Mean-Field Dynamics of the Bose–Hubbard Model in High Dimension

Shahnaz Farhat*, Denis Périce†, Sören Petrat‡

School of Science, Constructor University Bremen,
Campus Ring 1, 28759 Bremen, Germany

November 21, 2024

Abstract

The Bose–Hubbard model effectively describes bosons on a lattice with on-site interactions and nearest-neighbour hopping, serving as a foundational framework for understanding strong particle interactions and the superfluid to Mott insulator transition. This paper aims to rigorously establish the validity of a mean-field approximation for the dynamics of quantum systems in high dimension, using the Bose–Hubbard model on a square lattice as a case study. We prove a trace norm estimate between the first reduced density of the Schrödinger dynamics and the mean-field dynamics in the limit of large dimension. Here, the mean-field approximation is in the hopping amplitude and not the interaction, leading to a very rich and non-trivial mean-field equation. This mean-field equation does not only describe the condensate, as is the case when the mean-field description comes from a large particle number limit averaging out the interaction, but it allows for a phase transition to a Mott insulator since it contains the full non-trivial interaction. Our work is a rigorous justification of a simple case of the highly successful dynamical mean-field theory (DMFT) for bosons, which somewhat surprisingly yields many qualitatively correct results in three dimensions.

Contents

1 Introduction **2**
1.1 Model . . . 3
1.2 Main results . . . 4

2 Preliminaries **6**
2.1 Reduced densities matrices . . . 6
2.2 Well posedness . . . 9
2.3 Conservation Laws . . . 11
2.4 Gronwall estimate . . . 17

3 Proof using moment estimates **21**
3.1 Proof using moment method . . . 21

4 Proof using the excitation energy method **25**
4.1 Some ϵ stimates . . . 27
4.2 Proof of Grönwall Lemma . . . 31

References **38**

*Email: `sfarhat@constructor.university`
†Email: `dperice@constructor.university`, corresponding author.
‡Email: `spetrat@constructor.university`

1 Introduction

One of the big aspirations of mathematical physics is to advance our rigorous understanding of phase transitions. Within this research area lots of recent attention has been paid to the phenomenon of Bose–Einstein–Condensation (BEC), a phase of matter of cold Bose gases that has been predicted in 1924 by Bose [6] and Einstein [11, 12]. Since then, BEC has been studied extensively by theoretical physicists, and at least since the 1980's also by mathematical physicists with more rigorous methods. After the first experimental realizations in the labs of Cornell/Wieman [2] and Ketterle [10] in 1995 the study of BEC has received a new wave of attention throughout experimental, theoretical, and mathematical physics. As a recent highlight in mathematical physics, let us mention the rigorous derivation of the Lee–Huang–Yang formula by Fournais and Solovej [14, 15]. The motivation of our work comes from different perspectives:

1. Lots of recent effort has been put into understanding BEC at zero temperature, e.g., in the Gross–Pitaevskii limit [?] and thermodynamic limit [?]. This yields insight into the behaviour of the condensate and its excitations, e.g., a rigorous proof of Bogoliubov theory in the Gross–Pitaevskii regime [?], but is very far away from understanding the (thermodynamic) phase transition to BEC.

2. A very useful and simple method for studying many-body systems is the mean-field approximation. For bosonic cold atoms, one scales down the interaction potential with the inverse of the particle number N [?] (or density [?]), thus considering weak interaction. Then the inverse particle number can be regarded as a small parameter, and the interaction can be effectively replaced by its mean-field. In this mean-field limit, many physical effects can be rigorously established regarding the dynamics and the low-energy properties. In particular, one can prove the validity of Bogoliubov theory, and perturbative expansions beyond Bogoliubov. However, these types of mean-field models do not describe phase transitions.

3. For Bose gases in the continuum, one would ultimately like to prove a thermodynamics phase transition. However, Bose gases on the lattice offer a different possibility for a phase transition, namely a quantum phase transition between a BEC and a localized state usually called a Mott insulator; see, e.g., [28] for a review. There is only few mathematical rigorous works on this topic, e.g., [1].

Our work addresses these points in the following way. 1: We study a limit that may describe a phase transition. 2: Our limit is a mean-field limit, but not for large particle number but for large dimension. We hope and indeed show that some of the methods of large N mean-field limits are still relevant for this case. Since in our model the averaging is done over the hopping terms, and the interaction is treated non-perturbatively, our mean-field model is strongly interacting. 3: Our microscopic model is the Bose–Hubbard model, which is a lattice model that has been successfully used to describe the BEC-Mott transition.

DMFT

The purpose of this paper is to rigorously justify the validity of the mean-field approximation for a quantum system in large dimension. We choose the Bose–Hubbard model to illustrate this statement both for its remarkable usefulness in physics and for the technical simplifications it offers as a lattice model.

The Bose–Hubbard model is a popular model used to describe bosons on a lattice with on-site interactions, allowing hopping between nearest-neighbour lattice sites. It is well-known for capturing strong interactions between particles [5] and providing one of the simplest descriptions of the Mott transition to date, see [13] and later [21], see also [18, 19].

The so-called dynamical mean-field theory (DMFT) of the Bose–Hubbard model has been well-known for its description of the Mott-insulator/superfluid phase transition [20] [13]. DMFT is typically justified in the physics literature by stating that mean-field theories become exact in the limit of infinite dimensions [23]. This is also the case for fermions [22]. A remarkable fact is that DMFT tends to provide accurate results in three dimensions but are less reliable in lower dimensions [16].

In the mathematical literature, mean-field limits are typically considered as many-particle limits for the Bose–Hubbard model [25] or, more generally, for continuous models where the Hartree equation is obtained as effective dynamics (see [3, 4, 9, 17, 24, 26] for a review). This approach requires dividing the interaction term by the number of particles to ensure that the kinetic energy and the interaction energy of the ground state remain of the same order.

Our goal is to provide a rigorous justification, in the $d \to \infty$ limit, that the DMFT theory is a good approximation of the Schrödinger equation in the context of the Bose–Hubbard model. It is interesting to note that, in the large d limit, the roles of the kinetic energy and the interaction between particles are inverted compared to the usual mean-field limit $N \to \infty$. The interactions we aim to average in our regime are the interactions between nearest-neighbour sites due to the hopping term in the kinetic energy. Since we only consider on-site interactions, the interaction between particles acts as a one-site operator and therefore do not contributes to correlations between two different lattice sites. For our setting, the basic idea behind the mean-field approximation is that the coordination number of the lattice (the number of nearest neighbours) increases with the dimension. This means we have a mean-field picture locally around every site, which allows us to control the correlations between sites.

A major advantage of this mean-field scaling is that we do not need to artificially divide the interaction term by the number of particles. As the impressive apparent validly of the DMFT in $d = 3$ seems to suggest, we hope that in further works, the dimension-dependent estimates could be improved to such a level that the approximation would be pertinent in three dimensions. Such advances could stem from the fact that, at fixed density, the number of particles grows exponentially with the dimensions. Therefore, if one obtains polynomial estimates in terms of the number of particles, we achieve an exponential convergence rate in terms of the dimension, which may provide a decent level of approximation for dimensions as small as 3.

1.1 Model

We consider the d-dimensional square lattice with periodic boundary conditions $\Lambda \coloneqq (\mathbb{Z}/L\mathbb{Z})^d$ of volume $|\Lambda| \coloneqq L^d$. We denote $x \sim y$ if $x, y \in \Lambda$ are nearest neighbours. The one site Hilbert space is $l^2(\mathbb{C})$ and its canonical Hilbert basis is denoted $(|n\rangle)_{n\in\mathbb{N}}$. We define the standards annihilation and creation operators a, a^* with

$$\begin{aligned} a\,|0\rangle = 0, \forall n \in \mathbb{N}^*, a\,|n\rangle \coloneqq \sqrt{n}\,|n-1\rangle \\ \forall n \in \mathbb{N}, a^*\,|n\rangle \coloneqq \sqrt{n+1}\,|n+1\rangle \end{aligned}$$

The Number of particles operator is $\mathcal{N} \coloneqq a^* a$. We consider an order on Λ such that $\forall x \in \Lambda$

$$\#\{y \in \Lambda | y > x \text{ and } x \sim y\} = \#\{y \in \Lambda | x > y \text{ and } x \sim y\} = d.$$

For example, the lexicographic order does the job. The Fock space is

$$\mathcal{F} \coloneqq \ell^2(\mathbb{C})^{\otimes|\Lambda|} \cong \mathcal{F}_+\left(L^2(\Lambda, \mathbb{C})\right) \coloneqq \bigoplus_{k\in\mathbb{N}} L^2(\Lambda, \mathbb{C})^{\otimes_+ k}$$

Given a one site operator A and $x \in \Lambda$,

$$A_x \coloneqq \left(\bigotimes_{y<x} \mathbb{1}\right) A_x \left(\bigotimes_{y>x} \mathbb{1}\right)$$

With $\langle x, y\rangle$ we mean that $x, y \in \Lambda, x \sim y$ and $x < y$. The kinetic energy is given by the second quantized Laplacian

$$-\,\mathrm{d}\Gamma(\Delta_d) \coloneqq \sum_{\langle x,y\rangle} (a_x^* - a_y^*)(a_x - a_y) = -\sum_{\langle x,y\rangle} \left(a_x^* a_y + a_y^* a_x\right) + d \sum_{x\in\Lambda} (\mathcal{N}_x + \mathcal{N}_y)$$

$$= -\sum_{\langle x,y\rangle}\left(a_x^* a_y + a_y^* a_x\right) + 2d\sum_{x\in\Lambda}\mathcal{N}_x.$$

Let $\mathcal{N}_{\mathcal{F}}$ denotes the number of particles operator on Fock space,

$$\mathcal{N}_{\mathcal{F}} := d\Gamma(1) = \sum_{x\in\Lambda}\mathcal{N}_x.$$

Given hopping amplitude $J \in \mathbb{R}$, chemical potential $\mu \in \mathbb{R}$, and coupling constant $U \in \mathbb{R}$, we define the Bose–Hubbard Hamiltonian

$$\begin{aligned} H_d &:= -\frac{J}{2d}\,\mathrm{d}\Gamma(\Delta_d) + (J-\mu)\mathcal{N}_{\mathcal{F}} + \frac{U}{2}\sum_{x\in\Lambda}\mathcal{N}_x(\mathcal{N}_x - 1) \\ &= -\frac{J}{2d}\sum_{\langle x,y\rangle}\left(a_x^* a_y + a_y^* a_x\right) + (J-\mu)\sum_{x\in\Lambda}\mathcal{N}_x + \frac{U}{2}\sum_{x\in\Lambda}\mathcal{N}_x(\mathcal{N}_x-1). \end{aligned} \tag{1}$$

Here, we have scaled down the hopping term with the inverse of the dimension d. We are interested in a large d limit.

The time-dependent Schrödinger equation for $\Psi_d \in \mathcal{F}$ is

$$i\frac{d}{dt}\Psi_d(t) = H_d\Psi_d(t). \tag{2}$$

We call a d-site density matrix a positive trace 1 operator $\gamma_b \in \mathcal{L}^1(\mathcal{F})$. The Schrödinger equation for γ_d is

$$i\partial_t\gamma_d = [H_d, \gamma_d] \tag{3}$$

We also define it's first reduced density matrix as

$$\gamma_{\mathrm{d}}^{(1)} := \frac{1}{|\Lambda|}\sum_{x\in\Lambda}\mathrm{Tr}_{\Lambda\setminus\{x\}}(\gamma_d)$$

For the mean-field description we introduce the one-lattice-site wave function $\varphi \in l^2(\mathbb{C})$ and the associated order parameter $\alpha_\varphi := \langle\varphi, a\varphi\rangle$. We introduce the mean-field Hamiltonian

$$h^\varphi := -J\left(\alpha_\varphi a^* + \overline{\alpha_\varphi}a - |\alpha_\varphi|^2\right) + (J-\mu)\mathcal{N} + \frac{U}{2}\mathcal{N}(\mathcal{N}-1), \tag{4}$$

and the (nonlinear) mean-field equation

$$i\partial_t\varphi = h^\varphi\varphi. \tag{5}$$

We introduce the corresponding orthogonal projections $p = p^\varphi := |\varphi\rangle\langle\varphi|$ and $q = q^\varphi := 1 - p^\varphi$.

1.2 Main results

Theorem 1. *Assume that γ_d follows (3) with $\gamma_d(0)$ a d-site density matrix and φ solves (5) with $\varphi(0) \in l^2(\mathbb{C})$ such that $\|\varphi\|_{l^2} = 1$. If $\exists a, c > 0$ such that*

$$\forall n \in \mathbb{N}, \mathrm{Tr}\left(p_\varphi(0)\mathbb{1}_{\mathcal{N}=n}\right) \leqslant ce^{-\frac{n}{a}} \text{ and } \mathrm{Tr}\left(\gamma_d^{(1)}(0)\mathbb{1}_{\mathcal{N}=n}\right) \leqslant ce^{-\frac{n}{a}} \tag{6}$$

then $\forall t \in \mathbb{R}_+$,

$$\left\|\gamma_d^{(1)}(t) - p_\varphi(t)\right\|_{\mathcal{L}^1} \leqslant \sqrt{2}\left(\left\|\gamma_{\mathrm{d}}^{(1)}(0) - p_\varphi(0)\right\|_{\mathcal{L}^1} + \frac{C_2e^{C_1t} + \mathrm{Tr}\left(p(0)\mathcal{N}\right)^{\frac{1}{2}}}{d\left(C_4 + 2\left(\sqrt{2(a+e)}e^{\frac{C_1}{2}t} + 1\right)\sqrt{\ln(d+1)}\right)}\right)^{\frac{1}{2}}$$

$$e^{JC_3\left(C_4+2\left(\sqrt{2(a+e)}e^{\frac{C_1}{2}t}+1\right)\sqrt{\ln(d+1)}\right)t}$$

with the following constant independent of d and t:

$$\begin{aligned}
C_1 &:= 2eJ\max\left(\mathrm{Tr}\left(p_\varphi(0)\mathcal{N}\right),1\right)\\
C_2 &:= 4\left(c(1+a)+e^{-1}\right)(2+4(a+e))\\
C_3 &:= \left(\mathrm{Tr}\left(p_\varphi(0)\mathcal{N}\right)+1\right)^{\frac{1}{2}}\\
C_4 &:= 4\mathrm{Tr}\left(p(0)\mathcal{N}\right)^{\frac{1}{2}}+2.
\end{aligned}$$

Note that the d-dependent terms are small when $d\to\infty$, indeed:

$$\frac{1}{\left(d\sqrt{\ln(d+1)}\right)^{\frac{1}{2}}}e^{2JC_3\left(\sqrt{2(a+e)}e^{\frac{C_1}{2}t}+1\right)\sqrt{\ln(d+1)}t}=e^{2JC_3\left(\sqrt{2(a+e)}e^{\frac{C_1}{2}t}+1\right)\sqrt{\ln(d+1)}t-\frac{1}{2}\ln(d)-\frac{1}{4}\ln(\ln(d+1))}$$

$$\underset{d\to\infty}{\longrightarrow}0.$$

Theorem 2. *Let $\varphi\in\ell^2(\mathbb{C})$ be a solution of (5) such that $\|\varphi\|_{\ell^2}=1$ and $\Psi_d\in\mathcal{F}$ be a solution of (2) such that $\|\Psi_d\|=1$. Assume that there is $C>0$ such that*

$$\mathrm{Tr}(p(0)\mathcal{N}^4)\leqslant C.$$

Then, for $U>0$ and for all $t\in\mathbb{R}$, we have

$$\begin{aligned}
&\left\|\gamma_d^{(1)}(t)-p(t)\right\|_{\mathscr{L}^1}\\
&\leqslant\left(\frac{1}{d}\frac{1}{U}+C(J,\mu,U)\left(1+\frac{1}{U^2}\right)e^{C(J,\mu,U)\sum_{j=1}^{7}t^j}\left(\mathrm{Tr}\left(\gamma_d^{(1)}(0)\left(q(0)\mathcal{N}^2q(0)+q(0)\right)\right)+\frac{1}{d}\right)\right)^{1/2}
\end{aligned}\tag{7}$$

where $C(J,\mu,U)>0$ is polynomailly dependent on the parameters of our model J, μ and U.

The idea make a mean-field approximation for the interaction between the lattice sites implies technical issues because the hopping term has singularities due to the unboundedness of creation and annihilation operators. We therefore have to deal with a very singular interaction. Indeed, this unboundedness is worse than what is usually covered for the large number of particles mean-field limits where the singularities of the interaction potential can be smoothed out against the density due to the convolution structure of the interaction term in the energy.

discuss briefly well posedness of Hartree and Schrödinger equations

2 Preliminaries

Proposition 3. *Let $\alpha \in \mathbb{C}$, we have the decomposition*

$$H_d = -\frac{J}{2d} \sum_{\substack{x,y\in\Lambda \\ x\sim y}} a_x^* a_y + (J-\mu) \sum_{x\in\Lambda} \mathcal{N}_x + \frac{U}{2} \sum_{x\in\Lambda} \mathcal{N}_x(\mathcal{N}_x - 1) \tag{8}$$

$$= \sum_{x\in\Lambda} h_x^{\varphi} - \frac{J}{2d} \sum_{\substack{x,y\in\Lambda \\ x\sim y}} \left(a_x^* - \overline{\alpha}\right)\left(a_y - \alpha\right) \tag{9}$$

and the energy upper bound

$$\frac{1}{|\Lambda|} \inf_{\psi\in\mathcal{F}} \langle \psi, H\psi \rangle \leqslant \inf_{\varphi\in l^2(\mathbb{C})} \langle \varphi, h^{\varphi} \varphi \rangle$$

Proof. By summing

$$a_x^* a_y = \left(a_x^* - \overline{\alpha}\right)\left(a_y - \alpha\right) + \alpha a_x^* + \overline{\alpha} a_y - |\alpha|^2$$

over $x \sim y$, we obtain

$$\sum_{\substack{x,y\in\Lambda \\ x\sim y}} a_x^* a_y = 2d \sum_{x\in\Lambda} \left(\alpha a^* + \overline{\alpha} a - |\alpha|^2\right)_x + \sum_{\substack{x,y\in\Lambda \\ x\sim y}} \left(a_x^* - \overline{\alpha}\right)\left(a_y - \alpha\right).$$

Inserting this in (1) and identifying (4) we obtain (9):

$$\begin{aligned} H_d &= -\frac{J}{2d} \sum_{\substack{x,y\in\Lambda \\ x\sim y}} a_x^* a_y + (J-\mu) \sum_{x\in\Lambda} \mathcal{N}_x + \frac{U}{2} \sum_{x\in\Lambda} \mathcal{N}_x(\mathcal{N}_x - 1) \\ &= \sum_{x\in\Lambda} \left(-J\left(\alpha a^* + \overline{\alpha} a - |\alpha|^2\right) + (J-\mu)\mathcal{N} + \frac{U}{2}\mathcal{N}(\mathcal{N}-1) \right)_x - \frac{J}{2d} \sum_{\substack{x,y\in\Lambda \\ x\sim y}} \left(a_x^* - \overline{\alpha}\right)\left(a_y - \alpha\right). \end{aligned}$$

□

2.1 Reduced densities matrices

Given a d-site density matrix, we define it's second reduced density matrices as

$$\gamma_d^{(2)} := \frac{1}{d\,|\Lambda|} \sum_{\langle x,y \rangle} \mathrm{Tr}_{\Lambda\setminus\{x,y\}}\left(\gamma_d\right) = \frac{1}{2d\,|\Lambda|} \sum_{\substack{x,y\in\Lambda \\ x\sim y}} \mathrm{Tr}_{\Lambda\setminus\{x,y\}}\left(\gamma_d\right).$$

Note that the 2-site density matrix is symmetrized over all interacting pairs of sites and not over all pairs of sites. The normalisation factor $2d\,|\Lambda|$ is indeed the number of interacting pairs of sites.

Let $x_{1:k} \in \Lambda^k$, $A^{(k)} \in \mathcal{K}(l^2(\mathbb{C})^{\otimes k})$ we denote $A^{(k)}_{x_{1:k}} \in \mathcal{K}(\mathcal{F})$ the operator acting on the coordinates $x_{1:k}$ as $A^{(k)}$. We recall that the partial trace is characterised by the following identity for $x_1 < \cdots < x_k$,

$$\mathrm{Tr}\left(\mathrm{Tr}_{\Lambda\setminus\{x_{1:k}\}}\left(\gamma_d\right) A^{(k)}\right) = \mathrm{Tr}\left(\gamma_d A^{(k)}_{x_{1:k}}\right).$$

Proposition 4. *If γ_d be a d-site density matrix, then $\gamma_{\mathrm{d}}^{(2)}$ is symmetric and reduces to $\gamma_{\mathrm{d}}^{(1)}$:*

$$\begin{aligned} &\forall A, B \in \mathcal{K}(l^2(\mathbb{C})), \mathrm{Tr}\left(\gamma_{\mathrm{d}}^{(2)}\, A \otimes B\right) = \mathrm{Tr}\left(\gamma_{\mathrm{d}}^{(2)}\, B \otimes A\right), \\ &\mathrm{Tr}_1\left(\gamma_{\mathrm{d}}^{(2)}\right) = \mathrm{Tr}_2\left(\gamma_{\mathrm{d}}^{(2)}\right) = \gamma_{\mathrm{d}}^{(1)}. \end{aligned}$$

Moreover, if $C \in \mathcal{L}(l^2(\mathbb{C}))$ *and* $D \in \mathcal{L}(l^2(\mathbb{C})^{\otimes 2})$*, then*

$$\frac{1}{|\Lambda|}\sum_{x\in\Lambda}\mathrm{Tr}\left(\gamma_d C_x\right) = \mathrm{Tr}\left(\gamma_{\mathrm{d}}^{(1)}\, C\right)$$
$$\frac{1}{2d\,|\Lambda|}\sum_{\substack{x,y\in\Lambda\\ x\sim y}}\mathrm{Tr}\left(\gamma_d D_{x,y}\right) = \mathrm{Tr}\left(\gamma_{\mathrm{d}}^{(2)}\, D\right)$$

Proof. Noticing that

$$\gamma_{\mathrm{d}}^{(2)} = \frac{1}{2d\,|\Lambda|}\sum_{\substack{x,y\in\Lambda\\ x\sim y}}\mathrm{Tr}_{\Lambda\setminus\{x,y\}}\left(\gamma_d\right)$$

we can exchange the sum indices x, y:

$$\mathrm{Tr}\left(\gamma_{\mathrm{d}}^{(2)}\, A\otimes B\right) = \frac{1}{2d\,|\Lambda|}\sum_{\substack{x,y\in\Lambda\\ x\sim y}}\mathrm{Tr}\left(\gamma_d A_x B_y\right) = \frac{1}{2d\,|\Lambda|}\sum_{\substack{x,y\in\Lambda\\ x\sim y}}\mathrm{Tr}\left(\gamma_d A_y B_x\right) = \mathrm{Tr}\left(\gamma_{\mathrm{d}}^{(2)}\, B\otimes A\right).$$

Let $K \in \mathcal{K}(l^2(\mathbb{C}))$, using the symmetry of $\gamma_{\mathrm{d}}^{(2)}$,

$$\mathrm{Tr}\left(\mathcal{K}\mathrm{Tr}_1\left(\gamma_{\mathrm{d}}^{(2)}\right)\right) = \mathrm{Tr}\left(K_1\,\gamma_{\mathrm{d}}^{(2)}\right) = \frac{1}{d\,|\Lambda|}\sum_{\langle x,y\rangle}\mathrm{Tr}\left(\gamma_d K_x\right) = \frac{1}{|\Lambda|}\sum_{x\in\Lambda}\mathrm{Tr}\left(\gamma_d K_x\right) = \mathrm{Tr}\left(K\,\gamma_{\mathrm{d}}^{(1)}\right).$$

Finally we compute

$$\frac{1}{|\Lambda|}\sum_{x\in\Lambda}\mathrm{Tr}\left(\gamma_d C_x\right) = \frac{1}{|\Lambda|}\sum_{x\in\Lambda}\mathrm{Tr}\left(\mathrm{Tr}_{\Lambda\setminus\{x\}}\left(\gamma_d\right)C\right) = \mathrm{Tr}\left(\gamma_{\mathrm{d}}^{(1)}\, C\right)$$
$$\frac{1}{2d\,|\Lambda|}\sum_{\substack{x,y\in\Lambda\\ x\sim y}}\mathrm{Tr}\left(\gamma_d D_{x,y}\right) = \frac{1}{2d\,|\Lambda|}\sum_{\substack{x,y\in\Lambda\\ x\sim y}}\mathrm{Tr}\left(\mathrm{Tr}_{\Lambda\setminus\{x,y\}}\left(\gamma_d\right)D\right) = \mathrm{Tr}\left(\gamma_{\mathrm{d}}^{(2)}\, D\right).$$

□

Here are useful formulas for the energies.

Proposition 5. *If* γ_d *be a d-site density matrix, then*

$$\frac{\mathrm{Tr}\left(\gamma_d H_d\right)}{|\Lambda|} = \mathrm{Tr}\left(\gamma_{\mathrm{d}}^{(1)}\left((J-\mu)\mathcal{N} + \frac{U}{2}\mathcal{N}\left(\mathcal{N}-1\right)\right)\right) - J\mathrm{Tr}\left(\gamma_{\mathrm{d}}^{(2)}\, a^*\otimes a\right). \tag{10}$$

If $\varphi \in l^2(\mathbb{C})$*, then*

$$\langle\varphi, h^{\varphi}\varphi\rangle = J\left(\langle\varphi,\mathcal{N}\varphi\rangle - |\alpha_\varphi|^2\right) - \mu\langle\varphi,\mathcal{N}\varphi\rangle + \frac{U}{2}\langle\varphi,\mathcal{N}(\mathcal{N}-1)\varphi\rangle. \tag{11}$$

Proof. Starting from (8) and using Proposition 4 we get (10).
(11) is a consequence of (4) and $\alpha_\varphi = \langle\varphi, a\varphi\rangle$. □

The following Hamiltonian bounds give us a control the Bose–Hubbard energy and the mean-field energy in term of moments of moments of the number of excitations operator.

Proposition 6 (Hamiltonian bounds)**.** *Let* γ_d *be a d-site density matrix and* $\varphi \in l^2(\mathbb{C})$*.*
If $U = 0$*, then*

$$-\left(J_- + \mu\right)\langle\varphi,\mathcal{N}\varphi\rangle \leqslant \langle\varphi, h^{\varphi}\varphi\rangle \leqslant \left(J_+ - \mu\right)\langle\varphi,\mathcal{N}\varphi\rangle \tag{12}$$

$$-|J|-(2J_-+\mu)\operatorname{Tr}\left(\gamma_{\mathrm{d}}^{(1)}\mathcal{N}\right)\leqslant\frac{\operatorname{Tr}(\gamma_d H_d)}{|\Lambda|}\leqslant|J|+(2J_+-\mu)\operatorname{Tr}\left(\gamma_{\mathrm{d}}^{(1)}\mathcal{N}\right)\tag{13}$$

otherwise

$$-\frac{\left(J_-+\mu+\frac{U}{2}\right)_+^2}{|U|}+\frac{2U-|U|}{4}\langle\varphi,\mathcal{N}^2\varphi\rangle\leqslant\langle\varphi,h^\varphi\varphi\rangle\leqslant\frac{\left(J_+-\mu-\frac{U}{2}\right)_+^2}{|U|}+\frac{2U+|U|}{4}\langle\varphi,\mathcal{N}^2\varphi\rangle\tag{14}$$

$$-|J|-\frac{\left(2J_-+\mu+\frac{U}{2}\right)_+^2}{|U|}+\frac{2U-|U|}{4}\operatorname{Tr}\left(\gamma_{\mathrm{d}}^{(1)}\mathcal{N}^2\right)\leqslant\frac{\operatorname{Tr}(\gamma_d H_d)}{|\Lambda|}$$
$$\leqslant|J|+\frac{\left(2J_+-\mu-\frac{U}{2}\right)_+^2}{|U|}+\frac{2U+|U|}{4}\operatorname{Tr}\left(\gamma_{\mathrm{d}}^{(1)}\mathcal{N}^2\right)\tag{15}$$

Proof. **Mean-field energy**

Using Cauchy-Schwarz's inequality we have

$$0\leqslant|\alpha_\varphi|^2=|\langle\varphi,a\varphi\rangle|^2\leqslant\|\varphi\|_{l^2}^2\,\|a\varphi\|_{l^2}^2=\langle a\varphi,a\varphi\rangle=\langle\varphi,a^*a\varphi\rangle=\langle\varphi,\mathcal{N}\varphi\rangle\tag{16}$$

so

$$-J_-\langle\varphi,\mathcal{N}\varphi\rangle\leqslant J\left(\langle\varphi,\mathcal{N}\varphi\rangle-|\alpha_\varphi|^2\right)\leqslant J_+\langle\varphi,\mathcal{N}\varphi\rangle.$$

Inserting this in (11) we get

$$-\left(J_-+\mu+\frac{U}{2}\right)\langle\varphi,\mathcal{N}\varphi\rangle+\frac{U}{2}\langle\varphi,\mathcal{N}^2\varphi\rangle\leqslant\langle\varphi,h^\varphi\varphi\rangle\leqslant\left(J_+-\mu-\frac{U}{2}\right)\langle\varphi,\mathcal{N}\varphi\rangle+\frac{U}{2}\langle\varphi,\mathcal{N}^2\varphi\rangle.$$

Taking $U=0$ gives us (12), otherwise we have

$$-\left(J_-+\mu+\frac{U}{2}\right)_+\langle\varphi,\mathcal{N}\varphi\rangle+\frac{U}{2}\langle\varphi,\mathcal{N}^2\varphi\rangle\leqslant\langle\varphi,h^\varphi\varphi\rangle\leqslant\left(J_+-\mu-\frac{U}{2}\right)_+\langle\varphi,\mathcal{N}\varphi\rangle+\frac{U}{2}\langle\varphi,\mathcal{N}^2\varphi\rangle$$

Finally, we obtain (14) by inserting

$$\left(J_+-\mu-\frac{U}{2}\right)_+\langle\varphi,\mathcal{N}\varphi\rangle\leqslant\frac{\left(J_+-\mu-\frac{U}{2}\right)_+^2}{|U|}+\frac{|U|}{4}\langle\varphi,\mathcal{N}^2\varphi\rangle$$
$$\left(J_-+\mu+\frac{U}{2}\right)_+\langle\varphi,\mathcal{N}\varphi\rangle\leqslant\frac{\left(J_-+\mu+\frac{U}{2}\right)_+^2}{|U|}+\frac{|U|}{4}\langle\varphi,\mathcal{N}^2\varphi\rangle.$$

d-site energy

We estimate the 2-site term with

$$\left|\operatorname{Tr}\left(\gamma_{\mathrm{d}}^{(2)}a^*\otimes a\right)\right|\leqslant\operatorname{Tr}\left(\gamma_{\mathrm{d}}^{(1)}\mathcal{N}\right)^{\frac{1}{2}}\operatorname{Tr}\left(\gamma_{\mathrm{d}}^{(1)}(\mathcal{N}+1)\right)^{\frac{1}{2}}\leqslant\operatorname{Tr}\left(\gamma_{\mathrm{d}}^{(1)}\mathcal{N}\right)+1$$

so

$$-1\leqslant\operatorname{Tr}\left(\gamma_{\mathrm{d}}^{(1)}\mathcal{N}\right)-\operatorname{Tr}\left(\gamma_{\mathrm{d}}^{(2)}a^*\otimes a\right)\leqslant2\operatorname{Tr}\left(\gamma_{\mathrm{d}}^{(1)}\mathcal{N}\right)+1$$

and

$$-J_+-J_-\left(2\operatorname{Tr}\left(\gamma_{\mathrm{d}}^{(1)}\mathcal{N}\right)+1\right)\leqslant J\left(\operatorname{Tr}\left(\gamma_{\mathrm{d}}^{(1)}\mathcal{N}\right)-\operatorname{Tr}\left(\gamma_{\mathrm{d}}^{(2)}a^*\otimes a\right)\right)\leqslant J_+\left(2\operatorname{Tr}\left(\gamma_{\mathrm{d}}^{(1)}\mathcal{N}\right)+1\right)+J_-$$

since $J_++J_-=|J|$, with (10) we get

$$-|J|-\left(2J_-+\mu+\frac{U}{2}\right)\operatorname{Tr}\left(\gamma_{\mathrm{d}}^{(1)}\mathcal{N}\right)+\frac{U}{2}\operatorname{Tr}\left(\gamma_{\mathrm{d}}^{(1)}\mathcal{N}^2\right)\leqslant\frac{\operatorname{Tr}(\gamma_d H_d)}{|\Lambda|}$$
$$\leqslant|J|+\left(2J_+-\mu-\frac{U}{2}\right)\operatorname{Tr}\left(\gamma_{\mathrm{d}}^{(1)}\mathcal{N}\right)+\frac{U}{2}\operatorname{Tr}\left(\gamma_{\mathrm{d}}^{(1)}\mathcal{N}^2\right)$$

And obtain (13) (15) with the same arguments as for h^φ. $\square$

2.2 Well posedness

We refer to [7, 8, 27]. The mean-field dynamic is

$$\partial_t \varphi(t) = v(\varphi(t)) \tag{17}$$

where $v : \ell^2(\mathbb{C}) \to \ell^2(\mathbb{C})$ is the nonlinear operator defined as follows

$$v(\varphi) := \left(iJ \left(\langle \varphi, a\varphi \rangle a^* + \langle \varphi, a^*\varphi \rangle a - |\langle \varphi, a\varphi \rangle|^2 \right) - i(J-\mu)\mathcal{N} - i\frac{U}{2}\mathcal{N}(\mathcal{N}-1) \right) \varphi. \tag{18}$$

Now, any solution to the mean-field equation (17) satisfies

$$\varphi(t) = \varphi_0 + \int_0^t v(\varphi(s)) ds \tag{19}$$

or equivalently by looking at $\tilde{\varphi}(t) = e^{itf(\mathcal{N})}\varphi(t)$ with $f(\mathcal{N}) := (J-\mu)\mathcal{N} + \frac{U}{2}\mathcal{N}(\mathcal{N}-1)$

$$\varphi(t) = e^{-itf(\mathcal{N})}\varphi_0 + iJ \int_0^t e^{-i(t-s)f(\mathcal{N})} \left(\alpha_\varphi(s) a^* + \overline{\alpha_\varphi(s)} a - |\alpha_\varphi(s)|^2 \right) \varphi(s)) ds$$

We know that if φ is a solution to the mean-field equation (17) and we start from the initial data $\varphi(0) \in \mathcal{D}(\mathcal{N}^k)$, then by (61), for all $t \leqslant T$, the solution $\varphi(t) \in \mathcal{D}(\mathcal{N}^K)$. Now, let $\varphi(0) \in \mathcal{D}(\mathcal{N}^2)$ and let $\varphi(\cdot)$ satisfies (17), then it satisfies (19) and we get

$$\|\varphi(t)\| \leqslant \|\varphi(0)\| + \int_0^t \|v(\varphi(s))\| ds$$

and we have for $t \leqslant T$

$$\|v(\varphi(s))\| \lesssim \|\mathcal{N}^2 \varphi(s)\| < +\infty$$

which implies

$$\|\varphi(t)\| \lesssim \|\varphi(0)\| + T \sup_{t \in [0,T]} \|\mathcal{N}^2 \varphi(t)\| < +\infty.$$

First approach: To prove existence of solution to above nonlinear equation, we might apply Cauchy-Lipschitz theorem in Banach spaces. To this end, we may consider the space $X = \mathcal{C}([0,T]; \ell^2(\mathbb{C}))$ or $X = \mathcal{C}([0,T]; \mathcal{D}(\mathcal{N}^2))$ endowed with the following norm

$$\|\varphi\| := \sup_{t \in [0,T]} \|\varphi(t)\|_{\ell^2}.$$

And, let $\tilde{\varphi}_0(t) := e^{-itf(\mathcal{N})}\varphi_0$ and we consider the associated ball

$$B_X(\tilde{\varphi}_0, R] := \{\varphi \in X; \quad \|\varphi - \tilde{\varphi}_0\| \leqslant R\}.$$

Then, we consider the map $\Gamma : B_X \to B_X$ such that

$$\Gamma(\varphi)(t) = \tilde{\varphi}_0(t) + iJ \int_0^t e^{-i(t-s)f(\mathcal{N})} \left(\alpha_\varphi(s) a^* + \overline{\alpha_\varphi(s)} a - |\alpha_\varphi(s)|^2 \right) \varphi(s) ds$$

with $\alpha_\varphi = \langle \varphi, a\varphi \rangle$.

Proposition 7 (Lipschitz condition)**.** *The vector field $v : \ell^2(\mathbb{C}) \to \ell^2(\mathbb{C})$ satisfies the following Lipschitz condition: for all $\varphi_1, \varphi_2 \in \mathcal{D}(\mathcal{N}^2)$*

$$\|v(\varphi_1) - v(\varphi_2)\| \leqslant C \|\mathcal{N}^2(\varphi_1 - \varphi_2)\|$$

Proof. We first remark that

$$\begin{aligned}|\langle\varphi_1,a\varphi_1\rangle-\langle\varphi_2,a\varphi_2\rangle| &= |\langle\varphi_1-\varphi_2,a\varphi_1\rangle+\langle\varphi_2,a(\varphi_1-\varphi_2)\rangle| \\ &\leqslant \|\varphi_1-\varphi_2\|\|a\varphi_1\|+\|a^*\varphi_2\|\|\varphi_1-\varphi_2\| \\ &= (\|a\varphi_1\|+\|a^*\varphi_2\|)\,\|\varphi_1-\varphi_2\|\end{aligned} \tag{20}$$

and as well

$$\begin{aligned}&\left|\langle\varphi_1,a\varphi_1\rangle\overline{\langle\varphi_1,a\varphi_1\rangle}-\langle\varphi_2,a\varphi_2\rangle\overline{\langle\varphi_2,a\varphi_2\rangle}\right| \\ &= \left|\langle\varphi_1,a\varphi_1\rangle\overline{(\langle\varphi_1,a\varphi_1\rangle-\langle\varphi_2,a\varphi_2\rangle)}+(\langle\varphi_1,a\varphi_1\rangle-\langle\varphi_2,a\varphi_2\rangle)\,\overline{\langle\varphi_2,a\varphi_2\rangle}\right| \\ &\leqslant (|\langle\varphi_1,a\varphi_1\rangle|+|\langle\varphi_2,a\varphi_2\rangle|)\,(\|a\varphi_1\|+\|a^*\varphi_2\|)\,\|\varphi_1-\varphi_2\|.\end{aligned} \tag{21}$$

This gives

$$\begin{aligned}&\|\langle\varphi_1,a\varphi_1\rangle a^*\varphi_1-\langle\varphi_2,a\varphi_2\rangle a^*\varphi_2\| \\ &= \|\langle\varphi_1,a\varphi_1\rangle a^*\,(\varphi_1-\varphi_2)+(\langle\varphi_1,a\varphi_1\rangle-\langle\varphi_2,a\varphi_2\rangle)\,a^*\varphi_2\| \\ &\leqslant |\langle\varphi_1,a\varphi_1\rangle|\|a^*\,(\varphi_1-\varphi_2)\,\|+(\|a\varphi_1\|+\|a^*\varphi_2\|)\,\|a^*\varphi_2\|\|\varphi_1-\varphi_2\|\end{aligned} \tag{22}$$

and similarly for the other term

$$\begin{aligned}\|\overline{\langle\varphi_1,a\varphi_1\rangle}a\varphi_1-\overline{\langle\varphi_2,a\varphi_2\rangle}a\varphi_2\| \leqslant{}& |\langle\varphi_1,a\varphi_1\rangle|\|a\,(\varphi_1-\varphi_2)\,\| \\ &+(\|a\varphi_1\|+\|a^*\varphi_2\|)\,\|a\varphi_2\|\|\varphi_1-\varphi_2\|.\end{aligned} \tag{23}$$

Moreover, we have

$$\begin{aligned}&\||\langle\varphi_1,a\varphi_1\rangle|^2\varphi_1-|\langle\varphi_2,a\varphi_2\rangle|^2\varphi_2\| \\ &= \||\langle\varphi_1,a\varphi_1\rangle|^2\,(\varphi_1-\varphi_2)+\left(|\langle\varphi_1,a\varphi_1\rangle|^2-|\langle\varphi_2,a\varphi_2\rangle|^2\right)\varphi_2\| \\ &\leqslant |\langle\varphi_1,a\varphi_1\rangle|^2\|\varphi_1-\varphi_2\|+||\langle\varphi_1,a\varphi_1\rangle|^2-|\langle\varphi_2,a\varphi_2\rangle|^2|\|\varphi_2\| \\ &\leqslant \left(|\langle\varphi_1,a\varphi_1\rangle|^2+\|\varphi_2\|\,(|\langle\varphi_1,a\varphi_1\rangle|+|\langle\varphi_2,a\varphi_2\rangle|)\,(\|a\varphi_1\|+\|a^*\varphi_2\|)\right)\|\varphi_1-\varphi_2\|.\end{aligned} \tag{24}$$

So, we get

$$\begin{aligned}\|v(\varphi_1)-v(\varphi_2)\| \leqslant{}& C_1\|\varphi_1-\varphi_2\|+C_2\,(\|a(\varphi_1-\varphi_2)\|+\|a^*(\varphi_1-\varphi_2)\|) \\ &+|J-\mu|\|\mathcal{N}(\varphi_1-\varphi_2)\|+\frac{|U|}{2}\|\mathcal{N}(\mathcal{N}-1)(\varphi_1-\varphi_2)\|\end{aligned} \tag{25}$$

where we have introduced $C_1, C_2>0$

$$\begin{aligned}C_1 :={}& |J|\,(\|a\varphi_1\|+\|a^*\varphi_2\|)\,(\|a^*\varphi_2\|+\|a\varphi_2\|) \\ &+(|\langle\varphi_1,a\varphi_1\rangle|+|\langle\varphi_2,a\varphi_2\rangle|)\,(\|a\varphi_1\|+\|a^*\varphi_2\|)\end{aligned} \tag{26}$$

$$C_2 := |\langle\varphi_1,a\varphi_1\rangle|. \tag{27}$$

□

In our case, our vector field satisfies some sort of Lipschitz condition! Here both terms $f(\mathcal{N})$ and the remaining part of mean-field Hamiltonian are unbounded operators. And, we want to have a map Γ which maps the space X to the same space X!
Second approach: We want to prove existence of solution to the mean-field equation

$$i\partial_t p = [h^{\alpha_\varphi}, p] \tag{28}$$

with $p=|\varphi\rangle\langle\varphi|$ where φ will be the solution to the mean-field dynamic (17). We can also apply the fixed point argument

$$S := \left\{p \in \mathcal{L}^1(\ell^2(\mathbb{C}));\ \sqrt{f(\mathcal{N})}p\sqrt{f(\mathcal{N})} \in \mathcal{L}^1(\ell^2(\mathbb{C}))\right\},$$

endowed with the following norm

$$\|p\|_S := \|\sqrt{f(\mathcal{N})}p\sqrt{f(\mathcal{N})}\|_{\mathcal{L}^1}$$

Now, using $\tilde{p}(t) := e^{itf(\mathcal{N})}p(t)e^{-itf(\mathcal{N})}$, then any solution to (28) satisfies

$$p(t) = e^{-itf(\mathcal{N})}p(0)e^{itf(\mathcal{N})} - i\int_0^t e^{-i(t-s)f(\mathcal{N})}[h_0^{\alpha_\varphi}, p(s)]e^{i(t-s)f(\mathcal{N})}ds$$

where $h_0^{\alpha_\varphi} = -J\left(\mathrm{Tr}(pa)a^* + \mathrm{Tr}(pa^*)a - |\mathrm{Tr}(pa)|^2\right)$. Then, define the space $Y := \mathcal{C}([0,T], S)$ endowed with the following norm

$$\|p\| := \sup_{[0,T]} \|p(t)\|_S$$

and the map $\Gamma : Y \to Y$ as follows

$$\Gamma(p)(t) := e^{-itf(\mathcal{N})}p(0)e^{itf(\mathcal{N})} - i\int_0^t e^{-i(t-s)f(\mathcal{N})}[h_0^{\alpha_\varphi}, p(s)]e^{i(t-s)f(\mathcal{N})}ds$$

We want to apply the fixed point argument version in infinite dimensional space to this map Γ.
Third approach: Our Mean-field dynamics could be written as

$$\frac{d\varphi}{dt} = A\varphi + K(\varphi)$$

where in our case A is unbounded skew-adjoint linear operator and $K(\varphi)$ is as well unbounded skew-adjoint but nonlinear operator

$$\begin{aligned}
A\varphi &:= -i(J-\mu)\mathcal{N}\varphi - i\frac{U}{2}\mathcal{N}(\mathcal{N}-1)\varphi \\
K(\varphi) &:= iJ\left(\langle\varphi, a\varphi\rangle a^*\varphi + \langle\varphi, a^*\varphi\rangle a\varphi - |\langle\varphi, a\varphi\rangle|^2\varphi\right)
\end{aligned}$$

We note that $\mathcal{D}(\mathcal{A}) = \mathcal{D}(\mathcal{N}^2)$. We have the following results which is proved in the work of Sigal.

Theorem 8 (I. Sigal [27])**.** *Let $(B, \|\cdot\|)$ be a Banach space and let A be an operator in B generating a continuous one-Parameter semi-group. Assume that $\mathcal{D}(\mathcal{A})$ is dense in B and $(\mathcal{D}(\mathcal{A}), \|A\cdot\|)$ is complete. Let $K : (\mathcal{D}(\mathcal{A}), \|A\cdot\|) \to B$ is differential where $\frac{\partial k(u)}{\partial u}v$ is a Lipschitz function of u and v where $u \in (\mathcal{D}(\mathcal{A}), \|A\cdot\|)$ and $v \in (\mathcal{D}(\mathcal{A}), \|\cdot\|)$. Let $u_0 \in \mathcal{D}(A)$, then the differential equation*

$$\frac{d\varphi}{dt} = A\varphi + K(\varphi), \quad \varphi(0) = \varphi_0$$

has a continuous solution $\varphi : [0,\epsilon] \to (\mathcal{D}(\mathcal{A}), \|A\cdot\|)$.

We have $\mathcal{D}(\mathcal{A}) = \mathcal{D}(\mathcal{N}^2)$ which is dense in $B = \ell^2(\mathbb{C})$. Moreover in our case, we have $K : (\mathcal{D}(\mathcal{N}^2), \|\mathcal{N}^2\cdot\|) \to (B, \|\cdot\|)$ is defined and Lipschitz function.

2.3 Conservation Laws

We now prove that the mean-field particle number and the mean-field energy are conserved.

Proposition 9 (Conservation laws for the mean-field dynamics)**.** *Let φ solves (5) with $\varphi(0) \in l^2(\mathbb{C})$.*

- *If $\langle\varphi(0), \mathcal{N}\varphi(0)\rangle < \infty$, then $\langle\varphi, \mathcal{N}\varphi\rangle$ is finite and conserved.*
- *If $\left\langle\varphi(0), \mathcal{N}^2\varphi(0)\right\rangle < \infty$, then $\langle\varphi, h^\varphi\varphi\rangle$ is finite and conserved.*

- *Let $k \in \mathbb{N}/2, k \geqslant 1$ and $t \in \mathbb{R}_+$. If $p_\varphi(0)\mathcal{N}^k \in \mathcal{L}^1(l^2(\mathbb{C}))$,*

$$\mathrm{Tr}\left(p_\varphi(t)\mathcal{N}^k\right) \leqslant \left(\mathrm{Tr}\left(p(0)\mathcal{N}^k\right) + e^{-1}k^k\right) e^{2eJk\mathrm{Tr}(p_\varphi(0)\mathcal{N})^{\frac{1}{2}}t} \tag{29}$$

$$\mathrm{Tr}\left(p_\varphi(t)\mathcal{N}^k\right) \leqslant \sum_{l=0}^{2(k-1)} \binom{2k}{l} \left(J\mathrm{Tr}\left(p_\varphi(0)\mathcal{N}\right)^{\frac{1}{2}} t\right)^l \mathrm{Tr}\left(p_\varphi(0)\left(\mathcal{N}+l\right)^{k-\frac{l}{2}}\right). \tag{30}$$

Proof. **Preliminaries**

Let $n \in \mathbb{N}$ and $\mathcal{A} \in \mathcal{L}^1\left(l^2(\mathbb{C})\right)$ be the positive operator such that

$$\mathcal{A}^{k-1} := (\mathcal{N}+n+1)^k - (\mathcal{N}+n)^k \leqslant k\left(\mathcal{N}+n+1\right)^{k-1}. \tag{31}$$

Noticing that $a\left(\mathcal{N}+n\right)^k = \left(\mathcal{N}+n+1\right)^k a$,

$$\left\langle \varphi, \left[a, (\mathcal{N}+n)^k\right]\varphi\right\rangle = \left\langle \varphi, \mathcal{A}^{k-1}a\varphi\right\rangle = \left\langle \mathcal{A}^{\frac{k}{2}-\frac{1}{4}}\varphi, \mathcal{A}^{\frac{k}{2}-\frac{3}{4}}a\varphi\right\rangle$$

so with Cauchy-Schwarz's inequality,

$$\begin{aligned}
\left|\left\langle \varphi, \left[a, (\mathcal{N}+n)^k\right]\varphi\right\rangle\right| &\leqslant \left\langle \varphi, \mathcal{A}^{k-\frac{1}{2}}\varphi\right\rangle^{\frac{1}{2}} \left\langle \varphi, a^*\mathcal{A}^{k-\frac{3}{2}}a\varphi\right\rangle^{\frac{1}{2}} \\
&\leqslant k\left\langle \varphi, (\mathcal{N}+n+1)^{k-\frac{1}{2}}\varphi\right\rangle^{\frac{1}{2}} \left\langle \varphi, a^*\left(\mathcal{N}+n+1\right)^{k-\frac{3}{2}}a\varphi\right\rangle^{\frac{1}{2}} \\
&= k\left\langle \varphi, (\mathcal{N}+n+1)^{k-\frac{1}{2}}\varphi\right\rangle^{\frac{1}{2}} \left\langle \varphi, (\mathcal{N}+n)^{k-\frac{3}{2}}\mathcal{N}\varphi\right\rangle^{\frac{1}{2}} \\
&\leqslant k\left\langle \varphi, (\mathcal{N}+n+1)^{k-\frac{1}{2}}\varphi\right\rangle
\end{aligned}$$

Recalling (5),

$$\begin{aligned}
i\partial_t\left\langle \varphi, (\mathcal{N}+n)^k\varphi\right\rangle &= \left\langle \varphi, \left[(\mathcal{N}+n)^k, h^\varphi\right]\varphi\right\rangle = -J\left\langle \varphi, \left[(\mathcal{N}+n)^k, \alpha_\varphi a^* + \overline{\alpha_\varphi}a\right]\varphi\right\rangle \\
&= J\alpha_\varphi\left\langle \varphi, \left[a^*, (\mathcal{N}+n)^k\right]\varphi\right\rangle + J\overline{\alpha_\varphi}\left\langle \varphi, \left[a, (\mathcal{N}+n)^k\right]\varphi\right\rangle \\
&= 2iJ\ \mathrm{Im}\left[\overline{\alpha_\varphi}\left\langle \varphi, \left[a, (\mathcal{N}+n)^k\right]\varphi\right\rangle\right].
\end{aligned}$$

Hence with (16),

$$\left|\partial_t\left\langle \varphi, (\mathcal{N}+n)^k\varphi\right\rangle\right| \leqslant 2Jk\left\langle \varphi, \mathcal{N}\varphi\right\rangle^{\frac{1}{2}}\left\langle \varphi, (\mathcal{N}+n+1)^{k-\frac{1}{2}}\varphi\right\rangle. \tag{32}$$

Conservation of the mean-field number of particles

Taking $k = 1$ and $n = 0$ in (32) we obtain

$$\left|\partial_t\left\langle \varphi, \mathcal{N}\varphi\right\rangle\right| \leqslant 2J\left\langle \varphi, \mathcal{N}\varphi\right\rangle^{\frac{1}{2}}\left\langle \varphi, \sqrt{\mathcal{N}+1}\varphi\right\rangle \leqslant 2J\left(\left\langle \varphi, \mathcal{N}\varphi\right\rangle + \left\langle \varphi, \mathcal{N}\varphi\right\rangle^{\frac{1}{2}}\right) \leqslant 4J\left\langle \varphi, \mathcal{N}\varphi\right\rangle + \frac{J}{2}$$

so by Gronwall lemma, $\forall t \in \mathbb{R}_+$,

$$\left\langle \varphi(t), \mathcal{N}\varphi(t)\right\rangle \leqslant \left(\left\langle \varphi(0), \mathcal{N}\varphi(0)\right\rangle + \frac{1}{8}\right)e^{4Jt} - \frac{1}{8}.$$

Recalling (16), we conclude that α_φ is bounded for all time and therefore

$$\begin{aligned}
i\partial_t\left\langle \varphi, \mathcal{N}\varphi\right\rangle = \left\langle \varphi, \left[\mathcal{N}, h^\varphi\right]\varphi\right\rangle &= -J\left(\alpha_\varphi\left\langle \varphi, \left[\mathcal{N}, a^*\right]\varphi\right\rangle + \overline{\alpha_\varphi}\left\langle \varphi, \left[\mathcal{N}, a\right]\varphi\right\rangle\right) \\
&= -J\left(\alpha_\varphi\left\langle \varphi, a^*\varphi\right\rangle - \overline{\alpha_\varphi}\left\langle \varphi, a\varphi\right\rangle\right) = -J\left(\left|\alpha_\varphi\right|^2 - \left|\alpha_\varphi\right|^2\right) = 0.
\end{aligned}$$

Proof of (30)
By induction on k, we prove that, $\left\langle \varphi(0), \mathcal{N}^k \varphi(0) \right\rangle < \infty \implies \forall n \in \mathbb{N}$,

$$\left\langle \varphi(t), (\mathcal{N}+n)^k \varphi(t) \right\rangle \leqslant \sum_{l=0}^{2(k-1)} \binom{2k}{l} \left(J \left\langle \varphi(0), \mathcal{N}\varphi(0) \right\rangle^{\frac{1}{2}} t \right)^l \left\langle \varphi(0), (\mathcal{N}+n+l)^{k-\frac{l}{2}} \varphi(0) \right\rangle . \tag{33}$$

The inequality is indeed true for $k = 1$ since $\langle \varphi, (\mathcal{N}+n)\, \varphi \rangle$ is conserved. Assume (33) holds and

$$\left\langle \varphi(0), \mathcal{N}^{k+\frac{1}{2}} \varphi(0) \right\rangle < \infty.$$

Let $n \in \mathbb{N}$, using the conservation of $\langle \varphi, \mathcal{N}\varphi \rangle$, with $k + \frac{1}{2}$ instead of k, (32) becomes

$$\left| \partial_t \left\langle \varphi, (\mathcal{N}+n)^{k+\frac{1}{2}} \varphi \right\rangle \right| \leqslant J(2k+1) \left\langle \varphi(0), \mathcal{N}\varphi(0) \right\rangle^{\frac{1}{2}} \left\langle \varphi, (\mathcal{N}+n+1)^k \varphi \right\rangle .$$

Integrating over time and inserting (33),

$$\begin{aligned}
&\left\langle \varphi(t), (\mathcal{N}+n)^{k+\frac{1}{2}} \varphi(t) \right\rangle \\
&\leqslant \left\langle \varphi(0), (\mathcal{N}+n)^{k+\frac{1}{2}} \varphi(0) \right\rangle + J(2k+1) \left\langle \varphi(0), \mathcal{N}\varphi(0) \right\rangle^{\frac{1}{2}} \int_0^t \left\langle \varphi(\tau), (\mathcal{N}+n+1)^k \varphi(\tau) \right\rangle d\tau \\
&= \left\langle \varphi(0), (\mathcal{N}+n)^{k+\frac{1}{2}} \varphi(0) \right\rangle + J(2k+1) \left\langle \varphi(0), \mathcal{N}\varphi(0) \right\rangle^{\frac{1}{2}} \\
&\cdot \sum_{l=0}^{2(k-1)} \binom{2k}{l} \left(J \left\langle \varphi(0), \mathcal{N}\varphi(0) \right\rangle^{\frac{1}{2}} \right)^l \left\langle \varphi(0), (\mathcal{N}+n+l+1)^{k-\frac{l}{2}} \varphi(0) \right\rangle \int_0^t \tau^l d\tau \\
&= \left\langle \varphi(0), (\mathcal{N}+n)^{k+\frac{1}{2}} \varphi(0) \right\rangle \\
&+ \sum_{l=0}^{2(k-1)} \binom{2k+1}{l+1} \left(J \left\langle \varphi(0), \mathcal{N}\varphi(0) \right\rangle^{\frac{1}{2}} \tau \right)^{l+1} \left\langle \varphi(0), (\mathcal{N}+n+l+1)^{k-\frac{l}{2}} \varphi(0) \right\rangle \\
&= \left\langle \varphi(0), (\mathcal{N}+n)^{k+\frac{1}{2}} \varphi(0) \right\rangle \\
&+ \sum_{l=1}^{2(k-1)+1} \binom{2k+1}{l} \left(J \left\langle \varphi(0), \mathcal{N}\varphi(0) \right\rangle^{\frac{1}{2}} \tau \right)^{l} \left\langle \varphi(0), (\mathcal{N}+n+l)^{k+\frac{1}{2}-\frac{l}{2}} \varphi(0) \right\rangle \\
&\leqslant \sum_{l=0}^{2\left(k+\frac{1}{2}-1\right)} \binom{2\left(k+\frac{1}{2}\right)}{l} \left(J \left\langle \varphi(0), \mathcal{N}\varphi(0) \right\rangle^{\frac{1}{2}} t \right)^{l} \left\langle \varphi(0), (\mathcal{N}+n+l)^{k+\frac{1}{2}-\frac{l}{2}} \varphi(0) \right\rangle .
\end{aligned}$$

which concludes the induction. Noticing that

$$\left\| p_\varphi(0)\mathcal{N}^k \right\|_{\mathcal{L}^1} = \left\| \mathcal{N}^k p_\varphi(0) \right\|_{\mathcal{L}^1} = \left\langle \varphi(0), \mathcal{N}^{2k} \varphi(0) \right\rangle < \infty,$$

we infer that

$$\forall l \in [1, k]\,, \left\| p_\varphi(0)\mathcal{N}^l \right\|_{\mathcal{L}^1} = \left\langle \varphi(0), \mathcal{N}^{2l} \varphi(0) \right\rangle < \infty.$$

Therefore, taking $2k$ instead of k in (33) we see that $p_\varphi(t)\mathcal{N}^k \in \mathcal{L}^1(l^2(\mathbb{C}))$. We conclude (30) holds from (33) with $n = 0$.
Proof of (29)
Since

$$N \geqslant 1 \implies (N+1)^k = N^k e^{k \ln\left(1+\frac{1}{N}\right)} \leqslant N^k e^{\frac{k}{N}}$$

we notice that

$$N \geqslant k \implies (N+1)^k \leqslant e N^k$$

Introducing a cutoff in (32),

$$\begin{aligned}\left|\partial_t \mathrm{Tr}\left(p_\varphi(t)\mathcal{N}^k\right)\right| &\leqslant 2Jk\mathrm{Tr}\left(p_\varphi(0)\mathcal{N}\right)^{\frac{1}{2}} \mathrm{Tr}\left(p_\varphi(t)\left(\mathcal{N}+1\right)^{k-\frac{1}{2}}\right) \leqslant 2Jk\mathrm{Tr}\left(p_\varphi(0)\mathcal{N}\right)^{\frac{1}{2}} \mathrm{Tr}\left(p_\varphi(t)\left(\mathcal{N}+1\right)^{k}\right) \\ &= 2Jk\mathrm{Tr}\left(p_\varphi(0)\mathcal{N}\right)^{\frac{1}{2}} \mathrm{Tr}\left(p_\varphi(t)\left(\mathcal{N}+1\right)^{k}\left(\mathbb{1}_{\mathcal{N}<k} + \mathbb{1}_{\mathcal{N}\geqslant k}\right)\right) \qquad (34) \\ &\leqslant 2Jk\mathrm{Tr}\left(p_\varphi(0)\mathcal{N}\right)^{\frac{1}{2}}\left(\mathrm{Tr}\left(p_\varphi(t)k^k\right) + e\mathrm{Tr}\left(p_\varphi(t)\mathcal{N}^k\right)\right) \\ &= 2Jk\mathrm{Tr}\left(p_\varphi(0)\mathcal{N}\right)^{\frac{1}{2}}\left(k^k + e\mathrm{Tr}\left(p_\varphi(t)\mathcal{N}^k\right)\right). \qquad (35)\end{aligned}$$

With Gronwall lemma we conclude that

$$\begin{aligned}\mathrm{Tr}\left(p(t)\mathcal{N}^k\right) &\leqslant \left(\mathrm{Tr}\left(p(0)\mathcal{N}^k\right) + e^{-1}k^k\right) e^{2eJk\mathrm{Tr}(p_\varphi(0)\mathcal{N})^{\frac{1}{2}}t} - e^{-1}k^k \\ &\leqslant \left(\mathrm{Tr}\left(p(0)\mathcal{N}^k\right) + e^{-1}k^k\right) e^{2eJk\mathrm{Tr}(p_\varphi(0)\mathcal{N})^{\frac{1}{2}}t}.\end{aligned}$$

Conservation of the mean-field energy
Using $\left[\mathcal{N}^2, a\right] = -2\mathcal{N}a - a$, we compute

$$\begin{aligned}i\partial_t \alpha_\varphi = \langle \varphi, [h^\varphi, a]\,\varphi\rangle &= \left\langle \varphi, \left(-J\alpha_\varphi\,[a^*, a] + \left(J - \mu - \frac{U}{2}\right)[\mathcal{N}, a] + \frac{U}{2}\left[\mathcal{N}^2, a\right]\right)\varphi\right\rangle \\ &= \left\langle \varphi, \left(J\alpha_\varphi + \left(-J + \mu + \frac{U}{2}\right)a + \frac{U}{2}\left(-2\mathcal{N}a - a\right)\right)\varphi\right\rangle = \mu\alpha_\varphi - U\langle\varphi, \mathcal{N}a\varphi\rangle\end{aligned}$$

so

$$\begin{aligned}|\partial_t\alpha_\varphi| &\leqslant |\mu|\,\langle\varphi,\mathcal{N}\varphi\rangle^{\frac{1}{2}} + U\left\langle\varphi,\mathcal{N}^{\frac{3}{2}}\varphi\right\rangle\left\langle\varphi, a^*\mathcal{N}^{\frac{1}{2}}a\varphi\right\rangle \\ &= |\mu|\,\langle\varphi,\mathcal{N}\varphi\rangle^{\frac{1}{2}} + U\left\langle\varphi,\mathcal{N}^{\frac{3}{2}}\varphi\right\rangle^{\frac{1}{2}}\left\langle\varphi,(\mathcal{N}-1)^{\frac{1}{2}}\mathcal{N}\varphi\right\rangle^{\frac{1}{2}} \leqslant |\mu|\,\langle\varphi,\mathcal{N}\varphi\rangle^{\frac{1}{2}} + U\left\langle\varphi,\mathcal{N}^{\frac{3}{2}}\varphi\right\rangle\end{aligned}$$

and (33) for $k = \frac{3}{2}$ implies that $\partial_t\alpha_\varphi$ stays bounded, so

$$\begin{aligned}i\partial_t\langle\varphi, h^\varphi\varphi\rangle = \langle\varphi, \partial_t h^\varphi\varphi\rangle &= -J\left\langle\varphi, \left(\partial_t\alpha_\varphi a^* + \partial_t\overline{\alpha_\varphi}a - \overline{\alpha_\varphi}\partial_t\alpha_\varphi - \alpha_\varphi\partial_t\overline{\alpha_\varphi}\right)\varphi\right\rangle \\ &= -J\left(\overline{\alpha_\varphi}\partial_t\alpha_\varphi + \alpha_\varphi\partial_t\overline{\alpha_\varphi} - \overline{\alpha_\varphi}\partial_t\alpha_\varphi - \alpha_\varphi\partial_t\overline{\alpha_\varphi}\right) = 0.\end{aligned}$$

□

Proposition 10 (Conservation laws of the Bose–Hubbard dynamics)**.** *Let γ_d follows* (3) *with $\gamma_d(0)$ a d-site density matrix.*

Then, $\forall t \in \mathbb{R}, \gamma_d(t)$ is a d-site density matrix. Moreover, if

$$\mathrm{Tr}\left(\gamma_{\mathrm{d}}^{(1)}(0)\mathcal{N}^4\right) < \infty \qquad (36)$$

then $\mathrm{Tr}\left(\gamma_d H_d\right)$ and $\mathrm{Tr}\left(\gamma_{\mathrm{d}}^{(1)}\mathcal{N}\right)$ are finite and conserved and

$$i\partial_t\,\gamma_{\mathrm{d}}^{(1)} = \left[h^\varphi, \gamma_{\mathrm{d}}^{(1)}\right] - J\mathrm{Tr}_2\left(\left[\left(a^* - \overline{\alpha}\right)\otimes\left(a - \alpha\right) + \left(a - \alpha\right)\otimes\left(a^* - \overline{\alpha}\right), \gamma_{\mathrm{d}}^{(2)}\right]\right). \qquad (37)$$

Let $k \in \mathbb{N}/2, k \geqslant 1$ and $t \in \mathbb{R}_+$, if $\gamma_{\mathrm{d}}^{(1)}(0)\mathcal{N}^k \in \mathcal{L}^1(l^2(\mathbb{C}))$ then

$$\mathrm{Tr}\left(\gamma_{\mathrm{d}}^{(1)}(t)\mathcal{N}^k\right) \leqslant \left(\mathrm{Tr}\left(\gamma_{\mathrm{d}}^{(1)}(0)\mathcal{N}^k\right) + e^{-1}k^k\right)e^{2eJkt}. \qquad (38)$$

Lemma 11. *If γ_d be a d-site density matrix. Let $A \in \mathcal{L}(l^2(\mathbb{C}))$ self adjoint such that $A \geqslant 0$ or $\gamma_{\mathrm{d}}^{(1)} A \in \mathcal{L}^1(l^2(\mathbb{C}))$, then*

$$\left[\sum_{x\in\Lambda} A_x, \gamma_d\right]^{(1)} = \left[A, \gamma_{\mathrm{d}}^{(1)}\right]. \tag{39}$$

Let $B \in \mathcal{L}(l^2(\mathbb{C})^{\otimes 2})$ self adjoint such that $B \geqslant 0$ or $\gamma_{\mathrm{d}}^{(2)} A \in \mathcal{L}^1(l^2(\mathbb{C})^{\otimes 2})$, then

$$\frac{1}{2d}\left[\sum_{\substack{x,y\in\Lambda\\ x\sim y}} B_{x,y}, \gamma_d\right]^{(1)} = \mathrm{Tr}_1\left(\left[B, \gamma_{\mathrm{d}}^{(2)}\right]\right) + \mathrm{Tr}_2\left(\left[B, \gamma_{\mathrm{d}}^{(2)}\right]\right) \tag{40}$$

Proof. Everything is well defined since REF ?

$$\left\|\gamma_d \sum_{x\in\Lambda} A_x\right\|_{\mathcal{L}^1} = |\Lambda| \left\|\gamma_d^{(1)} A\right\|_{\mathcal{L}^1}$$
$$\left\|\gamma_d \sum_{\substack{x,y\in\Lambda\\ x\sim y}} B_{x,y}\right\|_{\mathcal{L}^1} = 2d\,|\Lambda| \left\|\gamma_d^{(2)} B\right\|_{\mathcal{L}^1}$$

Let $K \in \mathcal{K}(l^2(\mathbb{C}))$,

$$\begin{aligned}\mathrm{Tr}\left(\left[\sum_{x\in\Lambda} A_x, \gamma_d\right]^{(1)} K\right) &= \frac{1}{|\Lambda|}\sum_{z\in\Lambda}\mathrm{Tr}\left(\left[\sum_{x\in\Lambda} A_x, \gamma_d\right] K_z\right) = \frac{1}{|\Lambda|}\sum_{z\in\Lambda}\mathrm{Tr}\left(\left[K_z, \sum_{x\in\Lambda} A_x\right]\gamma_d\right)\\ &= \frac{1}{|\Lambda|}\sum_{x\in\Lambda}\mathrm{Tr}\left([K,A]_x\,\gamma_d\right) = \mathrm{Tr}\left([K,A]\,\gamma_{\mathrm{d}}^{(1)}\right) = \mathrm{Tr}\left(\left[A, \gamma_{\mathrm{d}}^{(1)}\right] K\right).\end{aligned}$$

And similarly,

$$\begin{aligned}\frac{1}{2d}\mathrm{Tr}\left(\left[\sum_{\substack{x,y\in\Lambda\\ x\sim y}} B_{x,y}, \gamma_d\right]^{(1)} K\right) &= \frac{1}{2d\,|\Lambda|}\sum_{z\in\Lambda}\mathrm{Tr}\left(\left[\sum_{\substack{x,y\in\Lambda\\ x\sim y}} B_{x,y}, \gamma_d\right] K_z\right)\\ &= \frac{1}{2d\,|\Lambda|}\sum_{z\in\Lambda}\mathrm{Tr}\left(\left[K_z, \sum_{\substack{x,y\in\Lambda\\ x\sim y}} B_{x,y}\right]\gamma_d\right)\\ &= \frac{1}{2d\,|\Lambda|}\sum_{\substack{x,y\in\Lambda\\ x\sim y}}\mathrm{Tr}\left(\left[K_x + K_y, \frac{B_{x,y}+B_{y,x}}{2}\right]\gamma_d\right)\\ &= \mathrm{Tr}\left(\left[K_1 + K_2, \frac{B_{1,2}+B_{2,1}}{2}\right]\gamma_{\mathrm{d}}^{(2)}\right) = \mathrm{Tr}\left(\left[B, \gamma_{\mathrm{d}}^{(2)}\right](K_1+K_2)\right)\\ &= \mathrm{Tr}\left(\left(\mathrm{Tr}_2\left(\left[B, \gamma_{\mathrm{d}}^{(2)}\right]\right) + \mathrm{Tr}_1\left(\left[B, \gamma_{\mathrm{d}}^{(2)}\right]\right)\right) K\right).\end{aligned}$$

Since $\mathcal{L}^1(l^2(\mathbb{C})) = \mathcal{K}(l^2(\mathbb{C}))^*$ we deduce (39) and (40). □

Proof. of Proposition 10

The fact that γ_d remains a d-site density matrix follows from the unitary character of the dynamics.

H_d is self-adjoint, so $H_d^2 \geqslant 0$ and $\mathrm{Tr}\left(\gamma_d(0)H_d^2\right)$ is well defined. This quantity can be infinite, however, under the assumption (36) we see with Proposition (6) that it is finite. Since H_d does not depend on time, $\mathrm{Tr}\left(\gamma_d H_d\right)$ is well defined, finite and conserved since. Similarly, the same conclusions hold for

$$\mathrm{Tr}\left(\gamma_{\mathrm{d}}^{(1)}\mathcal{N}\right)=\frac{\mathrm{Tr}\left(\gamma_d\mathcal{N}_{\mathcal{F}}\right)}{|\Lambda|}$$

noticing that $[H_d,\mathcal{N}_{\mathcal{F}}]=0$.

Proof of (37)

Inserting (9) then using (39) and (40),

$$\begin{aligned}
i\partial_t\,\gamma_{\mathrm{d}}^{(1)}=[H,\gamma_d]^{(1)}&=\left[\sum_{x\in\Lambda}h_x^{\varphi},\gamma_d\right]^{(1)}-\frac{J}{2d}\left[\sum_{\substack{x,y\in\Lambda\\ x\sim y}}\left(a_x^*-\overline{\alpha}\right)\left(a_y-\alpha\right),\gamma_d\right]^{(1)}\\
&=\left[h^{\varphi},\gamma_{\mathrm{d}}^{(1)}\right]-J\left(\mathrm{Tr}_1\left(\left[\left(a^*-\overline{\alpha}\right)\otimes\left(a-\alpha\right),\gamma_{\mathrm{d}}^{(2)}\right]\right)+\mathrm{Tr}_2\left(\left[\left(a^*-\overline{\alpha}\right)\otimes\left(a-\alpha\right),\gamma_{\mathrm{d}}^{(2)}\right]\right)\right)\\
&=\left[h^{\varphi},\gamma_{\mathrm{d}}^{(1)}\right]-J\mathrm{Tr}_2\left(\left[\left(a^*-\overline{\alpha}\right)\otimes\left(a-\alpha\right)+\left(a-\alpha\right)\otimes\left(a^*-\overline{\alpha}\right),\gamma_{\mathrm{d}}^{(2)}\right]\right).
\end{aligned}$$

Controlling $\mathrm{Tr}\left(\gamma_{\mathrm{d}}^{(1)}\mathcal{N}^k\right)$

Starting from (37),

$$\begin{aligned}
i\partial_t\mathrm{Tr}\left(\gamma_{\mathrm{d}}^{(1)}\mathcal{N}^k\right)&=\mathrm{Tr}\left(\left[h^{\mathrm{mf},\varphi},\gamma_{\mathrm{d}}^{(1)}\right]\mathcal{N}^k\right)-J\mathrm{Tr}\left(\left[\left(a^*-\overline{\alpha}\right)\otimes\left(a-\alpha\right)+\left(a-\alpha\right)\otimes\left(a^*-\overline{\alpha}\right),\gamma_{\mathrm{d}}^{(2)}\right]\mathcal{N}_1^k\right)\\
&=\mathrm{Tr}\left(\gamma_{\mathrm{d}}^{(1)}\left[\mathcal{N}^k,h^{\mathrm{mf},\varphi}\right]\right)+J\mathrm{Tr}\left(\gamma_{\mathrm{d}}^{(2)}\left[\left(a^*-\overline{\alpha}\right)\otimes\left(a-\alpha\right)+\left(a-\alpha\right)\otimes\left(a^*-\overline{\alpha}\right),\mathcal{N}_1^k\right]\right)\\
&=J\mathrm{Tr}\left(\gamma_{\mathrm{d}}^{(1)}\left[\alpha a^*+\overline{\alpha}a,\mathcal{N}^k\right]\right)+J\mathrm{Tr}\left(\gamma_{\mathrm{d}}^{(2)}\left[a^*\otimes\left(a-\alpha\right)+a\otimes\left(a^*-\overline{\alpha}\right),\mathcal{N}_1^k\right]\right)\\
&=J\mathrm{Tr}\left(\gamma_{\mathrm{d}}^{(2)}\left[a^*\otimes a+a\otimes a^*,\mathcal{N}_1^k\right]\right)=2iJ\mathrm{Im}\left[\mathrm{Tr}\left(\gamma_{\mathrm{d}}^{(2)}\left[a_1,\mathcal{N}_1^k\right]a_2\right)\right]
\end{aligned}$$

Using (31) again and the Cauchy-Schwarz inequality,

$$\begin{aligned}
\left|\partial_t\mathrm{Tr}\left(\gamma_{\mathrm{d}}^{(1)}\mathcal{N}^k\right)\right|&\leqslant 2J\left|\mathrm{Tr}\left(\gamma_{\mathrm{d}}^{(2)}\left[a_1,\mathcal{N}_1^k\right]a_2\right)\right|=2J\left|\mathrm{Tr}\left(\gamma_{\mathrm{d}}^{(2)}a_2\mathcal{A}_1^{k-1}a_1\right)\right|\\
&\leqslant 2J\mathrm{Tr}\left(\gamma_{\mathrm{d}}^{(2)}a_2\mathcal{A}_1^{k-1}a_2^*\right)^{\frac{1}{2}}\mathrm{Tr}\left(\gamma_{\mathrm{d}}^{(2)}a_1^*\mathcal{A}_1^{k-1}a_1\right)^{\frac{1}{2}}\\
&\leqslant 2Jk\mathrm{Tr}\left(\gamma_{\mathrm{d}}^{(2)}\left(\mathcal{N}_1+1\right)^{k-1}\left(\mathcal{N}_2+1\right)\right)^{\frac{1}{2}}\mathrm{Tr}\left(\gamma_{\mathrm{d}}^{(1)}a^*\left(\mathcal{N}+1\right)^{k-1}a\right)^{\frac{1}{2}}\\
&=2Jk\mathrm{Tr}\left(\gamma_{\mathrm{d}}^{(2)}\left(\mathcal{N}_1+1\right)^{k-1}\left(\mathcal{N}_2+1\right)\right)^{\frac{1}{2}}\mathrm{Tr}\left(\gamma_{\mathrm{d}}^{(1)}\mathcal{N}^k\right)^{\frac{1}{2}}
\end{aligned}$$

Since $\left[\left(\mathcal{N}_1+1\right)^{k-1},\left(\mathcal{N}_2+1\right)\right]=0$, by Young's inequality,

$$\left(\mathcal{N}_1+1\right)^{k-1}\left(\mathcal{N}_2+1\right)\leqslant\left(1-\frac{1}{k}\right)\left(\mathcal{N}_1+1\right)^k+\frac{1}{k}\left(\mathcal{N}_2+1\right)^k$$

Thus with the same cutoff as (34) we conclude that

$$\begin{aligned}
\left|\partial_t\mathrm{Tr}\left(\gamma_{\mathrm{d}}^{(1)}\mathcal{N}^k\right)\right|&\leqslant 2Jk\left(\left(1-\frac{1}{k}\right)\mathrm{Tr}\left(\gamma_{\mathrm{d}}^{(2)}\left(\mathcal{N}_1+1\right)^k\right)+\frac{1}{k}\mathrm{Tr}\left(\gamma_{\mathrm{d}}^{(2)}\left(\mathcal{N}_2+1\right)^k\right)\right)^{\frac{1}{2}}\mathrm{Tr}\left(\gamma_{\mathrm{d}}^{(1)}\mathcal{N}^k\right)^{\frac{1}{2}}\\
&=2Jk\mathrm{Tr}\left(\gamma_{\mathrm{d}}^{(1)}\left(\mathcal{N}+1\right)^k\right)^{\frac{1}{2}}\mathrm{Tr}\left(\gamma_{\mathrm{d}}^{(1)}\mathcal{N}^k\right)^{\frac{1}{2}}\\
&\leqslant 2Jk\mathrm{Tr}\left(\gamma_{\mathrm{d}}^{(1)}\mathcal{N}^k\right)^{\frac{1}{2}}\left(k^k+e\mathrm{Tr}\left(\gamma_{\mathrm{d}}^{(1)}\mathcal{N}^k\right)\right)^{\frac{1}{2}}\\
&\leqslant 2Jk\left(k^k+e\mathrm{Tr}\left(\gamma_{\mathrm{d}}^{(1)}\mathcal{N}^k\right)\right).
\end{aligned}$$

Small improvement on the constant via

$$\mathrm{Tr}\left(\gamma_{\mathrm{d}}^{(1)}\mathcal{N}^k\right)^{\frac{1}{2}}\left(k^k+e\mathrm{Tr}\left(\gamma_{\mathrm{d}}^{(1)}\mathcal{N}^k\right)\right)^{\frac{1}{2}}\leqslant \mathrm{Tr}\left(\gamma_{\mathrm{d}}^{(1)}\mathcal{N}^k\right)^{\frac{1}{2}}k^{\frac{k}{2}}+\sqrt{e}\mathrm{Tr}\left(\gamma_{\mathrm{d}}^{(1)}\mathcal{N}^k\right)$$
$$\leqslant k^k+\left(\frac{1}{4}+\sqrt{e}\right)\mathrm{Tr}\left(\gamma_{\mathrm{d}}^{(1)}\mathcal{N}^k\right)$$

and $\left(\frac{1}{4}+\sqrt{e}\right)<e$. With Gronwall lemma we conclude that

$$\mathrm{Tr}\left(\gamma_{\mathrm{d}}^{(1)}(t)\mathcal{N}^k\right)\leqslant\left(\mathrm{Tr}\left(\gamma_{\mathrm{d}}^{(1)}(0)\mathcal{N}^k\right)+e^{-1}k^k\right)e^{2eJkt}-e^{-1}k^k\leqslant\left(\mathrm{Tr}\left(\gamma_{\mathrm{d}}^{(1)}(0)\mathcal{N}^k\right)+e^{-1}k^k\right)e^{2eJkt}.$$

□

2.4 Gronwall estimate

The following quantity is appropriate from Gronwall estimates.

Proposition 12. *Let p be a rank one projection and γ a positive trace 1 operator on $l^2(\mathbb{C})$ and $q \coloneqq 1-p$. Then,*

$$2\mathrm{Tr}\left(\gamma q\right)\leqslant\|\gamma-p\|_{\mathcal{L}^1}\leqslant 2\sqrt{2}\sqrt{\mathrm{Tr}\left(\gamma q\right)} \tag{41}$$
$$\mathrm{Tr}\left(\gamma q\right)\leqslant\|\gamma-p\|_{\mathcal{L}^2}\leqslant \sqrt{2}\sqrt{\mathrm{Tr}\left(\gamma q\right)}. \tag{42}$$

Proof. In order to get the upper bound in (41), we first notice that since $\gamma\leqslant 1$ and $\mathrm{Tr}\left(\gamma\right)=\mathrm{Tr}\left(p\right)=1$,

$$\|p\gamma p-p\|_{\mathcal{L}^1}=\mathrm{Tr}\left(\left(1-\gamma\right)p\right)=1-\mathrm{Tr}\left(\gamma p\right)=\mathrm{Tr}\left(\gamma(1-p)\right)=\mathrm{Tr}\left(\gamma q\right)$$

so

$$\begin{aligned}\|\gamma-p\|_{\mathcal{L}^1}&=\|(p+q)\gamma(p+q)-p\|_{\mathcal{L}^1}\leqslant 2\mathrm{Tr}\left(\gamma q\right)+2\left\|q\gamma p\right\|_{\mathcal{L}^1}\\&\leqslant 2\mathrm{Tr}\left(\gamma q\right)+2\sqrt{\mathrm{Tr}\left(\gamma q\right)}\sqrt{\mathrm{Tr}\left(\gamma p\right)}\\&=2\sqrt{\mathrm{Tr}\left(\gamma q\right)}\left(\sqrt{\mathrm{Tr}\left(\gamma q\right)}+\sqrt{1-\mathrm{Tr}\left(\gamma q\right)}\right)\leqslant 2\sqrt{2}\sqrt{\mathrm{Tr}\left(\gamma q\right)}\end{aligned}$$

using $0\leqslant x\leqslant 1 \implies \sqrt{x}+\sqrt{1-x}\leqslant\sqrt{2}$. Then using

$$\mathrm{Tr}\left(\left(p-\gamma\right)\left(p-q\right)\right)=1-\mathrm{Tr}\left(\gamma p\right)+\mathrm{Tr}\left(\gamma q\right)=2\mathrm{Tr}\left(\gamma q\right)$$

and the fact that p is a rank 1 projection,

$$2\mathrm{Tr}\left(\gamma q\right)\leqslant\|\gamma-p\|_{\mathcal{L}^1}\|2p-1\|_{\mathcal{L}^\infty}=\|\gamma-p\|_{\mathcal{L}^1}$$

we get the lower bound. The the upper bound of (42) is given by

$$\|\gamma-p\|_{\mathcal{L}^2}=\mathrm{Tr}\left((\gamma-p)^2\right)\leqslant 2-2\mathrm{Tr}\left(\gamma p\right)=2\mathrm{Tr}\left(\gamma q\right)$$

and finally, the lower bound is derived through

$$\mathrm{Tr}\left(\gamma q\right)=1-\mathrm{Tr}\left(\gamma p\right)=\mathrm{Tr}\left(\left(p-\gamma\right)p\right)\leqslant\|\gamma-p\|_{\mathcal{L}^2}.$$

□

Proposition 13 (Gronwall estimate tentative)**.** *Let γ_d follows (3) with $\gamma_d(0)$ a d-site density matrix and φ solves (5) with $\varphi(0)\in l^2(\mathbb{C})$. We denote $p\coloneqq|\varphi\rangle\langle\varphi|$ and $q\coloneqq 1-p$. Then,*

$$\begin{aligned}&\left|\partial_t\mathrm{Tr}\left(\gamma_{\mathrm{d}}^{(1)}q\right)\right|\\&\leqslant J\left(\mathrm{Tr}\left(p\mathcal{N}\right)+1\right)^{\frac{1}{2}}\left(8\mathrm{Tr}\left(p\mathcal{N}\right)^{\frac{1}{2}}\mathrm{Tr}\left(\gamma_{\mathrm{d}}^{(1)}q\right)+4\mathrm{Tr}\left(\gamma_{\mathrm{d}}^{(1)}q\right)^{\frac{1}{2}}\mathrm{Tr}\left(\gamma_{\mathrm{d}}^{(1)}q\left(\mathcal{N}+1\right)q\right)^{\frac{1}{2}}+\frac{\mathrm{Tr}\left(p\mathcal{N}\right)^{\frac{1}{2}}}{d}\right)\end{aligned}$$

Proof. **Decomposition**

Introduce the following self-adjoint operator

$$A \coloneqq (a^* - \overline{\alpha_\varphi}) \otimes (a - \alpha_\varphi) + (a - \alpha_\varphi) \otimes (a^* - \overline{\alpha_\varphi})\,.$$

With 37, we start by computing

$$\begin{aligned} i\partial_t \mathrm{Tr}\left(\gamma_{\mathrm{d}}^{(1)}\, q\right) &= \mathrm{Tr}\left(\left[h^\varphi, \gamma_{\mathrm{d}}^{(1)}\right] q\right) - J\mathrm{Tr}\left(\left[A, \gamma_{\mathrm{d}}^{(2)}\right] q_1\right) + \mathrm{Tr}\left(\gamma_{\mathrm{d}}^{(1)}\,[h^\varphi, q]\right) = J\mathrm{Tr}\left(\gamma_{\mathrm{d}}^{(2)}\,[A, q_1]\right) \\ &= 2iJ\mathrm{Im}\left[\mathrm{Tr}\left(\gamma_{\mathrm{d}}^{(2)}\, A q_1\right)\right] \end{aligned} \tag{43}$$

Inserting resolution of identities,

$$\begin{aligned} &\mathrm{Tr}\left(\gamma_{\mathrm{d}}^{(2)}\, A q_1\right) \\ &= \mathrm{Tr}\left(\gamma_{\mathrm{d}}^{(2)}\, p_1p_2Aq_1p_2\right) + \mathrm{Tr}\left(\gamma_{\mathrm{d}}^{(2)}\, p_1p_2Aq_1q_2\right) + \mathrm{Tr}\left(\gamma_{\mathrm{d}}^{(2)}\, p_1q_2Aq_1p_2\right) + \mathrm{Tr}\left(\gamma_{\mathrm{d}}^{(2)}\, p_1q_2Aq_1q_2\right) \\ &+ \mathrm{Tr}\left(\gamma_{\mathrm{d}}^{(2)}\, q_1p_2Aq_1p_2\right) + \mathrm{Tr}\left(\gamma_{\mathrm{d}}^{(2)}\, q_1p_2Aq_1q_2\right) + \mathrm{Tr}\left(\gamma_{\mathrm{d}}^{(2)}\, q_1q_2Aq_1p_2\right) + \mathrm{Tr}\left(\gamma_{\mathrm{d}}^{(2)}\, q_1q_2Aq_1q_2\right) \end{aligned}$$

$q_1p_2Aq_1p_2$ and $q_1q_2Aq_1q_2$ are self adjoint and do not contribute to 43. This is also the case for $q_1p_2Aq_1q_2$ and $q_1q_2Aq_1p_2$ which are conjugated. $p_1p_2Aq_1p_2 = 0$ by definition of A. Then by symmetry, we see that the $p_1q_2Aq_1p_2$ is also not contributing:

$$\mathrm{Tr}\left(\gamma_{\mathrm{d}}^{(2)}\, p_1q_2Aq_1p_2\right) = \mathrm{Tr}\left(\gamma_{\mathrm{d}}^{(2)}\, q_1p_2Ap_1q_2\right) = \overline{\mathrm{Tr}\left(\gamma_{\mathrm{d}}^{(2)}\, p_1q_2Aq_1p_2\right)}.$$

Thus, we are left with

$$i\partial_t \mathrm{Tr}\left(\gamma_{\mathrm{d}}^{(1)}\, q\right) = 2iJ\mathrm{Im}\left[\mathrm{Tr}\left(\gamma_{\mathrm{d}}^{(2)}\, p_1p_2Aq_1q_2\right)\right] + 2iJ\mathrm{Im}\left[\mathrm{Tr}\left(\gamma_{\mathrm{d}}^{(2)}\, p_1q_2Aq_1q_2\right)\right]. \tag{44}$$

Estimation of the $p_1p_2Aq_1q_2$ term

Since $pq = 0$,

$$\mathrm{Tr}\left(\gamma_{\mathrm{d}}^{(2)}\, p_1p_2Aq_1q_2\right) = \mathrm{Tr}\left(\gamma_{\mathrm{d}}^{(2)}\, p_1p_2(a_1^*a_2 + a_1a_2^*)q_1q_2\right)$$

and by symmetry of $\gamma_{\mathrm{d}}^{(2)}$

$$\mathrm{Tr}\left(\gamma_{\mathrm{d}}^{(2)}\, p_1p_2a_1^*a_2q_1q_2\right) = \mathrm{Tr}\left(\gamma_{\mathrm{d}}^{(2)}\, p_1p_2a_1a_2^*q_1q_2\right).$$

Thus, we estimate with Cauchy-Schwarz's inequality

$$\begin{aligned} \left|\mathrm{Tr}\left(\gamma_{\mathrm{d}}^{(2)}\, p_1p_2Aq_1q_2\right)\right| &= 2\left|\mathrm{Tr}\left(\gamma_{\mathrm{d}}^{(2)}\, p_1p_2a_1^*a_2q_1q_2\right)\right| = \frac{1}{d\,|\Lambda|}\left|\sum_{\substack{x,y\in\Lambda \\ x\sim y}} \mathrm{Tr}\left(\gamma_d p_xp_ya_x^*a_yq_xq_y\right)\right| \\ &\leqslant \frac{1}{d\,|\Lambda|}\sum_{x\in\Lambda}\left|\mathrm{Tr}\left(q_x\gamma_d^{\frac{1}{2}}\cdot\gamma_d^{\frac{1}{2}}\sum_{y\in\Lambda|x\sim y} p_xp_ya_x^*a_yq_y\right)\right| \\ &\leqslant \frac{1}{2d\epsilon\,|\Lambda|}\sum_{x\in\Lambda}\mathrm{Tr}\left(q_x\gamma_d\right) + \frac{\epsilon}{2d\,|\Lambda|}\sum_{x\in\Lambda}\mathrm{Tr}\left(\gamma_d\sum_{\substack{y\in\Lambda|x\sim y \\ z\in\Lambda|x\sim z}} p_xp_ya_x^*a_yq_yq_za_z^*a_xp_zp_x\right) \\ &= \frac{1}{2d\epsilon}\mathrm{Tr}\left(\gamma_{\mathrm{d}}^{(1)}\, q\right) + \epsilon\frac{\mathrm{Tr}\left(p\mathcal{N}\right)}{2d\,|\Lambda|}\sum_{\substack{x,y\in\Lambda \\ x\sim y}}\sum_{z\in\Lambda|x\sim z}\mathrm{Tr}\left(\gamma_dp_xp_ya_yq_yq_za_z^*p_z\right) \end{aligned}$$

$$= \frac{1}{2d\epsilon}\mathrm{Tr}\left(\gamma_{\mathrm{d}}^{(1)}\, q\right) + \epsilon\frac{\mathrm{Tr}\,(p\mathcal{N})}{2d\,|\Lambda|}\sum_{\substack{x,y\in\Lambda\\ x\sim y}}\mathrm{Tr}\left(\gamma_d p_x p_y a_y q_y q_z a_y^* p_y\right)$$

$$+ \epsilon\frac{\mathrm{Tr}\,(p\mathcal{N})}{2d\,|\Lambda|}\sum_{\substack{x,y\in\Lambda\\ x\sim y}}\sum_{\substack{z\in\Lambda|x\sim z\\ z\neq y}}\mathrm{Tr}\left(q_y\gamma_d q_z p_x p_y a_y a_z^* p_z\right)$$

Inserting

$$\sum_{\substack{x,y\in\Lambda\\ x\sim y}}\mathrm{Tr}\left(\gamma_d p_x p_y a_y q_y a_y^* p_y\right) \leqslant \sum_{\substack{x,y\in\Lambda\\ x\sim y}}\mathrm{Tr}\left(\gamma_d p_x p_y a_y a_y^* p_y\right) = \mathrm{Tr}\,(p(\mathcal{N}+1))\sum_{\substack{x,y\in\Lambda\\ x\sim y}}\mathrm{Tr}\left(\gamma_d p_x p_y\right)$$
$$\leqslant 2d\,|\Lambda|\left(\mathrm{Tr}\,(p\mathcal{N})+1\right)$$

along with

$$\begin{aligned}\sum_{\substack{x,y\in\Lambda\\ x\sim y}}\sum_{\substack{z\in\Lambda|x\sim z\\ z\neq y}}\mathrm{Tr}\left(q_y\gamma_d q_z p_x p_y a_y a_z^* p_z\right) &\leqslant \sum_{\substack{x,y\in\Lambda\\ x\sim y}}\sum_{\substack{z\in\Lambda|x\sim z\\ z\neq y}}\mathrm{Tr}\left(\gamma_d q_z p_x p_y a_y a_y^* p_y\right)^{\frac{1}{2}}\mathrm{Tr}\left(\gamma_d q_y p_z a_z a_z^* p_z\right)^{\frac{1}{2}}\\ &= \mathrm{Tr}\,(p(\mathcal{N}+1))\sum_{\substack{x,y\in\Lambda\\ x\sim y}}\sum_{\substack{z\in\Lambda|x\sim z\\ z\neq y}}\mathrm{Tr}\left(\gamma_d q_z p_x p_y\right)^{\frac{1}{2}}\mathrm{Tr}\left(\gamma_d q_y p_z\right)^{\frac{1}{2}}\\ &\leqslant \left(\mathrm{Tr}\,(p\mathcal{N})+1\right)\sum_{\substack{x,y\in\Lambda\\ x\sim y}}\sum_{z\in\Lambda|x\sim z}\mathrm{Tr}\left(\gamma_d q_z\right)^{\frac{1}{2}}\mathrm{Tr}\left(\gamma_d q_y\right)^{\frac{1}{2}}\\ &= \left(\mathrm{Tr}\,(p\mathcal{N})+1\right)\sum_{x\in\Lambda}\left(\sum_{y\in\Lambda|x\sim y}\mathrm{Tr}\left(\gamma_d q_y\right)^{\frac{1}{2}}\right)^2\\ &\leqslant 2d\left(\mathrm{Tr}\,(p\mathcal{N})+1\right)\sum_{\substack{x,y\in\Lambda\\ x\sim y}}\mathrm{Tr}\left(\gamma_d q_y\right)\\ &= 4d^2\,|\Lambda|\left(\mathrm{Tr}\,(p\mathcal{N})+1\right)\mathrm{Tr}\left(\gamma_{\mathrm{d}}^{(1)}\, q\right)\end{aligned}$$

and choosing $\epsilon^{-1} \coloneqq 2d\mathrm{Tr}\,(p\mathcal{N})^{\frac{1}{2}}\left(\mathrm{Tr}\,(p\mathcal{N})+1\right)^{\frac{1}{2}}$, we obtain

$$\begin{aligned}\left|\mathrm{Tr}\left(\gamma_{\mathrm{d}}^{(2)}\, p_1p_2Aq_1q_2\right)\right| &\leqslant \frac{1}{2d\epsilon}\mathrm{Tr}\left(\gamma_{\mathrm{d}}^{(1)}\, q\right) + \epsilon\mathrm{Tr}\,(p\mathcal{N})\left(\mathrm{Tr}\,(p\mathcal{N})+1\right)\\ &\quad + 2d\epsilon\mathrm{Tr}\,(p\mathcal{N})\left(\mathrm{Tr}\,(p\mathcal{N})+1\right)\mathrm{Tr}\left(\gamma_{\mathrm{d}}^{(1)}\, q\right)\\ &= \mathrm{Tr}\,(p\mathcal{N})^{\frac{1}{2}}\left(\mathrm{Tr}\,(p\mathcal{N})+1\right)^{\frac{1}{2}}\left(2\mathrm{Tr}\left(\gamma_{\mathrm{d}}^{(1)}\, q\right)+\frac{1}{2d}\right).\end{aligned} \tag{45}$$

Estimation of the $p_1q_2Aq_1q_2$ term
Since $pq=0$,

$$\mathrm{Tr}\left(\gamma_{\mathrm{d}}^{(2)}\, p_1q_2Aq_1q_2\right) = \mathrm{Tr}\left(\gamma_{\mathrm{d}}^{(2)}\, p_1q_2a_1^*a_2q_1q_2\right) + \mathrm{Tr}\left(\gamma_{\mathrm{d}}^{(2)}\, p_1q_2a_1a_2^*q_1q_2\right) - \alpha_\varphi\mathrm{Tr}\left(\gamma_{\mathrm{d}}^{(2)}\, p_1q_2a_1^*q_1q_2\right)$$
$$- \overline{\alpha_\varphi}\mathrm{Tr}\left(\gamma_{\mathrm{d}}^{(2)}\, p_1q_2a_1q_1q_2\right)$$

Inserting

$$\begin{aligned}\left|\mathrm{Tr}\left(\gamma_{\mathrm{d}}^{(2)}\, p_1q_2a_1^*a_2q_1q_2\right)\right| &\leqslant \mathrm{Tr}\left(\gamma_{\mathrm{d}}^{(2)}\, p_1q_2\mathcal{N}_1p_1\right)^{\frac{1}{2}}\mathrm{Tr}\left(\gamma_{\mathrm{d}}^{(2)}\, q_2q_1a_2a_2^*q_2\right)^{\frac{1}{2}}\\ &= \mathrm{Tr}\,(p\mathcal{N})^{\frac{1}{2}}\,\mathrm{Tr}\left(\gamma_{\mathrm{d}}^{(2)}\, p_1q_2\right)^{\frac{1}{2}}\mathrm{Tr}\left(\gamma_{\mathrm{d}}^{(2)}\, q_1q_2\left(\mathcal{N}_2+1\right)q_2\right)^{\frac{1}{2}}\end{aligned}$$

$$\leqslant \operatorname{Tr}(p\mathcal{N})^{\frac{1}{2}} \operatorname{Tr}\left(\gamma_{\mathrm{d}}^{(1)}\, q\right)^{\frac{1}{2}} \operatorname{Tr}\left(\gamma_{\mathrm{d}}^{(1)}\, q\,(\mathcal{N}+1)\, q\right)^{\frac{1}{2}}$$

$$\begin{aligned}
\left|\operatorname{Tr}\left(\gamma_{\mathrm{d}}^{(2)}\, p_1 q_2 a_1^* q_1 q_2\right)\right| &\leqslant \operatorname{Tr}\left(\gamma_{\mathrm{d}}^{(2)}\, p_1 q_2 \mathcal{N}_1 p_1\right)^{\frac{1}{2}} \operatorname{Tr}\left(\gamma_{\mathrm{d}}^{(2)}\, q_1 q_2\right)^{\frac{1}{2}} \\
&= \operatorname{Tr}(p\mathcal{N})^{\frac{1}{2}} \operatorname{Tr}\left(\gamma_{\mathrm{d}}^{(2)}\, p_1 q_2\right)^{\frac{1}{2}} \operatorname{Tr}\left(\gamma_{\mathrm{d}}^{(2)}\, q_1 q_2\right)^{\frac{1}{2}} \leqslant \operatorname{Tr}(p\mathcal{N})^{\frac{1}{2}} \operatorname{Tr}\left(\gamma_{\mathrm{d}}^{(1)}\, q\right)
\end{aligned}$$

and similarly

$$\begin{aligned}
\left|\operatorname{Tr}\left(\gamma_{\mathrm{d}}^{(2)}\, p_1 q_2 a_1 a_2^* q_1 q_2\right)\right| &\leqslant \left(\operatorname{Tr}(p\mathcal{N})+1\right)^{\frac{1}{2}} \operatorname{Tr}\left(\gamma_{\mathrm{d}}^{(1)}\, q\right)^{\frac{1}{2}} \operatorname{Tr}\left(\gamma_{\mathrm{d}}^{(1)}\, q\mathcal{N}q\right)^{\frac{1}{2}} \\
\left|\operatorname{Tr}\left(\gamma_{\mathrm{d}}^{(2)}\, p_1 q_2 a_1 q_1 q_2\right)\right| &\leqslant \left(\operatorname{Tr}(p\mathcal{N})+1\right)^{\frac{1}{2}} \operatorname{Tr}\left(\gamma_{\mathrm{d}}^{(1)}\, q\right)
\end{aligned}$$

we get

$$\begin{aligned}
&\left|\operatorname{Tr}\left(\gamma_{\mathrm{d}}^{(2)}\, p_1 q_2 A q_1 q_2\right)\right| \qquad (46)\\
&\leqslant 2\left(\operatorname{Tr}(p\mathcal{N})+1\right)^{\frac{1}{2}} \left(|\alpha_\varphi| \operatorname{Tr}\left(\gamma_{\mathrm{d}}^{(1)}\, q\right) + \operatorname{Tr}\left(\gamma_{\mathrm{d}}^{(1)}\, q\right)^{\frac{1}{2}} \operatorname{Tr}\left(\gamma_{\mathrm{d}}^{(1)}\, q\,(\mathcal{N}+1)\, q\right)^{\frac{1}{2}}\right) \\
&\leqslant 2\left(\operatorname{Tr}(p\mathcal{N})+1\right)^{\frac{1}{2}} \left(\operatorname{Tr}(p\mathcal{N})^{\frac{1}{2}} \operatorname{Tr}\left(\gamma_{\mathrm{d}}^{(1)}\, q\right) + \operatorname{Tr}\left(\gamma_{\mathrm{d}}^{(1)}\, q\right)^{\frac{1}{2}} \operatorname{Tr}\left(\gamma_{\mathrm{d}}^{(1)}\, q\,(\mathcal{N}+1)\, q\right)^{\frac{1}{2}}\right). \qquad (47)
\end{aligned}$$

Conclusion

Inserting (45) and (47) in (44) we obtain

$$\begin{aligned}
&\left|\partial_t \operatorname{Tr}\left(\gamma_{\mathrm{d}}^{(1)}\, q\right)\right| \\
&\leqslant 2J\operatorname{Tr}(p\mathcal{N})^{\frac{1}{2}} \left(\operatorname{Tr}(p\mathcal{N})+1\right)^{\frac{1}{2}} \left(2\operatorname{Tr}\left(\gamma_{\mathrm{d}}^{(1)}\, q\right) + \frac{1}{2d}\right) \\
&+ 4J\left(\operatorname{Tr}(p\mathcal{N})+1\right)^{\frac{1}{2}} \left(\operatorname{Tr}(p\mathcal{N})^{\frac{1}{2}} \operatorname{Tr}\left(\gamma_{\mathrm{d}}^{(1)}\, q\right) + \operatorname{Tr}\left(\gamma_{\mathrm{d}}^{(1)}\, q\right)^{\frac{1}{2}} \operatorname{Tr}\left(\gamma_{\mathrm{d}}^{(1)}\, q\,(\mathcal{N}+1)\, q\right)^{\frac{1}{2}}\right) \\
&= J\left(\operatorname{Tr}(p\mathcal{N})+1\right)^{\frac{1}{2}} \left(8\operatorname{Tr}(p\mathcal{N})^{\frac{1}{2}} \operatorname{Tr}\left(\gamma_{\mathrm{d}}^{(1)}\, q\right) + 4\operatorname{Tr}\left(\gamma_{\mathrm{d}}^{(1)}\, q\right)^{\frac{1}{2}} \operatorname{Tr}\left(\gamma_{\mathrm{d}}^{(1)}\, q\,(\mathcal{N}+1)\, q\right)^{\frac{1}{2}} + \frac{\operatorname{Tr}(p\mathcal{N})^{\frac{1}{2}}}{d}\right)
\end{aligned}$$

□

3 Proof using moment estimates

The main idea to close the Gronwall estimate we started in Proposition 13 is use the followings estimates obtained by iterating the Cauchy-Schwarz inequality along with the moments bounds we obtained in Section 2.3.

Lemma 14. *Let $k \in \mathbb{N}$ and $\gamma, p \in \mathcal{L}^1(l^2(\mathbb{C}))$. We assume that $0 \leqslant \gamma \leqslant 1$, p is a rank one projection and $p\mathcal{N}^k, \gamma\mathcal{N}^k \in \mathcal{L}^1(l^2(\mathbb{C}))$. We denote $q := 1 - p$, then*

$$\operatorname{Tr}\left(\gamma q\left(\mathcal{N}+1\right)q\right) \leqslant \operatorname{Tr}\left(\gamma q\right)^{1-2^{-k}} \operatorname{Tr}\left(\gamma q\left(\mathcal{N}+1\right)^{2^k} q\right)^{2^{-k}} \tag{48}$$

$$\operatorname{Tr}\left(\gamma q\mathcal{N}^k q\right) \leqslant 2\operatorname{Tr}\left(\gamma\mathcal{N}^k\right) + 2\operatorname{Tr}\left(p\mathcal{N}^k\right) \tag{49}$$

Proof. **Proof of** (48)

We proceed by induction on k. The inequality is trivial for $k = 0$. With the Cauchy-Schwarz inequality,

$$\operatorname{Tr}\left(\gamma q\left(\mathcal{N}+1\right)^{2^k} q\right) \leqslant \operatorname{Tr}\left(\gamma q\right)^{\frac{1}{2}} \operatorname{Tr}\left(\gamma q\left(\mathcal{N}+1\right)^{2^{k+1}} q\right)^{\frac{1}{2}}$$

so assuming the result we prove

$$\begin{aligned}
\operatorname{Tr}\left(\gamma q\left(\mathcal{N}+1\right)q\right) &\leqslant \operatorname{Tr}\left(\gamma q\right)^{1-2^{-k}} \operatorname{Tr}\left(\gamma q\left(\mathcal{N}+1\right)^{2^k} q\right)^{2^{-k}} \\
&\leqslant \operatorname{Tr}\left(\gamma q\right)^{1-2^{-k}+2^{-(k+1)}} \operatorname{Tr}\left(\gamma q\left(\mathcal{N}+1\right)^{2^{k+1}} q\right)^{2^{-(k+1)}} \\
&= \operatorname{Tr}\left(\gamma q\right)^{1-2^{-(k+1)}} \operatorname{Tr}\left(\gamma q\left(\mathcal{N}+1\right)^{2^{k+1}} q\right)^{2^{-(k+1)}}.
\end{aligned}$$

Proof of (49)

With the Cauchy-Schwarz inequality,

$$\begin{aligned}
\operatorname{Tr}\left(\gamma q\mathcal{N}^k q\right) &= \operatorname{Tr}\left(\gamma\mathcal{N}^k\right) - \operatorname{Tr}\left(\gamma p\mathcal{N}^k p\right) - \operatorname{Tr}\left(\gamma p\mathcal{N}^k q\right) - \operatorname{Tr}\left(\gamma q\mathcal{N}^k p\right) \\
&= \operatorname{Tr}\left(\gamma\mathcal{N}^k\right) - \operatorname{Tr}\left(\gamma p\mathcal{N}^k p\right) + 2\sqrt{\operatorname{Tr}\left(\gamma p\mathcal{N}^k p\right)}\sqrt{\operatorname{Tr}\left(\gamma q\mathcal{N}^k q\right)} \\
&\leqslant \operatorname{Tr}\left(\gamma\mathcal{N}^k\right) + \operatorname{Tr}\left(\gamma p\mathcal{N}^k p\right) + \frac{1}{2}\operatorname{Tr}\left(\gamma q\mathcal{N}^k q\right)
\end{aligned}$$

so

$$\operatorname{Tr}\left(\gamma q\mathcal{N}^k q\right) \leqslant 2\operatorname{Tr}\left(\gamma\mathcal{N}^k\right) + 2\operatorname{Tr}\left(\gamma p\mathcal{N}^k p\right) \leqslant 2\operatorname{Tr}\left(\gamma\mathcal{N}^k\right) + 2\operatorname{Tr}\left(p\mathcal{N}^k\right)$$

□

3.1 Proof using moment method

Lemma 15 (Decay of large particle number probability)**.** *Let $(u_n)_{n\in\mathbb{N}} \subset \mathbb{R}_+$,*

$$\exists a > 0 | \forall n \in \mathbb{N}, u_n \leqslant e^{-\frac{n}{a}} \implies \forall k \in \mathbb{N}, \sum_{n\in\mathbb{N}} n^k u_n \leqslant (1+a)a^k k! \tag{50}$$

and conversely,

$$\exists b > 0 | \forall k \in \mathbb{N}, \sum_{n\in\mathbb{N}} n^k u_n \leqslant b^k k! \implies \forall M \in \mathbb{N}, \sum_{n=M}^{\infty} (n+1)u_n \leqslant (2+4b)e^{-\frac{M}{2b}} \tag{51}$$

Proof. **Proof of** (50)

The function

$$f_a : \begin{array}{ccc} \mathbb{R}_+ & \rightarrow & \mathbb{R}_+ \\ x & \mapsto & x^k e^{-\frac{x}{a}} \end{array}$$

is increasing up to ak and decreasing afterwards. Thus by series-integral comparison,

$$\begin{aligned}\sum_{n\in\mathbb{N}} f_a(n) &\leqslant \int_{\mathbb{R}_+} f_a(x)dx + f_a\left(\lfloor ak\rfloor\right) + f_a\left(\lceil ak\rceil\right) = a^k\left(ak! + f_1\left(a^{-1}\lfloor ak\rfloor\right) + f_1\left(a^{-1}\lceil ak\rceil\right)\right)\\ &\leqslant a^k\left(ak! + 2f_1(k)\right) = a^k\left(ak! + 2\left(\frac{k}{e}\right)^k\right).\end{aligned}$$

If $k \geqslant 1$, inserting the Stirling lower approximation,

$$\sqrt{2\pi k}\left(\frac{k}{e}\right)^k \leqslant k! \tag{52}$$

we get

$$\sum_{n\in\mathbb{N}} n^k e^{-\frac{n}{a}} \leqslant a^k k!\left(a + \sqrt{\frac{2}{\pi k}}\right) \leqslant (1+a)\, a^k k!.$$

This also holds for $k = 0$ since

$$\sum_{n\in\mathbb{N}} e^{-\frac{n}{a}} \leqslant 1 + \int_{\mathbb{R}_+} e^{-\frac{x}{a}} dx = 1 + a.$$

Proof of (51)

Let $0 < a < \frac{1}{b}, M \in \mathbb{N}$,

$$\begin{aligned}\sum_{n=M}^{\infty}(n+1)u_n e^{aM} &\leqslant \sum_{n\in\mathbb{N}}(n+1)u_n e^{an} = \sum_{n,k\in\mathbb{N}}(n+1)\frac{(an)^k}{k!}u_n = \sum_{k\in\mathbb{N}}\frac{a^k}{k!}\left(\sum_{n\in\mathbb{N}} n^{k+1}u_n + \sum_{n\in\mathbb{N}} n^k u_n\right)\\ &\leqslant \sum_{k\in\mathbb{N}}\frac{a^k}{k!}\left(b^{k+1}(k+1)! + b^k k!\right) = \sum_{k\in\mathbb{N}}\left(b(k+1)(ab)^k + (ab)^k\right) = \frac{b}{(1-ab)^2} + \frac{1}{1-ab}\end{aligned}$$

Choosing $a \coloneqq \frac{1}{2b}$,

$$\sum_{n=M}^{\infty}(n+1)u_n \leqslant \frac{1-ab+b}{(1-ab)^2}e^{-aM} = (2+4b)e^{-\frac{M}{2b}}$$

□

Proof. Proof of 1

Controlling $\mathrm{Tr}\left(\gamma_{\mathrm{d}}^{(1)}\, q\left(\mathcal{N}+1\right)\mathbb{1}_{\mathcal{N}\geqslant M} q\right)$ **with moments**

Let us denote $p \coloneqq p_\varphi$ and $q \coloneqq q_\varphi$. Let $k \in \mathbb{N}$, applying (50) to $u_n \coloneqq \mathrm{Tr}\left(p(0)\mathbb{1}_{\mathcal{N}=n}\right)$ and then $u_n \coloneqq \mathrm{Tr}\left(\gamma_d^{(1)}(0)\mathbb{1}_{\mathcal{N}=n}\right)$ while using the assumption (6), we obtain that

$$\mathrm{Tr}\left(p(0)\mathcal{N}^k\right) \leqslant c(1+a)a^k k!$$
$$\mathrm{Tr}\left(p(0)\mathcal{N}^k\right) \leqslant c(1+a)a^k k!.$$

With (49), (29) (38) and (52), if $k \geqslant 1$,

$$
\begin{aligned}
\sum_{n\in\mathbb{N}} n^k \mathrm{Tr}\left(\gamma_{\mathrm{d}}^{(1)}(t) q \mathbb{1}_{\mathcal{N}=n} q\right) &= \mathrm{Tr}\left(\gamma_{\mathrm{d}}^{(1)}(t) q(t) \mathcal{N}^k q(t)\right) \leqslant 2\mathrm{Tr}\left(\gamma_{\mathrm{d}}^{(1)}(t)\mathcal{N}^k\right) + 2\mathrm{Tr}\left(p(t)\mathcal{N}^k\right) \\
&\leqslant 2\left(\mathrm{Tr}\left(p(0)\mathcal{N}^k\right) + e^{-1}k^k\right) e^{2eJk\mathrm{Tr}(p(0)\mathcal{N})^{\frac{1}{2}} t} \\
&+ 2\left(\mathrm{Tr}\left(\gamma_{\mathrm{d}}^{(1)}(0)\mathcal{N}^k\right) + e^{-1}k^k\right) e^{2eJkt} \\
&\leqslant 2\left(\mathrm{Tr}\left(p(0)\mathcal{N}^k\right) + \mathrm{Tr}\left(\gamma_{\mathrm{d}}^{(1)}(0)\mathcal{N}^k\right) + 2e^{-1}k^k\right) e^{C_1 kt} \\
&\leqslant 4\left(c(1+a)a^k k! + e^{-1}k^k\right) e^{C_1 kt} \leqslant 4\left(c(1+a)a^k + \frac{e^{k-1}}{\sqrt{2\pi k}}\right) k! e^{C_1 kt} \\
&\leqslant 4\left(c(1+a) + e^{-1}\right)\left(a^k + e^k\right) k! e^{C_1 kt} \\
&\leqslant 4\left(c(1+a) + e^{-1}\right)\left((a+e)e^{C_1 t}\right)^k k!.
\end{aligned}
$$

This is also valid for $k = 0$, so by (51),

$$
\begin{aligned}
\mathrm{Tr}\left(\gamma_{\mathrm{d}}^{(1)}(t) q(t) (\mathcal{N}+1) \mathbb{1}_{\mathcal{N}\geqslant M} q(t)\right) &= \sum_{n=M}^{\infty} (n+1)\mathrm{Tr}\left(\gamma_{\mathrm{d}}^{(1)}(t) q(t) \mathbb{1}_{\mathcal{N}=n} q(t)\right) \\
&\leqslant 4\left(c(1+a)+e^{-1}\right)\left(2 + 4(a+e)e^{C_1 t}\right) e^{-\frac{M}{2(a+e)}e^{-C_1 t}} \\
&\leqslant C_2 e^{C_1 t - \frac{M}{2(a+e)}e^{-C_1 t}}.
\end{aligned} \tag{53}
$$

Conclusion

Let $M \in \mathbb{N}^*$, using proposition 13 while introducing a cutoff on $\mathcal{N}$ and then Proposition 9

$$
\begin{aligned}
&\left|\partial_t \mathrm{Tr}\left(\gamma_{\mathrm{d}}^{(1)} q\right)\right| \\
&\leqslant JC_3 \left(8\mathrm{Tr}\left(p\mathcal{N}\right)^{\frac{1}{2}} \mathrm{Tr}\left(\gamma_{\mathrm{d}}^{(1)} q\right) + 4\mathrm{Tr}\left(\gamma_{\mathrm{d}}^{(1)} q\right)^{\frac{1}{2}} \mathrm{Tr}\left(\gamma_{\mathrm{d}}^{(1)} q (\mathcal{N}+1)\left(\mathbb{1}_{\mathcal{N}<M} + \mathbb{1}_{\mathcal{N}\geqslant M}\right) q\right)^{\frac{1}{2}}\right. \\
&\left. + \frac{\mathrm{Tr}\left(p\mathcal{N}\right)^{\frac{1}{2}}}{d}\right) \\
&\leqslant JC_3 \left(\left(8\mathrm{Tr}\left(p\mathcal{N}\right)^{\frac{1}{2}} + 4\sqrt{M}\right) \mathrm{Tr}\left(\gamma_{\mathrm{d}}^{(1)} q\right) + 4\mathrm{Tr}\left(\gamma_{\mathrm{d}}^{(1)} q\right)^{\frac{1}{2}} \mathrm{Tr}\left(\gamma_{\mathrm{d}}^{(1)} q (\mathcal{N}+1) \mathbb{1}_{\mathcal{N}\geqslant M} q\right)^{\frac{1}{2}} + \frac{\mathrm{Tr}\left(p\mathcal{N}\right)^{\frac{1}{2}}}{d}\right) \\
&\leqslant JC_3 \left(\left(8\mathrm{Tr}\left(p\mathcal{N}\right)^{\frac{1}{2}} + 4\sqrt{M} + 4\epsilon^{-1}\right) \mathrm{Tr}\left(\gamma_{\mathrm{d}}^{(1)} q\right) + \epsilon\mathrm{Tr}\left(\gamma_{\mathrm{d}}^{(1)} q (\mathcal{N}+1) \mathbb{1}_{\mathcal{N}\geqslant M} q\right) + \frac{\mathrm{Tr}\left(p\mathcal{N}\right)^{\frac{1}{2}}}{d}\right).
\end{aligned}
$$

We insert (53), use the conservation of the mean-field number of particles and choose $\epsilon \coloneqq \dfrac{e^{\frac{M}{2(a+e)}e^{-C_1 t}}}{d}$:

$$
\begin{aligned}
&\left|\partial_t \mathrm{Tr}\left(\gamma_{\mathrm{d}}^{(1)}(t) q(t)\right)\right| \\
&\leqslant JC_3 \left(\left(8\mathrm{Tr}\left(p(0)\mathcal{N}\right)^{\frac{1}{2}} + 4\sqrt{M} + 4\epsilon^{-1}\right) \mathrm{Tr}\left(\gamma_{\mathrm{d}}^{(1)}(t) q(t)\right) + \epsilon C_2 e^{C_1 t - \frac{M}{2(a+e)}e^{-C_1 t}} + \frac{\mathrm{Tr}\left(p(0)\mathcal{N}\right)^{\frac{1}{2}}}{d}\right) \\
&\leqslant JC_3 \left(\left(8\mathrm{Tr}\left(p(0)\mathcal{N}\right)^{\frac{1}{2}} + 4\sqrt{M} + 4de^{-\frac{M}{2(a+e)}e^{-C_1 t}}\right) \mathrm{Tr}\left(\gamma_{\mathrm{d}}^{(1)}(t) q(t)\right) + \frac{C_2 e^{C_1 t} + \mathrm{Tr}\left(p(0)\mathcal{N}\right)^{\frac{1}{2}}}{d}\right).
\end{aligned}
$$

Choose

$$
M \coloneqq \left\lceil 2(a+e)e^{C_1 t} \ln\left(\frac{d}{\sqrt{\ln(d+1)}}\right)\right\rceil
$$

so that

$$\sqrt{M}+de^{-\frac{M}{2(a+e)}e^{-C_1t}} \leqslant \left(2(a+e)e^{C_1t}\ln\left(\frac{d}{\sqrt{\ln(d+1)}}\right)+1\right)^{\frac{1}{2}}+\sqrt{\ln(d+1)}$$
$$\leqslant \left(\sqrt{2(a+e)}e^{\frac{C_1}{2}t}+1\right)\sqrt{\ln(d+1)}+1$$

noticing that

$$d\geqslant 1 \implies \ln\left(\frac{d}{\sqrt{\ln(d+1)}}\right)\leqslant \ln(d+1).$$

Then

$$\left|\partial_t \mathrm{Tr}\left(\gamma_{\mathrm{d}}^{(1)}(t)q(t)\right)\right| \leqslant JC_3\left(\left(2C_4+4\left(\sqrt{2(a+e)}e^{\frac{C_1}{2}t}+1\right)\sqrt{\ln(d+1)}\right)\mathrm{Tr}\left(\gamma_{\mathrm{d}}^{(1)}(t)q(t)\right)\right.$$
$$\left.+\frac{C_2e^{C_1t}+\mathrm{Tr}\left(p(0)\mathcal{N}\right)^{\frac{1}{2}}}{d}\right).$$

Noticing that time dependent coefficients in the above expression are non-decreasing in time, we can use Gronwall lemma an obtain

$$\mathrm{Tr}\left(\gamma_{\mathrm{d}}^{(1)}(t)q(t)\right) \leqslant \left(\mathrm{Tr}\left(\gamma_{\mathrm{d}}^{(1)}(0)q(0)\right)+\frac{C_2e^{C_1t}+\mathrm{Tr}\left(p(0)\mathcal{N}\right)^{\frac{1}{2}}}{d\left(2C_4+4\left(\sqrt{2(a+e)}e^{\frac{C_1}{2}t}+1\right)\sqrt{\ln(d+1)}\right)}\right)$$
$$e^{JC_3\left(2C_4+4\left(\sqrt{2(a+e)}e^{\frac{C_1}{2}t}+1\right)\sqrt{\ln(d+1)}\right)t}.$$

We conclude with (41):

$$\left\|\gamma_{\mathrm{d}}^{(1)}(t)-p(t)\right\|_{\mathcal{L}^1}$$
$$\leqslant 2\sqrt{2}\sqrt{\mathrm{Tr}\left(\gamma_{\mathrm{d}}^{(1)}(t)q(t)\right)}$$
$$\leqslant 2\sqrt{2}\left(\mathrm{Tr}\left(\gamma_{\mathrm{d}}^{(1)}(0)q(0)\right)+\frac{C_2e^{C_1t}+\mathrm{Tr}\left(p(0)\mathcal{N}\right)^{\frac{1}{2}}}{d\left(2C_4+4\left(\sqrt{2(a+e)}e^{\frac{C_1}{2}t}+1\right)\sqrt{\ln(d+1)}\right)}\right)^{\frac{1}{2}}$$
$$e^{JC_3\left(C_4+2\left(\sqrt{2(a+e)}e^{\frac{C_1}{2}t}+1\right)\sqrt{\ln(d+1)}\right)t}.$$
$$\leqslant \sqrt{2}\left(\left\|\gamma_{\mathrm{d}}^{(1)}(0)-p(0)\right\|_{\mathcal{L}^1}+\frac{C_2e^{C_1t}+\mathrm{Tr}\left(p(0)\mathcal{N}\right)^{\frac{1}{2}}}{d\left(C_4+2\left(\sqrt{2(a+e)}e^{\frac{C_1}{2}t}+1\right)\sqrt{\ln(d+1)}\right)}\right)^{\frac{1}{2}}$$
$$e^{JC_3\left(C_4+2\left(\sqrt{2(a+e)}e^{\frac{C_1}{2}t}+1\right)\sqrt{\ln(d+1)}\right)t}.$$

□

Let us comment on the choice of the cutoff parameter: optimising in M requires to solve for $x\geqslant 0$,

$$\sqrt{x}=de^{-\frac{x}{2(a+e)}e^{-C_1t}} \iff e^{\frac{x}{a+e}e^{-C_1t}}x=d^2 \iff e^{\frac{x}{a+e}e^{-C_1t}}\frac{x}{a+e}e^{-C_1t}=\frac{d^2}{a+e}e^{-C_1t}$$
$$\iff \frac{x}{a+e}e^{-C_1t}=W_0\left(\frac{d^2}{a+e}e^{-C_1t}\right) \iff x=(a+e)e^{C_1t}W_0\left(\frac{d^2}{a+e}e^{-C_1t}\right),$$

where W_0 is the principal branch of the Lambert W function. Our choice of M comes from the fact that

$$W_0(x)\underset{x\to\infty}{=}\ln\left(\frac{x}{\ln(x)}\right)+o(1).$$

4 Proof using the excitation energy method

The Bose–Hubbard Hamiltonian H can be written as a sum of two time-dependent quantities:

$$H_d = \sum_{x\in\Lambda} h_x^{\alpha_\varphi}(t) + \tilde{H}(t),$$

where $\alpha_\varphi(t) := \langle \varphi(t), a\varphi(t)\rangle$ and $h_x^{\alpha_\varphi}(t)$ and $\tilde{H}(t)$ are defined below

$$h_x^{\alpha_\varphi}(t) := -J\Big[\alpha_\varphi(t)a_x^* + \overline{\alpha_\varphi(t)}a_x - |\alpha_\varphi(t)|^2\Big] + (J-\mu)\mathcal{N}_x + \frac{U}{2}\mathcal{N}_x(\mathcal{N}_x - 1), \tag{54}$$

$$\begin{aligned}\tilde{H}(t) := &-\frac{J}{2d}\sum_{<x,y>}\big(p_x p_y K_{x,y}^{(2)} q_x q_y + p_x q_y K_{x,y}^{(2)} q_x p_y\big) + h.c.\\ &-\frac{J}{2d}\sum_{<x,y>} p_x q_y K_{x,y}^{(3)}(t) q_x q_y + h.c.\\ &-\frac{J}{2d}\sum_{<x,y>} q_x q_y K_{x,y}^{(4)}(t) q_x q_y\end{aligned} \tag{55}$$

and with

$$K_{x,y}^{(2)} := a_x^* a_y + a_y^* a_x, \tag{56}$$

$$K_{x,y}^{(3)}(t) := K_{x,y}^{(2)} - \alpha_\varphi(t)a_x^* - \overline{\alpha_\varphi(t)}a_x, \tag{57}$$

$$K_{x,y}^{(4)}(t) := K_{x,y}^{(3)}(t) - \alpha_\varphi(t)a_y^* - \overline{\alpha_\varphi(t)}a_y + 2|\alpha_\varphi(t)|^2, \tag{58}$$

where the script i in the expression $K_{x,y}^{(i)}$ refers to the number of q that accompany it in the expression of $\tilde{H}$ in (55). And, here $K_{x,y}^{(2)}$ does not depend on t whereas the other terms $K_{x,y}^{(3)}$ and $k_{x,y}^{(4)}$ do through the term $\alpha_\varphi(t)$. For the sake of simplicity, we will often omit the time dependence in the quantities. Let $c > 0$ be large enough and define

$$f(t) := \frac{1}{|\Lambda|}\left\langle \Psi_d(t), \left(H + \sum_{x\in\Lambda}\big(q_x(t)h_x^{\alpha_\varphi}(t)q_x(t) - h_x^{\alpha_\varphi}(t) + cq_x(t)\big)\right)\Psi_d(t)\right\rangle. \tag{59}$$

Since the mean-field particle number is conserved, we note that

$$|\alpha_\varphi(t)| = |\langle\varphi(t), a\varphi(t)\rangle| \leqslant \|a\varphi(t)\| = \sqrt{\langle\varphi(0), \mathcal{N}\varphi(0)\rangle}. \tag{60}$$

In our analysis, we require at most the fourth-order moment $\langle\varphi(t), \mathcal{N}^4\varphi(t)\rangle$. Therefore, we provide an estimate for this moment below.

Proposition 16 (Moment bounds). *We have*

$$\big\langle\varphi(t), \mathcal{N}^4\varphi(t)\big\rangle \leqslant \sum_{j=0}^{6}\left(8J\sqrt{\langle\varphi(0),\mathcal{N}\varphi(0)\rangle}\right)^j \left\langle\varphi(0), (\mathcal{N}+j)^{4-\frac{j}{2}}\varphi(0)\right\rangle \frac{t^j}{j!}. \tag{61}$$

Proof. In general, we have for some $C > 0$

$$\begin{aligned}\left|\partial_t\left\langle\varphi(t), \mathcal{N}^k\varphi(t)\right\rangle\right| &\leqslant CJk\sqrt{\langle\varphi(0),\mathcal{N}\varphi(0)\rangle}\left|\left\langle\varphi(t), a\mathcal{N}^{\frac{k}{2}-\frac{3}{4}}\mathcal{N}^{\frac{k}{2}-\frac{1}{4}}\varphi(t)\right\rangle\right|\\ &\leqslant CJk\sqrt{\langle\varphi(0),\mathcal{N}\varphi(0)\rangle}\langle\varphi(t), (\mathcal{N}+1)^{k-\frac{1}{2}}\varphi(t)\rangle.\end{aligned}$$

In particular for $k = 4$, we compute the derivative

$$\begin{aligned}\partial_t\left\langle\varphi(t), \mathcal{N}^4\varphi(t)\right\rangle &= i\left\langle\varphi(t), [h^{\alpha_\varphi}(t), \mathcal{N}^4]\varphi(t)\right\rangle\\ &= -iJ\alpha_\varphi(t)\left\langle\varphi(t), [a^*, \mathcal{N}^4]\varphi(t)\right\rangle - iJ\overline{\alpha_\varphi(t)}\left\langle\varphi(t), [a, \mathcal{N}^4]\varphi(t)\right\rangle\\ &= -2J\mathrm{Im}\left(\overline{\alpha_\varphi(t)}\left\langle\varphi(t), [a^*, \mathcal{N}^4]\varphi(t)\right\rangle\right)\end{aligned}$$

which gives

$$
\begin{aligned}
\left|\partial_t \left\langle\varphi(t), \mathcal{N}^4\varphi(t)\right\rangle\right| &\leqslant 8J\sqrt{\langle\varphi(0), \mathcal{N}\varphi(0)\rangle}\left|\langle\varphi(t), a^*\mathcal{N}^3\varphi(t)\rangle\right| \\
&\leqslant 8J\sqrt{\langle\varphi(0), \mathcal{N}\varphi(0)\rangle}\left\langle\varphi(t), (\mathcal{N}+1)^{7/2}\varphi(t)\right\rangle.
\end{aligned}
$$

Then by iterating six times, the identity (61) follows. More generally, one could have as well

$$
\left\langle\varphi(t), \mathcal{N}^k\varphi(t)\right\rangle \leqslant \sum_{j=0}^{2k-2}\left(CJk\sqrt{\langle\varphi(0), \mathcal{N}\varphi(0)\rangle}\right)^j \left\langle\varphi(0), (\mathcal{N}+j)^{k-\frac{j}{2}}\varphi(0)\right\rangle \frac{t^j}{j!}. \tag{62}
$$

□

Notation. Throughout our estimates, we use the quantities $C > 0$, $C(J, \mu, U) > 0$, and $\tilde{C}(t) > 0$ with the following definitions:

- C: a positive constant that is independent of all parameters in the model;
- $C(J, \mu, U)$: a positive constant that depends polynomially only on a set of parameters (J, μ, U), but is independent of the initial conditions and time t;
- $\tilde{C}(t)$: a positive quantity that depends on $C(J, \mu, U)$, the initial data $\langle\varphi(0), \mathcal{N}^j\varphi(0)\rangle$, $\quad j \leqslant 4$, and time t , with a polynomial dependence on t.

These quantities may change from one line to the next in the subsequent computations.

Proposition 17 (Grönwall's estimate)**.** *Let $c > 0$ be as in* (81)*. Then, for all $t \in \mathbb{R}$, we have*

$$
f(t) \leqslant e^{\int_0^t \tilde{C}(s)ds} f(0) + \frac{1}{d}\int_0^t \left(1 + \tilde{C}(s)\right) e^{\int_s^t \tilde{C}(r)dr} ds, \tag{63}
$$

where $\tilde{C}(t)$ is a polynomial in t of order six and depends on J, μ, U and on the initial data as follows

$$
\begin{aligned}
\tilde{C}(t) := \frac{C(J, \mu, U)}{U}&\left(1 + \frac{1}{U} + \langle\varphi(0), \mathcal{N}\varphi(0)\rangle^2\right)\Bigg(1 \\
&+ \sum_{j=0}^{6}\left(8J\langle\varphi(0), \mathcal{N}\varphi(0)\rangle^{1/2}\right)^j \left\langle\varphi(0), (\mathcal{N}+j)^{4-\frac{j}{2}}\varphi(0)\right\rangle \frac{t^j}{j!}\Bigg),
\end{aligned} \tag{64}
$$

where $C(J, \mu, U) > 0$ is a polynomial in our parameters model J, μ and U.

Proposition 18. *For all $t \in \mathbb{R}$, for some $C > 0$, we have*

$$
\begin{aligned}
&\frac{1}{|\Lambda|}\sum_{x\in\Lambda}\langle\Psi_d(t), q_x(t)\Psi_d(t)\rangle \leqslant \frac{1}{d}\left(\frac{4}{U}\left(1 + e^{\int_0^t \tilde{C}(s)ds}\right) + \frac{4}{U}\int_0^t (1 + \tilde{C}(s))e^{\int_s^t \tilde{C}(r)dr} ds\right) \\
&+ Ce^{\int_0^t \tilde{C}(s)ds}\left(1 + J^2 + U + \left(J - \mu - \frac{U}{2}\right)^2\right)\left(1 + \frac{1}{U} + \langle\varphi(0), \mathcal{N}\varphi(0)\rangle^2\right) \\
&\frac{1}{|\Lambda|}\sum_{x\in\Lambda}\left\langle\Psi_d(0), \left(q_x(0)\mathcal{N}_x^2 q_x(0) + q_x(0)\right)\Psi_d(0)\right\rangle.
\end{aligned} \tag{65}
$$

The proof of Propositions 17 and 18 is postponed to the end of the section. We need first to consider some intermediate steps summarized in the following propositions.

4.1 Some ϵ stimates

In this paragraph, we present some useful estimates that establish relation between the quantities f and $\frac{1}{|\Lambda|}\sum_{x\in\Lambda}\langle\Psi_d, q_x\Psi_d\rangle$.

Proposition 19 (ϵ estimates). *For all $\epsilon > 0$, for some $C > 0$, we have*

$$\begin{aligned}
&\frac{1}{|\Lambda|}\left|\left\langle \Psi_d, \left(\tilde{H} + \sum_{x\in\Lambda} q_x\left(h_x^{\alpha_\varphi} - \frac{U}{2}\mathcal{N}_x^2\right)q_x\right)\Psi_d\right\rangle\right| \\
&\leqslant C\left(1 + J^2 + \left(J - \mu - \frac{U}{2}\right)^2\right)\left(1 + \frac{1}{\epsilon} + \langle\varphi(0), \mathcal{N}\varphi(0)\rangle^2\right)\frac{1}{|\Lambda|}\sum_{x\in\Lambda}\langle\Psi_d, q_x\Psi_d\rangle \qquad (66)\\
&+ \epsilon\frac{1}{|\Lambda|}\sum_{x\in\Lambda}\langle\Psi_d, q_x\mathcal{N}_x^2 q_x\Psi_d\rangle + \frac{1}{d},
\end{aligned}$$

Proof. Recall the definition of $\tilde{H}$ in (55), we have

$$\begin{aligned}
\frac{1}{|\Lambda|}\langle\Psi_d, \tilde{H}\Psi_d\rangle &= -\frac{J}{2d}\frac{1}{|\Lambda|}\sum_{<x,y>}\langle\Psi_d, p_x p_y K_{x,y}^{(2)} q_x q_y\Psi_d\rangle + h.c. &(67)\\
&- \frac{J}{2d}\frac{1}{|\Lambda|}\sum_{<x,y>}\langle\Psi_d, p_x q_y K_{x,y}^{(2)} q_x p_y\Psi_d\rangle + h.c. &(68)\\
&- \frac{J}{d}\frac{1}{|\Lambda|}\sum_{<x,y>}\langle\Psi_d, p_x q_y K_{x,y}^{(3)} q_x q_y\Psi_d\rangle + h.c. &(69)\\
&- \frac{J}{2d}\frac{1}{|\Lambda|}\sum_{<x,y>}\langle\Psi_d, q_x q_y K_{x,y}^{(4)} q_x q_y\Psi_d\rangle, &(70)
\end{aligned}$$

where the terms $K_{x,y}^{(2)}$, $K_{x,y}^{(3)}$ and $K_{x,y}^{(4)}$ are defined in (56), (57) and (58). Let us estimate the above equation term by term. We start by estimating $pp - qq$ terms inside (67):

$$\begin{aligned}
&\left|-\frac{J}{2d}\frac{1}{|\Lambda|}\sum_{x\in\Lambda}\sum_{\substack{y\in\Lambda\\x\sim y}}\langle\Psi_d, p_x p_y a_x^* a_y q_x q_y\Psi_d\rangle\right| = \left|-\frac{J}{2d}\frac{1}{|\Lambda|}\sum_{x\in\Lambda}\langle a_x p_x\sum_{\substack{y\in\Lambda\\x\sim y}}(q_y a_y^* p_y)\Psi_d, q_x\Psi_d\rangle\right| \\
&\leqslant \frac{J}{2d}\frac{1}{|\Lambda|}\sum_{x\in\Lambda}\|q_x\Psi_d\|\left\|a_x p_x\sum_{\substack{y\in\Lambda\\x\sim y}}\left(q_y a_y^* p_y\right)\Psi_d\right\| \\
&\leqslant \frac{J}{|\Lambda|}\sum_{x\in\Lambda}\langle\varphi(0), (\mathcal{N}+1)\varphi(0)\rangle\|q_x\Psi_d\|\left(\frac{1}{2d} + \frac{1}{(2d)^2}\sum_{\substack{y,z\in\Lambda\\y\neq z, x\sim y, x\sim z}}\|q_z\Psi_d\|\|q_y\Psi_d\|\right)^{1/2} \\
&\leqslant \frac{J}{|\Lambda|}\sum_{x\in\Lambda}\langle\varphi(0), (\mathcal{N}+1)\varphi(0)\rangle\|q_x\Psi_d\|\left(\frac{1}{2d} + \frac{1}{4d}\sum_{\substack{y\in\Lambda\\x\sim y}}\|q_y\Psi_d\|^2 + \frac{1}{4d}\sum_{\substack{z\in\Lambda\\x\sim z}}\|q_z\Psi_d\|^2\right)^{1/2} \\
&\leqslant \frac{J^2}{2}\langle\varphi(0), (\mathcal{N}+1)\varphi(0)\rangle^2\frac{1}{|\Lambda|}\sum_{x\in\Lambda}\|q_x\Psi_d\|^2 + \frac{1}{4d} + \frac{1}{8d}\frac{1}{|\Lambda|}\underbrace{\sum_{x\in\Lambda}\sum_{\substack{y\in\Lambda\\x\sim y}}\|q_y\Psi_d\|^2}_{=2d\sum_{x\in\Lambda}\|q_x\Psi_d\|^2} + \frac{1}{8d}\frac{1}{|\Lambda|}\underbrace{\sum_{x\in\Lambda}\sum_{\substack{z\in\Lambda\\x\sim z}}\|q_z\Psi_d\|^2}_{=2d\sum_{x\in\Lambda}\|q_x\Psi_d\|^2} \\
&\leqslant \left(\frac{1}{2} + \frac{J^2}{2}\langle\varphi(0), (\mathcal{N}+1)\varphi(0)\rangle^2\right)\frac{1}{|\Lambda|}\sum_{x\in\Lambda}\langle\Psi_d, q_x\Psi_d\rangle + \frac{1}{4d}.
\end{aligned}$$

In the above computations, we applied the Cauchy–Schwarz inequality and the identity $ab \leqslant \frac{a^2}{2} + \frac{b^2}{2}$. We provided detailed calculations for just one term in $K_{x,y}^{(2)}$ from (56), specifically $a_x^* a_y$; the computation for the other term, $a_y^* a_x$, can be handled in a similar way. As a result, we obtain:

$$|(67)| \leqslant \big(1 + J^2 \langle \varphi(0), (\mathcal{N}+1)\varphi(0)\rangle^2\big) \frac{1}{|\Lambda|} \sum_{x\in\Lambda} \langle \Psi_d, q_x \Psi_d \rangle + \frac{1}{d}. \tag{71}$$

Next, we proceed to estimate the $pq - qp$ terms within (68):

$$\begin{aligned}
&\left| -\frac{J}{2d} \frac{1}{|\Lambda|} \sum_{x\in\Lambda} \sum_{\substack{y\in\Lambda \\ x\sim y}} \langle \Psi_d, p_x q_y a_x^* a_y q_x p_y \Psi_d \rangle \right| = \left| -\frac{J}{2d} \frac{1}{|\Lambda|} \sum_{x\in\Lambda} \sum_{\substack{y\in\Lambda \\ x\sim y}} \langle a_x p_x q_y \Psi_d, a_y p_y q_x \Psi_d \rangle \right| \\
&\leqslant \frac{J}{2d} \frac{1}{|\Lambda|} \sum_{x\in\Lambda} \sum_{\substack{y\in\Lambda \\ x\sim y}} \langle \varphi(0), \mathcal{N}\varphi(0)\rangle \|q_x \Psi_d\| \|q_y \Psi_d\| \\
&\leqslant J \langle \varphi(0), \mathcal{N}\varphi(0)\rangle \frac{1}{|\Lambda|} \sum_{x\in\Lambda} \langle \Psi_d, q_x \Psi_d \rangle,
\end{aligned}$$

where we once again employed the same techniques as in the previous case. By repeating the same arguments for the remaining terms in (68), we obtain

$$|(68)| \leqslant 2J \langle \varphi(0), \mathcal{N}\varphi(0)\rangle \frac{1}{|\Lambda|} \sum_{x\in\Lambda} \langle \Psi_d, q_x \Psi_d \rangle. \tag{72}$$

We now proceed to estimate the $pq - qq$ terms within (69), yielding:

$$\begin{aligned}
&\left| \frac{J}{d} \frac{1}{|\Lambda|} \sum_{x\in\Lambda} \sum_{\substack{y\in\Lambda \\ x\sim y}} \langle \Psi_d, p_x q_y a_x^* a_y q_x q_y \Psi_d \rangle \right| \leqslant \left(J + \frac{J^2}{2\epsilon}\right) \frac{1}{|\Lambda|} \sum_{x\in\Lambda} \langle \Psi_d, q_x \Psi_d \rangle + \frac{\epsilon}{2} \frac{1}{|\Lambda|} \sum_{x\in\Lambda} \langle \Psi_d, q_x \mathcal{N}_x^2 q_x \Psi_d \rangle, \\
&\left| \frac{J}{d} \frac{1}{|\Lambda|} \sum_{x\in\Lambda} \sum_{\substack{y\in\Lambda \\ x\sim y}} \langle \Psi_d, p_x q_y a_y^* a_x q_x q_y \Psi_d \rangle \right| \leqslant J^2 \frac{1}{2\epsilon} \frac{1}{|\Lambda|} \sum_{x\in\Lambda} \langle \Psi_d, q_x \Psi_d \rangle + \frac{\epsilon}{2} \frac{1}{|\Lambda|} \sum_{x\in\Lambda} \langle \Psi_d, q_x \mathcal{N}_x^2 q_x \Psi_d \rangle, \\
&\left| \frac{J}{d} \frac{1}{|\Lambda|} \sum_{x\in\Lambda} \sum_{\substack{y\in\Lambda \\ x\sim y}} \langle \Psi_d, p_x q_y \alpha_\varphi a_x^* q_x q_y \Psi_d \rangle \right| \leqslant 2J \langle \varphi(0), \mathcal{N}\varphi(0)\rangle \frac{1}{|\Lambda|} \sum_{x\in\Lambda} \langle \Psi_d, q_x \Psi_d \rangle, \\
&\left| \frac{J}{d} \frac{1}{|\Lambda|} \sum_{x\in\Lambda} \sum_{\substack{y\in\Lambda \\ x\sim y}} \langle \Psi_d, p_x q_y \overline{\alpha_\varphi} a_x q_x q_y \Psi_d \rangle \right| \leqslant 2J \langle \varphi(0), (\mathcal{N}+1)\varphi(0)\rangle \frac{1}{|\Lambda|} \sum_{x\in\Lambda} \langle \Psi_d, q_x \Psi_d \rangle.
\end{aligned}$$

Combining the above estimates, we get

$$|(69)| \leqslant 2\left(3J + \frac{J^2}{\epsilon} + 4J\langle \varphi(0), \mathcal{N}\varphi(0)\rangle\right) \frac{1}{|\Lambda|} \sum_{x\in\Lambda} \langle \Psi_d, q_x \Psi_d \rangle + 2\epsilon \frac{1}{|\Lambda|} \sum_{x\in\Lambda} \langle \Psi_d, q_x \mathcal{N}_x^2 q_x \Psi_d \rangle. \tag{73}$$

The final and most difficult part is (70), as it involves $qq - qq$ terms. Here, we provide the computations only for the terms $\alpha_\varphi a_x^*$ and $a_x^* a_y$ in $K_{x,y}^{(4)}$ from (58). We begin with the first term

$$\begin{aligned}
&\left|\frac{J}{2d}\frac{1}{|\Lambda|}\sum_{x\in\Lambda}\sum_{\substack{y\in\Lambda\\ x\sim y}}\langle\Psi_d, q_x q_y \alpha_\varphi a_x^* q_x q_y \Psi_d\rangle\right| = \left|\frac{J}{2d}\frac{1}{|\Lambda|}\sum_{x\in\Lambda}\sum_{\substack{y\in\Lambda\\ x\sim y}}\alpha_\varphi\langle a_x q_x\Psi_d, q_y q_x \Psi_d\rangle\right| \\
&\leqslant \frac{J}{2d}\frac{1}{|\Lambda|}\sqrt{\langle\varphi(0),\mathcal{N}\varphi(0)\rangle}\sum_{x\in\Lambda}\sum_{\substack{y\in\Lambda\\ x\sim y}}\|q_x\Psi_d\|\|a_x q_x\Psi_d\| \\
&\leqslant J\sqrt{\langle\varphi(0),\mathcal{N}\varphi(0)\rangle}\frac{1}{|\Lambda|}\sum_{x\in\Lambda}\|q_x\Psi_d\|\left(\frac{\epsilon}{2}\|\mathcal{N}_x q_x\Psi_d\|^2+\frac{1}{2\epsilon}\|q_x\Psi_d\|^2\right)^{1/2} \\
&\leqslant \left(\frac{1}{4\epsilon}+\frac{J^2}{2}\langle\varphi(0),\mathcal{N}\varphi(0)\rangle\right)\frac{1}{|\Lambda|}\sum_{x\in\Lambda}\langle\Psi_d, q_x\Psi_d\rangle+\frac{\epsilon}{4}\frac{1}{|\Lambda|}\sum_{x\in\Lambda}\langle\Psi_d, q_x\mathcal{N}_x^2 q_x\Psi_d\rangle.
\end{aligned}$$

In the above calculations, we applied the Cauchy–Schwarz inequality twice in the first three lines, utilized the bound (60) on α_φ, and used the identity $ab \leqslant \frac{1}{\epsilon}\frac{a^2}{2}+\epsilon\frac{b^2}{2}$, which is valid for all $\epsilon>0$. Next, we proceed to estimate the second term involving $a_x^* a_y$:

$$\begin{aligned}
&\left|-\frac{J}{2d}\frac{1}{|\Lambda|}\sum_{x\in\Lambda}\sum_{\substack{y\in\Lambda\\ x\sim y}}\langle\Psi_d, q_x q_y a_x^* a_y q_x q_y\Psi_d\rangle\right| = \left|\frac{J}{2d}\frac{1}{|\Lambda|}\sum_{x\in\Lambda}\sum_{\substack{y\in\Lambda\\ x\sim y}}\langle a_x q_x q_y\Psi_d, a_y q_y q_x\Psi_d\rangle\right| \\
&\leqslant \frac{J}{2d}\frac{1}{|\Lambda|}\sum_{x\in\Lambda}\sum_{\substack{y\in\Lambda\\ x\sim y}}\sqrt{\langle\Psi_d, q_y q_x\mathcal{N}_x q_x\Psi_d\rangle}\sqrt{\langle\Psi_d, q_x q_y\mathcal{N}_y q_y\Psi_d\rangle} \\
&\leqslant \frac{1}{2d}\frac{1}{|\Lambda|}\sum_{x\in\Lambda}\sum_{\substack{y\in\Lambda\\ x\sim y}}\sqrt{J\|q_x\Psi_d\|\|\mathcal{N}_x q_x\Psi_d\|}\sqrt{J\|q_y\Psi_d\|\|\mathcal{N}_y q_y\Psi_d\|} \\
&\leqslant \frac{1}{2d}\frac{1}{|\Lambda|}\sum_{x\in\Lambda}\sum_{\substack{y\in\Lambda\\ x\sim y}}\left(\frac{J^2}{2\epsilon}\|q_x\Psi_d\|^2+\frac{\epsilon}{2}\|\mathcal{N}_x q_x\Psi_d\|^2\right)^{1/2}\left(\frac{J^2}{2\epsilon}\|q_y\Psi_d\|^2+\frac{\epsilon}{2}\|\mathcal{N}_y q_y\Psi_d\|^2\right)^{1/2} \\
&\leqslant \left(\frac{J^2}{2\epsilon}\frac{1}{|\Lambda|}\sum_{x\in\Lambda}\|q_x\Psi_d\|^2+\frac{\epsilon}{2}\frac{1}{|\Lambda|}\sum_{x\in\Lambda}\|\mathcal{N}_x q_x\Psi_d\|^2\right)^{1/2}\left(\frac{J^2}{2\epsilon}\frac{1}{|\Lambda|}\sum_{x\in\Lambda}\|q_x\Psi_d\|^2+\frac{\epsilon}{2}\frac{1}{|\Lambda|}\sum_{x\in\Lambda}\|\mathcal{N}_x q_x\Psi_d\|^2\right)^{1/2} \\
&\leqslant \frac{J^2}{2\epsilon}\frac{1}{|\Lambda|}\sum_{x\in\Lambda}\langle\Psi_d, q_x\Psi_d\rangle+\frac{\epsilon}{2}\frac{1}{|\Lambda|}\sum_{x\in\Lambda}\langle\Psi_d, q_x\mathcal{N}_x^2 q_x\Psi_d\rangle,
\end{aligned}$$

where again in the above computations, we have applied Cauchy–Schwarz two times in the first three lines and $ab \leqslant \frac{1}{\epsilon}\frac{a^2}{2}+\epsilon\frac{b^2}{2}$ for all $\epsilon>0$. Then, we get, for some $C>0$

$$|(70)| \leqslant C\left(\frac{1}{\epsilon}+\frac{J^2}{\epsilon}+J^2\langle\varphi(0),\mathcal{N}\varphi(0)\rangle\right)\frac{1}{|\Lambda|}\sum_{x\in\Lambda}\langle\Psi_d, q_x\Psi_d\rangle + C\epsilon\frac{1}{|\Lambda|}\sum_{x\in\Lambda}\langle\Psi_d, q_x\mathcal{N}_x^2 q_x\Psi_d\rangle. \tag{74}$$

Then combining all bounds, we get for some $C>0$

$$\begin{aligned}
\left|\frac{1}{|\Lambda|}\langle\Psi_d,\tilde{H}\Psi_d\rangle\right| &\leqslant C\epsilon\frac{1}{|\Lambda|}\sum_{x\in\Lambda}\langle\Psi_d, q_x\mathcal{N}_x^2 q_x\Psi_d\rangle+\frac{1}{d} \\
&\quad + C\left(1+J^2\right)\left(1+\frac{1}{\epsilon}+\langle\varphi(0),\mathcal{N}\varphi(0)\rangle+\langle\varphi(0),\mathcal{N}\varphi(0)\rangle^2\right)\frac{1}{|\Lambda|}\sum_{x\in\Lambda}\langle\Psi_d, q_x\Psi_d\rangle.
\end{aligned}\tag{75}$$

Similary, we can prove for some $C > 0$

$$
\begin{aligned}
&\left| \frac{1}{|\Lambda|} \sum_{x\in\Lambda} \langle \Psi_d, q_x \left(h_x^{\alpha_\varphi} - \frac{U}{2}\mathcal{N}_x^2 \right) q_x \Psi_d \rangle \right| \\
&= \left| \frac{1}{|\Lambda|} \sum_{x\in\Lambda} \left\langle \Psi_d, q_x \left(-J\alpha_\varphi a_x^* - J\overline{\alpha_\varphi} a_x + J|\alpha_\varphi|^2 + \left(J - \mu - \frac{U}{2} \right) \mathcal{N}_x \right) q_x \Psi_d \right\rangle \right| \\
&\leqslant C \left(1 + J^2 + \left(J - \mu - \frac{U}{2} \right)^2 \right) \left(\frac{1}{\epsilon} + \langle \varphi(0), \mathcal{N}\varphi(0) \rangle \right) \frac{1}{|\Lambda|} \sum_{x\in\Lambda} \langle \Psi_d, q_x \Psi_d \rangle \\
&+ \epsilon \frac{1}{|\Lambda|} \sum_{x\in\Lambda} \langle \Psi_d, q_x \mathcal{N}_x^2 q_x \Psi_d \rangle.
\end{aligned}
\tag{76}
$$

Now, choosing the right ϵ, we get the bound (66).

□

Proposition 20 (Equivalence)**.** *Define g as follows*

$$
g := \frac{1}{|\Lambda|} \sum_{x\in\Lambda} \left\langle \Psi_d, \left(q_x \mathcal{N}_x^2 q_x + q_x \right) \Psi_d \right\rangle. \tag{77}
$$

Then, for $U > 0$ and for some $C > 0$, we have the following equivalence

$$
\left(\frac{U}{4} \right) g - \frac{1}{d} \leqslant f \leqslant C \left(1 + J^2 + U + \left(J - \mu - \frac{U}{2} \right)^2 \right) \left(1 + \frac{1}{U} + \langle \varphi(0), \mathcal{N}\varphi(0) \rangle^2 \right) g + \frac{1}{d}. \tag{78}
$$

Proof. We start to prove the left hand side of (78). Indeed, the bound (66) in Proposition 19 imply that

$$
\begin{aligned}
&\frac{1}{|\Lambda|} \left\langle \Psi_d, \left(\tilde{H} + \sum_{x\in\Lambda} q_x \left(h_x^{\alpha_\varphi} - \frac{U}{2}\mathcal{N}_x^2 \right) q_x \right) \Psi_d \right\rangle \geqslant -\epsilon \frac{1}{|\Lambda|} \sum_{x\in\Lambda} \langle \Psi_d, q_x \mathcal{N}_x^2 q_x \Psi_d \rangle \\
&- C \left(1 + J^2 + \left(J - \mu - \frac{U}{2} \right)^2 \right) \left(1 + \frac{1}{\epsilon} + \langle \varphi(0), \mathcal{N}\varphi(0) \rangle^2 \right) \frac{1}{|\Lambda|} \sum_{x\in\Lambda} \langle \Psi_d, q_x \Psi_d \rangle - \frac{1}{d}.
\end{aligned}
\tag{79}
$$

Using the above estimate (79), we get

$$
\begin{aligned}
f &= \frac{1}{|\Lambda|} \left\langle \Psi_d, \left(\tilde{H} + \sum_{x\in\Lambda} q_x \left(h_x^{\alpha_\varphi} - \frac{U}{2}\mathcal{N}_x^2 \right) q_x \right) \Psi_d \right\rangle \\
&+ \frac{U}{2} \frac{1}{|\Lambda|} \sum_{x\in\Lambda} \langle \Psi_d, q_x \mathcal{N}_x^2 q_x \Psi_d \rangle + \frac{c}{|\Lambda|} \sum_{x\in\Lambda} \langle \Psi_d, q_x \Psi_d \rangle \\
&\geqslant (\frac{U}{2} - \epsilon) \frac{1}{|\Lambda|} \sum_{x\in\Lambda} \langle \Psi_d, q_x \mathcal{N}_x^2 q_x \Psi_d \rangle \\
&+ \left(c - C \left(1 + J^2 + \left(J - \mu - \frac{U}{2} \right)^2 \right) \left(1 + \frac{1}{\epsilon} + \langle \varphi(0), \mathcal{N}\varphi(0) \rangle^2 \right) \right) \frac{1}{|\Lambda|} \sum_{x\in\Lambda} \langle \Psi_d, q_x \Psi_d \rangle - \frac{1}{d}.
\end{aligned}
$$

Thus, we have to choose $\epsilon > 0$ small enough and $c > 0$ large enough such that

$$
\frac{U}{2} - \epsilon > 0, \qquad c - C \left(1 + J^2 + \left(J - \mu - \frac{U}{2} \right)^2 \right) \left(1 + \frac{1}{\epsilon} + \langle \varphi(0), \mathcal{N}\varphi(0) \rangle^2 \right) > 0. \tag{80}
$$

Take the following choice of ϵ and c

$$
c = C \left(1 + J^2 + \left(J - \mu - \frac{U}{2} \right)^2 \right) \left(1 + \frac{1}{\epsilon} + \langle \varphi(0), \mathcal{N}\varphi(0) \rangle^2 \right) + \frac{U}{4}, \qquad \epsilon = \frac{U}{4}. \tag{81}
$$

Now, with the special choice of ϵ and c in (81) which implies

$$\frac{U}{2} - \epsilon = \frac{U}{4} > 0, \qquad c - C\left(1 + J^2 + \left(J - \mu - \frac{U}{2}\right)^2\right)\left(1 + \frac{1}{\epsilon} + \langle\varphi(0), \mathcal{N}\varphi(0)\rangle^2\right) = \frac{U}{4} > 0$$

we get

$$f \geqslant \frac{U}{4}\frac{1}{|\Lambda|}\sum_{x\in\Lambda}\langle\Psi_d, \left(q_x + q_x\mathcal{N}_x^2 q_x\right)\Psi_d\rangle - \frac{1}{d}.$$

In a similar way as in Proposition 19, we can prove with some $C > 0$

$$\left|\frac{1}{|\Lambda|}\langle\Psi_d, \tilde{H}\Psi_d\rangle\right| \leqslant C(1 + J^2)\left(1 + \langle\varphi(0), \mathcal{N}\varphi(0)\rangle^2\right)\frac{1}{|\Lambda|}\sum_{x\in\Lambda}\langle\Psi_d, (q_x\mathcal{N}_x^2 q_x + q_x)\Psi_d\rangle + \frac{1}{d} \tag{82}$$

and as well

$$\begin{aligned} &\left|\frac{1}{|\Lambda|}\sum_{x\in\Lambda}\langle\Psi_d, q_x h_x^{\alpha_\varphi} q_x\Psi_d\rangle\right| \\ &= \left|\frac{1}{|\Lambda|}\sum_{x\in\Lambda}\left\langle\Psi_d, q_x\left(-J\alpha_\varphi a_x^* - J\overline{\alpha_\varphi}a_x + J|\alpha_\varphi|^2 + (J-\mu)\mathcal{N}_x + \frac{U}{2}\mathcal{N}_x(\mathcal{N}_x - 1)\right)q_x\Psi_d\right\rangle\right| \\ &\leqslant C\left(1 + |J| + U + \left|J - \mu - \frac{U}{2}\right|\right)\left(1 + \langle\varphi(0), \mathcal{N}\varphi(0)\rangle\right)\frac{1}{|\Lambda|}\sum_{x\in\Lambda}\langle\Psi_d, (q_x\mathcal{N}_x^2 q_x + q_x)\Psi_d\rangle. \end{aligned} \tag{83}$$

Now, to prove the right hand side of (78), we exploit (82), (83) and the special choice of c in (81) which lead to

$$\begin{aligned} |f| &= \left|\frac{1}{|\Lambda|}\langle\Psi_d, (\underbrace{H_d - \sum_{x\in\Lambda} h_x^{\alpha_\varphi}}_{=\tilde{H}})\Psi_d\rangle + \frac{1}{|\Lambda|}\sum_{x\in\Lambda}\langle\Psi_d, q_x h_x^{\alpha_\varphi} q_x\Psi_d\rangle + \frac{c}{|\Lambda|}\sum_{x\in\Lambda}\langle\Psi_d, q_x\Psi_d\rangle\right| \\ &= \left|\frac{1}{|\Lambda|}\langle\Psi_d, \tilde{H}\Psi_d\rangle + \frac{1}{|\Lambda|}\sum_{x\in\Lambda}\langle\Psi_d, q_x h_x^{\alpha_\varphi} q_x\Psi_d\rangle + \frac{c}{|\Lambda|}\sum_{x\in\Lambda}\langle\Psi_d, q_x\Psi_d\rangle\right| \\ &\leqslant C\left(1 + J^2 + U + \left(J - \mu - \frac{U}{2}\right)^2\right)\left(1 + \frac{1}{U} + \langle\varphi(0), \mathcal{N}\varphi(0)\rangle^2\right)\frac{1}{|\Lambda|}\sum_{x\in\Lambda}\langle\Psi_d, (q_x\mathcal{N}_x^2 q_x + q_x)\Psi_d\rangle \\ &+ \frac{1}{d}. \end{aligned}$$

Then since $f \in \mathbb{R}$, the bound on the right hand side of (78) follows.

□

4.2 Proof of Grönwall Lemma

Due to the change in definition of $\langle x, y\rangle$, it might be that there is 1/2 missing in the definition of $\mathcal{R}$, I will check that.

Proposition 21. *The expectation of $\dot{\tilde{H}}$ could be written as*

$$\begin{aligned} \frac{1}{|\Lambda|}\langle\Psi_d, \dot{\tilde{H}}\Psi_d\rangle &= -\frac{i}{|\Lambda|}\sum_{x\in\Lambda}\langle\Psi_d, [\tilde{H}, q_x h_x^{\alpha_\varphi} q_x - h_x^{\alpha_\varphi}]\Psi_d\rangle + \mathcal{R} \\ &= \frac{J}{2d}\frac{i}{|\Lambda|}\sum_{<x,y>}\langle\Psi_d, [\tilde{H}_{x,y}, q_x h_x^{\alpha_\varphi} q_x - h_x^{\alpha_\varphi} + q_y h_y^{\alpha_\varphi} q_y - h_y^{\alpha_\varphi}]\Psi_d\rangle + \mathcal{R}, \end{aligned} \tag{84}$$

with $\tilde{H}_{x,y}$ refers to the terms in (55) *such that $\tilde{H} = \sum_{<x,y>} \tilde{H}_{x,y}$ and where we have introduced the rest terms $\mathcal{R} \equiv \mathcal{R}(t)$ as*

$$\mathcal{R} := -\frac{J}{d}\frac{i}{|\Lambda|}\sum_{<x,y>}\left\langle \Psi_d, \Big(p_x h_x^{\alpha_\varphi} p_y K^{(2)}_{x,y} q_x q_y + q_x h_x^{\alpha_\varphi} p_x q_y K^{(2)}_{x,y} p_x p_y\Big)\Psi_d\right\rangle + h.c. \tag{85}$$

$$-\frac{J}{d}\frac{i}{|\Lambda|}\sum_{<x,y>}\left\langle \Psi_d, q_y p_x\Big(h_x^{\alpha_\varphi} + h_y^{\alpha_\varphi} p_y\Big) K^{(2)}_{x,y} q_x p_y \Psi_d\right\rangle + h.c. \tag{86}$$

$$\begin{aligned}+\frac{J}{d}\frac{i}{|\Lambda|}\sum_{<x,y>}\Big\langle \Psi_d, q_y\Big(&(h_x^{\alpha_\varphi} p_x - p_x h_x^{\alpha_\varphi} - p_x h_y^{\alpha_\varphi} p_y) K^{(3)}_{x,y} \\ &+ p_x K^{(3)}_{x,y}(p_x h_x^{\alpha_\varphi} + p_y h_y^{\alpha_\varphi})\Big) q_x q_y \Psi_d\Big\rangle + h.c.\end{aligned} \tag{87}$$

$$+\frac{J}{d}\frac{i}{|\Lambda|}\sum_{<x,y>}\langle \Psi_d, q_x q_y K^{(4)}_{x,y} p_x h_x^{\alpha_\varphi} q_x q_y \Psi_d\rangle + h.c. \tag{88}$$

$$-\frac{J}{2d}\frac{1}{|\Lambda|}\sum_{<x,y>}\left\langle \Psi_d, \left(p_x q_y \dot{K}^{(3)}_{x,y} q_x q_y + \frac{1}{2} q_x q_y \dot{K}^{(4)}_{x,y} q_x q_y\right)\Psi_d\right\rangle + h.c.. \tag{89}$$

Proof. We start by writing some useful computations

$$\dot{\alpha}_\varphi = i\mu\alpha_\varphi - iU\langle \varphi, \mathcal{N}a\varphi\rangle, \tag{90}$$

$$\overline{\alpha_\varphi}\dot{\alpha}_\varphi + \alpha_\varphi\dot{\overline{\alpha_\varphi}} = 2U\mathrm{Im}\left(\langle \varphi, \mathcal{N}a\varphi\rangle\overline{\alpha_\varphi}\right), \tag{91}$$

$$\dot{h}_x^{\alpha_\varphi} = -J\dot{\alpha}_\varphi a_x^* - J\dot{\overline{\alpha_\varphi}} a_x + 2JU\mathrm{Im}\left(\langle \varphi, \mathcal{N}a\varphi\rangle\overline{\alpha_\varphi}\right), \tag{92}$$

$$\dot{K}^{(2)}_{x,y} = 0, \qquad \dot{K}^{(3)}_{x,y} = -\dot{\alpha}_\varphi a_x^* - \dot{\overline{\alpha_\varphi}} a_x, \tag{93}$$

$$\dot{K}^{(4)}_{x,y} = -\dot{\alpha}_\varphi(a_x^* + a_y^*) - \dot{\overline{\alpha_\varphi}}(a_x + a_y) + 4U\mathrm{Im}\left(\langle \varphi, \mathcal{N}a\varphi\rangle\overline{\alpha_\varphi}\right). \tag{94}$$

Using (55), we compute the time derivative of $\tilde{H}$

$$\begin{aligned}\dot{\tilde{H}} = &-\frac{J}{2d}\sum_{<x,y>}\Big[\dot{p}_x p_y K^{(2)}_{x,y} q_x q_y + p_x \dot{p}_y K^{(2)}_{x,y} q_x q_y + p_x p_y K^{(2)}_{x,y} \dot{q}_x q_y + p_x p_y K^{(2)}_{x,y} q_x \dot{q}_y \\ &\qquad + \dot{p}_x q_y K^{(2)}_{x,y} q_x p_y + p_x \dot{q}_y K^{(2)}_{x,y} q_x p_y + p_x q_y K^{(2)}_{x,y} \dot{q}_x p_y + p_x q_y K^{(2)}_{x,y} q_x \dot{p}_y \\ &\qquad + \dot{q}_x q_y K^{(2)}_{x,y} p_x p_y + q_x \dot{q}_y K^{(2)}_{x,y} p_x p_y + q_x q_y K^{(2)}_{x,y} \dot{p}_x p_y + q_x q_y K^{(2)}_{x,y} p_x \dot{p}_y \\ &\qquad + \dot{q}_x p_y K^{(2)}_{x,y} p_x q_y + q_x \dot{p}_y K^{(2)}_{x,y} p_x q_y + q_x p_y K^{(2)}_{x,y} \dot{p}_x q_y + q_x p_y K^{(2)}_{x,y} p_x \dot{q}_y\Big] \\ &-\frac{J}{d}\sum_{<x,y>}\Big[\dot{p}_x q_y K^{(3)}_{x,y} q_x q_y + p_x \dot{q}_y K^{(3)}_{x,y} q_x q_y + p_x q_y K^{(3)}_{x,y} \dot{q}_x q_y + p_x q_y K^{(3)}_{x,y} q_x \dot{q}_y \\ &\qquad + \dot{q}_x q_y K^{(3)}_{x,y} p_x q_y + q_x \dot{q}_y K^{(3)}_{x,y} p_x q_y + q_x q_y K^{(3)}_{x,y} \dot{p}_x q_y + q_x q_y K^{(3)}_{x,y} p_x \dot{q}_y\Big] \\ &-\frac{J}{2d}\sum_{<x,y>}\Big[\dot{q}_x q_y K^{(4)}_{x,y} q_x q_y + q_x \dot{q}_y K^{(4)}_{x,y} q_x q_y + q_x q_y K^{(4)}_{x,y} \dot{q}_x q_y + q_x q_y K^{(4)}_{x,y} q_x \dot{q}_y\Big] \\ &-\frac{J}{d}\sum_{<x,y>}\Big[p_x q_y \dot{K}^{(3)}_{x,y} q_x q_y + q_x q_y \dot{K}^{(3)}_{x,y} p_x q_y\Big] \\ &-\frac{J}{2d}\sum_{<x,y>} q_x q_y \dot{K}^{(4)}_{x,y} q_x q_y.\end{aligned}$$

Now, using $i\dot{p}_x = [h_x^{\alpha_\varphi}, p_x]$, $i\dot{q}_x = [h_x^{\alpha_\varphi}, q_x]$ and the time-derivatives of $K^{(3)}_{x,y}$ and $K^{(4)}_{x,y}$ as in (93) and

(94), we arrive at

$$
\begin{aligned}
\dot{\tilde{H}} &= i\frac{J}{2d}\sum_{<x,y>} \\
&\Big[[h_x^{\alpha_\varphi},p_x]p_yK^{(2)}_{x,y}q_xq_y + p_x[h_y^{\alpha_\varphi},p_y]K^{(2)}_{x,y}q_xq_y + p_xp_yK^{(2)}_{x,y}[h_x^{\alpha_\varphi},q_x]q_y + p_xp_yK^{(2)}_{x,y}q_x[h_y^{\alpha_\varphi},q_y] \\
&+ [h_x^{\alpha_\varphi},p_x]q_yK^{(2)}_{x,y}q_xp_y + p_x[h_y^{\alpha_\varphi},q_y]K^{(2)}_{x,y}q_xp_y + p_xq_yK^{(2)}_{x,y}[h_x^{\alpha_\varphi},q_x]p_y + p_xq_yK^{(2)}_{x,y}q_x[h_y^{\alpha_\varphi},p_y] \\
&+ [h_x^{\alpha_\varphi},q_x]q_yK^{(2)}_{x,y}p_xp_y + q_x[h_y^{\alpha_\varphi},q_y]K^{(2)}_{x,y}p_xp_y + q_xq_yK^{(2)}_{x,y}[h_x^{\alpha_\varphi},p_x]p_y + q_xq_yK^{(2)}_{x,y}p_x[h_y^{\alpha_\varphi},p_y] \\
&+ [h_x^{\alpha_\varphi},q_x]p_yK^{(2)}_{x,y}p_xq_y + q_x[h_y^{\alpha_\varphi},p_y]K^{(2)}_{x,y}p_xq_y + q_xp_yK^{(2)}_{x,y}[h_x^{\alpha_\varphi},p_x]q_y + q_xp_yK^{(2)}_{x,y}p_x[h_y^{\alpha_\varphi},q_y]\Big] \\
&+ i\frac{J}{d}\sum_{<x,y>}\Big[[h_x^{\alpha_\varphi},p_x]q_yK^{(3)}_{x,y}q_xq_y + p_x[h_y^{\alpha_\varphi},q_y]K^{(3)}_{x,y}q_xq_y + p_xq_yK^{(3)}_{x,y}[h_x^{\alpha_\varphi},q_x]q_y + p_xq_yK^{(3)}_{x,y}q_x[h_y^{\alpha_\varphi},q_y] \\
&+ [h_x^{\alpha_\varphi},q_x]q_yK^{(3)}_{x,y}p_xq_y + q_x[h_y^{\alpha_\varphi},q_y]K^{(3)}_{x,y}p_xq_y + q_xq_yK^{(3)}_{x,y}[h_x^{\alpha_\varphi},p_x]q_y + q_xq_yK^{(3)}_{x,y}p_x[h_y^{\alpha_\varphi},q_y]\Big] \\
&+ i\frac{J}{2d}\sum_{<x,y>}\Big[[h_x^{\alpha_\varphi},q_x]q_yK^{(4)}_{x,y}q_xq_y + q_x[h_y^{\alpha_\varphi},q_y]K^{(4)}_{x,y}q_xq_y + q_xq_yK^{(4)}_{x,y}[h_x^{\alpha_\varphi},q_x]q_y + q_xq_yK^{(4)}_{x,y}q_x[h_y^{\alpha_\varphi},q_y]\Big] \\
&+ \frac{J}{d}\sum_{<x,y>}\Big[p_xq_y\big(\dot{\alpha}_\varphi a_x^* - \dot{\overline{\alpha_\varphi}}a_x\big)q_xq_y + q_xq_y\big(\dot{\alpha}_\varphi a_x^* - \dot{\overline{\alpha_\varphi}}a_x\big)p_xq_y\Big] \\
&+ \frac{J}{2d}\sum_{<x,y>} q_xq_y\big(\dot{\alpha}_\varphi(a_x^*+a_y^*) + \dot{\overline{\alpha_\varphi}}(a_x+a_y) - 4U\mathrm{Im}(\langle\varphi,\mathcal{N}a\varphi\rangle\overline{\alpha_\varphi})\big)q_xq_y.
\end{aligned}
$$

To obtain (84), we isolate the first part in the right hand side of (84) and keep the rest which we denoted by $\mathcal{R}$ in this case where the time derivative of α_φ is computed in (90). □

Proposition 22. *The rest term $\mathcal{R}(t)$ in Proposition 21 satisfies the following bound*

$$
|\mathcal{R}(t)| \leqslant \tilde{C}(t)\left(\frac{1}{|\Lambda|}\sum_{x\in\Lambda}\langle\Psi_d(t),(q_x(t)\mathcal{N}_xq_x(t)+q_x(t))\Psi_d(t)\rangle + \frac{1}{d}\right),
$$

where $\tilde{C}(t)$ depends on μ, U, J, $\langle\varphi(0),\mathcal{N}^j\varphi(0)\rangle$ for $j\leqslant 4$ and it is a polynomial of order six in t such that

$$
\begin{aligned}
\tilde{C}(t) := C(J,\mu,U)\left(1+\langle\varphi(0),\mathcal{N}\varphi(0)\rangle^2\right)\Big(1& \\
+\sum_{j=0}^{6}\left(8J\langle\varphi(0),\mathcal{N}\varphi(0)\rangle^{1/2}\right)^j\left\langle\varphi(0),(\mathcal{N}+j)^{4-\frac{j}{2}}\varphi(0)\right\rangle\frac{t^j}{j!}\Big),&
\end{aligned}
\tag{95}
$$

where $C(J,\mu,U)>0$ depends polynomially on the model parameters J, μ and U.

Proof. Let us estimate each term in $\mathcal{R}$. We start by explaining in details how to estimate one of the terms in (85). We have by Cauchy–Schwarz and Hölder's inequality

$$
\begin{aligned}
&\left| -\frac{J}{d}\frac{i}{|\Lambda|}\sum_{x\in\Lambda}\sum_{\substack{y\in\Lambda\\x\sim y}}\langle \Psi_d, p_x h_x^{\alpha_\varphi} p_y a_x^* a_y q_x q_y \Psi_d\rangle \right| \\
&\leqslant \frac{J}{d}\frac{1}{|\Lambda|}\sum_{x\in\Lambda}\|a_x^* q_x \Psi_d\| \left\| h_x^{\alpha_\varphi} p_x \left(\sum_{\substack{y\in\Lambda\\x\sim y}} q_y a_y^* p_y\right)\Psi_d \right\| \\
&\leqslant \frac{2J}{|\Lambda|}\sum_{x\in\Lambda}\sqrt{\langle\varphi(0),(\mathcal{N}+1)\varphi(0)\rangle}\sqrt{\langle\varphi,(h^{\alpha_\varphi})^2\varphi\rangle}\|a_x^* q_x\Psi_d\| \left(\frac{1}{2d}+\frac{1}{(2d)^2}\sum_{\substack{y\in\Lambda\\y\neq z, x\sim y}}\sum_{\substack{z\in\Lambda\\x\sim z}}\|q_z\Psi_d\|\|q_y\Psi_d\|\right)^{\frac{1}{2}} \\
&\leqslant \frac{2J}{|\Lambda|}\sum_{x\in\Lambda}\sqrt{\langle\varphi(0),(\mathcal{N}+1)\varphi(0)\rangle}\sqrt{\langle\varphi,(h^{\alpha_\varphi})^2\varphi\rangle}\|a_x^* q_x\Psi_d\| \left(\frac{1}{2d}+\frac{1}{4d}\sum_{\substack{y\in\Lambda\\y\neq z, x\sim y}}\|q_y\Psi_d\|^2+\frac{1}{4d}\sum_{\substack{z\in\Lambda\\x\sim z}}\|q_z\Psi_d\|^2\right)^{\frac{1}{2}} \\
&\leqslant CJ\sqrt{\langle\varphi(0),(\mathcal{N}+1)\varphi(0)\rangle}\sqrt{\langle\varphi,(h^{\alpha_\varphi})^2\varphi\rangle}\Big(\frac{1}{|\Lambda|}\sum_{x}\|a_x^* q_x\Psi_d\|^2+\frac{1}{d} \\
&\qquad\qquad + \frac{1}{d}\frac{1}{|\Lambda|}\underbrace{\sum_{x\in\Lambda}\sum_{\substack{y\in\Lambda\\x\sim y}}\|q_y\Psi_d\|^2}_{=2d\sum_{x\in\Lambda}\|q_x\Psi_d\|^2}+\frac{1}{d}\frac{1}{|\Lambda|}\underbrace{\sum_{x\in\Lambda}\sum_{\substack{z\in\Lambda\\x\sim z}}\|q_z\Psi_d\|^2}_{=2d\sum_{x\in\Lambda}\|q_x\Psi_d\|^2}\Big) \\
&\leqslant CJ\sqrt{\langle\varphi(0),(\mathcal{N}+1)\varphi(0)\rangle}\sqrt{\langle\varphi,(h^{\alpha_\varphi})^2\varphi\rangle}\left(\frac{1}{|\Lambda|}\sum_{x\in\Lambda}\langle\Psi_d,(q_x\mathcal{N}_x q_x+q_x)\Psi_d\rangle+\frac{1}{d}\right).
\end{aligned}
$$

The other terms in (85) could be done in similar manner. We get at the end

$$
|(85)| \leqslant CJ\left[1+\langle\varphi(0),(\mathcal{N})\varphi(0)\rangle\right]\sqrt{\langle\varphi,(h^{\alpha_\varphi})^2\varphi\rangle}\left(\frac{1}{|\Lambda|}\sum_{x\in\Lambda}\langle\Psi_d,(q_x\mathcal{N}_x q_x+q_x)\Psi_d\rangle+\frac{1}{d}\right).
$$

We will now provide estimates for some terms within (86), (87), and (88). We begin by estimating the terms involving two q inside (86):

$$
\begin{aligned}
&\left|\frac{J}{d}\frac{i}{|\Lambda|}\sum_{x\in\Lambda}\sum_{\substack{y\in\Lambda\\x\sim y}}\langle\Psi_d, q_y p_x h_x^{\alpha_\varphi} a_x^* a_y q_x p_y\Psi_d\rangle\right| \\
&= \left|\frac{J}{d}\frac{1}{|\Lambda|}\sum_{x\in\Lambda}\sum_{\substack{y\in\Lambda\\x\sim y}}\langle h_x^{\alpha_\varphi} p_x a_y^* q_y\Psi_d, a_x^* q_x p_y\Psi_d\rangle\right| \\
&\leqslant \frac{J}{d}\frac{1}{|\Lambda|}\sum_{x\in\Lambda}\sum_{\substack{y\in\Lambda\\x\sim y}}\sqrt{\langle\varphi,(h^{\alpha_\varphi})^2\varphi\rangle}\left(\|q_x\Psi_d\|^2+\|\mathcal{N}_x^{1/2}q_x\Psi_d\|^2\right)^{1/2}\left(\|q_y\Psi_d\|^2+\|\mathcal{N}_y^{1/2}q_y\Psi_d\|^2\right)^{1/2} \\
&\leqslant CJ\sqrt{\langle\varphi,(h^{\alpha_\varphi})^2\varphi\rangle}\frac{1}{|\Lambda|}\sum_{x\in\Lambda}\langle\Psi_d,(q_x\mathcal{N}_x q_x+q_x)\Psi_d\rangle.
\end{aligned}
$$

Treating the other terms similarly inside (86), we get

$$
|(86)| \leqslant CJ\left(1+\sqrt{\langle\varphi(0),(\mathcal{N})\varphi(0)\rangle}\right)\sqrt{\langle\varphi,(h^{\alpha_\varphi})^2\varphi\rangle}\frac{1}{|\Lambda|}\sum_{x\in\Lambda}\langle\Psi_d,(q_x\mathcal{N}_x q_x+q_x)\Psi_d\rangle.
$$

Next, we estimate the two terms involving three q inside (87). The first is

$$\begin{aligned}
&\left|\frac{J}{d}\frac{i}{|\Lambda|}\sum_{x\in\Lambda}\sum_{\substack{y\in\Lambda\\x\sim y}}\langle\Psi_d, q_y h_x^{\alpha_\varphi} p_x a_x^* a_y q_x q_y \Psi_d\rangle\right| = \left|\frac{J}{d}\frac{1}{|\Lambda|}\sum_{x\in\Lambda}\sum_{\substack{y\in\Lambda\\x\sim y}}\langle a_y^* q_y\Psi_d, h_x^{\alpha_\varphi} p_x a_x^* q_x q_y \Psi_d\rangle\right| \\
&\leqslant \frac{J}{d}\frac{1}{|\Lambda|}\sum_{x\in\Lambda}\sum_{\substack{y\in\Lambda\\x\sim y}}\|a_y^* q_y\Psi_d\|\Big[\langle\Psi_d, q_x a_x \underbrace{p_x(h_x^{\alpha_\varphi})^2 p_x}_{=\langle\varphi,(h^{\alpha_\varphi})^2\varphi\rangle p_x} a_x^* q_x\Psi_d\rangle\Big]^{\frac{1}{2}} \\
&\leqslant \frac{J}{d}\frac{1}{|\Lambda|}\sum_{x\in\Lambda}\sum_{\substack{y\in\Lambda\\x\sim y}}\sqrt{\langle\varphi,(h^{\alpha_\varphi})^2\varphi\rangle}\Big[\langle\Psi_d, q_y(\mathcal{N}_y+1)q_y\Psi_d\rangle\Big]^{\frac{1}{2}}\Big[\langle\Psi_d, q_x(\mathcal{N}_x+1)q_x\Psi_d\rangle\Big]^{\frac{1}{2}} \\
&\leqslant CJ\sqrt{\langle\varphi,(h^{\alpha_\varphi})^2\varphi\rangle}\frac{1}{|\Lambda|}\sum_{x\in\Lambda}\langle\Psi_d,(q_x\mathcal{N}_x q_x + q_x)\Psi_d\rangle
\end{aligned}$$

and the second is

$$\begin{aligned}
&\left|-\frac{J}{d}\frac{i}{|\Lambda|}\sum_{x\in\Lambda}\sum_{\substack{y\in\Lambda\\x\sim y}}\langle\Psi_d, q_y p_x h_y^{\alpha_\varphi} p_y a_x^* a_y q_x q_y \Psi_d\rangle\right| = \left|-\frac{J}{d}\frac{i}{|\Lambda|}\sum_{x\in\Lambda}\sum_{\substack{y\in\Lambda\\x\sim y}}\langle q_y a_x p_x\Psi_d, h_y^{\alpha_\varphi} p_y a_y q_y q_x \Psi_d\rangle\right| \\
&\leqslant \frac{J}{d}\frac{1}{|\Lambda|}\sum_{x\in\Lambda}\sum_{\substack{y\in\Lambda\\x\sim y}}\sqrt{\langle\varphi(0),\mathcal{N}\varphi(0)\rangle}\|q_y\Psi_d\|\left(\left\langle\Psi_d, q_x q_y a_y^* \underbrace{p_y(h_y^{\alpha_\varphi})^2 p_y}_{=\langle\varphi,(h^{\alpha_\varphi})^2\varphi\rangle p_y} a_y q_y\Psi_d\right\rangle\right)^{1/2} \\
&= \frac{J}{d}\frac{1}{|\Lambda|}\sum_{x\in\Lambda}\sum_{\substack{y\in\Lambda\\x\sim y}}\sqrt{\langle\varphi,(h^{\alpha_\varphi})^2\varphi\rangle}\sqrt{\langle\varphi(0),\mathcal{N}\varphi(0)\rangle}\|q_y\Psi_d\|\left(\left\langle\Psi_d, q_x q_y a_y^* \underbrace{p_y}_{\leqslant 1} a_y q_y\Psi_d\right\rangle\right)^{1/2} \\
&\leqslant CJ\sqrt{\langle\varphi(0),\mathcal{N}\varphi(0)\rangle}\sqrt{\langle\varphi,(h^{\alpha_\varphi})^2\varphi\rangle}\frac{1}{|\Lambda|}\sum_{x\in\Lambda}\langle\Psi_d,(q_x\mathcal{N}_x q_x + q_x)\Psi_d\rangle.
\end{aligned}$$

And finally we estimate one term involving four qs

$$\begin{aligned}
&\left|\frac{J}{d}\frac{i}{|\Lambda|}\sum_{<x,y>}\langle\Psi_d, q_x q_y a_x^* a_y p_x h_x^{\alpha_\varphi} q_x q_y\Psi_d\rangle\right| \\
&= \left|\frac{J}{d}\frac{1}{|\Lambda|}\sum_{<x,y>}\langle h_x^{\alpha_\varphi} p_x a_x q_x q_y\Psi_d, a_y q_x q_y\Psi_d\rangle\right| \\
&\leqslant CJ\sqrt{\langle\varphi,(h^{\alpha_\varphi})^2\varphi\rangle}\frac{1}{|\Lambda|}\sum_{x\in\Lambda}\langle\Psi_d, q_x\mathcal{N}_x q_x\Psi_d\rangle.
\end{aligned}$$

Estimating the other terms similarly, we get

$$|(86)+(87)+(88)| \leqslant CJ\left(1+\langle\varphi(0),\mathcal{N}\varphi(0)\rangle\right)\sqrt{\langle\varphi,(h^{\alpha_\varphi})^2\varphi\rangle}\frac{1}{|\Lambda|}\sum_{x\in\Lambda}\langle\Psi_d,(q_x\mathcal{N}_x q_x + q_x)\Psi_d\rangle.$$

We also have

$$\begin{aligned}
|(89)| \leqslant & CJ\left(|\mu+U|\sqrt{\langle\varphi(0),\mathcal{N}\varphi(0)\rangle} + U\langle\varphi,\mathcal{N}^{3/2}\varphi\rangle + U\sqrt{\langle\varphi(0),\mathcal{N}\varphi(0)\rangle}\sqrt{\langle\varphi,(\mathcal{N}+1)^{3/2}\varphi\rangle}\right) \\
&\frac{1}{|\Lambda|}\sum_{x\in\Lambda}\langle\Psi_d,\big(q_x\mathcal{N}_x q_x + q_x\big)\Psi_d\rangle,
\end{aligned}$$

where we have used

$$
\begin{aligned}
|\dot{\alpha}| &\leqslant C\left(|\mu|\sqrt{\langle\varphi(0),\mathcal{N}\varphi(0)\rangle}+U\langle\varphi,(\mathcal{N}+1)^{3/2}\varphi\rangle\right)\\
|\dot{\overline{\alpha_\varphi}}\alpha_\varphi+\overline{\alpha_\varphi}\dot{\alpha}_\varphi| &\leqslant CU\sqrt{\langle\varphi(0),\mathcal{N}\varphi(0)\rangle}\langle\varphi,(\mathcal{N}+1)^{3/2}\varphi\rangle.
\end{aligned}
$$

Combining all arguments, we get the following bound on the expectation of $\mathcal{R}$

$$
|\mathcal{R}|\leqslant C(J,\mu,U)\left(1+\langle\varphi(0),\mathcal{N}\varphi(0)\rangle\right)\sqrt{\langle\varphi,(h^{\alpha_\varphi})^2\varphi\rangle}\left(\frac{1}{|\Lambda|}\sum_{x\in\Lambda}\langle\Psi_d,(q_x\mathcal{N}_xq_x+q_x)\Psi_d\rangle+\frac{1}{d}\right).
$$

where clearly $C(J,\mu,U)$ is a polynomial in J, μ and U. We also have by Cauchy–Schwarz

$$
\begin{aligned}
\langle\varphi,(h^{\alpha_\varphi})^2\varphi\rangle &= \left\langle\varphi,\left(-J\big(\alpha_\varphi a^*+\overline{\alpha_\varphi}a-|\alpha_\varphi|^2\big)+(J-\mu)\mathcal{N}+\frac{U}{2}\mathcal{N}(\mathcal{N}-1)\right)^2\varphi\right\rangle\\
&\leqslant C(J,\mu,U)\left(1+\langle\varphi(0),\mathcal{N}\varphi(0)\rangle^2\right)\left(1+\langle\varphi,\mathcal{N}^4\varphi\rangle\right)
\end{aligned}
\tag{96}
$$

which implies, by exploiting the bound (61), that

$$
\begin{aligned}
\sqrt{\langle\varphi,(h^{\alpha_\varphi})^2\varphi\rangle} &\leqslant C(J,\mu,U)\left(1+\langle\varphi(0),\mathcal{N}\varphi(0)\rangle\right)\left(1+\sqrt{\langle\varphi,\mathcal{N}^4\varphi\rangle}\right)\\
&\leqslant C(J,\mu,U)\left(1+\langle\varphi(0),\mathcal{N}\varphi(0)\rangle\right)\Bigg(1+\\
&\sum_{j=0}^{6}\frac{\left(8J\langle\varphi(0),\mathcal{N}\varphi(0)\rangle^{1/2}t\right)^j}{j!}\left\langle\varphi(0),(\mathcal{N}+j)^{4-\frac{j}{2}}\varphi(0)\right\rangle\Bigg)
\end{aligned}
\tag{97}
$$

where in the above estimate we have used the bound on α_φ in (60) and the estimate (61). Then, using the above bound on the estimate of $|\mathcal{R}|$, we get the desired result.

□

Proof of Proposition 17. Recall the definition of f in (59)*, its derivative is*

$$
\begin{aligned}
\dot{f} &= \frac{i}{|\Lambda|}\sum_{x\in\Lambda}\langle\Psi_d,[H_d,H_d+q_xh_x^{\alpha_\varphi}q_x-h_x^{\alpha_\varphi}+cq_x]\Psi_d\rangle+\frac{1}{|\Lambda|}\sum_{x\in\Lambda}\langle\Psi_d,(q_x\dot{h}_x^{\alpha_\varphi}q_x-\dot{h}_x^{\alpha_\varphi})\Psi_d\rangle\\
&-\frac{i}{|\Lambda|}\sum_{x\in\Lambda}\langle\Psi_d,[h_x^{\alpha_\varphi},q_xh_x^{\alpha_\varphi}q_x+cq_x]\Psi_d\rangle\\
&=\frac{i}{|\Lambda|}\sum_{x\in\Lambda}\langle\Psi_d,[\tilde{H},q_xh_x^{\alpha_\varphi}q_x-h_x^{\alpha_\varphi}+cq_x]\Psi_d\rangle+\frac{1}{|\Lambda|}\sum_{x\in\Lambda}\langle\Psi_d,(q_x\dot{h}_x^{\alpha_\varphi}q_x+\dot{\tilde{H}})\Psi_d\rangle,
\end{aligned}
$$

where we have used that the fact that H is time independent but it can be written as sum of two time dependent quantities which implies $\dot{H}_d=0=\sum_{x\in\Lambda}\dot{h}_x^{\alpha_\varphi}+\dot{\tilde{H}}$. Exploit Proposition 21, we get

$$
\dot{f}=\frac{i}{|\Lambda|}\sum_{x\in\Lambda}\langle\Psi_d,[\tilde{H},cq_x]\Psi_d\rangle+\frac{1}{|\Lambda|}\sum_{x\in\Lambda}\langle\Psi_d,q_x\dot{h}_x^{\alpha_\varphi}q_x\Psi_d\rangle+\mathcal{R}.
$$

We now estimate the remaining terms inside $\dot{f}$. So, we have

$$
\begin{aligned}
&\left|\frac{1}{|\Lambda|}\sum_{x\in\Lambda}\langle\Psi_d,q_x\dot{h}_x^{\alpha_\varphi}q_x\Psi_d\rangle\right|\\
&=\left|\frac{1}{|\Lambda|}\left\langle\Psi_d,q_x\left(-J\dot{\alpha}_\varphi a_x^*-J\dot{\overline{\alpha_\varphi}}a_x+2JU\mathrm{Im}(\langle\varphi,\mathcal{N}a\varphi\rangle\overline{\alpha_\varphi})\right)q_x\Psi_d\right\rangle\right|\\
&\leqslant C(J,\mu,U)\left(1+\langle\varphi(0),\mathcal{N}\varphi(0)\rangle\right)\left(1+\sqrt{\langle\varphi,\mathcal{N}^2\varphi\rangle}\right)\frac{1}{|\Lambda|}\sum_{x\in\Lambda}\left\langle\Psi_d,\left(q_x\mathcal{N}_x^2q_x+q_x\right)\Psi_d\right\rangle,
\end{aligned}
$$

and since c depends on J, μ, U and on the initial data as indicated in (81)*, we have*

$$\begin{aligned}&\left|\frac{1}{|\Lambda|}\sum_{x\in\Lambda}\langle\Psi_d,[\tilde{H},cq_x]\Psi_d\rangle\right|\\&\leqslant C(J,\mu,U)\left(1+\frac{1}{U}+\langle\varphi(0),\mathcal{N}\varphi(0)\rangle^2\right)\frac{1}{|\Lambda|}\sum_{x\in\Lambda}\left\langle\Psi_d,(q_x\mathcal{N}_x^2q_x+q_x)\Psi_d\right\rangle+\frac{1}{d}.\end{aligned}$$

Exploit the above two estimates and Proposition 22, then use Proposition 20 and the left hand side of (78) *which gives*

$$g=\frac{1}{|\Lambda|}\sum_{x\in\Lambda}\left\langle\Psi_d,(q_x\mathcal{N}_x^2q_x+q_x)\Psi_d\right\rangle\leqslant\frac{4}{U}(f+\frac{1}{d})$$

to get

$$\left|\frac{d}{dt}f(t)\right|\leqslant\tilde{C}(t)\left(f(t)+\frac{1}{d}\right)+\frac{1}{d},$$

where $\tilde{C}(t)$ depends now on the initial data and on the other parameters of our model as defined in (64)*. Now, using Grönwall lemma, we arrive at*

$$f(t)\leqslant e^{\int_0^t\tilde{C}(s)ds}f(0)+\frac{1}{d}\int_0^t\left(1+\tilde{C}(s)\right)e^{\int_s^t\tilde{C}(r)dr}ds. \tag{98}$$

□

Proof of Proposition 18. *We have*

$$\begin{aligned}&\frac{1}{|\Lambda|}\sum_{x\in\Lambda}\langle\Psi_d,q_x\Psi_d\rangle\leqslant\frac{1}{|\Lambda|}\sum_{x\in\Lambda}\langle\Psi_d,(q_x\mathcal{N}_x^2q_x+q_x)\Psi_d\rangle\\&\leqslant\frac{4}{U}\left(f+\frac{1}{d}\right)\\&\leqslant\frac{4}{U}e^{\int_0^t\tilde{C}(s)ds}f(0)+\frac{1}{d}\left(\frac{4}{U}+\frac{4}{U}\int_0^t\left(1+\tilde{C}(s)\right)e^{\int_s^t\tilde{C}(r)dr}ds\right)\\&\leqslant\frac{C}{U}\left(1+J^2+U+\left(J-\mu-\frac{U}{2}\right)^2\right)\left(1+\frac{1}{U}+\langle\varphi(0),\mathcal{N}\varphi(0)\rangle^2\right)e^{\int_0^t\tilde{C}(s)ds}\\&\frac{1}{|\Lambda|}\sum_{x\in\Lambda}\left\langle\Psi_d(0),(q_x(0)\mathcal{N}_x^2q_x(0)+q_x(0))\Psi_d(0)\right\rangle+\frac{1}{d}\frac{4}{U}\left(1+e^{\int_0^t\tilde{C}(s)ds}+\int_0^t(1+\tilde{C}(s))e^{\int_s^t\tilde{C}(r)dr}ds\right).\end{aligned} \tag{99}$$

where $\tilde{C}(t)$ is defined in (64)*. In the above computations, we have used the left hand side of the estimate* (78) *in the second inequality; then in third inequality, we have used the estimate* (63)*; then again in the fourth inequality, we have used the right hand side of* (78)*.* □

Proof of Theorem 2. *First note that since* $\mathrm{Tr}(p(0)\mathcal{N}^4)\leqslant C$*, we get that $\tilde{C}(t)$ defined in* (64) *satisfies*

$$\tilde{C}(t)\leqslant C(J,\mu,U)\left(1+\sum_{j=1}^{6}t^j\right)$$

where $C(J,\mu,U)>0$ depends polynomially on the paramters of our model J, μ and U. Exploit this in the equation in Proposition 18, we get

$$\begin{aligned}&\frac{1}{|\Lambda|}\sum_{x\in\Lambda}\langle\Psi_d(t),q_x(t)\Psi_d(t)\rangle\leqslant\frac{1}{d}\frac{1}{U}\\&+C(J,\mu,U)e^{C(J,\mu,U)\sum_1^7|t|^j}\left(1+\frac{1}{U^2}\right)\left(\frac{1}{|\Lambda|}\sum_{x\in\Lambda}\left\langle\Psi_d(0),\left(q_x(0)\mathcal{N}_x^2q_x(0)+q_x(0)\right)\Psi_d(0)\right\rangle+\frac{1}{d}\right).\end{aligned} \tag{100}$$

□

References

[1] *M. Aizenman, E. Lieb, R. Seiringer, J. Solovej, and J. Yngvason. Bose–Einstein quantum phase transition in an optical lattice model.* Phys. Rev. A*, 70:023612, 2004.*

[2] *M. H. Anderson, J. R. Ensher, M. R. Matthews, C. E. Wieman, and E. A. Cornell. Observation of Bose–Einstein condensation in a dilute atomic vapor.* Science*, 269(5221):198–201, 1995.*

[3] *V. Bach, S. Breteaux, S. Petrat, P. Pickl, and T. Tzaneteas. Kinetic energy estimates for the accuracy of the time-dependent hartree-fock approximation with coulomb interaction.* Journal de Mathématiques Pures et Appliquées*, 105:1–30, 2014.*

[4] *N. Benedikter, M. Porta, and B. Schlein. Mean–field evolution of fermionic systems.* Communications in Mathematical Physics*, 331(3):1087–1131, Apr. 2014.*

[5] *I. Bloch, J. Dalibard, and W. Zwerger. Many-body physics with ultracold gases.* Reviews of Modern Physics*, 80:885–964, 2007.*

[6] *Bose. Plancks Gesetz und Lichtquantenhypothese.* Z. Phys.*, 26:178–181, 1924.*

[7] *A. Bove, G. D. Prato, and G. Fano. On the hartree-fock time-dependent problem.* Communications in Mathematical Physics*, 49:25–33, 1976.*

[8] *J. Chadam. The time-dependent hartree-fock equations with coulomb two-body interaction.* Communications in Mathematical Physics*, 46:99–104, 1976.*

[9] *L. Chen, J. Lee, and M. Liew. Combined mean-field and semiclassical limits of large fermionic systems.* Journal of Statistical Physics*, 182(2), Jan. 2021.*

[10] *K. B. Davis, M.-O. Mewes, M. R. Andrews, N. J. van Druten, D. S. Durfee, D. M. Kurn, and W. Ketterle. Bose–Einstein condensation in a gas of sodium atoms.* Phys. Rev. Lett.*, 75(22):3969–3973, 1995.*

[11] *A. Einstein. Quantentheorie des einatomigen idealen Gases.* Sitzber. Kgl. Preuss. Akad. Wiss.*, pages 261–267, 1924.*

[12] *A. Einstein. Quantentheorie des einatomigen idealen Gases. Zweite Abhandlung.* Sitzber. Kgl. Preuss. Akad. Wiss.*, pages 3–14, 1925.*

[13] *M. P. A. Fisher, P. B. Weichman, G. Grinstein, and D. S. Fisher. Boson localization and the superfluid-insulator transition.* Phys. Rev. B*, 40:546–570, Jul 1989.*

[14] *S. Fournais and J. P. Solovej. The energy of dilute Bose gases.* Ann. Math.*, 192:893–976, 2020.*

[15] *S. Fournais and J. P. Solovej. The energy of dilute Bose gases ii: the general case.* Invent. Math.*, 232:863–994, 2022.*

[16] *A. Georges, G. Kotliar, W. Krauth, and M. Rozenberg. Dynamical mean-field theory of strongly correlated fermion systems and the limit of infinite dimensions.* Rev. Mod. Phys.*, 68:13–125, 01 1996.*

[17] *F. Golse.* On the Dynamics of Large Particle Systems in the Mean Field Limit*, page 1–144. Springer International Publishing, 2016.*

[18] *M. Greiner, O. Mandel, T. Esslinger, T. W. Hänsch, and I. Bloch. Quantum phase transition from a superfluid to a mott insulator in a gas of ultracold atoms.* Nature*, 415:39–44, 2002.*

[19] M. Greiner, O. Mandel, T. Rom, A. Altmeyer, A. Widera, T. Hänsch, and I. Bloch. *Quantum phase transition from a superfluid to a mott insulator in an ultracold gas of atoms.* Physica B: Condensed Matter, *329-333:11–12, 2003. Proceedings of the 23rd International Conference on Low Temperature Physics.*

[20] W.-J. Hu and N.-H. Tong. *Dynamical mean-field theory for the bose-hubbard model.* Phys. Rev. B, *80:245110, Dec 2009.*

[21] D. Jaksch, C. Bruder, J. I. Cirac, C. W. Gardiner, and P. Zoller. *Cold bosonic atoms in optical lattices.* Phys. Rev. Lett., *81:3108–3111, Oct 1998.*

[22] W. Metzner and D. Vollhardt. *Correlated lattice fermions in $d = \infty$ dimensions.* Phys. Rev. Lett., *62:324–327, Jan 1989.*

[23] E. Pavarini, E. Koch, A. Lichtenstein, and D. Vollhardt. Dynamical Mean-Field Theory of Correlated Electrons. *Institute for Advanced Simulation, 2022.*

[24] S. Petrat and P. Pickl. *A new method and a new scaling for deriving fermionic mean-field dynamics.* Mathematical Physics, Analysis and Geometry, *19, 2016.*

[25] E. Picari, A. Ponno, and L. Zanelli. *Mean field derivation of dnls from the bose–hubbard model.* Annales Henri Poincaré, *23, 10 2021.*

[26] M. Porta, S. Rademacher, C. Saffirio, and B. Schlein. *Mean field evolution of fermions with coulomb interaction.* Journal of Statistical Physics, *166(6):1345–1364, Jan. 2017.*

[27] I. E. Segal. *Non-linear semi-groups.* Annals of Mathematics, *78:339, 1963.*

[28] W. Zwerger. *Mott–Hubbard transition of cold atoms in optical lattices.* Journal of Optics B, *5:9–16, 2003.*