\gamma_b\mathscr{L}_b^k\left(\Pi_N + \Pi_{N+1}\right)\right] \tag{86}$$

Estimate of $\mathcal{E}_{1,b}$:

Following the same proof as for (60) we obtain instead of (68):

$$\left|\int_{\mathbb{R}^2}\varphi(z)\mathcal{E}_{1,b}(z)dz\right| \leqslant C\sqrt{|\mathrm{supp}(\varphi)|}\left\|\varphi\right\|_{L^\infty}\left\|W\right\|_{W^{3,\infty}}\left(\epsilon l_b^2\sum_{n=0}^N\frac{1}{(n+1)^{k-3}} + \frac{1}{\epsilon}\mathrm{Tr}\left[\gamma_b\mathscr{L}_b^k\right]\right)$$

With the asymptotic (43), taking $\epsilon \coloneqq \frac{1}{l_b}\sqrt{\frac{\mathrm{Tr}\left[\gamma_b \mathscr{L}_b^k\right]}{p_{k-4}(N)}}$ we have

$$\begin{aligned}\left|\int_{\mathbb{R}^2} \varphi(z)\mathcal{E}_{1,b}(z)dz\right| \leqslant & C\sqrt{|\mathrm{supp}(\varphi)|}\,\|\varphi\|_{L^\infty}\|W\|_{W^{3,\infty}}\left(\epsilon l_b^2 p_{k-4}(N) + \frac{1}{\epsilon}\mathrm{Tr}\left[\gamma_b \mathscr{L}_b^k\right]\right)\\ = & C\sqrt{|\mathrm{supp}(\varphi)|}\,\|\varphi\|_{L^\infty}\|W\|_{W^{3,\infty}}\sqrt{\mathrm{Tr}\left[\gamma_b \mathscr{L}_b^k\right]}\,l_b\sqrt{p_{k-4}(N)}\end{aligned} \tag{87}$$

Estimate of $\mathcal{E}_{2,b}$:

Following the same proof as for (61) we obtain instead of (72)

$$\left|\int_{\mathbb{R}^2} \varphi(z)\mathcal{E}_{2,b}(z)dz\right| \leqslant C\sqrt{|\mathrm{supp}(\varphi)|}\,\|\nabla\varphi\|_{L^\infty}\left\|d^2W\right\|_{L^\infty}\left(\epsilon\sum_{n=0}^{N}\frac{1}{(n+1)^{k-1}} + \frac{l_b^2}{\epsilon}\mathrm{Tr}\left[\gamma_b \mathscr{L}_b^k\right]\right)$$

With the asymptotic (43), taking $\epsilon \coloneqq l_b\sqrt{\frac{\mathrm{Tr}\left[\gamma_b \mathscr{L}_b^k\right]}{p_{k-2}(N)}}$ we have

$$\begin{aligned}\left|\int_{\mathbb{R}^2} \varphi(z)\mathcal{E}_{2,b}(z)dz\right| \leqslant & C\sqrt{|\mathrm{supp}(\varphi)|}\,\|\nabla\varphi\|_{L^\infty}\left\|d^2W\right\|_{L^\infty}\left(\epsilon p_{k-2}(N) + \frac{l_b^2}{\epsilon}\mathrm{Tr}\left[\gamma_b \mathscr{L}_b^k\right]\right)\\ = & C\sqrt{|\mathrm{supp}(\varphi)|}\,\|\nabla\varphi\|_{L^\infty}\left\|d^2W\right\|_{L^\infty}\sqrt{\mathrm{Tr}\left[\gamma_b \mathscr{L}_b^k\right]}\,l_b\sqrt{p_{k-2}(N)}\end{aligned} \tag{88}$$

Estimate of $\mathcal{E}_{3,b}$:

We only need to estimate terms of the form

$$\frac{\sqrt{N}}{l_b^3}\mathrm{Tr}\left[\gamma_b \left|\psi_{N,z}\right\rangle\left\langle\psi_{N+1,z}\right|(X_i - z_i)\right]$$

or

$$\frac{\sqrt{N}}{l_b^3}\mathrm{Tr}\left[\gamma_b \left|\psi_{N+1,z}\right\rangle\left\langle\psi_{N,z}\right|(X_i - z_i)\right]$$

with $i \in \{1,2\}$. With Young's inequality and the same inequality as in (36),

$$\begin{aligned}\left|\mathrm{Tr}\left[\gamma_b \left|\psi_{N+1,z}\right\rangle\left\langle\psi_{N,z}\right|(X_i - z_i)\right]\right| \leqslant & \epsilon\mathrm{Tr}\left[(X_i - z_i)^2\Pi_{N,z}\right] + \frac{1}{\epsilon}\mathrm{Tr}\left[\gamma_b^2\Pi_{N+1,z}\right]\\ \leqslant & 2\epsilon(N+1)l_b^2 + \frac{2\pi l_b^2}{\epsilon}\mathrm{Tr}\left[\gamma_b\Pi_{N+1,z}\right]\end{aligned} \tag{89}$$

and similarly

$$\mathrm{Tr}\left[\gamma_b \left|\psi_{N,z}\right\rangle\left\langle\psi_{N+1,z}\right|(X_i - z_i)\right] \leqslant 2\epsilon(N+2)l_b^2 + \frac{2\pi l_b^2}{\epsilon}\mathrm{Tr}\left[\gamma_b\Pi_{N,z}\right]$$

Since $N+1 \sim N+2 \sim N$, integrating against $\varphi \in C_c^\infty\left(\mathbb{R}^2\right)$, we obtain

$$\left|\int_{\mathbb{R}^2} \varphi\mathcal{E}_{3,b}(z)dz\right| \leqslant C\left\|d^2W\right\|_{L^\infty}\frac{\sqrt{N}}{l_b^3}\left(\|\varphi\|_{L^1}\epsilon N l_b^2 + \|\varphi\|_{L^\infty}\frac{l_b^4}{\epsilon}\left(\mathrm{Tr}\left[\gamma_b\Pi_{N+1}\right] + \mathrm{Tr}\left[\gamma_b\Pi_N\right]\right)\right)$$

$$\leqslant C\left\|\varphi\right\|_{L^\infty}\left\|d^2W\right\|_{L^\infty}\left(\left|\mathrm{supp}(\varphi)\right|\epsilon\frac{N^{\frac{3}{2}}}{l_b}+\frac{l_b\sqrt{N}}{\epsilon N^k}N^k\mathrm{Tr}\left[\gamma_b\left(\Pi_N+\Pi_{N+1}\right)\right]\right)$$

$$\leqslant C\left\|\varphi\right\|_{L^\infty}\left\|d^2W\right\|_{L^\infty}\left(\left|\mathrm{supp}(\varphi)\right|\epsilon\frac{N^{\frac{3}{2}}}{l_b}+\frac{l_b}{\epsilon N^{k-\frac{1}{2}}}\mathrm{Tr}\left[\gamma_b\mathscr{L}_b^k\left(\Pi_N+\Pi_{N+1}\right)\right]\right)$$

Taking $\epsilon := \dfrac{l_b}{\sqrt{|\mathrm{supp}(\varphi)|}N^{\frac{k+1}{2}}}\sqrt{\mathrm{Tr}\left[\gamma_b\mathscr{L}_b^k\left(\Pi_N+\Pi_{N+1}\right)\right]}$, we conclude that

$$\left|\int_{\mathbb{R}^2}\varphi\mathcal{E}_{3,b}(z)dz\right|\leqslant C\sqrt{|\mathrm{supp}(\varphi)|}\left\|\varphi\right\|_{L^\infty}\left\|d^2W\right\|_{L^\infty}N^{1-\frac{k}{2}}\sqrt{\mathrm{Tr}\left[\gamma_b\mathscr{L}_b^k\left(\Pi_N+\Pi_{N+1}\right)\right]}\tag{90}$$

Conclusion:

Exchanging x and y in (81) and (83) we obtain with the same arguments as for (88) and (90):

$$\left|\int_{\mathbb{R}^2}\varphi\widetilde{\mathcal{E}}_{2,b}(z)dz\right|\leqslant C\sqrt{|\mathrm{supp}(\varphi)|}\left\|\nabla\varphi\right\|_{L^\infty}\left\|d^2W\right\|_{L^\infty}\sqrt{\mathrm{Tr}\left[\gamma_b\mathscr{L}_b^k\right]}l_b\sqrt{p_{k-2}(N)}\tag{91}$$

$$\left|\int_{\mathbb{R}^2}\varphi\widetilde{\mathcal{E}}_{3,b}(z)dz\right|\leqslant C\sqrt{|\mathrm{supp}(\varphi)|}\left\|\varphi\right\|_{L^\infty}\left\|d^2W\right\|_{L^\infty}N^{1-\frac{k}{2}}\sqrt{\mathrm{Tr}\left[\gamma_b\mathscr{L}_b^k\left(\Pi_N+\Pi_{N+1}\right)\right]}\tag{92}$$

Putting together (86) (87) (88) (90) (91) (92) with (84) we get

$$\left|\int_{\mathbb{R}^2}\varphi(z)\left(\partial_t\rho_{\gamma_b}^{sc,\leqslant N}(z)+\nabla^\perp W(z)\cdot\nabla\rho_{\gamma_b}^{sc,\leqslant N}(z)\right)dz\right|$$

$$\leqslant C\left\|\varphi\right\|_{L^\infty}\left\|\nabla W\right\|_{L^\infty}\frac{1}{l_bN^{k-\frac{1}{2}}}\mathrm{Tr}\left[\gamma_b\mathscr{L}_b^k\left(\Pi_N+\Pi_{N+1}\right)\right]+C\sqrt{|\mathrm{supp}(\varphi)|}\left\|\varphi\right\|_{W^{1,\infty}}\left\|W\right\|_{W^{3,\infty}}\Bigg($$

$$\sqrt{\mathrm{Tr}\left[\gamma_b\mathscr{L}_b^k\right]}l_b\sqrt{p_{k-4}(N)}+\sqrt{\mathrm{Tr}\left[\gamma_b\mathscr{L}_b^k\right]}l_b\sqrt{p_{k-2}(N)}+N^{1-\frac{k}{2}}\sqrt{\mathrm{Tr}\left[\gamma_b\mathscr{L}_b^k\left(\Pi_N+\Pi_{N+1}\right)\right]}\Bigg)$$

Notice from (41) that if $x\geqslant e$,

$$\alpha<\beta\implies p_\alpha(x)\geqslant p_\beta(x)\tag{93}$$

so

$$p_{k-2}(N)\leqslant p_{k-4}(N)$$

and we conclude that

$$\left|\int_{\mathbb{R}^2}\varphi(z)\left(\partial_t\rho_{\gamma_b}^{sc,\leqslant N}(z)+\nabla^\perp W(z)\cdot\nabla\rho_{\gamma_b}^{sc,\leqslant N}(z)\right)dz\right|$$

$$\leqslant C\left\|\varphi\right\|_{L^\infty}\left\|\nabla W\right\|_{L^\infty}\frac{1}{l_bN^{k-\frac{1}{2}}}\mathrm{Tr}\left[\gamma_b\mathscr{L}_b^k\left(\Pi_N+\Pi_{N+1}\right)\right]+C\sqrt{|\mathrm{supp}(\varphi)|}\left\|\varphi\right\|_{W^{1,\infty}}\left\|W\right\|_{W^{3,\infty}}\Bigg($$

$$\sqrt{\mathrm{Tr}\left[\gamma_b\mathscr{L}_b^k\right]}l_b\sqrt{p_{k-4}(N)}+N^{1-\frac{k}{2}}\sqrt{\mathrm{Tr}\left[\gamma_b\mathscr{L}_b^k\left(\Pi_N+\Pi_{N+1}\right)\right]}\Bigg)$$

□

If we assume

$$\mathrm{Tr}\left[\gamma_b\mathscr{L}_b^k\right]=\sum_{n\in\mathbb{N}}\mathrm{Tr}\left[\gamma_b\mathscr{L}_b^k\Pi_n\right]<\infty$$

then up to a subsequence, whose choice will be the object to Lemma VI.1, we can assume

$$\mathrm{Tr}\left[\gamma_b\mathscr{L}_b^k\Pi_N\right]=\mathcal{O}\left(\frac{1}{N}\right)$$

This time we have a small error in (75) if

$$l_bN^{k+\frac{1}{2}}\gg 1,\quad N^{\frac{1-k}{2}}\ll 1,\quad l_bN^{2-\frac{k}{2}}\ll 1$$

When $1<k<4$, this is equivalent to

$$\frac{1}{l_b^{\frac{2}{2k+1}}}\ll N\ll\frac{1}{l_b^{\frac{2}{4-k}}}$$

which is a possible choice of N. Hence we will be able to control the dynamics of the truncated semi-classical density for $k>1$. One could conclude the proof of the main results with the conservation of the second moment of the kinetic energy. As we only obtained Proposition III.4, for the proof to really work with the kinetic energy we further expand W.

Proposition V.4: *Gyrokinetic equation for the truncated semi-classical density*
Let $t\in\mathbb{R}_+, k\geqslant 0, \gamma_b(t)\in\mathcal{L}^1\left(L^2\left(\mathbb{R}^2\right)\right), W\in W^{4,\infty}(\mathbb{R}^2)$ *and assume*

$$Tr\left[\gamma_b(t)\right]=1, 0\leqslant\gamma_b(t)\leqslant 2\pi l_b^2$$
$$\partial_t\gamma_b(t)=\frac{1}{il_b^2}\left[\mathscr{L}_b+W,\gamma_b(t)\right]$$

then $\forall\varphi\in C_c^\infty\left(\mathbb{R}^2\right)$,

$$\begin{aligned}&\left|\int_{\mathbb{R}^2}\varphi(z)\left(\partial_t\rho_{\gamma_b}^{\leqslant N}(t,z)+\nabla^\perp W(z)\cdot\nabla\rho_{\gamma_b}^{\leqslant N}(t,z)\right)dz\right|\\
\leqslant& c(\varphi,W)\left(\frac{1}{l_bN^{k-\frac{1}{2}}}Tr\left[\gamma_b(t)\mathscr{L}_b^k\Pi_{N-1:N+1}\right]+\left(N^{1-\frac{k}{2}}+l_bN^{\frac{3-k}{2}}\right)\sqrt{Tr\left[\gamma_b(t)\mathscr{L}_b^k\Pi_{N-1:N+1}\right]}\right.\\
&\left.+\,l_b^2N+\sqrt{Tr\left[\gamma_b(t)\mathscr{L}_b^k\right]}\cdot\begin{cases}l_bN^{1-\frac{k}{2}}+l_b^2N^{\frac{5-k}{2}} & if\ k<2\\ l_b\sqrt{\ln(N)}+l_b^2N^{\frac{3}{2}} & if\ k=2\\ l_b+l_b^2N^{\frac{5-k}{2}} & if\ 2<k<5\\ l_b+l_b^2\sqrt{\ln(N)} & if\ k=5\\ l_b & if\ k>5\end{cases}\right)\end{aligned}$$

where

$$c(\varphi,W)\coloneqq\left(1+|supp(\varphi)|\right)\|\varphi\|_{W^{1,\infty}}\|W\|_{W^{4,\infty}}$$

Proof: **a direct computation:**

Redefine

$$\mathcal{V}_z(x) \coloneqq W(x) - dW(z)(x-z) - \frac{1}{2}d^2W(z)(x-z,x-z) - \frac{1}{6}d^3W(z)(x-z,x-z,x-z)$$

Notice that

$$\begin{aligned}
&d^3W(z)(y-z,y-z,y-z)\\
=&d^3W(z)(y-x,y-z,y-z) + d^3W(z)(x-z,y-z,y-z)\\
=&d^3W(z)(y-x,y-z,y-z) + d^3W(z)(x-z,y-x,y-z) + d^3W(z)(x-z,x-z,y-z)\\
=&d^3W(z)(y-x,y-z,y-z) + d^3W(z)(x-z,y-x,y-z) + d^3W(z)(x-z,x-z,y-x)\\
&+ d^3W(z)(x-z,x-z,x-z)
\end{aligned}$$

so (58) becomes

$$\begin{aligned}
&\left[\Pi_{\leqslant N,z}, W - X\cdot\nabla W(z)\right](x,y) = \Pi_{\leqslant N,z}(x,y)\Bigg(\mathcal{V}_z(y) - \mathcal{V}_z(x) + \frac{d^2W(z)}{2}(y-x,y-z)\\
&+\frac{d^2W(z)}{2}(x-z,y-x) + \frac{d^3W(z)}{6}(y-x,y-z,y-z) + \frac{d^3W(z)}{6}(x-z,y-x,y-z)\\
&+\frac{d^3W(z)}{6}(x-z,x-z,y-x)\Bigg)
\end{aligned}$$

Then inserting (20),

$$\begin{aligned}
&[\Pi_{\leqslant N,z}, W - X\cdot\nabla W(z)]\\
&= [\Pi_{\leqslant N,z}, \mathcal{V}_z] + \frac{d^2W(z)}{2}(\left[\Pi_{\leqslant N,z}, X\right], X-z) + \frac{d^2W(z)}{2}(X-z, [\Pi_{\leqslant N,z}, X])\\
&\quad+\frac{d^3W(z)}{6}(\left[\Pi_{\leqslant N,z}, X\right], X-z, X-z) + \frac{d^3W(z)}{6}(X-z, \left[\Pi_{\leqslant N,z}, X\right], X-z)\\
&\quad+\frac{d^3W(z)}{6}(X-z, X-z, [\Pi_{\leqslant N,z}, X])\\
&= [\Pi_{\leqslant N,z}, \mathcal{V}_z] + \frac{il_b^2}{2}d^2W(z)(\nabla_z^\perp\Pi_{\leqslant N,z}, X-z) + \frac{il_b^2}{2}d^2W(z)(X-z, \nabla_z^\perp\Pi_{\leqslant N,z})\\
&\quad+\frac{il_b^2}{6}d^3W(z)(\nabla_z^\perp\Pi_{\leqslant N,z}, X-z, X-z) + \frac{il_b^2}{6}d^3W(z)(X-z, \nabla_z^\perp\Pi_{\leqslant N,z}, X-z)\\
&\quad+\frac{il_b^2}{6}d^3W(z)(X-z, X-z, \nabla_z^\perp\Pi_{\leqslant N,z})\\
&\quad+\frac{i\sqrt{N+1}l_b}{2\sqrt{2}}d^2W(z)\left(\begin{pmatrix}1 & 1\\ -i & i\end{pmatrix}\begin{pmatrix}|\psi_{N,z}\rangle\langle\psi_{N+1,z}|\\ |\psi_{N+1,z}\rangle\langle\psi_{N,z}|\end{pmatrix}, X-z\right)\\
&\quad+\frac{i\sqrt{N+1}l_b}{2\sqrt{2}}d^2W(z)\left(X-z, \begin{pmatrix}1 & 1\\ -i & i\end{pmatrix}\begin{pmatrix}|\psi_{N,z}\rangle\langle\psi_{N+1,z}|\\ |\psi_{N+1,z}\rangle\langle\psi_{N,z}|\end{pmatrix}\right)\\
&\quad+\frac{i\sqrt{N+1}l_b}{6\sqrt{2}}d^3W(z)\left(\begin{pmatrix}1 & 1\\ -i & i\end{pmatrix}\begin{pmatrix}|\psi_{N,z}\rangle\langle\psi_{N+1,z}|\\ |\psi_{N+1,z}\rangle\langle\psi_{N,z}|\end{pmatrix}, X-z, X-z\right)\\
&\quad+\frac{i\sqrt{N+1}l_b}{6\sqrt{2}}d^3W(z)\left(X-z, \begin{pmatrix}1 & 1\\ -i & i\end{pmatrix}\begin{pmatrix}|\psi_{N,z}\rangle\langle\psi_{N+1,z}|\\ |\psi_{N+1,z}\rangle\langle\psi_{N,z}|\end{pmatrix}, X-z\right)\\
&\quad+\frac{i\sqrt{N+1}l_b}{6\sqrt{2}}d^3W(z)\left(X-z, X-z, \begin{pmatrix}1 & 1\\ -i & i\end{pmatrix}\begin{pmatrix}|\psi_{N,z}\rangle\langle\psi_{N+1,z}|\\ |\psi_{N+1,z}\rangle\langle\psi_{N,z}|\end{pmatrix}\right)
\end{aligned}\tag{94}$$

Defining the errors terms

$$
\begin{aligned}
\mathcal{E}_{0,b}(z) &\coloneqq \frac{\sqrt{N+1}}{2\sqrt{2}\pi l_b^3}\nabla W(z)\cdot\begin{pmatrix}1 & 1\\ -i & i\end{pmatrix}\begin{pmatrix}\mathrm{Tr}\left[\gamma_b\left|\psi_{N,z}\right\rangle\left\langle\psi_{N+1,z}\right|\right]\\ \mathrm{Tr}\left[\gamma_b\left|\psi_{N+1,z}\right\rangle\left\langle\psi_{N,z}\right|\right]\end{pmatrix}\\
\mathcal{E}_{1,b}(z) &\coloneqq \frac{1}{2i\pi l_b^4}\mathrm{Tr}\left[\gamma_b\left[\Pi_{\leqslant N,z},\mathcal{V}_z\right]\right]\\
\mathcal{E}_{2,b}(z) &\coloneqq \frac{1}{4\pi l_b^2}\mathrm{Tr}\left[\gamma_b d^2W(z)\left(\nabla_z^\perp\Pi_{\leqslant N,z},X-z\right)\right]\\
\widetilde{\mathcal{E}}_{2,b}(z) &\coloneqq \frac{1}{4\pi l_b^2}\mathrm{Tr}\left[\gamma_b d^2W(z)\left(X-z,\nabla_z^\perp\Pi_{\leqslant N,z}\right)\right]\\
\mathcal{E}_{3,b}(z) &\coloneqq \frac{\sqrt{N+1}}{4\sqrt{2}\pi l_b^3}\mathrm{Tr}\left[\gamma_b d^2W(z)\left(\begin{pmatrix}1 & 1\\ -i & i\end{pmatrix}\begin{pmatrix}\left|\psi_{N,z}\right\rangle\left\langle\psi_{N+1,z}\right|\\ \left|\psi_{N+1,z}\right\rangle\left\langle\psi_{N,z}\right|\end{pmatrix},X-z\right)\right]\\
\widetilde{\mathcal{E}}_{3,b}(z) &\coloneqq \frac{\sqrt{N+1}}{4\sqrt{2}\pi l_b^3}\mathrm{Tr}\left[\gamma_b d^2W(z)\left(X-z,\begin{pmatrix}1 & 1\\ -i & i\end{pmatrix}\begin{pmatrix}\left|\psi_{N,z}\right\rangle\left\langle\psi_{N+1,z}\right|\\ \left|\psi_{N+1,z}\right\rangle\left\langle\psi_{N,z}\right|\end{pmatrix}\right)\right]\\
\mathcal{E}_{4,b}(z) &\coloneqq \frac{1}{12\pi l_b^2}\mathrm{Tr}\left[\gamma_b d^3W(z)\left(\nabla_z^\perp\Pi_{\leqslant N,z},X-z,X-z\right)\right]\\
\widetilde{\mathcal{E}}_{4,b}(z) &\coloneqq \frac{1}{12\pi l_b^2}\mathrm{Tr}\left[\gamma_b d^3W(z)\left(X-z,\nabla_z^\perp\Pi_{\leqslant N,z},X-z\right)\right]\\
\widetilde{\widetilde{\mathcal{E}}}_{4,b}(z) &\coloneqq \frac{1}{12\pi l_b^2}\mathrm{Tr}\left[\gamma_b d^3W(z)\left(X-z,X-z,\nabla_z^\perp\Pi_{\leqslant N,z}\right)\right]\\
\mathcal{E}_{5,b}(z) &\coloneqq \frac{\sqrt{N+1}}{12\sqrt{2}\pi l_b^3}\mathrm{Tr}\left[\gamma_b d^3W(z)\left(\begin{pmatrix}1 & 1\\ -i & i\end{pmatrix}\begin{pmatrix}\left|\psi_{N,z}\right\rangle\left\langle\psi_{N+1,z}\right|\\ \left|\psi_{N+1,z}\right\rangle\left\langle\psi_{N,z}\right|\end{pmatrix},X-z,X-z\right)\right] \qquad (95)\\
\widetilde{\mathcal{E}}_{5,b}(z) &\coloneqq \frac{\sqrt{N+1}}{12\sqrt{2}\pi l_b^3}\mathrm{Tr}\left[\gamma_b d^3W(z)\left(X-z,\begin{pmatrix}1 & 1\\ -i & i\end{pmatrix}\begin{pmatrix}\left|\psi_{N,z}\right\rangle\left\langle\psi_{N+1,z}\right|\\ \left|\psi_{N+1,z}\right\rangle\left\langle\psi_{N,z}\right|\end{pmatrix},X-z\right)\right]\\
\widetilde{\widetilde{\mathcal{E}}}_{5,b}(z) &\coloneqq \frac{\sqrt{N+1}}{12\sqrt{2}\pi l_b^3}\mathrm{Tr}\left[\gamma_b d^3W(z)\left(X-z,X-z,\begin{pmatrix}1 & 1\\ -i & i\end{pmatrix}\begin{pmatrix}\left|\psi_{N,z}\right\rangle\left\langle\psi_{N+1,z}\right|\\ \left|\psi_{N+1,z}\right\rangle\left\langle\psi_{N,z}\right|\end{pmatrix}\right)\right]
\end{aligned}
$$

and inserting (94) in (76) we have

$$
\begin{aligned}
&\partial_t\rho_{\gamma_b}^{\leqslant N}+\nabla^\perp W\cdot\nabla\rho_{\gamma_b}^{\leqslant N}\\
=&\mathcal{E}_{0,b}+\mathcal{E}_{1,b}+\mathcal{E}_{2,b}+\widetilde{\mathcal{E}}_{2,b}+\mathcal{E}_{3,b}+\widetilde{\mathcal{E}}_{3,b}+\mathcal{E}_{4,b}+\widetilde{\mathcal{E}}_{4,b}+\widetilde{\widetilde{\mathcal{E}}}_{4,b}+\mathcal{E}_{5,b}+\widetilde{\mathcal{E}}_{5,b}+\widetilde{\widetilde{\mathcal{E}}}_{5,b} \qquad (96)
\end{aligned}
$$

Estimate of $\mathcal{E}_{1,b}$:

Using that the fourth derivative of W is bounded, we get

$$\left|\mathcal{V}_z(x+\sqrt{2}l_bx)\right|\leqslant C\left\|W\right\|_{W^{4,\infty}}\left|l_bx\right|^4$$

so following the same proof as for (60) we obtain instead of (68)

$$\left|\int_{\mathbb{R}^2}\varphi(z)\mathcal{E}_{1,b}(z)dz\right|\leqslant C\sqrt{\left|\mathrm{supp}(\varphi)\right|}\left\|\varphi\right\|_{L^\infty}\left\|W\right\|_{W^{4,\infty}}\left(\epsilon l_b^4\sum_{n=0}^{N}\frac{1}{(n+1)^{k-4}}+\frac{1}{\epsilon}\mathrm{Tr}\left[\gamma_b\mathscr{L}_b^k\right]\right)$$

With the asymptotic (43), taking $\epsilon\coloneqq\frac{1}{l_b^2}\sqrt{\frac{\mathrm{Tr}\left[\gamma_b\mathscr{L}_b^k\right]}{p_{k-5}(N)}}$ we have

$$\left|\int_{\mathbb{R}^2}\varphi(z)\mathcal{E}_{1,b}(z)dz\right|\leqslant C\sqrt{\left|\mathrm{supp}(\varphi)\right|}\left\|\varphi\right\|_{L^\infty}\left\|W\right\|_{W^{4,\infty}}\left(\epsilon l_b^4p_{k-5}(N)+\frac{1}{\epsilon}\mathrm{Tr}\left[\gamma_b\mathscr{L}_b^k\right]\right)$$

$$=C\sqrt{|\mathrm{supp}(\varphi)|}\,\|\varphi\|_{L^\infty}\,\|W\|_{W^{4,\infty}}\sqrt{\mathrm{Tr}\left[\gamma_b\mathscr{L}_b^k\right]}l_b^2\sqrt{p_{k-5}(N)} \tag{97}$$

Estimate of $\mathcal{E}_{4,b}$:

By symmetry of the derivatives of W,

$$\mathcal{E}_{4,b}(z)=\frac{1}{12\pi l_b^2}\mathrm{Tr}\left[\gamma_b\nabla_z^\perp\Pi_{\leqslant N,z}\odot\nabla^{\otimes 3}W(z)\odot^2(X-z)^{\otimes 2}\right]$$

Next we notice that

$$\begin{aligned}
&\nabla_z^\perp\odot\nabla^{\otimes 3}W(z)\odot^2(X-z)^{\otimes 2}\\
=&\left(\nabla_z^\perp\odot\nabla^{\otimes 3}W(z)\right)\odot^2(X-z)^{\otimes 2}+\nabla^{\otimes 3}W(z)\odot^3\nabla_z^\perp\otimes(X-z)^{\otimes 2}\\
=&\nabla^{\otimes 3}W(z)\odot^3\begin{pmatrix}0&1\\-1&0\end{pmatrix}\otimes(X-z)+\nabla^{\otimes 3}W(z)\odot^3(X-z)\otimes\begin{pmatrix}0&1\\-1&0\end{pmatrix}\\
=&\left(\nabla^{\otimes 3}W(z)\odot^2\begin{pmatrix}0&1\\-1&0\end{pmatrix}\right)\odot(X-z)+\left(\nabla^{\otimes 3}W(z)\odot(X-z)\right)\odot\begin{pmatrix}0&1\\-1&0\end{pmatrix}=0
\end{aligned}$$

since $\nabla^{\otimes 3}W(z)$, $\nabla^{\otimes 3}W(z)\odot(X-z)$ are symmetric and $\begin{pmatrix}0&1\\-1&0\end{pmatrix}$ anti-symmetric. Hence with an integration by parts,

$$\int_{\mathbb{R}^2}\varphi(z)\mathcal{E}_{4,b}(z)dz=\frac{-1}{12\pi l_b^2}\int_{\mathbb{R}^2}\nabla^\perp\varphi(z)\odot d^3W(z)\odot^2\mathrm{Tr}\left[\gamma_b\Pi_{\leqslant N,z}(X-z)^{\otimes 2}\right] \tag{98}$$

From this point on, we follow the same proof as for (61) with one difference in (71) where we use

$$\mathrm{Tr}\left[\Pi_{n,z}\left|X-z\right|^4\right]\leqslant C(n+1)^2l_b^4$$

and obtain instead of (72),

$$\begin{aligned}
\left|\int_{\mathbb{R}^2}\varphi(z)\mathcal{E}_{2,b}(z)dz\right|\leqslant&C\sqrt{|\mathrm{supp}(\varphi)|}\,\|\nabla\varphi\|_{L^\infty}\left\|d^3W\right\|_{L^\infty}\left(\epsilon l_b^2\sum_{n=0}^N\frac{1}{(n+1)^{k-2}}+\frac{l_b^2}{\epsilon}\mathrm{Tr}\left[\gamma_b\mathscr{L}_b^k\right]\right)\\
\leqslant&C\sqrt{|\mathrm{supp}(\varphi)|}\,\|\nabla\varphi\|_{L^\infty}\left\|d^3W\right\|_{L^\infty}l_b^2\left(\epsilon p_{k-3}(N)+\frac{1}{\epsilon}\mathrm{Tr}\left[\gamma_b\mathscr{L}_b^k\right]\right)
\end{aligned}$$

and with $\epsilon:=\sqrt{\dfrac{\mathrm{Tr}\left[\gamma_b\mathscr{L}_b^k\right]}{p_{k-3}(N)}}$ we conclude that

$$\left|\int_{\mathbb{R}^2}\varphi(z)\mathcal{E}_{2,b}(z)dz\right|\leqslant C\sqrt{|\mathrm{supp}(\varphi)|}\,\|\nabla\varphi\|_{L^\infty}\left\|d^3W\right\|_{L^\infty}\sqrt{\mathrm{Tr}\left[\gamma_b\mathscr{L}_b^k\right]}\sqrt{p_{k-3}(N)}l_b^2 \tag{99}$$

Estimate of $\widetilde{\mathcal{E}}_{4,b}$:

Similarly as for $\mathcal{E}_{4,b}$ we proceed to an integration by parts and instead of (98) we have

$$\int_{\mathbb{R}^2}\varphi(z)\mathcal{E}_{4,b}(z)dz=\frac{-1}{12\pi l_b^2}\int_{\mathbb{R}^2}\nabla^\perp\varphi(z)\odot d^3W(z)\odot^2\mathrm{Tr}\left[\gamma_b(X-z)\otimes\Pi_{\leqslant N,z}(X-z)\right]$$

But

$$|\mathrm{Tr}\left[\gamma_b(X-z)\otimes\Pi_{\leqslant N,z}(X-z)\right]|\leqslant Cl_b^2\mathrm{Tr}\left[\Pi_{n,Z}\left|X-z\right|^2\right]\leqslant CNl_b^4$$

and

$$\left|\int_{\mathbb{R}^2}\varphi(z)\mathcal{E}_{4,b}(z)dz\right|\leqslant C\left|\mathrm{supp}(\varphi)\right|\|\nabla\varphi\|_{L^\infty}\left\|d^3W\right\|_{L^\infty}l_b^2N \tag{100}$$

Estimate of $\mathcal{E}_{5,b}$:

We follow the same proof as for (82), the only difference being in (89) that we replace with

$$\begin{aligned}\left|\mathrm{Tr}\left[\gamma_b\left|\psi_{N+1,z}\right\rangle\left\langle\psi_{N,z}\right|(X_i-z_i)^2\right]\right|\leqslant&\epsilon\mathrm{Tr}\left[(X_i-z_i)^4\Pi_{N,z}\right]+\frac{1}{\epsilon}\mathrm{Tr}\left[\gamma_b^2\Pi_{N+1,z}\right]\\ \leqslant&2\epsilon(N+1)^2l_b^4+\frac{2\pi l_b^2}{\epsilon}\mathrm{Tr}\left[\gamma_b\Pi_{N+1,z}\right]\end{aligned} \tag{101}$$

So we gain a factor $\sqrt{N}l_b$ and instead of (90) we obtain

$$\left|\int_{\mathbb{R}^2}\varphi\mathcal{E}_{5,b}(z)dz\right|\leqslant C\sqrt{\left|\mathrm{supp}(\varphi)\right|}\|\varphi\|_{L^\infty}\left\|d^3W\right\|_{L^\infty}l_bN^{\frac{3-k}{2}}\sqrt{\mathrm{Tr}\left[\gamma_b\mathscr{L}_b^k\left(\Pi_N+\Pi_{N+1}\right)\right]} \tag{102}$$

Estimate of $\widetilde{\mathcal{E}}_{5,b}$:

$$\widetilde{\mathcal{E}}_{5,b}(z)=\frac{\sqrt{N+1}}{12\sqrt{2}\pi l_b^3}\mathrm{Tr}\left[\gamma_bd^3W(z)\left(X-z,\begin{pmatrix}1&1\\-i&i\end{pmatrix}\begin{pmatrix}\left|\psi_{N,z}\right\rangle\left\langle\psi_{N+1,z}\right|\\\left|\psi_{N+1,z}\right\rangle\left\langle\psi_{N,z}\right|\end{pmatrix},X-z\right)\right]$$

We only need to estimate terms of the form

$$\mathrm{Tr}\left[\gamma_b(X_i-z_i)\left|\psi_{N+1,z}\right\rangle\left\langle\psi_{N,z}\right|(X_j-z_j)\right]$$

with $i,j\in\{1,2\}$. But with Young's inequality,

$$\begin{aligned}2\left|\mathrm{Tr}\left[\gamma_b(X_i-z_i)\left|\psi_{N+1,z}\right\rangle\left\langle\psi_{N,z}\right|(X_j-z_j)\right]\right|\leqslant&\mathrm{Tr}\left[\gamma_b(X_i-z_i)\Pi_{N+1,z}(X_i-z_i)\right]\\&+\mathrm{Tr}\left[\gamma_b(X_j-z_j)\Pi_{N,z}(X_j-z_j)\right]\end{aligned}$$

We are going to control the second term with $j=1$, all the others terms being estimated the same way. With (21),

$$\begin{aligned}&\mathrm{Tr}\left[\gamma_b(X_1-z_1)\Pi_{N,z}(X_1-z_1)\right]\\&\leqslant\mathrm{Tr}\left[\gamma_b\Pi_{N,z}(X_1-z_1)^2\right]+\mathrm{Tr}\left[\gamma_b\left[X_1,\Pi_{N,z}\right](X_1-z_1)\right]\\&\leqslant\mathrm{Tr}\left[\gamma_b\Pi_{N,z}(X_1-z_1)^2\right]+il_b^2\mathrm{Tr}\left[\gamma_b\partial_{z_2}\Pi_{N,z}(X_1-z_1)\right]\\&-i\frac{\sqrt{N+1}l_b}{\sqrt{2}}\mathrm{Tr}\left[\gamma_b\left(\left|\psi_{N,z}\right\rangle\left\langle\psi_{N+1,z}\right|+\left|\psi_{N+1,z}\right\rangle\left\langle\psi_{N,z}\right|\right)(X_1-z_1)\right]\\&+i\frac{\sqrt{N}l_b}{\sqrt{2}}\mathrm{Tr}\left[\gamma_b\left(\left|\psi_{N-1,z}\right\rangle\left\langle\psi_{N,z}\right|+\left|\psi_{N,z}\right\rangle\left\langle\psi_{N-1,z}\right|\right)(X_1-z_1)\right]\end{aligned}$$

The first term is of the same form as (95), for the second we proceed to an integration by parts and obtain two terms of the form (82). The last two terms are also of the form (82). So using (102) and (90) we obtain

$$\begin{aligned}
&\left|\int_{\mathbb{R}^2}\varphi(z)\widetilde{\mathcal{E}}_{5,b}(z)dz\right| \\
\leqslant& C\sqrt{|\mathrm{supp}(\varphi)|}\,\|\varphi\|_{L^\infty}\left\|d^3W\right\|_{L^\infty} l_bN^{\frac{3-k}{2}}\sqrt{\mathrm{Tr}\left[\gamma_b\mathscr{L}_b^k\Pi_N\right]} \\
&+C\sqrt{|\mathrm{supp}(\varphi)|}\,\|\varphi\|_{W^{1,\infty}}\|W\|_{W^{4,\infty}} l_b^2N^{1-\frac{k}{2}}\sqrt{\mathrm{Tr}\left[\gamma_b\mathscr{L}_b^k\Pi_N\right]} \\
&+C\sqrt{|\mathrm{supp}(\varphi)|}\,\|\varphi\|_{L^\infty}\left\|d^3W\right\|_{L^\infty} l_bN^{\frac{3-k}{2}}\sqrt{\mathrm{Tr}\left[\gamma_b\mathscr{L}_b^k\left(\Pi_{N-1}+\Pi_N+\Pi_{N+1}\right)\right]} \\
\leqslant& C\sqrt{|\mathrm{supp}(\varphi)|}\,\|\varphi\|_{W^{1,\infty}}\|W\|_{W^{4,\infty}} l_bN^{\frac{3-k}{2}}\sqrt{\mathrm{Tr}\left[\gamma_b\mathscr{L}_b^k\left(\Pi_{N-1}+\Pi_N+\Pi_{N+1}\right)\right]} \qquad (103)
\end{aligned}$$

Conclusion:

$\mathcal{E}_{0,b},\mathcal{E}_{2,b},\widetilde{\mathcal{E}}_{2,b},\mathcal{E}_{3,b},\widetilde{\mathcal{E}}_{3,b}$ are the same than in Proposition V.3. We control $\widetilde{\widetilde{\mathcal{E}}}_{4,b},\widetilde{\widetilde{\mathcal{E}}}_{5,b}$ with respectively the same arguments as for $\mathcal{E}_{4,b},\mathcal{E}_{5,b}$ by exchanging x and y. Therefore, putting together (86), (88), (90), (97), (99), (100), (102), (103) with (96) we obtain

$$\begin{aligned}
&\left|\int_{\mathbb{R}^2}\varphi(z)\left(\partial_t\rho_{\gamma_b}^{sc,\leqslant N}(z)+\nabla^\perp W(z)\cdot\nabla\rho_{\gamma_b}^{sc,\leqslant N}(z)\right)dz\right| \\
\leqslant& C\|\varphi\|_{L^\infty}\|\nabla W\|_{L^\infty}\frac{1}{l_bN^{k-\frac{1}{2}}}\mathrm{Tr}\left[\gamma_b\mathscr{L}_b^k\left(\Pi_N+\Pi_{N+1}\right)\right]+C\sqrt{|\mathrm{supp}(\varphi)|}\,\|\varphi\|_{W^{1,\infty}}\|W\|_{W^{4,\infty}}\Big(\\
&\sqrt{\mathrm{Tr}\left[\gamma_b\mathscr{L}_b^k\right]}l_b\sqrt{p_{k-2}(N)}+N^{1-\frac{k}{2}}\sqrt{\mathrm{Tr}\left[\gamma_b\mathscr{L}_b^k\left(\Pi_N+\Pi_{N+1}\right)\right]}+\sqrt{\mathrm{Tr}\left[\gamma_b\mathscr{L}_b^k\right]}l_b^2\sqrt{p_{k-5}(N)} \\
&+\sqrt{\mathrm{Tr}\left[\gamma_b\mathscr{L}_b^k\right]}l_b^2\sqrt{p_{k-3}(N)}+l_bN^{\frac{3-k}{2}}\sqrt{\mathrm{Tr}\left[\gamma_b\mathscr{L}_b^k\left(\Pi_N+\Pi_{N+1}\right)\right]} \\
&+l_bN^{\frac{3-k}{2}}\sqrt{\mathrm{Tr}\left[\gamma_b\mathscr{L}_b^k\left(\Pi_{N-1}+\Pi_N+\Pi_{N+1}\right)\right]}\Big)+C\,|\mathrm{supp}(\varphi)|\,\|\nabla\varphi\|_{L^\infty}\left\|d^3W\right\|_{L^\infty}l_b^2N
\end{aligned}$$

With (93), we get

$$\begin{aligned}
&\left|\int_{\mathbb{R}^2}\varphi(z)\left(\partial_t\rho_{\gamma_b}^{sc,\leqslant N}(z)+\nabla^\perp W(z)\cdot\nabla\rho_{\gamma_b}^{sc,\leqslant N}(z)\right)dz\right|\leqslant c(\varphi,W)\left(\frac{1}{l_bN^{k-\frac{1}{2}}}\mathrm{Tr}\left[\gamma_b\mathscr{L}_b^k\Pi_{N-1:N+1}\right]\right. \\
&+\sqrt{\mathrm{Tr}\left[\gamma_b\mathscr{L}_b^k\right]}\left(l_b\sqrt{p_{k-2}(N)}+l_b^2\sqrt{p_{k-5}(N)}\right) \\
&\left.+\left(N^{1-\frac{k}{2}}+l_bN^{\frac{3-k}{2}}\right)\sqrt{\mathrm{Tr}\left[\gamma_b\mathscr{L}_b^k\Pi_{N-1:N+1}\right]}+l_b^2N\right)
\end{aligned}$$

We conclude with

$$\begin{aligned}
&l_b\sqrt{p_{k-2}(N)}+l_b^2\sqrt{p_{k-5}(N)} \\
=&l_b\left(N^{1-\frac{k}{2}}\mathbb{1}_{k<2}+\sqrt{\ln(N)}\mathbb{1}_{k=2}+\mathbb{1}_{k>2}\right)+l_b^2\left(N^{\frac{5-k}{2}}\mathbb{1}_{k<5}+\sqrt{\ln(N)}\mathbb{1}_{k=5}+\mathbb{1}_{k>5}\right) \\
\leqslant&\left(l_bN^{1-\frac{k}{2}}+l_b^2N^{\frac{5-k}{2}}\right)\mathbb{1}_{k<2}+\left(l_b\sqrt{\ln(N)}+l_b^2N^{\frac{3}{2}}\right)\mathbb{1}_{k=2}+\left(l_b+l_b^2N^{\frac{5-k}{2}}\right)\mathbb{1}_{2<k<5} \\
&+\left(l_b+l_b^2\sqrt{\ln(N)}\right)\mathbb{1}_{k=5}+l_b\mathbb{1}_{k>5}
\end{aligned}$$

□

VI Proofs of the main results

This last section contains the conclusion of the proofs of Theorem I.3 and Theorem I.4. What is left to do is passing to the limit in the estimates of Section IV and Section V.

Lemma VI.1: *Choice of a convenient* N
Let $N_- := (N_{-,b})_{b>0}, N_+ := (N_{+,b})_{b>0} \subset \mathbb{N}^*$ *such that*

$$1 \ll N_- \ll N_+$$

Let $k \geqslant 0, T > 0, \gamma_b \in L^\infty\left([0,T], \mathcal{L}^1\left(L^2\left(\mathbb{R}^2\right)\right)\right)$ *and assume*

$$\forall t \in [0,T]\,, \gamma_b(t) \geqslant 0,\, Tr[\gamma_b(t)] = 1$$

$$\int_0^T Tr\left[\gamma_b(t)\mathscr{L}_b^k\right] dt < \infty$$

then $\exists N \in [\![N_-, N_+]\!]$ *such that*

$$\int_0^T Tr\left[\gamma_b(t)\mathscr{L}_b^k \Pi_{N-1:N+1}\right] dt \leqslant \frac{6}{N\ln\left(\frac{N_+}{N_-}\right)} \int_0^T Tr\left[\gamma_b(t)\mathscr{L}_b^k\right] dt$$

Proof: Assume for contradiction that

$$\forall N \in [\![N_-, N_+]\!]\,, \int_0^T \mathrm{Tr}\left[\gamma_b(t)\mathscr{L}_b^k \Pi_{N-1:N+1}\right] dt > \frac{6}{N\ln\left(\frac{N_+}{N_-}\right)} \int_0^T \mathrm{Tr}\left[\gamma_b(t)\mathscr{L}_b^k\right] dt$$

then

$$\begin{aligned}\int_0^T \mathrm{Tr}\left[\gamma_b(t)\mathscr{L}_b^k\right] dt \geqslant& \frac{1}{3}\sum_{N=N_-}^{N_+} \int_0^T \mathrm{Tr}\left[\gamma_b(t)\mathscr{L}_b^k \Pi_{N-1:N+1}\right] dt \\ &> \frac{2}{\ln\left(\frac{N_+}{N_-}\right)} \int_0^T \mathrm{Tr}\left[\gamma_b(t)\mathscr{L}_b^k\right] dt \sum_{N=N_-}^{N_+} \frac{1}{N} \underset{b\to\infty}{\sim} 2\int_0^T \mathrm{Tr}\left[\gamma_b(t)\mathscr{L}_b^k\right] dt\end{aligned}$$

which yields the desired contradiction. □

We may now conclude the proof of our main result.

Proof of Theorem I.3: With an integration by part,

$$\begin{aligned}&\int_{\mathbb{R}^2} \varphi(0,z)\rho_{\gamma_b}(0,z)dz - \int_{\mathbb{R}_+\times\mathbb{R}^2} \rho_{\gamma_b}(t,z)\mathrm{GYRO}_{\rho_{\gamma_b}}(\varphi)(t,z)dtdz \\ =& \int_{\mathbb{R}^2} \varphi(0,z)\rho_{\gamma_b}^{sc,\leqslant N}(0,z)dz - \int_{\mathbb{R}_+\times\mathbb{R}^2} \rho_{\gamma_b}^{sc,\leqslant N}(t,z)\mathrm{GYRO}_{\rho_{\gamma_b}}(\varphi)(t,z)dtdz \\ &+ \int_{\mathbb{R}^2} \varphi(0,z)\left(\rho_{\gamma_b} - \rho_{\gamma_b}^{sc,\leqslant N}\right)(0,z)dz - \int_{\mathbb{R}_+\times\mathbb{R}^2} \left(\rho_{\gamma_b} - \rho_{\gamma_b}^{sc,\leqslant N}\right)(t,z)\mathrm{GYRO}_{\rho_{\gamma_b}}(\varphi)(t,z)dtdz\end{aligned}$$

$$= \int\limits_{\mathbb{R}_+\times\mathbb{R}^2} \varphi \mathrm{GYRO}_{\rho_{\gamma_b}}(\rho_{\gamma_b}^{sc,\leqslant N}) dtdz$$
$$+ \int\limits_{\mathbb{R}^2} \varphi(0,z)\left(\rho_{\gamma_b} - \rho_{\gamma_b}^{sc,\leqslant N}\right)(0,z)dz - \int\limits_{\mathbb{R}_+\times\mathbb{R}^2} \left(\rho_{\gamma_b} - \rho_{\gamma_b}^{sc,\leqslant N}\right)\mathrm{GYRO}_{\rho_{\gamma_b}}(\varphi) dtdz \tag{104}$$

Let $T \geqslant 0$ be such that $\mathrm{supp}(\varphi) \subset [0,T]\times\mathbb{R}^2$. By Proposition III.1,

$$\forall t \in [0,T], \mathrm{Tr}\left[\gamma_b(t)\right] = 1 \text{ and } 0 \leqslant \gamma_b(t) \leqslant 2\pi l_b^2$$

Let $t \in [0,T]$, then by Proposition III.3 applied to $W = V + \frac{1}{2} w \star \rho_{\gamma_b(t)}$ and Proposition III.2,

$$\begin{aligned} \mathrm{Tr}\left[\gamma_b(t)\mathscr{L}_b\right] &\leqslant \left|\mathrm{Tr}\left[\gamma_b(t)H_b(t)\right]\right| + \left\|V + \frac{1}{2} w \star \rho_{\gamma_b(t)}\right\|_{L^\infty} \\ &= \left|\mathrm{Tr}\left[\gamma_b(0)H_b(0)\right]\right| + \left\|V + \frac{1}{2} w \star \rho_{\gamma_b(t)}\right\|_{L^\infty} \\ &\leqslant \left|\mathrm{Tr}\left[\gamma_b(0)H_b(0)\right]\right| + \|V\|_{L^\infty} + \frac{1}{2}\|w\|_{L^\infty} \leqslant C(V,w) \end{aligned} \tag{105}$$

and

$$\|V + w \star \rho_{\gamma_b}(t)\|_{W^{\infty,4}} \leqslant \|V\|_{W^{\infty,4}} + \|w\|_{W^{\infty,4}}$$

so from Proposition V.4 for $k = 1, W = V + w \star \rho_{\gamma_b}(t)$ we get

$$\left|\int\limits_{\mathbb{R}^2} \varphi(t,z)\mathrm{GYRO}_{\gamma_b}\left(\rho_{\gamma_b}^{\leqslant N}\right)(t,z)dz\right| \leqslant C(\varphi,V,w)\left(\frac{1}{l_b\sqrt{N}}\mathrm{Tr}\left[\gamma_b(t)\mathscr{L}_b\Pi_{N-1:N+1}\right]\right.$$
$$\left. + \left(\sqrt{N} + l_b N\right)\sqrt{\mathrm{Tr}\left[\gamma_b(t)\mathscr{L}_b\Pi_{N-1:N+1}\right]} + l_b^2 N + \sqrt{\mathrm{Tr}\left[\gamma_b(t)\mathscr{L}_b\right]}\left(l_b\sqrt{N} + l_b^2 N^2\right)\right)$$

Integrating over time,

$$\begin{aligned} \left|\int\limits_{\mathbb{R}_+\times\mathbb{R}^2} \varphi \mathrm{GYRO}_{\rho_{\gamma_b}}(\rho_{\gamma_b}^{sc,\leqslant N}) dtdz\right| &\leqslant C(\varphi,V,w)\left(\frac{1}{l_b\sqrt{N}}\int_0^T \mathrm{Tr}\left[\gamma_b(t)\mathscr{L}_b\Pi_{N-1:N+1}\right]dt\right. \\ &\quad + \left(\sqrt{N} + l_b N\right)\int_0^T \sqrt{\mathrm{Tr}\left[\gamma_b(t)\mathscr{L}_b\Pi_{N-1:N+1}\right]}dt + l_b^2 NT \\ &\quad \left. + \int_0^T \sqrt{\mathrm{Tr}\left[\gamma_b(t)\mathscr{L}_b\right]}\left(l_b\sqrt{N} + l_b^2 N^2\right)dt\right) \\ &\leqslant C(\varphi,V,w)\left(\frac{1}{l_b\sqrt{N}}\int_0^T \mathrm{Tr}\left[\gamma_b(t)\mathscr{L}_b\Pi_{N-1:N+1}\right]dt\right. \\ &\quad + \left(\sqrt{N} + l_b N\right)\sqrt{T}\left(\int_0^T \mathrm{Tr}\left[\gamma_b(t)\mathscr{L}_b\Pi_{N-1:N+1}\right]dt\right)^{\frac{1}{2}} + l_b^2 NT \end{aligned}$$

$$+\sqrt{T}\left(\int_0^T \mathrm{Tr}\left[\gamma_b(t)\mathscr{L}_b\right]dt\right)^{\frac{1}{2}}\left(l_b\sqrt{N}+l_b^2N^2\right)\Bigg)$$

Choosing N as in Lemma VI.1 and using (105), we have

$$\begin{aligned}
&\left|\int_{\mathbb{R}_+\times\mathbb{R}^2}\varphi \mathrm{GYRO}_{\rho_{\gamma_b}}(\rho_{\gamma_b}^{sc,\leqslant N})dtdz\right| \\
&\leqslant C(\varphi,V,w)\left(\frac{1}{l_bN^{\frac{3}{2}}\ln\left(\frac{N_+}{N_-}\right)}+\frac{1+l_b\sqrt{N}}{\sqrt{\ln\left(\frac{N_+}{N_-}\right)}}+l_b^2N+l_b\sqrt{N}+l_b^2N^2\right) \\
&\leqslant C(\varphi,V,w)\left(\frac{1}{l_bN_-^{\frac{3}{2}}\ln\left(\frac{N_+}{N_-}\right)}+\frac{1+l_b\sqrt{N_+}}{\sqrt{\ln\left(\frac{N_+}{N_-}\right)}}+l_b\sqrt{N_+}+l_b^2N_+^2\right)
\end{aligned}$$

We start by imposing the constrains

$$\frac{1}{l_b^{\frac{2}{3}}}\leqslant N_-,\quad N_+\leqslant\frac{1}{l_b^2}$$

so that

$$l_b\sqrt{N_+}\leqslant 1,\quad 1\leqslant l_bN_-^{\frac{3}{2}}\leqslant l_bN_+^{\frac{3}{2}},\quad l_b\sqrt{N_+}\leqslant l_b^2N_+^2$$

and

$$\left|\int_{\mathbb{R}_+\times\mathbb{R}^2}\varphi \mathrm{GYRO}_{\rho_{\gamma_b}}(\rho_{\gamma_b}^{sc,\leqslant N})dtdz\right|\leqslant C(\varphi,V,w)\left(\frac{1}{l_bN_-^{\frac{3}{2}}\ln\left(\frac{N_+}{N_-}\right)}+\frac{1}{\sqrt{\ln\left(\frac{N_+}{N_-}\right)}}+l_b^2N_+^2\right) \tag{106}$$

With Proposition IV.3 for $k=1$ and (105),

$$\begin{aligned}
\left|\int_{\mathbb{R}^2}\varphi(0,z)\left(\rho_{\gamma_b}-\rho_{\gamma_b}^{sc,\leqslant N}\right)(0,z)dz\right| \leqslant& C(\varphi)\left(\frac{1}{\sqrt{N}}\sqrt{\mathrm{Tr}\left[\gamma_b(0)\Pi_{>N}\mathscr{L}_b\right]}+\sqrt{\mathrm{Tr}\left[\gamma_b(0)\mathscr{L}_b\right]}l_b\sqrt{N}\right) \\
\leqslant& C(\varphi,V,w)\left(\frac{1}{\sqrt{N}}+l_b\sqrt{N}\right) \\
\leqslant& C(\varphi,V,w)\left(\frac{1}{\sqrt{N_-}}+l_b^2N_+^2\right)
\end{aligned} \tag{107}$$

Finally, since $\mathrm{GYRO}_{\rho_{\gamma_b}}(\varphi)\in C_c^\infty\left(\mathbb{R}^2\right)$, $\mathrm{supp}\left(\mathrm{GYRO}_{\rho_{\gamma_b}}(\varphi)\right)\subset\mathrm{supp}(\varphi)$, and $\forall t\in[0,T]$,

$$\left\|\mathrm{GYRO}_{\rho_{\gamma_b}}(\varphi)(t,\bullet)\right\|_{W^{1,\infty}}=\left\|\partial_t\varphi(t,\bullet)+\nabla^\perp(V+w\star\rho_{\gamma_b(t)})\cdot\nabla\varphi(t)\right\|_{W^{1,\infty}}$$

$$\leqslant \|\varphi\|_{W^{1,\infty}} \left(1 + \left\|V + w \star \rho_{\gamma_b(t)}\right\|_{W^{1,\infty}}\right)$$
$$\leqslant \|\varphi\|_{W^{1,\infty}} \left(1 + \|V\|_{W^{1,\infty}} + \|w\|_{W^{1,\infty}}\right)$$

thus similarly to (107) we obtain

$$\left| \int\limits_{\mathbb{R}_+ \times \mathbb{R}^2} \left(\rho_{\gamma_b} - \rho_{\gamma_b}^{sc,\leqslant N}\right) \mathrm{GYRO}_{\rho_{\gamma_b}}(\varphi) dt dz \right| \leqslant C(\varphi, V, w) \left(\frac{1}{\sqrt{N_-}} + l_b^2 N_+^2\right) \tag{108}$$

Inserting (106), (107), (108) in (104), and then taking

$$N_- := l_b^{-\alpha}, \quad N_+ := l_b^{-\beta}, \quad \text{with} \quad \frac{2}{3} < \alpha, \quad \alpha < \beta < 1 \tag{109}$$

we conclude that

$$\left| \int\limits_{\mathbb{R}^2} \varphi(0,z)\rho_{\gamma_b}(0,z)dz - \int\limits_{\mathbb{R}_+ \times \mathbb{R}^2} \rho_{\gamma_b}(t,z)\mathrm{GYRO}_{\rho_{\gamma_b}}(\varphi)(t,z)dtdz \right|$$
$$\leqslant C(\varphi, V, w) \left(\frac{1}{l_b N_-^{\frac{3}{2}} \ln\left(\frac{N_+}{N_-}\right)} + \frac{1}{\sqrt{\ln\left(\frac{N_+}{N_-}\right)}} + l_b^2 N_+^2 + \frac{1}{\sqrt{N_-}} \right)$$
$$= C(\varphi, V, w) \left(\frac{l_b^{\frac{3}{2}\alpha - 1}}{(\beta - \alpha)\ln\left(l_b^{-1}\right)} + \frac{1}{\sqrt{(\beta - \alpha)\ln\left(l_b^{-1}\right)}} + l_b^{2(1-\beta)} + l_b^{\frac{\alpha}{2}} \right) \leqslant C(\varphi, V, w) \frac{1}{\sqrt{\ln\left(l_b^{-1}\right)}}$$

□

Proof of Theorem I.4:

Weak limit:

With Proposition III.1,

$$\forall t \in \mathbb{R}_+, \rho_{\gamma_b}^{sc,\leqslant N}(t) \geqslant 0 \quad \left\|\rho_{\gamma_b}^{sc,\leqslant N}(t)\right\|_{L^1} = 1,$$

Let $T > 0$, then

$$\left\| {\rho_{\gamma_b}^{sc,\leqslant N}}_{|[0,T]} \right\|_{L^1} = T$$

The truncated semi-classical densities are bounded in $\mathcal{M}\left([0,T] \times \mathbb{R}^2\right)$ so up to a subsequence we have weak star convergence in the sense of measures:

$$\rho_{\gamma_b}^{sc,\leqslant N} \underset{b\to\infty}{\overset{*}{\rightharpoonup}} \rho_T \in \mathcal{M}\left([0,T] \times \mathbb{R}^2\right)$$

By uniqueness of the limit

$$T_2 \geqslant T_1 \implies \rho_{T_2|[0,T_1]} = \rho_{T_1}$$

hence we constructed a limit $\rho \in \mathcal{M}\left([0,T] \times \mathbb{R}^2\right)$ such that $\forall \varphi \in C_c^0\left(\mathbb{R}_+ \times \mathbb{R}^2\right)$,

$$\int\limits_{\mathbb{R}_+ \times \mathbb{R}^2} \rho_{\gamma_b}^{sc, \leqslant N} \varphi \underset{b \to \infty}{\to} \int\limits_{\mathbb{R}_+ \times \mathbb{R}^2} \rho \varphi \tag{110}$$

Let $(\Omega_n)_{n \in \mathbb{N}}$ be an increasing sequence of open bounded subsets of $\mathbb{R}_+^* \times \mathbb{R}^2$, such that

$$\bigcup_{n \in \mathbb{N}} \Omega_n = \mathbb{R}_+^* \times \mathbb{R}^2$$

By [35, 1. theorem 6], the embedding $\mathcal{M}(\Omega_n) \subset W^{-1,1}(\Omega_n)$ is compact. So after a diagonal extraction we can have

$$\forall n \in \mathbb{N}, \rho_{\gamma_b}^{sc, \leqslant N} \underset{b \to \infty}{\to} \rho_n \in W^{-1,1}(\Omega_n)$$

but $W^{1,\infty}(\Omega_n) \subset C^0(\Omega_n)$, so $\rho_n = \rho_{|\Omega_n}$. Hence, $\forall n \in \mathbb{N}$,

$$\left\| \rho - \rho_{\gamma_b}^{sc, \leqslant N} \right\|_{W^{-1,1}(\Omega_n)} \underset{b \to \infty}{\to} 0 \tag{111}$$

$\left(\rho_{\gamma_b}^{sc, \leqslant N}(0)\right)_{b>0}$ is also bounded in $\mathcal{M}\left(\mathbb{R}^2\right)$, so after another extraction one has

$$\rho_{\gamma_b}^{sc, \leqslant N}(0) \underset{b \to \infty}{\overset{*}{\rightharpoonup}} \rho_0 \in \mathcal{M}\left(\mathbb{R}^2\right) \tag{112}$$

in the sense of measures.

Error decomposition:

Let $\varphi \in C_c^\infty\left(\mathbb{R}_+^\times \mathbb{R}^2\right)$,

$$\begin{aligned}
&\int\limits_{\mathbb{R}^2} \varphi(0)\rho_0 - \int\limits_{\mathbb{R}_+ \times \mathbb{R}^2} \rho \mathrm{GYRO}_\rho(\varphi) \\
&= \int\limits_{\mathbb{R}^2} \varphi(0)\rho_0 - \int\limits_{\mathbb{R}_+ \times \mathbb{R}^2} \rho_{\gamma_b}^{sc, \leqslant N} \mathrm{GYRO}_\rho(\varphi) + \int\limits_{\mathbb{R}_+ \times \mathbb{R}^2} \left(\rho_{\gamma_b}^{sc, \leqslant N} - \rho\right) \mathrm{GYRO}_\rho(\varphi) \\
&= \int\limits_{\mathbb{R}^2} \varphi(0)\rho_0 - \int\limits_{\mathbb{R}_+ \times \mathbb{R}^2} \rho_{\gamma_b}^{sc, \leqslant N} \mathrm{GYRO}_{\rho_{\gamma_b}}(\varphi) + \int\limits_{\mathbb{R}_+ \times \mathbb{R}^2} \rho_{\gamma_b}^{sc, \leqslant N} \left(\mathrm{GYRO}_{\rho_{\gamma_b}}(\varphi) - \mathrm{GYRO}_\rho(\varphi)\right) \\
&\quad + \int\limits_{\mathbb{R}_+ \times \mathbb{R}^2} \left(\rho_{\gamma_b}^{sc, \leqslant N} - \rho\right) \mathrm{GYRO}_\rho(\varphi) \\
&= \int\limits_{\mathbb{R}_+ \times \mathbb{R}^2} \varphi \mathrm{GYRO}_{\rho_{\gamma_b}}\left(\rho_{\gamma_b}^{sc, \leqslant N}\right) + \int\limits_{\mathbb{R}^2} \varphi(0)(\rho_0 - \rho_{\gamma_b}^{sc, \leqslant N}(0)) + \int\limits_{\mathbb{R}_+ \times \mathbb{R}^2} \rho_{\gamma_b}^{sc, \leqslant N} \nabla_z \varphi \cdot (\rho_{\gamma_b} - \rho) \star \nabla^\perp w \\
&\quad + \int\limits_{\mathbb{R}_+ \times \mathbb{R}^2} \left(\rho_{\gamma_b}^{sc, \leqslant N} - \rho\right) \mathrm{GYRO}_\rho(\varphi) \\
&= \int\limits_{\mathbb{R}_+ \times \mathbb{R}^2} \varphi \mathrm{GYRO}_{\rho_{\gamma_b}}\left(\rho_{\gamma_b}^{sc, \leqslant N}\right) + \int\limits_{\mathbb{R}^2} \varphi(0)(\rho_0 - \rho_{\gamma_b}^{sc, \leqslant N}(0))
\end{aligned}$$

$$+\int\limits_{\mathbb{R}_+\times\mathbb{R}^2}(\rho_{\gamma_b}-\rho)\left(\nabla_z\varphi\cdot\rho_{\gamma_b}^{sc,\leqslant N}\right)\star\nabla^\perp w+\int\limits_{\mathbb{R}_+\times\mathbb{R}^2}\left(\rho_{\gamma_b}^{sc,\leqslant N}-\rho\right)\mathrm{GYRO}_\rho(\varphi)$$

$$=\int\limits_{\mathbb{R}_+\times\mathbb{R}^2}\varphi\mathrm{GYRO}_{\rho_{\gamma_b}}\left(\rho_{\gamma_b}^{sc,\leqslant N}\right)+\int\limits_{\mathbb{R}^2}\varphi(0)(\rho_0-\rho_{\gamma_b}^{sc,\leqslant N}(0))$$

$$+\int\limits_{\mathbb{R}_+\times\mathbb{R}^2}\left(\rho_{\gamma_b}-\rho_{\gamma_b}^{sc,\leqslant N}\right)\left(\nabla_z\varphi\cdot\rho_{\gamma_b}^{sc,\leqslant N}\right)\star\nabla^\perp w+\int\limits_{\mathbb{R}_+\times\mathbb{R}^2}\left(\rho_{\gamma_b}^{sc,\leqslant N}-\rho\right)\left(\nabla_z\varphi\cdot\rho_{\gamma_b}^{sc,\leqslant N}\right)\star\nabla^\perp w$$

$$+\int\limits_{\mathbb{R}_+\times\mathbb{R}^2}\left(\rho_{\gamma_b}^{sc,\leqslant N}-\rho\right)\mathrm{GYRO}_\rho(\varphi) \tag{113}$$

$\left(\nabla_z\varphi\cdot\rho_{\gamma_b}^{sc,\leqslant N}\right)\star\nabla^\perp w\in W^{1,\infty}\left(\mathbb{R}_+\times\mathbb{R}^2\right)$**:**

The goal of this part is to prove that $\left(\nabla_z\varphi\cdot\rho_{\gamma_b}^{sc,\leqslant N}\right)\star\nabla^\perp$ is bounded in $W^{1,\infty}\left(\mathbb{R}_+\times\mathbb{R}^2\right)$ uniformly in t and b. Let $\varphi\in C_c^\infty\left(\mathbb{R}_+^*\times\mathbb{R}^2\right)$, then $\forall t\in\mathbb{R}_+$,

$$\left\|\left(\nabla\varphi(t)\cdot\rho_{\gamma_b}^{sc,\leqslant N}(t)\right)\star\nabla^\perp w\right\|_{W^{1;\infty}}\leqslant\left\|\nabla\varphi(t)\rho_{\gamma_b}^{sc,\leqslant N}(t)\right\|_{L^1}\|w\|_{W^{1;\infty}}\leqslant\|\varphi\|_{W^{1,\infty}}\|w\|_{W^{1;\infty}} \tag{114}$$

and with $\partial_t\varphi$ instead of φ

$$\left\|\left(\nabla\partial_t\varphi(t)\cdot\rho_{\gamma_b}^{sc,\leqslant N}(t)\right)\star\nabla^\perp w\right\|_{L^\infty}\leqslant\left\|\nabla\partial_t\varphi(t)\rho_{\gamma_b}^{sc,\leqslant N}(t)\right\|_{L^1}\|w\|_{L^\infty}\leqslant\|\varphi\|_{W^{1,\infty}}\|w\|_{L^\infty} \tag{115}$$

Let $x\in\mathbb{R}^2$, with an integration by parts,

$$\begin{aligned}
&\left(\nabla\varphi(t)\cdot\partial_t\rho_{\gamma_b}^{sc,\leqslant N}(t)\right)\star\nabla^\perp w(x)\\
&=\int\limits_{\mathbb{R}^2}\partial_t\rho_{\gamma_b}^{sc,\leqslant N}(t,z)\nabla_z\varphi(t,z)\cdot\nabla^\perp w(x-z)dz\\
&=-\int\limits_{\mathbb{R}^2}\nabla^\perp\left(V+w\star\rho_{\gamma_b(t)}\right)(z)\cdot\nabla_z\rho_{\gamma_b}(t,z)\nabla_z\varphi(t,z)\cdot\nabla^\perp w(x-z)dz\\
&\quad+\int\limits_{\mathbb{R}^2}\mathrm{GYRO}_{\rho_{\gamma_b}}(\rho_{\gamma_b}^{sc,\leqslant N})(t,z)\nabla_z\varphi(t,z)\cdot\nabla^\perp w(x-z)dz\\
&=\int\limits_{\mathbb{R}^2}\rho_{\gamma_b}(t,z)\nabla^\perp\left(V+w\star\rho_{\gamma_b(t)}\right)(z)\cdot\nabla_z\left(\nabla_z\varphi(t,z)\cdot\nabla^\perp w(x-z)\right)dz\\
&\quad+\int\limits_{\mathbb{R}^2}\mathrm{GYRO}_{\rho_{\gamma_b}}(\rho_{\gamma_b}^{sc,\leqslant N})(t,z)\nabla_z\varphi(t,z)\cdot\nabla^\perp w(x-z)dz\\
&=\int\limits_{\mathbb{R}^2}\rho_{\gamma_b}(t,z)\nabla^\perp\left(V+w\star\rho_{\gamma_b(t)}\right)(z)\cdot\nabla_z^{\otimes 2}\varphi(t,z)\nabla^\perp w(x-z)dz\\
&\quad+\int\limits_{\mathbb{R}^2}\rho_{\gamma_b}(t,z)\nabla^\perp\left(V+w\star\rho_{\gamma_b(t)}\right)(z)\cdot\nabla_z\otimes\nabla_z^\perp w(x-z)\nabla_z\varphi(t,z)dz\\
&\quad+\int\limits_{\mathbb{R}^2}\mathrm{GYRO}_{\rho_{\gamma_b}}(\rho_{\gamma_b}^{sc,\leqslant N})(t,z)\nabla_z\varphi(t,z)\cdot\nabla^\perp w(x-z)dz
\end{aligned}\tag{116}$$

For the two first terms in (116) we use Hölder inequality, and for the last term Proposition V.4 with $\nabla_z\varphi(t)\cdot\nabla^\perp w(x-\bullet)$ as test function. Hence with the choices (109),

$$\begin{aligned}&\left\|\left(\nabla\varphi(t)\cdot\partial_t\rho_{\gamma_b}^{sc,\leqslant N}(t)\right)\star\nabla^\perp w\right\|_{L^\infty}\leqslant\left\|\nabla V+\nabla w\star\rho_{\gamma_b(t)}\right\|_{L^\infty}\left\|d^2\varphi(t)\right\|_{L^\infty}\|\nabla w\|_{L^\infty}\\&+\left\|\nabla V+\nabla w\star\rho_{\gamma_b(t)}\right\|_{L^\infty}\left\|d^2 w\right\|_{L^\infty}\|\nabla\varphi(t)\|_{L^\infty}\\&+C\left(1+|\mathrm{supp}(\varphi)|\right)\left\|\nabla\varphi(t)\cdot\nabla^\perp w(x-\bullet)\right\|_{W^{1,\infty}}\left\|V+w\star\rho_{\gamma_b(t)}\right\|_{W^{4,\infty}}\frac{1}{\sqrt{\ln\left(l_b^{-1}\right)}}\end{aligned}$$

and $\forall t\in\mathbb{R}_+$,

$$\begin{aligned}&\left\|\nabla\varphi(t)\cdot\nabla^\perp w(x-\bullet)\right\|_{W^{1,\infty}}\leqslant\|\varphi(t)\|_{W^{2,\infty}}\|w\|_{W^{2,\infty}}\\&\|\varphi(t)\|_{W^{k,\infty}}\leqslant\|\varphi\|_{W^{k,\infty}}\end{aligned}$$

hence

$$\left\|\left(\nabla\varphi(t)\cdot\partial_t\rho_{\gamma_b}^{sc,\leqslant N}(t)\right)\star\nabla^\perp w\right\|_{L^\infty}\leqslant C(\varphi,V,w)\tag{117}$$

With (114), (115), (117) we conclude that

$$\left\|\left(\nabla_z\varphi\cdot\rho_{\gamma_b}^{sc,\leqslant N}\right)\star\nabla^\perp w\right\|_{W^{1,\infty}}\leqslant C(\varphi,V,w)\tag{118}$$

Conclusion:

With the choices (109), by (106),

$$\left|\int\limits_{\mathbb{R}_+\times\mathbb{R}^2}\varphi\mathrm{GYRO}_{\rho_{\gamma_b}}(\rho_{\gamma_b}^{sc,\leqslant N})dtdz\right|\leqslant C(\varphi,V,w)\frac{1}{\sqrt{\ln\left(l_b^{-1}\right)}}\tag{119}$$

With (112), since $\varphi(0)\in C_c^0\left(\mathbb{R}^2\right)$,

$$\int\limits_{\mathbb{R}^2}\varphi(0)(\rho_0-\rho_{\gamma_b}^{sc,\leqslant N}(0))\underset{b\to\infty}{\to}0\tag{120}$$

Using (114), similarly as for (107) we have

$$\left|\int\limits_{\mathbb{R}_+\times\mathbb{R}^2}\left(\rho_{\gamma_b}-\rho_{\gamma_b}^{sc,\leqslant N}\right)\left(\nabla_z\varphi\cdot\rho_{\gamma_b}^{sc,\leqslant N}\right)\star\nabla^\perp w\right|\leqslant C(\varphi,V,w)\frac{1}{\sqrt{\ln\left(l_b^{-1}\right)}}\tag{121}$$

Using (118) and (111) for n large enough so $\mathrm{supp}(\varphi)\subset\Omega_n$,

$$\begin{aligned}&\left|\int\limits_{\mathbb{R}_+\times\mathbb{R}^2}\left(\rho_{\gamma_b}^{sc,\leqslant N}-\rho\right)\left(\nabla_z\varphi\cdot\rho_{\gamma_b}^{sc,\leqslant N}\right)\star\nabla^\perp w\right|\\&\leqslant\left\|\rho-\rho_{\gamma_b}^{sc,\leqslant N}\right\|_{W^{-1,1}(\Omega_n)}\left\|\left(\nabla_z\varphi\cdot\rho_{\gamma_b}^{sc,\leqslant N}\right)\star\nabla^\perp w\right\|_{W^{1,\infty}}\underset{b\to\infty}{\to}0\end{aligned}\tag{122}$$

And since $\mathrm{GYRO}_\rho(\varphi)\in C_c^0\left(\mathbb{R}\times\mathbb{R}^2\right)$, (110) gives

$$\int\limits_{\mathbb{R}_+\times\mathbb{R}^2}\left(\rho_{\gamma_b}^{sc,\leqslant N}-\rho\right)\mathrm{GYRO}_\rho(\varphi)\underset{b\to\infty}{\to}0\tag{123}$$

Hence the five terms in (113) converge to 0 respectively because of (119), (120), (121), (122), (123) hence the conclusion

$$\int_{\mathbb{R}^2} \varphi(0)\rho_0 - \int_{\mathbb{R}_+ \times \mathbb{R}^2} \rho \mathrm{GYRO}_\rho(\varphi) = 0$$

□

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