PHD Thesis

Semi-classical limits of 2D fermions under high magnetic field

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Résumé en français

Cette thèse porte sur l'analyse mathématique de systèmes de fermions dans la limite où le nombre de particules tend vers l'infini. L'étude de ce genre de systèmes nécessite de calculer les énergies et densités des états physiques. Ce problème est extrêmement difficile à résoudre même numériquement en raison des interactions et des corrélations entre les particules. En effet avec un grand nombre de particules l'espace de Hilbert représentant l'espace des états possibles du système est immense. Cependant, l'objectif général est d'obtenir des théories effectives où l'énergie ne dépend que de la densité de particules (théorie fonctionnelle de la densité). Dans le cadre de la thèse, nous nous intéressons plus précisément à des fermions sans spin confinés dans un espace à deux dimensions soumis à un fort champ magnétique transverse et homogène. C'est à dire que l'amplitude du champ magnétique tend vers l'infini en même temps que le nombre de particules. Ce contexte physique est motivé par l'effet Hall quantique. Les principaux outils proviennent de l'analyse semi-classique mais aussi plus généralement de la théorie spectrale et de l'analyse fonctionnelle.

Après un premier chapitre introductif situant les résultats principaux par rapport à la bibliographie, le deuxième chapitre de la thèse reprend le contenu de la prépublication "Multiple Landau level filling for a mean field limit of 2D fermions". Dans ce travail, nous étudions la limite de champ moyen, couplée à une limite semi-classique et fort champ magnétique, pour l'état fondamental du système de fermions. En physique classique, les particules chargées soumises à un champ magnétique transverse et homogène décrivent des orbites. En mécanique quantique, l'énergie cinétique est quantifiée en niveaux d'énergie discrets appelés niveaux de Landau, séparés par un gap constant. Les résultats principaux de cet article étendent des travaux de Lieb-Solovej-Yngvason et Fournais-Madsen en considérant un régime où le gap entre niveaux de Landau est l'échelle d'énergie dominante. De plus, les particules sont placées dans un domaine borné, permettant à la dégénérescence des niveaux de Landau d'être finie. Nous obtenons ainsi un modèle limite où un nombre arbitraire de niveaux de Landau sont remplis. Les travaux existants, utilisant un confinement par un potentiel de piégeage, ne peuvent décrire cette situation. Nous discutons aussi de la physique du modèle limite dans le dernier niveau de Landau partiellement rempli.

Le dernier chapitre présente un résultat sur la dynamique dans le même contexte que le chapitre précédent. Notre point de départ est l'équation de Hartree pour la première matrice densité. Il est bien connu que cette équation peut être obtenue par une approximation de champ moyen de la dynamique de Schrödinger à N corps pour des fermions en interaction. Nous étudions une limite fort champ magnétique, couplée avec une limite semi-classique et prouvons que la densité converge vers une solution d'une équation de transport gyro-cinétique. Dans un cadre de mécanique classique, des travaux de Golse-St Raymond obtiennent des résultats similaires avec comme point de départ l'équation de Vlasov. Plus récemment un travail dans la limite semi-classique de Ben Porat traite le cas où le gap entre niveaux de Landau est très petit devant les interactions. La nouveauté principale de notre résultat est de considérer le régime quantique au départ où le gap entre niveaux de Landau est du même ordre de grandeur que l'énergie d'interaction. La preuve consiste à construire une densité semi-classique dont l'évolution en temps est obtenue puis comparée à l'équation cible.

Abstract in English

This thesis is about the mathematical analysis of fermionic systems in the large number of particles limit. The study of this kind of systems requires computing the energies and densities of the physical states. This problem is extremely difficult to solve, even numerically, due to the interactions and correlations between particles. Indeed, with a large number of particles, the Hilbert space representing the space of possible states of the system is very large. However, the general goal is to obtain effective theories where the energy only depends on the particle density (density functional theory). More precisely, we consider spinless fermions confined in a two-dimensional space subjected to a strong transverse and homogeneous magnetic field. That is to say the amplitude of the magnetic field goes to infinity simultaneously with the number of particles. This physical context is motivated by the quantum Hall effect. The main tools come from semi-classical analysis but also more generally from spectral theory and functional analysis.

After a first introductory chapter contextualising the main results in the bibliography, the second chapter is based on the content of the pre-publication “Multiple Landau level filling for a mean field limit of 2D fermions”. In this work, we study the mean field limit, coupled with a semi-classical and strong magnetic field limit, of the fermionic system's ground state. In classical physics, charged particles subjected to a transverse and homogeneous magnetic field describe orbits. In quantum mechanics, kinetic energy is quantized into discrete energy levels called Landau levels, separated by a constant gap. The main results of this paper extend work by Lieb-Solovej-Yngvason and Fournais-Madsen by considering a regime where the gap between Landau levels is the dominant energy scale. Moreover, the particles are placed in a bounded domain, allowing the degeneracy of the Landau levels to be finite. We thus obtain a limit model where an arbitrary number of Landau levels are filled. Existing works, using confinement by a trapping potential, cannot describe this situation. We also discuss the physics of the limit model in the last partially filled Landau level.

The last chapter presents a result on the dynamics in the same context as the previous chapter. Our starting point is the Hartree equation for the first density matrix. It is well known that this equation can be obtained by a mean-field approximation of the many-body Schrödinger dynamics for interacting fermions. We study a strong magnetic field limit, coupled with a semi-classical limit and prove that the density converges to a solution of a gyrokinetic transport equation. In a classical mechanics framework, Golse-St Raymond obtained similar results with the Vlasov equation as starting point. More recently, a work of Ben Porat in the semi-classical limit deals with the case where the gap between Landau levels is very small compared to the interactions. The main novelty of our result is to consider a quantum regime where the gap between Landau levels is of the same order as the interaction energy. The proof consists in constructing a semi-classical density for which the dynamic is obtained and then compared to the target equation.

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Chapter I

Introduction

I.1 Physique quantique à N corps

I.1.1 Contexte

Le problème à N corps quantique correspond à l'étude d'un système de N particules en interaction régies par les lois de la mécanique quantique. On s'intéresse typiquement aux énergies possibles, aux densités et à la dynamique du système. En mécanique quantique, une particule placée dans un domaine Ω est décrite par un vecteur de l'espace de Hilbert $L^2(\Omega, \mathbb{C})$. Un système de N particules est quant à lui décrit par un élément de

$$L^2(\Omega, \mathbb{C})^{\otimes N} \cong L^2(\Omega^N, \mathbb{C})$$

L'énergie d'un système quantique est décrite par un opérateur auto-adjoint agissant sur l'espace de Hilbert des états, appelé Hamiltonien, noté $\mathcal{H}$. Lorsque l'on mesure l'énergie d'un système quantique, le résultat est une valeur propre de l'Hamiltonien, puis, l'état se retrouve projeté sur le sous espace propre associé. Étant donné un état physique ψ, la densité de particules correspond au module carré de cet état, elle est ainsi intégrable. La dynamique est quant à elle régie par l'équation de Schrödinger

$$i\hbar\partial_t\psi = \mathcal{H}\psi$$

$\hbar$ est la constante de Planck réduite, elle définit l'échelle des phénomènes quantiques. Dans un système d'unités pour lequel la masse des particule est choisie égale à 1/2, l'Hamiltonien d'un système de N particules sans spin prend la forme :

$$\mathcal{H}_N \coloneqq \sum_{i=1}^{N} \left(-\hbar^2\Delta_i + V(x_i)\right) + \sum_{i<j} w(x_i - x_j) \tag{I.1.1}$$

L'opérateur $-\Delta$ correspond à l'énergie cinétique du système. $V : \Omega \to \mathbb{R}$ est le potentiel extérieur auquel sont soumis les particules. w représente le potentiel d'interaction entre une paire de particules. Les potentiels agissent multiplicativement sur les états physiques. Les notations $\Delta_i, V(x_i)$ signifient que ces opérateurs n'agissent que sur la $i^{\text{ième}}$ particule, autrement dit sur la $i^{\text{ième}}$ coordonnée de $L^2(\Omega^N, \mathbb{C})$. $w(x_i - x_j)$ agit quant à lui sur la paire de particules (i, j), c'est-à-dire sur la $i^{\text{ième}}$ et la $j^{\text{ième}}$ coordonnée de $L^2(\Omega^N, \mathbb{C})$. L'Hamiltonien à N corps comprend alors la somme des énergies cinétique et potentielle de chaque particule ainsi que la somme des interactions entre chaque paire de particules.

Dans un état physique à N corps $\psi_N \in L^2(\Omega^N, \mathbb{C})$, les densités des particules sont corrélées si ψ_N n'est pas factorisable sous la forme

$$\psi_N = u^{\otimes N}, \text{ avec } u \in L^2(\Omega, \mathbb{C})$$

La résolution du problème quantique à N corps, c'est-à-dire la diagonalisation de l'Hamiltonien à N corps est en général particulièrement difficile à cause des interactions et des corrélations entre les particules. Même numériquement, l'étude de ce problème devient vite impossible pour de grandes valeurs de N car l'espace de Hilbert des états devient immense. Le nombre d'électrons dans les systèmes physiques est typiquement de l'ordre de 10^{10} électrons par mm^2 d'après [43]. Comprendre comment les particules interagissent, se distribuent dans l'espace et évoluent dans le temps dans un système à N corps est essentiel pour de nombreux domaines de la physique, tels que la physique des particules, la chimie quantique et l'étude des matériaux quantiques. Dans un premier temps on peut s'intéresser à la valeur propre la plus basse (l'énergie de l'état fondamental) qui est l'énergie du système sans excitation (à température nulle). Du côté des mathématiques, l'objectif général est l'obtention de théories efficaces où l'énergie ne dépend que de la densité physique des particules. La densité de l'état fondamental est ainsi donnée par le minimiseur et l'énergie fondamentale par le minimum de la fonctionnelle. Une telle théorie est appelée une théorie fonctionnelle de la densité. La minimisation de l'énergie se fait donc sur un espace fonctionnel des densités, plus simple que l'espace de Hilbert de départ. Cela vient cependant aux dépens d'une non-linéarité dans le terme d'interaction. Le modèle le plus simple de ce type est le modèle de Thomas Fermi (voir Subsection I.2.1). Les résultats mathématiques sur les théories fonctionnelles de la densité sont connus pour être utiles dans les méthodes numériques pour l'étalonnage des paramètres libres. Les théories fonctionnelles de la densité sont les principaux outils pour les calculs numériques quantiques, par exemple en chimie quantique où l'on veut calculer la densité électronique dans les systèmes moléculaires.

Par exemple, la théorie d'Hartree-Fock, qui sera détaillée en Subsection II.6.1, fait l'hypothèse de champ moyen. Cela revient à supposer que les particules ne sont pas corrélées, c'est-à-dire que chaque particule ne voit que le potentiel moyen généré par les autres particules. Mathématiquement, cette hypothèse revient à supposer que les densités des particules sont indépendantes. Ainsi, une expression de l'énergie qui ne dépend que de la densité peut être obtenue. Certaines théories fonctionnelles de la densité ajoutent des termes pour prendre en compte les corrélations améliorant ainsi la théorie d'Hartree-Fock. Cela est possible grâce à différentes approximations dont la plus simple est l'approximation de densité locale. Cette approximation suppose que le terme d'échange-corrélation dans la fonctionnelle est localement égal à celui pouvant être obtenu analytiquement pour un gaz homogène d'électrons.

Mathématiquement, il est intéressant d'apporter une justification rigoureuse à ce genre d'approximations en montrant l'égalité du résultat apporté par les modèles approchés avec celui fourni par l'état fondamental de l'Hamiltonien quand $N \to \infty$ au vu du nombre de particules dans les systèmes physiques concernés.

I.1.2 Fermions

Les particules physiques étant indiscernables, la densité physique de particules doit être invariante par permutation des coordonnées. Il y a deux manières canoniques d'imposer cette symétrie sur $L^2(\Omega^N, \mathbb{C})$ permettant de modéliser des bosons ou des fermions. En effet, une fonction symétrique ou antisymétrique sous l'échange de deux coordonnées est respectivement appelée bosonique ou fermionique.

Notation I.1.1

On note $H \coloneqq L^2(\Omega)$ l'espace des états modélisant une particule. On utilisera la notation bra-ket, aussi appelée formalisme de Dirac. Un vecteur $\psi \in H$ sera aussi noté $|\psi\rangle$, et son dual, c'est-à-dire la projection sur ψ, $\langle\psi|$. Le produit scalaire sur H sera quant à lui noté $\langle \bullet | \bullet \rangle$.

On considère l'action du groupe symétrique S_N sur $H^{\otimes N}$, définie par

$$\begin{array}{ccc} S_N \times H^{\otimes N} & \to & H^{\otimes N} \\ \sigma, u := \bigotimes_{i=1}^N u_i & \mapsto & \mathcal{U}_\sigma u := \bigotimes_{i=1}^N u_{\sigma^{-1}(i)} \end{array}$$

On définit alors l'espace de Hilbert symétrique/antisymétrique à N corps comme

$$H^{\otimes_\pm N} := \bigcap_{\sigma \in S_N} \operatorname{Ker}\left(\mathcal{U}_\sigma - \epsilon_\pm(\sigma)\right)$$

Avec cette convention,

$$\mathcal{U}_\sigma u(x_{1:N}) = \prod_{i=1}^N u_{\sigma^{-1}(i)}(x_i) = \prod_{i=1}^N u_i(x_{\sigma(i)}) = u(x_{\sigma(i)}, \dots x_{\sigma(N)})$$

donc

$$H^{\otimes_\pm N} = \left\{\psi_N \in H^{\otimes N} | \forall \sigma \in S_N, \psi_N(x_{\sigma(i)}, \dots x_{\sigma(N)}) = \psi_N(x_{1:N})\right\} \tag{I.1.2}$$

Dans la suite on utilise la convention suivante pour les listes :

$$x_{1:N} := (x_1, \dots, X_N)$$

Si $\psi_N \in H^{\otimes N}$ a une densité invariante par permutation des coordonnées, $\sigma \cdot |\psi_N|^2 = |\psi_N|^2$ et donc $\sigma \cdot \psi_N$ et ψ_N ne diffèrent que d'une phase :

$$\forall x_{1:N} \in \Omega^N, \sigma \cdot \psi_N(x_{1:N}) = e^{i\phi_N(\sigma; x_{1:N})}\psi_n(x_{1:N})$$

Maintenant, en supposant l'invariance par translation du système, c'est-à-dire en supposant que la phase est la même après translation de ψ_N,

$$\mathcal{U}_\sigma \cdot \psi_N(x_{1:N} + y_{1:N}) = e^{i\phi_N(\sigma; x_{1:N})}\psi_n(x_{1:N} + y_{1:N})$$

alors la phase est constante. Physiquement, l'échange de deux particules ne dépend pas de l'origine spatiale choisie. De plus,

$$\mathcal{U}_{\sigma\tau} = \mathcal{U}_\sigma \mathcal{U}_\tau$$

implique que $\mathcal{U}_\sigma$ est unitaire d'inverse $\mathcal{U}_{\sigma^{-1}}$ et que

$$e^{i\phi_N(\sigma\tau)} = e^{i\phi_N(\sigma)}e^{i\phi_N(\tau)}$$

Cela revient à dire que la phase ϕ_n est une représentation scalaire, donc unitaire, du groupe symétrique. Or, il existe seulement deux représentations scalaires irréductibles du groupe symétrique, données par le morphisme de signature et l'identité :

$$\epsilon_\pm(\sigma) := (\pm 1)^{|\{i<j|\sigma(i)>\sigma(j)\}|}$$

L'antisymétrie encode le principe de Pauli : on ne peut pas mettre deux particules dans le même état. En effet si $\psi, \chi \in H$,

$$\psi = \chi \implies \psi \wedge \chi = \psi \otimes \chi - \chi \otimes \psi = 0$$

Il s'agit en fait d'un argument par l'absurde. Si l'on a un état antisymétrique à deux corps avec deux particules dans le même état, la densité à deux corps est nulle, donc il n'y a pas de particules, ce qui est absurde car l'état doit être normalisé. La classe de particules décrites par des états antisymétriques est appelée fermions, la classe de particules décrites par des états symétriques est appelée bosons.

Le résultat de cet argument diffère si l'on essaye d'imposer la symétrie de permutation avant la quantification. En partant de l'espace de configuration classique pour N particules Ω^N, puis en imposant la symétrie en identifiant les orbites sous l'action du groupe symétrique, un échange de particules est alors une boucle dans cet espace topologique. Comprendre les manières possibles d'échanger des particules revient alors à calculer son groupe fondamental. En dimension $d \geqslant 3$, ce groupe fondamental est à nouveau S_N, ainsi la quantification (recherche de représentations scalaires irréductibles) donne des bosons et des fermions. Mais en dimension $d = 2$, le groupe fondamental obtenu est le groupe des tresses B_N qui a une structure plus riche et donne lieu à une autre classe de particules appelée anyons. Lors de la transposition de deux particules anyoniques, leur fonction d'onde change d'une phase $e^{i\pi\alpha}$, avec $0 < \alpha < 1$. Il s'agit en quelque sorte d'une interpolation entre les bosons et les fermions. Les anyons semblent apparaître comme des quasi-particules dans l'effet Hall quantique [10].

On définit une projection conique sur l'espace des tenseurs symétriques/antisymétriques $H^{\otimes_\pm N}$.

Proposition I.1.2: *Projection sur les espaces bosonique/fermionique*

Soit

$$\mathcal{P}_\pm = \frac{1}{N!} \sum_{\sigma \in S_N} \epsilon_\pm(\sigma) \mathcal{U}_\sigma$$

est une projection orthogonale de $H^{\otimes N}$ vers $H^{\otimes_\pm N}$. Définissons le produit tensoriel symétrique/antisymétrique

$$\bigotimes_{i=1}^{N}{}_{\pm} u_i = \sum_{\sigma \in S_N} \epsilon_\pm(\sigma) u_{\sigma(i)} = N! \mathcal{P}_\pm \bigotimes_{i=1}^{N} u_i$$

Soit $u_{1:N} \in H$, si $u_{1:N}$ est liée, leur produit antisymétrique est nul et si $u_{1:N}$ est une famille orthogonale,

$$\left\| \frac{1}{\sqrt{N!}} \bigotimes_{i=1}^{N}{}_{\pm} u_i \right\| = 1$$

Proof:

En utilisant $\epsilon_\pm(\sigma)\epsilon_\pm(\tau) = \epsilon_\pm(\sigma\tau)$ et $\mathcal{U}_\sigma \mathcal{U}_\tau = \mathcal{U}_{\sigma\tau}$,

$$\mathcal{P}_\pm^2 = \frac{1}{N!^2} \sum_{\sigma, \tau \in S_N} \epsilon_\pm(\sigma\tau) \mathcal{U}_{\sigma\tau} = \mathcal{P}_\pm$$

après le changement de variable $\sigma := \sigma\tau$. Donc $\mathcal{P}_\pm$ est une projection orthogonale sur

$\mathrm{Rg}\,(\mathcal{P}_\pm)$, et avec le même changement de variable,

$$\mathcal{U}_\tau \mathcal{P}_\pm = \frac{1}{N!} \sum_{\sigma \in S_N} \epsilon_\pm(\sigma) \mathcal{U}_{\tau\sigma} = \epsilon_\pm(\tau) \mathcal{P}_\pm$$

Donc $\mathrm{Rg}\,(\mathcal{P}_\pm) \subset H^{\otimes_\pm N}$. Pour l'autre inclusion,

$$\mathcal{P}_{\pm | H^{\otimes_\pm N}} = \frac{1}{N!} \sum_{\sigma \in S_N} \epsilon_\pm(\sigma) \mathcal{U}_{\sigma | H^{\otimes_\pm N}} = \frac{1}{N!} \sum_{\sigma \in S_N} \epsilon_\pm(\sigma)^2 = 1$$

Soit $\chi, \psi \in H^{\otimes N}$, avec le changement de variables $\sigma \coloneqq \sigma^{-1}$,

$$\langle \chi | \mathcal{P}_\pm \psi \rangle = \frac{1}{N!} \sum_{\sigma \in S_N} \epsilon_\pm(\sigma) \langle \chi | \mathcal{U}_\sigma \psi \rangle = \frac{1}{N!} \sum_{\sigma \in S_N} \epsilon_\pm(\sigma) \langle \mathcal{U}_{\sigma^{-1}} \chi | \psi \rangle = \langle \mathcal{P}_\pm \chi | \psi \rangle$$

Si $u_j = u_k$ for $j \neq k$, avec le changement de variables $\sigma \coloneqq \tau_{j,k}\sigma$,

$$\bigotimes_{i=1}^{N}{}_{-}\, u_i = \bigotimes_{i=1}^{N}{}_{-}\, u_{\tau_{j,k}(i)} = \sum_{\sigma \in S_N} \epsilon_-(\sigma) u_{\tau_{j,k}\sigma(i)} = \sum_{\sigma \in S_N} \epsilon_-(\tau_{j,k}\sigma) u_{\sigma(i)} = -\bigotimes_{i=i}^{N}{}_{\pm}\, u_i$$

donc si $u_{1:N}$ est liée, le produit antisymétrique est nul et si $u_{1:N}$ est orthonormale, on peut calculer la norme

$$\begin{aligned}\left\langle \bigotimes_{i=i}^{N}{}_{\pm}\, u_i \middle| \bigotimes_{i=i}^{N}{}_{\pm}\, u_i \right\rangle &= \sum_{\sigma,\tau \in S_N} \epsilon_\pm(\sigma\tau) \left\langle \bigotimes_{i=1}^{N} u_\sigma(i) \middle| \bigotimes_{i=1}^{N} u_\tau(i) \right\rangle = \sum_{\sigma,\tau \in S_N} \epsilon_\pm(\sigma\tau) \prod_{i=1}^{N} \langle u_\sigma(i) | u_\tau(i) \rangle \\ &= \sum_{\sigma,\tau \in S_N} \epsilon_\pm(\sigma\tau) \prod_{i=1}^{N} \delta_{\sigma(i),\tau(i)} = \sum_{\sigma,\tau \in S_N} \epsilon_\pm(\sigma\tau) \delta_{\sigma,\tau} = \sum_\sigma \epsilon_\pm \left(\sigma^2\right) = N!\end{aligned}$$

I.1.3 Régime de champ moyen et limite semi-classique

Étudier un système quantique de N fermions revient donc à diagonaliser l'Hamiltonien (I.1.1) agissant sur l'espace $H^{\otimes_- N}$ défini en (I.1.2). Mais, pour avoir une limite intéressante quand $N \to \infty$, il est nécessaire de s'assurer du fait que toutes les contributions dans l'énergie soient du même ordre de grandeur. Pour cela, remarquons que le nombre de termes d'interaction est de l'ordre $\mathcal{O}(N^2)$. En divisant le terme d'interaction par un facteur N on obtient l'Hamiltonien de champ moyen

$$\mathcal{H}_N \coloneqq \sum_{i=1}^{N} \left(-\hbar^2 \Delta_i + V(x_i)\right) + \frac{1}{N} \sum_{i<j} w(x_i - x_j) \tag{I.1.3}$$

Ainsi, les termes d'interaction et de potentiel extérieur sont tous les deux d'ordre N. Il reste à imposer la même chose pour le terme d'énergie cinétique. Dans le cas de bosons, pour l'état fondamental, ce terme est d'ordre N car toutes les particules peuvent minimiser leur énergie cinétique en se plaçant dans le même état. À cause du principe de Pauli, la situation est radicalement différente dans le cas des fermions. Toutes les particules ne pouvant pas minimiser leur énergie cinétique simultanément, l'ordre de grandeur de cette dernière se trouve être plus élevé que dans le cas bosonique. L'outil permettant de déterminer cet ordre de grandeur est l'inégalité de Lieb-Thirring.

Pour énoncer cette inégalité il faut introduire la notation de matrice densité. Dans la limite $N \to \infty$, le nombre de variables des fonctions de $L^2(\mathbb{R}^d)^{\otimes - N}$ diverge. Pour espérer obtenir des résultats de convergence, il est nécessaire de s'intéresser aux densités réduites qui appartiennent à un espace avec un nombre fini de variables.

Notation I.1.3: *Matrices densités*

On note $\mathcal{L}^1$ l'espace des opérateurs à trace. Une matrice densité à N corps est un élément $\gamma_N \in \mathcal{L}^1\left(H^{\otimes_\pm N}\right)$ positif de trace 1.
Soit Tr_I la trace partielle qui prend la trace sur les coordonnées dans $I \subset [\![1, N]\!]$ de $H^{\otimes N}$, définie par

$$\forall A_{1:N} \in \mathcal{L}^1\left(H\right)^N, \mathrm{Tr}_I\left[\bigotimes_{i=1}^{N} A_i\right] \coloneqq \mathrm{Tr}\left[\bigotimes_{i\in I} A_i\right]\bigotimes_{i\notin I} A_i$$

puis étendue par linéarité sur $\mathcal{L}^1\left(H\right)^{\otimes N}$. On associe à γ_N la $k^{\text{ième}}$ matrice densité réduite

$$\gamma_N^{(k)} \coloneqq \mathrm{Tr}_{k+1:N}\left[\gamma_N\right]$$

et la $k^{\text{ième}}$ densité réduite

$$\forall x_{1:k} \in \Omega^k, \rho_{\gamma_N^{(k)}}(x_{1:k}) \coloneqq \gamma_N^{(k)}(x_{1:k}, x_{1:k})$$

En définissant les densités réduites comme la diagonale des matrices densité réduites, dans le cas d'une projection

$$\gamma_N = \left|\psi_N\right\rangle\left\langle\psi_N\right|, \text{ avec } \psi_N \in L^2\left(H^{\otimes_\pm N}\right)$$

on retrouve bien

$$\rho_{\gamma_N} = \left|\psi_N\right|^2$$

De plus, si on identifie les opérateurs avec leurs noyaux intégraux, la $k^{\text{ième}}$ matrice densité réduite peut s'exprimer comme

$$\gamma_N^{(k)}(x_{1:k}, y_{1:k}) \coloneqq \int\limits_{\Omega^{(N-k)}} \gamma_N(x_{1:k}, x_{k+1:N}; y_{1:k}, x_{k+1:N}) dx_{k+1:N}$$

Remarquons aussi qu'avec ces notations si $\mu^{(k)}$ désigne la $k^{\text{ième}}$ marginale de μ, on a

$$\rho_{\gamma_N^{(k)}} = \rho_{\gamma_N}^{(k)}$$

Notation I.1.4: *Convention pour les constantes*

Tout au long du texte, on utilisera la notation C pour les constantes. C'est-à-dire qu'avec une famille de paramètres $(\lambda_i)_i$ et deux fonctions positives f, g la relation

$$f(\bullet, (\lambda_i)_i) = \mathcal{O}(g(\bullet, (\lambda_i)_i))$$

sera notée

$$f(\bullet, (\lambda_i)_i) \leqslant C\left((\lambda_i)_i\right) g(\bullet, (\lambda_i)_i)$$

Nous sommes maintenant prêts pour l'énoncé de l'inégalité de Lieb-Thirring.

Theorem I.1.5: *Inégalité de Lieb-Thirring*

Soit $\psi_N \in L^2(\mathbb{R}^d)^{\otimes - N}$,

$$\left\langle \psi_N \middle| \sum_{i=1}^{N} (-\Delta_i) \psi_N \right\rangle \geqslant C N^{1+\frac{2}{d}} \int_{\mathbb{R}^d} \left(\rho_{\gamma_N}^{(1)}\right)^{1+\frac{2}{d}} \tag{I.1.4}$$

Le livre [26, theorem 4.3] fournit une preuve de cette inégalité. La principale étape de cette preuve est l'obtention d'une borne inférieure pour la somme des valeurs propres négatives de l'Hamiltonien à un corps. Cette inégalité permet d'utiliser le contrôle sur l'énergie cinétique pour montrer que la première densité réduite n'est pas juste intégrable mais dans $L^{1+\frac{d}{2}}$. Une inégalité de ce genre sera démontrée dans le second chapitre en Section II.4 et adaptée au contexte du chapitre, c'est-à-dire avec un champ magnétique sur un domaine borné.

L'inégalité de Lieb-Thirring donne donc une borne inférieure pour l'ordre de grandeur du terme d'énergie cinétique dans (I.1.3). De plus, pour des électrons sans interaction : $w = 0$, confinés dans un boîte : $V = 0$ dans $[0, L]^d$ et $V = \infty$ en dehors, un calcul explicite de l'énergie fondamentale peut être réalisé. En effet, les vecteurs propres de l'énergie cinétique sont de la forme $e^{ik\cdot\bullet}$ où k est le vecteur d'onde dans le réseau réciproque de la maille $[0, d]^d$:

$$k \in \frac{2\pi}{L}\mathbb{Z}^d \tag{I.1.5}$$

En sommant les N plus petites valeurs propres k^2 de l'énergie cinétique, on obtient

$$\mathcal{O}\left(N^{1+\frac{d}{2}}\right)$$

comme ordre de grandeur. Pour ramener le terme d'énergie cinétique au même ordre que les contributions du potentiel extérieur et des interactions et dans (I.1.3) on prend un régime où la constante de Planck réduite est petite. Explicitement

$$\hbar \coloneqq N^{-\frac{1}{d}} \tag{I.1.6}$$

ainsi par (I.1.4),

$$\left\langle \psi_N \middle| \sum_{i=1}^{N} (-\hbar^2 \Delta_i) \psi_N \right\rangle \geqslant C N \int_{\mathbb{R}^d} \left(\rho_{\gamma_N}^{(1)}\right)^{1+\frac{2}{d}}$$

Ce régime est dit semi-classique car dans la limite $\hbar \to 0$ un système quantique converge vers son équivalent en mécanique classique. Le régime de champ moyen est alors naturellement couplé à une limite semi-classique. Il est connu, voir [17], que l'énergie fondamentale de l'Hamiltonien à N corps converge alors vers le minimum de la fonctionelle de Thomas-Fermi :

$$\mathcal{E}_{TF}(\rho) \coloneqq \frac{d}{d+2} C_{TF} \int_{\mathbb{R}^d} \rho(x)^{1+\frac{d}{2}} dx + \int_{\mathbb{R}^d} \rho(x) V(x) + \frac{1}{2} \iint_{\mathbb{R}^d \times \mathbb{R}^d} \rho(x)\rho(y) w(x-y) dx dy$$

avec comme constante

$$C_{TF} \coloneqq 4\pi^2 \left(\frac{d}{|S^{d-1}|} \right)^{\frac{2}{d}}$$

Ici, l'énergie cinétique est donnée par la même puissance de ρ dans dans l'inégalité de Lieb-Thirring. Cette intégrale peut être obtenue en regardant la limite de la somme des N plus petites valeurs propres $\hbar^2 k^2$ de l'énergie cinétique pour k dans le réseau réciproque (I.1.5).

Le modèle d'un atome neutre est un exemple de système physique respectant le régime champ moyen/semi-classique. Il prend en compte un noyau chargé positivement de charge $Z \coloneqq N$ où N est le numéro atomique de l'atome. Les fermions sont ici N électrons de charge -1 interagissant entre eux avec un potentiel coulombien, placés dans un potentiel coulombien créé par le noyau. L'Hamiltonien est alors :

$$H_N \coloneqq \sum_{i=1}^{N} \left(-\Delta_i - \frac{Z}{|x_i|} \right) + \sum_{1 \leqslant i < j \leqslant N} \frac{1}{|x_j - x_i|} \tag{I.1.7}$$

En dimension $d = 2$,

$$\frac{H_N}{N} \coloneqq \sum_{i=1}^{N} \left(-\left(N^{-\frac{1}{2}} \right)^2 \Delta_i - \frac{1}{|x_i|} \right) + \frac{1}{N} \sum_{1 \leqslant i < j \leqslant N} \frac{1}{|x_j - x_i|} \tag{I.1.8}$$

À un coefficient multiplicatif près, l'Hamiltonien de la forme (I.1.3) avec (I.1.6).

I.1.4 Effet Hall quantique

L'effet Hall est un phénomène qui se produit dans certains matériaux conducteurs lorsqu'ils sont soumis à un champ magnétique transverse au matériau. Si l'on impose un courant électrique I à travers le matériau, l'effet Hall se manifeste par l'apparition d'une tension électrique transversale au courant dans le matériau, appelée tension de Hall, notée V_H. Elle est due à la déviation des porteurs de charge, électrons ou trous, par la force magnétique. La conductance de Hall est alors

$$\sigma_{Hall} \coloneqq \frac{I}{V_H}$$

Pour des systèmes classiques cette conductance est simplement proportionelle à l'amplitude du champ magnétique. Le phénomène peut alors s'expliquer par le fait que des particules chargées soumises à un champ magnétique transverse décrivent des orbites, appelés orbites cyclotron. En mécanique quantique l'énergie cinétique d'une particule dans un champ magnétique transverse et homogène est quantifiée en niveau d'énergie discrets, séparés par un gap constant, nommés niveaux de Landau. Pour des systèmes aux échelles de la mécanique quantique, la conductance de Hall se retrouve quantifiée en multiples du quantum de conductance:

$$\sigma_{Hall} = \nu \frac{e^2}{2\pi\hbar}$$

où e est la charge élémentaire, ν est appelé facteur de remplissage et correspond aux nombre de niveaux de Landau remplis dans le système. ν peut prendre des valeurs entières mais aussi certaines valeurs fractionnaires. L'effet Hall fractionnaire est dû aux interactions entre particules [30]. La quantification de la conductivité de Hall a la propriété d'être extrêmement précise lors de mesure expérimentales. Ainsi l'effet Hall quantique trouve des applications en métrologie.

Le gap entre niveaux de Landau étant proportionnel à l'intensité du champ magnétique, le nombre de niveaux remplis dépend de l'intensité du champ magnétique. Lorsque le champ magnétique est très fort, le gap entre le plus bas niveau de Landau et le niveau suivant est très grand, ainsi toutes les particules se retrouvent dans le plus bas niveau afin de minimiser leur énergie cinétique. Dans cette thèse, les champ magnétiques seront d'une intensité permettant un mixage entre niveaux de Landau dans le but de fournir un contexte mathématique dans lequel le facteur de remplissage peut prendre des valeurs entières et fractionnaires. Cependant l'effet Hall fractionnaire reste hors de la portée des résultats présentés dans ce manuscrit car ce phénomène apparaît au delà des limites de champ moyen et semi-classiques qui sont ici étudiées.

I.2 Statique

I.2.1 Un théorème de Lieb, Solojev et Yngvason

On considère un système de N fermions sans spin dans un espace à deux dimensions. On se donne maintenant un champ magnétique uniforme d'intensité b, transverse au domaine des fermions. Pour un champ magnétique assez fort, les spins sont généralement considérés alignés avec le champ magnétique et donc ne sont pas à prendre en compte. Ce problème a initialement été étudié par Lieb Solovej et Yngvason in [43], [47], [44], [45], [42] et plus récemment par Fournais, Lewin and Madsen in [17], [16]. Dans ce qui suit, nous allons énoncer le résultat principal de [42]. On se place en jauge symétrique :

$$A(x, y) \coloneqq \frac{1}{2}\begin{pmatrix} -y \\ x \end{pmatrix}$$

Le potentiel vecteur A engendre le champ magnétique

$$\nabla \wedge A = \nabla \wedge \begin{pmatrix} -y \\ x \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

En présence d'un champ magnétique, on remplace le Laplacien par le Laplacien magnétique, (I.1.1) devient alors

$$\mathcal{H}_N \coloneqq \sum_{i=1}^{N}\left((-i\hbar\nabla_i + bA(x_i))^2 + V(x_i)\right) + \frac{1}{N}\sum_{i<j} w(x_i - x_j)$$

et agit sur $L^2\left(\mathbb{R}^2\right)^{\otimes_- N}$. On suppose que le potentiel extérieur est confinant

$$V(x) \underset{|x|\to\infty}{\to} \infty$$

et que le potentiel d'interaction est le potentiel Coulombien :

$$w(x - y) = \frac{1}{|x - y|}$$

L'hypothèse du potentiel confinant empêche les particules de s'échapper à l'infini. Concernant l'effet Hall quantique, les particules sont confinées dans des puits quantiques, par exemple grâce à des semi-conducteurs. On définit la longueur magnétique

$$l_b \coloneqq \sqrt{\frac{\hbar}{b}}$$

En mécanique classique, l_b correspond au rayon d'une orbite cyclotron. En mécanique quantique la longueur magnétique est la distance sur laquelle les états propres de l'énergie cinétique sont localisés. L'état fondamental et l'énergie fondamentale de l'Hamiltonien sont

$$\psi_N^0 \coloneqq \operatorname{argmin}\left\{\langle\psi_N|H_N\psi_N\rangle, \psi_N \in L^2\left(\mathbb{R}^2\right)^{\otimes_- N} \text{ tel que } \|\Psi_N\|_{L^2} = 1\right\}$$

$$E_N^0 \coloneqq \min \left\langle \psi_N^0 \middle| H_N \psi_N^0 \right\rangle$$

On définit maintenant les modèles limite, à commencer par le modèle électrostatique :

$$\mathcal{E}_{class}(\rho) \coloneqq \int_{\mathbb{R}^2} \rho(x)V(x)dx + \frac{1}{2} \iint_{\mathbb{R}^2\times\mathbb{R}^2} \frac{\rho(x)\rho(y)}{|x-y|}dxdy$$
$$E_{class}^0 \coloneqq \min \left\{\mathcal{E}_{class}(\rho) \text{ tel que } \rho \in L^1\left(\mathbb{R}^2\right), \rho \geqslant 0, \|\rho\|_{L^1} = 1\right\}$$

Ce modèle est purement classique, le premier terme correspond à l'énergie potentielle et le second à l'énergie d'interaction entre les particules. Ensuite, on considère le modèle de Thomas-Fermi :

$$\mathcal{E}_{TF}(\rho) \coloneqq \int_{\mathbb{R}^2} \rho(x)^2 dx + \mathcal{E}_{class}(\rho)$$
$$E_{TF}^0 \coloneqq \min \left\{\mathcal{E}_{TF}(\rho) \text{ tel que } \rho \in L^1\left(\mathbb{R}^2\right), \rho \geqslant 0, \|\rho\|_{L^1} = 1\right\}$$

Finalement, on définit le modèle de Thomas-Fermi magnétique où le terme d'énergie cinétique est légèrement modifié par rapport au modèle de Thomas-Fermi :

$$\mathcal{E}_{TFM}(\rho) \coloneqq \int_{\mathbb{R}^2} j_b(\rho(x))dx + \mathcal{E}_{class}(\rho)$$
$$E_{TFM}^0 \coloneqq \min \left\{\mathcal{E}_{TFM}(\rho) \text{ tel que } \rho \in L^1\left(\mathbb{R}^2\right), \rho \geqslant 0, \|\rho\|_{L^1} = 1\right\}$$

avec $j_b \in C^0(\mathbb{R}_+)$ défini par

$$j_b(0) = 0$$
$$j_b'(\rho) = 2\hbar b\left(n + \frac{1}{2}\right) \text{ si } \frac{n}{2\pi l_b^2} < \rho < \frac{n+1}{2\pi l_b^2}$$

Dans ce contexte, l'énergie du niveau de Landau $n \in \mathbb{N}$ est $2\hbar b\left(n + \frac{1}{2}\right)$. Le gap entre deux niveaux voisins est donc $2\hbar b$. Il est bien connu, [30] ou Subsection II.1.4, que la dégénérescence par unité d'aire à l'intérieur d'un niveau de Landau est d'ordre $1/(2\pi l_b^2)$. Ceci justifie la définition de $j_b(\rho(x))$ car les niveaux de Landau les plus bas sont remplis en priorité. Ainsi, pour $x \in \mathbb{R}^2$ tel que

$$\frac{n}{2\pi l_b^2} < \rho(x) < \frac{n+1}{2\pi l_b^2}$$

n niveaux de Landau sont remplis localement en x et rajouter une particule en x coûte une énergie $2\hbar b\left(n + \frac{1}{2}\right)$, c'est-à-dire l'énergie d'une particule dans le $(n+1)^{\text{ième}}$ niveau de Landau. Notons aussi que la fonction j_b ainsi définie est une approximation de la fonction carré lorsque $l_b \to \infty$. Dans cette limite, le modèle de Thomas-Fermi magnétique se réduit à celui de Thomas-Fermi.

Le résultat de [42, Théorèmes 1.2 et 1.3] est alors le suivant :

Theorem I.2.1: *Ground states of large quantum dots in magnetic fields (LSY'95)*

Si $V \in C^{1,\alpha}_{loc}$ pour un $0 < \alpha \leqslant 1$. En notant ρ_N la première densité réduite de $|\psi_N^0\rangle\langle\psi_N^0|$, uniformément en b,

$$\begin{aligned} \frac{E_N^0}{N} &\underset{N\to\infty}{\longrightarrow} E^0_{MTF} \\ \frac{\rho_N}{N} &\overset{*}{\rightharpoonup} \rho^0_{TFM} \text{ au sens des mesures} \end{aligned}$$

puis si $l_b^2 N \to \infty$,

$$\begin{aligned} \frac{E_N^0}{N} &\underset{N\to\infty}{\longrightarrow} E^0_{TF} \\ \frac{\rho_N}{N} &\overset{*}{\rightharpoonup} \rho^0_{TF} \text{ au sens des mesures} \end{aligned}$$

et si $l_b^2 N \to 0$,

$$\begin{aligned} \frac{E_N^0}{N} &\underset{N\to\infty}{\longrightarrow} E^0_{class} \\ \frac{\rho_N}{N} &\overset{*}{\rightharpoonup} \rho^0_{class} \text{ au sens des mesures} \end{aligned}$$

En se plaçant dans un potentiel confinant, le système reste dans un volume borné. Ainsi, tout dépend de la limite du ratio

$$\frac{\frac{1}{2\pi l_b^2}}{N} = \frac{1}{2\pi l_b^2 N}$$

de la densité d'état dans un niveau de Landau divisé par le nombre de particules :

- Si $l_b^2 N \to 0$, la densité d'état est suffisamment élevée pour permettre d'accueillir toutes les particules dans le plus bas niveau de Landau. Comme le gap entre niveaux de Landau est très grand, il est impossible pour les particules d'être excités dans des niveaux supérieurs. L'énergie cinétique du système à N corps est alors triviale : $N\hbar b$ où $\hbar b$ est l'énergie du plus bas niveau de Landau. Et le modèle converge alors vers un modèle électrostatique décrit par $\mathcal{E}_{class}$.

- Si $l_b^2 N \to \infty$, tous les niveaux de Landau sont occupés, le gap entre niveaux est petit par rapport au nombre de particules. Le modèle limite, celui de Thomas-Fermi, est alors le même que l'on obtient en l'absence de champ magnétique.

- Si $l_b^2 N = \mathcal{O}(1)$, le nombre de niveau de Landau remplis varie en fonction de $\rho(x)$. La limite obtenue est le modèle de Thomas Fermi magnétique. Le facteur de remplissage au point x étant donné par $2\pi l_b^2 \rho(x)$. Comme ρ est a priori non borné, dans ce régime, des particules se retrouvent dans tous les niveaux de Landau. Le gap d'énergie est comparable aux termes d'énergie potentielle et d'interaction, les particules optimisent à la fois leur niveau de Landau et leur position dans les potentiels.

I.2.2 Résultats

On souhaite traiter la situation intermédiaire par rapport à [42] où seul un nombre fini de niveaux de Landau sont complètement remplis. Précisément, notre résultat est une limite où les q premiers niveaux Landau sont entièrement remplis, le niveau Landau suivant est partiellement rempli avec un taux de remplissage $r < 1$ et tous les niveaux Landau supérieurs sont vides. Cette configuration est physiquement motivée par l'effet Hall quantique qui se produit principalement dans un niveau Landau partiellement rempli tandis que les niveaux Landau inférieurs sont remplis et inertes, et les niveaux supérieurs sont vides, voir [30].

Dans cette perspective, on fixe le rapport entre le nombre de particules et la dégénérescence des niveaux de Landau. Sur tout l'espace $\mathbb{R}^2$ cette dégénérescence est infinie. Pour s'assurer de la finitude de la dégénérescence des niveaux de Landau, on travaille sur un domaine borné. Pour simplifier la situation, l'idéal serait de considérer des conditions aux bords périodiques. Mais, en présence d'un champ magnétique, les conditions aux bords périodiques doivent être modifiées. C'est un problème bien connu, par exemple voir [30, Section 3.9]. On définit alors des opérateurs de translation magnétique qui commutent avec le moment magnétique. Ces opérateurs de translations magnétiques définissent les conditions aux bords périodiques magnétiques. La conséquence de ce choix de domaine est qu'il faut adapter le formalisme semi-classique standard.

Notation I.2.2: *Modèle*

On se place sur $\Omega \coloneqq [0,L]^2$ de taille $L > 0$. On travaille en jauge de Coulomb :

$$\nabla \cdot A = 0$$

où $A \in C^\infty(\mathbb{R}^2, \mathbb{R}^2)$ est le potentiel vecteur générant le champ magnétique transverse et homogène. Le Laplacien magnétique est

$$\mathscr{L}_{\hbar,b} \coloneqq (i\hbar\nabla + bA)^2$$

En identifiant $\mathbb{R}^2$ avec $\mathbb{C}$ on utilise une notation complexe pour le vecteur $(x,y) \in \mathbb{R}^2$, c'est-à-dire

$$(x,y) = x + iy \in \mathbb{C}$$

Soit $z_0 \in \mathbb{C}$,

$$\nabla \wedge (A - A(\bullet - z_0)) = 0$$

on peut donc choisir $\varphi_{z_0} \in C^\infty(\mathbb{R}^2, \mathbb{R})$ tel que

$$A - A(\bullet - z_0) \eqqcolon l_b^2 \nabla \varphi_{z_0}$$

Pour une fonction $\psi \in L^2(\Omega)$ les conditions magnétiques périodiques s'écrivent

$$\forall t \in [0,L], \begin{cases} \psi(L+it) = e^{i\varphi_L(L+it)}\psi(it) \\ \psi(t+iL) = e^{i\varphi_{iL}(t+iL)}\psi(t) \end{cases}$$

Et le domaine du Laplacien magnétique est

$$\mathrm{Dom}\,(\mathscr{L}_{\hbar,b}) \coloneqq \left\{\psi \in H^2(\Omega) \text{ tel que (II.1.4) soit vérifié}\right\}$$

L'Hamiltonien à N-corps est

$$\mathscr{H}_N := \sum_{j=1}^{N} \left(\left(i\hbar\nabla_{x_j} + bA(x_j) \right)^2 + V(x_j) \right) + \frac{2}{N-1} \sum_{1 \leqslant j < k \leqslant N} w(x_j - x_k)$$

agissant sur $L^2(\Omega)^{\otimes - N}$. Le potentiel d'interaction est supposé radial pour la métrique sur le tore :

$$w(x-y) =: \widetilde{w}(d(x,y)) \text{ avec } d(x,y) := \min_{r \in L\mathbb{Z}^2} |x - y + r|$$

Le domaine de $\mathscr{H}_N$ est

$$\mathrm{Dom}\,(\mathscr{H}_N) := \bigwedge^{N} \mathrm{Dom}\,(\mathscr{L}_{\hbar,b})$$

et son énergie fondamentale

$$E_N^0 := \inf \left\{ \langle \psi_N | \mathscr{H}_N \psi_N \rangle, \psi_N \in \mathrm{Dom}(\mathscr{H}_N) \text{ tel que } \|\psi_N\|_{L^2} = 1 \right\} \tag{I.2.1}$$

Le fait que $\mathscr{H}_N$ soit auto-adjoint demande que le flux magnétique passant au travers du domaine, bL^2, soit quantifié :

$$\exists d \in \mathbb{N} \text{ tel que } 2\pi d = \frac{b}{\hbar} L^2 = \frac{L^2}{l_b^2}$$

Ici, d se retrouve être la dégénérescence des niveaux de Landau. On peut donc fixer le nombre de niveaux remplis en choisissant un régime où N/d est fixé.

Notation I.2.3: *Régime semi-classique*

On choisit la constante de Planck $\hbar := (\hbar_N)_{N \in \mathbb{N}}$ telle que

$$N^{-\frac{1}{2}} \ll \hbar \ll N^{-\frac{1}{4}} \tag{I.2.2}$$

Soit $q \in \mathbb{N}, r \in [0,1), b := (b_N)_{N \in \mathbb{N}}$ tel que

$$d := \frac{L^2}{2\pi l_b^2} \subset \mathbb{N}^*$$

et

$$\frac{N}{d} \underset{N \to \infty}{=} q + r + o\left(\frac{1}{\hbar b} \right) \tag{I.2.3}$$

q donnera le nombre de niveaux Landau entièrement remplis et r le taux de remplissage du $q^{\text{ième}}$ niveau de Landau. Avec cette notation,

$$\frac{N}{d} = \frac{2\pi l_b^2 N}{L^2} \underset{N \to \infty}{\to} q + r \quad \text{et} \quad \frac{1}{l_b^2} = \frac{b}{\hbar} \underset{N \to \infty}{\sim} \frac{2\pi N}{(q+r)L^2}$$

Dans ce régime, l'ordre de grandeur du champ magnétique est $b = \mathcal{O}(\hbar N)$. Donc l'ordre de grandeur de l'énergie cinétique est

$$\hbar b = \mathcal{O}(\hbar^2 N) \gg 1$$

La contribution de l'énergie cinétique doit être de premier ordre par rapport aux termes potentiels si l'on veut imposer le nombre de niveaux de Landau remplis :

$$\hbar^2 N \gg 1$$

d'où la borne supérieure dans (I.2.2). $\hbar \gg N^{-\frac{1}{2}}$ est une condition technique pour contrôler un terme d'erreur venant de l'énergie cinétique. La vitesse de convergence (I.2.3) est imposée pour la même raison.

On obtient un modèle semi-classique où l'énergie ne dépend plus de la fonction d'onde mais de la densité dans l'espace des phases $\mathbb{N} \times \Omega$. Cela signifie que les particules ont deux degrés de liberté : le premier est $n \in \mathbb{N}$ le nombre quantique représentant l'indice de niveau de Landau et $x \in \Omega$ représentant la position des particules dans l'espace. En mécanique classique, x s'interprète comme le centre de l'orbite cyclotron. Ce modèle est semi-classique dans le sens où le principe de Pauli est toujours valable comme borne sur la densité.

Notation I.2.4: *Fonctionnelle semi-classique*

On considère la mesure suivante sur l'espace des phases :

$$\eta := \left(\sum_{n \in \mathbb{N}} \delta_n \right) \otimes \lambda_\Omega$$

où λ_Ω est la mesure de Lebesgue sur Ω. Pour une densité sur l'espace des phases $m \in L^1(\mathbb{N} \times \Omega, \mathbb{R}_+)$, l'énergie semi-classique est

$$\mathcal{E}_{sc,\hbar b}[m] := \int_{\mathbb{N}\times\Omega} E_n m(n,R) d\eta(n,R) + \int_{\mathbb{N}\times\Omega} V m d\eta + \int_{(\mathbb{N}\times\Omega)^2} w m^{\otimes 2} d\eta^{\otimes 2}$$

où

$$E_n := 2\hbar b \left(n + \frac{1}{2} \right)$$

est l'énergie du niveau de Landau n. Le domaine de cette fonctionelle est

$$\mathcal{D}_{sc} := \left\{ m \in L^1(\mathbb{N} \times \Omega) \text{ tel que } \int_{\mathbb{N}\times\Omega} m d\eta = 1 \text{ and } 0 \leqslant m \leqslant \frac{1}{(q+r)L^2} \right\}$$

$$E^0_{sc,\hbar b} := \inf_{m \in \mathcal{D}_{sc}} \mathcal{E}_{sc,\hbar b}[m]$$

On définit également le modèle électrostatique pour le niveau de Landau partiellement rempli qui ne dépend que de la densité.

Notation I.2.5: *Modèle électrostatique pour le niveau de Landau partiellement rempli*

$$\mathcal{E}_{qLL}[\rho] := \int_{\Omega} V\rho + \iint_{\Omega^2} w(x-y)\rho(x)\rho(y)dxdy \tag{I.2.4}$$

$$\mathcal{D}_{qLL} := \left\{ \rho \in L^1(\Omega) \text{ tel que } \int_{\Omega} \rho = \frac{r}{q+r} \text{ and } 0 \leqslant \rho \leqslant \frac{1}{(q+r)L^2} \right\} \tag{I.2.5}$$

$$E^0_{qLL} := \inf_{\mathcal{D}_{qLL}} \mathcal{E}_{qLL}$$

Soit les énergies

$$\begin{aligned} E^{q,r} &:= \frac{q^2+2qr+r}{q+r} \\ E^{q,r}_V &:= \frac{q}{q+r} \fint_{\Omega} V \\ E^{q,r}_w &:= \frac{q^2+2qr}{(q+r)^2} \fint_{\Omega^2} w \end{aligned}$$

et $\rho \in \mathcal{D}_{qLL}$,

$$m_\rho(n,x) := \mathbb{1}_{n<q} \frac{1}{L^2(q+r)} + \mathbb{1}_{n=q}\rho(x)$$

m_ρ est une densité sur l'espace des phases construite avec les q plus bas niveaux de Landau maximisant la densité sous la contrainte du principe de Pauli dans (I.2.5) et une densité ρ dans le niveau de Landau suivant, partiellement rempli. La proportion de particules dans le niveau partiellement rempli est

$$\frac{r}{q+r}$$

ce qui correspond à la constante de normalisation dans (I.2.5). Notons que le principe de Pauli est bien la borne correcte pour avoir

$$\int_{\mathbb{N}\times\Omega} m_\rho d\eta = 1$$

On démontrera dans la Proposition II.6.1 que

$$\mathcal{E}_{sc,\hbar b}\left[m_\rho\right] = \hbar b E^{q,r} + E^{q,r}_V + E^{q,r}_w + \mathcal{E}_{qLL}\left[\rho\right]$$

où

- $\hbar b E^{q,r}$ est l'énergie cinétique des $q+1$ plus bas niveaux de Landau
- $E^{q,r}_V$ est l'énergie potentielle des q plus bas niveaux de Landau

- $E_w^{q,r}$ est l'énergie d'interaction contenant toutes les interactions exceptées celles à l'intérieur du niveau partiellement rempli.

On peut maintenant énoncer notre résultat principal sur la convergence de l'énergie fondamentale.

Theorem I.2.6: *Convergence de l'énergie fondamentale*

Si $V, w \in L^2(\Omega)$,

$$\frac{E_N^0}{N} \underset{N\to\infty}{=} \hbar b E^{q,r} + E_V^{q,r} + E_w^{q,r} + E_{qll}^0 + o(1)$$

Le premier ordre de l'énergie quantique fondamentale est l'énergie cinétique du système $\hbar b E^{q,r}$. Pour les termes d'ordre 1, les seules contributions non triviales à l'énergie fondamentale sont l'énergie potentielle et d'interaction à l'intérieur du niveau de Landau partiellement rempli. Les niveaux inférieurs sont totalement remplis, ainsi leurs contribution à l'énergie sont constante. L'interaction du niveau partiellement rempli avec tous les autres niveaux est également une constante. Pour les niveaux de Landau supérieurs, leur contribution à l'énergie est nulle car ils sont vides.

Le théorème pour la convergence des densités est :

Theorem I.2.7: *Convergence des densités*

Si $V, w \in L^2(\Omega)$, et $(\psi_N)_{N\in\mathbb{N}}$ est une suite de minimiseurs de (I.2.1). En notant $\rho_N^{(k)}$ la $k^{\text{ième}}$ densité réduite de $|\psi_N\rangle\langle\psi_N|$, $\exists\mu \in \mathcal{P}(\mathcal{D}_{qLL})$ tel que

- μ ne donne de la masse qu'aux minimiseurs de (I.2.4)
- $\forall k \in \mathbb{N}^*$, au sens des mesures,

$$\rho_N^{(k)} \underset{N\to\infty}{\overset{*}{\rightharpoonup}} \int_{\mathcal{D}_{qLL}} \left(\frac{q}{L^2(q+r)} + \rho\right)^{\otimes k} d\mu(\rho)$$

La densité de particules converge vers une somme convexe de densités de la forme

$$\frac{q}{L^2(q+r)} + \rho$$

De par la forme du principe de Pauli, il est visible que la constante dans ce terme correspond aux particules dans les niveaux remplis. Ensuite, dans le niveau de Landau partiellement rempli, la densité est donnée par un minimiseur de (I.2.4).

I.2.3 Perspectives

Les hypothèses de régularité sur les potentiels ne sont pas optimales. Nous nous attendons à ce que ce résultat reste vrai si les potentiels ont une partie positive L^1 et une partie négative L^2. Sous ces hypothèses, il faut prouver que les particules ne se concentreront pas dans les L^1 singularités positives des potentiels. Ceci a été fait dans [42] pour le potentiel de Coulomb.

Une autre piste intéressante à explorer serait de regarder le cas d'un champ magnétique non homogène (toujours transverse). En choisissant un champ magnétique suffisamment régulier, à l'échelle l_b, le champ magnétique se comporte comme une constante. Ainsi l'énergie cinétique reste localement quantifiée en niveaux de Landau.

Le modèle pourrait aussi être complexifié en ajoutant les spins, ce qui revient à rajouter les matrices de Pauli dans le terme d'énergie cinétique de l'Hamiltonien à N corps.

Remplacer le Laplacien magnétique par l'opérateur de Dirac permettrait une description du graphène. En effet, dans le graphène, la relation de dispersion au niveau des cônes de Dirac dans la structure de bandes rend le transport des électrons relativiste. Le graphène est connu comme étant un bon matériau pour l'observation expérimentale de l'effet Hall quantique [37]. Il est donc intéressant de considérer un modèle pour des fermions $2d$ dans un cadre de relativité restreinte. L'opérateur de Dirac n'étant pas positif, on ferait l'hypothèse de la mer de Dirac, supposant que dans le vide, tous les états d'énergie négative sont occupés. Une difficulté pour ce résultat serait de gérer des potentiels généraux : en effet, cela demanderait d'adapter une inégalité de Lieb-Thirring pour l'opérateur de Dirac avec condition aux bords magnétiques périodiques.

I.3 Dynamique

I.3.1 Un petit tour de la bibliographie

Cette partie porte maintenant sur la dynamique donnée par l'Hamiltonien à N corps (I.1.3) et l'équation de Schrödinger

$$i\hbar\partial_t\psi_N = \mathcal{H}_N\psi_N \tag{I.3.1}$$

pour $\psi_N \in L^\infty\left(\mathbb{R}_+, L^2(\Omega)^{\otimes_\pm N}\right)$.

Il est bien connu que l'équation de Hartree peut être obtenue par une limite de champ moyen lorsque $N \to \infty$. Si ψ_N est une solution de (I.3.1), des résultats tels que [25] [19] prouvent que la première matrice densité réduite de $|\psi_N\rangle\langle\psi_N|$, satisfait presque l'équation de Hartree :

$$i\hbar\partial_t\gamma = \left[-\hbar^2\Delta + V + w \star \rho_\gamma, \gamma\right] \tag{I.3.2}$$

avec $\gamma \in L^\infty\left(\mathbb{R}_+, \mathcal{L}^1\left(L^2(\Omega)\right)\right)$.

On considère la densité sur l'espace des phases $\mathbb{R}^d \times \mathbb{R}^d$, position-quantité de mouvement, aussi appelée fonction de Husimi, ici notée μ, associée à la matrice densité solution de l'équation de Hartree (I.3.2). Dans la limite semi-classique on obtient l'équation de Vlasov [46] [24] [14] :

$$\partial_t\mu(t,x,p) + p\cdot\nabla_x\mu(t,x,p) + F(t,x)\cdot\nabla_p\mu(t,x,p)$$

avec un champ de force donné par

$$F(t,x) := -\nabla_x\left(V + \rho\star w\right)(t,x)$$

et la densité physique

$$\rho(t,x) := \int_{\mathbb{R}^d}\mu(t,x,p)dp$$

Les limites de champ moyen et semi-classique peuvent être combinées pour obtenir directement l'équation de Vlasov de celle de Schrödinger à N corps [7] [8] [3]. Certains résultats plus récents ont permis de gérer des potentiels singuliers [18] [23] [6]. Ces résultats existent dans les cas fermionique et bosonique. Si l'on part de la dynamique de Schrödinger à N-corps, la caractéristique fermionique du système est imposée par l'antisymétrie des fonctions. Au niveau de l'équation de Hartree, l'aspect fermionique est imposé directement via le principe de Pauli :

$$\frac{\gamma}{\mathrm{Tr}\left[\gamma\right]} \leqslant \frac{1}{N}$$

Tous ces résultats s'adaptent en présence d'un champ magnétique, en dimension $d = 2$ ou $d = 3$, dans le cas où l'amplitude du champ magnétique n'est pas trop élevée comme dans le cas Thomas-Fermi de [42]. Cela vient du fait que l'espace des phases pour l'approximation semi-classique est toujours $\mathbb{R}^d \times \mathbb{R}^d$. L'équation de Hartree devient alors

$$i\hbar\partial_t\gamma = \left[(i\hbar\nabla + bA)^2 + V + w\star\rho_\gamma, \gamma\right] \tag{I.3.3}$$

Dans l'équation de Vlasov, il est nécessaire d'ajouter la force de Lorentz

$$F(t,x) = p \wedge b(0,0,1) - \nabla_x \left(V + \rho \star w\right)(t,x)$$

Du côté de la littérature concernant les systèmes classiques, en partant de l'équation de Vlasov, une équation gyro-cinétique de transport peut être obtenue pour la densité physique dans une limite fort champ magnétique :

$$\partial_t \rho(t,x) + \nabla_x^\perp \left(V + w \star \rho\right)(t,x) \cdot \nabla_x \rho(t,x) = 0 \tag{I.3.4}$$

Voir [40] [33] [39] [27] [32] [29] [20] [11] pour quelques références. Certains résultats commencent même de la dynamique de Newton [12].

Un résultat semi-classique [4] obtient l'équation d'Euler pour la vorticité depuis la dynamique de Schrödinger à N corps. Ce théorème concerne un régime dans lequel le gap entre les niveaux de Landau est petit comparé aux interactions. Ainsi l'espace des phases pour l'approximation semi-classique est toujours $\mathbb{R}^d \times \mathbb{R}^d$.

I.3.2 Résultats

L'objet d'étude est toujours un système de fermions $2d$ soumis à un champ magnétique transverse et homogène. Notre point de départ est l'équation de Hartree (I.3.3), appropriée pour un système fermionique d'un grand nombre de particules. L'équation gyro-cinétique (I.3.4) est obtenue dans une limite semi-classique couplée à une limite fort champ magnétique. Contrairement à [4], le régime est choisi pour que tous les termes de l'Hamiltonien

$$\left(i\hbar\nabla + bA\right)^2 + V + w \star \rho_\gamma$$

soient d'ordre 1. Cela entraîne un grand nombre de conséquences sur la méthode de preuve. En effet, vu que le gap entre niveaux de Landau $\hbar b$ doit être pris d'ordre 1, l'espace des phases pour l'approximation semi-classique est alors $\mathcal{N} \times \mathbb{R}^2$. Autrement dit, la quantification de l'énergie cinétique continue de jouer un rôle important dans la limite. Un fait marquant est que les fonctions de Wigner et la quantification de Weyl, utilisées par exemple dans [46], n'ont pas d'équivalent sur cet espace des phases. Les travaux [7] [8] [3] utilisent plutôt les fonctions de Husimi, qui ont un équivalent sur $\mathcal{N} \times \mathbb{R}^2$. Ces fonctions de Husimi sont construites à partir des états cohérents et seront introduites en Section II.5. Une difficulté principale vient du fait que les états cohérents ne sont pas bien localisés pour les niveaux de Landau élevés.

Notation I.3.1: *Définition du régime*

On considère une limite fort champ magnétique

$$b \to +\infty$$

et semi-classique

$$\hbar \underset{b\to\infty}{\to} 0$$

telle que

$$\hbar b \underset{b\to\infty}{\to} 1$$

Soit $\gamma \in L^\infty\left(\mathbb{R}_+, \mathcal{L}^1\left(L^2(\mathbb{R}^2)\right)\right)$, tel que

$$\operatorname{Tr}[\gamma(0)] = 1 \quad \text{and} \quad 0 \leqslant \gamma(0) \leqslant 2\pi l_b^2 \tag{I.3.5}$$

on définit la matrice densité remise à la bonne échelle de temps :

$$\forall t \in \mathbb{R}_+, \gamma_b(t) \coloneqq \gamma(bt)$$

Le choix de l'échelle de temps s'explique grâce à une étude classique Subsection III.1.4 qui nous montre que le mouvement du centre de l'orbite cyclotron s'effectue sur une échelle de temps d'ordre b. Ainsi

$$\partial_t \gamma_b = \frac{b}{i\hbar}\left[\mathscr{L}_b + V + w \star \rho_{\gamma_b}, \gamma_b\right] = \frac{1}{il_b^2}\left[\mathscr{L}_b + V + w \star \rho_{\gamma_b}, \gamma_b\right] \tag{I.3.6}$$

La contrainte (I.3.5) se propage en temps (voir Proposition III.3.1). De plus

$$\int_{\mathbb{R}^2} \rho_{\gamma_b}(t) = \operatorname{Tr}[\gamma_b(t)]$$

Donc le principe de Pauli $\gamma_b \leqslant 2\pi l_b^2$ garantit que le système occupe un volume d'ordre 1 dans la limite. En effet, un état fermionique typique satisfaisant (I.3.5) est un produit antisymétrique de N fonctions orthogonales avec

$$N \coloneqq \mathcal{O}\left(\frac{1}{2\pi l_b^2}\right)$$

Un tel état à N particules occupe un volume d'ordre

$$\frac{N}{l_b^{-2}} = \mathcal{O}(1)$$

Cela montre que tous les termes de l'Hamiltonien $\mathscr{L}_b + V + w \star \rho_\gamma$ sont bien d'ordre 1. On énonce maintenant notre résultat principal.

Theorem I.3.2: *Gyro-cinétique de la solution de l'équation d'Hartree*

Soit $\gamma_b \in L^\infty\left(\mathbb{R}_+, \mathcal{L}^1\left(L^2\left(\mathbb{R}^2\right)\right)\right)$ une solution de (I.3.6) telle que

$$\operatorname{Tr}[\gamma_b(0)] = 1, 0 \leqslant \gamma_b(0) \leqslant 2\pi l_b^2$$
$$\operatorname{Tr}\left[\gamma_b(0)\left(\mathscr{L}_{\hbar,b} + V + \frac{1}{2} w \star \rho_{\gamma_b}(0)\right)\right] < \infty$$

Si $V, w \in W^{4,\infty}\left(\mathbb{R}^2\right)$, alors $\forall \varphi \in C_c^\infty\left(\mathbb{R}_+ \times \mathbb{R}^2\right)$,

$$\left| \int_{\mathbb{R}^2} \varphi(0,z)\rho_{\gamma_b}(0,z)dz - \int_{\mathbb{R}_+ \times \mathbb{R}^2} \rho_{\gamma_b}(t,z)\mathrm{GYRO}_{\rho_{\gamma_b}}(\varphi)(t,z)dtdz \right| \leqslant C(\varphi, V, w)\frac{1}{\sqrt{\ln\left(l_b^{-1}\right)}}$$

De plus, la première densité réduite converge vers une solution de l'équation gyro-cinétique.

Theorem I.3.3: *Convergence des densités*

Sous les mêmes hypothèses que Theorem I.3.2, après extraction d'une sous suite, ρ_{γ_b} converge au sens des mesures :

$$\rho_{\gamma_b} \underset{b\to\infty}{\overset{*}{\rightharpoonup}} \rho \in \mathcal{M}(\mathbb{R}_+ \times \mathbb{R}^2)$$
$$\rho_{\gamma_b}(0) \underset{b\to\infty}{\overset{*}{\rightharpoonup}} \rho_0 \in \mathcal{M}(\mathbb{R}^2)$$

vers une solution faible de (I.3.4), c'est-à-dire que $\forall \varphi \in C_c^\infty(\mathbb{R}_+ \times \mathbb{R}^2)$,

$$\int\limits_{\mathbb{R}^2} \varphi(0)\rho_0 - \int\limits_{\mathbb{R}_+\times\mathbb{R}^2} \rho\left(\partial_t\varphi + \nabla^\perp (V + w \star \rho) \cdot \nabla_z \varphi\right) = 0$$

I.3.3 Perspectives

Une première amélioration serait de réduire la régularité demandée sur les potentiels. Dans ce but, on pourrait utiliser des méthodes similaires à [6]. Pour cela, il faudrait établir l'équation pour la dynamique à l'intérieur d'un niveau de Landau, afin de pouvoir contrôler l'équation pour la fonction de Husimi directement et non la densité physique. Pour la dynamique de la fonction de Husimi, on s'attend à obtenir, en plus de l'équation gyro-cinétique, un terme de saut entre niveaux de Landau. Ainsi on obtiendrait un système d'équations couplées avec une équation pour chaque niveau de Landau. Aussi, l'obtention de la propagation des moments de l'énergie cinétique d'ordre supérieur, faciliterait le contrôle des erreurs dans la dynamique de la première densité réduite. La propagation de certains commutateurs, entre γ_b et l'énergie cinétique ou des moments de l'opérateur position, pourraient aussi être utile.

Ensuite, un but principal est de prendre comme point de départ l'équation de Schrödinger à N-corps. Une première avancée dans cette direction serait de considérer l'équation de Hartree Fock à la place de celle de Hartree, c'est-à-dire chercher à contrôler le terme d'échange dans la limite de fort champ magnétique. Ce terme ne devrait pas poser de problème car il est petit dans un régime semi-classique.

En étudiant le problème $3d$, on pourrait obtenir pour la dynamique un terme supplémentaire de transport dans la direction du champ magnétique.

I.4 Quelques outils

I.4.1 Matrices densités

En reprenant les Notation I.1.3, remarquons que les matrices densités étant positives, ce sont des opérateurs auto-adjoints. De plus, comme ce sont des opérateurs à trace, elles sont aussi compactes et ainsi diagonalisable par le théorème spectral. Voici quelques propriétés supplémentaires des matrices densités qui seront utilisées tout au long de la thèse:

Property I.4.1: *Matrices densités*

Soit $\gamma_N \in \mathcal{L}^1(H^{\otimes_\pm N})$ tel que $\mathrm{Tr}\left[\gamma_N\right] = 1$ et $\gamma_N \geqslant 0$,

$$\mathrm{Tr}\left[\gamma_N^{(k)}\right] = 1, \quad 0 \leqslant \gamma_N^{(k)}$$

Soit $k \in [\![1:N]\!]$, si $A_k \in \mathcal{L}^1\left(H^{\otimes k}\right)$, alors

$$\mathrm{Tr}\left[\gamma_N A_k \otimes \mathrm{Id}_H^{\otimes(N-k)}\right] = \mathrm{Tr}\left[\gamma_N^{(k)} A_k\right] \tag{I.4.1}$$

Proof:

On peut supposer $\gamma_N = \Gamma_k \otimes \Gamma_{N-k}$, où $\Gamma_k \in \mathcal{L}^1\left(H^{\otimes k}\right), \Gamma_{N-k} \in \mathcal{L}^1\left(H^{\otimes(N-k)}\right)$ puis revenir au cas général par linéarité. Ainsi,

$$\begin{aligned}\mathrm{Tr}\left[\gamma_N A_k \otimes \mathrm{Id}_H^{\otimes(N-k)}\right] &= \mathrm{Tr}\left[(\Gamma_k A_k) \otimes \Gamma_{N-k}\right] = \mathrm{Tr}\left[\Gamma_k A_k\right]\mathrm{Tr}\left[\Gamma_{N-k}\right] = \mathrm{Tr}\left[\mathrm{Tr}\left[\Gamma_{N-k}\right]\Gamma_k A_k\right] \\ &= \mathrm{Tr}\left[\mathrm{Tr}_{k+1:N}\left[\Gamma_k \otimes \Gamma_{N-k}\right] A_k\right] = \mathrm{Tr}\left[\mathrm{Tr}_{k+1:N}\left[\gamma_N\right] A_k\right] = \mathrm{Tr}\left[\gamma_N^{(k)} A_k\right]\end{aligned}$$

La trace et la positivité des matrices densités réduites est une conséquence directe de ces propriétés pour γ_N. Par (I.4.1) appliqué à $A_k \coloneqq \mathrm{Id}_{H^{\otimes k}}$,

$$\mathrm{Tr}\left[\gamma_N^{(k)}\right] = \mathrm{Tr}\left[\mathrm{Tr}_{k+1:N}\left[\gamma_N\right]\right] = \mathrm{Tr}\left[\gamma_N\right] = 1$$

Puis, si $\psi \in H^{\otimes k}$,

$$\langle \psi | \gamma_N^{(k)} \psi \rangle = \langle \psi \otimes \mathbb{1}^{\otimes(N-k)} \gamma_N \psi \otimes \mathbb{1}^{\otimes(N-k)} \rangle \geqslant 0$$

Dans le cas où la matrice densité est un projecteur sur un produit symétrique/antisymétrique, il existe une formule exacte permettant d'exprimer les densités réduites en fonctions des vecteurs du produit.

Theorem I.4.2: *Théorème de Wick*

Si $\gamma_N \coloneqq |\psi_N\rangle\langle\psi_N|$, avec

$$\psi_N = \frac{1}{\sqrt{N!}} \bigotimes_{i=1}^{N}{}_{\pm}\, \phi_j$$

où $(\phi_j)_j$ est une famille orthonormale, alors

$$\gamma_N^{(k)} = \frac{(N-k)!}{N!} \sum_{1 \leqslant i_1 \neq \cdots \neq i_k \leqslant N} \left| \bigotimes_{j=1}^{k} \phi_{i_j} \right\rangle \left\langle \sum_{\sigma \in S_k} \epsilon_{\pm}(\sigma) \bigotimes_{j=1}^{k} \phi_{i_{\sigma(j)}} \right|$$

Proof:

Premièrement,

$$\gamma_N = \frac{1}{N!} \sum_{\sigma,\tau \in S_N} \epsilon_{\pm}(\sigma\tau) \left| \bigotimes_{j=1}^{N} \phi_{\sigma(j)} \right\rangle \left\langle \bigotimes_{j=1}^{N} \phi_{\tau(j)} \right| = \frac{1}{N!} \sum_{\sigma,\tau \in S_N} \epsilon_{\pm}(\sigma\tau) \bigotimes_{j=1}^{N} \left|\phi_{\sigma(j)}\right\rangle \left\langle \phi_{\tau(j)}\right|$$

Donc

$$\begin{aligned}
\gamma_N^{(k)} = \mathrm{Tr}_{k+1:N}\left[\gamma_N\right] &= \frac{1}{N!} \sum_{\sigma,\tau \in S_N} \epsilon_{\pm}(\sigma\tau) \left(\bigotimes_{j=1}^{k} \left|\phi_{\sigma(j)}\right\rangle \left\langle \phi_{\tau(j)}\right| \right) \prod_{j=k+1}^{N} \mathrm{Tr}\left[\left|\phi_{\sigma(j)}\right\rangle \left\langle \phi_{\tau(j)}\right|\right] \\
&= \frac{1}{N!} \sum_{\sigma,\tau \in S_N} \epsilon_{\pm}(\sigma\tau) \left(\bigotimes_{j=1}^{k} \left|\phi_{\sigma(j)}\right\rangle \left\langle \phi_{\tau(j)}\right| \right) \prod_{j=k+1}^{N} \delta_{\sigma(j),\tau(j)}
\end{aligned}$$

Avec le changement de variables $\tau \coloneqq \sigma^{-1}\tau$ on obtient

$$\begin{aligned}
\gamma_N^{(k)} &= \frac{1}{N!} \sum_{\sigma \in S_N, \tau \in S_N} \epsilon_{\pm}(\sigma\sigma\tau) \left(\bigotimes_{j=1}^{k} \left|\phi_{\sigma(j)}\right\rangle \left\langle \phi_{\sigma\tau(j)}\right| \right) \prod_{j=k+1}^{N} \delta_{\sigma(j),\sigma\tau(j)} \\
&= \frac{1}{N!} \sum_{\sigma \in S_N, \tau \in S_k} \epsilon_{\pm}(\tau) \left(\bigotimes_{j=1}^{k} \left|\phi_{\sigma(j)}\right\rangle \left\langle \phi_{\sigma\tau(j)}\right| \right) \\
&= \frac{1}{N!} \sum_{1 \leqslant i_1 \neq \cdots \neq i_k \leqslant N} \sum_{\sigma \in S_N, \tau \in S_k} \epsilon_{\pm}(\tau) \left(\bigotimes_{j=1}^{k} \left|\phi_{\sigma(j)}\right\rangle \left\langle \phi_{\sigma\tau(j)}\right| \right) \prod_{j=1}^{k} \delta_{\sigma(j),i_j} \\
&= \frac{(N-k)!}{N!} \sum_{1 \leqslant i_1 \neq \cdots \neq i_k \leqslant N} \sum_{\tau \in S_k} \epsilon_{\pm}(\tau) \bigotimes_{j=1}^{k} \left|\phi_{i_j}\right\rangle \left\langle \phi_{i_{\tau(j)}}\right| \\
&= \frac{(N-k)!}{N!} \sum_{1 \leqslant i_1 \neq \cdots \neq i_k \leqslant N} \left| \bigotimes_{j=1}^{k} \phi_{i_j} \right\rangle \left\langle \sum_{\tau \in S_k} \epsilon_{\pm}(\tau) \bigotimes_{j=1}^{k} \phi_{i_{\tau(j)}} \right|
\end{aligned}$$

Du Théorème de Wick peut être déduit le principe de Pauli pour la $k^{\text{ième}}$ matrice densité réduite.

Theorem I.4.3: *Principe de Pauli*

Soit $\gamma_N \in \mathcal{L}^1(H^{\otimes_{\pm} N})$ tel que $\mathrm{Tr}\left[\gamma_N\right] = 1$ et $\gamma_N \geqslant 0$,

$$0 \leqslant \gamma_N^{(k)} \leqslant \frac{(N-k)!k!}{N!}$$

De plus la borne supérieure est atteinte pour produit symétriques/antisymétriques.

Proof:

On peut supposer $\gamma_N = |\psi_N\rangle\langle\psi_N|$, avec

$$\psi_N = \frac{1}{\sqrt{N!}} \bigotimes_{i=1}^{N}{}_{\pm}\, \phi_j$$

où $(\phi_j)_j$ est une famille orthonormale puis conclure par linéarité. Soit

$$\psi_k := \frac{1}{\sqrt{k!}} \bigotimes_{j=1}^{k}{}_{\pm}\, \phi_j$$

En utilisant Theorem I.4.2, on calcule alors

$$\begin{aligned}
\left\langle \psi_k \middle| \gamma_N^{(k)} \psi_k \right\rangle &= \frac{(N-k)!}{N!k!} \sum_{1\leqslant i_1\neq\cdots\neq i_k\leqslant N} \sum_{\sigma,\tau,\upsilon\in S_k} \epsilon_{\pm}(\sigma\tau\upsilon) \left\langle \bigotimes_{j=1}^{k} \phi_{\tau(j)} \middle| \bigotimes_{j=1}^{k} \phi_{i_j} \right\rangle \left\langle \bigotimes_{j=1}^{k} \phi_{i_{\sigma(j)}} \middle| \bigotimes_{j=1}^{k} \phi_{\upsilon(j)} \right\rangle \\
&= \frac{(N-k)!}{N!k!} \sum_{\sigma,\tau,\upsilon,\omega\in S_k} \epsilon_{\pm}(\sigma\tau\upsilon) \left\langle \bigotimes_{j=1}^{k} \phi_{\tau(j)} \middle| \bigotimes_{j=1}^{k} \phi_{\omega(j)} \right\rangle \left\langle \bigotimes_{j=1}^{k} \phi_{\omega\sigma(j)} \middle| \bigotimes_{j=1}^{k} \phi_{\upsilon(j)} \right\rangle \\
&= \frac{(N-k)!}{N!k!} \sum_{\sigma,\tau,\upsilon,\omega\in S_k} \epsilon_{\pm}(\sigma\tau\upsilon)\delta_{\tau,\omega}\delta_{\omega\sigma,\upsilon} = \frac{(N-k)!}{N!k!} \sum_{\sigma,\omega\in S_k} \epsilon_{\pm}(\sigma\omega\omega\sigma) = \frac{(N-k)!k!}{N!}
\end{aligned}$$

On peut construire une base de Hilbert de H en complétant $(\phi_j)_{1\leqslant j\leqslant N}$. Si ψ_k est un produit d'éléments de la famille $(\phi_j)_{1\leqslant j\leqslant N}$, quitte à réordonner les ϕ_j, on se ramène au calcul ci-dessus. On conclut alors en remarquant que si ψ_k est un produit contenant des termes dans $\operatorname{span}\left((\phi_j)_{1\leqslant j\leqslant N}\right)^{\perp}$, le résultat du calcul ci-dessus est 0.

I.4.2 États cohérents

Les états cohérents sont indispensables dans l'étude de limites semi-classiques. Soit $a^\dagger$ et a des opérateurs de création et d'annihilation :

$$\left[a, a^\dagger\right] = \mathrm{Id}_H$$

où H est l'espace de Hilbert généré par application successives de $a^\dagger$: $\left({a^\dagger}^n|0\rangle\right)_{n\in\mathbb{N}}$ sur le vide $|0\rangle := 1_{\mathbb{C}}$. Soit $\alpha\in\mathbb{C}$, l'opérateur état cohérent est

$$D(\alpha) := e^{\alpha a^\dagger - \overline{\alpha} a}$$

Par la formule de Baker-Campbell-Hausdorff,

$$\mathscr{D}(\alpha) = e^{-\frac{1}{2}\left[\alpha a^\dagger, -\overline{\alpha}a\right]} e^{\alpha a^\dagger} e^{-\overline{\alpha}a} = e^{-\frac{|\alpha|^2}{2}} e^{\alpha a^\dagger} e^{-\overline{\alpha}a}$$

On définit alors les états cohérents

$$\psi_\alpha := \mathscr{D}(\alpha)|0\rangle = e^{-\frac{|\alpha|^2}{2}} e^{\alpha a^\dagger}|0\rangle = e^{-\frac{|\alpha|^2}{2}} \sum_{n\in\mathbb{N}} \frac{\alpha^n}{n!} {a^\dagger}^n|0\rangle = e^{-\frac{|\alpha|^2}{2}} \sum_{n\in\mathbb{N}} \frac{\alpha^n}{\sqrt{n}} |n\rangle$$

où $|n\rangle \coloneqq \frac{a^{\dagger n}}{\sqrt{n!}}|0\rangle$ est normé. L'opérateur nombre d'excitations est

$$\mathcal{N} \coloneqq a^\dagger a$$

On a les propriétés suivantes pour les états cohérents :

Proposition I.4.4

$$\begin{aligned}
&\langle\psi_\alpha|\psi_\alpha\rangle = 1\\
&[a, \mathscr{D}(\alpha)] = \alpha\\
&a\psi_\alpha = \alpha\psi_\alpha\\
&\langle\psi_\alpha|\mathcal{N}\psi_\alpha\rangle = |\alpha|^2\\
&\Delta\mathcal{N} \coloneqq \sqrt{\langle\psi_\alpha|\mathcal{N}^2\psi_\alpha\rangle - \langle\psi_\alpha|\mathcal{N}\psi_\alpha\rangle^2} = |\alpha|\\
&\frac{1}{\pi}\int\limits_{\mathbb{C}} |\psi_\alpha\rangle\langle\psi_\alpha|\,d\alpha = \mathrm{Id}_H
\end{aligned} \tag{I.4.2}$$

Proof:

Par un calcul direct,

$$\langle\psi_\alpha|\psi_\alpha\rangle = e^{-|\alpha|^2}\sum_{n\in\mathbb{N}}\frac{\overline{\alpha}\alpha}{n!} = 1$$

et

$$\begin{aligned}
[a, \mathscr{D}(\alpha)] =& e^{-\frac{|\alpha|^2}{2}}\left[a, e^{\alpha a^\dagger}\right]e^{-\overline{\alpha}a} = e^{-\frac{|\alpha|^2}{2}}\left(\sum_{n\in\mathbb{N}}\frac{\alpha^n}{n!}\left[a, a^{\dagger n}\right]\right)e^{-\overline{\alpha}a}\\
=& e^{-\frac{|\alpha|^2}{2}}\left(\sum_{n\in\mathbb{N}^*}\frac{\alpha^n}{n!}na^{\dagger n-1}\right)e^{-\overline{\alpha}a} = \alpha\mathscr{D}(\alpha)
\end{aligned}$$

donc

$$\begin{aligned}
&a\psi_\alpha = a\mathscr{D}(\alpha)|0\rangle = \mathscr{D}(\alpha)a|0\rangle + \alpha\mathscr{D}(\alpha)|0\rangle = \alpha\psi_\alpha\\
&\langle\psi_\alpha|\mathcal{N}\psi_\alpha\rangle = \langle a\psi_\alpha|a\psi_\alpha\rangle = |\alpha|^2\\
&\left\langle\psi_\alpha\middle|\mathcal{N}^2\psi_\alpha\right\rangle = \left\langle a\psi_\alpha\middle|aa^\dagger a\psi\alpha\right\rangle = \langle aa\psi_\alpha|aa\psi_\alpha\rangle + \langle a\psi_\alpha|a\psi_\alpha\rangle = |\alpha|^4 + |\alpha|^2\\
&\Delta\mathcal{N} = \sqrt{|\alpha|^4 + |\alpha|^2 - |\alpha|^4} = |\alpha|
\end{aligned}$$

Finalement,

$$\begin{aligned}
\int\limits_{\mathbb{C}} |\psi_\alpha\rangle\langle\psi_\alpha|\,d\alpha =& \sum_{n,m\in\mathbb{N}}\int\limits_{\mathbb{C}} e^{-|\alpha|^2}\frac{\alpha^n\overline{\alpha}^m}{\sqrt{n!m!}}|n\rangle\langle m|\,d\alpha = \sum_{n\in\mathbb{N}}|\psi_n\rangle\langle\psi_n|\int\limits_{\mathbb{C}} e^{-|\alpha|^2}\frac{|\alpha|^{2n}}{n!}d\alpha\\
=& 2\pi\sum_{n\in\mathbb{N}}|n\rangle\langle n|\frac{1}{n!}\int\limits_{\mathbb{R}_+} r^{2n+1}e^{-r^2}dr = \pi\sum_{n\in\mathbb{N}}|n\rangle\langle n|\frac{1}{n!}\int\limits_{\mathbb{R}_+} r^n e^{-r}dr
\end{aligned}$$

$$=\pi \sum_{n\in\mathbb{N}} |n\rangle \langle n| = \pi \mathrm{Id}_H$$

Avec (I.4.2), en interprétant a comme un opérateur de position dans $\mathbb{C}$, ψ_α est un état physique localisé autour α.

Chapter II

Multiple Landau level filling for a large magnetic field limit of 2D fermions

Abstract :

Motivated by the quantum hall effect, we study N two dimensional interacting fermions in a large magnetic field limit. We work in a bounded domain, ensuring finite degeneracy of the Landau levels. In our regime, several levels are fully filled and inert: the density in these levels is constant. We derive a limiting mean-field and semi classical description of the physics in the last, partially filled Landau level.

II.1 Context and result

II.1.1 Model

We consider a system of N interacting fermionic particles in two dimensions. They are placed under a homogeneous magnetic field perpendicular to the domain. In this context the kinetic energy of the particles is quantized into discrete energy levels called Landau levels, separated by a finite energy gap. This problem has initially been studied by Lieb Solovej and Yngvason in [43], [47], [44], [45], [42] and more recently by Fournais, Lewin and Madsen in [17], [16].

Our goal is to study the mean field and semi-classical limit under high magnetic field so the Landau level quantization plays an important role. This setup is related to that of [42] where three regimes are studied. In the first one, the energy gap is small with respect to the potential contributions in the energy so particles occupy all Landau levels and a standard Thomas–Fermi model is obtained in the limit. In the second one, the energy gap is comparable to the potential energy terms, particles optimise both their Landau level and their position in the potentials and the limit is a magnetic Thomas-Fermi model. For the last scaling, the gap is large compared to the potential energies so all particles occupy the lowest Landau level and the limit is described with a classical continuum electrostatic theory in this level. We want to deal with the intermediate situation where only a finite number of Landau levels are completely filled. Precisely, our result is a limit where the q first Landau Level are fully filled, the next Landau level is partially filled with filling ratio $r < 1$ and all higher Landau levels are empty. We also provide a model for the physics in the partially filled Landau level. This setup is physically motivated by the quantum Hall effect which mostly takes place in a partially filled Landau level while lower Landau levels are filled and inert, and higher levels are empty, see [30].

In this perspective we want to fix the limit ratio of the number of particles to the degeneracy of Landau levels. On the whole space $\mathbb{R}^2$ this degeneracy is infinite. To ensure finiteness of the degeneracy of the Landau levels (see Proposition II.2.11), we work on a bounded domain. For simplicity, we would like to consider a torus with periodic boundary conditions. But, in the presence of a magnetic field the periodic boundary conditions must be modified. This is a well known issue, for example see [30, Section 3.9]. As explained in Subsection II.2.1, we define magnetic translation operators to ensure commutation with the magnetic momentum. These magnetic translations operators define the so called magnetic periodic boundary conditions.

Notation II.1.1: *Model*

We work on the domain $\Omega := [0, L]^2$ of fixed size $L > 0$. The one body kinetic energy operator, also called magnetic Laplacian, is

$$\mathscr{L}_{\hbar,b} := (i\hbar\nabla + bA)^2 \tag{II.1.1}$$

We work in the Coulomb gauge:

$$\nabla \cdot A = 0$$

where $A \in C^\infty(\mathbb{R}^2, \mathbb{R}^2)$ is the vector potential generating the constant magnetic field: identifying $\mathbb{R}^2$ with $\mathbb{R}^2 \times \{0\} \subset \mathbb{R}^3$, we assume

$$\nabla \wedge A = (0, 0, 1) \tag{II.1.2}$$

b is the magnetic field amplitude with associated magnetic length

$$l_b := \sqrt{\frac{\hbar}{b}}$$

We identify $\mathbb{R}^2$ with $\mathbb{C}$ and use complex notation for the variables $(x, y) \in \mathbb{R}^2$ namely

$$(x, y) = x + iy \in \mathbb{C}$$

Let $z_0 \in \mathbb{C}$, by (II.1.2)

$$\nabla \wedge (A - A(\bullet - z_0)) = 0$$

so we can choose $\varphi_{z_0} \in C^\infty(\mathbb{R}^2, \mathbb{R})$ such that

$$A - A(\bullet - z_0) =: l_b^2 \nabla \varphi_{z_0} \tag{II.1.3}$$

For some usual expressions see (II.2.3) and (II.2.4). As detailed in Subsection II.2.1, for a wave-function $\psi \in L^2(\Omega)$ the magnetic periodic boundary conditions are

$$\forall t \in [0, L], \begin{cases} \psi(L + it) = e^{i\varphi_L(L+it)}\psi(it) \\ \psi(t + iL) = e^{i\varphi_{iL}(t+iL)}\psi(t) \end{cases} \tag{II.1.4}$$

and the domain of the magnetic Laplacian is

$$\mathrm{Dom}\left(\mathscr{L}_{\hbar,b}\right) := \left\{\psi \in H^2(\Omega) \text{ such that (II.1.4) holds}\right\} \tag{II.1.5}$$

Now, the N-body Hamiltonian is

$$\mathscr{H}_N := \sum_{j=1}^{N} \left(\left(i\hbar\nabla_{x_j} + bA(x_j)\right)^2 + V(x_j) \right) + \frac{2}{N-1} \sum_{1 \leqslant j < k \leqslant N} w(x_j - x_k) \tag{II.1.6}$$

acting on the space of N-body fermionic wave-functions

$$L^2_-\left(\Omega^N\right) := \bigwedge^N L^2(\Omega).$$

We denote $\mathbb{T} := \mathbb{R}^2/L\mathbb{Z}^2$. $V \in L^2(\mathbb{T})$ is the external potential and $w \in L^2(\mathbb{T})$ the interaction potential assumed to be radial for the metric on the torus:

$$w(x - y) =: \widetilde{w}(d(x, y)) \text{ with } d(x, y) := \min_{r \in L\mathbb{Z}^2} |x - y + r|$$

The domain of the N-body Hamiltonian (II.1.6) is

$$\mathrm{Dom}\left(\mathscr{H}_N\right) := \bigwedge^N \mathrm{Dom}\left(\mathscr{L}_{\hbar,b}\right)$$

We define the N-body ground state energy

$$E_N^0 := \inf\left\{\langle\psi_N|\mathscr{H}_N\psi_N\rangle, \psi_N \in \mathrm{Dom}(\mathscr{H}_N) \text{ such that } \|\psi_N\|_{L^2} = 1\right\} \tag{II.1.7}$$

When we change the gauge, wave-function change by a phase thus the energy is in dependant of the choice of gauge. There are $N(N-1)/2$ interacting pairs of fermions. Thus, we divide the interactions term by $(N-1)/2$ so that the order of the contribution coming from interactions is $\mathcal{O}(N)$ and comparable to the contribution coming from the external potential.

As we will see in Subsection II.2.2, the self adjointness of the magnetic Laplacian and the existence of its eigenvectors require the magnetic field flux bL^2 going through the domain to be quantized in multiples of $2\pi\hbar$:

$$\exists d \in \mathbb{N} \text{ such that } 2\pi d = \frac{b}{\hbar}L^2 = \frac{L^2}{l_b^2}$$

We will prove in Proposition II.2.11 that d is the degeneracy of Landau levels. Now, we can fix the number of filled Landau levels by choosing a scaling for which the ratio N/d is fixed.

Notation II.1.2: *Scaling*

We take Planck's constant to be a sequence $\hbar := (\hbar_N)_{N\in\mathbb{N}}$ such that

$$N^{-\frac{1}{2}} \ll \hbar \ll N^{-\frac{1}{4}} \tag{II.1.8}$$

Let $q \in \mathbb{N}, r \in [0,1), b := (b_N)_{N\in\mathbb{N}}$ be such that

$$d := \frac{L^2}{2\pi l_b^2} \subset \mathbb{N}^* \tag{II.1.9}$$

and

$$\frac{N}{d} \underset{N\to\infty}{=} q + r + o\left(\frac{1}{\hbar b}\right) \tag{II.1.10}$$

where $E^* := E\backslash\{0\}$ for $E \subset \mathbb{R}$.

q will give the number of fully filled Landau levels and r the filling ratio of the q^{th} Landau level. Note that the lowest Landau level index is 0 in our convention. With this notation,

$$\frac{N}{d} = \frac{2\pi l_b^2 N}{L^2} \underset{N\to\infty}{\longrightarrow} q + r \quad \text{and} \quad \frac{1}{l_b^2} = \frac{b}{\hbar} \underset{N\to\infty}{\sim} \frac{2\pi N}{(q+r)L^2} \tag{II.1.11}$$

With this scaling, we find that the order of the magnetic field is $b = \mathcal{O}(\hbar N)$. It is known (II.2.18), that the order or the kinetic energy is

$$\hbar b = \mathcal{O}(\hbar^2 N) \gg 1 \tag{II.1.12}$$

The kinetic energy contribution needs to be of leading order compared to the potential terms if we want to impose the number of filled Landau level and this is true if and only if

$$\hbar^2 N \gg 1$$

hence the upper bound in (II.1.8). The condition $\hbar \gg N^{-\frac{1}{2}}$ is necessary in our approach to control some error terms coming from the kinetic energy. This is also the reason why we impose the convergence rate in (II.1.10). This scaling is a semi-classical limit because Planck's constant goes to 0.

To satisfy (II.1.10), one can take for example

$$d \coloneqq \left\lfloor \frac{N}{q+r} \right\rfloor$$

so

$$\frac{N}{q+r} - 1 \leqslant d \leqslant \frac{N}{q+r} \implies q + r \leqslant \frac{N}{d} \leqslant \frac{N}{\frac{N}{q+r} - 1} = \frac{q+r}{1 - \frac{q+r}{N}} \underset{N\to\infty}{=} q + r + \mathcal{O}\left(\frac{1}{N}\right)$$

Note that if r is rational one can take sequences such that there is no error in (II.1.10), and if r is irrational (see [1, Proposition 1.4]) it is always possible to have

$$\frac{N}{d} \underset{N\to\infty}{=} q + r + \mathcal{O}\left(\frac{1}{N^2}\right)$$

II.1.2 Semi-classical limit model

In the limit, we obtain a semi-classic model where the energy no longer depend on the wave-function but on the density in phase space. This comes with a non linearity in the interaction term. The phase space is $\mathbb{N} \times \Omega$. This means that particles have two degrees of freedom: the first one is $n \in \mathbb{N}$ the quantum number representing the Landau Level index and $R \in \Omega$ representing the position of particles in space. In classical mechanics, one can think of R as the center of the cyclotron orbit of the particles and n as the index of the quantized angular velocity of the cyclotron orbit. This model is semi-classical in the sense that the Pauli principle still holds as a bound on the density.

Notation II.1.3: *Semi-classical functional*

We consider the measure on phase space

$$\eta \coloneqq \left(\sum_{n\in\mathbb{N}} \delta_n\right) \otimes \lambda_\Omega$$

where λ_Ω is the Lebesgue measure on Ω. For a phase space density $m \in L^1(\mathbb{N} \times \Omega, \mathbb{R}_+)$, the semi-classical energy is

$$\mathcal{E}_{sc,\hbar b}[m] \coloneqq \int_{\mathbb{N}\times\Omega} E_n m(n,R) d\eta(n,R) + \int_{\mathbb{N}\times\Omega} V m d\eta + \int_{(\mathbb{N}\times\Omega)^2} w m^{\otimes 2} d\eta^{\otimes 2} \tag{II.1.13}$$

where, as we will see in Section II.2,

$$E_n \coloneqq 2\hbar b\left(n + \frac{1}{2}\right) \tag{II.1.14}$$

is the energy of the n^{th} Landau level. Define the semi-classical domain

$$\mathcal{D}_{sc} := \left\{ m \in L^1(\mathbb{N} \times \Omega) \text{ such that } \int_{\mathbb{N}\times\Omega} m d\eta = 1 \text{ and } 0 \leqslant m \leqslant \frac{1}{(q+r)L^2} \right\} \tag{II.1.15}$$

and the semi-classical ground state energy

$$E^0_{sc,\hbar b} := \inf_{m \in \mathcal{D}_{sc}} \mathcal{E}_{sc,\hbar b}[m]$$

We also define the electrostatic model for the partially filled Landau level that only depends on the density.

Notation II.1.4: *Electrostatic model for the partially filled level*

Define

$$\mathcal{E}_{qLL}[\rho] := \int_{\Omega} V\rho + \iint_{\Omega^2} w(x-y)\rho(x)\rho(y)dxdy \tag{II.1.16}$$

with domain

$$\mathcal{D}_{qLL} := \left\{ \rho \in L^1(\Omega) \text{ such that } \int_{\Omega} \rho = \frac{r}{q+r} \text{ and } 0 \leqslant \rho \leqslant \frac{1}{(q+r)L^2} \right\} \tag{II.1.17}$$

The associated ground state energy is

$$E^0_{qLL} := \inf_{\mathcal{D}_{qLL}} \mathcal{E}_{qLL}$$

We define the following energies:

$$E^{q,r} := \frac{q^2+2qr+r}{q+r} \tag{II.1.18}$$

$$E^{q,r}_V := \frac{q}{q+r} \fint_{\Omega} V \tag{II.1.19}$$

$$E^{q,r}_w := \frac{q^2+2qr}{(q+r)^2} \fint_{\Omega^2} w \tag{II.1.20}$$

Let $\rho \in \mathcal{D}_{qLL}$, define

$$m_\rho(n,x) := \mathbb{1}_{n<q} \frac{1}{L^2(q+r)} + \mathbb{1}_{n=q}\rho(x) \tag{II.1.21}$$

m_ρ is a phase space density constructed with the qLL lowest Landau levels saturating the Pauli principle in (II.1.15) and (II.1.17) and with the density ρ in the partially filled Landau level. The ratio of particles in the partially filled Landau level is

$$\frac{r}{q+r}$$

This corresponds to the normalization constraint in (II.1.17). With this we see that the Pauli principle is indeed the correct bound on the densities to have

$$\int_{\mathbb{N}\times\Omega} m_\rho d\eta = 1$$

We will see in Proposition II.6.1 by a direct computation that

$$\mathcal{E}_{sc,\hbar b}\left[m_\rho\right] = \hbar b E^{q,r} + E_V^{q,r} + E_w^{q,r} + \mathcal{E}_{qLL}\left[\rho\right]$$

where

- $\hbar b E^{q,r}$ is the kinetic energy contribution from the $q+1$ lowest Landau levels
- $E_V^{q,r}$ is the external potential energy contribution from the q lowest Landau levels
- $E_w^{q,r}$ is the energy contribution from interactions between the q lowest Landau levels and the interactions between the q lowest Landau levels and the $(q+1)^{th}$ Landau level. in other words, it contains all the interactions except the ones inside the partially filled level.

The particles in the partially filled Landau level try to optimise their localisation with respect to the self consistent potential $V + w * \rho$:

$$\mathcal{E}_{qLL}\left[\rho\right] = \int_\Omega \left(V + w * \rho\right)\rho$$

II.1.3 Main results

We can now state our main result:

Theorem II.1.5: *Mean field limit with magnetic periodic conditions*

With the definitions introduced in Notation II.1.1, Notation II.1.2, Notation II.1.3, Notation II.1.4,

$$\frac{E_N^0}{N} \underset{N\to\infty}{=} \hbar b E^{q,r} + E_V^{q,r} + E_w^{q,r} + E_{qll}^0 + o(1)$$

This means that in the limit, the first order in the quantum many body energy per particle is the trivial energy $\hbar b E^{q,r}$. Then for terms of order 1, the only non trivial contribution to the energy are the external potential term and the interaction term inside the partially filled Landau level. The lower Landau levels are totally filled and therefore their contribution to the energy is constant. The interaction of the partially filled level with all other level will also be a constant. For higher Landau levels, their contribution to the energy is null because they are totally empty.

The regularity assumptions on the potentials are not minimal, we expect this result to hold true if potentials have an L^1 positive part and an L^2 negative part. Under these assumptions, one needs to prove that the particles will not concentrate in the L^1 positive singularities of the potentials. This has been done in [42] for the repulsive $1/|x|$ Coulomb potential. We will not deal with this issue in this paper.

The number of variables of the densities is going to infinity in our limit. As usual for a large number of particles, obtaining a convergence of densities requires to work in a space with a finite number of variables and therefore look at reduced densities.

Notation II.1.6: *Reduced densities*

We denote $\mathcal{L}^p$ the set of p-Schatten class operators along with $\|\bullet\|_{\mathcal{L}^p}$ the p-Schatten norm. Let $\gamma_N \in \mathcal{L}^1\left(L^2_-\left(\Omega^N\right)\right)$ a positive operator (thus self adjoint) of trace 1. We call such an operator an N-body density matrix. By the spectral theorem, γ_N is diagonalizable in a Hilbert basis of $L^2_-\left(\Omega^N\right)$:

$$\gamma_N = \sum_{i\in\mathbb{N}} \lambda_i \left|u_i\right\rangle \left\langle u_i\right| \text{ with } 0 \leqslant \lambda_i \leqslant 1 \text{ and } \sum_{i\in\mathbb{N}} \lambda_i = 1$$

We will denote in the same way operators and their integral kernel. We introduce compact notation for lists:

$$\begin{aligned} 1:n &\coloneqq (1,\ldots,n) \\ x_{1:n} &\coloneqq (x_1,\ldots,x_n) \end{aligned}$$

The density associated to γ_N is

$$\rho_{\gamma_N}(x_{1:N}) \coloneqq \gamma_N(x_{1:N}, x_{1:N})$$

Let Tr_I be the partial trace that traces out coordinates in $I \subset [\![1,N]\!]$ of $L^2(\Omega)^{\otimes N}$, it is defined as the linear operator on $\mathcal{L}^1\left(L^2\left(\Omega^N\right)\right)$ satisfying

$$\forall A_{1:N} \in \mathcal{L}^1\left(L^2(\Omega)\right), \mathrm{Tr}_I\left[\bigotimes_{i=1}^N A_i\right] \coloneqq \mathrm{Tr}\left[\bigotimes_{i\in I} A_i\right] \bigotimes_{i\notin I} A_i$$

Let $1 \leqslant k < N$, we define the k^{th} reduced density matrix associated to γ_N by

$$\gamma_N^{(k)} = \mathrm{Tr}_{k+1:N}\left[\gamma_N\right] \tag{II.1.22}$$

with the convention that $\gamma_N^{(N)} \coloneqq \gamma_N$. For an N variables symmetric density ρ_N we denote $\rho_N^{(k)}$ its k^{th} marginal. If one starts from a wave-function $\psi_N \in L^2_-\left(\Omega^N\right)$ we use the notation

$$\begin{aligned} \gamma_{\psi_N} &\coloneqq \left|\psi_N\right\rangle \left\langle \psi_N\right| \\ \rho_{\psi_N} &\coloneqq \rho_{\gamma_{\psi_N}} = \left|\psi_N\right|^2 \end{aligned} \tag{II.1.23}$$

Note that with this notation

$$\rho_{\gamma_N^{(k)}} = \rho_{\gamma_N}^{(k)} \tag{II.1.24}$$

In (II.1.22), we have integrated the last $N-k$ variables but the result does not depend of the choice of these variables. Indeed, a permutation of coordinates brings a sign $\pm$ in front of each $|u_i\rangle$ and this keeps γ_N invariant.

The domain Ω is a locally compact metric space, the set of Radon measures on it is the dual of continuous and compactly supported functions

$$\mathcal{M}(\Omega) = C_c^0(\Omega)^*$$

In our case, since Ω is compact this is just the dual of continuous functions. We denote $\mathcal{M}_+(\Omega)$ the set of positive Radon measures. Let $\mathcal{P}(\Omega)$ be the set of probabilities on Ω:

$$\mathcal{P}(\Omega) \coloneqq \{\mu \in \mathcal{M}_+(\Omega) \text{ such that } \mu(\Omega) = 1\}$$

On this space the weak star topology is metrizable using a Wasserstein metric [38, Section 7.12]. Moreover $\mathcal{P}(\Omega)$ is also locally compact [41, section 17.E], thus it is possible to iterate and define the space of probability measures on $\mathcal{P}(\Omega)$ namely $\mathcal{P}\left(\mathcal{P}(\Omega)\right)$.

Now, we have the following theorem for the convergence of reduced densities:

Theorem II.1.7: *Densities convergence with magnetic periodic conditions*

With the definitions introduced in Notation II.1.1, Notation II.1.2, Notation II.1.3, Notation II.1.4, Notation II.1.6, if (ψ_N) is a sequence of minimizers of (II.1.7), then $\exists \mu \in \mathcal{P}\left(\mathcal{D}_{qLL}\right)$ such that

- μ only charges minimizers of the limit energy functional (II.1.16)
- $\forall k \in \mathbb{N}^*$, in the sense of Radon measures

$$\rho^{(k)}_{\psi_N} \xrightarrow[N\to\infty]{*} \int_{\mathcal{D}_{qLL}} \left(\frac{q}{L^2(q+r)} + \rho\right)^{\otimes k} d\mu(\rho) \tag{II.1.25}$$

The density of particles converge to a convex combination of densities of the form

$$\frac{q}{L^2(q+r)} + \rho$$

From the Pauli principle in (II.1.17) we see that the constant term in this expression corresponds to particles in the q lowest and fully filled Landau levels. Then the density of particles in the partially filled Landau level is given by a minimizer ρ of the limit functional (II.1.16).

II.1.4 Scaling

Another way to obtain the scaling in Notation II.1.2 is to observe that we have two characteristic length-scales:

- $\dfrac{L}{\sqrt{N}}$, measuring the mean distance between particles

- l_b, the magnetic length, which, in classical mechanics corresponds to the radius of a cyclotron orbit. Due to the Pauli principle, l_b will be the order of the minimum distance between particles inside a Landau level. More precisely the Pauli principle takes the form of an upper bound on the density in phase space.

The square ratio of these length is

$$\frac{L^2}{Nl_b^2} = \frac{bL^2}{\hbar N} \tag{II.1.26}$$

If this ratio goes to zero, the mean distance between particles is very small compared to the minimal length-scale between two particles in a fixed Landau level. This implies that the particles must fill many Landau levels and this corresponds to the scaling in [42] where the energy gap between Landau level is small compared to the potential terms.

If this ratio goes to infinity, the mean distance between particles is very large compared to the minimal length-scale between two particles in a fixed Landau Level. As a consequence, all particles can be placed in the lowest Landau level and this corresponds to the regime in [42] where particles only occupy the lowest Landau level and do not feel the Pauli principle.

In the limit we study, we see from (II.1.11) that the ratio (II.1.26) has been fixed to be

$$\frac{L^2}{Nl_b^2} \underset{N\to\infty}{\longrightarrow} \frac{2\pi}{q+r} \tag{II.1.27}$$

in order to fill a finite number of Landau levels. In our limit we fixed L and took l_b going to zero, but one can also ensure (II.1.27) by fixing a magnetic length $\widetilde{l_b} > 0$ and taking a domain length $\widetilde{L}$ going to infinity as

$$\widetilde{L} := \frac{\widetilde{l_b}}{l_b} L \tag{II.1.28}$$

In this limit the density of particles in the domain Ω is fixed:

$$\frac{\widetilde{L}^2}{N} \underset{N\to\infty}{\longrightarrow} \widetilde{l_b}^2 \frac{2\pi}{q+r} \tag{II.1.29}$$

The two scalings can be unified by saying that we use a small dimensionless parameter

$$\alpha := \frac{l_b}{L} = \frac{\widetilde{l_b}}{\widetilde{L}} \tag{II.1.30}$$

Indeed, taking a small magnetic length as in the first scaling or a large domain length in the second scaling implies that α goes to zero as in (II.1.30). One can go from a limit to the other by unitary transformation of the Hamiltonian.

Proposition II.1.8: *Re-scaling*

Define the unitary operator

$$U_\tau : \begin{array}{ccc} L^2([0,L]^2) & \to & L^2\left(\left[0,\widetilde{L}\right]^2\right) \\ u & \mapsto & u_\tau := \tau u\,(\tau\bullet) \end{array} \text{ with } \tau := \frac{L}{\widetilde{L}}$$

then the N-body Hamiltonian (II.1.6) is unitarily equivalent to

$$U_\tau^{\otimes N}\mathscr{H}_N U_\tau^{*\otimes N} = \frac{1}{\tau}\mathscr{H}_{N,\tau}$$

where

$$\mathscr{H}_{N,\tau} := \sum_{j=1}^{N}\left((i\hbar_\tau\nabla_j + b_\tau A_\tau(x_j))^2 + V_\tau(x_j)\right) + \frac{2}{N-1}\sum_{1\leqslant j<k\leqslant N} w_\tau(x_j - x_k) \tag{II.1.31}$$

and

$$\hbar_\tau := \frac{\hbar}{\sqrt{\tau}} \qquad A_\tau := \frac{1}{\tau}A\,(\tau\bullet) \qquad b_\tau := \tau^{\frac{3}{2}}b \qquad V_\tau := \tau V\,(\tau\bullet) \qquad w_\tau := \tau w\,(\tau\bullet)$$

Proof:

First, we check unitarity.
Let $u \in L^2(\Omega)$,

$$\|u_\tau\|^2 = \tau^2\int_{\left[0,\widetilde{L}\right]^2} u\,(\tau\cdot x)^2\,dx = \int_{[0,L]^2} u(x)^2 dx = \|u\|^2$$

With the same change of variable we see that the adjoint of U_τ is

$$U_\tau^* : \begin{array}{ccc} L^2\left(\left[0,\widetilde{L}\right]^2\right) & \to & L^2([0,L]^2) \\ u & \mapsto & \frac{1}{\tau}u\left(\frac{1}{\tau}\bullet\right) \end{array}$$

The transformation of the kinetic momentum is

$$\begin{aligned} U_\tau\,(i\hbar\nabla + bA)\,U_\tau^* u &= \frac{1}{\tau}U_\tau\,(i\hbar\nabla + bA)\,u\left(\frac{1}{\tau}\bullet\right) \\ &= \frac{1}{\tau}U_\tau\left(i\hbar\frac{1}{\tau}\nabla u\left(\frac{1}{\tau}\bullet\right) + bA(\bullet)u\left(\frac{1}{\tau}\bullet\right)\right) \\ &= i\hbar\frac{1}{\tau}\nabla u(\bullet) + bA\,(\tau\bullet)\,u(\bullet) = \frac{1}{\sqrt{\tau}}\,(i\hbar_\tau\nabla + b_\tau A_\tau)\,u \end{aligned}$$

The action of U_τ on the potentials is

$$U_\tau V(x)U_\tau^* = V\,(\tau\cdot x)$$

$$U_\tau^{\otimes 2} w(x) U_\tau^{*\otimes 2} = w\left(\tau \cdot x\right)$$

We conclude by putting together this two computations and factorising by $\dfrac{1}{\tau}$

We confirm that the new magnetic length is

$$\sqrt{\frac{\hbar_\tau}{b_\tau}} = \frac{\widetilde{L} l_b}{L} = \widetilde{l_b}$$

Moreover if one chooses a linear A (like (II.2.3) or (II.2.4)) and

$$V(x) = \varrho * \frac{1}{|x|} \qquad w(x) = \frac{1}{|x|}$$

then the vector potential and the interaction potential are not rescaled :

$$A_\tau = A \qquad w_\tau = w$$

If we assume that the external potential is generated by a background charge density $\varrho \in L^1(\Omega)$ it transforms as

$$V_\tau(x) = \int_\Omega \tau \varrho(\tau x - y) \frac{1}{|y|} dy = \int_{\left[0,\widetilde{L}\right]^2} \tau^2 \varrho(\tau\left(y - x\right)) \frac{1}{|y|} dy =: \varrho_\tau * \frac{1}{|x|}$$

The re-scaling preserves the total charge

$$\int_{\left[0,\widetilde{L}\right]^2} \varrho_\tau dx = \int_\Omega \varrho dx$$

and

$$\mathscr{H}_{N,\tau} = \sum_{j=1}^N \left(\left(i\hbar_\tau \nabla_j + b_\tau A(x_j)\right)^2 + \rho_\tau * \frac{1}{|x|} \right) + \frac{2}{N-1} \sum_{1 \leqslant j < k \leqslant N} w(x_j - x_k)$$

We conclude that our initial scaling is equivalent to a thermodynamic limit.

II.1.5 Organisation of the paper

The next two sections contain preparations and necessary tools. Section II.2 is about the diagonalisation of the magnetic Laplacian (II.1.1). In Section II.3 we define the orthogonal projection on Landau levels and localise it in space, this will be the central object in the definition of the semi-classical densities. Then we prove a Lieb-Thirring inequality in Section II.4 to deal with L^2 potentials. The last two sections contain the proof of Theorem II.1.5 and Theorem II.1.7. In Section II.5 we justify the semi-classical approximation and express the energy in terms of semi-classical densities. Finally, in Section II.6 we prove the mean-field approximation giving an upper and a lower energy bound.

II.2 Quantization

In this Section, we recall the diagonalization of the magnetic Laplacian (II.1.1). We construct an orthonormal basis of $L^2(\Omega)$ adapted to the Landau levels in terms of magnetic periodic eigenstates of $\mathscr{L}_{\hbar,b}$. This result is stated in Proposition II.2.16. This fact is already well known in the literature, see [35], [36] or in [30, section 3.9]. Thus the reader may go directly to Proposition II.2.16 and accept its statement.

To prove Proposition II.2.16 we will see that on a finite domain, the degeneracy of Landau levels is finite in Proposition II.2.11. We use the fact that the Landau levels are isomorphic and we study the lowest Landau level for which we prove the following properties:

- the wave-functions have a finite number of zeros inside the domain (Proposition II.2.10)
- the degeneracy is the number of zeros of the wave-functions (Proposition II.2.11)

Then, we will prove another expression for the eigenfunctions in Proposition II.2.18 using the Poisson summation formula.

II.2.1 Magnetic translation operators

In this subsection we explain the definition of the boundary conditions (II.1.4). If T_{z_0} is the translation operator by z_0 and we try to commute the magnetic momentum with a translation we get

$$[i\hbar\nabla + bA, T_{z_0}] = b\,[A, T_{z_0}] = b(A - A(\bullet - z_0))T_{z_0}$$

Thus we cannot impose periodic boundary conditions, which would mean finding joint eigenfunctions of $\mathscr{L}_{\hbar,b}$ and of the translation operators with eigenvalue 1. The remedy is to compose the translation operator with a change of phase chosen to ensure commutation with $\mathscr{P}_{\hbar,b}$. Thus, $\mathscr{L}_{\hbar,b}$ and the magnetic translations can be diagonalized jointly. This means that we can

Notation II.2.1

Let $z_0 \in \mathbb{C}$, define the translation operator on $u \in L^2_{loc}(\mathbb{R}^2)$ by

$$T_{z_0}u \coloneqq u(\bullet - z_0)$$

We define the magnetic translation operators as

$$\tau_{z_0} \coloneqq e^{i\varphi_{z_0}}T_{z_0} \tag{II.2.1}$$

They define the conditions (II.1.4) on $\partial\Omega$. Let $k \geqslant 1$, we define the magnetic periodic Sobolev spaces as

$$H^k_{mp}(\Omega) \coloneqq \left\{\psi \in H^k(\Omega) \text{ such that (II.1.4) hold}\right\}$$

We will use similar notation for other usual functional spaces where the subscript mp stands for magnetic periodic and p for periodic. The domain of the magnetic momentum

$$\mathscr{P}_{\hbar,b} \coloneqq i\hbar\nabla + bA$$

is

$$\mathrm{Dom}(\mathscr{P}_{\hbar,b}) \coloneqq H^1_{mp}(\Omega)$$

On Coulomb gauge, there exists $\phi \in C^\infty(\mathbb{R}^2, \mathbb{R})$ such that the vector potential satisfies

$$A = \nabla^\perp \phi \coloneqq \begin{pmatrix} -\partial_y \phi \\ \partial_x \phi \end{pmatrix} \tag{II.2.2}$$

For $k > 1, H^k(\Omega) \hookrightarrow C^0(\Omega)$, so the conditions (II.1.4) are well defined. For $k = 1$ they are defined with the trace operator T and $\psi_{|\Omega} \coloneqq T\psi$.

For some examples of Coulomb gauges, one can take the symmetric gauge:

$$\phi_{sym} \coloneqq \frac{|z|^2}{4} \qquad A_{Sym} \coloneqq \frac{1}{2}(x,y)^\perp \coloneqq \frac{1}{2}(-y,x) \qquad \varphi_{z_0,sym} \coloneqq \frac{x_0 y - y_0 x}{2l_b^2} \tag{II.2.3}$$

or the Landau gauge:

$$\phi_{Lan} \coloneqq \frac{y^2}{2} \qquad A_{Lan} \coloneqq (-y, 0) \qquad \varphi_{z_0,Lan} \coloneqq -\frac{y_0 x}{l_b^2} \tag{II.2.4}$$

If we insert the Landau gauge (II.2.4) in (II.1.4) we get the boundary conditions in Landau gauge:

$$\forall t \in [0, L], \begin{cases} \psi(L + it) = \psi(it) \\ \psi(t + iL) = e^{-i\frac{Lt}{l_b^2}} \psi(t) \end{cases} \tag{II.2.5}$$

In complex notation, the vector potential Definition (II.2.2) becomes

$$2i\partial_{\bar{z}}\phi = A \tag{II.2.6}$$

and with (II.1.2),

$$\Delta \phi = 1 \tag{II.2.7}$$

In the next proposition we also emphasize the importance of the flux quantization (II.1.9). The magnetic translations in the two directions L and iL defining the lattice commute if and only if the flux is quantized. Therefore when the flux is quantized, we are able to impose magnetic periodic boundary conditions in both directions L and iL.

Proposition II.2.2: *Commutation between magnetic Laplacian and magnetic translations*

We have the commutation relation

$$[\tau_L, \tau_{iL}] = 0 \iff \frac{L^2}{l_b^2} \in 2\pi\mathbb{Z}$$

and in this case,

$$\forall k \geqslant 1, H^k_{mp}(\Omega) = \left\{ \psi_{|\Omega}, \psi \in H^k_{loc}\left(\mathbb{R}^2\right) \text{ such that } \tau_L \psi = \tau_{iL}\psi = \psi \right\} \tag{II.2.8}$$

moreover, the magnetic Laplacian (II.1.1) commutes with the magnetic translations defined in (II.1.3) and (II.2.1):

$$[\mathscr{P}_{\hbar,b}, \tau_{z_0}] = 0 \text{ on } H^1_{mp}(\Omega) \text{ and } [\mathscr{L}_{\hbar,b}, \tau_{z_0}] = 0 \text{ on } H^2_{mp}(\Omega)$$

In view of (II.2.8) we will identify $\psi \in H^k_{mp}(\Omega)$ and its extension on $\mathbb{R}^2$ from now on.

Proof of Proposition II.2.2:

We compute

$$[\tau_L, \tau_{iL}] = e^{i\varphi_L} T_L e^{i\varphi_{iL}} T_{iL} - e^{i\varphi_{iL}} T_{iL} e^{i\varphi_L} T_L = \left(e^{i(\varphi_L + T_L\varphi_{iL})} - e^{i(\varphi_{iL}+T_{iL}\varphi_L)}\right) T_L T_{iL} \tag{II.2.9}$$

So

$$[\tau_L, \tau_{iL}] = 0 \iff \exists d \in \mathbb{Z} \text{ such that } \varphi_L + T_L\varphi_{iL} - \varphi_{iL} - T_{iL}\varphi_L = 2\pi d$$

and it is sufficient to prove

$$\varphi_L + T_L\varphi_{iL} - \varphi_{iL} - T_{iL}\varphi_L = \frac{L^2}{l_b^2} \tag{II.2.10}$$

With the Stokes theorem:

$$\int\limits_{\partial\Omega} A.dl = \int\limits_{\Omega} dS = L^2 \tag{II.2.11}$$

Using (II.1.3) we get another computation for this integral

$$\begin{aligned}
\int\limits_{\partial\Omega} A.dl &= \int\limits_0^L (A(u) - A(u+iL)) \cdot (1,0)du + i\int\limits_0^L (A(L+iu) - A(iu)) \cdot (0,1)du \\
&= l_b^2 \int\limits_0^L (-\partial_x\varphi_{iL}(u+iL) + i\partial_y\varphi_L(L+iu))\, du = l_b^2\left[-\varphi_{iL}(u+iL) + \varphi_L(L+iu)\right]_0^L \\
&= l_b^2\left(\varphi_L + T_L\varphi_{iL} - \varphi_{iL} - T_{iL}\varphi_L\right)(L+iL)
\end{aligned}$$

but because of (II.2.11) this quantity is constant. Dividing by l_b^2 gives (II.2.10).
The right to left inclusion in (II.2.8) is just the evaluation of $\tau_L\psi = \tau_{iL}\psi = \psi$ on $\partial\Omega$. The left to right inclusion is obtained by extending ψ with the magnetic translations τ_L, τ_{iL} and the extension is well defined due to the commutation relation $[\tau_L, \tau_{iL}] = 0$.
Next we compute

$$\mathscr{P}_{\hbar,b} e^{i\varphi_{z_0}} = e^{i\varphi_{z_0}}(i\hbar\nabla + bA - \hbar\nabla\varphi_{z_0})$$

so

$$e^{i\varphi_{z_0}} T_{z_0} \mathscr{P}_{\hbar,b} = (i\hbar\nabla + bA(\bullet - z_0) + \hbar\nabla\varphi_{z_0}) e^{i\varphi_{z_0}} T_{z_0}$$

With the definitions (II.1.3), (II.2.1) this ensure that $[\mathscr{P}_{\hbar,b}, \tau_{z_0}] = 0$.

To emphasize even more the importance of the flux quantisation, we state that

$$\left\{\psi_{|\Omega}, \psi \in H^k_{loc}\left(\mathbb{R}^2\right) \text{ such that } \tau_L \psi = \tau_{iL}\psi = \psi\right\}$$

is trivial without it.

Property II.2.3

$\left\{\psi_{|\Omega}, \psi \in H^k_{loc}\left(\mathbb{R}^2\right) \text{ such that } \tau_L \psi = \tau_{iL}\psi = \psi\right\}$ is dense in $L^2(\Omega)$ if and only if (II.1.9) holds. Otherwise

$$\left\{\psi_{|\Omega}, \psi \in H^k_{loc}\left(\mathbb{R}^2\right) \text{ such that } \tau_L \psi = \tau_{iL}\psi = \psi\right\} = \{0\}$$

Proof:

If (II.1.9) holds, by Proposition II.2.2,

$$C^\infty_c(\Omega) \subset H^k_{mp}(\Omega) = \left\{\psi_{|\Omega}, \psi \in H^k_{loc}\left(\mathbb{R}^2\right) \text{ such that } \tau_L \psi = \tau_{iL}\psi = \psi\right\}$$

hence the density in $L^2(\Omega)$.
If (II.1.9) does not hold, let $\psi \in H^k_{loc}\left(\mathbb{R}^2\right)$ such that $\tau_L \psi = \tau_{iL}\psi = \psi$, then

$$[\tau_L, \tau_{iL}]\psi = 0$$

but using (II.2.9) and (II.2.10)

$$\begin{aligned}
[\tau_L, \tau_{iL}]\psi =& e^{i(\varphi_{iL} + T_{iL}\varphi_L)}\left(e^{i(\varphi_L + T_L\varphi_{iL}) - \varphi_{iL} - T_{iL}\varphi_L} - 1\right) T_L T_{iL}\psi \\
=& e^{i(\varphi_{iL} + T_{iL}\varphi_L)}\left(e^{i\frac{L^2}{l_b^2}} - 1\right) T_L T_{iL}
\end{aligned}$$

and since the flux is not quantized,

$$\frac{L^2}{l_b^2} \notin 2\pi\mathbb{Z}$$

so $T_L T_{iL}\psi = 0$ and $\psi = 0$.

The next lemma proves that

$$\left\{u \in L^2(\Omega) \text{ such that } \mathcal{L}_{\hbar,b} u \in L^2(\Omega)\right\} = H^2(\Omega)$$

Indeed,

Lemma II.2.4

Let $\psi \in L^2(\Omega)$, assume $\mathscr{L}_{\hbar,b}\psi \in L^2(\Omega)$, then $\psi \in H^2(\Omega)$ and

$$\|\psi\|_{H^2} \leqslant C(b,\hbar)\,\|\mathscr{L}_{\hbar,b}\psi\|$$

Note that the converse embedding is also true because A and its gradient are bounded.

Proof of Lemma II.2.4:

$$\begin{aligned}\|\mathscr{L}_{\hbar,b}\psi\| &\geqslant \hbar^2 \|\Delta\psi\| - \left\|(b^2A^2 + i\hbar b\nabla\cdot A)\psi\right\| - \|2\hbar bA\cdot\nabla\psi\| \\ &\geqslant \hbar^2\|\Delta\psi\| - C(b)\|\psi\| - 2\hbar\|A\|_\infty\|\nabla\psi\|\end{aligned}$$

But the gradient term can be controlled by the Laplacian. Let $D > 0$, using that

$$\forall k \in \frac{1}{L}\mathbb{Z}^2, k^2 \leqslant C(b,\hbar,D) + Dk^4$$

with Parseval's theorem we obtain

$$\|\nabla\psi\|^2 \leqslant C(b,\hbar,D)\|\psi\|^2 + D\|\Delta\psi\|^2$$

Choose $D = \left(\frac{1}{2\hbar\|A\|_\infty}\cdot\frac{\hbar^2}{2}\right)^2$ so that

$$2\hbar\|A\|_\infty\|\nabla\psi\| \leqslant C(b,\hbar)\|\psi\| + 2\hbar\|A\|_\infty\sqrt{D}\|\Delta\psi\| = C(b,\hbar)\|\psi\| + \frac{\hbar^2}{2}\|\Delta\psi\| \tag{II.2.12}$$

and we can conclude that

$$\|\mathscr{L}_{\hbar,b}\psi\| \geqslant \frac{\hbar^2}{2}\|\Delta\psi\| - C(b,\hbar)\|\psi\| \tag{II.2.13}$$

Finally $\|\psi\|$ can also be controlled:

$$\|\mathscr{L}_{\hbar,b}\psi\|^2 = (\hbar b)^2\left\|\left(\mathcal{N} + \frac{\mathbb{1}}{2}\right)\psi\right\|^2 \geqslant \left(\frac{\hbar b}{2}\right)^2\|\psi\|^2 \tag{II.2.14}$$

since $\mathcal{N}$ is positive (II.2.19). Combining (II.2.12), (II.2.13), (II.2.14) gives the desired embedding.

II.2.2 Landau Level quantization

In this subsection, we set up the usual formalism for the description of the magnetic Laplacian in term of annihilation and creation operators. More details about these operators and the properties of Landau levels can be found in [13].

Notation II.2.5

We denote by π_x, π_y the coordinates of the magnetic momentum:

$$\mathscr{P}_{\hbar,b} =: \begin{pmatrix} i\hbar\partial_x + bA_x \\ i\hbar\partial_y + bA_y\end{pmatrix} =: \begin{pmatrix}\pi_x \\ \pi_y\end{pmatrix}$$

and define the annihilation and creation operators respectively as

$$a := \frac{\pi_y - i\pi_x}{\sqrt{2\hbar b}} \qquad a^\dagger := \frac{\pi_y + i\pi_x}{\sqrt{2\hbar b}} \tag{II.2.15}$$

and the number of excitation operator $\mathcal{N} \coloneqq a^\dagger a$.

The quantization of the magnetic Laplacian comes from the following commutation relations:

$$\begin{aligned} [\pi_x, \pi_y] &= i\hbar b && \text{(II.2.16)} \\ \left[a, a^\dagger\right] &= \mathrm{Id} \text{ (canonical commutation relation)} \\ [\tau_{z_0}, a] &= \left[\tau_{z_0}, a^\dagger\right] = 0 && \text{(II.2.17)} \end{aligned}$$

and the magnetic Laplacian is diagonal in terms of creation and annihilation operators:

$$\mathscr{L}_{\hbar,b} = 2\hbar b\left(\mathcal{N} + \frac{\mathrm{Id}}{2}\right) \tag{II.2.18}$$

In the next lemma we prove that the magnetic Laplacian $\mathscr{L}_{\hbar,b}$ defines a Sobolev space whose norm turns out to be equivalent to the $H^1(\Omega)$ norm.

Lemma II.2.6

$\mathscr{L}_{\hbar,b}$ defines the Sobolev space $(\mathrm{Dom}\left(\mathscr{L}_{\hbar,b}\right), \langle \bullet \rangle_{\mathscr{L}})$ with

$$\langle \chi | \psi \rangle_{\mathscr{L}} \coloneqq \langle \mathscr{L}_{\hbar,b}\chi | \psi \rangle$$

which is equivalent to $(H^2_{mp}(\Omega), \langle \bullet \rangle_{H^1})$. The quadratic form defined by $\langle \bullet \rangle_{\mathscr{L}}$ is continuous, Hermitian and coercive on $\mathrm{Dom}\left(\mathscr{L}_{\hbar,b}\right)$.

Proof:

First, we prove a Green formula for the magnetic momentum. Let $\chi \in H^1(\Omega), \vec{\psi} \in H^1(\Omega, \mathbb{C}^2)$, we use the Stokes theorem:

$$\int_{\partial\Omega} \chi \vec{\psi}^{\perp} \cdot \vec{n}(x) dx = \int_{\Omega} \nabla \cdot \left(\chi \vec{\psi}\right) = \int_{\Omega} \nabla \chi \cdot \vec{\psi} + \int_{\Omega} \chi \nabla \cdot \vec{\psi}$$

where $\vec{n}(x)$ is the outer normal vector of Ω when $x \in \partial\Omega$. So

$$\begin{aligned} \langle \chi | \mathscr{P}_{\hbar,b} \cdot \vec{\psi} \rangle &= \int_{\Omega} \overline{\chi(x)} \left(i\hbar\nabla + bA\right) \cdot \vec{\psi}(x) dx \\ &= \int_{\Omega} \vec{\psi}(x) \cdot \overline{\left(i\hbar\nabla + bA\right)\chi(x)} dx + i\hbar \int_{\partial\Omega} \overline{\chi(x)} \vec{\psi}^{\perp}(x) \cdot \vec{n}(x) dx \\ &= \overline{\langle \vec{\psi} | \mathscr{P}_{\hbar,b}\chi \rangle} + i\hbar \int_{\partial\Omega} \overline{\chi} \vec{\psi}^{\perp} \cdot \vec{n} \end{aligned}$$

Further assume $\chi, \vec{\psi}$ are magnetic periodic, then $\overline{\chi}\vec{\psi}$ is periodic so the boundary term vanishes. Thus $\mathscr{P}_{\hbar,b}$ is symmetric. The symmetry of $\mathscr{L}_{\hbar,b}$ follows from

$$\mathscr{P}_{\hbar,b} H^2_{mp}(\Omega) \subset H^1_{mp}(\Omega, \mathbb{C}^2)$$

Indeed, if $\psi \in H^2_{mp}(\Omega)$, $\mathscr{P}_{\hbar,b}\psi \in H^1(\Omega, \mathbb{C}^2)$ and $\mathscr{P}_{\hbar,b}\psi$ is magnetic periodic since the magnetic translations commute with $\mathscr{P}_{\hbar,b}$. We deduce that π_x and π_y are symmetric on $H^2_{mp}(\Omega)$, so a and $a^\dagger$ are adjoint of one another and

$$\langle \psi | \mathcal{N} \psi \rangle = \langle a\psi | a\psi \rangle \geqslant 0 \tag{II.2.19}$$

Let $\psi \in H^2_{mp}(\Omega)$, now we prove that the norm

$$\|\psi\|_{\mathscr{L}} := \sqrt{\langle \psi | \psi \rangle_{\mathscr{L}}} = \|\mathscr{P}_{\hbar,b}\psi\|_{L^2}$$

is equivalent to the H^1 norm. A and its gradient are bounded so $\left(H^2_{mp}(\Omega), \langle \bullet \rangle_{H^1}\right)$ is continuously embedded in $(H^2_{mp}(\Omega), \|\bullet\|_{\mathscr{L}})$. Moreover

$$\|\mathscr{P}_{\hbar,b}\psi\|_{L^2} \geqslant \hbar \|\nabla\psi\|_{L^2} - \|bA\psi\|_{L^2} \geqslant \hbar \|\nabla\psi\|_{L^2} - b \|A\|_{L^\infty} \|\psi\|_{L^2}$$

And $\|\psi\|_{L^2}$ can be controlled with (II.2.18) and (II.2.19)

$$\|\psi\|^2_{\mathscr{L}} = \langle \psi | \left(2\hbar b \mathcal{N} + \hbar b\right) \psi \rangle \geqslant \hbar b \|\psi\|^2_{L^2} \tag{II.2.20}$$

so

$$\|\hbar \nabla \psi\|_{L^2} \leqslant \|\psi\|_{\mathscr{L}} + \frac{b}{\sqrt{\hbar b}} \|A\|_{L^\infty} \|\psi\|_{\mathscr{L}}$$

Therefore we have the desired continuous embedding:

$$\|\psi\|_{H^1} \leqslant C(b, \hbar) \|\psi\|_{\mathscr{L}} \tag{II.2.21}$$

Finally to prove that $(\mathrm{Dom}\left(\mathscr{L}_{\hbar,b}\right), \langle \bullet \rangle_{\mathscr{L}})$ is a Hilbert space we need to prove that it is closed in $H^2(\Omega)$. Let $\psi_n \in \mathrm{Dom}\left(\mathscr{L}_{\hbar,b}\right)$ such that $\psi_n \to \psi$ in $H^2(\Omega)$, the limit also satisfies magnetic periodic boundary conditions because

$$\begin{aligned} \tau_{z_0}\psi_n = \psi_n \implies & \|\tau_{z_0}\psi - \psi\|_{L^2} \leqslant \|\tau_{z_0}\psi - \tau_{z_0}\psi_n\|_{L^2} + \|\psi_n - \psi\|_{L^2} = 2\|\psi_n - \psi\|_{L^2} \underset{n\to\infty}{\longrightarrow} 0 \\ \implies & \tau_{z_0}\psi = \psi \end{aligned}$$

Continuity and coercivity are trivial, if $\chi, \psi \in \mathrm{Dom}\left(\mathscr{L}_{\hbar,b}\right)$,

$$\begin{aligned} &|\langle \chi | \psi \rangle|_{\mathscr{L}} = |\langle \mathscr{P}_{\hbar,b}\chi | \mathscr{P}_{\hbar,b}\psi \rangle| \leqslant \|\mathscr{P}_{\hbar,b}\chi\|_{L^2} \|\mathscr{P}_{\hbar,b}\psi\|_{L^2} = \|\chi\|_{\mathscr{L}} \|\psi\|_{\mathscr{L}} \\ &\langle \psi | \psi \rangle_{\mathscr{L}} = \|\psi\|^2_{\mathscr{L}} \geqslant \|\psi\|^2_{\mathscr{L}} \end{aligned} \tag{II.2.22}$$

The proposition implies spectral properties of $\mathscr{L}_{\hbar,b}$.

Corollary II.2.7: *Spectral analysis of the magnetic Laplacian*

$\mathscr{L}_{\hbar,b}$ is a closed positive self-adjoint operator and the embedding $\mathrm{Dom}\left(\mathscr{L}_{\hbar,b}\right) \hookrightarrow L^2(\Omega)$ is continuous and compact.

Proof:

The positivity of $\mathscr{L}_{\hbar,b}$ follows from that of $\mathcal{N}$. Using the Lax-Milgram theorem, see results of [9, Section 2.5], and Lemma II.2.6 the operator $\mathcal{L}$ of domain

$$\mathrm{Dom}(\mathcal{L}) := \left\{u \in \mathrm{Dom}\left(\mathscr{L}_{\hbar,b}\right), \text{ such that } \forall v \in \mathrm{Dom}\left(\mathscr{L}_{\hbar,b}\right) \left|\langle u|v\rangle_{\mathscr{L}}\right| \leqslant C(u) \left\|v\right\|_{L^2}\right\}$$

defined by

$$\forall v \in \mathrm{Dom}(\mathcal{L}), u \in \mathrm{Dom}\left(\mathscr{L}_{\hbar,b}\right), \langle u|v\rangle_{\mathscr{L}} =: \langle \mathcal{L}u|v\rangle$$

is closed and self adjoint. But this operator is equal to $\left(\mathscr{L}_{\hbar,b}, \mathrm{Dom}\left(\mathscr{L}_{\hbar,b}\right)\right)$ because it coincides with $\mathscr{L}_{\hbar,b}$ on $\mathrm{Dom}\left(\mathscr{L}_{\hbar,b}\right)$ and the required inequality in the definition of $\mathrm{Dom}(\mathcal{L})$ is satisfied taking $C(u) := \left\|\mathcal{L}_{\hbar,bA}u\right\|_{L^2}$, thus $\mathrm{Dom}\left(\mathscr{L}_{\hbar,b}\right) = \mathrm{Dom}(L)$.
The continuity of $\mathrm{Dom}\left(\mathscr{L}_{\hbar,b}\right) \hookrightarrow L^2(\Omega)$ has been proved in (II.2.20). Then, we have the canonical embeddings

$$\left(\mathrm{Dom}\left(\mathscr{L}_{\hbar,b}\right) \hookrightarrow L^2(\Omega)\right) = \left(H^1(\Omega) \hookrightarrow L^2(\Omega)\right) \circ \left(\mathrm{Dom}\left(\mathscr{L}_{\hbar,b}\right) \hookrightarrow H^1(\Omega)\right)$$

The boundary of Ω is Lipschitz so the left embedding is compact due to the Rellich–Kondrachov theorem and the right one is continuous from Lemma II.2.6. Thus, the composition is compact.

$H^2_{mp}(\Omega)$ contains the smooth and compactly supported functions, so it is dense in $L^2(\Omega)$. We can conclude using the Lax-Milgram theorem [9, Corollary 4.26] that the resolvent of $\mathscr{L}_{\hbar,b}$ is well defined and compact. Applying the spectral theorem to the resolvent of $\mathscr{L}_{\hbar,b}$ proves that its spectrum is punctual and $L^2(\Omega)$ is a Hilbertian direct sum of the eigenspaces of $\mathscr{L}_{\hbar,b}$. The same conclusions also holds for the N-body Hamiltonian (II.1.6) since the magnetic Laplacian is of dominant order in it.

$\mathcal{N}$ inherits the properties of $\mathscr{L}_{\hbar,b}$ in Corollary II.2.7 and it is well known that

$$\mathrm{sp}(\mathcal{N}) = \mathbb{N}$$

Notation II.2.8: *Landau levels*

We define the n^{th} Landau level as the eigenspace associated to $n \in \mathbb{N}$:

$$\mathrm{nLL} := \left\{\psi \in \mathrm{Dom}\left(\mathscr{L}_{\hbar,b}\right) \text{ such that } \mathcal{N}\psi = n\psi\right\}$$

The ground level, denoted LLL for *Lowest Landau Level* has energy $E_0 = \hbar b$.

It is well known that the Landau levels are isomorphic, and that the operator $a^\dagger/\sqrt{n+1}$ is a unitary mapping from nLL to (n+1)LL of inverse $a/\sqrt{n+1}$. Using the creation operator, if we find a basis of the lowest Landau level we will be able to generate a basis of any Landau level. This is why, in the next session we start with a study of the lowest level.

II.2.3 Lowest Landau level

We start with the following characterisation:

Proposition II.2.9: *Lowest Landau level*

Denote by $\mathcal{O}(\Omega)$ the set of holomorphic functions, then

$$\mathrm{LLL} \subset \ker(a) \subset \mathcal{O}(\Omega) e^{-\frac{\phi}{l_b^2}}$$

where ϕ is defined in (II.2.2).

Proof:

Take $\psi \in \mathrm{LLL}$, then $a\psi = 0$, using (II.2.6)

$$(\pi_y - i\pi_x)\psi = (i\hbar\partial_y + \hbar\partial_x + bA_y - ibA_x)\psi = (2\hbar\partial_{\bar{z}} - ibA)\psi = 2(\hbar\partial_{\bar{z}} + b\partial_{\bar{z}}\phi)\psi = 0$$

So $\exists f \in \mathcal{O}(\Omega)$ such that $\psi = fe^{-\frac{\phi}{l_b^2}}$.

This proves that the zeros of a wave-function of LLL are given by the zeros of an holomorphic function. Since zeros of an holomorphic function must be isolated, the compactness of the domain implies that wave-functions have a finite number of zeros. Actually, the next proposition says that this number of zeros is d defined in (II.1.9), and therefore independent of the choice of wave-function. One can see [35, section 1] as a reference.

Proposition II.2.10: *Zeros of LLL wave-functions*

If $\psi \in \mathrm{LLL}$, then ψ has exactly d zeros inside Ω.

Proof:

Let

$$\psi := fe^{-\frac{\phi}{l_b^2}} \in \mathrm{LLL}$$

and n_0 be the number of zeros of ψ which can be computed with the logarithmic derivative through

$$n_0 = \frac{1}{2i\pi}\int_{\partial\Omega} \partial_z ln(f) dz = \frac{1}{2i\pi}\int_{\partial\Omega} \frac{\partial_z f}{f} dz \tag{II.2.23}$$

With Definitions (II.2.2) and (II.1.3)

$$A - T_{z_0}A = \begin{pmatrix} -\partial_y\phi + T_{z_0}\partial_y\phi \\ \partial_x\phi - T_{z_0}\partial_x\phi \end{pmatrix} = l_b^2 \nabla\varphi_{z_0} \tag{II.2.24}$$

If $\tau_{z_0}\psi = \psi$, the boundary condition on f is

$$f = e^{\frac{\phi}{l_b^2}}\psi = e^{\frac{\phi}{l_b^2} + i\varphi_{z_0}} T_{z_0}\psi = e^{\frac{\phi - T_{z_0}\phi}{l_b^2} + i\varphi_{z_0}} T_{z_0} f \tag{II.2.25}$$

Using equation (II.2.24), we get

$$l_b^2 \partial_z \varphi_{z_0} = l_b^2 \frac{\partial_x\varphi_{z_0} - i\partial_y\varphi_{z_0}}{2} = \frac{-i\partial_x - \partial_y}{2}\phi + T_{z_0}\frac{i\partial_x + \partial_y}{2}\phi = -i\partial_z\phi + iT_{z_0}\partial_z\phi \tag{II.2.26}$$

With equations (II.2.25) and (II.2.26), the boundary condition on $\partial_z f$ takes the form

$$l_b^2 \partial_z f = e^{\frac{\phi - T_{z_0}\phi}{l_b^2} + i\varphi_{z_0}} \left(T_{z_0} l_b^2 \partial_z f + 2\left(\partial_z \phi - T_{z_0} \partial_z \phi\right) T_{z_0} f \right)$$

As for $l_b^2 \frac{\partial_z f}{f}$, we have

$$l_b^2 \frac{\partial_z f}{f} = T_{z_0} l_b^2 \frac{\partial_z f}{f} + 2\partial_z(\phi - T_{z_0}\phi)$$

Finally, we can compute the integral in (II.2.23):

$$\begin{aligned}
2i\pi l_b^2 n_0 &= l_b^2 \int_0^L \left(\frac{\partial_z f}{f}(t) + i\frac{\partial_z f}{f}(L+it) - \frac{\partial_z f}{f}(t+iL) - i\frac{\partial_z f}{f}(it) \right) dt \\
&= -\int_0^L 2\partial_z(\phi - T_{iL}\phi)(t+iL)dt + i\int_0^L 2\partial_z(\phi - T_L\phi)(L+it)dt \\
&= \int_0^L \big(2\partial_z\phi(t) - 2\partial_z\phi(t+iL) + 2i\partial_z\phi(L+it) - 2i\partial_z\phi(it)\big)dt \\
&= \int_{\partial\Omega} 2\partial_z\phi dz = i\int_{\partial\Omega} \overline{A(z)}dz
\end{aligned} \tag{II.2.27}$$

where the last equality comes from Equation (II.2.6) which implies $2\partial_z\varphi = i\overline{A}$. But the integral of $\overline{A}$ over a complex loop can be related to the integral of A over a loop in $\mathbb{R}^2$. Let $\gamma \coloneqq \gamma_x + i\gamma_y : [0,1] \to \partial\Omega$ be a parametrization of $\partial\Omega$:

$$\begin{aligned}
\int_{\partial\Omega} \overline{A(z)}dz &= \int_0^1 \overline{A(\gamma(u))}\gamma'(u)du \\
&= \int_0^1 A(\gamma(u)) \cdot \gamma'(u)du + i\int_0^1 \big(A_x(\gamma(u))\gamma_y'(u) - A_y(\gamma(u))\gamma_x'(u)\big)\,du \\
&= \int_{\partial\Omega} A \cdot dl + i\int_0^1 \big(A_x(\gamma(u))\gamma_y'(u) - A_y(\gamma(u))\gamma_x'(u)\big)\,du = \int_{\partial\Omega} A \cdot dl + i\int_{\partial\Omega} A^{\perp} \cdot dl \\
&= \int_{\partial\Omega} A \cdot dl + i\int_{\Omega} \nabla \cdot A = \int_{\partial\Omega} A \cdot dl
\end{aligned}$$

Combining this with (II.2.27) and (II.2.11) we get

$$2\pi n_0 = \frac{l_b^2}{L^2}$$

so with (II.1.9), we conclude that $n_0 = d$.

An elliptic function can be expressed as a rational function in terms of the Weierstrass elliptic function and its derivative. In the case of magnetic periodic boundary conditions we will see that we have a decomposition in terms of theta functions from which we construct our basis of LLL. A similar proof of the following proposition can be found in [36] or [51, Chapter V Theorem 8].

Proposition II.2.11: *Degeneracy of Landau levels*

Landau levels have a finite degeneracy and

$$\forall n \in \mathbb{N}, \mathrm{Dim}(\mathrm{nLL}) = d$$

Proof:

Since all Landau levels are isomorphic, a proof for the lowest Landau level is sufficient. The Landau level dimension is independent of the gauge, for simplicity we use the Landau gauge in this proof.
In Landau gauge (II.2.4), the boundary condition on f in Equation (II.2.25) becomes

$$f(z) = e^{\frac{y^2-(y-y_0)^2}{2l_b^2} - i\frac{y_0 x}{l_b^2}} f(z - z_0) = e^{-\frac{y_0^2}{2l_b^2} - i\frac{y_0 z}{l_b^2}} f(z - z_0)$$

using equation (II.1.9), this can be rewritten as

$$\begin{aligned} f(z - L) &= f(z) \\ f(z - iL) &= f(z)e^{\frac{L^2}{2l_b^2} + i\frac{Lz}{l_b^2}} = f(z)e^{\pi d + 2i\pi d\frac{z}{L}} \end{aligned} \tag{II.2.28}$$

The periodicity along the real axis allows us to expand in Fourier series:

$$f(z) = \sum_{k\in\mathbb{Z}} c_k(y)e^{2i\pi k\frac{x}{L}} \tag{II.2.29}$$

The holomorphy of f implies that

$$2\partial_{\bar{z}} f = 0 = \sum_k \left(\frac{2i\pi k}{L}c_k(y) + ic_k'(y)\right) e^{2i\pi k\frac{x}{L}}$$

Solving the EDO in y that we obtain by identifying the Fourier coefficients gives

$$c_k(y) = c_k(0)e^{-2\pi k\frac{y}{L}}$$

Plugging this in (II.2.29) leads to

$$f(z) = \sum_k c_k(0)e^{2i\pi k\frac{z}{L}} \tag{II.2.30}$$

Finally we impose the pseudo-periodicity along the imaginary axis (II.2.28):

$$\sum_k c_k(0)e^{2i\pi k\frac{z}{L} + 2\pi k} = \sum_k c_k(0)e^{\pi d + 2i\pi\frac{z}{L}(k+d)}$$

Identifying the Fourier coefficients

$$c_k e^{2\pi k} = c_{k-d}e^{\pi d} \tag{II.2.31}$$

implies that they are only d independent Fourier coefficients.

We will prove in the next Section that the above relation between Fourier coefficients gives a decomposition of f in terms of theta functions.

II.2.4 Magnetic periodic eigenfunctions

This Section contains computations of the eigenfunctions of $\mathscr{L}_{\hbar,b}$ with magnetic periodic boundary conditions.

Notation II.2.12: *Theta functions*

Let τ be a complex parameter in the upper half plane, we define

$$\theta(z,\tau) \coloneqq \sum_{k\in\mathbb{Z}} e^{i\pi\tau k^2 + 2i\pi k z}$$

Theta functions are pseudo-periodic:

$$\begin{aligned}\theta(z+1,\tau) &= \theta(z,\tau)\\ \theta(z+\tau,\tau) &= \theta(z,\tau)e^{-i\pi(\tau+2z)}\end{aligned}$$

We complete the computation of Proposition II.2.11 and express the wave-functions of the magnetic Laplacian (II.1.1) in term of theta functions.

Proposition II.2.13: *LLL wave-functions*

The following family, indexed by $l \in [\![0, d-1]\!]$, is an orthonormal basis of the lowest Landau level in Landau gauge:

$$\psi_{0l}(z) \coloneqq \frac{\pi^{-\frac14}}{\sqrt{Ll_b}} e^{2i\pi l\frac{x}{L}} \sum_{k\in\mathbb{Z}} e^{2i\pi kd\frac{x}{L} - \frac{1}{2l_b^2}\left(y+kL+l\frac{L}{d}\right)^2} \tag{II.2.32}$$

$$= \frac{\pi^{-\frac14}}{\sqrt{Ll_b}} e^{-\frac{\pi l^2}{d} - \frac{y^2}{2l_b^2} + 2i\pi l\frac{z}{L}} \theta\left(d\frac{z}{L} + il, id\right) \tag{II.2.33}$$

Proof:

With the same notation as in the proof of Proposition II.2.11, we prove by induction that

$$c_{l+kd} = c_l e^{-2\pi kl - \pi dk^2} \tag{II.2.34}$$

This is satisfied for $k = 0$. Using (II.2.31) and assuming the relation (II.2.34) for $k \in \mathbb{N}$,

$$c_{l+(k+1)d} = c_{l+kd}e^{\pi d - 2\pi(l+[k+1]d)} = c_l e^{-2\pi kl - \pi dk^2 - \pi d - 2\pi(l+kd)} = c_l e^{-2\pi(k+1)l - \pi d(k+1)^2}$$

at this point we are done for $k \geqslant 0$, but $c_l = c_{l+kd}e^{2\pi kl + \pi dk^2}$ so

$$c_{l-kd} = c_l e^{2\pi k(l-kd)+\pi dk^2} = c_l e^{2\pi kl - \pi dk^2} = c_l e^{-2\pi(-k)l - \pi d(-k)^2}$$

and we obtain (II.2.34) for $k \leqslant 0$. Inserting (II.2.34) in (II.2.30) gives

$$f(z) = \sum_{l=0}^{d-1} e^{2i\pi l\frac{z}{L}} \sum_{k\in\mathbb{Z}} c_{l+kd} e^{2i\pi kd\frac{z}{L}} = \sum_{l=0}^{d-1} c_l e^{2i\pi l\frac{z}{L}} \sum_{k\in\mathbb{Z}} e^{-2\pi kl - \pi dk^2 + 2i\pi kd\frac{z}{L}}$$

$$= \sum_{l=0}^{d-1} c_l e^{2i\pi l \frac{z}{L}} \sum_{k\in\mathbb{Z}} e^{i\pi(id)k^2 + 2i\pi k\left(d\frac{z}{L}+il\right)} = \sum_{l=0}^{d-1} c_l e^{2i\pi l\frac{z}{L}} \theta\left(d\frac{z}{L} + il, id\right)$$

We found a family of the lowest Landau level indexed by $l \in [\![0, d-1]\!]$ with expression in Landau gauge. We need to normalise this family and to verify that the wave-functions are orthogonal. We start by proving that

$$e^{-\frac{y^2}{2l_b^2}+2i\pi l\frac{z}{L}} \theta\left(d\frac{z}{L} + il, id\right) = e^{\frac{\pi l^2}{d}+2i\pi l\frac{x}{L}} \sum_{k\in\mathbb{Z}} e^{2i\pi kd\frac{x}{L} - \frac{1}{2l_b^2}\left(y+kL+l\frac{L}{d}\right)^2}$$

Using (II.1.9),

$$\begin{aligned}
e^{-\frac{y^2}{2l_b^2}+2i\pi l\frac{z}{L}} \theta\left(d\frac{z}{L} + il, id\right) &= e^{-\frac{y^2}{2l_b^2}+2i\pi l\frac{z}{L}} \sum_{k\in\mathbb{Z}} e^{-\pi dk^2 - 2\pi kl + 2i\pi kd\frac{z}{L}} \\
&= e^{2i\pi l\frac{x}{L}} \sum_{k\in\mathbb{Z}} e^{-\frac{y^2}{2l_b^2} - 2\pi l\frac{y}{L} - \pi dk^2 - 2\pi kl - 2\pi kd\frac{y}{L} + 2i\pi kd\frac{x}{L}} \\
&= e^{2i\pi l\frac{x}{L}} \sum_{k\in\mathbb{Z}} e^{2i\pi kd\frac{x}{L} - \frac{1}{2l_b^2}\left(y^2 + 2ly\frac{L}{d} + L^2k^2 + 2kl\frac{L^2}{d} + 2kLy\right)} \\
&= e^{\frac{\pi l^2}{d}+2i\pi l\frac{x}{L}} \sum_{k\in\mathbb{Z}} e^{2i\pi kd\frac{x}{L} - \frac{1}{2l_b^2}\left(y+kL+l\frac{L}{d}\right)^2}
\end{aligned}$$

Finally we check the orthonormality. Let $0 \leqslant l \leqslant q < d$,

$$\begin{aligned}
\langle\psi_{0l}|\psi_{0q}\rangle &= \frac{1}{\sqrt{\pi}Ll_b} \sum_{k,p\in\mathbb{Z}} \int_\Omega e^{2i\pi(q-l)\frac{x}{L} + 2i\pi d(p-k)\frac{x}{L} - \frac{1}{2l_b^2}\left(y+kL+l\frac{L}{d}\right)^2 - \frac{1}{2l_b^2}\left(y+pL+q\frac{L}{d}\right)^2} dxdy \\
&= \frac{1}{\sqrt{\pi}Ll_b} \sum_{k,p\in\mathbb{Z}} \int_0^L e^{2i\pi(q-l+d[p-k])\frac{x}{L}} dx \int_0^L e^{-\frac{1}{2l_b^2}\left(y+kL+l\frac{L}{d}\right)^2 - \frac{1}{2l_b^2}\left(y+pL+q\frac{L}{d}\right)^2} dy
\end{aligned}$$

Since $0 \leqslant q - l < d$ we have a simplification:

$$\int_0^L e^{2i\pi(q-l+d[p-k])\frac{x}{L}} dx = L\delta_{lq}\delta_{kp}$$

Therefore

$$\langle\psi_{0l}|\psi_{0q}\rangle = \delta_{lq} \frac{1}{\sqrt{\pi}l_b} \sum_{k\in\mathbb{Z}} \int_0^L e^{-\frac{1}{l_b^2}\left(y+kL+l\frac{L}{d}\right)^2} dy = \delta_{lq}$$

One can check that the above wave-functions satisfy the boundary conditions (II.2.5). Using (II.2.32) we observe the L-periodicity along the real axis. Along the imaginary axis we increment the index k by 1:

$$\psi_{0l}(z+iL) = \frac{\pi^{-\frac{1}{4}}}{\sqrt{Ll_b}} e^{2i\pi l\frac{x}{L}} \sum_{k\in\mathbb{Z}} e^{2i\pi kd\frac{x}{L} - \frac{1}{2l_b^2}\left(y+(k+1)L+l\frac{L}{d}\right)^2} = e^{-2i\pi d\frac{x}{L}}\psi_{0l}(z)$$

and obtain the magnetic periodic boundary conditions in Landau gauge (II.2.4). The lowest Landau level is generated by successive magnetic translations:

Corollary II.2.14: *Generation of nLL with magnetic translations*

If $l \in [\![0, d-1]\!]$,

$$\psi_{0l} = \left(\tau_{-i\frac{L}{d}}\right)^l \psi_{00} = \tau_{-il\frac{L}{d}}\psi_{00} \tag{II.2.35}$$

Proof:

$\varphi_{,Lan} = -\dfrac{y_0 x}{l_b^2}$ defined in (II.2.4) is linear in z_0 and independent of y, thus with (II.2.1),

$$\tau_{-i\frac{L}{d}} = e^{i\frac{Lx}{dl_b^2}} T_{-i\frac{L}{d}} = e^{2i\pi\frac{x}{L}} T_{-i\frac{L}{d}}$$

and

$$\tau_{-il\frac{L}{d}} = \left(\tau_{-i\frac{L}{d}}\right)^l = e^{2i\pi l\frac{x}{L}} \left(T_{-i\frac{L}{d}}\right)^l$$

With this, (II.2.32) can be written as (II.2.35).

In order to obtain a full basis of L^2, we only need to apply successively $a^\dagger$ to generate the Landau levels and $\tau_{-i\frac{L}{d}}$ to generate the wave-functions inside a Landau level. The successive applications of $a^\dagger$ bring out Hermite polynomials.

Notation II.2.15: *Hermite polynomials*

For $n \in \mathbb{N}$, we define the n^{th} Hermite polynomial by

$$H_n \coloneqq (-1)^n e^{x^2} \left(\frac{d}{dx}\right)^n e^{-x^2} \tag{II.2.36}$$

We recall some basis properties of Hermite polynomials that will be useful: $H_0 = 1$ and for all $n \in \mathbb{N}$ we have the relations

$$H_{n+1} = 2xH_n - H_n' \tag{II.2.37}$$
$$H_n(-x) = (-1)^n H_n(x) \tag{II.2.38}$$
$$H_n' = 2nH_{n-1} \tag{II.2.39}$$

Using this, we can give expressions for the full basis.

Proposition II.2.16: *nLL wave-functions*

The following family indexed by $(n, l) \in \mathbb{N} \times [\![0, d-1]\!]$ is a Hilbert basis of eigenfunctions of $\mathscr{L}_{\hbar,b}$ in Landau gauge:

$$\begin{aligned}\psi_{nl} &:= \frac{a^{\dagger n}}{\sqrt{n!}} \left(\tau_{-i\frac{L}{d}}\right)^l \psi_{00} \\ &= \frac{c_n}{\sqrt{Ll_b}} e^{2i\pi l \frac{x}{L}} \sum_{k\in\mathbb{Z}} H_n\left(\frac{1}{l_b}\left[y + kL + l\frac{L}{d}\right]\right) e^{2i\pi kd\frac{x}{L} - \frac{1}{2l_b^2}\left(y+kL+l\frac{L}{d}\right)^2}\end{aligned} \tag{II.2.40}$$

with the normalization factor

$$c_n := \frac{1}{\pi^{\frac{1}{4}}\sqrt{n!}} \left(\frac{-i}{\sqrt{2}}\right)^n$$

Proof:

Due to (II.2.17) the order of application of the magnetic translations and the creation operators does not matter. Also, due to Corollary II.2.14, it is enough to deal with $l = 0$. In order to lighten the computations we define the dimensionless variable

$$y_k = \frac{y + kL}{l_b}$$

We proceed by induction in n. The initialisation is given by $H_0 = 1$ and (II.2.32):

$$\psi_{00}(z) = \frac{c_0}{\sqrt{Ll_b}} \sum_{k\in\mathbb{Z}} H_0(y_k) e^{2i\pi kd\frac{x}{L} - \frac{y_k^2}{2}}$$

In complex notation $a^\dagger$ (II.2.15) becomes in Landau gauge

$$a^\dagger = \frac{-2\hbar\partial_z - iby}{\sqrt{2\hbar b}} = -\sqrt{2}l_b\partial_z - \frac{iy}{\sqrt{2}l_b} = \frac{-i}{\sqrt{2}}\left(-l_b(i\partial_x + \partial_y) + \frac{y}{l_b}\right)$$

so

$$\begin{aligned}\frac{a^{\dagger n+1}}{\sqrt{(n+1)!}}\psi_{00}(z) &= \frac{a^\dagger}{\sqrt{n+1}}\psi_{n,0} = \frac{c_{n+1}}{\sqrt{Ll_b}} \sum_{k\in\mathbb{Z}} \left(-l_b(i\partial_x + \partial_y) + \frac{y}{l_b}\right) H_n(y_k) e^{2i\pi kd\frac{x}{L} - \frac{y_k^2}{2}} \\ &= \frac{c_{n+1}}{\sqrt{Ll_b}} \sum_{k\in\mathbb{Z}} \left(\left[2\pi k\frac{dl_b}{L} + y_k + \frac{y}{l_b}\right] H_n(y_k) - H_n'(y_k)\right) e^{2i\pi kd\frac{x}{L} - \frac{y_k^2}{2}} \\ &= \frac{c_{n+1}}{\sqrt{Ll_b}} \sum_{k\in\mathbb{Z}} \left[2y_k H_n(y_k) - H_n'(y_k)\right] e^{2i\pi kd\frac{x}{L} - \frac{y_k^2}{2}} \\ &= \frac{c_{n+1}}{\sqrt{Ll_b}} \sum_{k\in\mathbb{Z}} H_{n+1}(y_k) e^{2i\pi kd\frac{x}{L} - \frac{y_k^2}{2}}\end{aligned}$$

where the last equality uses (II.2.37).

As expected with our boundary conditions the modulus of the wave-functions:

$$|\psi_{nl}| = \left| \frac{c_n}{\sqrt{Ll_b}} \sum_{k\in\mathbb{Z}} H_n\left(\frac{1}{l_b}\left[y + kL + l\frac{L}{d}\right]\right) e^{2i\pi kd\frac{x}{L} - \frac{1}{2l_b^2}\left(y+kL+l\frac{L}{d}\right)^2} \right| \tag{II.2.41}$$

is periodic on the lattice $L\mathbb{Z}^2$, but the periodicity along the real axis is even shorter. Indeed we see in (II.2.41) that $|\psi_{nl}|$ is L/d-periodic in x.

We can write another useful form of equation (II.2.40) using the Poisson summation formula. The advantage of the expression in Proposition II.2.18 is the fact that the index l is decoupled from the polynomials and the Gaussian factors which is not the case in (II.2.40). This will simplify the computation of the Landau level's projector when we will sum over l in (II.3.5).

Notation II.2.17: *Fourier transform*

We use the convention

$$\mathcal{F}g(\nu) \coloneqq \hat{g}(\nu) \coloneqq \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} g(x)e^{-i\nu x}dx$$

for which $\mathcal{F}$ is unitary on $L^2(\mathbb{R})$. And denote the Hermite function

$$h_n(x) \coloneqq H_n(x)e^{-\frac{x^2}{2}} \tag{II.2.42}$$

In this convention the Poisson summation formula is

$$\sum_{k\in\mathbb{Z}} g(k) = \sqrt{2\pi} \sum_{k\in\mathbb{Z}} \hat{g}\left(2\pi k\right) \tag{II.2.43}$$

h_n are the eigenfunctions of the one dimensional harmonic oscillator and of the Fourier transform:

$$\mathcal{F}h_n = (-i)^n h_n \tag{II.2.44}$$

with the following normalization

$$\|h_n\|_{L^2}^2 = \sqrt{\pi}2^n n! \tag{II.2.45}$$

With this we are ready for the next computation:

Proposition II.2.18: *Poisson summation of eigenfunctions*

$$\psi_{nl}(z) = \tilde{c}_n \frac{\sqrt{l_b}}{L^{\frac{3}{2}}} e^{-i\frac{xy}{l_b^2}} \sum_{k\in\mathbb{Z}} H_n\left(\frac{1}{l_b}\left[x + k\frac{L}{d}\right]\right) e^{-2i\pi k\left(\frac{y}{L}+\frac{l}{d}\right) - \frac{1}{2l_b^2}\left(x+k\frac{L}{d}\right)^2}$$

with the normalization factor

$$\tilde{c}_n \coloneqq \frac{\pi^{\frac{1}{4}}(-1)^n 2^{\frac{1-n}{2}}}{\sqrt{n!}}$$

Proof:

We start from (II.2.40) expressed in terms of h_n:

$$\psi_{nl}(z) = \frac{c_n}{\sqrt{Ll_b}} e^{2i\pi l \frac{x}{L}} \sum_{k\in\mathbb{Z}} h_n\left(\frac{1}{l_b}\left[y + kL + l\frac{L}{d}\right]\right) e^{2i\pi kd\frac{x}{L}}$$

Define

$$g(u) := h_n\left(\frac{1}{l_b}\left[y + uL + l\frac{L}{d}\right]\right) e^{2i\pi du\frac{x}{L}}$$

so we have

$$\psi_{nl}(z) = \frac{c_n}{\sqrt{Ll_b}} e^{2i\pi l\frac{x}{L}} \sum_{k\in\mathbb{Z}} g(k) \tag{II.2.46}$$

in order to apply the Poisson summation formula to g. To do so, we compute $\hat{g}$ with a change of variable and equations (II.1.9) and (II.2.44):

$$\begin{aligned}
\hat{g}(\nu) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} h_n\left(\frac{1}{l_b}\left[y + uL + l\frac{L}{d}\right]\right) e^{-iu\left(\nu - 2\pi d\frac{x}{L}\right)} du \\
&= \frac{l_b}{L\sqrt{2\pi}} e^{i\left(\frac{y}{L}+\frac{l}{d}\right)\left(\nu - 2\pi d\frac{x}{L}\right)} \int_{\mathbb{R}} h_n(u) e^{-i\frac{ul_b}{L}\left(\nu - 2\pi d\frac{x}{L}\right)} du \\
&= \frac{l_b}{L\sqrt{2\pi}} e^{i\left(\frac{y}{L}+\frac{l}{d}\right)\left(\nu - 2\pi d\frac{x}{L}\right)} \widehat{h_n}\left(\frac{l_b}{L}\nu - \frac{x}{l_b}\right) = \frac{(-i)^n l_b}{L} e^{i\left(\frac{y}{L}+\frac{l}{d}\right)\left(\nu - 2\pi d\frac{x}{L}\right)} h_n\left(\frac{l_b}{L}\nu - \frac{x}{l_b}\right)
\end{aligned}$$

so by using (II.1.9) again:

$$\begin{aligned}
\hat{g}(2\pi k) &= \frac{(-i)^n l_b}{L} e^{i\left(\frac{y}{L}+\frac{l}{d}\right)\left(2\pi k - 2\pi d\frac{x}{L}\right)} h_n\left(2\pi k\frac{l_b}{L} - \frac{x}{l_b}\right) \\
&= \frac{(-i)^n l_b}{L} e^{-i\frac{xy}{l_b^2} - 2i\pi l\frac{x}{L}} e^{2i\pi k\left(\frac{y}{L}+\frac{l}{d}\right)} h_n\left(\frac{1}{l_b}\left[k\frac{L}{d} - x\right]\right)
\end{aligned}$$

To conclude the computation we insert this after applying the Poisson summation formula (II.2.43) to (II.2.46):

$$\begin{aligned}
\psi_{nl}(z) &= \frac{c_n}{\sqrt{Ll_b}} e^{2i\pi l\frac{x}{L}} \sqrt{2\pi} \sum_{k\in\mathbb{Z}} \hat{g}\left(2\pi k\right) \\
&= \frac{c_n}{\sqrt{Ll_b}} \cdot \frac{\sqrt{2\pi}(-i)^n l_b}{L} e^{-i\frac{xy}{l_b^2}} \sum_{k\in\mathbb{Z}} H_n\left(\frac{1}{l_b}\left[k\frac{L}{d} - x\right]\right) e^{2i\pi k\left(\frac{y}{L}+\frac{l}{d}\right) - \frac{1}{2l_b^2}\left(k\frac{L}{d} - x\right)^2} \\
&= \widetilde{c}_n \frac{\sqrt{l_b}}{L^{\frac{3}{2}}} e^{-i\frac{xy}{l_b^2}} \sum_{k\in\mathbb{Z}} H_n\left(\frac{1}{l_b}\left[x + k\frac{L}{d}\right]\right) e^{-2i\pi k\left(\frac{y}{L}+\frac{l}{d}\right) - \frac{1}{2l_b^2}\left(x + k\frac{L}{d}\right)^2}
\end{aligned}$$

by changing the sum index k to $-k$, using the parity of Hermite polynomials and the relation

$$c_n\sqrt{2\pi}(-i)^n = \widetilde{c}_n$$

II.3 Projectors on Landau levels

From the construction of an $L^2(\Omega)$ basis adapted to Landau levels, we define the projectors on Landau levels in Notation II.3.1. Since the phase space is $\mathbb{N} \times \Omega$ we also want to localise the projectors in space. Then we prove some properties of the projector that will be needed for the semi-classical analysis. In Proposition II.3.4 we give an equivalent for the diagonal of the projector's integral kernel, and in Corollary II.3.6 an equivalent for its trace.

II.3.1 nLL projectors

Notation II.3.1: *Projectors*

The orthogonal projector on nLL is

$$\Pi_n \coloneqq \sum_{l=0}^{d-1} |\psi_{nl}\rangle \langle\psi_{nl}|$$

Let $g \in C^\infty(\mathbb{R}^2, \mathbb{R}_+)$ radial with support included in the ball $B(0, L/2)$ such that $\|g\|_{L^2} = 1$. Let $\lambda \geqslant 1$, define the localizer $g_\lambda \in C^\infty(\mathbb{T})$ defined by

$$g_\lambda(x) \coloneqq \begin{cases} \lambda g(\lambda x) & \text{if } x \in B\left(0, \frac{L}{2\lambda}\right) \\ 0 & \text{else} \end{cases}$$

Note that

$$\|g_\lambda\|_{L^2} = 1$$

Then define the localised projector

$$\Pi_{n,R} \coloneqq g_\lambda(\bullet - R)\Pi_n g_\lambda(\bullet - R) \tag{II.3.1}$$

We assume the following scaling for $\lambda \coloneqq (\lambda_N)_N$:

$$1 \ll \lambda \ll \frac{N^{-\frac{1}{2}}}{\hbar^2} \tag{II.3.2}$$

This localised projector was introduced by Lieb and Yngvason in [48] and [49] where it has been called coherent operator. We take the bounds (II.3.2) in order to have $g_\lambda^2 \overset{*}{\rightharpoonup} \delta$ so the projector is well localised and

$$\frac{\hbar^2}{l_b}\lambda = \hbar b \lambda l_b \ll 1$$

This is necessary because $\hbar b \lambda l_b$ is the order of some error terms coming from the kinetic energy (for example in Proposition II.6.7).

Property II.3.2

Π_n and $\Pi_{n,R}$ are positive and satisfy the following resolution of identity:

$$\sum_{n\in\mathbb{N}} \Pi_n = \mathrm{Id} \qquad \int_{\mathbb{N}\times\Omega} \Pi_X d\eta(X) = \mathrm{Id} \tag{II.3.3}$$

Proof:

Let $\psi \in L^2(\Omega)$

$$\langle \psi | \Pi_{n,R}\psi\rangle = \langle g(\bullet - R)\psi | \Pi_n g(\bullet - R)\psi\rangle \geqslant 0$$

because Π_n is a projector. The first resolution of the identity is a consequence of the completeness of the basis and implies the second one:

$$\int_{\mathbb{N}\times\Omega} \Pi_X d\eta(X) = \int_{\Omega} g_\lambda(\cdot - R)\left(\sum_{n\in\mathbb{N}} \Pi_n\right) g_\lambda(\cdot - R) dR = \int_{\Omega} g_\lambda(\cdot - R)^2 dR = \mathrm{Id}$$

II.3.2 Integral kernels of the projectors

The computations of Section II.2 lead to the following expression of the nLL-projector's kernel.

Proposition II.3.3: *Kernel of the nLL-projector*

With notation (II.2.42) and $x \coloneqq x_1 + ix_2$, $y = y_1 + iy_2$,

$$\Pi_n(x,y) = \frac{1}{\|h_n\|^2_{L^2} L l_b} e^{i\frac{y_1y_2 - x_1x_2}{l_b^2}} \sum_{k,q\in\mathbb{Z}} H_n\left(\frac{1}{l_b}\left[x_1 + k\frac{L}{d}\right]\right) H_n\left(\frac{1}{l_b}\left[y_1 + qL + k\frac{L}{d}\right]\right)$$
$$\cdot e^{2i\pi k\frac{y_2 - x_2}{L} + 2i\pi d q\frac{y_2}{L} - \frac{1}{2l_b^2}\left(x_1 + k\frac{L}{d}\right)^2 - \frac{1}{2l_b^2}\left(y_1 + qL + k\frac{L}{d}\right)^2} \tag{II.3.4}$$

Proof:

From Proposition II.2.18:

$$\Pi_n(x,y) = \tilde{c}_n^2 \frac{l_b}{L^3} e^{i\frac{y_1y_2 - x_1x_2}{l_b^2}} \sum_{l=0}^{d-1} \sum_{k,p\in\mathbb{Z}} H_n\left(\frac{1}{l_b}\left[x_1 + k\frac{L}{d}\right]\right) H_n\left(\frac{1}{l_b}\left[y_1 + p\frac{L}{d}\right]\right)$$
$$\cdot e^{2i\pi l\frac{p-k}{d} + 2i\pi\frac{py_2 - kx_2}{L} - \frac{1}{2l_b^2}\left(x_1 + k\frac{L}{d}\right)^2 - \frac{1}{2l_b^2}\left(y_1 + p\frac{L}{d}\right)^2}$$

Then, we use

$$\sum_{l=0}^{d-1} e^{2i\pi l\frac{p-k}{d}} = d\mathbb{1}_{p=k\,(\mathrm{mod}\,d)} \tag{II.3.5}$$

to conclude. The computation of the normalization factor can be performed with (II.2.45):

$$\widetilde{c}_n^2 \frac{l_b}{L^3} \cdot d = \frac{\widetilde{c}_n^2}{2\pi} \cdot \frac{1}{Ll_b} = \frac{1}{\sqrt{\pi} 2^n n!} \cdot \frac{1}{Ll_b} = \frac{1}{\|h_n\|_{L^2}^2 Ll_b}$$

The above simplification for the sum in l is the reason why we used the Poisson formula on wave-functions. The argument does not work on the expression in (II.2.40) since the Gaussian terms depend on l.

If we consider the same setup on the whole space $\mathbb{R}^2$ instead of Ω, the expression of the projector in Landau gauge becomes (see [30, Section 3.2]):

$$\Pi_0^\infty(x,y) \coloneqq \frac{1}{2\pi l_b^2} e^{-\frac{|x-y|}{2l_b^2} + i\frac{\mathrm{Im}[x\overline{y}]}{4l_b^2} + i\frac{y_1 y_2 - x_2 x_1}{2l_b^2}}$$

The next proposition states that the diagonal of the projector's kernel on Ω converges to that of the projector on the whole space. This is expected since the limit is equivalent to a scaling where the size of the domain goes to infinity. This result will be important to estimate the trace of $\Pi_{n,R}$.

Proposition II.3.4: *Convergence of the integral kernel*

The kernel (II.3.4) satisfies

$$\Pi_n(z,z) \underset{b\to\infty}{\sim} \frac{1}{2\pi l_b^2}$$

uniformly in z with the convergence rate

$$\left\| \Pi_n(z,z) - \frac{1}{2\pi l_b^2} \right\|_{L^\infty} \leqslant \frac{C(n)}{Ll_b} \tag{II.3.6}$$

Moreover with notation (II.2.42),

$$\left\| (\mathscr{P}_{\hbar,b}\Pi_n)(z,z) - \frac{b}{l_b} \cdot \frac{1}{2\pi \|h_n\|_{L^2}^2} \int_{\mathbb{R}} \begin{pmatrix} ih_n'(u) \\ uh_n(u) \end{pmatrix} h_n(u) e^{-u^2} du \right\|_{L^\infty} \leqslant C(n)b \tag{II.3.7}$$

The proof needs the following technical lemma.

Lemma II.3.5

Let $m \in \mathbb{N}, c > 0$, the following series are uniformly bounded in α, a, b:

$$\forall \alpha \in \mathbb{R}_+, a, b \in [-1,1], \alpha \sum_{q\in\mathbb{Z}^*} |a+b+\alpha q|^m e^{-c(a+\alpha q)^2} \leqslant C(c,m) \tag{II.3.8}$$

$$\forall \alpha \in [0,1], a \in \mathbb{R}, b \in [-1,1], \alpha \sum_{k\in\mathbb{Z}} |a+b+\alpha k|^m e^{-c(a+\alpha k)^2} \leqslant C(c,m) \tag{II.3.9}$$

Moreover, if P_n, Q_n are complex polynomials of degree n, the function

$$\Xi(z) \coloneqq \sum_{k,q\in\mathbb{Z}} P_n\left(\frac{1}{l_b}\left[x+k\frac{L}{d}\right]\right) Q_n\left(\frac{1}{l_b}\left[x+qL+k\frac{L}{d}\right]\right) e^{2i\pi qd\frac{y}{L}-\frac{1}{2l_b^2}\left(x+k\frac{L}{d}\right)^2-\frac{1}{2l_b^2}\left(x+qL+k\frac{L}{d}\right)^2}$$

is of order $\dfrac{1}{l_b}$ and can be uniformly approximated as

$$\left\|\Xi(z)-\frac{L}{2\pi l_b}\int_{\mathbb{R}} P_n(u)Q_n(u)e^{-u^2}du\right\|_{L^\infty} \leqslant C(n) \tag{II.3.10}$$

Proof of Proposition II.3.4:

We start from (II.3.4):

$$\Pi_n(z,z) = \frac{1}{\|h_n\|_{L^2}^2 Ll_b}\sum_{k,q\in\mathbb{Z}} h_n\left(\frac{1}{l_b}\left[x+k\frac{L}{d}\right]\right) h_n\left(\frac{1}{l_b}\left[x+qL+k\frac{L}{d}\right]\right) e^{2i\pi qd\frac{y}{L}}$$

We apply Lemma II.3.5 and thus compute

$$\frac{1}{\|h_n\|_{L^2}^2 Ll_b}\int_{\mathbb{R}} h_n\left(\frac{x}{l_b}+2\pi u\frac{l_b}{L}\right)^2 du = \frac{1}{Ll_b}\cdot\frac{L}{2\pi l_b} = \frac{1}{2\pi l_b^2}$$

and obtain (II.3.6). Starting again from (II.3.4) and using notation (II.3.11), we compute in Landau gauge

$$\begin{aligned}
&(\mathscr{P}_{\hbar,b}\Pi_n)(x,y)\\
&=\begin{pmatrix} i\hbar\partial_{x_1}-bx_2\\ i\hbar\partial_{x_2}\end{pmatrix}\frac{1}{\|h_n\|_{L^2}^2 Ll_b}e^{i\frac{y_1y_2-x_1x_2}{l_b^2}}\sum_{k,q\in\mathbb{Z}} h_n(k_{b,x_1})\,h_n(k_{b,y_1+qL})\cdot e^{2i\pi k\frac{y_2-x_2}{L}+2i\pi dq\frac{y_2}{L}}\\
&=\frac{1}{\|h_n\|_{L^2}^2 Ll_b}e^{i\frac{y_1y_2-x_1x_2}{l_b^2}}\sum_{k,q\in\mathbb{Z}}\frac{\hbar}{l_b}\begin{pmatrix} ih_n'(k_{b,x_1})\\ k_{b,x_1}h_n(k_{b,x_1})\end{pmatrix} h_n(k_{b,y_1+qL})\cdot e^{2i\pi k\frac{y_2-x_2}{L}+2i\pi dq\frac{y_2}{L}}
\end{aligned}$$

So

$$(\mathscr{P}_{\hbar,b}\Pi_n)(z,z) = \frac{b}{\|h_n\|_{L^2}^2 L}\sum_{k,q\in\mathbb{Z}}\begin{pmatrix} ih_n'(k_{b,x})\\ k_{b,x}h_n(k_{b,x})\end{pmatrix} h_n(k_{b,x+qL})\,e^{2i\pi dq\frac{y}{L}}$$

and with Lemma II.3.5,

$$\left\|(\mathscr{P}_{\hbar,b}\Pi_n)(z,z)-\frac{L}{2\pi l_b}\cdot\frac{b}{\|h_n\|_{L^2}^2 L}\int_{\mathbb{R}}\begin{pmatrix} ih_n'(u)\\ uh_n(u)\end{pmatrix} h_n(u)e^{-u^2}du\right\|_{L^\infty} \leqslant C(n)b$$

Finally, we compare the trace of $\Pi_{n,R}$ to the trace of the projector on the whole space.

Corollary II.3.6: *Approximation of the projector's trace*

$$\left|\mathrm{Tr}\left[\Pi_{n,R}\right]-\frac{1}{2\pi l_b^2}\right|\leqslant\frac{C(n)}{l_b}$$

Proof:

This is a direct consequence of Proposition II.3.4 after integrating on $z\in\Omega$:

$$\mathrm{Tr}\left[\Pi_{n,R}\right]=\int_\Omega\Pi_{n,R}(z,z)dz=\frac{1}{2\pi l_b^2}\int_\Omega g_\lambda(z-R)^2dz+\mathcal{O}\left(\frac{1}{l_b}\right)=\frac{1}{2\pi l_b^2}+\mathcal{O}\left(\frac{1}{l_b}\right)$$

We end this section with the proof of the technical Lemma.

Proof of Lemma II.3.5:

Let $\alpha\in\mathbb{R}_+,a,b\in[-1,1]$. If $q\geqslant2$ then $q\leqslant2(q-1)$ so

$$\forall u\in[q-1,q),\left|a+b+\alpha q\right|^m e^{-c(a+\alpha q)^2}\leqslant(2+2\alpha u)^m e^{-c(a+\alpha u)^2}$$

and

$$\alpha\sum_{q\geqslant2}\left|a+b+\alpha q\right|^m e^{-c(a+\alpha q)^2}\leqslant\int_1^\infty(2+2\alpha u)^m e^{-c(a+\alpha u)^2}\leqslant\int_{\mathbb{R}}(2+2u)^m e^{-c(a+u)^2}du\leqslant C(c,m)$$

the term for $q=1$ is

$$\alpha\left|a+b+\alpha\right|^m e^{-c(a+\alpha)^2}\leqslant C$$

for the negative q, we see that

$$\alpha\sum_{q\leqslant-1}\left|a+b+\alpha q\right|^m e^{-c(a+\alpha q)^2}=\alpha\sum_{q\geqslant1}\left|-a-b+\alpha q\right|^m e^{-c(-a+\alpha q)^2}\leqslant C(c,m)$$

because $-a,-b\in[-1,1]$.
For (II.3.9), let $\alpha\in[0,1],a\in\mathbb{R},b\in[-1,1]$. We see that the series is α-periodic in a so we can assume $0\leqslant a\leqslant1$ and use (II.3.8) and for $k=0$:

$$\alpha\left|a+b\right|^m e^{-ca^2}\leqslant2^m$$

Now we use this result to prove the approximation of Ξ. Due to the Gaussian factor, all terms for which $q\neq0$ have a fair chance to vanish when $l_b\to0$. Thus, we focus first on the term indexed by $q=0$. To simplify notation we introduce

$$u_{b,x}\coloneqq\frac{1}{l_b}\left(x+u\frac{L}{d}\right)=\frac{x}{l_b}+2\pi u\frac{l_b}{L}\tag{II.3.11}$$

$$\xi(u)\coloneqq P_n\left(u_{b,x}\right)Q_n\left(u_{b,x}\right)e^{-u_{b,x}{}^2}$$

$$\Xi_{|q\neq0}(z)\coloneqq\Xi(z)-\sum_{k\in\mathbb{Z}}\xi(k)$$

so

$$\sum_{k\in\mathbb{Z}}\xi(k)=\sum_{k\in\mathbb{Z}}P_n\left(\frac{1}{l_b}\left[x+k\frac{L}{d}\right]\right)Q_n\left(\frac{1}{l_b}\left[x+k\frac{L}{d}\right]\right)e^{-\frac{1}{l_b^2}\left(x+k\frac{L}{d}\right)^2}$$

is the term for $q=0$ and $\Xi_{|q\neq 0}(z)$ contains the other terms.
Note that Ξ is L/d-periodic in x so we can choose $x\in[0,L/d]$ and

$$\frac{x}{l_b}\leqslant 2\pi\frac{l_b}{L}\underset{N\to\infty}{\longrightarrow}0 \tag{II.3.12}$$

For $q=0$, if we replace the sum in k by the associated integral we obtain:

$$\int_{\mathbb{R}}\xi(u)du=\frac{L}{2\pi l_b}\int_{\mathbb{R}}P_n(u)Q_n(u)e^{-u^2}du$$

which is the approximation in (II.3.10). For the convergence of the Riemann sum, we compute the derivative of the integrand. There exists R_n a polynomial of degree $2n+1$ such that

$$\xi'(u)=2\pi\frac{l_b}{L}R_n\left(u_{b,x}\right)e^{-{u_{b,x}}^2}$$

Now, use the mean value theorem:

$$\left|\sum_{k\in\mathbb{Z}}\xi(k)-\int_{\mathbb{R}}\xi(u)du\right|\leqslant\sum_{k\in\mathbb{Z}}\int_k^{k+1}|\xi(k)-\xi(u)|\,du\leqslant 2\pi\frac{l_b}{L}\sum_{k\in\mathbb{Z}}\sup_{k\leqslant u\leqslant k+1}\left|R_n\left(u_{b,x}\right)\right|e^{-{u_{b,x}}^2} \tag{II.3.13}$$

To control this we only need to control monomials. If $k\leqslant u\leqslant k+1$,

$$\begin{aligned}&\left|u_{b,x}\right|^m e^{-{u_{b,x}}^2}\\ &\leqslant\left|k_{b,x}\right|^m e^{-{k_{b,x}}^2}+\left|(k+1)_{b,x}\right|^m e^{-{(k+1)_{b,x}}^2}+\left|k_{b,x}\right|^m e^{-{(k+1)_{b,x}}^2}+\left|(k+1)_{b,x}\right|^m e^{-{k_{b,x}}^2}\\ &=\left|k_{b,x}\right|^m e^{-{k_{b,x}}^2}+\left|(k+1)_{b,x}\right|^m e^{-{(k+1)_{b,x}}^2}+\left|(k+1)_{b,x-\frac{L}{d}}\right|^m e^{-{(k+1)_{b,x}}^2}+\left|k_{b,x+\frac{L}{d}}\right|^m e^{-{k_{b,x}}^2}\end{aligned}$$

Thus after some change of indices,

$$2\pi\frac{l_b}{L}\sum_{k\in\mathbb{Z}}\sup_{k\leqslant u\leqslant k+1}\left|u_{b,x}\right|^m e^{-{u_{b,x}}^2}\leqslant 2\pi\frac{l_b}{L}\sum_{k\in\mathbb{Z}}\left(2\left|k_{b,x}\right|^m+\left|k_{b,x-\frac{L}{d}}\right|^m+\left|k_{b,x+\frac{L}{d}}\right|^m\right)e^{-{k_{b,x}}^2}$$

Using (II.3.9) with $\alpha=2\pi\frac{l_b}{L}\to 0$, $a=\frac{x}{l_b}$, $b\in\left\{0,2\pi\frac{l_b}{L},-2\pi\frac{l_b}{L}\right\}$, $c=1$:

$$\left|\sum_{k\in\mathbb{Z}}\xi(k)-\int_{\mathbb{R}}\xi(u)du\right|\leqslant C(n)$$

We next control $\Xi_{|q\neq 0}$. Let $\epsilon>0$, with Young's inequality:

$$-\left(\frac{x}{l_b}+2\pi k\frac{l_b}{L}\right)\cdot q\frac{L}{l_b}\leqslant\frac{\epsilon}{2}\left(\frac{x}{l_b}+2\pi k\frac{l_b}{L}\right)^2+\frac{1}{2\epsilon}\left(q\frac{L}{l_b}\right)^2$$

so

$$e^{-\frac{1}{2}\left(\frac{x}{l_b}+2\pi k\frac{l_b}{L}\right)^2-\frac{1}{2}\left(\frac{x}{l_b}+2\pi k\frac{l_b}{L}+q\frac{L}{l_b}\right)^2} \leqslant e^{-\left(1-\frac{\epsilon}{2}\right)\left(\frac{x}{l_b}+2\pi k\frac{l_b}{L}\right)^2-\left(\frac{1}{2}-\frac{1}{2\epsilon}\right)\left(q\frac{L}{l_b}\right)^2}$$

We take $\epsilon = 3/2$. As in (II.3.13), we need to deal with monomial terms of the form

$$\sum_{q\neq 0,k}\left|\frac{x}{l_b}+2\pi k\frac{l_b}{L}\right|^m\left|q\frac{L}{l_b}\right|^{\widetilde{m}} e^{-\frac{1}{4}\left(\frac{x}{l_b}+2\pi k\frac{l_b}{L}\right)^2-\frac{1}{6}\left(q\frac{L}{l_b}\right)^2}$$

by using

- (II.3.8) for the sum in q with $\alpha = \dfrac{L}{l_b}, a = 0, b = 0, c = \dfrac{1}{6}$
- (II.3.9) for the sum in k with $\alpha = 2\pi\dfrac{l_b}{L} \to 0, a = \dfrac{x}{l_b} \to 0, b = 0, c = \dfrac{1}{4}$

We conclude that

$$\left|\Xi_{|q\neq 0}\right| \leqslant C(n)\frac{L}{l_b}\cdot\frac{l_b}{L} = C(n)$$

II.4 A Lieb-Thirring inequality

In this section we prove a Lieb-Thirring inequality for the magnetic Laplacian with magnetic periodic boundary conditions:

Theorem II.4.1: *Kinetic energy inequality*

Let $\gamma \in \mathcal{L}^1(L^2(\Omega))$ a positive operator, then

$$\int_\Omega \rho_\gamma^2 \leqslant \frac{C \|\gamma\|_{\mathcal{L}^\infty}}{\hbar^2} \mathrm{Tr}\left[\mathscr{L}_{\hbar,b}\gamma\right] \tag{II.4.1}$$

Moreover if $\psi_N \in L^2_-(\Omega^N)$ with $\|\psi_N\|_{L^2} = 1$,

$$\left\|\rho_{\psi_N}^{(1)}\right\|_{L^2}^2 \leqslant \frac{C}{\hbar b}\mathrm{Tr}\left[\mathscr{L}_{\hbar,b}(x_i)\gamma_{\psi_N}^{(1)}\right] \text{ and } \left|\int_\Omega V\rho_{\psi_N}^{(1)}\right| \leqslant \frac{C}{\hbar b}\mathrm{Tr}\left[\mathscr{L}_{\hbar,b}(x_i)\gamma_{\psi_N}^{(1)}\right]\|V\|_{L^2} \tag{II.4.2}$$

$$\int_{\Omega^2} w\rho_{\psi_N}^{(2)} \leqslant \frac{C}{\hbar b}\mathrm{Tr}\left[\mathscr{L}_{\hbar,b}(x_i)\gamma_{\psi_N}^{(1)}\right]\|w\|_{L^2} \tag{II.4.3}$$

We follow the proof of [26, Chapter 4]. To achieve this goal we prove the following sequence of inequalities: a Kato inequality (Lemma II.4.2), a diamagnetic inequality for Green functions (Proposition II.4.5), a Lieb-Thirring inequality (Theorem II.4.6) from which we deduce the inequality on the kinetic energy (Theorem II.4.1).

II.4.1 Reduced densities

We give some usual properties of the reduced density matrices, see Notation II.1.6. Let γ_N be an N-body density matrix, since the Hamiltonian only contains one-body and two-body terms, the quantum N-body energy in the state γ_N only depends on the two first reduced densities:

$$\frac{\mathrm{Tr}\left[\mathscr{H}_N\gamma_N\right]}{N} = \mathrm{Tr}\left[\left(\mathscr{L}_{\hbar,b} + V\right)\gamma_N^{(1)}\right] + \mathrm{Tr}\left[w\gamma_N^{(2)}\right] \tag{II.4.4}$$

moreover, reduced densities inherit trace and Pauli principle from γ_N:

$$\mathrm{Tr}\left[\gamma_N^{(k)}\right] = 1, \quad 0 \leqslant \gamma_N^{(k)} \leqslant \frac{k!\,(N-k)!}{N!} \tag{II.4.5}$$

We can also express the reduced density matrices in term of integral kernels:

$$\gamma_N^{(k)}(x_{1:k}, y_{1:k}) \coloneqq \int_{\Omega^{N-k}} \gamma_N(x_{1:k}, x_{k+1:N}; y_{1:k}, x_{k+1:N})dx_{k+1:N} \tag{II.4.6}$$

The reduced density matrices are symmetric under permutation of coordinates:

$$\forall \sigma \in S_k, \gamma_N^{(k)}\left(x_{\sigma(1:k)}, y_{\sigma(1:k)}\right) = \gamma_N^{(k)}\left(x_{1:k}, y_{1:k}\right)$$

and consistent:

$$\forall q \in [\![1:k]\!], \gamma_N^{(q)}(x_{1:q}, y_{1:q}) = \int_{\Omega^{k-q}} \gamma_N(x_{1:q}, x_{q+1:k}; y_{1:q}, x_{q+1:k}) dx_{q+1:k}$$

Note that the reduced densities $\rho_{\gamma_N}^{(k)}$ inherit the symmetry and the consistency from the reduced density matrices.

II.4.2 A Kato inequality with periodic boundary conditions

One can look at [53, Theorem X.27] for a proof of the Kato inequality in the non magnetic case.

Lemma II.4.2: *Periodic Kato inequality*

Define the complex sign

$$s(u) := \begin{cases} \dfrac{\overline{u}}{|u|} & \text{if} \quad u \neq 0 \\ 0 & \text{if} \quad u = 0 \end{cases}$$

Let $u \in C^\infty(\Omega)$ then $|u| \in H^1(\Omega)$ and

$$|\hbar \nabla |u|| \leqslant |\mathscr{P}_{\hbar,b} u| \tag{II.4.7}$$

Moreover if $|u|$ is periodic, then

$$-\hbar^2 \Delta |u| \leqslant \mathrm{Re}\left[s(u) \mathscr{L}_{\hbar,b} u\right] \tag{II.4.8}$$

in the weak sense on $C_p^\infty(\Omega)$, or equivalently, $\forall \varphi \in C_p^\infty(\Omega, \mathbb{R}_+)$,

$$-\hbar^2 \int_\Omega |u| \Delta \varphi \leqslant \int_\Omega \mathrm{Re}\left[s(u) \mathscr{L}_{\hbar,b} u\right] \varphi$$

Proof:

$$\frac{1}{2} \hbar \nabla |u|^2 = \frac{1}{2} \hbar \nabla (\overline{u} u) = \mathrm{Re}\left[\overline{u} \hbar \nabla u\right] = \mathrm{Re}\left[\overline{u}\left(\hbar \nabla - ibA\right) u\right]$$

so taking absolute values

$$\left|\hbar \frac{\nabla |u|^2}{2}\right| \leqslant |u| \, |\mathscr{P}_{\hbar,b} u| \tag{II.4.9}$$

Define

$$u_\epsilon = \sqrt{|u|^2 + \epsilon^2} \in C_p^\infty(\Omega, \mathbb{R}_+^*)$$

Using (II.4.9),

$$|\hbar \nabla u_\epsilon| = \frac{\left|\hbar \nabla |u|^2\right|}{2 u \epsilon} \leqslant \frac{|u|}{u_\epsilon} |\mathscr{P}_{\hbar,b} u| \leqslant |\mathscr{P}_{\hbar,b} u| \tag{II.4.10}$$

So ∇u_ϵ is bounded in $L^2(\Omega, \mathbb{R}^2)$ and converges weakly to $v \in L^2(\Omega, \mathbb{R}^2)$ up to sequence of ϵ. Let $\varphi \in C_c^\infty(\mathrm{int}(\Omega), \mathbb{R}^2)$, since $\varphi \in L^2(\Omega, \mathbb{R}^2)$ and $0 \leqslant u_\epsilon - |u| \leqslant \epsilon$

$$\int_\Omega v \cdot \varphi = \lim \int_\Omega \nabla u_\epsilon \cdot \varphi = -\lim \int_\Omega u_\epsilon \nabla \cdot \varphi = -\int_\Omega |u| \nabla \cdot \varphi$$

so $v = \nabla |u|$ and the bound (II.4.10) passes to the limit and we obtain (II.4.7).
To prove (II.4.8) we use polar coordinates

$$u =: |u| e^{i\theta}$$

Let $x \in \Omega$, if $|u|(x) \neq 0$, $|u|$ is smooth on a neighbourhood V_x of x where $|u| > 0$ and thus

$$\nabla u_\epsilon = \frac{|u|}{u_\epsilon} \nabla |u| \to \nabla |u| \text{ pointwise on } V_x$$

$e^{i\theta} = u/|u|$ is also smooth and up to another restriction of V_x we can invert the complex exponential so θ is smooth. Under these conditions, we can do a direct computation and use Cauchy-Schwarz:

$$\begin{aligned}
\mathrm{Re}\left[s(u)\mathscr{L}_{\hbar,b}u\right] =& \mathrm{Re}\left[e^{-i\theta}\left(-\hbar^2\Delta + 2i\hbar bA\cdot\nabla + i\hbar b(\nabla.A) + |bA|^2\right)|u| e^{i\theta}\right] \\
=& -\hbar^2\Delta|u| + \mathrm{Re}\left[-|u| e^{-i\theta}\hbar^2\Delta e^{i\theta} - 2i\hbar^2\nabla|u|\cdot\nabla\theta + 2i\hbar bA\cdot\nabla|u|\right] \\
& - 2\hbar b|u| A\cdot\nabla\theta + |u| |bA|^2 \\
=& -\hbar^2\Delta|u| + \mathrm{Re}\left[-i\hbar^2|u|\Delta\theta + |u|\hbar^2|\nabla\theta|^2\right] - 2\hbar b|u| A\cdot\nabla\theta + |u| |bA|^2 \\
=& -\hbar^2\Delta|u| + \hbar^2|u| |\nabla\theta|^2 - 2\hbar b|u| A\cdot\nabla\theta + |u| |bA|^2 \geqslant -\hbar^2\Delta|u|
\end{aligned}$$

Note that if $u(x) = 0$ then x is a local minimum of u_ϵ so

$$\Delta u_\epsilon(x) \geqslant 0$$

Let $\varphi \in C_p^\infty(\Omega, \mathbb{R}_+)$, since u_ϵ and φ are periodic, the boundary terms vanish in the following integration by parts:

$$\int_\Omega u_\epsilon \Delta\varphi = \int_\Omega \varphi\Delta u_\epsilon \geqslant \int_{\Omega\backslash u^{-1}(\{0\})} \varphi\Delta u_\epsilon \tag{II.4.11}$$

Now we take $\epsilon \to 0$, u_ϵ converges uniformly to $|u|$ so

$$\int_\Omega u_\epsilon\Delta\varphi \underset{\epsilon\to 0}{\to} \int_\Omega |u| \Delta\varphi \tag{II.4.12}$$

Using $|u| \leqslant u_\epsilon$, when $u(x) \neq 0$,

$$\Delta u_\epsilon = \nabla\cdot\frac{|u|\nabla|u|}{u_\epsilon} = \frac{|\nabla|u||^2 + |u|\Delta|u|}{u_\epsilon} - \frac{|u|^2|\nabla|u||^2}{u_\epsilon^3} \geqslant \frac{|u|}{u_\epsilon}\Delta|u| \tag{II.4.13}$$

so (II.4.11) implies that

$$\int_{\Omega} u_\epsilon \Delta\varphi \geqslant \int_{\Omega\backslash u^{-1}(\{0\})} \varphi \frac{|u|}{u_\epsilon} \Delta |u| \tag{II.4.14}$$

With dominated convergence using inequality (II.4.13),

$$\int_{\Omega\backslash u^{-1}(\{0\})} \varphi \frac{|u|}{u_\epsilon} \Delta |u| \underset{\epsilon\to 0}{\to} \int_{\Omega\backslash u^{-1}(\{0\})} \varphi\Delta |u| \tag{II.4.15}$$

With (II.4.14), (II.4.12) and (II.4.15) we have

$$\int_{\Omega} |u| \, \Delta\varphi \geqslant \int_{\Omega\backslash u^{-1}(\{0\})} \varphi\Delta |u|$$

we can conclude that

$$-\hbar^2 \int_{\Omega} |u| \, \Delta\varphi \leqslant -\hbar^2 \int_{\Omega\backslash u^{-1}(\{0\})} \varphi\Delta |u| \leqslant \int_{\Omega\backslash u^{-1}(\{0\})} \operatorname{Re}\left[s(u)\mathscr{L}_{\hbar,b} u\right] \varphi = \int_{\Omega} \operatorname{Re}\left[s(u)\mathscr{L}_{\hbar,b} u\right] \varphi$$

II.4.3 Diamagnetic inequality

The main lemma for the Lieb-Thirring inequality in the magnetic case is the diamagnetic inequality in term of Green functions because it allows to restrict ourselves to the non magnetic case.

Notation II.4.3: *Green functions*

The resolvents of $-\hbar^2\Delta$ with periodic boundary conditions and $\mathscr{L}_{\hbar,b}$ are well defined for $\lambda > 0$:

$$G_{bA,\lambda} \coloneqq (\mathscr{L}_{\hbar,b} + \lambda)^{-1} \quad G_\lambda \coloneqq (-\hbar^2\Delta + \lambda)^{-1}$$

Their integral kernels define the corresponding Green functions.

They have the following properties:

Property II.4.4

Let $x \in \Omega$, then $G_{bA,\lambda}(x, \bullet), G_\lambda(x, \bullet) \in L^2(\Omega)$ and

$$G_\lambda(x, y) = G_\lambda(x-y) = G_\lambda(y-x) = \frac{1}{L^2} \sum_{k\in\frac{2\pi\hbar}{L}\mathbb{Z}^2} \frac{1}{k^2+\lambda} e^{ik\cdot(x-y)} \geqslant 0$$

Proof:

The periodic Laplacian is diagonalizable in the plane wave basis indexed by $k \in \frac{2\pi}{L}\mathbb{Z}^2$:

$$e_k(x) \coloneqq \frac{1}{L} e^{ik\cdot x}$$

Indeed

$$-\hbar^2\Delta + \lambda = \sum_{k\in\frac{2\pi}{L}\mathbb{Z}^2} \left(\hbar^2 k^2 + \lambda\right) |e_k\rangle \langle e_k|$$

so

$$G_\lambda(x,y) = \frac{1}{L^2} \sum_{k\in\frac{2\pi}{L}\mathbb{Z}^2} \frac{1}{\hbar^2 k^2 + \lambda} e^{ik\cdot(x-y)}$$

A change of index $k \coloneqq -k$ gives $G_\lambda(x,y) \in \mathbb{R}$. Let $f \in L^2(\Omega)$, since $G_\lambda f$ solves

$$(-\hbar^2\Delta + \lambda)u = f, u \in H^2_p(\Omega)$$

by the Lax-Milgram theorem, $G_\lambda f$ is the unique minimizer of the following functional

$$\mathcal{J}(u) \coloneqq \int_\Omega \left(\hbar^2 |\nabla u|^2 + \lambda |u|^2 - fu\right) dx$$

Assuming $f \geqslant 0$, we see that $\mathcal{J}(u) \geqslant \mathcal{J}(|u|)$ and conclude that $G_\lambda f \geqslant 0$. This implies $G_\lambda(x,y) \geqslant 0$. Finally,

$$-\hbar^2\Delta + \lambda \geqslant \lambda \text{ and } \mathscr{L}_{\hbar,b} + \lambda \geqslant \lambda$$

so

$$\|G_\lambda\|_{\mathcal{L}^\infty} \leqslant \frac{1}{\lambda} \text{ and } \|G_{bA,\lambda}\|_{\mathcal{L}^\infty} \leqslant \frac{1}{\lambda}$$

and $G_{bA,\lambda}(x,\bullet), G_\lambda(\bullet,y) \in L^2(\Omega)$.

Now we prove a diamagnetic inequality:

Proposition II.4.5: *Diamagnetic inequality for Green functions*

For all $x \in \Omega$ and for almost every $y \in \Omega$,

$$|G_{bA,\lambda}(x,y)| \leqslant G_\lambda(x,y)$$

Proof:

Let $\varphi \in C^{\infty}(\Omega)$, by definition

$$\int_{\Omega} G_{bA,\lambda}(x,\bullet)\left(\mathscr{L}_{\hbar,b}+\lambda\right)\varphi = G_{bA,\lambda}\left(\mathscr{L}_{\hbar,b}+\lambda\right)\varphi = \varphi$$

so, in the distributional sense

$$\left(\mathscr{L}_{\hbar,b}+\lambda\right)G_{bA,\lambda}(x,\bullet) = \delta_x \tag{II.4.16}$$

Let $\rho \in C^{\infty}(\mathbb{R}^2,\mathbb{R}_+)$ radial with support included in the ball $B(0,L/2)$ such that $\|\rho\|_{L^1} = 1$. Let $n \in \mathbb{N}^*$, define the localizer $\rho_n \in C^{\infty}(\mathbb{T})$ defined by

$$\rho_n(x) := \begin{cases} n^2\rho(nx) & \text{if } x \in B\left(0,\frac{L}{2n}\right) \\ 0 & \text{else} \end{cases}$$

Since ρ_n is periodic, the regularisation

$$u_x := G_{bA,\lambda}(x,\bullet) * \rho_n \in C^{\infty}_p(\Omega)$$

Thus, equation (II.4.16) becomes

$$\left(\mathscr{L}_{\hbar,b}+\lambda\right)u_x = \delta_x * \rho_n = \rho_n(x-\bullet) \tag{II.4.17}$$

We estimate

$$\operatorname{Re}\left[s(u_x)\left(\mathscr{L}_{\hbar,b}+\lambda\right)u_x\right] = \operatorname{Re}\left[s(u_x)\rho_n(x-\bullet)\right] \leqslant \rho_n(x-\bullet)$$

Then we apply Kato's inequality (II.4.7) to u_x, use $s(u_x)u_x = |u_x|$ and obtain

$$\left(-\hbar^2\Delta+\lambda\right)|u_x| \leqslant \operatorname{Re}\left[s(u_x)\mathscr{L}_{\hbar,b}u_x\right] + \lambda|u_x| \leqslant \rho_n(x-\bullet) \tag{II.4.18}$$

in a weak sense on $C^{\infty}_p(\Omega)^*$.
Similarly as (II.4.17),

$$(-\hbar^2\Delta+\lambda)\rho_n * G_{\lambda}(\bullet,y) = \rho_n(y-\bullet)$$

Thus testing inequality (II.4.18) on $\rho_n * G_{\lambda}(\bullet,y) \in C^{\infty}_p(\Omega,\mathbb{R}_+)$ we get

$$\int_{\Omega}|u_x(z)|\,\rho_n(y-z)dz \leqslant \int_{\Omega}\rho_n(x-z)\rho_n * G_{\lambda}(\bullet,y)(z)dz$$

With the changes of variables $t := t+x-y$, $z := z+x-y$ and Property II.4.4,

$$\begin{aligned}|G_{bA,\lambda}(x,\bullet) * \rho_n| * \rho_n(y) &\leqslant \iint_{\Omega^2}\rho_n(x-z)\rho_n(z-t)G_{\lambda}(t-y)dzdt \\ &= \iint_{\Omega^2}\rho_n(2x-y-z)\rho_n(z-t)G_{\lambda}(t-x)dzdt\end{aligned}$$

$$=\rho_n * \rho_n * G_\lambda(x,\bullet)(2x-y) \tag{II.4.19}$$

If $\varphi_n \to \varphi$ in $L^2(\Omega)$, by Young's inequality

$$\begin{aligned}\|\rho_n * \varphi_n - \varphi\|_{L^2} &\leqslant \|\rho_n * (\varphi_n - \varphi)\|_{L^2} + \|\rho_n * \varphi - \varphi\|_{L^2} \\ &\leqslant \|\rho_n\|_{L^1} \|\varphi_n - \varphi\|_{L^2} + \|\rho_n * \varphi - \varphi\|_{L^2} \underset{n\to\infty}{\to} 0\end{aligned}$$

Fix $x \in \Omega$, with Property II.4.4 up to a subsequence, $|\rho_n * G_{bA,\lambda}(x,\bullet)| \to |G_{bA,\lambda}(x,\bullet)|$ in $L^2(\Omega)$ so

$$|G_{bA,\lambda}(x,\bullet) * \rho_n| * \rho_n \underset{n\to\infty}{\to} |G_{bA,\lambda}(x,\bullet)|$$

in $L^2(\Omega)$ and up to another subsequence almost everywhere. Similarly, almost everywhere

$$\rho_n * \rho_n * G_\lambda(x,\bullet) \underset{n\to\infty}{\to} G_\lambda(x,\bullet)$$

So passing to the limit in (II.4.19), for almost every $y \in \Omega$,

$$|G_{bA,\lambda}(x,y)| \leqslant G_\lambda(x,2x-y) = G_\lambda(y-x) = G_\lambda(x,y)$$

II.4.4 Lieb-Thirring inequality

We would like to prove a kinetic inequality of the form

$$\mathrm{Tr}\left[\gamma \mathscr{L}_{\hbar,b}\right] \geqslant \frac{C}{\|\gamma\|_{\mathcal{L}^\infty}} \int_\Omega \rho_\gamma^2$$

with $\gamma \in \mathcal{L}^1(L^2(\Omega))$ a positive operator. We will deal with the magnetic field with the diamagnetic inequality and use Lieb-Thirring inequality for the Laplacian. But the previous inequality for the Laplacian is false if we take $\gamma := |e_0\rangle \langle e_0|$,

$$\mathrm{Tr}\left[-\Delta\gamma\right] = 0 < \frac{C}{L^2} = \frac{C}{\|\gamma\|_{\mathcal{L}^\infty}} \int_\Omega \rho_\gamma^2$$

To avoid this, we add 1 to the Laplacian so that the constant mode has a non-zero energy.

Theorem II.4.6: *Lieb-Thirring Inequality*

Let $\mathcal{V} \in L^2(\Omega, \mathbb{R}_+)$,

$$-\mathrm{Tr}\left[\mathbb{1}_{(\mathscr{L}_{\hbar,b}+1-\mathcal{V})\leqslant 0}\left(\mathscr{L}_{\hbar,b}+1-\mathcal{V}\right)\right] \leqslant \frac{C_{LT}}{\hbar^2}\int_\Omega \mathcal{V}(x)^2 dx \tag{II.4.20}$$

Proof:

We denote N_λ the number of eigenvalues of $\mathscr{L}_{\hbar,b}+1$ less than or equal to λ. From [26, Section 4.3],

$$-\mathrm{Tr}\left[\mathbb{1}_{(\mathscr{L}_{\hbar,b}+1-\mathcal{V})\leqslant 0}\left(\mathscr{L}_{\hbar,b}+1-\mathcal{V}\right)\right]=\int\limits_{\mathbb{R}_+}N_\lambda d\lambda$$

Define the Birman-Schwinger operator

$$K_\lambda \coloneqq \sqrt{\mathcal{V}}G_{bA,\lambda}\sqrt{\mathcal{V}}$$

and let B_λ be the number of eigenvalues of K_λ greater or equal to 1. We use the diamagnetic inequality to restrict to the non magnetic case. Since $G_{bA,\lambda}$ is positive, we can define its square root. Using the arguments of [26, Theorem 4.4] we can deduce from Proposition II.4.5 the diamagnetic inequality for $\sqrt{G_{bA,\lambda}}$:

$$\left|\sqrt{G_{bA,\lambda}}(x,y)\right|\leqslant\sqrt{G_\lambda}(x,y)$$

Hence with Proposition II.4.5,

$$\left|G^{\frac{3}{2}}_{bA,\lambda}(x,y)\right|=\left|\int\limits_\Omega G_{bA,\lambda}(x,z)\sqrt{G_{bA,\lambda}}(z,y)dz\right|\leqslant\int\limits_\Omega G_\lambda(x,z)\sqrt{G_\lambda}(z,y)dz=G^{\frac{3}{2}}_\lambda(x,y)$$

So taking $m\coloneqq 3/2$ and using an inequality on the traces of powers (see [26, Theorem 4.5]),

$$N_\lambda=B_\lambda\leqslant\mathrm{Tr}\left[K^m_\lambda\right]\leqslant\mathrm{Tr}\left[\mathcal{V}^{\frac{m}{2}}K^m_\lambda\mathcal{V}^{\frac{m}{2}}\right]\leqslant\int\limits_\Omega\mathcal{V}(x)^m\left|G^m_{A,\lambda+1}(x,x)\right|dx$$
$$\leqslant\int\limits_\Omega\mathcal{V}(x)^mG^m_{\lambda+1}(x,x)dx$$

So we obtain

$$-\mathrm{Tr}\left[\mathbb{1}_{(\mathscr{L}_{\hbar,b}+1-\mathcal{V})\leqslant 0}\left(\mathscr{L}_{\hbar,b}+1-\mathcal{V}\right)\right]\leqslant\int\limits_\Omega\int\limits_1^\infty\mathcal{V}(x)^mG^m_\lambda(x,x)dxd\lambda \tag{II.4.21}$$

The kernel of G^m_λ is

$$G^m_\lambda(x)=\frac{1}{L^2}\sum_{k\in\frac{2\pi\hbar}{L}\mathbb{Z}^2}\frac{1}{(k^2+\lambda)^m}e^{i\frac{k\cdot x}{\hbar}}$$

We use the integral bound for the sum

$$\sum_{k\in\mathbb{Z}}\frac{1}{(k^2+\lambda)^m}\leqslant\lambda^{-m}+\int\limits_{\mathbb{R}}\frac{1}{(u^2+\lambda)^m}du$$

so

$$G_\lambda^m(0) = \frac{1}{L^2}\sum_{k,q\in\mathbb{Z}} \frac{1}{\left(\left(\frac{2\pi\hbar}{L}\right)^2(k^2+q^2)+\lambda\right)^m} = \frac{1}{L^2}\left(\frac{L}{2\pi\hbar}\right)^{2m}\sum_{k,q\in\mathbb{Z}}\frac{1}{\left(k^2+q^2+\left(\frac{L}{2\pi\hbar}\right)^2\lambda\right)^m}$$
$$\leqslant \frac{1}{L^2}\left(\frac{L}{2\pi\hbar}\right)^{2m}\sum_{k\in\mathbb{Z}}\left(\frac{1}{\left(k^2+\left(\frac{L}{2\pi\hbar}\right)^2\lambda\right)^m}+\int_{\mathbb{R}}\frac{1}{\left(k^2+u^2+\left(\frac{L}{2\pi\hbar}\right)^2\lambda\right)^m}du\right)$$

We estimate the integral

$$\int_{\mathbb{R}}\frac{1}{\left(k^2+u^2+\left(\frac{L}{2\pi\hbar}\right)^2\lambda\right)^m}du = \left(k^2+\left(\frac{L}{2\pi\hbar}\right)^2\lambda\right)^{-m}\int_{\mathbb{R}}\frac{1}{\left(\frac{u^2}{k^2+\left(\frac{L}{2\pi\hbar}\right)^2\lambda}+1\right)^m}du$$
$$= \frac{I(m)}{\left(k^2+\left(\frac{L}{2\pi\hbar}\right)^2\lambda\right)^{m-\frac{1}{2}}}$$

with

$$m > \frac{1}{2} \implies I(m) \coloneqq \int_{\mathbb{R}}\frac{1}{(1+u^2)^m}du < \infty$$

Similarly

$$G_\lambda^m(0) \leqslant \frac{1}{L^2}\left(\frac{L}{2\pi\hbar}\right)^{2m}\sum_{k\in\mathbb{Z}}\left(\frac{1}{\left(k^2+\left(\frac{L}{2\pi\hbar}\right)^2\lambda\right)^m}+\frac{I(m)}{\left(k^2+\left(\frac{L}{2\pi\hbar}\right)^2\lambda\right)^{m-\frac{1}{2}}}\right)$$
$$\leqslant \frac{\lambda^{-m}}{L^2}+\frac{I(m)}{2\pi\hbar L}\lambda^{-m+\frac{1}{2}}$$
$$+\frac{1}{L^2}\left(\frac{L}{2\pi\hbar}\right)^{2m}\left(\int_{\mathbb{R}}\frac{1}{\left(u^2+\left(\frac{L}{2\pi\hbar}\right)^2\lambda\right)^m}du+\int_{\mathbb{R}}\frac{I(m)}{\left(u^2+\left(\frac{L}{2\pi\hbar}\right)^2\lambda\right)^{m-\frac{1}{2}}}du\right)$$
$$\leqslant \frac{\lambda^{-m}}{L^2}+\frac{I(m)}{\pi\hbar L}\lambda^{-m+\frac{1}{2}}+\frac{1}{(2\pi\hbar)^2}I(m)I\left(m-\frac{1}{2}\right)\lambda^{-m+1}$$
$$\leqslant \frac{C(m)}{\hbar^2}\lambda^{-m+1}$$

since $\lambda \geqslant 1$. We need $m > 1$ for the integrals to converge. We use the same trick as [26] changing the potential to

$$\mathcal{W}_\lambda(x) \coloneqq \max\left(\mathcal{V}-\frac{\lambda}{2},0\right)$$

Combining this with (II.4.21) and the change of variable

$$\mu \coloneqq \frac{2V(x)}{\lambda}, d\lambda = -\frac{2V(x)}{\mu^2} d\mu$$

we obtain

$$\begin{aligned}
-\mathrm{Tr}\left[\mathbb{1}_{(\mathscr{L}_{\hbar,b}+1-\mathcal{V})\leqslant 0}\left(\mathscr{L}_{\hbar,b}+1-\mathcal{V}\right)\right] &\leqslant \frac{C(m)}{\hbar^2} \int_1^\infty \int_\Omega \lambda^{-m+1} \max\left(V(x) - \frac{\lambda}{2}, 0\right)^m d\lambda dx \\
&\leqslant \frac{C(m)}{\hbar^2} \int_\Omega \left(\int_0^{2V(x)} \lambda \left(\frac{2V(x)}{\lambda} - 1 \right)^m d\lambda \right) dx \\
&= \frac{C(m)}{\hbar^2} \int_\Omega \left(\int_1^\infty \frac{2V(x)}{\mu} (\mu - 1)^m \cdot \frac{2V(x)}{\mu^2} d\mu \right) dx \\
&= \frac{C(m)}{\hbar^2} \int_\Omega V(x)^2 \left(\int_1^\infty \frac{(\mu-1)^m}{\mu^3} d\mu \right) dx
\end{aligned}$$

The integral in μ converges if $3 - m > 1$. To conclude we need $1 < m < 2$ hence the choice $m = 3/2$ is convenient.

This leads to proof of the Fundamental inequality of kinetic energy:

Proof of Theorem II.4.1:

With the variational principle and the Lieb-Thirring inequality (II.4.20),

$$\begin{aligned}
\mathrm{Tr}\left[\left(\mathscr{L}_{\hbar,b}+1\right)\gamma\right] =& \mathrm{Tr}\left[\left(\mathscr{L}_{\hbar,b}+1-\mathcal{V}\right)\gamma\right] + \mathrm{Tr}\left[\mathcal{V}\gamma\right] \\
&\geqslant \|\gamma\|_{\mathcal{L}^\infty} \mathrm{Tr}\left[\mathbb{1}_{(\mathscr{L}_{\hbar,b}+1-\mathcal{V})\leqslant 0}\left(\mathscr{L}_{\hbar,b}+1-\mathcal{V}\right)\right] + \mathrm{Tr}\left[\mathcal{V}\gamma\right] \\
&\geqslant -\frac{C_{LT}\|\gamma\|_{\mathcal{L}^\infty}}{\hbar^2} \int_\Omega \mathcal{V}^2 + \int_\Omega \mathcal{V}\rho_\gamma
\end{aligned}$$

Then choose $\mathcal{V} \coloneqq C_N \mathbb{1}_{\rho_\gamma \leqslant c}\rho_\gamma$:

$$\mathrm{Tr}\left[\left(\mathscr{L}_{\hbar,b}+1\right)\gamma\right] \geqslant C_N \left(1 - C_N \frac{C_{LT}\|\gamma\|_{\mathcal{L}^\infty}}{\hbar^2}\right) \int_{\rho_\gamma \leqslant c} {\rho_\gamma}^2$$

The constant preceding the integral is maximal when

$$C_N \coloneqq \frac{\hbar^2}{2C_{LT}\|\gamma\|_{\mathcal{L}^\infty}}$$

and we get

$$\mathrm{Tr}\left[\left(\mathscr{L}_{\hbar,b}+1\right)\gamma\right] \geqslant \frac{\hbar^2}{4C_{LT}\|\gamma\|_{\mathcal{L}^\infty}} \int_{\rho_\gamma \leqslant c} {\rho_\gamma}^2 \tag{II.4.22}$$

Since $\mathscr{L}_{\hbar,b} \geqslant \hbar b$,

$$\operatorname{Tr}\left[\mathscr{L}_{\hbar,b}\gamma\right] \geqslant \hbar b \operatorname{Tr}\left[\gamma\right]$$

so because $\hbar b \to \infty$,

$$\operatorname{Tr}\left[\left(\mathscr{L}_{\hbar,b}+1\right)\gamma\right] \leqslant \left(1+\frac{1}{\hbar b}\right)\operatorname{Tr}\left[\mathscr{L}_{\hbar,b}\gamma\right] \leqslant C\operatorname{Tr}\left[\mathscr{L}_{\hbar,b}\gamma\right]$$

With this and monotone convergence we take the limit $c \to \infty$ in inequality (II.4.22) and obtain (II.4.1). Applying this to (II.4.5), we have

$$\frac{1}{\hbar b}\operatorname{Tr}\left[\mathscr{L}_{\hbar,b}\gamma^{(1)}_{\psi_N}\right] \geqslant C\frac{l_b^2}{\left\|\gamma^{(1)}_{\psi_N}\right\|_{\mathcal{L}^\infty}}\left\|\rho^{(1)}_{\psi_N}\right\|^2_{L^2} \geqslant CNl_b^2\left\|\rho^{(1)}_{\psi_N}\right\|^2_{L^2} \geqslant C\left\|\rho^{(1)}_{\psi_N}\right\|^2_{L^2}$$

For the second reduced density, by symmetry

$$\begin{aligned}
&N\left(\operatorname{Tr}\left[\mathscr{L}_{\hbar,b}\gamma^{(1)}_{\psi_N}\right]-\operatorname{Tr}\left[w\gamma^{(2)}_{\psi_N}\right]\right) = \left\langle \psi_N \middle| \left(\sum_{i=1}^N \mathscr{L}_{\hbar,b}(x_i) - \frac{2}{N-1}\sum_{i<j} w(x_i-w_j)\right)\psi_N\right\rangle \\
&= \left\langle \psi_N \middle| \left(\sum_{i=1}^N \mathscr{L}_{\hbar,b}(x_i) - \frac{N}{N-1}\sum_{j=2}^N w(x_1-w_j)\right)\psi_N\right\rangle \\
&\geqslant \left\langle \psi_N \middle| \sum_{j=2}^N \left(\mathscr{L}_{\hbar,b}(x_i) - \frac{N}{N-1} w(x_1-x_j)\right)\psi_N\right\rangle \\
&= \int_\Omega \left\langle \psi_N(x,\bullet) \middle| \sum_{j=2}^N \left(\mathscr{L}_{\hbar,b}(x_i) - \frac{N}{N-1} w(x-x_j)\right)\psi_N(x,\bullet)\right\rangle dx \\
&= \int_\Omega \left(\left\langle \psi_N(x,\bullet) \middle| \sum_{j=2}^N \mathscr{L}_{\hbar,b}(x_i)\psi_N(x,\bullet)\right\rangle - N\int_\Omega w(x-y)\rho^{(1)}_{\psi_N(x,\bullet)}dy\right)dx
\end{aligned}$$

Then using (II.4.2) and then Young's inequality,

$$\begin{aligned}
&\operatorname{Tr}\left[\mathscr{L}_{\hbar,b}\gamma^{(1)}_{\psi_N}\right]-\operatorname{Tr}\left[w\gamma^{(2)}_{\psi_N}\right] \\
&\geqslant \frac{1}{N}\int_\Omega\left(C\hbar b(N-1)\left\|\rho^{(1)}_{\psi_N(x,\bullet)}\right\|^2_{L^2} - N\int_\Omega w(x-y)\rho^{(1)}_{\psi_N(x,\bullet)}dy\right)dx \\
&\geqslant \int_\Omega\left(C\hbar b\left\|\rho^{(1)}_{\psi_N(x,\bullet)}\right\|^2_{L^2} - \int_\Omega w(x-y)\rho^{(1)}_{\psi_N(x,\bullet)}dy\right)dx \\
&\geqslant \int_\Omega\left(C\hbar b\left\|\rho^{(1)}_{\psi_N(x,\bullet)}\right\|^2_{L^2} - \frac{1}{2}\left(\frac{1}{2C\hbar b}\left\|w\right\|^2_{L^2} + 2C\hbar b\left\|\rho^{(1)}_{\psi_N(x,\bullet)}\right\|^2_{L^2}\right)\right)dx \geqslant -\frac{C}{\hbar b}\left\|w\right\|^2_{L^2}
\end{aligned}$$

Changing w to ϵw, dividing by ϵ and using (II.4.4) gives

$$\int_{\Omega^2} w\rho^{(2)}_{\psi_N} \leqslant \frac{1}{\epsilon}\mathrm{Tr}\left[\mathscr{L}_{\hbar,b}\gamma^{(1)}_{\psi_N}\right] + \frac{C}{\hbar b}\epsilon\left\|w\right\|^2_{L^2}$$

To optimise in ϵ, we choose $\epsilon \coloneqq \dfrac{\hbar b}{\left\|w\right\|_{L^2}}$, we get

$$\int_{\Omega^2} w\rho^{(2)}_{\psi_N} \leqslant \left(\frac{1}{\hbar b}\mathrm{Tr}\left[\mathscr{L}_{\hbar,b}\gamma^{(1)}_{\psi_N}\right] + C\right)\left\|w\right\|_{L^2} \leqslant \frac{C}{\hbar b}\mathrm{Tr}\left[\mathscr{L}_{\hbar,b}\gamma^{(1)}_{\psi_N}\right]\left\|w\right\|_{L^2}$$

because $\mathscr{L}_{\hbar,b} \geqslant \hbar b$. Similarly with Young's inequality and (II.4.2),

$$\left|\int_{\Omega} V\rho^{(1)}_{\psi_N}\right| \leqslant \frac{C}{\hbar b}\mathrm{Tr}\left[\mathscr{L}_{\hbar,b}\gamma^{(1)}_{\psi_N}\right]\left\|V\right\|_{L^2}$$

II.5 Semi-classical limit

In this section we introduce the Husimi functions representing densities in the phase space. The fundamentals properties of these functions can be found in Property II.5.3. Then we prove that the N-body quantum energy can be approximated by a semi-classical functional depending only on Husimi functions in Proposition II.5.4.

II.5.1 Husimi functions

Notation II.5.1

Let $k \in \mathbb{N}^*, \gamma_k \in \mathcal{L}^1(L^2(\Omega^k))$, recalling Notation II.3.1 and (II.1.23) we define the associated Husimi functions or lower symbol as

$$m_{\gamma_k}(X_{1:k}) := \operatorname{Tr}\left[\gamma_k \bigotimes_{i=1}^{k} \Pi_{X_i}\right] \text{ with } X_{1:k} \in (\mathbb{N} \times \Omega)^k$$

Conversely, if $m_k \in L^1\left((\mathbb{N} \times \Omega)^k\right)$, define the associated density matrix

$$\gamma_{m_k} := (2\pi l_b^2)^k \int_{(\mathbb{N}\times\Omega)^k} m_k(X_{1:k}) \bigotimes_{i=1}^{k} \Pi_{X_i} d\eta^{\otimes k}(X_{1:k})$$

We call m_k the upper symbol of γ_{m_k}. We also associate a density to m_k:

$$\rho_{m_k} := \sum_{n_{1:k} \in \mathbb{N}^k} m_k(n_{1:k}; \bullet)$$

we extend the definition (II.1.13) to Husimi functions, if $k \geqslant 2$:

$$\mathcal{E}_{sc,\hbar b}\left[m_k\right] := \int_{\mathbb{N}\times\Omega} E_n m_k^{(1)}(n,R) d\eta(n,R) + \int_{\mathbb{N}\times\Omega} V m_k^{(1)} d\eta + \int_{(\mathbb{N}\times\Omega)^2} w m_k^{(2)} d\eta^{\otimes 2} \tag{II.5.1}$$

and we also extend (II.1.16) to $\rho_k \in L^1\left(\Omega^k\right)$:

$$\mathcal{E}_{qLL}\left[\rho_k\right] = \int_{\Omega} V \rho_k^{(1)} + \int_{\Omega^2} w \rho_k^{(2)} \tag{II.5.2}$$

If one starts from a wave-function $\psi_N \in L^2_-\left(\Omega^N\right)$ we use the notation

$$m_{\psi_N} := m_{\gamma_{\psi_N}}$$

For another discussion and further references about lower and upper symbols one can look at [15, Definition 3.13].

The k-body Husimi function is the joint probability distribution for k particles in phase space. Similarly as for (II.1.24), we have

$$m^{(k)}_{\gamma_N} = m_{\gamma^{(k)}_N} \text{ and } \rho^{(k)}_{m_{\gamma_N}} = \rho_{m^{(k)}_{\gamma_N}}$$

The next lemma provides a translation between Husimi functions and density matrices.

Lemma II.5.2: *Relations between Husimi functions and reduced densities*

Let $\gamma_k \in \mathcal{L}^1\left(L^2\left(\Omega^k\right)\right)$ be a positive operator, then $m_{\gamma_k} \in L^1\left((\mathbb{N}\times\Omega)^k\right)$ and

$$0 \leqslant m_{\gamma_k} \leqslant \frac{\|\gamma_k\|_{\mathcal{L}^\infty}}{(2\pi l_b^2)^k}(1+\mathcal{O}(l_b)) \qquad \int\limits_{(\mathbb{N}\times\Omega)^k} m_{\gamma_k} d\eta^{\otimes k} = \mathrm{Tr}\left[\gamma_k\right]$$

Conversely if $m_k \in L^1\left((\mathbb{N}\times\Omega)^k\right)$ is positive, then $\gamma_{m_k} \in \mathcal{L}^1(L^2(\Omega^k))$ and

$$0 \leqslant \gamma_{m_k} \leqslant \left(2\pi l_b^2\right)^k \|m_k\|_{L^\infty} \qquad \mathrm{Tr}\left[\gamma_{m_k}\right] = \|m_k\|_{L^1} + \mathcal{O}(l_b)$$

Moreover if $\gamma_N \in \mathcal{L}^1\left(L^2_-\left(\Omega^k\right)\right)$ and $1 \leqslant k \leqslant N$, then

$$m^{(k)}_{\gamma_N} \leqslant \frac{(N-k)!}{\left(2\pi l_b^2\right)^k N!}\mathrm{Tr}\left[\gamma_N\right]\left(1+\mathcal{O}\left(l_b\right)\right)$$

Proof:

m_{γ_k} is positive because $\forall X \in \mathbb{N}\times\Omega$, Π_X and γ_k are positive. With Corollary II.3.6,

$$m_{\gamma_k}(X_{1:k}) \leqslant \|\gamma_k\|_{\mathcal{L}^\infty}\prod_{i=1}^k \mathrm{Tr}\left[\Pi_{X_i}\right] = \|\gamma_k\|_{\mathcal{L}^\infty}\left(\frac{1}{2\pi l_b^2}+\mathcal{O}\left(\frac{1}{l_b}\right)\right)^k = \frac{\|\gamma_k\|_{\mathcal{L}^\infty}}{(2\pi l_b^2)^k}(1+\mathcal{O}(l_b))$$

Then, with the resolution of identity (II.3.3) we have

$$\int\limits_{(\mathbb{N}\times\Omega)^k} m_{\gamma_k} d\eta^{\otimes k} = \mathrm{Tr}\left[\gamma_k\right]$$

Since $\forall X \in \mathbb{N}\times\Omega$, Π_X and m_k are positive γ_{m_k} is also positive. (II.3.3) also implies

$$\gamma_{m_k} \leqslant \left(2\pi l_b^2\right)^k \|m_k\|_{L^\infty} \int\limits_{(\mathbb{N}\times\Omega)^k} \bigotimes_{i=1}^k \Pi_{X_i} d\eta^{\otimes k}(X_{1:k}) = \left(2\pi l_b^2\right)^k \|m_k\|_{L^\infty}$$

Finally, using Corollary II.3.6,

$$\begin{aligned}\mathrm{Tr}\left[\gamma_{m_k}\right] &= (2\pi l_b^2)^k \int\limits_{(\mathbb{N}\times\Omega)^k} m_k(X_{1:k})\mathrm{Tr}\left[\bigotimes_{i=1}^k \Pi_{X_i}\right] d\eta^{\otimes k}(X_{1:k}) = \int\limits_{(\mathbb{N}\times\Omega)^k} m_k d\eta^{\otimes k} + \mathcal{O}(l_b) \\ &= \|m_k\|_{L^1} + \mathcal{O}(l_b)\end{aligned}$$

$\Pi_X \in \mathcal{L}^1(L^2(\Omega))$ is positive, thus it can be diagonalized:

$$\Pi_X = \sum_{i\in\mathbb{N}} \lambda_{i,X}\left|\psi_{i,X}\right\rangle\left\langle\psi_{i,X}\right| \text{ with } \lambda_{i,X} \geqslant 0 \text{ and } \sum_{i\in\mathbb{N}}\lambda_{i,X} = \mathrm{Tr}\left[\Pi_X\right]$$

We have

$$\begin{aligned} m_{\gamma_N^{(k)}}(X_{1:k}) &= \sum_{i_{1:k}\in\mathbb{N}^k} \left(\prod_{j=1}^k \lambda_{i_j,X_j}\right) \operatorname{Tr}\left[\gamma_N^{(k)} \left|\bigotimes_{j=1}^k \psi_{i_j,X_j}\right\rangle \left\langle \bigotimes_{j=1}^k \psi_{i_j,X_j}\right|\right] \\ &= \sum_{i_{1:k}\in\mathbb{N}^k} \left(\prod_{j=1}^k \lambda_{i_j,X_j}\right) \operatorname{Tr}\left[\gamma_N \left|\bigotimes_{j=1}^k \psi_{i_j,X_j}\right\rangle \left\langle \bigotimes_{j=1}^k \psi_{i_j,X_j}\right| \otimes \mathrm{Id}_{L^2(\Omega^{N-k})}\right] \end{aligned} \tag{II.5.3}$$

Let $\psi_{1:N} \in L^2(\Omega)$ be an orthonormal family, we claim that

$$\left|\bigotimes_{i=1}^k \psi_i\right\rangle \left\langle \bigotimes_{i=1}^k \psi_i\right| \otimes \mathrm{Id}_{L^2(\Omega^{N-k})} \leqslant \frac{(N-k)!}{N!} \text{ on } \mathcal{L}^1\left(L^2_-\left(\Omega^N\right)\right) \tag{II.5.4}$$

Indeed, if we consider the Slater state

$$\chi_N := \frac{1}{\sqrt{N!}} \sum_{\sigma\in S_N} \epsilon(\sigma) \bigotimes_{j=1}^N \psi_{\sigma(j)}$$

then

$$\begin{aligned} &\left\langle \chi_N \middle| \left(\left|\bigotimes_{i=1}^k \psi_i\right\rangle \left\langle \bigotimes_{i=1}^k \psi_i\right| \otimes \mathrm{Id}_{L^2(\Omega^{N-k})}\right) \chi_N \right\rangle \\ =&\frac{1}{N!} \sum_{\sigma,\tau\in S_N} \epsilon(\sigma\tau) \left\langle \bigotimes_{i=1}^k \psi_i \middle| \bigotimes_{i=1}^k \psi_{\tau(i)} \right\rangle \left\langle \bigotimes_{i=1}^N \psi_{\sigma(i)} \middle| \left(\bigotimes_{i=1}^k \psi_i\right) \otimes \bigotimes_{i=k+1}^N \psi_{\tau(i)} \right\rangle \\ =&\frac{1}{N!} \sum_{\sigma,\tau\in S_N} \epsilon(\sigma\tau) \left(\prod_{i=1}^k \delta_{\sigma(i),i}\delta_{\tau(i),i}\right) \prod_{i=k+1}^N \delta_{\sigma(i),\tau(i)} = \frac{1}{N!} \sum_{\sigma\in S_N} \prod_{i=1}^k \delta_{\sigma(i),i} = \frac{(N-k)!}{N!} \end{aligned}$$

If the Slater determinant does not contain the $\psi_{1:k}$ then the result of this computation is 0, thus we obtain (II.5.4). Then with (II.5.3) and Corollary II.3.6,

$$\begin{aligned} m_{\gamma_N^{(k)}}(X_{1:k}) &\leqslant \frac{(N-k)!}{N!} \sum_{i_{1:k}\in\mathbb{N}^k} \left(\prod_{j=1}^k \lambda_{i_j,X_j}\right) \operatorname{Tr}\left[\gamma_N\right] = \frac{(N-k)!}{N!} \operatorname{Tr}\left[\gamma_N\right] \prod_{j=1}^k \operatorname{Tr}\left[\Pi_{X_j}\right] \\ &= \frac{(N-k)!}{\left(2\pi l_b^2\right)^k N!} \operatorname{Tr}\left[\gamma_N\right] \left(1+\mathcal{O}\left(l_b\right)\right) \end{aligned}$$

We have the following properties for the Husimi functions coming from reduced density matrices.

Property II.5.3: *Husimi functions*

Let γ_N be an N-body density matrix, then $m^{(k)}_{\gamma_N}$ are symmetric, consistent and satisfy

$$0 \leqslant m^{(k)}_{\gamma_N} \leqslant \frac{(N-k)!}{(2\pi l_b^2)^k N!} + \mathcal{O}(l_b) \tag{II.5.5}$$

$$\int_{(\mathbb{N}\times\Omega)^k} m_{\gamma_N}^{(k)} d\eta^{\otimes k} = \left\| m_{\gamma_N}^{(k)} \right\|_{L^1} = 1 \tag{II.5.6}$$

$$\rho_{m_{\gamma_N}}^{(k)} = (g_\lambda^2)^{\otimes k} * \rho_{\gamma_N}^{(k)} \tag{II.5.7}$$

Proof:

Let $k > q \geqslant 1$ and $X_{1:q} \in (\mathbb{N} \times \Omega)^q$. Recalling the results of Subsection II.4.1, we prove that the N-body Husimi functions are consistent marginals using (II.3.3):

$$\begin{aligned}\int_{(\mathbb{N}\times\Omega)^{k-q}} m_{\gamma_N}^{(k)}(X_{1:k}) d\eta(X_{q+1:k}) &= \operatorname{Tr}\left[\int_{(\mathbb{N}\times\Omega)^{k-q}} \gamma_N^{(k)} \bigotimes_{i=1}^{k} \Pi_{X_i} d\eta(X_{q+1:k})\right] \\ &= \operatorname{Tr}\left[\gamma_N^{(k)} \bigotimes_{i=1}^{q} \Pi_{X_i} \otimes \mathrm{Id}_{\mathbb{N}\times\Omega}^{\otimes(k-q)}\right] = \operatorname{Tr}\left[\gamma_N^{(q)} \bigotimes_{i=1}^{q} \Pi_{X_i}\right] \\ &= m_{\gamma_N}^{(q)}(X_{1:q})\end{aligned} \tag{II.5.8}$$

Let $\sigma \in S_k$, the symmetry of the Husimi measures follows from the symmetry of the reduced density matrices:

$$\begin{aligned}\sigma^{-1} \cdot m_{\gamma_N}^{(k)}(X_{1:k}) &= \operatorname{Tr}\left[\gamma_N^{(k)} \bigotimes_{i=1}^{q} \Pi_{X_{\sigma(i)}}\right] = \operatorname{Tr}\left[\left(\sigma \cdot \gamma_N^{(k)}\right) \bigotimes_{i=1}^{q} \Pi_{X_i}\right] = \operatorname{Tr}\left[\gamma_N^{(k)} \bigotimes_{i=1}^{q} \Pi_{X_i}\right] \\ &= m_{\gamma_N}^{(k)}(X_{1:k})\end{aligned}$$

(II.5.5) and (II.5.6) follow from

$$\operatorname{Tr}\left[\gamma_N^{(k)}\right] = 1$$

and Lemma II.5.2.
For the last point we perform a straightforward computation:

$$\begin{aligned}\sum_{n_{1:k}\geqslant 0} m_{\gamma_N}^{(k)}(n_{1:k}; R_{1:k}) &= \operatorname{Tr}\left[\gamma_N^{(k)} \sum_{n_{1:k}\geqslant 0} \bigotimes_{i=1}^{k} \Pi_{n_i,R_i}\right] = \int_{\Omega^k} \gamma_N^{(k)}(x_{1:k}, x_{1:k}) \prod_{i=1}^{k} g_\lambda(x_i - R_i)^2 dx_{1:k} \\ &= (g_\lambda^2)^{\otimes k} * \rho_{\gamma_N}^{(k)}(R_{1:k})\end{aligned}$$

Equation (II.5.7) tells us that summing the densities inside every Landau level approximate the total density.

II.5.2 Semi-classical energy

We now prove that the quantum energy can be approximated by the following semi-classical energy, only depending on the one body and two body Husimi functions.

Proposition II.5.4: *Semi-classical approximation*

Let $\psi_N \in L^2_-(\Omega^N)$, $\|\psi_N\|_{L^2} = 1$, the quantum energy can be approximated with the semi-classical energy (II.5.1)

$$\frac{\langle\psi_N|\mathcal{H}_N\psi_N\rangle}{N} \underset{N\to\infty}{=} \mathcal{E}_{sc,\hbar b}\left[m_{\psi_N}\right] + \mathcal{O}\left(\frac{f(\lambda)}{\hbar b}\mathrm{Tr}\left[\mathscr{L}_{\hbar,b}\gamma_N^{(1)}\right]\right) + \mathcal{O}\left((\hbar\lambda)^2\right) \tag{II.5.9}$$

where

$$f(\lambda) := \max\left(\left\|g_\lambda^2 * V - V\right\|_{L^2}, \left\|(g_\lambda^2)^{\otimes 2} * w - w\right\|_{L^2}\right) \underset{\lambda\to\infty}{\to} 0 \tag{II.5.10}$$

The term kinetic energy

$$\frac{1}{\hbar b}\mathrm{Tr}\left[\mathscr{L}_{\hbar,b}\gamma_{\psi_N}^{(1)}\right]$$

will be bounded when we will take a sequence of minimizers of the N-body quantum energy. Recalling (II.1.8) and (II.1.11),

$$bl_b = \mathcal{O}\left(\hbar N l_b\right) = \mathcal{O}\left(\hbar N^{\frac{1}{2}}\right) \gg 1$$

so with (II.3.2)

$$(\hbar\lambda)^2 \ll \hbar\lambda \ll \hbar b\lambda l_b \ll 1 \tag{II.5.11}$$

Moreover, $\lambda \to \infty$ so the error terms in (II.5.9) will be small.

Proof of Proposition II.5.4:

We start with the kinetic term. Inserting the resolution of identity (II.3.3) we have

$$\mathrm{Tr}\left[\mathscr{L}_{\hbar,b}\gamma_{\psi_N}^{(1)}\right] = \int\limits_{\mathbb{N}\times\Omega} \mathrm{Tr}\left[\mathscr{L}_{\hbar,b} g_\lambda(\bullet - R)\Pi_n g_\lambda(\bullet - R)\gamma_{\psi_N}^{(1)}\right] d\eta(n,R)$$

Now, we use the diagonalization of the magnetic Laplacian $\mathscr{L}_{\hbar,b}\Pi_n = E_n\Pi_n$ by commuting $\mathscr{L}_{\hbar,b}$ with $g_\lambda(\bullet - R)$ to obtain

$$\begin{aligned}
\mathrm{Tr}\left[\mathscr{L}_{\hbar,b}\gamma_{\psi_N}^{(1)}\right] =& \mathrm{Tr}\left[\int\limits_{\mathbb{N}\times\Omega} E_n\Pi_{n,R}\gamma_{\psi_N}^{(1)} d\eta(n,R)\right] \\
&+ \mathrm{Tr}\left[\gamma_{\psi_N}^{(1)} \int\limits_{\mathbb{N}\times\Omega} \left[\mathscr{L}_{\hbar,b}, g_\lambda(\bullet - R)\right]\Pi_n g_\lambda(\bullet - R) dR\right] \\
=& \int\limits_{\mathbb{N}\times\Omega} E_n m_{\psi_N}^{(1)}(n,R) d\eta(n,R) + \mathrm{Tr}\left[\gamma_{\psi_N}^{(1)} \int\limits_{\Omega} \left[\mathscr{L}_{\hbar,b}, g_\lambda(\bullet - R)\right] g_\lambda(\bullet - R) dR\right]
\end{aligned} \tag{II.5.12}$$

We compute

$$[\mathscr{P}_{\hbar,b}, g_\lambda(\bullet - R)] = i\hbar\nabla g_\lambda(\bullet - R)$$

and

$$\begin{aligned}
[\mathscr{L}_{\hbar,b}, g_\lambda(\bullet - R)] &= [\mathscr{P}_{\hbar,b}, g_\lambda(\bullet - R)] \cdot \mathscr{P}_{\hbar,b} + \mathscr{P}_{\hbar,b} \cdot [\mathscr{P}_{\hbar,b}, g_\lambda(\bullet - R)] \\
&= 2i\hbar\nabla g_\lambda(\bullet - R) \cdot \mathscr{P}_{\hbar,b} - \hbar^2\Delta g_\lambda(\bullet - R) \\
&= \mathscr{P}_{\hbar,b} \cdot 2i\hbar\nabla g_\lambda(\bullet - R) + \hbar^2\Delta g_\lambda(\bullet - R) \qquad \text{(II.5.13)}
\end{aligned}$$

inserting this in (II.5.12), we find

$$\begin{aligned}
\mathrm{Tr}\left[\mathscr{L}_{\hbar,b}\gamma^{(1)}_{\psi_N}\right] = &\int\limits_{\mathbb{N}\times\Omega} E_n m^{(1)}_{\psi_N}(n,R)d\eta(n,R) \\
&+ 2i\hbar\mathrm{Tr}\left[\gamma^{(1)}_{\psi_N}\mathscr{P}_{\hbar,b} \cdot \int\limits_{\Omega} \nabla g_\lambda(\bullet - R)g_\lambda(\bullet - R)dR\right] \\
&+ \hbar^2\mathrm{Tr}\left[\gamma^{(1)}_{\psi_N}\int\limits_{\Omega} \Delta g_\lambda(\bullet - R)g_\lambda(\bullet - R)dR\right]
\end{aligned}$$

But because g has a fixed L^2 norm and is periodic

$$\nabla\int\limits_{\Omega} g_\lambda(\bullet - R)^2 dR = 0 = 2\int\limits_{\Omega} \nabla g_\lambda(\bullet - R)g_\lambda(\bullet - R)dR$$

Moreover

$$\int\limits_{\Omega} \Delta g_\lambda(\bullet - R)g_\lambda(\bullet - R)dR = -\int\limits_{\Omega} (\nabla g_\lambda)^2 = -\lambda^4\int\limits_{\Omega} (\nabla g(\lambda x))^2 dx = -\lambda^2\|\nabla g\|^2_{L^2} \qquad \text{(II.5.14)}$$

Therefore

$$\mathrm{Tr}\left[\mathscr{L}_{\hbar,b}\gamma^{(1)}_{\psi_N}\right] = \int\limits_{\mathbb{N}\times\Omega} E_n m^{(1)}_{\psi_N}(n,R)d\eta(n,R) - (\hbar\lambda)^2\|\nabla g\|^2_{L^2} \qquad \text{(II.5.15)}$$

If we take a k variable potential $V_k \in L^1\left(\Omega^k\right)$,

$$\mathrm{Tr}\left[V_k\gamma^{(k)}_{\psi_N}\right] = \int\limits_{\Omega^k} \gamma^{(k)}_{\psi_N}(x_{1:k}; x_{1:k})\, V_k(x_{1:k})dx_{1:k} = \int\limits_{\Omega^k} \rho^{(k)}_{\psi_N} V_k$$

To express this in terms of Husimi functions we use (II.5.7):

$$\mathrm{Tr}\left[V_k\gamma^{(k)}_{\psi_N}\right] = \int\limits_{\Omega^k} \rho^{(k)}_{m_{\gamma_N}} V_k + \int\limits_{\Omega^k} \left(\rho^{(k)}_{\psi_N} - (g^2_\lambda)^{\otimes k} * \rho^{(k)}_{\psi_N}\right) V_k$$

$$= \int_{\Omega^k} \rho^{(k)}_{m_{\gamma_N}} V_k + \int_{\Omega^k} \rho^{(k)}_{\psi_N} \left(V_k - (g_\lambda^2)^{\otimes k} * V_k\right)$$

Thus applying (II.4.4) and using (II.5.15),

$$\begin{aligned}
\frac{\langle \psi_N | \mathcal{H}_N \psi_N \rangle}{N} &= \mathrm{Tr}\left[(\mathscr{L}_{\hbar,b} + V)\gamma^{(1)}_{\psi_N}\right] + \mathrm{Tr}\left[w\gamma^{(2)}_{\psi_N}\right] \\
&= \int_{\mathbb{N}\times\Omega} E_n m^{(1)}_{\psi_N}(n,R) d\eta(n,R) + \int_{\Omega} \rho^{(1)}_{\psi_N} V + \int_{\Omega^2} \rho^{(2)}_{\psi_N} w - (\hbar\lambda)^2 \left\|\nabla g\right\|^2_{L^2} \\
&= \mathcal{E}_{sc,\hbar b}\left[m_{\psi_N}\right] + \int_{\Omega} \rho^{(1)}_{\psi_N} \left[V - g_\lambda^2 * V\right] + \int_{\Omega^2} \rho^{(2)}_{\psi_N} \left[w - (g_\lambda^2)^{\otimes 2} * w\right] - (\hbar\lambda)^2 \left\|\nabla g\right\|^2_{L^2}
\end{aligned}$$

Using $V, w \in L^2(\Omega)$ and the fact that w and thus

$$(g_\lambda^2)^{\otimes 2} * w(x,y) = \iint_{\Omega^2} g_\lambda^2(z) g_\lambda^2(t) w(x - y + t - z) dz dt$$

only depends on $x - y$ we can use the kinetic energy inequalities (II.4.2) and (II.4.3) to control the errors terms:

$$\begin{aligned}
\left|\frac{\langle \psi_N | \mathcal{H}_N \psi_N \rangle}{N} - \mathcal{E}_{sc,\hbar b}\left[m_{\psi_N}\right]\right| \leqslant& \left|\int_{\omega} \rho^{(1)}_{\psi_N} \left[V - g_\lambda^2 * V\right]\right| + \left|\int_{\Omega^2} \rho^{(2)}_{\psi_N} \left[w - (g_\lambda^2)^{\otimes 2} * w\right]\right| \\
&+ (\hbar\lambda)^2 \left\|\nabla g\right\|^2_{L^2} \\
\leqslant& \frac{C}{\hbar b} \mathrm{Tr}\left[\mathscr{L}_{\hbar,b}\gamma^{(1)}_N\right] f(\lambda) + (\hbar\lambda)^2 \left\|\nabla g\right\|^2_{L^2}
\end{aligned}$$

and we have

$$f(\lambda) \underset{\lambda\to\infty}{\to} 0$$

II.6 Mean field limit

In Section II.5 we went from the quantum N-body energy to the semi-classical energy $(II.5.1)$ (Proposition II.5.4). The last step needed to obtain the limit models (II.1.13) and (II.1.16) out of (II.5.1) and (II.5.2) is to remove correlations. Indeed we see that for $m \in L^1(\mathbb{N} \times \Omega)$ and $\rho \in L^1(\Omega)$

$$\begin{aligned}
\mathcal{E}_{sc,\hbar b}\left[m^{\otimes 2}\right] &= \mathcal{E}_{sc,\hbar b}\left[m\right] \\
\mathcal{E}_{qLL}\left[\rho^{\otimes 2}\right] &= \mathcal{E}_{qLL}\left[\rho\right]
\end{aligned}$$

For fermionic states there are always some correlations due to anti-symmetry. Therefore the objective of this section is to prove that all other correlations are negligible. Neglecting correlations except those coming from the anti-symmetry is called the mean field approximation. We prove that this approximation holds in the mean field limit using Lieb's variational principle (Theorem II.6.5) for the energy upper bound in Subsection II.6.1 and the De Finetti Theorem II.6.11 for the lower bound in Subsection II.6.2.

The next proposition is a computation of the semi-classical energy when the low Landau levels are saturated. In this case the semi-classical energy is a sum of constant energies and of the semi-classical functional (II.1.16) for particles in qLL.

Proposition II.6.1: *Saturated low Landau level energy*

Let $\rho \in \mathcal{D}_{qLL}$, using Notation II.1.4 and Notation II.5.1

$$\mathcal{E}_{sc,\hbar b}\left[m_\rho\right] = \hbar b E^{q,r} + E_V^{q,r} + E_w^{q,r} + \mathcal{E}_{qLL}[\rho] = \hbar b E^{q,r} + \mathcal{E}_{qLL}[\rho_{m_\rho}]$$

Proof:

With a straightforward computation:

$$\begin{aligned}
\mathcal{E}_{sc,\hbar b}\left[m_\rho\right] &= \sum_{n\in\mathbb{N}} E_n \int_\Omega m_\rho(n,x)dx + \sum_{n\in\mathbb{N}} \int_\Omega V(x) m_\rho(n,x)dx \\
&+ \sum_{n,\tilde{n}\in\mathbb{N}} \int_{\Omega^2} w(x-y) m_\rho(n,x) m_\rho(\tilde{n},y)dxdy \\
&= \frac{1}{q+r}\sum_{n=0}^{q-1} E_n + \frac{r}{q+r}E_q + \frac{q}{(q+r)} \fint_\Omega V + \int_\Omega V\rho + \frac{q^2}{(q+r)^2}\fint_{\Omega^2} w \\
&+ \frac{2q}{L^2(q+r)} \iint_{\Omega^2} w(x-y)\rho(x)dxdy + \int_{\Omega^2} w(x-y)\rho(x)\rho(y)dxdy \\
&= \mathcal{E}_{qLL}[\rho] + \frac{2\hbar b}{q+r}\cdot\frac{q}{2}\cdot\left(q-1+\frac{1}{2}+\frac{1}{2}\right) + \frac{r2\hbar b}{q+r}\cdot\left(q+\frac{1}{2}\right) \\
&+ \frac{q}{q+r}\fint_\Omega V + \frac{q^2}{(q+r)^2}\fint_\Omega w + \frac{2qr}{(q+r)^2}\fint_\Omega w
\end{aligned}$$

$$
\begin{aligned}
&= \mathcal{E}_{qLL}[\rho] + \frac{q^2+2qr+r}{q+r}\hbar b + \frac{q}{q+r}\fint_{\Omega} V + \frac{q^2+2qr}{(q+r)^2}\fint_{\Omega} w \\
&= \mathcal{E}_{qLL}[\rho] + \hbar b E^{q,r} + E_V^{q,r} + E_w^{q,r}
\end{aligned}
$$

We obtain the second equality with

$$
\begin{aligned}
\mathcal{E}_{qLL}\left[\rho_{m_\rho}\right] &= \sum_{n\in\mathbb{N}}\int_{\Omega} V(x) m_\rho(n,x)dx + \sum_{n,\tilde{n}\in\mathbb{N}}\int_{\Omega^2} w(x-y)m_\rho(n,x)m_\rho(\tilde{n},y)dxdy \\
&= \mathcal{E}_{qLL}[\rho] + E_V^{q,r} + E_w^{q,r}
\end{aligned}
$$

II.6.1 Energy upper bound

In this part we prove the energy upper bound:

Proposition II.6.2: *Upper energy bound*

$$
\frac{E_N^0}{N} \leqslant \hbar b E^{q,r} + E_V^{q,r} + E_w^{q,r} + \mathcal{E}_{qLL}[\rho] + \hbar b \mathcal{O}\left(1 - \frac{d(q+r)}{N}\right) + \mathcal{O}\left(f(\lambda)\right) + \mathcal{O}\left(\hbar b \lambda l_b\right)
$$

The main tool for this proof is the Hartree-Fock theory obtained when one only consider Slater trial states. For Slater states, many energy computation are simplified (see Wick's Theorem II.6.4): the second reduced density matrix can be reconstructed from the first reduced density matrix. The second reduced density matrix is given in term of a perfectly uncorrelated term and an exchange term that will reduce the energy in the case of repulsive interactions. The exchange term contains the correlations due to anti-symmetry, these are the minimal correlations fermionic states can have. Thus Hartree Fock theory is a way to assume that all other correlations are negligible. The Hartree-Fock energy gives a canonical upper bound for the N-body quantum energy since the variational ensemble is restricted to Slater sates. Hartree-Fock theory can be extended to one body operator (see Notation II.6.3), and using Lieb's variational principle (Theorem II.6.5) one can show that the theory still provides an approximate upper bound for the N-body quantum energy. Then we show that the Hartree-Fock energy is an approximation for the semi-classical energy (Proposition II.6.7).

Notation II.6.3: *Hartree Fock theory*

Let $s,t,u,v \in L^2(\Omega)$, if one define the exchange operator on $\mathcal{L}^1\left(L^2\left(\Omega^2\right)\right)$ as

$$
\mathrm{Ex}\left|s\otimes t\right\rangle\left\langle u\otimes v\right| \coloneqq \left|s\otimes t\right\rangle\left\langle v\otimes u\right| \tag{II.6.1}
$$

Let $\gamma \in \mathcal{L}\left(L^2(\Omega)\right)$, define

$$
\gamma_2 \coloneqq \frac{N}{N-1}\left(1-\mathrm{Ex}\right)\gamma^{\otimes 2} \tag{II.6.2}
$$

Define the Hartree-Fock energy

$$
\mathcal{E}_{HF}\left[\gamma\right] \coloneqq \mathrm{Tr}\left[\left(\mathscr{L}_{\hbar,b} + V\right)\gamma\right] + \mathrm{Tr}\left[w\gamma_2\right] \tag{II.6.3}
$$

With Wick's theorem definitions (II.6.2) and (II.6.3) are actual statements for Slater states.

Theorem II.6.4: *Wick's theorem*

Let $\psi_N = \dfrac{1}{\sqrt{N!}} \bigwedge_{j=1}^{N} \phi_j \in L^2_-(\Omega^N)$ with $(\phi_j)_j$ an orthonormal family, then

$$\gamma^{(1)}_{\psi_N} = \frac{1}{N}\sum_{i=1}^{N} |\phi_i\rangle\langle\phi_i|$$

and

$$\begin{aligned}\gamma^{(2)}_{\psi_N} &= \frac{N}{N-1}(1-\mathrm{Ex})\left(\gamma^{(1)}_{N}\right)^{\otimes 2}\\ &= \frac{1}{N(N-1)}\sum_{i,j=1}^{N} |\phi_i\otimes\phi_j\rangle\langle\phi_i\otimes\phi_j - \phi_j\otimes\phi_i|\end{aligned}$$

Thus for a Slater state γ_{ψ_N}

$$\left(\gamma^{(1)}_{\psi_N}\right)_2 = \gamma^{(2)}_{\psi_N}$$

and the Hartree-Fock energy is exactly what we obtain for the quantum N-body energy:

$$\mathcal{E}_{HF}\left[\gamma^{(1)}_{\psi_N}\right] = \mathrm{Tr}\left[(\mathscr{L}_{\hbar,b}+V)\,\gamma^{(1)}_{\psi_N}\right] + \mathrm{Tr}\left[w\gamma^{(2)}_{\psi_N}\right]$$

Lieb's theorem [52] extends the usual variational principle for operators of the form (II.6.2).

Theorem II.6.5: *Lieb's variational principle*

Let $\gamma \in \mathcal{L}\left(L^2(\Omega)\right)$ satisfying

$$\mathrm{Tr}\left[\gamma\right] = 1 \qquad\qquad 0 \leqslant \gamma \leqslant \frac{1}{N}$$

there exits an N-body density matrix γ_N and a positive operator L_2 such that

$$\begin{aligned}\gamma^{(1)}_N &= \gamma\\ \gamma^{(2)}_N &= \gamma_2 - L_2\end{aligned}$$

We start with Lieb's variational principle to get an energy upper bound in term of the operator γ_2. An important remark here is that we don't assume that the interaction potential is repulsive to get the upper bound as it is usually done when dealing with Lieb's variational principle. The reason why we where able to relax the assumption $w \geqslant 0$ is the computation (II.6.6). Lieb's variational principle has also been recently generalised in [5].

Proposition II.6.6

Let $\gamma \in \mathcal{L}\left(L^2(\Omega)\right)$ such that $\operatorname{Tr}[\gamma]=1$ and $0 \leqslant \gamma \leqslant \frac{1}{N}$, then

$$\frac{E_N^0}{N} \leqslant \mathcal{E}_{HF}[\gamma] + \frac{\operatorname{Tr}[\mathscr{L}_{\hbar,b}\gamma]}{\hbar b}\mathcal{O}(l_b)$$

Proof:

First we prove a lower bound for the interaction term. Using The Gagliardo-Nirenberg inequality for $\psi \in L^2(\Omega)$,

$$\|\psi\|_{L^4}^2 \leqslant C_{GN}\left(\sqrt{\|\psi\|_{L^2}\|\nabla\psi\|}_{L^2} + \|\psi\|_{L^2}\right)$$

along with Hölder's, Young's and Kato's (II.4.7) inequalities,

$$\begin{aligned}
|\langle\psi|\mathcal{V}\psi\rangle| &\leqslant \|\psi\|_{L^4}\|\mathcal{V}\psi\|_{L^{\frac{4}{3}}} \leqslant \|\mathcal{V}\|_{L^2}\|\psi\|_{L^4}^2 \leqslant C_{GN}\|\mathcal{V}\|_{L^2}\left(\|\psi\|_{L^2}\|\nabla|\psi|\|_{L^2} + \|\psi\|_{L^2}^2\right)\\
&\leqslant C_{GN}\|\mathcal{V}\|_{L^2}\left(\frac{1}{\hbar}\|\psi\|_{L^2}\|\mathscr{P}_{\hbar,b}\psi\|_{L^2} + \|\psi\|_{L^2}^2\right)\\
&\leqslant C_{GN}\|\mathcal{V}\|_{L^2}\left(\epsilon\|\mathscr{P}_{\hbar,b}\psi\|_{L^2}^2 + \left(1+\frac{1}{4\epsilon\hbar^2}\right)\|\psi\|_{L^2}^2\right)
\end{aligned}$$

So for $\psi_2 \in L^2(\Omega)\otimes \operatorname{Dom}(\mathscr{L}_{\hbar,b})$,

$$\begin{aligned}
|\langle\psi_2|w\psi_2\rangle| &\leqslant \int_{\Omega^2}|w(x-y)|\,|\psi_2(x,y)|^2\,dxdy \leqslant \|w\|_{L^2}\int_{\Omega}\|\psi_2(x,\bullet)\|_{L^4}^2\\
&\leqslant C_{GN}\|w\|_{L^2}\int_{\Omega}\left(\epsilon\|\mathscr{P}_{\hbar,b}\psi_2(x,\bullet)\|_{L^2}^2 + \left(1+\frac{1}{4\epsilon\hbar^2}\right)\|\psi_2(x,\bullet)\|_{L^2}^2\right)dx \qquad \text{(II.6.4)}\\
&= C_{GN}\|w\|_{L^2}\left(\epsilon\|1\otimes\mathscr{P}_{\hbar,b}\psi_2\|_{L^2}^2 + \left(1+\frac{1}{4\epsilon\hbar^2}\right)\|\psi_2\|_{L^2}^2\right)
\end{aligned}$$

Thus

$$\begin{aligned}
\left\langle\psi_2\middle|\left(C_{GN}\|w\|_{L^2}\epsilon\left(\mathrm{Id}_{L^2(\Omega)}\otimes\mathscr{L}_{\hbar,b}\right)+w\right)\psi_2\right\rangle &= C_{GN}\|w\|_{L^2}\epsilon\|1\otimes\mathscr{P}_{\hbar,b}\psi_2\|_{L^2}^2 + \langle\psi_2|w\psi_2\rangle\\
&\geqslant -C_{GN}\|w\|_{L^2}\left(1+\frac{1}{4\epsilon\hbar^2}\right)\|\psi_2\|_{L^2}^2
\end{aligned}$$

and

$$\epsilon C_{GN}\|w\|_{L^2}\left(\mathrm{Id}_{L^2(\Omega)}\otimes\mathscr{L}_{\hbar,b}\right) + w \geqslant -C_{GN}\|w\|_{L^2}\left(1+\frac{1}{4\epsilon\hbar^2}\right) \qquad \text{(II.6.5)}$$

Let γ_N and L_2 be the operators in Theorem II.6.5. Now we use (II.4.4), and (II.6.5):

$$\frac{E_N^0}{N} \leqslant \frac{\operatorname{Tr}[\mathscr{H}_N\gamma_N]}{N} = \operatorname{Tr}\left[\left(\mathscr{L}_{\hbar,b}+V\right)\gamma_N^{(1)}\right] + \operatorname{Tr}\left[w\gamma_N^{(2)}\right]$$

$$
\begin{aligned}
&= \mathrm{Tr}\left[\left(\mathscr{L}_{\hbar,b}+V\right)\gamma\right]+\mathrm{Tr}\left[w\left(\gamma_2-L_2\right)\right]=\mathcal{E}_{HF}\left[\gamma\right]-\mathrm{Tr}\left[wL_2\right]\\
&\leqslant \mathcal{E}_{HF}\left[\gamma\right]+C_{GN}\left\|w\right\|_{L^2}\left(\left(1+\frac{1}{4\epsilon\hbar^2}\right)\mathrm{Tr}\left[L_2\right]+\epsilon\mathrm{Tr}\left[\left(\mathrm{Id}_{L^2(\Omega)}\otimes\mathscr{L}_{\hbar,b}\right)L_2\right]\right)
\end{aligned}
\tag{II.6.6}
$$

To conclude we need to estimate the error terms. If A is an operator on $L^2(\Omega)$ it follows from (II.6.1) that

$$
\mathrm{Tr}\left[\left(\mathrm{Id}_{L^2(\Omega)}\otimes A\right)\mathrm{Ex}\gamma^{\otimes 2}\right]=\mathrm{Tr}\left[A\gamma^2\right]
\tag{II.6.7}
$$

Indeed, if we decompose γ in an orthonormal family:

$$
\gamma =: \sum_{i\in\mathbb{N}}\lambda_i\left|u_i\right\rangle\left\langle u_i\right|
$$

then

$$
\begin{aligned}
\mathrm{Tr}\left[\left(\mathrm{Id}_{L^2(\Omega)}\otimes A\right)\mathrm{Ex}\gamma^{\otimes 2}\right]&=\sum_{i,j\in\mathbb{N}}\lambda_i\lambda_j\mathrm{Tr}\left[\mathrm{Id}_{L^2(\Omega)}\otimes A\left|u_i\otimes u_j\right\rangle\left\langle u_j\otimes u_i\right|\right]\\
&=\sum_{i,j\in\mathbb{N}}\lambda_i\lambda_j\mathrm{Tr}\left[\left(\left|u_i\right\rangle\left\langle u_j\right|\right)\otimes\left(A\left|u_j\right\rangle\left\langle u_i\right|\right)\right]\\
&=\sum_{i,j\in\mathbb{N}}\lambda_i\lambda_j\mathrm{Tr}\left[\left|u_i\right\rangle\left\langle u_j\right|\right]\mathrm{Tr}\left[A\left|u_j\right\rangle\left\langle u_i\right|\right]=\sum_{i\in\mathbb{N}}\lambda_i^2\mathrm{Tr}\left[A\left|u_i\right\rangle\left\langle u_i\right|\right]\\
&=\mathrm{Tr}\left[A\gamma^2\right]
\end{aligned}
$$

Taking $A \coloneqq \mathrm{Id}_{L^2(\Omega)}$, we obtain

$$
\mathrm{Tr}\left[\mathrm{Ex}\gamma^{\otimes 2}\right]=\mathrm{Tr}\left[\gamma^2\right]
$$

and since γ is positive, with (II.6.2) we can estimate

$$
\begin{aligned}
\mathrm{Tr}\left[L_2\right]=\mathrm{Tr}\left[\gamma_2\right]-\mathrm{Tr}\left[\gamma_N^{(2)}\right]&=\frac{N}{N-1}\mathrm{Tr}\left[\gamma^{\otimes 2}-\mathrm{Ex}\gamma^{\otimes 2}\right]-1=\frac{N}{N-1}-\frac{N}{N-1}\mathrm{Tr}\left[\gamma^2\right]-1\\
&\leqslant\frac{1}{N-1}
\end{aligned}
$$

If $\epsilon\to 0$, we can control the first error term in (II.6.6) with

$$
0\leqslant\left(1+\frac{1}{4\epsilon\hbar^2}\right)\mathrm{Tr}\left[L_2\right]\leqslant\frac{C}{N\epsilon\hbar^2}
\tag{II.6.8}
$$

For the second error term, using Theorem II.6.5, (II.6.2) and (II.6.7) for $A \coloneqq \mathscr{L}_{\hbar,b}$,

$$
\begin{aligned}
0\leqslant&\mathrm{Tr}\left[\left(\mathrm{Id}_{L^2(\Omega)}\otimes\mathscr{L}_{\hbar,b}\right)L_2\right]=\mathrm{Tr}\left[\left(\mathrm{Id}_{L^2(\Omega)}\otimes\mathscr{L}_{\hbar,b}\right)\left(\gamma_2-\gamma_N^{(2)}\right)\right]\\
=&\frac{N}{N-1}\mathrm{Tr}\left[\left(\mathrm{Id}_{L^2(\Omega)}\otimes\mathscr{L}_{\hbar,b}\right)\left(1-\mathrm{Ex}\right)\gamma^{\otimes 2}\right]-\mathrm{Tr}\left[\left(\mathrm{Id}_{L^2(\Omega)}\otimes\mathscr{L}_{\hbar,b}\right)\gamma_N^{(2)}\right]\\
=&\frac{N}{N-1}\mathrm{Tr}\left[\mathscr{L}_{\hbar,b}\gamma\right]-\frac{N}{N-1}\mathrm{Tr}\left[\left(\mathrm{Id}_{L^2(\Omega)}\otimes\mathscr{L}_{\hbar,b}\right)\mathrm{Ex}\gamma^{\otimes 2}\right]-\mathrm{Tr}\left[\mathscr{L}_{\hbar,b}\gamma\right]
\end{aligned}
$$

$$=\frac{1}{N-1}\mathrm{Tr}\left[\mathscr{L}_{\hbar,b}\gamma\right]-\frac{N}{N-1}\mathrm{Tr}\left[\mathscr{L}_{\hbar,b}\gamma^2\right]\leqslant\frac{1}{N-1}\mathrm{Tr}\left[\mathscr{L}_{\hbar,b}\gamma\right]$$

When the kinetic energy is minimised $\mathrm{Tr}\left[\mathscr{L}_{\hbar,b}\gamma\right]$ is of order $\hbar b$ so the second error term in (II.6.6) will be of order:

$$0\leqslant\epsilon\mathrm{Tr}\left[\left(\mathrm{Id}_{L^2(\Omega)}\otimes\mathscr{L}_{\hbar,b}\right)L_2\right]\leqslant C\frac{\epsilon\hbar b}{N}\cdot\frac{\mathrm{Tr}\left[\mathscr{L}_{\hbar,b}\gamma\right]}{\hbar b}\tag{II.6.9}$$

We optimise in ϵ so the bounds in (II.6.8) and (II.6.9) are of the same order:

$$\frac{1}{N\epsilon\hbar^2}=\frac{\epsilon\hbar b}{N}\implies\epsilon=\frac{1}{\sqrt{\hbar^3 b}}=N^{2\delta-\frac{1}{2}}=o(1)\implies\frac{1}{N\epsilon\hbar^2}=\frac{\epsilon\hbar b}{N}=\frac{1}{l_bN}=\mathcal{O}(l_b)\tag{II.6.10}$$

so (II.6.6) becomes

$$\frac{E_N^0}{N}\leqslant\mathcal{E}_{HF}\left[\gamma\right]+\left(1+\frac{\mathrm{Tr}\left[\mathscr{L}_{\hbar,b}\gamma\right]}{\hbar b}\right)\mathcal{O}\left(l_b\right)=\mathcal{E}_{HF}\left[\gamma\right]+\frac{\mathrm{Tr}\left[\mathscr{L}_{\hbar,b}\gamma\right]}{\hbar b}\mathcal{O}\left(l_b\right)$$

Now we go from Hartree-Fock energy to the semi-classical energy.

Proposition II.6.7: *Semi-classical approximation of Hartree-Fock energy*

Let $n_0\in\mathbb{N}$, $m\in L^1(\mathbb{N}\times\Omega)$ such that $\forall n>n_0, m(n,\bullet)=0$ and

$$0\leqslant m\leqslant\frac{1}{2\pi l_b^2N}\tag{II.6.11}$$

then

$$\mathcal{E}_{HF}\left[\gamma_m\right]=\mathcal{E}_{sc,\hbar b}\left[m\right]+\mathcal{O}\left(f(\lambda)\right)+\mathcal{O}\left(\hbar b\lambda l_b\right)$$

Proof:

We start by proving that we recover the semi-classical functional from the direct terms. We compute the kinetic term using the commutation relation (II.5.13) and Corollary II.3.6:

$$\begin{aligned}\mathrm{Tr}\left[\mathscr{L}_{\hbar,b}\gamma_m\right]=&2\pi l_b^2\int_{\mathbb{N}\times\Omega}m(X)\mathrm{Tr}\left[\mathscr{L}_{\hbar,b}\Pi_X\right]d\eta(X)\\=&2\pi l_b^2\int_{\mathbb{N}\times\Omega}m(X)E_n\mathrm{Tr}\left[\Pi_X\right]d\eta(X)\\&+2\pi l_b^2\int_{\mathbb{N}\times\Omega}m(n,R)\mathrm{Tr}\left[\left[\mathscr{L}_{\hbar,b},g_\lambda(\bullet-R)\right]\Pi_ng_\lambda(\bullet-R)\right]d\eta(n,R)\\=&\int_{\mathbb{N}\times\Omega}E_nm(X)d\eta(X)+\mathcal{O}(\hbar bl_b)\end{aligned}$$

$$\begin{aligned}
&+ 2\pi l_b^2 \int\limits_{\mathbb{N}\times\Omega} m(n,R)\mathrm{Tr}\left[2i\hbar\nabla g_\lambda(\bullet - R)\mathscr{P}_{\hbar,b}\Pi_n g_\lambda(\bullet - R)\right] d\eta(n,R)\\
&- 2\pi l_b^2 \int\limits_{\mathbb{N}\times\Omega} m(n,R)\mathrm{Tr}\left[\hbar^2\Delta g_\lambda(\bullet - R)\Pi_n g_\lambda(\bullet - R)\right] d\eta(n,R)
\end{aligned} \tag{II.6.12}$$

Using (II.3.6), $\exists \mathcal{E} : \mathbb{N}\times\Omega \to \mathbb{R}$ such that

$$\begin{aligned}
&2\pi l_b^2 \Pi_n(x,x) = 1 + l_b \mathcal{E}(n,x)\\
&|\mathcal{E}(n,x)| \leqslant C(n)
\end{aligned}$$

With (II.5.14),

$$\begin{aligned}
&- 2\pi l_b^2 \int\limits_{\mathbb{N}\times\Omega} m(n,R)\mathrm{Tr}\left[\hbar^2\Delta g_\lambda(\bullet - R)\Pi_n g_\lambda(\bullet - R)\right] d\eta(n,R)\\
&= -\hbar^2 \int\limits_{\mathbb{N}\times\Omega} m(n,R)\left(\int\limits_\Omega \Delta g_\lambda(x-R)\left(1 + l_b\mathcal{E}(n,x)\right) g_\lambda(x-R)dx\right) d\eta(n,R)\\
&= (\hbar\lambda)^2 \|\nabla g\|_{L^2}^2 \|m\|_{L^1} - \hbar^2 l_b \int\limits_{\mathbb{N}\times\Omega} m(n,R)\left(\int\limits_\Omega \lambda^3\Delta g(\lambda x)\mathcal{E}(n,x+R)\lambda g(\lambda x)dx\right) d\eta(n,R)\\
&= (\hbar\lambda)^2 \|\nabla g\|_{L^2}^2 \|m\|_{L^1} + (\hbar\lambda)^2 \mathcal{O}(l_b) = \mathcal{O}\left((\hbar\lambda)^2\right)
\end{aligned} \tag{II.6.13}$$

And by (II.3.7), $\exists \widetilde{\mathcal{E}} : \mathbb{N}\times\Omega \to \mathbb{R}$ such that

$$\begin{aligned}
&\mathscr{P}_{\hbar,b}\Pi_n(x,x) = \frac{b}{l_b}C(n) + b\widetilde{\mathcal{E}}(n,x)\\
&|\mathcal{E}(n,x)| \leqslant \widetilde{C}(n)
\end{aligned}$$

so

$$\begin{aligned}
&2\pi l_b^2 \int\limits_{\mathbb{N}\times\Omega} m(n,R)\mathrm{Tr}\left[2i\hbar\nabla g_\lambda(\bullet - R)\mathscr{P}_{\hbar,b}\Pi_n g_\lambda(\bullet - R)\right] d\eta(n,R)\\
&=4i\pi l_b^2\hbar \int\limits_{\mathbb{N}\times\Omega} m(n,R)\left(\int\limits_\Omega \nabla g_\lambda(x-R)\left(C(n)\frac{b}{l_b} + b\widetilde{\mathcal{E}}(n,R)\right) g_\lambda(x-R)dx\right) d\eta(n,R)\\
&=\mathcal{O}\left(\hbar b\lambda l_b\right)
\end{aligned} \tag{II.6.14}$$

Inserting (II.6.13) and (II.6.14) in (II.6.12), we obtain

$$\mathrm{Tr}\left[\mathscr{L}_{\hbar,b}\gamma_m\right] = \int\limits_{\mathbb{N}\times\Omega} E_n m(X)d\eta(X) + \mathcal{O}\left(\hbar b\lambda l_b\right) + \mathcal{O}\left((\hbar\lambda)^2\right) \tag{II.6.15}$$

Let $k \in \mathbb{N}^*$ and $W_k \in L^2\left(\Omega^k\right)$, with the Fubini theorem

$$\begin{aligned}
\operatorname{Tr}\left[W_k \gamma_m^{\otimes k}\right] =&(2\pi l_b^2)^k \int\limits_{(\mathbb{N}\times\Omega)^k} m^{\otimes k}(X_{1:k}) \operatorname{Tr}\left[W_k \bigotimes_{i=1}^k \Pi_{X_i}\right] d\eta^{\otimes k}\left(X_{1:k}\right) \\
=& \left(2\pi l_b^2\right)^k \int\limits_{(\mathbb{N}\times\Omega)^k} m^{\otimes k}(X_{1:k}) \int\limits_{\Omega^k} W_k(x_{1:k}) \left(\bigotimes_{i=1}^k \Pi_{X_i}\right)(x_{1:k}, x_{1:k}) dx_{1:k} d\eta^{\otimes k}\left(X_{1:k}\right) \\
=& \int\limits_{\Omega^k} W_k(x_{1:k}) \left(\prod_{i=1}^k 2\pi l_b^2 \int\limits_{(\mathbb{N}\times\Omega)} m(X) \Pi_X(x_i, x_i) d\eta\left(X\right)\right) dx_{1:k} \\
=& \int\limits_{\Omega^k} W_k(x_{1:k}) \left(\prod_{i=1}^k \int\limits_{(\mathbb{N}\times\Omega)} m(n,R) g_\lambda^2(x_i - R)\left(1 + l_b \mathcal{E}(n, x_i)\right) d\eta\left(n,R\right)\right) dx_{1:k} \\
=& \int\limits_{\Omega^k} W_k \left(\rho_m^{\otimes k} * (g_\lambda^2)^{\otimes k}\right) dx \\
&+ l_b \int\limits_{\Omega^k} W_k(x_{1:k}) \left(\prod_{i=1}^k \int\limits_{\Omega} g_\lambda^2(x_i - R) \sum_{n=0}^{n_0} m(n,R) \mathcal{E}(n, x_i) dR\right) dx_{1:k}
\end{aligned}$$

But m has finitely many filled Landau level so with the Pauli principle (II.6.11), $\rho_m \in L^\infty(\Omega)$ and

$$\operatorname{Tr}\left[W_k \gamma_m^{\otimes k}\right] = \int\limits_{\Omega^k} W_k \rho_m^{\otimes k} + \mathcal{O}\left(\left\|W_k - W_k * (g_\lambda^2)^{\otimes k}\right\|_{L^1}\right) + \mathcal{O}\left(l_b\right) \tag{II.6.16}$$

Now we need to control the exchange term. It follows from (II.6.1) that

$$\operatorname{Ex}\gamma_m^{\otimes 2}(x, y; z, t) = \gamma_m(x,t)\gamma_m(y,z)$$

so with (II.6.4) for $\gamma_m \in L^2(\Omega) \otimes \operatorname{Dom}\left(\mathscr{L}_{\hbar,b}\right)$ as an integral kernel,

$$\begin{aligned}
\left|\operatorname{Tr}\left[w \operatorname{Ex}\gamma_m^{\otimes 2}\right]\right| =& \left|\int\limits_{\Omega^2} w(x-y) \left|\gamma_m(x,y)\right|^2 dxdy\right| \\
\leqslant& C_{GN} \left\|w\right\|_{L^2} \int\limits_{\Omega} \left(\epsilon \left\|\mathscr{P}_{\hbar,b}\gamma_m(x,\bullet)\right\|_{L^2}^2 + \left(1 + \frac{1}{4\epsilon\hbar^2}\right) \left\|\gamma_m(x,\bullet)\right\|_{L^2}^2\right) dx
\end{aligned} \tag{II.6.17}$$

With an integration by part,

$$\int\limits_{\Omega} \left\|\mathscr{P}_{\hbar,b}\gamma_m(x,\bullet)\right\|_{L^2}^2 dx = \int\limits_{\Omega^2} \mathscr{P}_{\hbar,b}\gamma_m(x,\bullet)(y) \cdot \overline{\mathscr{P}_{\hbar,b}\gamma_m(x,\bullet)(y)} dxdy$$

$$= \int_{\Omega^2} \mathscr{L}_{\hbar,b}\gamma_m(x,\bullet)(y)\overline{\gamma_m(x,y)}dxdy = \int_{\Omega^2} \overline{\gamma_m}(x,y)\mathscr{L}_{\hbar,b}\overline{\gamma_m}(\bullet,x)(y)dxdy$$

$$= \int_{\Omega^2} \overline{\gamma_m}(x,y)\left(\mathscr{L}_{\hbar,b}\overline{\gamma_m}\right)(y,x)dxdy = \mathrm{Tr}\left[\overline{\gamma_m}\mathscr{L}_{\hbar,b}\overline{\gamma_m}\right]$$

Inserting this in (II.6.17), using the cyclicity of the trace we get

$$\begin{aligned}\left|\mathrm{Tr}\left[w\mathrm{Ex}\gamma_m^{\otimes 2}\right]\right| = \left|\mathrm{Tr}\left[w\mathrm{Ex}\overline{\gamma_m}^{\otimes 2}\right]\right| &\leqslant C_{GN}\left\|w\right\|_{L^2}\left(\epsilon \mathrm{Tr}\left[\mathscr{L}_{\hbar,b}\gamma_m^2\right] + \left(1+\frac{1}{4\epsilon\hbar^2}\right)\mathrm{Tr}\left[\gamma_m^2\right]\right)\\ &\leqslant \frac{C_{GN}\left\|w\right\|_{L^2}}{N}\left(\epsilon\mathrm{Tr}\left[\mathscr{L}_{\hbar,b}\gamma_m\right] + \left(1+\frac{1}{4\epsilon\hbar^2}\right)\mathrm{Tr}\left[\gamma_m\right]\right)\end{aligned}$$

With (II.6.15), $\mathrm{Tr}\left[\mathscr{L}_{\hbar,b}\gamma_m\right] = \mathcal{O}\left(\hbar b\right)$ and using Lemma II.5.2, $\mathrm{Tr}\left[\gamma_m\right] = \left\|m\right\|_{L^1} + \mathcal{O}(l_b)$ so the choice of ϵ is the same as in (II.6.10) thus

$$\mathrm{Tr}\left[w\mathrm{Ex}\gamma_m^{\otimes 2}\right] = \mathcal{O}(l_b)$$

To conclude, with (II.6.3) and (II.6.2) then (II.6.15) and (II.6.16) applied to V and w

$$\begin{aligned}\mathcal{E}_{HF}\left[\gamma_m\right] =& \mathrm{Tr}\left[\mathscr{L}_{\hbar,b}\gamma_m\right] + \mathrm{Tr}\left[V\gamma_m\right] + \frac{N}{N-1}\mathrm{Tr}\left[w\gamma_m^{\otimes 2}\right] + \frac{N}{N-1}\mathrm{Tr}\left[w\mathrm{Ex}\gamma_m^{\otimes 2}\right]\\ =& \int_{\mathbb{N}\times\Omega} E_n m(X)d\eta(X) + \int_\Omega V\rho_m + \frac{N}{N-1}\int_{\Omega^2} w\rho_m^{\otimes 2} + \mathcal{O}\left(\left\|V - V*(g_\lambda^2)\right\|_{L^1}\right)\\ &+ \frac{N}{N-1}\mathcal{O}\left(\left\|w - w*(g_\lambda^2)^{\otimes 2}\right\|_{L^1}\right) + \mathcal{O}\left(l_b\right) + \mathcal{O}\left(\hbar b\lambda l_b\right) + \mathcal{O}\left((\hbar\lambda)^2\right)\end{aligned}$$

Recalling (II.5.10), the semi-classical energy expression (II.1.13), (II.5.11) and $\hbar b\lambda \gg 1$,

$$\mathcal{E}_{HF}\left[\gamma_m\right] = \mathcal{E}_{sc,\hbar}\left[m\right] + f(\lambda) + \mathcal{O}\left(\hbar b\lambda l_b\right)$$

With the notation of equation (II.1.21), we would like to define a one body operator with saturated low Landau levels:

$$\gamma_\rho := \frac{L^2(q+r)}{N}\int_{\Omega\times\mathbb{N}} m_\rho(X)\Pi_X d\eta(X)$$

We need to prove that the direct term gives the limit model for qLL and to control the exchange terms. But we cannot apply directly Lieb's principle because with Lemma II.5.2 we have an error on the trace

$$\mathrm{Tr}\left[\gamma_\rho\right] = 1 + o(1) \text{ and } 0 \leqslant \gamma_\rho \leqslant \frac{1}{N}$$

To cure this we modify m_ρ slightly.

Proposition II.6.8: *Corrected Husimi function*

Let $n_0 \in \mathbb{N}$, $m \in L^1(\mathbb{N} \times \Omega)$ such that $\forall n > n_0, m(n, \bullet) = 0, \|m\|_{L^1} = 1 + o(1)$ and

$$0 \leqslant m \leqslant \frac{1}{2\pi l_b^2 N}$$

there exist $\widetilde{m} \in L^1(\mathbb{N} \times \Omega), n_1 \in \mathbb{N}$ such that $\forall n > n_1, m(n, \bullet) = 0$ such that

$$\operatorname{Tr}\left[\gamma_m\right] = 1 \qquad\qquad 0 \leqslant \gamma_m \leqslant \frac{1}{N}$$

and

$$\mathcal{E}_{sc,\hbar b}\left[\widetilde{m}\right] = \mathcal{E}_{sc,\hbar b}\left[m\right] + \mathcal{O}\left(\hbar b l_b\right) + \mathcal{O}\left(\hbar b\left(1 - \|m\|_{L^1}\right)\right) \tag{II.6.18}$$

Proof:

First, by Corollary II.3.6, $\exists \mathcal{E} : \mathbb{N} \times \Omega \to \mathbb{R}$ such that

$$\begin{aligned} &2\pi l_b^2 \operatorname{Tr}\left[\Pi_X\right] = 1 + l_b \mathcal{E}(X) \\ &|\mathcal{E}(n, R)| \leqslant C(n) \end{aligned}$$

So

$$\operatorname{Tr}\left[\gamma_m\right] = \int\limits_{\mathbb{N}\times\Omega} m(X)\left(1 + l_b \mathcal{E}(X)\right) d\eta(X)$$

If $\operatorname{Tr}\left[\gamma_m\right] = 1$ then m has the desired properties. If $\operatorname{Tr}\left[\gamma_m\right] < 1$ we add some mass to m where it is possible without breaking the Pauli principle. Let $n_1 \in \mathbb{N}$ and

$$0 \leqslant \tau \leqslant \frac{1}{2\pi l_b^2 N}$$

we define

$$\widetilde{m}(\tau, n_1) \coloneqq m + \min\left(\tau, \frac{1}{2\pi l_b^2 N} - m\right) \mathbb{1}_{n \leqslant n_1}$$

By construction

$$0 \leqslant m \leqslant \widetilde{m}(\tau, n_1) \leqslant \frac{1}{2\pi l_b^2 N} \text{ and } \tau \mathbb{1}_{n \leqslant n_1} \leqslant \widetilde{m}(\tau, n_1) \tag{II.6.19}$$

We choose $n_1 > n_0$ and remark that

$$\operatorname{Tr}\left[\gamma_{\widetilde{m}}\right]\left(\frac{1}{2\pi l_B^2 N}, n_1\right) = \frac{1}{2\pi l_B^2 N} \int\limits_{\mathbb{N}\times\Omega} \mathbb{1}_{n \leqslant n_1}\left(1 + l_b \mathcal{E}(X)\right) d\eta(X) \geqslant \frac{L^2}{2\pi l_b^2 N} n_1 - l_b C(n_1)$$

Since $\exists n_1 \in \mathbb{N}$ such that

$$\frac{L^2}{2\pi l_b^2 N} n_1 > 1$$

for large enough N,

$$\operatorname{Tr}\left[\gamma_{\widetilde{m}}\right]\left(\frac{1}{2\pi l_B^2 N}, n_1\right) > 1$$

and

$$\operatorname{Tr}\left[\gamma_{\widetilde{m}}\right](0, n_1) = \operatorname{Tr}\left[\gamma_m\right] < 1$$

and $\operatorname{Tr}\left[\gamma_{\widetilde{m}}\right]$ is Lipschitz in τ, so by the intermediate value theorem we can conclude $\exists \tau \geqslant 0$ such that if we define

$$\widetilde{m} \coloneqq \widetilde{m}(\tau, n_1)$$

then

$$\operatorname{Tr}\left[\gamma_{\widetilde{m}}\right] = \int\limits_{\mathbb{N}\times\Omega} \widetilde{m}(X)\left(1 + l_b \mathcal{E}(X)\right) d\eta(X) = 1$$

Thus we can estimate

$$\begin{aligned}\sum_{n \leqslant n_1} \int\limits_{\Omega} \min\left(\tau, \frac{1}{2\pi l_b^2 N} - m(n,x)\right) dx &= \int\limits_{\mathbb{N}\times\Omega} \left(\widetilde{m} - m\right) d\eta = 1 - l_b \int\limits_{\mathbb{N}\times\Omega} \widetilde{m}\mathcal{E} d\eta - \int\limits_{\mathbb{N}\times\Omega} m d\eta \\ &= \mathcal{O}(l_b) + \mathcal{O}\left(1 - \|m\|_{L^1}\right)\end{aligned}$$

so

$$\tau = \frac{1}{L^2} \int\limits_{\Omega} \min\left(\tau, \frac{1}{2\pi l_b^2 N} - m(n_1, x)\right) dx \leqslant \mathcal{O}(l_b) + \mathcal{O}\left(1 - \|m\|_{L^1}\right) \tag{II.6.20}$$

Now if $\operatorname{Tr}\left[\gamma_m\right] > 1$ we remove some mass to m:

$$\widetilde{m}(\tau) \coloneqq \max\left(0, m - \tau\right) = m - \min(m, \tau)$$

by construction

$$0 \leqslant \widetilde{m} \leqslant m \leqslant \frac{1}{2\pi l_b^2 N} \tag{II.6.21}$$

We see that

$$\operatorname{Tr}\left[\gamma_{\widetilde{m}}\right](0) = \operatorname{Tr}\left[\gamma_m\right] > 1 \text{ and } \operatorname{Tr}\left[\gamma_{\widetilde{m}}\right]\left(\frac{1}{2\pi l_b^2 N}\right) = 0$$

so $\exists \tau \geqslant 0$ such that if one defines

$$\widetilde{m} \coloneqq \widetilde{m}(\tau)$$

we find that

$$\mathrm{Tr}\left[\gamma_{\widetilde{m}}\right]=\int\limits_{\mathbb{N}\times\Omega}\widetilde{m}(X)\left(1+l_b\mathcal{E}(X)\right)d\eta(X)=1$$

and like before,

$$\begin{aligned}\int\limits_{\mathbb{N}\times\Omega}\min(m,\tau)d\eta&=\int\limits_{\mathbb{N}\times\Omega}\left(m-\widetilde{m}\right)d\eta=\|m\|_{L^1}-1+l_b\int\limits_{\mathbb{N}\times\Omega}\widetilde{m}\mathcal{E}d\eta=\mathcal{O}(l_b)+\mathcal{O}\left(1-\|m\|_{L^1}\right)\\&=\int\limits_{m<\tau}md\eta+\int\limits_{\tau\leqslant m}\tau d\eta=\|m\|_{L^1}+\int\limits_{\tau\leqslant m}(\tau-m)d\eta\end{aligned}$$

So

$$\|m\|_{L^1}+\mathcal{O}(l_b)+\mathcal{O}\left(1-\|m\|_{L^1}\right)=\int\limits_{\tau\leqslant m}(m-\tau)d\eta\leqslant\frac{1}{\pi l_b^2N}\left|\mathbb{1}_{\tau\leqslant m}\right|$$

and

$$\begin{aligned}\tau\leqslant&\frac{1}{\left|\mathbb{1}_{\tau\leqslant m}\right|}\int\limits_{\mathbb{N}\times\Omega}\min(m,\tau)d\eta=\frac{1}{\left|\mathbb{1}_{\tau\leqslant m}\right|}\left(\mathcal{O}(l_b)+\mathcal{O}\left(1-\|m\|_{L^1}\right)\right)\\\leqslant&\frac{1}{\pi l_b^2N}\cdot\frac{\mathcal{O}(l_b)+\mathcal{O}\left(1-\|m\|_{L^1}\right)}{\|m\|_{L^1}+\mathcal{O}(l_b)+\mathcal{O}\left(1-\|m\|_{L^1}\right)}=\mathcal{O}(l_b)+\mathcal{O}\left(1-\|m\|_{L^1}\right)\end{aligned}\tag{II.6.22}$$

With inequalities (II.6.20) and (II.6.22) we know that

$$\|m-\widetilde{m}\|_{L^\infty}=\mathcal{O}(l_b)+\mathcal{O}\left(1-\|m\|_{L^1}\right)\tag{II.6.23}$$

Finally we prove the estimate on semi-classical energies (II.6.18):

$$\begin{aligned}\left|\mathcal{E}_{sc,\hbar b}\left[\widetilde{m}\right]-\mathcal{E}_{sc,\hbar b}\left[m\right]\right|\leqslant&\sum_{n=0}^{n_1}E_n\int\limits_{\Omega}\left|\widetilde{m}(n,\bullet)-m(n,\bullet)\right|+\sum_{n=0}^{n_1}\int\limits_{\Omega}|V|\left|\widetilde{m}(n,\bullet)-m(n,\bullet)\right|\\&+\sum_{n,\tilde{n}=0}^{n_1}\int\limits_{\Omega^2}|w(x-y)|\left|\widetilde{m}(n,x)\widetilde{m}(\tilde{n},y)-m(n,x)m(\tilde{n},y)\right|dxdy\\\leqslant&L^2\sum_{n=0}^{n_1}E_n\|m-\widetilde{m}\|_{L^\infty}+(n_1+1)\|V\|_{L^1}\|m-\widetilde{m}\|_{L^\infty}\\&+L^2\|w\|_{L^1}\sum_{n,\tilde{n}=0}^{n_1}\left\|\widetilde{m}(n,\bullet)\widetilde{m}(\tilde{n},\bullet)-m(n,\bullet)m(\tilde{n},\bullet)\right\|_{L^\infty}\end{aligned}$$

Moreover

$$\begin{aligned}\left\|\widetilde{m}(n,\bullet)\widetilde{m}(\tilde{n},\bullet)-m(n,\bullet)m(\tilde{n},\bullet)\right\|_{L^\infty}\leqslant&\left\|\widetilde{m}(n,\bullet)\right\|_{L^\infty}\left\|\widetilde{m}(\tilde{n},\bullet)-m(\tilde{n},\bullet)\right\|_{L^\infty}\\&+\left\|m(\tilde{n},\bullet)\right\|_{L^\infty}\left\|\widetilde{m}(n,\bullet)-m(n,\bullet)\right\|_{L^\infty}\end{aligned}$$

$$\leqslant \|\widetilde{m}\|_{L^\infty} \|\widetilde{m} - m\|_{L^\infty} + \|m\|_{L^\infty} \|\widetilde{m} - m\|_{L^\infty}$$

so with (II.6.19) and (II.6.21)

$$\left|\mathcal{E}_{sc,\hbar b}[\widetilde{m}] - \mathcal{E}_{sc,\hbar b}[m]\right| \leqslant \left(L^2 \sum_{n=0}^{n_1} E_n + (n_1+1)\|V\|_{L^1} + \frac{L^2}{\pi l_b^2 N}\|w\|_{L^1}(n_1+1)^2\right) \cdot \|m - \widetilde{m}\|_{L^\infty}$$

We conclude with (II.6.23).

Putting all of this together we obtain the upper bound.

Proof of Proposition II.6.2:

Recalling (II.1.21), let $\rho \in \mathcal{D}_{qLL}$ and define

$$m_{\rho,N} \coloneqq \frac{d(q+r)}{N} m_\rho \tag{II.6.24}$$

then

$$0 \leqslant m_{\rho,N} \leqslant \frac{d}{L^2 N} = \frac{1}{2\pi l_b^2 N}$$
$$\int\limits_{\mathbb{N}\times\Omega} m_{\rho,N} d\eta = \frac{d(q+r)}{N} = 1 + o(1)$$

We consider $\widetilde{m}_{\rho,N}$ the corrected Husimi function in Proposition II.6.8 associated with $m_{\rho,N}$, it satisfies

$$\mathcal{E}_{sc,\hbar b}[\widetilde{m}_{\rho,N}] = \mathcal{E}_{sc,\hbar b}[m_{\rho,N}] + \mathcal{O}(\hbar b l_b) + \hbar b \mathcal{O}\left(1 - \frac{d(q+r)}{N}\right) \tag{II.6.25}$$

and $\mathrm{Tr}\left[\gamma_{m_{\rho,N}}\right] = 1, 0 \leqslant \gamma_{m_{\rho,N}} \leqslant \frac{1}{N}$. Moreover by (II.6.15),

$$\mathrm{Tr}\left[\mathscr{L}_{\hbar,b}\gamma_{m_{\rho,N}}\right] = \mathcal{O}(\hbar b)$$

Thus, we can apply Propositions Proposition II.6.6, Proposition II.6.7 and (II.6.25):

$$\begin{aligned}
\frac{E_N^0}{N} \leqslant & \mathcal{E}_{HF}\left[\gamma_{m_{\rho,N}}\right] + \mathcal{O}(l_b) = \mathcal{E}_{sc,\hbar b}[\widetilde{m}_{\rho,N}] + \mathcal{O}(f(\lambda)) + \mathcal{O}(\hbar b \lambda l_b) \\
= & \mathcal{E}_{sc,\hbar b}[m_{\rho,N}] + \hbar b \mathcal{O}\left(1 - \frac{d(q+r)}{N}\right) + \mathcal{O}(f(\lambda)) + \mathcal{O}(\hbar b \lambda l_b) \\
= & \hbar b E^{q,r} + E_V^{q,r} + E_w^{q,r} + \mathcal{E}_{qll}\left[\frac{d(q+r)}{N}\rho\right] + \hbar b \mathcal{O}\left(1 - \frac{d(q+r)}{N}\right) + \mathcal{O}(f(\lambda)) + \mathcal{O}(\hbar b \lambda l_b) \\
= & \hbar b E^{q,r} + E_V^{q,r} + E_w^{q,r} + \mathcal{E}_{qLL}[\rho] + \hbar b \mathcal{O}\left(1 - \frac{d(q+r)}{N}\right) + \mathcal{O}(f(\lambda)) + \mathcal{O}(\hbar b \lambda l_b)
\end{aligned}$$

For the last equality we use the estimate

$$\left|\mathcal{E}_{qll}\left[\frac{d(q+r)}{N}\rho\right]-\mathcal{E}_{qll}\left[\rho\right]\right|\leqslant\left|1-\frac{d(q+r)}{N}\right|\|V\|_{L^2}\|\rho\|_{L^2}+\left(1-\left(\frac{d(q+r)}{N}\right)^2\right)\|w\|_{L^2}\|\rho\|_{L^2}^2$$

and

$$\left|\left(1-\left(\frac{d(q+r)}{N}\right)^2\right)\right|\leqslant C\left|1-\frac{d(q+r)}{N}\right|$$

II.6.2 Energy lower bound

In this part we prove the Energy lower bound :

Proposition II.6.9: *Lower bound*

Let $(\psi_N)_N$ be a sequence of minimizers of (II.1.7),

$$\mathcal{E}_{sc,\hbar b}\left[m_{\psi_N}\right]\geqslant\hbar bE^{q,r}+E_V^{q,r}+E_w^{q,r}+\mathcal{E}_{qLL}^0+o(1)$$

The main tool here is the De Finetti Theorem II.6.11. Husimi functions are symmetric and consistent measures. The De Finetti theorem states that such measures are indeed reduced to trivial measure of this kind, namely tensorized products of one body measures and their convex combinations. This result plays an important role in the justification of the decorrelation of densities for the lower bound.

We start by extracting some limit Husimi functions and give their fundamental properties. Similar arguments can be found in [17, Section 2]. With Notation II.5.1,

Proposition II.6.10

Let $(\psi_N)_N$ be a sequence of minimizers of (II.1.7), up to a subsequence

a) there exists limit Husimi functions $M^{(k)}\in L^\infty\left((\mathbb{N}\times\Omega)^k\right)$ such that

$$m_{\psi_N}^{(k)}\underset{N\to\infty}{\overset{*}{\rightharpoonup}}M^{(k)}\text{ in the weak star topology on }L^\infty\left((\mathbb{N}\times\Omega)^k\right)\tag{II.6.26}$$

$$0\leqslant M^{(k)}\leqslant\frac{1}{\left(L^2(q+r)\right)^k}\tag{II.6.27}$$

b) $M^{(1)}(q,\bullet)\in\mathcal{D}_{qLL}$ and

$$M^{(1)}(n,\bullet)=\mathbb{1}_{n<q}\frac{1}{L^2(q+r)}+\mathbb{1}_{n=q}M^{(1)}(q,\bullet)`\tag{II.6.28}$$

c) $M^{(k)}$ are the reduced densities of a symmetric measure M on $(\mathbb{N}\times\Omega)^{\mathbb{N}}$ and $\left\|M^{(k)}\right\|_{L^1}=1$

d) in the sense of Radon measures

$$\rho_{m_{\psi_N}}^{(k)}\underset{N\to\infty}{\overset{*}{\rightharpoonup}}\rho_{M^{(k)}}\tag{II.6.29}$$

e) we have convergence of the potential terms:

$$\mathcal{E}_{qLL}\left[\rho_{m_{\psi_N}}\right] \underset{N\to\infty}{\rightarrow} \mathcal{E}_{qLL}\left[\rho_M\right] \tag{II.6.30}$$

Proof:

a) From inequality (II.5.5) the Husimi functions are uniformly bounded, with a diagonal extraction we obtain (II.6.26) and the bound (II.5.5) with (II.1.11) induce (II.6.27) in the limit.

b) Now since we took a minimizer of the energy, with the upper bound Proposition II.6.2 and the Kinetic energy inequalities (II.4.2) and (II.4.3),

$$\begin{aligned}\frac{E_N^0}{N} =&\mathrm{Tr}\left[\mathscr{L}_{\hbar,b}\gamma_{\psi_N}^{(1)}\right] + \int_{\Omega} V\rho_{\psi_N}^{(1)} + \int_{\Omega^2} w\rho_{\psi_N}^{(2)} = \mathrm{Tr}\left[\mathscr{L}_{\hbar,b}\gamma_{\psi_N}^{(1)}\right]\left(1+\mathcal{O}\left(\frac{1}{\hbar b}\right)\right)\\ \leqslant&\mathcal{E}_{sc,\hbar b}\left[m_\rho\right] + \hbar b\mathcal{O}\left(1-\frac{d(q+r)}{N}\right) + \mathcal{O}\left(f(\lambda)\right) + \mathcal{O}\left(\hbar b\lambda l_b\right)\end{aligned}$$

so by Proposition II.6.1 we know that

$$\mathrm{Tr}\left[\mathscr{L}_{\hbar,b}\gamma_{\psi_N}^{(1)}\right] = \mathcal{O}\left(\hbar b\right) \tag{II.6.31}$$

Since the contribution of the potential are bounded, the only thing we have to look at are the kinetic terms. Let m_ρ be the Husimi function with saturated low Landau levels defined here (II.1.21). We denote

$$c_{N,n} \coloneqq \int_{\Omega}\left(m_{\psi_N}^{(1)}(n,.) - m_\rho(n,.)\right)$$

By definition of m_ρ and Lemma II.5.2 we have

$$\begin{aligned}&\sum_{n\in\mathbb{N}} c_{N,n} = \int_{\mathbb{N}\times\Omega} m_{\psi_N}^{(1)} - \int_{\mathbb{N}\times\Omega} m_\rho = 1-1=0\\ &n<q \implies c_{N,n} \leqslant \frac{L^2}{2\pi l_b^2 N} + \mathcal{O}\left(l_b\right) - \frac{1}{q+r} = \mathcal{O}\left(l_b\right) + \mathcal{O}\left(1-\frac{d(q+r)}{N}\right)\\ &n>q \implies c_{N,n} = \left\|m_{\psi_N}^{(1)}(n,\bullet)\right\|_{L^1} \geqslant 0\end{aligned}$$

Since $(E_n)_n$ is increasing

$$\begin{aligned}\sum_{n\in\mathbb{N}} E_n c_{N,n} \geqslant \sum_{n=0}^{q} E_n c_{N,n} + E_q\sum_{n>q} c_{N,n} =& -\sum_{n=0}^{q-1}(E_q-E_n)c_{N,n}\\ \geqslant&\mathcal{O}\left(\hbar b l_b\right) + \hbar b\mathcal{O}\left(1-\frac{d(q+r)}{N}\right)\end{aligned} \tag{II.6.32}$$

Now we compute

$$
\begin{aligned}
\mathcal{E}_{sc,\hbar b}\left[m_{\psi_N}\right] - \mathcal{E}_{sc,\hbar b}\left[m_\rho\right] = \sum_{n\in\mathbb{N}} E_n c_{N,n} + \int\limits_{\mathbb{N}\times\Omega} V\left(m_{\psi_N}^{(1)} - m_\rho\right) d\eta \\
+ \int\limits_{(\mathbb{N}\times\Omega)^2} w\left(m_{\psi_N}^{(2)} - m_\rho^{\otimes 2}\right) d\eta^{\otimes 2}
\end{aligned}
\tag{II.6.33}
$$

From the semi-classical approximation (Proposition II.5.4), (II.6.31) and the upper bound (Proposition II.6.2),

$$
\begin{aligned}
\frac{E_N^0}{N} =& \frac{\langle\psi_N|\mathcal{H}_N\psi_N\rangle}{N} = \mathcal{E}_{sc,\hbar b}\left[m_{\psi_N}\right] + \mathcal{O}\left(f(\lambda)\right) + \mathcal{O}\left((\hbar\lambda)^2\right) \\
\leqslant& \mathcal{E}_{sc,\hbar b}\left[m_\rho\right] + \hbar b\mathcal{O}\left(1 - \frac{d(q+r)}{N}\right) + \mathcal{O}\left(f(\lambda)\right) + \mathcal{O}\left(\hbar b\lambda l_b\right)
\end{aligned}
$$

so with (II.5.11),

$$
\mathcal{E}_{sc,\hbar b}\left[m_{\psi_N}\right] - \mathcal{E}_{sc,\hbar b}\left[m_\rho\right] \leqslant \hbar b\mathcal{O}\left(1 - \frac{d(q+r)}{N}\right) + \mathcal{O}\left(f(\lambda)\right) + \mathcal{O}\left(\hbar b\lambda l_b\right) \tag{II.6.34}
$$

All the potential terms in (II.6.33) are of order 1, therefore the sum in (II.6.32) is bounded and we have

$$
\mathcal{O}\left(\hbar b l_b\right) + \hbar b\mathcal{O}\left(1 - \frac{d(q+r)}{N}\right) \leqslant -\sum_{n=0}^{q-1}(E_q - E_n)c_{N,n} \leqslant \sum_{n\in\mathbb{N}} E_n c_{N,n} \leqslant C
$$

So

$$
\sum_{n=0}^{q-1} \frac{E_n - E_q}{\hbar b} c_{N,n} = \mathcal{O}\left(\frac{1}{\hbar b}\right) \tag{II.6.35}
$$

With a similar inequality as (II.6.32) but with E_{q+1} instead of E_q we deduce

$$
\begin{aligned}
C \geqslant& \sum_{n\in\mathbb{N}} E_n c_{N,n} \geqslant \sum_{n=0}^{q} E_n c_{N,n} + E_{q+1}\sum_{n>q} c_{N,n} = \sum_{n=0}^{q}(E_n - E_{q+1})c_{N,n} \\
\geqslant& \sum_{n=0}^{q} E_n c_{N,n} + E_q \sum_{n>q} c_{N,n} \geqslant \mathcal{O}\left(\hbar b l_b\right) + \hbar b\mathcal{O}\left(1 - \frac{d(q+r)}{N}\right)
\end{aligned}
\tag{II.6.36}
$$

and therefore (II.6.35) implies

$$
c_{N,q} = \mathcal{O}\left(\frac{1}{\hbar b}\right)
$$

Then

$$
\sum_{n>q} \frac{E_n}{\hbar b} c_{N,n} = \sum_{n\in\mathbb{N}} \frac{E_n}{\hbar b} c_{N,n} - \sum_{n=0}^{q} \frac{E_n}{\hbar b} c_{N,n} = \mathcal{O}\left(\frac{1}{\hbar b}\right) \geqslant \sum_{n>q} \int\limits_{\Omega} m_{\psi_N}^{(1)}(n, \bullet)
$$

and

$$c_{N,q} = \int\limits_{\Omega} m_N^{(1)}(q,R)dR - \int\limits_{\Omega} \rho(R)dR = \left\| m_N^{(1)}(q,\bullet) \right\|_{L^1} - \frac{r}{q+r} = \mathcal{O}\left(\frac{1}{\hbar b}\right) \tag{II.6.37}$$

From the consistency of $m_{\psi_N}^{(k)}$ in Property II.5.3,

$$\begin{aligned} \left\| m_{\psi_N}^{(1)}(n_1,\bullet) \right\|_{L^1} &= \int\limits_{\Omega} \left(\int\limits_{(\mathbb{N}\times\Omega)^{k-1}} m_{\psi_N}^{(k)}(n_1,x_1;X_{2:k}) d\eta^{\otimes(k-1)}(X_{2:k}) \right) dx_1 \\ &= \sum_{n_{2:k}\in\mathbb{N}^{k-1}} \left\| m_{\psi_N}^{(k)}(n_{1:k},\bullet) \right\|_{L^1} \end{aligned} \tag{II.6.38}$$

Since

$$\mathbb{N}^k \backslash [\![0:q]\!]^k = \bigsqcup_{i=1}^{k} \mathbb{N}^{i-1} \times (\mathbb{N}\backslash [\![0:q]\!]) \times \mathbb{N}^{k-i}$$

by the symmetry of $m_{\psi_N}^{(k)}$, (II.6.38) and (II.6.35),

$$\sum_{n_{1:k}\in\mathbb{N}^k\backslash[\![0:q]\!]^k} \left\| m_{\psi_N}^{(k)}(n_{1:k},\bullet) \right\|_{L^1} = k \sum_{n_1>q} \left\| m_{\psi_N}^{(1)}(n_1,\bullet) \right\|_{L^1} = \mathcal{O}\left(\frac{1}{\hbar b}\right) \tag{II.6.39}$$

Ω is bounded, thus testing (II.6.26) against $\mathbb{1}_{\{n_{1:k}\}\times\Omega} \in L^1\left((\mathbb{N}\times\Omega)^k\right)$,

$$\left\| m_{\psi_N}^{(k)}(n_{1:k};\bullet) \right\|_{L^1} \underset{N\to\infty}{\longrightarrow} \left\| M^{(k)}(n_{1:k};\bullet) \right\|_{L^1}$$

So (II.6.37) gives

$$\left\| M^{(1)}(q,\bullet) \right\|_{L^1} = \frac{r}{q+r}$$

and with (II.6.39), if $n_{1:k} \in \mathbb{N}^k \backslash [\![0:q]\!]^k$, then $M^{(k)}(n_{1:k},\bullet) = 0$ and we see that the norm (II.5.6) passes to the limit:

$$\begin{aligned} \left\| M^{(k)} \right\|_{L^1} &= \sum_{n_{1:k}\in[\![0:q]\!]^k} \left\| M^{(k)}(n_{1:k},\bullet) \right\|_{L^1} = \lim_{N\to\infty} \sum_{n_{1:k}\in[\![0:q]\!]^k} \left\| m_{\psi_N}^{(k)}(n_{1:k},\bullet) \right\|_{L^1} \\ &= \lim_{N\to\infty} \left(\sum_{n_{1:k}\in[\![0:q]\!]^k} \left\| m_{\psi_N}^{(k)}(n_{1:k},\bullet) \right\|_{L^1} + \sum_{n_{1:k}\in\mathbb{N}^k\backslash[\![0:q]\!]^k} \left\| m_{\psi_N}^{(k)}(n_{1:k},\bullet) \right\|_{L^1} \right) \\ &= \lim_{N\to\infty} \left\| m_{\psi_N}^{(k)} \right\|_{L^1} = 1 \end{aligned}$$

If $n < 0$, by (II.6.35),

$$\left\| m_{\psi_N}^{(1)}(n,\bullet) - \frac{1}{L^2(q+r)} \right\|_{L^1} \leqslant \left\| m_{\psi_N}^{(1)}(n,\bullet) - \frac{1}{2\pi l_b^2 N} \right\|_{L^1} + \mathcal{O}\left(1 - \frac{d(q+r)}{N}\right)$$

$$
\begin{aligned}
&= \int_\Omega \left(\frac{1}{2\pi l_b^2 N} - m^{(1)}_{\psi_N}(n,\bullet) \right) + \mathcal{O}\left(1 - \frac{d(q+r)}{N}\right) \\
&= \int_\Omega \left(\frac{1}{L^2(q+r)} - m^{(1)}_{\psi_N}(n,\bullet) \right) + \mathcal{O}\left(1 - \frac{d(q+r)}{N}\right) \\
&= - C_{N,n} + \mathcal{O}\left(1 - \frac{d(q+r)}{N}\right) = \mathcal{O}\left(\frac{1}{\hbar b}\right) + \mathcal{O}\left(1 - \frac{d(q+r)}{N}\right)
\end{aligned}
$$

so $M^{(1)}(n,\bullet) = \dfrac{1}{L^2(q+r)}$.

c) Testing (II.5.8) against $\varphi_q \in C^0_c\left((\mathbb{N}\times\Omega)^q\right)$, we have

$$
\int_{(\mathbb{N}\times\Omega)^q} \varphi_q m^{(q)}_{\psi_N} d\eta^{\otimes q} = \int_{(\mathbb{N}\times\Omega)^k} \varphi_q(X_{1:q}) m^{(k)}_{\psi_N}(X_{1:k}) d\eta^{\otimes k}(X_{1:k}) \tag{II.6.40}
$$

Since $\varphi_q \in L^1\left((\mathbb{N}\times\Omega)^k\right)$, with (II.6.26),

$$
\int_{(\mathbb{N}\times\Omega)^q} \varphi_q m^{(q)}_{\psi_N} d\eta^{\otimes q} \underset{N\to\infty}{\to} \int_{(\mathbb{N}\times\Omega)^q} \varphi_q M^{(q)} d\eta^{\otimes q} \tag{II.6.41}
$$

In order to pass to the limit in the right term of (II.6.40), for the low Landau levels we use (II.6.26) on

$$
\mathbb{1}_{(\llbracket 0:q\rrbracket\times\mathbb{N})^k}\left(\varphi_q \otimes \mathrm{Id}_{(\mathbb{N}\times\Omega)^{k-q}}\right) \in L^1\left((\mathbb{N}\times\Omega)^k\right)
$$

and for the high Landau levels we use (II.6.39) and $\varphi_q \in L^\infty\left((\mathbb{N}\times\Omega)^k\right)$:

$$
\begin{aligned}
\int_{(\mathbb{N}\times\Omega)^k} \varphi_q(X_{1:q}) m^{(k)}_{\psi_N}(X_{1:k}) d\eta^{\otimes k}(X_{1:k}) &= \int_{\Omega^k} \mathbb{1}_{(\llbracket 0:q\rrbracket\times\mathbb{N})^k}\left(\varphi_q \otimes \mathrm{Id}_{(\mathbb{N}\times\Omega)^{k-q}}\right) m^{(k)}_{\psi_N} d\eta^{\otimes k} \\
&\quad + \sum_{n_{1:k}\in\mathbb{N}^k\setminus\llbracket 0:q\rrbracket^k} \int_{\Omega^k} \varphi_q(n_{1:q}, x_{1:q}) m^{(k)}_{\psi_N}(n_{1:k}, x_{1:k}) dx_{1:k} \\
&\underset{N\to\infty}{\to} \int_{\Omega^k} \mathbb{1}_{(\llbracket 0:q\rrbracket\times\mathbb{N})^k}\left(\varphi_q \otimes \mathrm{Id}_{(\mathbb{N}\times\Omega)^{k-q}}\right) M^{(k)} d\eta^{\otimes k} \\
&= \int_{(\mathbb{N}\times\Omega)^k} \varphi_q(X_{1:q}) M^{(k)}(X_{1:k}) d\eta^{\otimes k}(X_{1:k})
\end{aligned} \tag{II.6.42}
$$

Thus passing to the limit in (II.6.40) and inserting (II.6.41) and(II.6.42) we obtain

$$
\forall \varphi_q \in C^0_c\left((\mathbb{N}\times\Omega)^q\right), \int_{(\mathbb{N}\times\Omega)^q} \varphi_q M^{(q)} d\eta^{\otimes q} = \int_{(\mathbb{N}\times\Omega)^k} \varphi_q(X_{1:q}) M^{(k)}(X_{1:k}) d\eta^{\otimes k}(X_{1:k})
$$

and this proves that the limit Husimi functions are also consistent. Testing against φ_q, we also obtain that the symmetry of Husimi functions passes to the limit. Then we can conclude with the Kolmogorov extension theorem that there exists M a symmetric measure on $(\mathbb{N}\times\Omega)^{\mathbb{N}}$ whose marginals are $(M^{(k)})_k$.

d) Let $\varphi_k \in C^0(\Omega^k)$, φ_k is bounded and

$$\mathbb{1}_{[\![0:q]\!]^k} \otimes \varphi_k \in L^1\left((\mathbb{N}\times\Omega)^k\right)$$

so using (II.6.26) and (II.6.39)

$$\begin{aligned}\int\limits_{\Omega^k} \varphi_k \rho^{(k)}_{m_{\psi_N}} &= \int\limits_{(\mathbb{N}\times\Omega)^k} \left(\mathbb{1}_{[\![0:q]\!]^k} \otimes \varphi_k\right) m^{(k)}_{\psi_N} d\eta^{\otimes k} + \sum_{n_{1:k}\in\mathbb{N}^k\setminus[\![0:q]\!]^k} \int\limits_{\Omega^k} \varphi_k m^{(k)}_{\psi_N} \\ &\underset{N\to\infty}{\to} \int\limits_{(\mathbb{N}\times\Omega)^k} \left(\mathbb{1}_{[\![0:q]\!]^k} \otimes \varphi_k\right) M^{(k)} d\eta^{\otimes k} = \int\limits_{\Omega^k} \varphi_k \rho^{(k)}_M\end{aligned}$$

e) Let $V_k \in L^2(\Omega^k)$, and $(V_{k,n})_n \subset C^\infty(\Omega^k)$ a sequence regularised with a convolution to a regular function so that

$$\|V_k - V_{k,n}\|_{L^2} \underset{n\to\infty}{\to} 0$$

we have

$$\int\limits_{\Omega^k} V_k \left(\rho^{(k)}_{m_{\psi_N}} - \rho^{(k)}_M\right) = \int\limits_{\Omega^k} V_{k,n} \left(\rho^{(k)}_{m_{\psi_N}} - \rho^{(k)}_M\right) + \int\limits_{\Omega^k} \rho^{(k)}_{m_{\psi_N}} (V_k - V_{k,n}) + \int\limits_{\Omega^k} \rho^{(k)}_M (V_{k,n} - V_k)$$

For a fixed n, since $V_{k,n} \in C^0(\Omega^k)$ by (II.6.29) the first term goes to 0 when $N\to\infty$. For the second term we use (II.4.2) if $V_1 = V$, (II.4.3) if $V_2 = w$ and (II.5.7)

$$\begin{aligned}\left|\int\limits_{\Omega^k} \rho^{(k)}_{m_{\psi_N}} (V_k - V_{k,n})\right| &= \left|\int\limits_{\Omega^2} \left(\left(g^k_\lambda\right)^{\otimes k} * \rho^{(k)}_N\right) (V_k - V_{k,n})\right| \leqslant C \left\|(V_k - V_{k,n}) * \left(g^2_\lambda\right)^{\otimes k}\right\|_{L^2} \\ &\leqslant C \|(V_k - V_{k,n})\|_{L^2}\end{aligned}$$

For the third term we use Hölder's inequality since $\rho^{(k)}_M \in L^\infty\left(\Omega^k\right)$ so we have

$$\lim_{N\to\infty} \left|\int\limits_{\Omega^k} V_k \left(\rho^{(k)}_{m_{\psi_N}} - \rho^{(k)}_M\right)\right| \leqslant C \|V_k - V_{n,k}\|_{L^2} \underset{n\to\infty}{\to} 0$$

Now we want to apply the De Finetti theorem to M:

Theorem II.6.11: *De Finetti or Hewitt-Savage*

Let X be a metric space, $\mu \in \mathcal{P}_s(X^{\mathbb{N}})$ be a symmetric probability measure with marginals $\left(\mu^{(n)}\right)_{n\geqslant 1}$.
$\exists P_\mu \in \mathcal{P}(\mathcal{P}(X))$ such that:

$$\forall n \in \mathbb{N}^*, \mu^{(n)} = \int\limits_{\mathcal{P}(\Omega)} \rho^{\otimes n} dP_\mu(\rho) \tag{II.6.43}$$

For a proof of this via the the Diaconis-Freedman theorem see [21, Section 2.1.] and for some further context one can look at [15, Section 2.2.].

Recalling the definition of the semi-classical domain (II.1.15), we obtain:

Proposition II.6.12: *Low Landau level filling of the limit factorised densities*

There exists $\mathcal{P}_M \in \mathcal{P}\left(\mathcal{D}_{sc}\right)$ such that

$$\forall k \in \mathbb{N}^*, M^{(k)} = \int\limits_{\mathcal{D}_{sc}} m^{\otimes k} d\mathcal{P}_M(m) \tag{II.6.44}$$

Let μ be the push-forward measure of $\mathcal{P}_M$ by the application

$$\begin{array}{ccc} L^1\left(\mathbb{N} \times \Omega\right) & \rightarrow & L^1(\Omega) \\ m & \mapsto & m(q, \bullet) \end{array}$$

then $\mu \in \mathcal{P}\left(\mathcal{D}_{qLL}\right)$ and

$$\rho_M^{(k)} = \int\limits_{\mathcal{D}_{qLL}} \left(\frac{q}{L^2(q+r)} + \rho\right)^{\otimes k} d\mu(\rho) \tag{II.6.45}$$

$$\mathcal{E}_{qLL}\left[\rho_M\right] = \int\limits_{\mathcal{D}_{qLL}} \mathcal{E}_{qLL}\left[\frac{q}{L^2(q+r)} + \rho\right] d\mu(\rho) = E_V^{q,r} + E_w^{q,r} + \int\limits_{\mathcal{D}_{qLL}} \mathcal{E}_{qLL}\left[\rho\right] d\mu(\rho) \tag{II.6.46}$$

Proof:

Applying Theorem II.6.11 to M obtained in Proposition II.6.10 gives the existence of $\mathcal{P}_M \in \mathcal{P}\left(\mathcal{P}\left(\mathbb{N} \times \Omega\right)\right)$ such that

$$\forall k \in \mathbb{N}^*, M^{(k)} = \int\limits_{\mathcal{P}(\mathbb{N}\times\Omega)} m^{\otimes k} d\mathcal{P}_M(m) \tag{II.6.47}$$

Let $\varphi \in C_c^0(\mathbb{N} \times \Omega, \mathbb{R}_+), \varphi \neq 0, \epsilon > 0, k \in \mathbb{N}^*$, and

$$A_\epsilon(\varphi) \coloneqq \left\{ m \in \mathcal{P}\left(\mathbb{N} \times \Omega\right) \middle| \int\limits_{\mathbb{N}\times\Omega} \varphi dm \geqslant \frac{1+\epsilon}{L^2(q+r)} \int\limits_{\mathbb{N}\times\Omega} \varphi \right\}$$

If $m \in A_\epsilon(\varphi)$, then

$$1 \leqslant \frac{L^2(q+r)}{(1+\epsilon)\|\varphi\|_{L^1(\eta)}} \int\limits_{\mathbb{N}\times\Omega} \varphi dm \leqslant \left(\frac{L^2(q+r)}{(1+\epsilon)\|\varphi\|_{L^1(\eta)}} \int\limits_{\mathbb{N}\times\Omega} \varphi dm\right)^k$$

so with (II.6.27),

$$\begin{aligned}
\mathcal{P}_M\left(A_\epsilon(\varphi)\right) &= \int\limits_{\mathcal{P}(\mathbb{N}\times\Omega)} \mathbb{1}_{A_\epsilon(\varphi)} d\mathcal{P}_M \leqslant \int\limits_{\mathcal{P}(\mathbb{N}\times\Omega)} \left(\frac{L^2(q+r)}{(1+\epsilon)\|\varphi\|_{L^1(\eta)}} \int\limits_{\mathbb{N}\times\Omega} \varphi dm\right)^k d\mathcal{P}_M(m) \\
&= \left(\frac{L^2(q+r)}{(1+\epsilon)\|\varphi\|_{L^1(\eta)}}\right)^k \int\limits_{\mathcal{P}(\mathbb{N}\times\Omega)} \left(\int\limits_{(\mathbb{N}\times\Omega)^k} \varphi^{\otimes k} dm^{\otimes k}\right) d\mathcal{P}_M(m) \\
&= \left(\frac{L^2(q+r)}{(1+\epsilon)\|\varphi\|_{L^1(\eta)}}\right)^k \int\limits_{(\mathbb{N}\times\Omega)^k} \varphi^{\otimes k} dM^{(k)} \leqslant \left(\frac{1}{1+\epsilon}\right)^k \underset{k\to\infty}{\to} 0
\end{aligned}$$

we proved that $\mathcal{P}_M\left(A_\epsilon(\varphi)\right) = 0$ and therefore

$$\mathcal{P}_M\left(\bigcap_{\substack{\varphi\in C_c^0(\mathbb{N}\times\Omega,\mathbb{R}_+)\\ \epsilon>0}} \mathcal{P}\left(\mathbb{N}\times\Omega\right)\backslash A_\epsilon(\varphi)\right) = 1 - \mathcal{P}_M\left(\bigcup_{\substack{\varphi\in C_c^0(\mathbb{N}\times\Omega,\mathbb{R}_+)\\ \epsilon>0}} A_\epsilon(\varphi)\right) = 1$$

therefore for $\mathcal{P}_m$ almost every $m \in \mathcal{P}\left(\mathbb{N}\times\Omega\right)$,

$$\forall \varphi \in C_c^0\left(\mathbb{N}\times\Omega,\mathbb{R}_+\right), \epsilon > 0, \int\limits_{\mathbb{N}\times\Omega} \varphi dm < \frac{1+\epsilon}{L^2(q+r)} \int\limits_{\mathbb{N}\times\Omega} \varphi \tag{II.6.48}$$

So for $\mathcal{P}_m$ almost every $m \in \mathcal{P}\left(\mathbb{N}\times\Omega\right)$, m is the density of a probability measure thus a positive function such that $\|m\|_{L^1} = 1$ and by (II.6.48), $m \in L^\infty(\mathbb{N}\times\Omega)$ and

$$m \leqslant \frac{1}{L^2(q+r)} \tag{II.6.49}$$

We have shown $\mathcal{P}_M \in \mathcal{P}\left(\mathcal{D}_{sc}\right)$, therefore (II.6.47) implies (II.6.44).
Moreover if $n < q$ by (II.6.28),

$$\int\limits_\Omega \frac{1}{L^2(q+r)} dx = \int\limits_{\mathbb{N}\times\Omega} \mathbb{1}_{\{n\}\times\Omega} dM^{(1)} = \int\limits_{\mathcal{P}(\mathbb{N}\times\Omega)} \left(\int\limits_\Omega m(n,x)dx\right) d\mathcal{P}_M(m)$$

so

$$\int\limits_{\mathcal{P}(\mathbb{N}\times\Omega)} \left(\int\limits_\Omega \left(\frac{1}{L^2(q+r)} - m(n,x)\right) dx\right) d\mathcal{P}_M(m) = 0$$

By (II.6.49) the integrand is positive thus null $\mathcal{P}_M$ almost everywhere, we conclude that for $\mathcal{P}_M$ almost every m

$$n < q \implies m(n, \bullet) = \frac{1}{L^2(q+r)} \tag{II.6.50}$$

If $n > q$ by (II.6.28),

$$0 = \int_{\mathbb{N}\times\Omega} \mathbb{1}_{\{n\}\times\Omega} dM^{(1)} = \int_{\mathcal{P}(\mathbb{N}\times\Omega)} \left(\int_{\Omega} m(n, x)dx \right) d\mathcal{P}_M(m)$$

Once again by (II.6.49) the right integrand is positive and thus null so for $\mathcal{P}_M$ almost every m

$$n > q \implies m(n, \bullet) = 0 \tag{II.6.51}$$

Finally if $n = q$, since $m \in \mathcal{P}(\mathbb{N} \times \Omega)$ we conclude using (II.6.51) and (II.6.50): for $\mathcal{P}_M$ almost everywhere m

$$\int_{\Omega} m(q, \bullet) = \int_{\mathbb{N}\times\Omega} m - \sum_{n<q} \int_{\Omega} m(n, \bullet) - \sum_{n>q} \int_{\Omega} m(n, \bullet) = 1 - \frac{q}{q+r} = \frac{r}{q+r} \tag{II.6.52}$$

Gathering (II.6.49), (II.6.50), (II.6.51) and (II.6.52), we now know that for $\mathcal{P}_M$ almost every m we have $m(q, \bullet) \in \mathcal{D}_{qLL}$. This means that $\mu \in \mathcal{P}(\mathcal{D}_{qLL})$.
Finally we compute

$$\begin{aligned}
\rho_M^{(k)} &= \sum_{n_{1:k}} M^{(k)}(n_{1:k}; \bullet) = \int_{\mathcal{D}_{sc}} \sum_{n_{1:k}} m^{\otimes k}(n_{1:k}; \bullet) d\mathcal{P}_M(m) = \int_{\mathcal{D}_{sc}} \left(\sum_{n\in\mathbb{N}} m(n; \bullet) \right)^{\otimes k} d\mathcal{P}_M(m) \\
&= \int_{\mathcal{D}_{sc}} \left(\frac{q}{L^2(q+r)} + m(q; \bullet) \right)^{\otimes k} d\mathcal{P}_M(m) = \int_{\mathcal{D}_{qLL}} \left(\frac{q}{L^2(q+r)} + \rho \right)^{\otimes k} d\mu(\rho) \\
&= \int_{\mathcal{D}_{qLL}} \left(\frac{q}{L^2(q+r)} + \rho \right)^{\otimes k} d\mu(\rho)
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{E}_{qLL}[\rho_M] = \int_{\mathcal{D}_{qLL}} \mathcal{E}_{qLL}\left[\frac{q}{L^2(q+r)} + \rho \right] d\mu(\rho) = & \int_{\mathcal{D}_{qLL}} \left(E_V^{q,r} + E_w^{q,r} + \mathcal{E}_{qLL}[\rho] \right) d\mu(\rho) \\
= & E_V^{q,r} + E_w^{q,r} + \int_{\mathcal{D}_{qLL}} \mathcal{E}_{qLL}[\rho] \, d\mu(\rho)
\end{aligned}$$

Now we are ready for the proof of the lower bound.

Proof of Proposition II.6.9:

Let $\rho \in \mathcal{D}_{qLL}$, starting from (II.6.33), using inequality (II.6.36) and Proposition II.6.1 we have

$$\begin{aligned}\mathcal{E}_{sc,\hbar b}\left[m_{\psi_N}\right] \geqslant& \mathcal{E}_{sc,\hbar b}\left[m_\rho\right] + \mathcal{E}_{qLL}\left[\rho_{m_{\psi_N}}\right] - \mathcal{E}_{qLL}\left[\rho_{m_\rho}\right] + \mathcal{O}\left(\hbar b l_b\right) + \hbar b \mathcal{O}\left(1 - \frac{d(q+r)}{N}\right)\\ =&\hbar b E_{q,r} + \mathcal{E}_{qLL}\left[\rho_{m_{\psi_N}}\right] + \mathcal{O}\left(\hbar b l_b\right) + \hbar b \mathcal{O}\left(1 - \frac{d(q+r)}{N}\right)\end{aligned}$$

We conclude with (II.6.30) and (II.6.46) and that

$$\begin{aligned}\mathcal{E}_{sc,\hbar b}\left[m_{\psi_N}\right] \geqslant&\hbar b E_{q,r} + \mathcal{E}_{qLL}\left[\rho_{m_{\psi_N}}\right] + \mathcal{O}\left(\hbar b l_b\right) + \hbar b \mathcal{O}\left(1 - \frac{d(q+r)}{N}\right)\\ =&\hbar b E_{q,r} + \mathcal{E}_{qLL}\left[\rho_M\right] + o(1) = \hbar b E^{q,r} + E_V^{q,r} + E_w^{q,r} + \int\limits_{\mathcal{D}_{qLL}} \mathcal{E}_{qLL}\left[\rho\right] d\mu(\rho) + o(1) \\ \geqslant&\hbar b E^{q,r} + E_V^{q,r} + E_w^{q,r} + \mathcal{E}_{qLL}^0 + o(1)\end{aligned} \tag{II.6.53}$$

II.6.3 Conclusion

Proof of Theorem II.1.5:

Let $(\psi_N)_N$ be a sequence of minimizers of (II.1.7), by (II.5.9)

$$\frac{E(N)}{N} = \frac{\langle \psi_N | \mathscr{H}_N \psi_N \rangle}{N} = \mathcal{E}_{sc,\hbar b}\left[m_{\psi_N}\right] + o(1)$$

Since the lower bound is true up to a subsequence for which the have Proposition II.6.10, for every adherence value of $E(N)/N$ we conclude by gathering Proposition II.6.2 and Proposition II.6.9.

Proof of Theorem II.1.7:

With (II.6.45) and (II.6.29) we get

$$\rho^{(k)}_{m_{\psi_N}} \underset{N\to\infty}{\overset{*}{\rightharpoonup}} \int\limits_{\mathcal{D}_{qLL}} \left(\frac{q}{L^2(q+r)} + \rho\right)^{\otimes k} d\mu(\rho)$$

Let $\varphi \in C^\infty\left(\Omega^k\right)$ with (II.5.7),

$$\int\limits_{\Omega^k} \varphi\left(\rho^{(k)}_{m_{\psi_N}} - \rho^{(k)}_{\psi_N}\right) = \int\limits_{\Omega^k} \varphi\left(\left(g_\lambda^2\right)^{\otimes k} * \rho^{(k)}_{\psi_N} - \rho^{(k)}_{\psi_N}\right) = \int\limits_{\Omega^k} \rho^{(k)}_{\psi_N}\left(\left(g_\lambda^2\right)^{\otimes k} * \varphi - \varphi\right) \underset{N\to\infty}{\to} 0 \tag{II.6.54}$$

by Hölder's inequality since

$$\left\|\rho^{(k)}_{\psi_N}\right\|_{L^1} = 1$$

and φ is Lipschitz. Up to a subsequence $\rho^{(k)}_{\psi_N}$ converges $\forall k \in \mathbb{N}^*$ in the sense of Radon measures. But with (II.6.54) this limit coincides with the one of $\rho^{(k)}_{m_{\psi_N}}$ so we obtain (II.1.25). Moreover by (II.6.53) and Proposition II.6.2

$$\mathcal{E}^0_{qLL} \geqslant \int_{\mathcal{D}_{qLL}} \mathcal{E}_{qLL}\left[\rho\right] d\mu(\rho) + o(1)$$

thus μ only gives mass to minimizers of $\mathcal{E}_{qLL}$.

Chapter III

Semi-classical limit of the 2D Hartree equation in a large magnetic field

Abstract:

We study the dynamic of two dimensional fermionic particles submitted to a magnetic field, assumed to be transverse to the domain and homogeneous. A large magnetic field regime where the gap between Landau levels is of the same order as the other energy contributions is considered. We start from the Hartree equation for the first reduced density matrix, describing the mean field behaviour of a large fermionic system, and derive a gyrokinetic transport equation for the first reduced density. We define a semi-classical density for which the dynamic is computed and compared to the limit gyrokinetic equation. It is shown that the first reduced density almost satisfies the gyrokinetic equation and converges in the sense of measures to a weak solution of this equation.

III.1 Context and results

III.1.1 Model

We consider a large system of interacting fermionic particles in two dimensions. They are placed under a homogeneous magnetic field perpendicular to the domain. In this context the kinetic energy of the particles is quantized into discrete energy levels called Landau levels, separated by a finite energy gap. Our goal is to study the semi-classical limit of the dynamics under high magnetic field. This setup is physically motivated by the quantum hall effect, see [30] for some physical context. We will start from the Hartree equation appropriate for a large system of fermions, and obtain a gyrokinetic transport equation for the density.

Notation III.1.1: *Model*

We work on $\mathbb{R}^2$. The one body kinetic energy operator, also called magnetic Laplacian, is

$$\mathscr{L}_b \coloneqq (i\hbar\nabla + bA)^2$$

With

$$\operatorname{Dom}(\mathscr{L}_b) \coloneqq \left\{\psi \in L^2\left(\mathbb{R}^2\right) \middle| \mathscr{L}_b\psi \in L^2\left(\mathbb{R}^2\right)\right\}$$

We work in symmetric gauge, namely the vector potential is

$$A = \frac{1}{2}X^\perp \tag{III.1.1}$$

where X is the position operator in $\mathbb{R}^2$. b is the magnetic field amplitude, we associate to it the magnetic length

$$l_b \coloneqq \sqrt{\frac{\hbar}{b}}$$

Let V be the external potential and w the interaction potential assumed to be radial:

$$w(x-y) =: \widetilde{w}(|x-y|)$$

We denote $\mathcal{L}^p$ the p^{th} Schatten space. Let $\gamma \in L^\infty\left(\mathbb{R}_+, \mathcal{L}^1\left(L^2\left(\mathbb{R}^2\right)\right)\right)$, and ρ_γ be the associated reduced density

$$\rho_\gamma(t,x) \coloneqq \gamma(t)(x,x) \tag{III.1.2}$$

Here we identified γ with its integral kernel and we will use this convention for the rest of the text. Our goal is to obtain from the Hartree equation

$$i\hbar\partial_t\gamma = [\mathscr{L}_b + V + w \star \rho_\gamma, \gamma] \tag{III.1.3}$$

the following gyrokinetic transport equation for a density $\rho : \mathbb{R}_+ \times \mathbb{R}^2 \to \mathbb{R}_+$,

$$\partial_t\rho + \nabla^\perp(V + w \star \rho)) \cdot \nabla\rho = 0 \tag{III.1.4}$$

Let $W : \mathbb{R}^2 \to \mathbb{R}$ be a potential, we introduce a notation for the gyrokinetic differential operator:

$$\mathrm{GYRO}_W(\mu)(t,z) \coloneqq \partial_t \mu(t,z) + \nabla^\perp W(z) \cdot \nabla \mu(t,z)$$
$$\mathrm{GYRO}_\rho(\mu)(t,z) \coloneqq \mathrm{GYRO}_{V+w\star\rho(t)}(\mu)(t,z) = \partial_t \mu(t,z) + \nabla^\perp \left(V + w \star \rho(t)\right)(z) \cdot \nabla \mu(t,z)$$

Finally denote

$$H_b(t) \coloneqq \mathscr{L}_b + V + \frac{1}{2} w \star \rho_{\gamma_b(t)} \tag{III.1.5}$$

III.1.2 Scaling

In classical mechanics, the movement of a fermion in a transverse magnetic field is decomposed in a cyclotron orbit and the motion of the orbit center. As exposed in Subsection III.1.4 the latter takes place on a time scale of order b. As recalled in Subsection III.2.2, the order of magnitude of the kinetic energy is $\hbar b$. Our plan is to look at a scaling where all the terms in the Hamiltonian are of order 1 and the time scale is of order b.

Notation III.1.2: *scaling*

We take a high magnetic field limit

$$b \to +\infty$$

coupled with a semi-classical scaling

$$\hbar \underset{b\to\infty}{\longrightarrow} 0$$

such that the magnetic kinetic energy is of order 1:

$$\hbar b \underset{b\to\infty}{\longrightarrow} 1 \tag{III.1.6}$$

Let $\gamma \in L^\infty\left(\mathbb{R}_+, \mathcal{L}^1\left(L^2(\mathbb{R}^2)\right)\right)$, such that

$$\operatorname{Tr}\left[\gamma(0)\right] = 1 \quad \text{and} \quad 0 \leqslant \gamma(0) \leqslant 2\pi l_b^2 \tag{III.1.7}$$

define the time rescaled density matrix

$$\forall t \in \mathbb{R}_+, \gamma_b(t) \coloneqq \gamma(bt)$$

If γ satisfies (III.1.3), the equation for the time-rescaled density matrix is

$$\partial_t \gamma_b = \frac{b}{i\hbar}\left[\mathscr{L}_b + V + w \star \rho_{\gamma_b}, \gamma_b\right] = \frac{1}{i l_b^2}\left[\mathscr{L}_b + V + w \star \rho_{\gamma_b}, \gamma_b\right] \tag{III.1.8}$$

Due to the constraint (III.1.7) known to propagate in time (see Proposition III.3.1), from (III.1.2) we see that

$$\int_{\mathbb{R}^2} \rho_{\gamma_b}(t) = \mathrm{Tr}\left[\gamma_b(t)\right]$$

Moreover the Pauli principle $\gamma_b \leqslant 2\pi l_b^2$ guarantees that the system occupies a volume of order 1 in the limit

$$l_b \underset{b\to\infty}{\longrightarrow} 0$$

Indeed it is known, [30] or [2, subsection I.4], that the degeneracy per area inside a Landau level is of order l_b^{-2}. The typical way of constructing a fermionic state satisfying (III.1.7) is with a N-body Slater determinant from N orthonormal one body wave-functions with

$$N := \mathcal{O}\left(\frac{1}{2\pi l_b^2}\right)$$

Such a N-particles state occupies a volume of order

$$\frac{N}{l_b^{-2}} = \mathcal{O}(1)$$

Hence with (III.1.6) this confirms that all the terms in the Hamiltonian $\mathscr{L}_b + V + w \star \rho_\gamma$ are of order 1. If one starts from the N-body Schrödinger dynamics, the fermionic characteristic of the system is imposed by the anti-symmetry of the wave-functions. Starting from the mean field dynamics, the fermionic characteristic is imposed directly via the Pauli principle $\gamma_b \leqslant 2\pi l_b^2$.

As a remark, we give an equivalent formulation of this scaling. If one takes exactly $\hbar = 1/b$, then (III.1.8) is equivalent to

$$i\partial_t \gamma = \left[\left(i\nabla + b^2 A\right) + b^2(V + w \star \rho_{\gamma_b}), \gamma_b\right]$$

In other words with the new scaling

$$\widetilde{b} := b^2$$
$$\widetilde{\gamma}_b := \frac{b^2}{2\pi}\gamma$$

we have

$$\mathrm{Tr}\left[\widetilde{\gamma}_b\right] = \frac{\widetilde{b}}{2\pi}, \quad \widetilde{\gamma}_b \leqslant 1$$
$$i\partial_t \widetilde{\gamma}_b = \left[\left(i\nabla + \widetilde{b}A\right)^2 + \widetilde{b}V + w \star \rho_{\widetilde{\gamma}_b}, \widetilde{\gamma}_b\right]$$

where all the terms in the Hamiltonian $\left(i\nabla + \widetilde{b}A\right)^2 + \widetilde{b}V + w \star \rho_{\widetilde{\gamma}_b}$ are of order $\widetilde{b}$.

III.1.3 Results

The classical counterpart of this work, starting from the Vlasov equation, has been well studied [40] [33] [39] [27] [32] [29] [20] [11]. Some results start from Newton's dynamics [12]. In the quantum literature it is known that the Hartree equation can be obtained by a mean field limit from the N-body Schrödinger dynamics [25] [19] [22]. It is also known that the Vlasov equation can be derived by a semi-classical limit from the Hartree equation [24] [14]. The mean field and semi-classical limits can be coupled to obtain directly the Vlasov equation from the N-body Schrödinger dynamics [7] [8] [3]. More recent results have been dealing with singular potentials [18] [23] [6]. We also refer to a semi-classical work [4] obtaining Euler's vorticity equation from the N-body Schrödinger dynamics in a regime where the gap between Landau levels is small compared to the interactions. Note that for large magnetic fields, the limit of the fundamental energy of the N-body Hamiltonian and the associated densities have been well studied [42], [43], [44], [45], [47] [16], [17] [2].

Now, with Notation III.1.1 and Notation III.1.2, we can state our main results. The first reduced density approximately satisfies a gyrokinetic dynamics:

Theorem III.1.3: *Gyrokinetic of the Hartree solution*

Let $\gamma_b \in L^\infty\left(\mathbb{R}_+, \mathcal{L}^1\left(L^2\left(\mathbb{R}^2\right)\right)\right)$ be a solution of (III.1.8) and assume

$$\begin{aligned}&\operatorname{Tr}\left[\gamma_b(0)\right] = 1, 0 \leqslant \gamma_b(0) \leqslant 2\pi l_b^2\\&\operatorname{Tr}\left[\gamma_b(0) H_b(0)\right] < \infty\end{aligned}$$

If $V, w \in W^{4,\infty}\left(\mathbb{R}^2\right)$, then $\forall \varphi \in C_c^\infty\left(\mathbb{R}_+ \times \mathbb{R}^2\right)$,

$$\left| \int_{\mathbb{R}^2} \varphi(0,z)\rho_{\gamma_b}(0,z)dz - \int_{\mathbb{R}_+\times\mathbb{R}^2} \rho_{\gamma_b}(t,z)\mathrm{GYRO}_{\rho_{\gamma_b}}(\varphi)(t,z)dtdz \right| \leqslant C(\varphi, V, w)\frac{1}{\sqrt{\ln\left(l_b^{-1}\right)}}$$

Moreover the reduced density converges to a weak solution of the gyrokinetic transport equation.

Theorem III.1.4: *Convergence of densities*

Under the same assumptions as Theorem III.1.3, up to a subsequence, ρ_{γ_b} converges in the sense of measures:

$$\begin{aligned}&\rho_{\gamma_b} \underset{b\to\infty}{\overset{*}{\rightharpoonup}} \rho \in \mathcal{M}(\mathbb{R}_+ \times \mathbb{R}^2)\\&\rho_{\gamma_b}(0) \underset{b\to\infty}{\overset{*}{\rightharpoonup}} \rho_0 \in \mathcal{M}(\mathbb{R}^2)\end{aligned}$$

to a weak solution of (III.1.4), meaning that $\forall \varphi \in C_c^\infty\left(\mathbb{R}_+ \times \mathbb{R}^2\right)$,

$$\int_{\mathbb{R}^2} \varphi(0)\rho_0 - \int_{\mathbb{R}_+\times\mathbb{R}^2} \rho\left(\partial_t\varphi + \nabla^\perp\left(V + w \star \rho\right)\cdot\nabla_z\varphi\right) = 0$$

III.1.4 Classical orbits

For a particle of charge -1 in a transverse magnetic field of amplitude b in a force field F, Newton's fundamental equation of dynamics gives

$$Z''(t) = F(t, Z(t)) + bZ'(t)^{\perp} \tag{III.1.9}$$

After one integration, assuming $Z(0) = 0$

$$Z'(t) = Z'(0) + \int_0^t F(\tau, Z(\tau))d\tau + bZ(t)^{\perp}$$

Inserting this in (III.1.9) we obtain

$$Z''(t) + b^2 Z(t) = F(t, Z(t)) + b\int_0^t F(\tau, Z(\tau))^{\perp}d\tau + bZ'(0)^{\perp}$$

For a constant and homogeneous force field we get

$$Z(t) = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} cos(bt) \\ sin(bt) \end{pmatrix} + \frac{F}{b^2} + \frac{Z'(0)^{\perp}}{b} + \frac{tF^{\perp}}{b}$$

where the matrix coefficient are determined through

$$Z(0) = 0 = \begin{pmatrix} \alpha \\ \gamma \end{pmatrix} + \frac{F}{b^2} + \frac{Z'(0)^{\perp}}{b}$$

$$Z'(0) = b\begin{pmatrix} \beta \\ \delta \end{pmatrix} + \frac{F^{\perp}}{b}$$

We can decompose the motion as

$$Z(t) = Z_{orb}(t) + Z_c(t)$$

with the cyclotron orbit

$$Z_{orb}(t) \coloneqq \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} cos(bt) \\ sin(bt) \end{pmatrix}$$

and the orbit center position

$$Z_c(t) \coloneqq \frac{F}{b^2} + \frac{Z'(0)^{\perp}}{b} + \frac{tF^{\perp}}{b}$$

The evolution equation for the orbit center is

$$Z_c'(t) = \frac{F^{\perp}}{b}$$

The characteristic time of the shift is of order b. For the orbit, the characteristic is $1/b$. Indeed if one define

$$Z_{c,b}(t) \coloneqq Z_c(bt)$$

the shift evolution is of order 1:

$$Z_{c,b}'(t) = F^{\perp}$$

III.1.5 Organisation of the paper

Section III.2 and Section III.3 are preliminaries respectively about the magnetic Laplacian and the conserved properties of the dynamics. In Section III.4 we introduce the semi-classical densities and prove that they approximate the physical density. Then we study the dynamics of the semi-classical densities in Section III.5. Section III.6 contains the conclusion of the proofs of the main theorems. Our method is similar to [7] [8] [3] the main difference being about the phase space. Our phase space is $\mathbb{N} \times \mathbb{R}^2$ because the gap between Landau level is of the same order as the other energy contributions, and thus doesn't disappears in the limit. For comparison, when one does semi-classical approximations with a small gap between Landau levels, the phase space is the usual position momentum phase space $\mathbb{R}^2 \times \mathbb{R}^2$.

III.2 Quantization

III.2.1 Spectral analysis of the magnetic Laplacian

In symmetric gauge, the magnetic Laplacian is

$$\mathscr{L}_b = \left(i\hbar\nabla + \frac{b}{2}\begin{pmatrix} -x_2 \\ x_1 \end{pmatrix} \right)^2$$

After conjugation by $e^{i\frac{x_1 x_2}{2l_b^2}}$, we obtain the magnetic Laplacian in Landau gauge:

$$\begin{aligned} e^{i\frac{x_1x_2}{2l_b^2}} \left(i\hbar\nabla + \frac{b}{2}\begin{pmatrix} -x_2 \\ x_1 \end{pmatrix} \right)^2 e^{-i\frac{x_1x_2}{2l_b^2}} &= \left(i\hbar\nabla + \frac{b}{2}\begin{pmatrix} -x_2 \\ x_1 \end{pmatrix} + \frac{\hbar}{2l_b^2}\begin{pmatrix} x_2 \\ x_1 \end{pmatrix} \right)^2 = \left(i\hbar\nabla + b\begin{pmatrix} 0 \\ x_1 \end{pmatrix} \right)^2 \\ &= \left(i\hbar\partial_{x_1}\right)^2 + \left(i\hbar\partial_{x_2} + bx_1\right)^2 \end{aligned}$$

Using the unitary Fourier convention

$$\mathcal{F}u(\nu) \coloneqq \widehat{u}(\nu) \coloneqq \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} u(x) e^{-i\nu x} dx$$

one conjugation by the unitary Fourier transform in the variable x_2, we obtain

$$\mathcal{F}_{x_2} \left(\left(i\hbar\partial_{x_1}\right)^2 + \left(i\hbar\partial_{x_2} + bx_1\right)^2 \right) \mathcal{F}_{x_2}^{-1} = \left(i\hbar\partial_{x_1}\right)^2 + \left(-\hbar\nu_2 + bx_1\right)^2$$

Defining the translation by $r \in \mathbb{R}$ operator in the x_1 variable

$$T_r u(x_1) \coloneqq u(x_1 - r)$$

after one last conjugation,

$$T_{-l_b^2\nu_2} \left(\left(i\hbar\partial_{x_1}\right)^2 + \left(-\hbar\nu_2 + bx_1\right)^2 \right) T_{l_b^2\nu_2} = \left(i\hbar\partial_{x_1}\right)^2 + \left(bx_1\right)^2$$

we see that $\mathscr{L}_b$ is unitary equivalent to the 1D harmonic oscillator. And the 1D harmonic oscillator is well known to have a compact resolvent because of the following embedding.

Proposition III.2.1

$H^1(\mathbb{R}) \cap L^2(\mathbb{R}, |x|\, dx)$ is compactly embedded is $L^2(\mathbb{R})$.

Proof:

Let $(u_n)_{n\in\mathbb{N}}$ be bounded in $H^1(\mathbb{R}) \cap L^2(\mathbb{R}, |x|\, dx)$, in others words:

$$\|u_n\|_{L^2} + \||\bullet|\, u_n\|_{L^2} + \|u'\|_{L^2} \leqslant C \tag{III.2.1}$$

Since $(u_n)_{n\in\mathbb{N}}$ and $((1 + |\bullet|)u_n)_{n\in\mathbb{N}}$ are bounded in $L^2(\Omega)$, after extraction, we can assume

$$\begin{aligned} u_n &\underset{n\to\infty}{\rightharpoonup} u \in L^2(\mathbb{R}) \\ (1 + |\bullet|)u_n &\underset{n\to\infty}{\rightharpoonup} \widetilde{u} \in L^2(\mathbb{R}) \end{aligned}$$

One checks that $\widetilde{u} = (1 + |\bullet|)u$ by testing both limits against test functions in $C_c^\infty(\mathbb{R})$ which is dense in $L^2(\mathbb{R})$. Then, notice that

$$\begin{aligned}\widehat{u_n}(\nu) =& \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} u_n(x) e^{-i\nu x} dx = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} (1+|x|) u_n(x) \frac{e^{-i\nu x}}{1+|x|} dx \\ \underset{n\to\infty}{\to} & \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} (1+|x|) u(x) \frac{e^{-i\nu x}}{1+|x|} dx = \widehat{u}(\nu) \end{aligned} \tag{III.2.2}$$

because $\dfrac{e^{-i\nu\bullet}}{1+|\bullet|} \in L^2(\mathbb{R})$. Moreover,

$$|\widehat{u_n}(\nu)| \leqslant \frac{1}{\sqrt{2\pi}} \left\|(1+|\bullet|)u_n\right\|_{L^2} \left\| \frac{e^{-i\nu\bullet}}{1+|\bullet|} \right\|_{L^2} \leqslant C \left\| \frac{1}{1+|\bullet|} \right\|_{L^2} \tag{III.2.3}$$

Let $R > 0$, with Parseval's formula

$$\|u_n - u\|_{L^2}^2 = \|\widehat{u_n} - \widehat{u}\|^2 = \int_{|x|\leqslant R} |\widehat{u_n}(x) - \widehat{u}(x)|^2 \, dx + \int_{|x|>R} |\widehat{u_n}(x) - \widehat{u}(x)|^2 \, dx$$

The fist term goes to 0 because of (III.2.2), (III.2.3) and dominated convergence. The second term goes to 0 when $R \to \infty$ uniformly in n because $\widehat{u} \in L^2(\mathbb{R})$ and $(\widehat{u_n})_{n\in\mathbb{N}}$ is a tight sequence because from (III.2.1),

$$\|u_n\|_{H^1} = \int_{\mathbb{R}} \left(1+|\nu|^2\right) |\widehat{u_n}(\nu)|^2 \, du \leqslant C$$

We conclude that

$$\|u_n - u\|_{L^2} \underset{n\to\infty}{\to} 0$$

Therefore we known that $\mathscr{L}_b$ has a purely punctual spectrum and that $L^2(\Omega)$ is a Hilbertian direct sum of its eigenspaces.

III.2.2 Landau quantization

In this subsection, we set up the usual formalism for the description of the magnetic Laplacian in terms of annihilation and creation operators. More details about these operators and the properties of Landau levels can be found in [13].

Notation III.2.2

We denote by p_1, p_2 the coordinates of the magnetic momentum:

$$\mathscr{P}_{\hbar,b} =: \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} =: -\begin{pmatrix} i\hbar\partial_{x_1} + bA_1 \\ i\hbar\partial_{x_2} + bA_2 \end{pmatrix}$$

and define the annihilation and creation operators respectively as

$$a \coloneqq \frac{p_1 + ip_2}{\sqrt{2\hbar b}} \quad a^\dagger \coloneqq \frac{p_1 - ip_2}{\sqrt{2\hbar b}}$$

and the number of excitation operator $\mathcal{N} \coloneqq a^\dagger a$.

The following is standard:

Property III.2.3: *Commutation relations*

We have the commutation relations:

$$\begin{aligned} [p_1, p_2] &= i\hbar b \\ \left[a, a^\dagger\right] &= \mathrm{Id} \text{ (canonical commutation relation)} \end{aligned}$$

and

$$\begin{aligned} \mathscr{L}_b &= 2\hbar b \left(\mathcal{N} + \frac{\mathrm{Id}}{2}\right) \\ \mathrm{sp}(\mathcal{N}) &= \mathbb{N} \end{aligned}$$

Proof:

Start with

$$[p_1, p_2] = i\hbar\left(\partial_{x_1} A_2 - \partial_{x_2} A_1\right) = i\hbar b$$

so

$$\left[a, a^\dagger\right] = \frac{[p_1, -ip_2] + [ip_2, p_1]}{2\hbar b} = \frac{-2i\,[p_1, p_2]}{2\hbar b} = \mathrm{Id}$$

and we compute

$$\mathcal{N} = \frac{p_1^2 + p_2^2 + i[p_1, p_2]}{2\hbar b} = \frac{\mathscr{L}_b}{2\hbar b} - \frac{\mathrm{Id}}{2}$$

Then,

$$\left[\mathcal{N}, a^\dagger\right] = a^\dagger a a^\dagger - a^\dagger a^\dagger a = a^\dagger \left[a, a^\dagger\right] = a^\dagger$$

and similarly

$$[\mathcal{N}, a] = -a$$

Let $n \geqslant 0$ and assume u_n is a normalised eigenvector of eigenvalue n, then

$$\mathcal{N} a^\dagger u_n = a^\dagger a a^\dagger u_n = a^\dagger (a^\dagger a + \mathbb{1}) u_n = a^\dagger (\mathcal{N} + \mathbb{1}) u_n = (n+1) a^\dagger u_n$$

and compute the norm

$$\langle a^\dagger u_n | a^\dagger u_n \rangle = \langle u_n | a a^\dagger u_n \rangle = \langle u_n | (\mathcal{N} + \mathbb{1}) n \rangle = n + 1$$

Therefore

$$u_{n+1} := \frac{a^\dagger}{\sqrt{n+1}} u_n$$

is a normalised eigenvector of egeinvalue $n+1$. Similarly,

$$\mathcal{N} a u_n = (n-1) a u_n \tag{III.2.4}$$

so if $n \neq 0$

$$u_{n-1} := \frac{a}{\sqrt{n}} u_n$$

is also a normalised eigenvector of eigenvalue $n-1$. Iterating this process we must have $n \in \mathbb{N}$ otherwise we construct eigenvectors of negative eigenvalues which is impossible since $\mathcal{N}$ is positive. Thus $\mathrm{sp}(\mathcal{N}) \subset \mathbb{N}$.
Since the spectrum is non empty it contains one eigenvector of integer eigenvalue. And successive applications of a and $a^\dagger$ to this vector generates the whole spectrum $\mathbb{N}$. Moreover, if u_0 is an eigenvector of egeinvalue 0, au_0 satisfies

$$\mathcal{N} a u_0 = -a u_0$$

but au_0 cannot be an eigenvector so

$$a u_0 = 0 \tag{III.2.5}$$

Notation III.2.4: *Landau levels*

We define the n^{th} Landau level as the eigenspace associated to $n \in \mathbb{N}$:

$$\mathrm{nLL} := \{\psi \in \mathrm{Dom}\,(\mathscr{L}_b) \text{ such that } \mathcal{N}\psi = n\psi\}$$

The ground level, denoted LLL for *Lowest Landau Level* has energy $E_0 = \hbar b$.

Next, we state that the Landau levels are isomorphic.

Proposition III.2.5: *Mapping between Landau levels*

The Landau levels are isomorphic, and the operator $\dfrac{a^\dagger}{\sqrt{n+1}}$ is a unitary mapping from nLL to (n+1)LL of inverse $\dfrac{a}{\sqrt{n+1}}$.

Proof:

Let $\psi \in$ (n+1)LL,

$$\frac{a^\dagger}{\sqrt{n+1}} \cdot \frac{a}{\sqrt{n+1}} \psi = \psi$$

Because of (III.2.4),

$$\mathscr{L}_b a\psi = 2\hbar b\left(n+\frac{1}{2}\right)a\psi$$

so $\mathscr{L}_b a\psi \in L^2(\Omega)$ and $a\psi \in H^2(\Omega)$. Thus $a\psi \in$ nLL and $\dfrac{a^\dagger}{\sqrt{n+1}}$: nLL $\to$ (n+1)LL is surjective. Finally the mapping is unitary:

$$\forall \chi,\psi \in \text{nLL}, \frac{1}{n+1}\langle a^\dagger\chi|a^\dagger\psi\rangle = \frac{1}{n+1}\langle\chi|aa^\dagger\psi\rangle = \frac{1}{n+1}\langle\chi|(\mathcal{N}+1)\psi\rangle = \langle\chi|\psi\rangle$$

Therefore we can extend a basis of LLL using $a^\dagger$ to higher Landau levels.

III.2.3 Landau levels

It is know [13] that the Lowest Landau level consists of holomorphic functions pondered by a Gaussian factor.

Notation III.2.6

For the rest of the text, we will identify

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2$$

with the complex notation $\mathbf{x} := x_1 + ix_2$ and use

$$\partial_{\mathbf{x}} := \frac{\partial_{x_1} - i\partial_{x_2}}{2} \qquad \partial_{\overline{\mathbf{x}}} := \frac{\partial_{x_1} + i\partial_{x_2}}{2}$$

Proposition III.2.7: *Lowest Landau level*

Denote by $\mathcal{O}$ the set of holomorphic functions, then

$$\text{LLL} \subset \ker(a) \subset \mathcal{O}e^{-\frac{|x|^2}{4l_b^2}}$$

Proof:

Take $\psi \in$ LLL, we saw in (III.2.5) that $a\psi = 0$ so

$$\begin{aligned} 0 =&(p_1 + ip_2)\psi = -(i\hbar\partial_{x_1} + A_1 - \hbar\partial_{x_2} + ibA_2)\psi = -\left(2i\hbar\partial_{\overline{\mathbf{x}}} - \frac{b}{2}x_2 + i\frac{b}{2}x_1\right)\psi \\ =& -2i\hbar\left(\partial_{\overline{\mathbf{x}}} + \frac{\mathbf{x}}{4l_b^2}\right)\psi \end{aligned} \tag{III.2.6}$$

Hence $\exists f \in \mathcal{O}$, such that

$$\psi = fe^{-\frac{|x|^2}{4l_b^2}}$$

In classical mechanics, a charged particle in a 2D plane subjected to a transverse magnetic field follows a cyclotron orbit. In the previous section, we quantized the kinetic energy of the cyclotron orbit. To finish the quantization, one can try to quantize the wave functions as a function of the position of the cyclotron orbit, called the guiding center, in the 2D plane. This approach is justified by the fact that in what follows, we will need to reintroduce the potentials V and w, which depend on the position in space.

Notation III.2.8

We introduce the following position operators

$$\begin{aligned} r &:= \begin{pmatrix} r_1 \\ r_2 \end{pmatrix} := \frac{\mathscr{P}^{\perp}_{\hbar,b}}{b} = \frac{1}{b}\begin{pmatrix} -p_2 \\ p_1 \end{pmatrix} \\ R &:= X - r \end{aligned}$$

We can therefore define creation and annihilation operators by:

$$b = \frac{R_1 - iR_2}{\sqrt{2}l_b} \quad b^\dagger = \frac{R_1 + iR_2}{\sqrt{2}l_b}$$

r represents the position of particles in the center of orbit frame. The classical physics meaning of this definition is that, since particles trajectories' are orbits, their momentum is perpendicular to their relative position with respect to the center to the orbit. Electrons are describing clock orbits thus the momentum rotated of $\pi/2$ gives us r. Moreover, r is related to the quantization of the cyclotron pulsation of the orbit because

$$\begin{aligned} a &= \frac{p_1 + ip_2}{\sqrt{2\hbar b}} = \frac{r_2 - ir_1}{\sqrt{2}l_b} = \frac{-i\mathbf{r}}{\sqrt{2}l_b} \\ a^\dagger &= \frac{i\overline{\mathbf{r}}}{\sqrt{2}l_b} \end{aligned}$$

From the definition of r, the position R of the orbit center is indeed

$$X = R + r$$

and related to the second harmonic oscillator

$$\begin{aligned} b &= \frac{\overline{\mathbf{R}}}{\sqrt{2}l_b} \\ b^\dagger &= \frac{\mathbf{R}}{\sqrt{2}l_b} \end{aligned}$$

We have the following commutation relations:

Proposition III.2.9

r, R and X commutes with one another. Moreover

$$\begin{aligned} [r_1, r_2] &= il_b^2 \\ [R_1, R_2] &= -il_b^2 \\ \left[b, b^\dagger\right] &= \mathrm{Id} \end{aligned}$$

$$\left[a,b\right] = \left[a,b^{\dagger}\right] = \left[a^{\dagger},b\right] = \left[a^{\dagger},b^{\dagger}\right] = 0$$

Proof:

With direct computations

$$\begin{aligned}
\left[\widetilde{R}_1,\widetilde{R}_2\right] &= \frac{1}{b^2}\left[-p_2,p_1\right] = \frac{i\hbar b}{b^2} = il_b^2 \\
\left[x,\widetilde{R}_1\right] &= \frac{1}{b}\left[x,-p_2\right] = 0 = \left[y,\widetilde{R}_2\right] \\
\left[R_1,\widetilde{R}_1\right] &= \left[x,\widetilde{R}_1\right] = 0 = \left[R_2,\widetilde{R}_2\right] \\
\left[R_1,\widetilde{R}_2\right] &= \left[x-\widetilde{R}_1,\widetilde{R}_2\right] = \frac{1}{b}\left[x,p_1\right] - \left[\widetilde{R}_1,\widetilde{R}_2\right] = i\frac{\hbar}{b} - il_b^2 = 0 = \left[R_2,\widetilde{R}_1\right] \\
\left[R_1,R_2\right] &= \left[x-\widetilde{R}_1,y-\widetilde{R}_2\right] = \frac{1}{b}\left(\left[x,-p_1\right]+\left[p_2,y\right]\right) + \left[\widetilde{R}_1,\widetilde{R}_2\right] = -il_b^2 \\
\left[b,b^{\dagger}\right] &= \frac{i}{l_b^2}\left[R_1,R_2\right] = \mathrm{Id}
\end{aligned}$$

We therefore have two independent harmonic oscillators. By successively applying the creation operators $a^{\dagger}$ and $b^{\dagger}$ we obtain the desired basis. One can look at [34] [31] [28] [13] for some references of the following expressions.

Proposition III.2.10

In symmetric gauge (III.1.1), the family defined by

$$\varphi_{n,m} \coloneqq \frac{{a^{\dagger}}^n {b^{\dagger}}^m}{\sqrt{n!m!}}\varphi_{0,0}$$

with

$$\varphi_{0,0}(x) = \frac{1}{\sqrt{2\pi}l_b} e^{\frac{-|x|^2}{4l_b^2}}$$

is an orthonormal Hilbert basis of $L^2\left(\mathbb{R}^2\right)$. The full expression, is

$$\varphi_{n,m}(x) = \frac{\left(\left(-2il_b^2\partial_{\mathbf{x}} + i\overline{\mathbf{x}}\right)^n \mathbf{x}^m\right)}{\sqrt{\pi n!m!}\left(\sqrt{2}l_b\right)^{n+m+1}} e^{\frac{-|x|^2}{4l_b^2}} \tag{III.2.7}$$

Proof:

In complex coordinates, using the computation in (III.2.6),

$$\begin{aligned}
a &= \frac{-\left(2i\hbar\partial_{\overline{\mathbf{x}}} + ib\frac{\mathbf{x}}{2}\right)}{\sqrt{2\hbar b}} = -\sqrt{2}il_b\partial_{\overline{\mathbf{x}}} - i\frac{\mathbf{x}}{2\sqrt{2}l_b} \\
a^{\dagger} &= -\sqrt{2}il_b\partial_{\mathbf{x}} + i\frac{\overline{\mathbf{x}}}{2\sqrt{2}l_b}
\end{aligned}$$

$$
\begin{aligned}
b &= \frac{\overline{\mathbf{R}}}{\sqrt{2}l_b} = \frac{\overline{\mathbf{x}}}{\sqrt{2}l_b} - \frac{\overline{\mathbf{r}}}{\sqrt{2}l_b} = \frac{\overline{\mathbf{x}}}{\sqrt{2}l_b} + ia^\dagger = \sqrt{2}l_b\partial_{\mathbf{x}} + \frac{\overline{\mathbf{x}}}{2\sqrt{2}l_b} \\
b^\dagger &= -\sqrt{2}l_b\partial_{\overline{\mathbf{x}}} + \frac{\mathbf{x}}{2\sqrt{2}l_b}
\end{aligned}
$$

Let's determine the common ground state by solving

$$
bfe^{-\frac{|x|^2}{4l_b^2}} = 0
$$

We have

$$
\sqrt{2}l_b\partial_{\mathbf{x}}f + \left(-\frac{\mathbf{x}}{2\sqrt{2}l_b} + \frac{\mathbf{x}}{2\sqrt{2}l_b}\right)f = \sqrt{2}l_b\partial_{\mathbf{x}}f = 0
$$

and since f is holomorphic it is constant. Then

$$
\varphi_{0,0} \coloneqq C_{0,0}e^{-\frac{|x|^2}{4l_b^2}}
$$

Choose $C_{0,0}$ such that $\|\varphi_{0,0}\|_{L^2} = 1$:

$$
\|\varphi_{0,0}\|_{L^2}^2 = C_{0,0}^2\int_{\mathbb{R}^2} e^{-\frac{|x|^2}{2l_b^2}}\,dx = C_{0,0}^2 2\pi l_b^2\int_{\mathbb{R}_+} 2re^{-r^2}dr = C_{0,0}^2 2\pi l_b^2
$$

thus

$$
C_{0,0} \coloneqq \frac{1}{\sqrt{2\pi}l_b}
$$

Then the basis

$$
\varphi_{n,m}(x) \coloneqq \frac{C_{0,0}}{\sqrt{n!m!}}{a^\dagger}^n{b^\dagger}^m e^{-\frac{|x|^2}{4l_b^2}}
$$

is an orthonormal Hilbert basis of $L^2\left(\mathbb{R}^2\right)$. Since we are interested the creation and annihilation on the polynomial part of the wave-functions, define for an operator u

$$
\widetilde{u} = e^{\frac{|x|^2}{4l_b^2}}ue^{-\frac{|x|^2}{4l_b^2}}
$$

Therefore

$$
\varphi_{n,m} \coloneqq C_{0,0}P_{n,m}e^{-\frac{|x|^2}{4l_b^2}}
$$

with

$$
P_{n,m} \coloneqq \frac{\widetilde{a^\dagger}^n\widetilde{b^\dagger}^m}{\sqrt{n!m!}}\mathrm{Id}
$$

and

$$
\begin{aligned}
afe^{-\frac{|x|^2}{4l_b^2}} &= \left(-\sqrt{2}il_b\partial_{\overline{\mathbf{x}}}f\right)e^{-\frac{|x|^2}{4l_b^2}} \\
a^\dagger fe^{-\frac{|x|^2}{4l_b^2}} &= \left(-\sqrt{2}il_b\partial_{\mathbf{x}}f + \frac{i\overline{\mathbf{x}}}{\sqrt{2}l_b}f\right)e^{-\frac{|x|^2}{4l_b^2}} \\
bfe^{-\frac{|x|^2}{4l_b^2}} &= \left(\sqrt{2}l_b\partial_{\mathbf{x}}f\right)e^{-\frac{|x|^2}{4l_b^2}} \\
b^\dagger fe^{-\frac{|x|^2}{4l_b^2}} &= \left(-\sqrt{2}l_b\partial_{\overline{\mathbf{x}}}f + \frac{\mathbf{x}}{\sqrt{2}l_b}f\right)e^{-\frac{|x|^2}{4l_b^2}}
\end{aligned}
$$

so we get that

$$
\begin{aligned}
\widetilde{a} &= -\sqrt{2}il_b\partial_{\overline{\mathbf{x}}} \\
\widetilde{a^\dagger} &= -\sqrt{2}il_b\partial_{\mathbf{x}} + \frac{i\overline{\mathbf{x}}}{\sqrt{2}l_b} \\
\widetilde{b} &= \sqrt{2}l_b\partial_{\mathbf{x}} \\
\widetilde{b^\dagger} &= -\sqrt{2}l_b\partial_{\overline{\mathbf{x}}} + \frac{\mathbf{x}}{\sqrt{2}l_b}
\end{aligned}
$$

So we have

$$
\begin{aligned}
P_{n,m} =& \frac{1}{\sqrt{n!m!}}\left(-\sqrt{2}il_b\partial_{\mathbf{x}} + \frac{i\overline{\mathbf{x}}}{\sqrt{2}l_b}\right)^n\left(-\sqrt{2}l_b\partial_{\overline{\mathbf{x}}} + \frac{\mathbf{x}}{\sqrt{2}l_b}\right)^m \mathrm{Id} \\
=& \frac{\left(-2il_b^2\partial_{\mathbf{x}} + i\overline{\mathbf{x}}\right)^n\mathbf{x}^m}{\sqrt{n!m!}\left(\sqrt{2}l_b\right)^{n+m}}
\end{aligned}
$$

III.2.4 Coherent states

Definition III.2.11

Define the coherent state

$$\psi_{n,z} \coloneqq e^{\frac{\overline{\mathbf{z}}b^\dagger - \mathbf{z}b}{\sqrt{2}l_b}}\varphi_{n,0}$$

and the associated projector

$$\Pi_{n,z} \coloneqq |\psi_{n,z}\rangle\langle\psi_{n,z}|$$

the Landau level projector

$$\Pi_n \coloneqq \sum_{m\in\mathbb{N}} |\varphi_{n,m}\rangle\langle\varphi_{n,m}|$$

the localised projector

$$\Pi_z = \sum_{n\in\mathbb{N}} \Pi_{n,z}$$

Let $N_1 \leqslant N_2$, define the truncated projector

$$\Pi_{N_1:N_2} \coloneqq \sum_{n=N_1}^{N_2} \Pi_n$$
$$\Pi_{\leqslant N} \coloneqq \Pi_{0:N}$$

with similar definitions for $\Pi_{>N}$ and $\Pi_{\leqslant N,z}, \Pi_{>N,z}$.

We also have the following resolutions of identity [34],

Lemma III.2.12

$$\frac{1}{2\pi l_b^2} \int_{\mathbb{R}^2} \Pi_{n,z} dz = \Pi_n \tag{III.2.8}$$
$$\frac{1}{2\pi l_b^2} \sum_{n\in\mathbb{N}} \int_{\mathbb{R}^2} \Pi_{n,z} dz = \mathrm{Id} \tag{III.2.9}$$
$$\int_{\mathbb{R}^2} \Pi_z dz = \mathrm{Id}$$

Note that $\psi_{n,z}$ is localised around z since

$$\overline{\mathbf{R}}\psi_{n,z} = \overline{\mathbf{z}}\psi_{n,z}$$

We have the following expressions for the coherent states (see [34]):

Proposition III.2.13

$$\psi_{n,z}(x) = \frac{i^n}{\sqrt{2\pi n!}l_b} \left(\frac{\mathbf{x} - \mathbf{z}}{\sqrt{2}l_b}\right)^n e^{-\frac{|x-z|^2-2iz^\perp\cdot x}{4l_b^2}} \tag{III.2.10}$$
$$\Pi_{n,z}(x,y) = \frac{1}{2\pi n! l_b^2} \left(\frac{\left(\overline{\mathbf{x} - \mathbf{z}}\right)\left(\mathbf{y} - \mathbf{z}\right)}{2l_b^2}\right)^n e^{-\frac{|x-z|^2+|y-z|^2-2iz^\perp\cdot(x-y)}{4l_b^2}}$$
$$\Pi_z(x,y) = \frac{1}{2\pi l_b^2} e^{-\frac{|x-y|^2-2i\left(x^\perp\cdot y+2z^\perp\cdot(x-y)\right)}{4l_b^2}} \tag{III.2.11}$$

Proof:

By the properties of the coherent states

$$\psi_{n,z} = e^{-\frac{|z|^2}{4l_b^2}} \sum_{m\in\mathbb{N}} \frac{\overline{\mathbf{z}}^m}{\left(\sqrt{2}l_b\right)^m \sqrt{m!}} \varphi_{n,m}$$

so with (III.2.7),

$$\begin{aligned}\psi_{n,z}(x) =& e^{-\frac{|z|^2}{4l_b^2}}\sum_{m\in\mathbb{N}}\frac{\overline{\mathbf{z}}^m}{\left(\sqrt{2}l_b\right)^m\sqrt{m!}}\cdot\frac{\left(\left(-2il_b^2\partial_{\mathbf{x}}+i\overline{\mathbf{x}}\right)^n\mathbf{x}^m\right)}{\sqrt{\pi n!m!}\left(\sqrt{2}l_b\right)^{n+m+1}}e^{\frac{-|x|^2}{4l_b^2}}\\ =& e^{-\frac{|z|^2+|x|^2}{4l_b^2}}\frac{\left(-2il_b^2\partial_{\mathbf{x}}+i\overline{\mathbf{x}}\right)^n e^{\frac{\overline{\mathbf{z}}\mathbf{x}}{2l_b^2}}}{\sqrt{\pi n!}\left(\sqrt{2}l_b\right)^{n+1}}=\frac{i^n}{\sqrt{2\pi n!}l_b}\left(\frac{\overline{\mathbf{x}-\mathbf{z}}}{\sqrt{2}l_b}\right)^n e^{-\frac{|z|^2+|x|^2-2\overline{\mathbf{z}}\mathbf{x}}{4l_b^2}}\\ =& \frac{i^n}{\sqrt{2\pi n!}l_b}\left(\frac{\overline{\mathbf{x}-\mathbf{z}}}{\sqrt{2}l_b}\right)^n e^{-\frac{|x-z|^2-2i\mathrm{Im}[\overline{\mathbf{z}}\mathbf{x}]}{4l_b^2}}\end{aligned}$$

note that

$$\mathrm{Im}\left[\overline{\mathbf{z}}\mathbf{x}\right]=z_1x_2-z_2x_1=z^\perp\cdot x$$

so

$$\psi_{n,z}(x)=\frac{i^n}{\sqrt{2\pi n!}l_b}\left(\frac{\overline{\mathbf{x}-\mathbf{z}}}{\sqrt{2}l_b}\right)^n e^{-\frac{|x-z|^2-2iz^\perp\cdot x}{4l_b^2}}$$

We also obtain the kernel of the projector

$$\begin{aligned}\Pi_{n,z}(x,y) =& \psi_{n,z}(x)\overline{\psi_{n,z}(y)}=\frac{1}{2\pi n!l_b^2}\left(\frac{\left(\overline{\mathbf{x}-\mathbf{z}}\right)\left(\mathbf{y}-\mathbf{z}\right)}{2l_b^2}\right)^n e^{-\frac{|x-z|^2+|y-z|^2-2iz^\perp\cdot(x-y)}{4l_b^2}}\\ =& \frac{1}{2\pi n!l_b^2}\left(\frac{\left(\overline{\mathbf{x}-\mathbf{z}}\right)\left(\mathbf{y}-\mathbf{z}\right)}{2l_b^2}\right)^n e^{-\frac{2|z|^2+|x|^2+|y|^2-2(\overline{\mathbf{z}}\mathbf{x}+\mathbf{z}\overline{\mathbf{y}})}{4l_b^2}}\end{aligned}\tag{III.2.12}$$

Then compute

$$\begin{aligned}\Pi_z(x,y) =& \sum_{n\in\mathbb{N}}\frac{1}{2\pi n!l_b^2}\left(\frac{\left(\overline{\mathbf{x}-\mathbf{z}}\right)\left(\mathbf{y}-\mathbf{z}\right)}{2l_b^2}\right)^n e^{-\frac{2|z|^2+|x|^2+|y|^2-2(\overline{\mathbf{z}}\mathbf{x}+\mathbf{z}\overline{\mathbf{y}})}{4l_b^2}}\\ =& \frac{1}{2\pi l_b^2}e^{-\frac{2|z|^2+|x|^2+|y|^2-2(\overline{\mathbf{z}}\mathbf{x}+\mathbf{z}\overline{\mathbf{y}})-2\left(\overline{\mathbf{x}-\mathbf{z}}\right)(\mathbf{y}-\mathbf{z})}{4l_b^2}}\end{aligned}$$

and

$$\begin{aligned}& 2|z|^2+|x|^2+|y|^2-2(\overline{\mathbf{z}}\mathbf{x}+\mathbf{z}\overline{\mathbf{y}})-2\left(\overline{\mathbf{x}-\mathbf{z}}\right)(\mathbf{y}-\mathbf{z})\\ =& |x|^2+|y|^2-2\left(\overline{\mathbf{z}}\mathbf{x}+\mathbf{z}\overline{\mathbf{y}}+\overline{\mathbf{x}}\mathbf{y}-\overline{\mathbf{x}}\mathbf{z}-\overline{\mathbf{z}}\mathbf{y}\right)=|x-y|^2-2i\mathrm{Im}\left[\overline{\mathbf{x}}\mathbf{y}+2\overline{\mathbf{z}}(\mathbf{x}-\mathbf{y})\right]\end{aligned}$$

so

$$\Pi_z(x,y)=\frac{1}{2\pi l_b^2}e^{-\frac{|x-y|^2-2i\left(x^\perp\cdot y+2z^\perp\cdot(x-y)\right)}{4l_b^2}}$$

From (III.2.11) we immediately see that

$$\nabla_z^\perp\Pi_z(x,y)=\frac{y-x}{il_b^2}$$

or as operator identity

$$\nabla_z^{\perp}\Pi_z = \frac{1}{il_b^2}\left[\Pi_z, X\right] \tag{III.2.13}$$

This formula will play a key role in the computation of the spacial derivative of the density in Section III.5. The following lemma is an approximation of (III.2.13) for the truncated projector.

Lemma III.2.14

$$\nabla_z^{\perp}\Pi_{\leqslant N,z} = \frac{1}{il_b^2}\left[\Pi_{\leqslant N,z}, X\right] - \frac{\sqrt{N+1}}{\sqrt{2}l_b}\begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}\begin{pmatrix} \left|\psi_{N,z}\right\rangle\left\langle\psi_{N+1,z}\right| \\ \left|\psi_{N+1,z}\right\rangle\left\langle\psi_{N,z}\right| \end{pmatrix} \tag{III.2.14}$$

$$\begin{aligned}\nabla_z^{\perp}\Pi_{n,z} = {}& \frac{1}{il_b^2}\left[\Pi_{n,z}, X\right] - \frac{\sqrt{n+1}}{\sqrt{2}l_b}\begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}\begin{pmatrix} \left|\psi_{n,z}\right\rangle\left\langle\psi_{n+1,z}\right| \\ \left|\psi_{n+1,z}\right\rangle\left\langle\psi_{n,z}\right| \end{pmatrix} \\ & + \frac{\sqrt{n}}{\sqrt{2}l_b}\begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}\begin{pmatrix} \left|\psi_{n-1,z}\right\rangle\left\langle\psi_{n,z}\right| \\ \left|\psi_{n,z}\right\rangle\left\langle\psi_{n-1,z}\right| \end{pmatrix}\end{aligned} \tag{III.2.15}$$

and

$$\begin{aligned}&\nabla_z^{\perp}\otimes\nabla_z^{\perp}\Pi_{\leqslant N,z}(x,y) = \frac{-1}{l_b^4}(x-y)^{\otimes 2}\Pi_{\leqslant N,z}(x,y) + \frac{\mathrm{Id}_2}{l_b^2}\Pi_{N,z}(x,y) - \frac{\sqrt{N+1}}{\sqrt{2}l_b}\cdot \\ &\left(\nabla_z^{\perp}\otimes\begin{pmatrix}\psi_{N,z}(x)\overline{\psi_{N+1,z}(y)} \\ \psi_{N+1,z}(x)\overline{\psi_{N,z}(y)}\end{pmatrix}\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} + \left(\nabla_z^{\perp}\otimes\begin{pmatrix}\psi_{N,z}(x)\overline{\psi_{N+1,z}(y)} \\ \psi_{N+1,z}(x)\overline{\psi_{N,z}(y)}\end{pmatrix}\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}\right)^T\right) \\ &-\frac{N+1}{2l_b^2}\begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}\begin{pmatrix}\sqrt{\frac{N+2}{N+1}}\psi_{N,z}(x)\overline{\psi_{N+2,z}(y)} & \psi_{N+1,z}(x)\overline{\psi_{N+1,z}(y)} \\ \psi_{N+1,z}(x)\overline{\psi_{N+1,z}(y)} & \sqrt{\frac{N+2}{N+1}}\psi_{N+2,z}(x)\overline{\psi_{N,z}(y)}\end{pmatrix}\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}\end{aligned} \tag{III.2.16}$$

Proof:

Step 1: Proof of (III.2.14)

Using

$$\begin{aligned}\psi_{n,z}(x) &= \frac{-i}{\sqrt{n}}\frac{\mathbf{\overline{z\text{ - }x}}}{\sqrt{2}l_b}\psi_{n-1,z}(x) \\ \overline{\psi_{n,z}(y)} &= \frac{i}{\sqrt{n}}\frac{\mathbf{z\text{ - }y}}{\sqrt{2}l_b}\overline{\psi_{n-1,z}(y)}\end{aligned}$$

With (III.2.12) compute

$$\begin{aligned}\partial_{\mathbf{z}}\Pi_{n,z}(x,y) &= \frac{\mathbf{\overline{y\text{ - }z}}}{2l_b^2}\Pi_{n,z}(x,y) + \frac{\mathbf{\overline{z\text{ - }x}}}{2l_b^2}\Pi_{n-1,z}(x,y) \\ &= \frac{\mathbf{\overline{y\text{ - }x}}}{2l_b^2}\psi_{n,z}(x)\overline{\psi_{n,z}(y)} + \frac{\mathbf{\overline{x\text{ - }z}}}{2l_b^2}\psi_{n,z}(x)\overline{\psi_{n,z}(y)} + \frac{\mathbf{\overline{z\text{ - }x}}}{2l_b^2}\psi_{n-1,z}(x)\overline{\psi_{n-1,z}(y)} \\ &= \frac{\mathbf{\overline{y\text{ - }x}}}{2l_b^2}\psi_{n,z}(x)\overline{\psi_{n,z}(y)} - i\frac{\sqrt{n+1}}{\sqrt{2}l_b}\psi_{n+1,z}(x)\overline{\psi_{n,z}(y)} + i\frac{\sqrt{n}}{\sqrt{2}l_b}\psi_{n,z}(x)\overline{\psi_{n-1,z}(y)}\end{aligned}$$

and

$$\begin{aligned}
\partial_{\overline{\mathbf{z}}}\Pi_{n,z}(x,y) =& \frac{\mathbf{x} - \mathbf{z}}{2l_b^2}\Pi_{n,z}(x,y) + \frac{\mathbf{z} - \mathbf{y}}{2l_b^2}\Pi_{n-1,z}(x,y) \\
=& \frac{\mathbf{x} - \mathbf{y}}{2l_b^2}\psi_{n,z}(x)\overline{\psi_{n,z}(y)} + \frac{\mathbf{y} - \mathbf{z}}{2l_b^2}\psi_{n,z}(x)\overline{\psi_{n,z}(y)} + \frac{\mathbf{z} - \mathbf{y}}{2l_b^2}\psi_{n-1,z}(x)\overline{\psi_{n-1,z}(y)} \\
=& \frac{\mathbf{x} - \mathbf{y}}{2l_b^2}\psi_{n,z}(x)\overline{\psi_{n,z}(y)} + i\frac{\sqrt{n+1}}{\sqrt{2}l_b}\psi_{n,z}(x)\overline{\psi_{n+1,z}(y)} - i\frac{\sqrt{n}}{\sqrt{2}l_b}\psi_{n-1,z}(x)\overline{\psi_{n,z}(y)}
\end{aligned}$$

So

$$\begin{aligned}
\partial_{z_1}\Pi_{n,z}(x,y) = & \left(\partial_{\mathbf{z}} + \partial_{\overline{\mathbf{z}}}\right)\Pi_{n,z}(x,y) \\
=& i\frac{x_2 - y_2}{l_b^2}\Pi_{n,z}(x,y) + i\frac{\sqrt{n+1}}{\sqrt{2}l_b}\left(\psi_{n,z}(x)\overline{\psi_{n+1,z}(y)} - \psi_{n+1,z}(x)\overline{\psi_{n,z}(y)}\right) \\
& - i\frac{\sqrt{n}}{\sqrt{2}l_b}\left(\psi_{n-1,z}(x)\overline{\psi_{n,z}(y)} - \psi_{n,z}(x)\overline{\psi_{n-1,z}(y)}\right)
\end{aligned}$$

and

$$\begin{aligned}
\partial_{z_2}\Pi_{n,z}(x,y) =& i\left(\partial_{\mathbf{z}} - \partial_{\overline{\mathbf{z}}}\right)\Pi_{n,z}(x,y) \\
=& i\frac{y_1 - x_1}{l_b^2}\Pi_{n,z}(x,y) + \frac{\sqrt{n+1}}{\sqrt{2}l_b}\left(\psi_{n,z}(x)\overline{\psi_{n+1,z}(y)} + \psi_{n+1,z}(x)\overline{\psi_{n,z}(y)}\right) \\
& - \frac{\sqrt{n}}{\sqrt{2}l_b}\left(\psi_{n-1,z}(x)\overline{\psi_{n,z}(y)} + \psi_{n,z}(x)\overline{\psi_{n-1,z}(y)}\right)
\end{aligned}$$

We deduce that

$$\begin{aligned}
\nabla_z^{\perp}\Pi_{n,z}(x,y) =& i\frac{x-y}{l_b^2}\Pi_{n,z}(x,y) - \frac{\sqrt{n+1}}{\sqrt{2}l_b}\begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}\begin{pmatrix} \psi_{n,z}(x)\overline{\psi_{n+1,z}(y)} \\ \psi_{n+1,z}(x)\overline{\psi_{n,z}(y)} \end{pmatrix} \\
& + \frac{\sqrt{n}}{\sqrt{2}l_b}\begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}\begin{pmatrix} \psi_{n-1,z}(x)\overline{\psi_{n,z}(y)} \\ \psi_{n,z}(x)\overline{\psi_{n-1,z}(y)} \end{pmatrix}
\end{aligned}$$

After summation over n, we conclude that

$$\nabla_z^{\perp}\Pi_{\leqslant N,z}(x,y) = i\frac{x-y}{l_b^2}\Pi_{\leqslant N,z}(x,y) - \frac{\sqrt{N+1}}{\sqrt{2}l_b}\begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}\begin{pmatrix} \psi_{N,z}(x)\overline{\psi_{N+1,z}(y)} \\ \psi_{N+1,z}(x)\overline{\psi_{N,z}(y)} \end{pmatrix}$$

Step 2: Proof of (III.2.16)

Noticing that

$$\begin{aligned}
\psi_{N+1,z}(x) &= \frac{-i}{\sqrt{N+1}}\frac{\overline{\mathbf{z} - \mathbf{x}}}{\sqrt{2}l_b}\psi_{N,z}(x) \\
\overline{\psi_{N+1,z}(y)} &= \frac{i}{\sqrt{N+1}}\frac{\mathbf{z} - \mathbf{y}}{\sqrt{2}l_b}\overline{\psi_{N,z}(y)}
\end{aligned}$$

we get

$$\begin{pmatrix}\psi_{N,z}(x)\overline{\psi_{N+1,z}(y)}\\ \psi_{N+1,z}(x)\overline{\psi_{N,z}(y)}\end{pmatrix}=\frac{i}{\sqrt{N+1}\sqrt{2}l_b}\Pi_{N,z}(x,y)\begin{pmatrix}\textbf{z-y}\\ \overline{\textbf{x-z}}\end{pmatrix}$$

Thus inserting (III.2.14),

$$\begin{aligned}
&\nabla_z^\perp\otimes\begin{pmatrix}\psi_{N,z}(x)\overline{\psi_{N+1,z}(y)}\\ \psi_{N+1,z}(x)\overline{\psi_{N,z}(y)}\end{pmatrix}\\
=&\frac{i}{\sqrt{N+1}\sqrt{2}l_b}\left(\nabla_z^\perp\Pi_{N,z}(x,y)\right)\otimes\begin{pmatrix}\textbf{z-y}\\ \overline{\textbf{x-z}}\end{pmatrix}+\frac{i}{\sqrt{N+1}\sqrt{2}l_b}\Pi_{N,z}(x,y)\nabla_z^\perp\otimes\begin{pmatrix}\textbf{z-y}\\ \overline{\textbf{x-z}}\end{pmatrix}\\
=&i\frac{x-y}{l_b^2}\Pi_{\leqslant N,z}(x,y)\otimes\frac{i}{\sqrt{N+1}\sqrt{2}l_b}\begin{pmatrix}\textbf{z-y}\\ \overline{\textbf{x-z}}\end{pmatrix}\\
&-\frac{\sqrt{N+1}}{\sqrt{2}l_b}\begin{pmatrix}1&1\\-i&i\end{pmatrix}\begin{pmatrix}\psi_{N,z}(x)\overline{\psi_{N+1,z}(y)}\\ \psi_{N+1,z}(x)\overline{\psi_{N,z}(y)}\end{pmatrix}\otimes\frac{i}{\sqrt{N+1}\sqrt{2}l_b}\begin{pmatrix}\textbf{z-y}\\ \overline{\textbf{x-z}}\end{pmatrix}\\
&+\frac{i}{\sqrt{N+1}\sqrt{2}l_b}\Pi_{N,z}(x,y)\begin{pmatrix}-i&-i\\1&-1\end{pmatrix}\\
=&i\frac{x-y}{l_b^2}\otimes\begin{pmatrix}\psi_{N,z}(x)\overline{\psi_{N+1,z}(y)}\\ \psi_{N+1,z}(x)\overline{\psi_{N,z}(y)}\end{pmatrix}\\
&-\frac{\sqrt{N+1}}{\sqrt{2}l_b}\begin{pmatrix}1&1\\-i&i\end{pmatrix}\begin{pmatrix}\sqrt{\frac{N+2}{N+1}}\psi_{N,z}(x)\overline{\psi_{N+2,z}(y)}&\psi_{N+1,z}(x)\overline{\psi_{N+1,z}(y)}\\ \psi_{N+1,z}(x)\overline{\psi_{N+1,z}(y)}&\sqrt{\frac{N+2}{N+1}}\psi_{N+2,z}(x)\overline{\psi_{N,z}(y)}\end{pmatrix}\\
&+\frac{1}{\sqrt{N+1}\sqrt{2}l_b}\Pi_{N,z}(x,y)\begin{pmatrix}1&1\\i&-i\end{pmatrix}
\end{aligned}\tag{III.2.17}$$

so

$$\begin{aligned}
&\nabla_z^\perp\otimes\nabla_z^\perp\Pi_{\leqslant N,z}(x,y)\\
=&\nabla_z^\perp\Pi_{\leqslant N,z}(x,y)\otimes i\frac{x-y}{l_b^2}-\nabla_z^\perp\otimes\frac{\sqrt{N+1}}{\sqrt{2}l_b}\begin{pmatrix}1&1\\-i&i\end{pmatrix}\begin{pmatrix}\psi_{N,z}(x)\overline{\psi_{N+1,z}(y)}\\ \psi_{N+1,z}(x)\overline{\psi_{N,z}(y)}\end{pmatrix}\\
=&\frac{-1}{l_b^4}(x-y)^{\otimes 2}\Pi_{\leqslant N,z}(x,y)-\frac{\sqrt{N+1}}{\sqrt{2}l_b}\begin{pmatrix}1&1\\-i&i\end{pmatrix}\begin{pmatrix}\psi_{N,z}(x)\overline{\psi_{N+1,z}(y)}\\ \psi_{N+1,z}(x)\overline{\psi_{N,z}(y)}\end{pmatrix}\otimes i\frac{x-y}{l_b^2}\\
&-\frac{\sqrt{N+1}}{\sqrt{2}l_b}\nabla_z^\perp\otimes\begin{pmatrix}\psi_{N,z}(x)\overline{\psi_{N+1,z}(y)}\\ \psi_{N+1,z}(x)\overline{\psi_{N,z}(y)}\end{pmatrix}\begin{pmatrix}1&-i\\1&i\end{pmatrix}
\end{aligned}\tag{III.2.18}$$

From (III.2.17) we can isolate

$$\begin{aligned}
&\begin{pmatrix}\psi_{N,z}(x)\overline{\psi_{N+1,z}(y)}\\ \psi_{N+1,z}(x)\overline{\psi_{N,z}(y)}\end{pmatrix}\otimes i\frac{x-y}{l_b^2}\\
=&\left(\nabla_z^\perp\otimes\begin{pmatrix}\psi_{N,z}(x)\overline{\psi_{N+1,z}(y)}\\ \psi_{N+1,z}(x)\overline{\psi_{N,z}(y)}\end{pmatrix}\right)^T-\frac{1}{\sqrt{N+1}\sqrt{2}l_b}\Pi_{N,z}(x,y)\begin{pmatrix}1&i\\1&-i\end{pmatrix}
\end{aligned}$$

$$+\frac{\sqrt{N+1}}{\sqrt{2}l_b}\begin{pmatrix}\sqrt{\frac{N+2}{N+1}}\psi_{N,z}(x)\overline{\psi_{N+2,z}(y)} & \psi_{N+1,z}(x)\overline{\psi_{N+1,z}(y)}\\ \psi_{N+1,z}(x)\overline{\psi_{N+1,z}(y)} & \sqrt{\frac{N+2}{N+1}}\psi_{N+2,z}(x)\overline{\psi_{N,z}(y)}\end{pmatrix}\begin{pmatrix}1 & -i\\ 1 & i\end{pmatrix}$$

thus inserting this in (III.2.18),

$$\begin{aligned}
&\nabla_z^{\perp}\otimes\nabla_z^{\perp}\Pi_{\leqslant N,z}(x,y)\\
=&\frac{-1}{l_b^4}(x-y)^{\otimes 2}\Pi_{\leqslant N,z}(x,y)-\frac{\sqrt{N+1}}{\sqrt{2}l_b}\nabla_z^{\perp}\otimes\begin{pmatrix}\psi_{N,z}(x)\overline{\psi_{N+1,z}(y)}\\ \psi_{N+1,z}(x)\overline{\psi_{N,z}(y)}\end{pmatrix}\begin{pmatrix}1 & -i\\ 1 & i\end{pmatrix}\\
&-\frac{\sqrt{N+1}}{\sqrt{2}l_b}\begin{pmatrix}1 & 1\\ -i & i\end{pmatrix}\left(\nabla_z^{\perp}\otimes\begin{pmatrix}\psi_{N,z}(x)\overline{\psi_{N+1,z}(y)}\\ \psi_{N+1,z}(x)\overline{\psi_{N,z}(y)}\end{pmatrix}\right)^T+\frac{1}{2l_b^2}\begin{pmatrix}1 & 1\\ -i & i\end{pmatrix}\Pi_{N,z}(x,y)\begin{pmatrix}1 & i\\ 1 & -i\end{pmatrix}\\
&-\frac{N+1}{2l_b^2}\begin{pmatrix}1 & 1\\ -i & i\end{pmatrix}\begin{pmatrix}\sqrt{\frac{N+2}{N+1}}\psi_{N,z}(x)\overline{\psi_{N+2,z}(y)} & \psi_{N+1,z}(x)\overline{\psi_{N+1,z}(y)}\\ \psi_{N+1,z}(x)\overline{\psi_{N+1,z}(y)} & \sqrt{\frac{N+2}{N+1}}\psi_{N+2,z}(x)\overline{\psi_{N,z}(y)}\end{pmatrix}\begin{pmatrix}1 & -i\\ 1 & i\end{pmatrix}\\
=&\frac{-1}{l_b^4}(x-y)^{\otimes 2}\Pi_{\leqslant N,z}(x,y)+\frac{\mathrm{Id}_2}{l_b^2}\Pi_{N,z}(x,y)-\frac{\sqrt{N+1}}{\sqrt{2}l_b}\cdot\\
&\left(\nabla_z^{\perp}\otimes\begin{pmatrix}\psi_{N,z}(x)\overline{\psi_{N+1,z}(y)}\\ \psi_{N+1,z}(x)\overline{\psi_{N,z}(y)}\end{pmatrix}\begin{pmatrix}1 & -i\\ 1 & i\end{pmatrix}+\left(\nabla_z^{\perp}\otimes\begin{pmatrix}\psi_{N,z}(x)\overline{\psi_{N+1,z}(y)}\\ \psi_{N+1,z}(x)\overline{\psi_{N,z}(y)}\end{pmatrix}\begin{pmatrix}1 & -i\\ 1 & i\end{pmatrix}\right)^T\right)\\
&-\frac{N+1}{2l_b^2}\begin{pmatrix}1 & 1\\ -i & i\end{pmatrix}\begin{pmatrix}\sqrt{\frac{N+2}{N+1}}\psi_{N,z}(x)\overline{\psi_{N+2,z}(y)} & \psi_{N+1,z}(x)\overline{\psi_{N+1,z}(y)}\\ \psi_{N+1,z}(x)\overline{\psi_{N+1,z}(y)} & \sqrt{\frac{N+2}{N+1}}\psi_{N+2,z}(x)\overline{\psi_{N,z}(y)}\end{pmatrix}\begin{pmatrix}1 & -i\\ 1 & i\end{pmatrix}
\end{aligned}$$

III.3 Properties of the time evolution

The goal of this section is to expose some basic properties of the dynamics: conservation fermionic states (Proposition III.3.1), conservation of the energy (Proposition III.3.2) and of the kinetic energy (Proposition III.3.2).

III.3.1 Conservation of fermionic states

Proposition III.3.1

Assume $\gamma_b \in L^\infty\left(\mathbb{R}_+, \mathcal{L}^1\left(L^2\left(\mathbb{R}^2\right)\right)\right)$ solves

$$\partial_t \gamma_b(t) = \frac{1}{il_b^2}\left[\mathscr{L}_b + V + w \star \rho_{\gamma_b(t)}, \gamma_b\right]$$

and satisfies

$$\operatorname{Tr}\left[\gamma_b(0)\right] = 1, \quad 0 \leqslant \gamma_b(0) \leqslant 2\pi l_b^2$$

then $\forall t \in \mathbb{R}_+$,

$$\operatorname{Tr}\left[\gamma_b(t)\right] = 1, \quad 0 \leqslant \gamma_b(t) \leqslant 2\pi l_b^2$$

Proof:

First, the trace is preserved by the evolution because

$$\frac{d}{dt}\operatorname{Tr}\left[\gamma_b(t)\right] = \frac{1}{il_b^2}\operatorname{Tr}\left[\left[\mathscr{L}_b + V + W \star \rho_{\gamma_b(t)}, \gamma_b(t)\right]\right] = 0$$

Moreover

$$\gamma_b(t) = \mathcal{U}(t,0)\gamma_b(0)\mathcal{U}(0,t)$$

with

$$\mathcal{U}(t_2,t_1) \coloneqq e^{\frac{1}{il_b^2}\int_{t_1}^{t_2}\left(\mathscr{L}_b + V + w\star\rho_{\gamma_b(\tau)}\right)d\tau}$$

so

$$\begin{aligned}
&\gamma(0) \geqslant 0 \implies \gamma_b(t) = \mathcal{U}(t,0)\gamma_b(0)\mathcal{U}(t,0)^* \geqslant 0 \\
&\gamma(0) \leqslant 2\pi l_b^2 \implies 2\pi l_b^2 - \gamma_b(t) = \mathcal{U}(t,0)\left(2\pi l_b^2 - \gamma_b(0)\right)\mathcal{U}(t,0)^* \geqslant 0 \implies \gamma_b(t) \leqslant 2\pi l_b^2
\end{aligned}$$

III.3.2 Energy conservation

Recalling the notation (III.1.5), we state that the quantity $\operatorname{Tr}\left[\gamma_b H_b\right]$ is preserved through the dynamics.

Proposition III.3.2: *Energy conservation*

Assume $\gamma_b \in L^\infty\left(\mathbb{R}_+, \mathcal{L}^1\left(L^2\left(\mathbb{R}^2\right)\right)\right)$ solves

$$\partial_t \gamma_b(t) = \frac{1}{il_b^2}\left[\mathscr{L}_b + V + w \star \rho_{\gamma_b(t)}, \gamma_b\right]$$

then

$$\frac{d}{dt}\mathrm{Tr}\left[\gamma_b(t) H_b(t)\right] = 0$$

Proof:

First compute

$$\begin{aligned}\partial_t H_b(t)(z) =& \frac{1}{2}\frac{d}{dt}(w \star \rho_{\gamma_b(t)})(z) = \frac{1}{2}\frac{d}{dt}\mathrm{Tr}\left[\gamma_b(t) w(\bullet - z)\right]\\ =& \frac{1}{2il_b^2}\mathrm{Tr}\left[\left[\mathscr{L}_b + V + w \star \rho_{\gamma_b(t)}, \gamma_b(t)\right] w(\bullet - z)\right]\\ =& \frac{1}{2il_b^2}\mathrm{Tr}\left[\left[w(\bullet - z), \mathscr{L}_b + V + w \star \rho_{\gamma_b(t)}\right]\gamma_b(t)\right] = \frac{1}{2il_b^2}\mathrm{Tr}\left[\left[w(\bullet - z), \mathscr{L}_b\right]\gamma_b(t)\right]\end{aligned}$$

so

$$\begin{aligned}\mathrm{Tr}\left[\gamma_b(t)\partial_t H_b(t)\right] =& \int_{\mathbb{R}^2} \rho_{\gamma_b(t)}(z)\partial_t H_b(t)(z) dz = \frac{1}{2il_b^2}\int_{\mathbb{R}^2} \mathrm{Tr}\left[\left[\rho_{\gamma_b(t)}(z) w(\bullet - z), \mathscr{L}_b\right]\gamma_b(t)\right] dz\\ =& \frac{1}{2il_b^2}\mathrm{Tr}\left[\left[w \star \rho_{\gamma_b(t)}, \mathscr{L}_b\right]\gamma_b(t)\right] \qquad \text{(III.3.1)}\end{aligned}$$

we can compute

$$\begin{aligned}\mathrm{Tr}\left[\partial_t\gamma_b(t) H_b(t)\right] =& \frac{1}{il_b^2}\mathrm{Tr}\left[\left[\mathscr{L}_b + V + w \star \rho_{\gamma_b(t)}, \gamma_b(t)\right] H_b(t)\right]\\ =& \frac{1}{il_b^2}\mathrm{Tr}\left[\left[H_b(t), H_b(t) + \frac{1}{2}w \star \rho_{\gamma_b(t)}\right]\gamma_b(t)\right]\\ =& \frac{1}{2il_b^2}\mathrm{Tr}\left[\left[H_b(t), w \star \rho_{\gamma_b(t)}\right]\gamma_b(t)\right] = \frac{1}{2il_b^2}\mathrm{Tr}\left[\left[\mathscr{L}_b, w \star \rho_{\gamma_b(t)}\right]\gamma_b(t)\right] \quad \text{(III.3.2)}\end{aligned}$$

Then with (III.3.1) and (III.3.2) we conclude that

$$\frac{d}{dt}\mathrm{Tr}\left[\gamma_b(t) H_b(t)\right] = \mathrm{Tr}\left[\partial_t \gamma_b(t) H_b(t)\right] + \mathrm{Tr}\left[\gamma_b(t)\partial_t H_b(t)\right] = 0$$

Next we une the energy conservation to control the kinetic energy. In Section III.4 and Section III.5 we will estimate error terms using this control.

Proposition III.3.3: *Kinetic energy conservation*

Let $\gamma_b \in \mathcal{L}^1\left(L^2\left(\mathbb{R}^2\right)\right), W \in L^\infty(\mathbb{R}^2)$ and assume

$$\mathrm{Tr}\left[\gamma_b\right] = 1$$

then

$$\mathrm{Tr}\left[\gamma_b \mathscr{L}_b\right] \leqslant \left|\mathrm{Tr}\left[\gamma_b\left(\mathscr{L}_b + W\right)\right]\right| + \|W\|_{L^\infty}$$

Proof:

The kinetic energy is bounded by

$$\mathrm{Tr}\left[\gamma_b \mathscr{L}_b\right] = \mathrm{Tr}\left[\gamma_b\left(\mathscr{L}_b + W\right)\right] - \mathrm{Tr}\left[\gamma_b W\right] \leqslant \left|\mathrm{Tr}\left[\gamma_b\left(\mathscr{L}_b + W\right)\right]\right| + \|W\|_{L^\infty}$$

In Proposition III.5.4 we manage to control the dynamics of a semi-classical density using only the kinetic energy. But we also present some estimates in Proposition III.5.1 and Proposition III.5.3 that only work with higher moments of the kinetic energy. Conservation of higher moments of the kinetic energy is a physical assumption. The next proposition is an attempt at controlling the 2^{nd} moment, however conservation could not be obtained yet due to the presence of $1/l_b$ factor in the derivative.

Proposition III.3.4: *2^{nd} moment of the Kinetic energy bound*

Let $t \in \mathbb{R}_+, \gamma_b(t) \in \mathcal{L}^1\left(L^2\left(\mathbb{R}^2\right)\right), W \in W^{2,\infty}(\mathbb{R}^2)$ and assume

$$\begin{aligned} &\mathrm{Tr}\left[\gamma_b(t)\right] = 1, \quad 0 \leqslant \gamma_b(t) \\ &\partial_t \gamma_b(t) = \frac{1}{i l_b^2}\left[\mathscr{L}_b + W, \gamma_b(t)\right] \end{aligned}$$

then

$$\left|\frac{d}{dt}\mathrm{Tr}\left[\gamma_b(t)\mathscr{L}_b^2\right]\right| \leqslant C\left(\|\Delta W\|_{L^\infty} + \frac{\|\nabla W\|_{L^\infty}}{l_b}\right)\mathrm{Tr}\left[\gamma_b(t)\mathscr{L}_b^2\right]$$

Proof:

First we compute

$$\frac{d}{dt}\mathrm{Tr}\left[\gamma_b \mathscr{L}_b^2\right] = \frac{1}{i l_b^2}\mathrm{Tr}\left[\left[\mathscr{L}_b + W, \gamma_b\right]\mathscr{L}_b^2\right] = \frac{1}{i l_b^2}\mathrm{Tr}\left[\left[\mathscr{L}_b^2, W\right]\gamma_b\right] \tag{III.3.3}$$

With a direct computation

$$\left[\mathscr{P}_b, W\right] = \left[i\hbar\nabla + bA, W\right] = i\hbar\nabla W$$

so

$$\begin{aligned} \left[\mathscr{L}_b, W\right] = \left[\mathscr{P}_b^2, W\right] &= \mathscr{P}_b \cdot \left[\mathscr{P}_b, W\right] + \left[\mathscr{P}_b, W\right]\cdot\mathscr{P}_b = \mathscr{P}_b \cdot (i\hbar\nabla W) + i\hbar\nabla W \cdot \mathscr{P}_b \\ &= i\hbar\left[\mathscr{P}_b, \nabla W\right] + 2i\hbar\nabla W\cdot\mathscr{P}_b = -\hbar^2\Delta W + 2i\hbar\nabla W\cdot\mathscr{P}_b \end{aligned}$$

and

$$\begin{aligned}\operatorname{Tr}\left[\left[\mathscr{L}_b^2, W\right]\gamma_b\right] =& \operatorname{Tr}\left[\mathscr{L}_b\left[\mathscr{L}_b, W\right]\gamma_b\right] + \operatorname{Tr}\left[\left[\mathscr{L}_b, W\right]\mathscr{L}_b\gamma_b\right] \\ =& -\hbar^2\left(\operatorname{Tr}\left[\mathscr{L}_b\Delta W\gamma_b\right] + \operatorname{Tr}\left[\Delta W\mathscr{L}_b\gamma_b\right]\right) \\ &+ 2i\hbar\left(\operatorname{Tr}\left[\mathscr{L}_b\nabla W\cdot\mathscr{P}_b\gamma_b\right] + \operatorname{Tr}\left[\nabla W\cdot\mathscr{P}_b\mathscr{L}_b\gamma_b\right]\right)\end{aligned}$$

But

$$[\mathscr{P}_b, \nabla W] \coloneqq \mathscr{P}_b\cdot\nabla W - \nabla W\cdot\mathscr{P}_b = i\hbar\Delta W$$

so

$$\begin{aligned}\operatorname{Tr}\left[\left[\mathscr{L}_b^2, W\right]\gamma_b\right] =& \hbar^2\left(\operatorname{Tr}\left[\Delta W\mathscr{L}_b\gamma_b\right] - \operatorname{Tr}\left[\mathscr{L}_b\Delta W\gamma_b\right]\right) \\ &+ 2i\hbar\left(\operatorname{Tr}\left[\mathscr{L}_b\nabla W\cdot\mathscr{P}_b\gamma_b\right] + \operatorname{Tr}\left[\mathscr{P}_b\cdot\nabla W\mathscr{L}_b\gamma_b\right]\right)\end{aligned} \tag{III.3.4}$$

As operators

$$\Delta W\mathscr{L}_b - \mathscr{L}_b\Delta W \leqslant \epsilon\left|\Delta W\right|^2 + \frac{1}{\epsilon}\mathscr{L}_b^2 \leqslant \epsilon\left\|\Delta W\right\|_{L^\infty}^2 + \frac{1}{\epsilon}\mathscr{L}_b^2 \tag{III.3.5}$$

and with Cauchy-schwarz inequality

$$\begin{aligned}&\left|\operatorname{Tr}\left[\mathscr{L}_b\nabla W\cdot\mathscr{P}_b\gamma_b\right] + \operatorname{Tr}\left[\mathscr{P}_b\cdot\nabla W\mathscr{L}_b\gamma_b\right]\right| \\ =&\left|\operatorname{Tr}\left[\gamma_b^{\frac12}\mathscr{L}_b\nabla W\cdot\mathscr{P}_b\gamma_b^{\frac12}\right] + \operatorname{Tr}\left[\gamma_b^{\frac12}\mathscr{P}_b\cdot\nabla W\mathscr{L}_b\gamma_b^{\frac12}\right]\right| \\ \leqslant&\sqrt{\operatorname{Tr}\left[\gamma_b^{\frac12}\mathscr{L}_b\nabla W\cdot\left(\gamma_b^{\frac12}\mathscr{L}_b\nabla W\right)^*\right]\operatorname{Tr}\left[\mathscr{P}_b\gamma_b^{\frac12}\cdot\left(\mathscr{P}_b\gamma_b^{\frac12}\right)^*\right]} \\ &+\sqrt{\operatorname{Tr}\left[\gamma_b^{\frac12}\mathscr{P}_b\cdot\left(\gamma_b^{\frac12}\mathscr{P}_b\right)^*\right]\operatorname{Tr}\left[\nabla W\mathscr{L}_b\gamma_b^{\frac12}\cdot\left(\nabla W\mathscr{L}_b\gamma_b^{\frac12}\right)^*\right]} \\ \leqslant& 2\sqrt{\operatorname{Tr}\left[\gamma_b\mathscr{L}_b\left|\nabla W\right|^2\mathscr{L}_b\right]\operatorname{Tr}\left[\gamma_b\mathscr{L}_b\right]} \leqslant \left\|\nabla W\right\|_{L^\infty}\sqrt{\operatorname{Tr}\left[\gamma_b\mathscr{L}_b^2\right]\operatorname{Tr}\left[\gamma_b\mathscr{L}_b\right]}\end{aligned} \tag{III.3.6}$$

Inserting (III.3.5)with $\epsilon \coloneqq \dfrac{\hbar b}{\left\|\Delta W\right\|_{L^\infty}}$ and (III.3.6) in (III.3.4) we obtain

$$\left|\operatorname{Tr}\left[\left[\mathscr{L}_b^2, W\right]\gamma_b\right]\right| \leqslant \hbar^2\left\|\Delta W\right\|_{L^\infty}\left(\hbar b + \frac{1}{\hbar b}\operatorname{Tr}\left[\gamma_b\mathscr{L}_b^2\right]\right) + 2\hbar\left\|\nabla W\right\|_{L^\infty}\sqrt{\operatorname{Tr}\left[\gamma_b\mathscr{L}_b^2\right]\operatorname{Tr}\left[\gamma_b\mathscr{L}_b\right]}$$

Recalling that $\mathscr{L}_b \geqslant \hbar b = o(1)$ we get

$$\begin{aligned}\left|\operatorname{Tr}\left[\left[\mathscr{L}_b^2, W\right]\gamma_b\right]\right| &\leqslant \left(\frac{\hbar^2}{\hbar b}\left\|\Delta W\right\|_{L^\infty} + \frac{2\hbar}{\sqrt{\hbar b}}\left\|\nabla W\right\|_{L^\infty}\right)\operatorname{Tr}\left[\gamma_b\mathscr{L}_b^2\right] \\ &= \left(l_b^2\left\|\Delta W\right\|_{L^\infty} + 2l_b\left\|\nabla W\right\|_{L^\infty}\right)\operatorname{Tr}\left[\gamma_b\mathscr{L}_b^2\right]\end{aligned}$$

We conclude with (III.3.3).

III.4 Semi-classical approximations

In this section we introduce the semi-classical density (III.4.2) and the truncated semi-classical density (III.4.3) that only takes into account Landau level under a certain threshold. We then state in Proposition III.4.2 and Proposition III.4.3 that these semi-classical densities are good approximations of the physical density.

Definition III.4.1

Let $\gamma \in \mathcal{L}^1\left(L^2\left(\mathbb{R}^2\right)\right)$. We define the semi-classical phase space density associated to γ, so-called Husimi function by

$$m_\gamma(n,z) \coloneqq \frac{1}{2\pi l_b^2}\left\langle \psi_{n,z} | \gamma \psi_{n,z} \right\rangle = \frac{1}{2\pi l_b^2}\operatorname{Tr}\left[\Pi_{n,z}\gamma\right] \tag{III.4.1}$$

and the semi-classical density

$$\rho_\gamma^{sc}(z) \coloneqq \sum_{n\in\mathbb{N}} m_\gamma(n,z) = \frac{1}{2\pi l_b^2}\operatorname{Tr}\left[\Pi_z\gamma\right] \tag{III.4.2}$$

Let N be a sequence such that

$$N \underset{b\to\infty}{\to} \infty$$

and the truncated semi-classical density be

$$\rho_\gamma^{sc,\leqslant N}(z) \coloneqq \frac{1}{2\pi l_b^2}\operatorname{Tr}\left[\gamma \Pi_{\leqslant N,z}\right] \tag{III.4.3}$$

The parameter N will represent the number of Landau levels we take into account for the semi-classical approximations in Section III.4 and Section III.5.

III.4.1 Semi-classical density

Proposition III.4.2: *Convergence of the semi-classical density*

Let $k > 1, \gamma_b \in \mathcal{L}^1\left(L^2\left(\mathbb{R}^2\right)\right)$ and assume

$$\operatorname{Tr}\left[\gamma_b\right] = 1, 0 \leqslant \gamma_b \leqslant 2\pi l_b^2$$

then $\forall \varphi \in C_c^\infty\left(\mathbb{R}^2\right)$,

$$\left|\int_{\mathbb{R}^2} \varphi\left(\rho_{\gamma_b} - \rho_{\gamma_b}^{sc}\right)\right| \leqslant C\sqrt{|\operatorname{supp}(\varphi)|}\,\|\varphi\|_{W^{1,\infty}}\sqrt{\operatorname{Tr}\left[\gamma_b \mathscr{L}_b^k\right]}\cdot\begin{cases} l_b^{2k-3} & \text{if } k < 2 \\ l_b\sqrt{\ln\left(\frac{1}{l_b^2}\right)} & \text{if } k = 2 \\ l_b & \text{if } k > 2\end{cases}$$

Notice that this estimate requires $k > \dfrac{3}{2}$ for the error on the right side to be be small.

Proof of Proposition III.4.2:

Let $\varphi \in C_c^\infty(\mathbb{R}^2)$,

$$\begin{aligned}\int_{\mathbb{R}^2} \varphi\left(\rho_{\gamma_b} - \rho_{\gamma_b}^{sc}\right) =& \mathrm{Tr}\left[\varphi\gamma_b\right] - \frac{1}{2\pi l_b^2}\int_{\mathbb{R}^2} \varphi(z)\mathrm{Tr}\left[\Pi_z\gamma_b\right]dz = \frac{1}{2\pi l_b^2}\int_{\mathbb{R}^2}\mathrm{Tr}\left[(\varphi - \varphi(z))\,\Pi_z\gamma_b\right]dz \\ =& \frac{1}{2\pi l_b^2}\int_{\mathbb{R}^2}\sum_{n\in\mathbb{N}}\mathrm{Tr}\left[(\varphi-\varphi(z))\,\Pi_{n,z}^2\gamma_b\right]dz \qquad \text{(III.4.4)}\end{aligned}$$

so with Young's inequality

$$\begin{aligned}\left|\int_{\mathbb{R}^2}\varphi\left(\rho_{\gamma_b} - \rho_{\gamma_b}^{sc}\right)\right| \leqslant& \frac{1}{4\pi l_b^2}\int_{\mathbb{R}^2}\sum_{n\in\mathbb{N}}\left(\frac{1}{\epsilon_n}\mathrm{Tr}\left[|\varphi-\varphi(z)|^2\,\Pi_{n,z}\right] + \epsilon_n\mathrm{Tr}\left[\Pi_{n,z}\gamma_b^2\right]\right)dz \\ =& \sum_{n\in\mathbb{N}}\frac{1}{4\pi\epsilon_n l_b^2}\int_{\mathbb{R}^2}\mathrm{Tr}\left[|\varphi-\varphi(z)|^2\,\Pi_{n,z}\right]dz + \sum_{n\in\mathbb{N}}\frac{\epsilon_n}{2}\mathrm{Tr}\left[\Pi_n\gamma_b^2\right] \\ \leqslant& \sum_{n\in\mathbb{N}}\frac{1}{4\pi\epsilon_n l_b^2}\int_{\mathbb{R}^2}\mathrm{Tr}\left[|\varphi-\varphi(z)|^2\,\Pi_{n,z}\right]dz + \sum_{n\in\mathbb{N}}\pi\epsilon_n l_b^2\mathrm{Tr}\left[\Pi_n\gamma_b\right] \qquad \text{(III.4.5)}\end{aligned}$$

and with the change of variable $x \coloneqq \dfrac{x-z}{\sqrt{2}l_b}$

$$\begin{aligned}\mathrm{Tr}\left[|\varphi-\varphi(z)|^2\,\Pi_{n,z}\right] =& \int_{\mathbb{R}^2}|\varphi(x)-\varphi(z)|^2\,\Pi_{n,z}(x,x)dx \\ =& \frac{1}{2\pi n! l_b^2}\int_{\mathbb{R}^2}|\varphi(x)-\varphi(z)|^2\left|\frac{x-z}{\sqrt{2}l_b}\right|^{2n}e^{-\frac{|x-z|^2}{2l_b^2}}dx \\ =& \frac{1}{\pi n!}\int_{\mathbb{R}^2}\left|\varphi\left(z+\sqrt{2}l_b x\right)-\varphi(z)\right|^2|x|^{2n}e^{-|x|^2}dx \\ \leqslant& \frac{1}{\pi n!}2l_b^2\,\|\nabla\varphi\|_\infty^2\int_{\mathbb{R}^2}|x|^{2(n+1)}e^{-|x|^2}dx = 2(n+1)l_b^2\,\|\nabla\varphi\|_\infty^2 \qquad \text{(III.4.6)}\end{aligned}$$

We can also write

$$\mathrm{Tr}\left[|\varphi-\varphi(z)|^2\,\Pi_{n,z}\right] \leqslant 4\,\|\varphi\|_{L^\infty}^2\,\mathrm{Tr}\left[\Pi_{n,z}\right] = 4\,\|\varphi\|_{L^\infty}^2 \qquad \text{(III.4.7)}$$

Since

$$\Pi_{n,z}(x,x) = \Pi_{n,x}(z,z)$$

with (III.4.6) and (III.4.7), by seeing that the integrand is symmetric in x and z we get

$$\int_{\mathbb{R}^2}\mathrm{Tr}\left[|\varphi-\varphi(z)|^2\,\Pi_{n,z}\right]dz = \int_{(\mathrm{supp}(\varphi)\times\mathbb{R}^2)\cup(\mathbb{R}^2\times\mathrm{supp}(\varphi))}|\varphi(x)-\varphi(z)|^2\,\Pi_{n,z}(x,x)dxdz$$

$$
\begin{aligned}
&\leqslant 2 \int\limits_{\mathbb{R}^2\times \mathrm{supp}(\varphi)} |\varphi(x)-\varphi(z)|^2\,\Pi_{n,z}(x,x)dxdz\\
&=2 \int\limits_{\mathrm{supp}(\varphi)} \mathrm{Tr}\left[|\varphi-\varphi(z)|^2\,\Pi_{n,z}\right]dz\\
&\leqslant C\,|\mathrm{supp}(\varphi)|\,\|\varphi\|^2_{W^{1,\infty}}\min\left((n+1)l_b^2,1\right)
\end{aligned}
\tag{III.4.8}
$$

Next we identify the k^{nd} moment of the kinetic energy in the sum:

$$
\sum_{n\in\mathbb{N}}(n+1)^k\mathrm{Tr}\left[\gamma_b\Pi_n\right]=\sum_{n\in\mathbb{N}}\left(\frac{n+1}{2\hbar b\left(n+\dfrac{1}{2}\right)}\right)^k\mathrm{Tr}\left[\gamma_b\Pi_n\mathscr{L}_b^k\right]\leqslant\sum_{n\in\mathbb{N}}\mathrm{Tr}\left[\gamma_b\Pi_n\mathscr{L}_b^k\right]=\mathrm{Tr}\left[\gamma_b\mathscr{L}_b^k\right]
\tag{III.4.9}
$$

Inserting (III.4.8) in (III.4.5), taking $\epsilon_n \coloneqq \sqrt{|\mathrm{supp}(\varphi)|}\,\|\varphi\|_{W^{1,\infty}}\,\epsilon(n+1)^k$ and using (III.4.9) we get

$$
\begin{aligned}
&\left|\int\limits_{\mathbb{R}^2}\varphi\left(\rho_{\gamma_b}-\rho^{sc}_{\gamma_b}\right)\right|\\
&\leqslant C\left(|\mathrm{supp}(\varphi)|\,\|\varphi\|^2_{W^{1,\infty}}\sum_{n\in\mathbb{N}}\frac{1}{\epsilon_n}\min\left(n+1,\frac{1}{l_b^2}\right)+l_b^2\sum_{n\in\mathbb{N}}\epsilon_n\mathrm{Tr}\left[\Pi_n\gamma_b\right]\right)\\
&\leqslant C\sqrt{|\mathrm{supp}(\varphi)|}\,\|\varphi\|_{W^{1,\infty}}\left(\sum_{n\in\mathbb{N}}\frac{1}{\epsilon(n+1)^k}\min\left(n+1,\frac{1}{l_b^2}\right)+\epsilon l_b^2\mathrm{Tr}\left[\gamma_b\mathscr{L}_b^k\right]\right)\\
&\leqslant C\sqrt{|\mathrm{supp}(\varphi)|}\,\|\varphi\|_{W^{1,\infty}}\left(\sum_{(n+1)l_b^2\leqslant 1}\frac{1}{\epsilon(n+1)^{k-1}}+\frac{1}{\epsilon l_b^2}\sum_{(n+1)l_b^2>1}\frac{1}{(n+1)^k}+\epsilon l_b^2\mathrm{Tr}\left[\gamma_b\mathscr{L}_b^k\right]\right)
\end{aligned}
\tag{III.4.10}
$$

Introducing the notation

$$
p_\lambda(x)\coloneqq x^{-\lambda}\mathbb{1}_{\lambda<0}+\ln(x)\mathbb{1}_{\lambda=0}+\mathbb{1}_{\lambda>0}
\tag{III.4.11}
$$

we have the asymptotics

$$
\sum_{(n+1)l_b^2>1}\frac{1}{(n+1)^k}=\sum_{n>\frac{1}{l_b^2}}^{\infty}\frac{1}{n^k}=\mathcal{O}\left(\frac{1}{\left(\frac{1}{l_b^2}\right)^{k-1}}\right)=\mathcal{O}\left(l_b^{2(k-1)}\right)
\tag{III.4.12}
$$

$$
\sum_{(n+1)l_b^2\leqslant 1}\frac{1}{(n+1)^{k-1}}=\sum_{1\leqslant n\leqslant\frac{1}{l_b^2}}\frac{1}{n^{k-1}}=\begin{cases}\mathcal{O}\left(l_b^{2(k-2)}\right) & \text{if } k<2\\ \mathcal{O}\left(\ln\left(l_b^{-2}\right)\right) & \text{if } k=2\\ \mathcal{O}(1) & \text{if } k>2\end{cases}=\mathcal{O}\left(p_{k-2}\left(l_b^{-2}\right)\right)
\tag{III.4.13}
$$

We notice that

$$l_b^{2(k-2)} \leqslant p_{k-2}\left(l_b^{-2}\right) \tag{III.4.14}$$

Inserting (III.4.12), (III.4.13), (III.4.14) in (III.4.10) and taking $\epsilon := \frac{1}{l_b}\sqrt{\frac{p_{k-2}\left(l_b^{-2}\right)}{\mathrm{Tr}\left[\gamma_b \mathscr{L}_b^k\right]}}$ we obtain

$$\begin{aligned}
\left|\int_{\mathbb{R}^2} \varphi\left(\rho_{\gamma_b} - \rho_{\gamma_b}^{sc}\right)\right| \leqslant & C\sqrt{|\mathrm{supp}(\varphi)|}\,\|\varphi\|_{W^{1,\infty}} \left(\frac{p_{k-2}\left(l_b^{-2}\right)}{\epsilon} + \frac{l_b^{2(k-2)}}{\epsilon} + \epsilon l_b^2 \mathrm{Tr}\left[\gamma_b \mathscr{L}_b^k\right]\right) \\
\leqslant & C\sqrt{|\mathrm{supp}(\varphi)|}\,\|\varphi\|_{W^{1,\infty}} \left(\frac{p_{k-2}\left(l_b^{-2}\right)}{\epsilon} + \epsilon l_b^2 \mathrm{Tr}\left[\gamma_b \mathscr{L}_b^k\right]\right) \\
= & C\sqrt{|\mathrm{supp}(\varphi)|}\,\|\varphi\|_{W^{1,\infty}} \sqrt{\mathrm{Tr}\left[\gamma_b \mathscr{L}_b^k\right]}\, l_b \sqrt{p_{k-2}\left(l_b^{-2}\right)}
\end{aligned}$$

Note that this choice of ϵ is possible because

$$\sum_{n\in\mathbb{N}} (n+1)^k \mathrm{Tr}\left[\Pi_n \gamma_b\right] \geqslant \sum_{n\in\mathbb{N}} \mathrm{Tr}\left[\Pi_n \gamma_b\right] = \mathrm{Tr}\left[\gamma_b\right] = 1 \tag{III.4.15}$$

The second error term in (III.4.10) requires $k > 1$ for the convergence of the series. For the estimate to only work with the kinetic energy we need to consider the truncated semi-classical density (III.4.3) instead of (III.4.2). This is the goal of the next section.

III.4.2 Truncated semi-classical density

Proposition III.4.3: *Convergence of the truncated semi-classical density*

Let $k > 1, \gamma_b \in \mathcal{L}^1\left(L^2\left(\mathbb{R}^2\right)\right)$ and assume

$$\mathrm{Tr}\left[\gamma_b\right] = 1, 0 \leqslant \gamma_b \leqslant 2\pi l_b^2$$

then $\forall \varphi \in C_c^\infty\left(\mathbb{R}^2\right)$,

$$\begin{aligned}
\left|\int_{\mathbb{R}^2} \varphi\left(\rho_{\gamma_b} - \rho_{\gamma_b}^{sc,\leqslant N}\right)\right| \leqslant & \|\varphi\|_{L^\infty} N^{-\frac{k}{2}} \sqrt{\mathrm{Tr}\left[\gamma_b \Pi_{>N} \mathscr{L}_b^k\right]} \\
& + C\sqrt{|\mathrm{supp}(\varphi)|}\,\|\varphi\|_{W^{1,\infty}} \sqrt{\mathrm{Tr}\left[\gamma_b \mathscr{L}_b^k\right]} \cdot \begin{cases} N^{1-\frac{k}{2}} l_b & \text{if } k < 2 \\ \sqrt{\ln(N)} l_b & \text{if } k = 2 \\ l_b & \text{if } k > 2 \end{cases}
\end{aligned} \tag{III.4.16}$$

This time, the right-hand side term is small for $k = 1$ if we choose N such that $\sqrt{N} l_b \ll 1$. This constraint has a physical meaning. Indeed from the expression of the coherent state

(III.2.10) we can infer that the characteristic length for particles in nLL is $\sqrt{n}l_b$. Hence $\rho^{sc,\leqslant N}_{\gamma_b}$ taken with $\sqrt{N}l_b \ll 1$ is the semi-classical density of well localised particles, and good localisation of the coherent states is key in semi-classical approximations. The first term in the right-hand side of III.4.16 corresponds to high Landau levels. The estimates will use the information about the moments of the kinetic energy to control the number of particles inside high Landau levels.

Proof of Proposition III.4.3:

Similarly as in the proof of Proposition III.4.2, instead of (III.4.4) we have

$$\int_{\mathbb{R}^2} \varphi\left(\rho_{\gamma_b} - \rho^{sc,\leqslant N}_{\gamma_b}\right) = \frac{1}{2\pi l_b^2}\sum_{n=0}^{N}\int_{\mathbb{R}^2} \mathrm{Tr}\left[\left(\varphi - \varphi(z)\right)\Pi_{n,z}\gamma_b\right] dz + \sum_{n>N}\mathrm{Tr}\left[\varphi\Pi_n\gamma_b\right] \tag{III.4.17}$$

Using

$$\sum_{n\leqslant N}(n+1)^k\mathrm{Tr}\left[\gamma_b\Pi_n\right] \leqslant \sum_{n\in\mathbb{N}}(n+1)^k\mathrm{Tr}\left[\gamma_b\Pi_n\right]$$

we obtain instead of (III.4.10):

$$\begin{aligned}&\frac{1}{2\pi l_b^2}\sum_{n=0}^{N}\int_{\mathbb{R}^2}\left|\mathrm{Tr}\left[\left(\varphi - \varphi(z)\right)\Pi_{n,z}\gamma_b\right]\right| dz\\ \leqslant& C\sqrt{|\mathrm{supp}(\varphi)|}\,\|\varphi\|_{W^{1,\infty}}\left(\sum_{n=0}^{N}\frac{1}{\epsilon(n+1)^{k-1}} + \epsilon l_b^2\mathrm{Tr}\left[\gamma_b\mathscr{L}_b^k\right]\right)\end{aligned}$$

Recalling the notation in (III.4.11),

$$\sum_{n=0}^{N}\frac{1}{(n+1)^{k-1}} = \begin{cases}\mathcal{O}\left(N^{2-k}\right) & \text{if } k<2\\ \mathcal{O}\left(\ln(N)\right) & \text{if } k=2\\ \mathcal{O}\left(1\right) & \text{if } k>2\end{cases} = \mathcal{O}\left(p_{k-2}(N)\right)$$

With $\epsilon := \dfrac{1}{l_b}\sqrt{\dfrac{p_{k-2}(N)}{\mathrm{Tr}\left[\gamma_b\mathscr{L}_b^k\right]}}$ we have

$$\frac{1}{2\pi l_b^2}\sum_{n=0}^{N}\int_{\mathbb{R}^2}\left|\mathrm{Tr}\left[\left(\varphi - \varphi(z)\right)\Pi_{n,z}\gamma_b\right]\right| dz \leqslant C\sqrt{|\mathrm{supp}(\varphi)|}\,\|\varphi\|_{W^{1,\infty}}\sqrt{\mathrm{Tr}\left[\gamma_b\mathscr{L}_b^k\right]}l_b\sqrt{p_{k-2}(N)} \tag{III.4.18}$$

Moreover

$$\left|\sum_{n>\mathbb{N}}\mathrm{Tr}\left[\varphi\Pi_n\gamma_b\right]\right| = \left|\mathrm{Tr}\left[\varphi\Pi_{>N}\gamma_b\right]\right| \leqslant \sqrt{\mathrm{Tr}\left[\gamma_b\left|\varphi\right|^2\right]\mathrm{Tr}\left[\gamma_b\Pi_{>N}\right]} \leqslant \|\varphi\|_{L^\infty}\sqrt{\mathrm{Tr}\left[\gamma_b\Pi_{>N}\right]}$$

and

$$\mathrm{Tr}\left[\gamma_b\Pi_{>N}\right] = \sum_{n>N}\mathrm{Tr}\left[\gamma_b\Pi_n\right] \leqslant \sum_{n>N}\frac{n^k}{N^k}\mathrm{Tr}\left[\gamma_b\Pi_n\right] \leqslant \frac{1}{N^k}\sum_{n>N}\mathrm{Tr}\left[\gamma_b\Pi_n\mathscr{L}_b^k\right]$$

$$=\frac{1}{N^k}\mathrm{Tr}\left[\gamma_b\Pi_{>N}\mathscr{L}_b^k\right]$$

so

$$\left|\sum_{n>N}\mathrm{Tr}\left[\varphi\Pi_n\gamma_b\right]\right|\leqslant\|\varphi\|_{L^\infty}N^{-\frac{k}{2}}\sqrt{\mathrm{Tr}\left[\gamma_b\Pi_{>N}\mathscr{L}_b^k\right]}\tag{III.4.19}$$

We conclude by assembling (III.4.17), (III.4.18) and (III.4.17).

III.5 Gyrokinetic dynamics for semi-classical densities

The goal of this section if to prove that the semi-classical densities almost satisfies (III.1.4).

III.5.1 dynamics of the semi-classical density

Proposition III.5.1: *Gyrokinetic equation for the semi-classical density*

Let $t \in \mathbb{R}_+, k > 3, \gamma_b(t) \in \mathcal{L}^1\left(L^2\left(\mathbb{R}^2\right)\right), W \in W^{3,\infty}(\mathbb{R}^2)$ and assume

$$\operatorname{Tr}\left[\gamma_b(t)\right] = 1, 0 \leqslant \gamma_b(t) \leqslant 2\pi l_b^2 \tag{III.5.1}$$

$$\partial_t \gamma_b(t) = \frac{1}{i l_b^2}\left[\mathscr{L}_b + W, \gamma_b(t)\right] \tag{III.5.2}$$

then $\forall \varphi \in C_c^\infty\left(\mathbb{R}^2\right)$,

$$\begin{aligned}&\left|\int_{\mathbb{R}^2} \varphi(z)\left(\partial_t \rho_{\gamma_b}^{sc}(t,z) + \nabla^\perp W(z)\cdot \nabla \rho_{\gamma_b}^{sc}(t,z)\right) dz\right| \\ \leqslant &C\sqrt{|\mathrm{supp}(\varphi)|}\, \|\varphi\|_{W^{1,\infty}} \|W\|_{W^{3,\infty}} \sqrt{\operatorname{Tr}\left[\gamma_b(t)\mathscr{L}_b^k\right]}\cdot \begin{cases} l_b^{2k-7} & \text{if } k < 4 \\ l_b\sqrt{\ln\left(\frac{1}{l_b^2}\right)} & \text{if } k = 4 \\ l_b & \text{if } k > 4\end{cases}\end{aligned} \tag{III.5.3}$$

This estimate requires $k > 7/2$ for the right-hand side to be small. We will later in the text improve this constraint, but we start by Proposition III.5.1 in order to introduce our method of proof. We begin the proof with a technical Lemma.

Lemma III.5.2

Let $\alpha \in \mathbb{R}_+$, then

$$I_n(\alpha) \coloneqq \frac{1}{\pi n!}\int_{\mathbb{R}^2} |x|^{\alpha+2n} e^{-|x|^2} dx \underset{n\to\infty}{\sim} n^{\frac{\alpha}{2}}$$

and $\exists C > 0$ such that

$$\forall n \in \mathbb{N}, I_n(\alpha) \leqslant C(n+1)^{\frac{\alpha}{2}}$$

Proof:

With polar coordinates and the change of variable $x \coloneqq x^2$

$$\frac{1}{\pi}\int_{\mathbb{R}^2} |x|^\alpha e^{-|x|^2}dx = 2\int_{\mathbb{R}_+} x^{\alpha+1}e^{-x^2}dx = \int_{\mathbb{R}_+} x^{\frac{\alpha}{2}}e^{-u}du = \Gamma\left(\frac{\alpha}{2}+1\right)$$

where Γ is the Euler Gamma function,

$$\Gamma(z) \coloneqq \int_{\mathbb{R}_+} t^{z-1}e^{-t}dt$$

We have the following equivalent for the Euler Gamma function (as a direct consequence of the Stirling formula)

$$\frac{\Gamma(n+x)}{\Gamma(n)} \underset{x\to\infty}{\sim} n^x \tag{III.5.4}$$

so

$$I_n(\alpha) = \frac{1}{n!}\Gamma\left(n+\frac{\alpha}{2}+1\right) = \frac{\Gamma\left(n+\frac{\alpha}{2}+1\right)}{\Gamma(n+1)}$$

and by (III.5.4),

$$I_n(\alpha) \underset{n\to\infty}{\sim} (n+1)^{\frac{\alpha}{2}} \underset{n\to\infty}{\sim} n^{\frac{\alpha}{2}}$$

Proof of Proposition III.5.1:

Step 1: a direct computation

We start from (III.4.2) and (III.5.2):

$$\begin{aligned}\partial_t\rho^{sc}_{\gamma_b}(z) =& \frac{1}{2\pi l_b^2}\mathrm{Tr}\left[\partial_t\gamma_b\Pi_z\right] = \frac{1}{2i\pi l_b^4}\mathrm{Tr}\left[\left[\mathscr{L}_b+W,\gamma_b\right]\Pi_z\right] = \frac{1}{2i\pi l_b^4}\mathrm{Tr}\left[\gamma_b\left[\Pi_z,\mathscr{L}_b+W\right]\right]\\ =& \frac{1}{2i\pi l_b^4}\mathrm{Tr}\left[\gamma_b\left[\Pi_z,W\right]\right]\end{aligned} \tag{III.5.5}$$

On the other hand, by (III.2.13) and (III.4.2),

$$\begin{aligned}\nabla^\perp W(z)\cdot\nabla\rho^{sc}_{\gamma_b}(z) =& \frac{1}{2\pi l_b^2}\nabla^\perp W(z)\cdot\mathrm{Tr}\left[\gamma_b\nabla_z\Pi_z\right] = -\frac{1}{2\pi l_b^2}\nabla W(z)\cdot\mathrm{Tr}\left[\gamma_b\nabla_z^\perp\Pi_z\right]\\ =& -\frac{1}{2i\pi l_b^4}\mathrm{Tr}\left[\gamma_b\left[\Pi_z,X\cdot\nabla W(z)\right]\right]\end{aligned} \tag{III.5.6}$$

Putting together (III.5.5) and (III.5.6),

$$\partial_t\rho^{sc}_{\gamma_b}(z) + \nabla^\perp W(z)\cdot\nabla\rho^{sc}_{\gamma_b}(z) = \frac{1}{2i\pi l_b^4}\mathrm{Tr}\left[\gamma_b\left[\Pi_z,W-X\cdot\nabla W(z)\right]\right] \tag{III.5.7}$$

Step 2: Taylor expansion of the potential

We expand W to the second order. Define

$$\mathcal{V}_z(x) \coloneqq W(x) - dW(z)(x-z) - \frac{1}{2}d^2W(z)(x-z, x-z) \tag{III.5.8}$$

so that

$$\begin{aligned} W(y) - W(x) - (y-x)\cdot\nabla W(z) =& \mathcal{V}_z(y) - \mathcal{V}_z(x) \\ &+ \frac{1}{2}d^2W(z)(y-z, y-z) - \frac{1}{2}d^2W(z)(x-z, x-z) \end{aligned}$$

Where the notation d^nW stand for the n^{th} differential of W, meaning that $d^nW(z)$ is a n-linear form. Notice that

$$\begin{aligned} d^2W(z)(y-z, y-z) =& d^2W(z)(y-x, y-z) + d^2W(z)(x-z, y-z) \\ =& d^2W(z)(y-x, y-z) + d^2W(z)(x-z, y-x) \\ &+ d^2W(z)(x-z, x-z) \end{aligned}$$

so

$$\begin{aligned} &[\Pi_z, W - X\cdot\nabla W(z)](x,y) = \Pi_z(x,y)\left(W(y) - W(x) - (y-x)\cdot\nabla W(z)\right) \\ =& \Pi_z(x,y)\left(\mathcal{V}_z(y) - \mathcal{V}_z(x) + \frac{1}{2}d^2W(z)(y-x, y-z) + \frac{1}{2}d^2W(z)(x-z, y-x)\right) \\ =& [\Pi_z, \mathcal{V}_z](x,y) + \frac{1}{2}d^2W(z)(\Pi_z(x,y)(y-x), y-z) + \frac{1}{2}d^2W(z)(x-z, \Pi_z(x,y)(y-x)) \end{aligned} \tag{III.5.9}$$

Thus with (III.2.13), we have the operator identity

$$\begin{aligned} [\Pi_z, W - X\cdot\nabla W(z)] =& [\Pi_z, \mathcal{V}_z] + \frac{1}{2}d^2W(z)([\Pi_z, X], X-z) + \frac{1}{2}d^2W(z)(X-z, [\Pi_z, X]) \\ =& [\Pi_z, \mathcal{V}_z] + \frac{il_b^2}{2}d^2W(z)(\nabla_z^\perp\Pi_z, X-z) + \frac{il_b^2}{2}d^2W(z)(X-z, \nabla_z^\perp\Pi_z) \end{aligned} \tag{III.5.10}$$

Defining the error terms

$$\mathcal{E}_{1,b}(z) \coloneqq \frac{1}{2i\pi l_b^4}\mathrm{Tr}\left[\gamma_b[\Pi_z, \mathcal{V}_z]\right] \tag{III.5.11}$$

$$\mathcal{E}_{2,b}(z) \coloneqq \frac{1}{4\pi l_b^2}\mathrm{Tr}\left[\gamma_b d^2W(z)(\nabla_z^\perp\Pi_z, X-z)\right] \tag{III.5.12}$$

$$\widetilde{\mathcal{E}}_{2,b}(z) \coloneqq \frac{1}{4\pi l_b^2}\mathrm{Tr}\left[\gamma_b d^2W(z)(X-z, \nabla_z^\perp\Pi_z)\right] \tag{III.5.13}$$

and inserting (III.5.10) (III.5.11) (III.5.12) (III.5.13) in (III.5.7) we obtain

$$\partial_t\rho_{\gamma_b}^{sc}(z) + \nabla^\perp W(z)\cdot\nabla\rho_{\gamma_b}^{sc}(z) = \mathcal{E}_{1,b}(z) + \mathcal{E}_{2,b}(z) + \widetilde{\mathcal{E}}_{2,b}(z) \tag{III.5.14}$$

Step 3: Estimate of $\mathcal{E}_{1,b}(z)$

From (III.5.11),

$$\mathcal{E}_{1,b}(z) = \frac{1}{2i\pi l_b^4} \sum_{n\in\mathbb{N}} \mathrm{Tr}\left[\gamma_b\left[\Pi_{n,z}, \mathcal{V}_z\right]\right] \tag{III.5.15}$$

We introduce another projector:

$$\mathrm{Tr}\left[\gamma_b\left[\Pi_{n,z}, \mathcal{V}_z\right]\right] = \mathrm{Tr}\left[\gamma_b \Pi_{n,z}\left[\Pi_{n,z}, \mathcal{V}_z\right]\right] + \mathrm{Tr}\left[\Pi_{n,z}\gamma_b\left[\Pi_{n,z}, \mathcal{V}_z\right]\right]$$

Since $\Pi_{n,z}, \mathcal{V}_z, \gamma_b$ are self-adjoint, with Young's inequality

$$\begin{aligned}
\left|\mathrm{Tr}\left[\gamma_b\left[\Pi_{n,z}, \mathcal{V}_z\right]\right]\right| \leqslant & 2\sqrt{\mathrm{Tr}\left[\left[\Pi_{n,z}, \mathcal{V}_z\right]\left[\Pi_{n,z}, \mathcal{V}_z\right]^*\right]\mathrm{Tr}\left[\gamma_b\Pi_{n,z}\left(\gamma_b\Pi_{n,z}\right)^*\right]} \\
= & 2\left\|\left[\Pi_{n,z}, \mathcal{V}_z\right]\right\|_{\mathcal{L}^2}\sqrt{\mathrm{Tr}\left[\gamma_b^2\Pi_{n,z}\right]} \leqslant \epsilon_n \left\|\left[\Pi_{n,z}, \mathcal{V}_z\right]\right\|_{\mathcal{L}^2}^2 + \frac{1}{\epsilon_n}\mathrm{Tr}\left[\gamma_b^2\Pi_{n,z}\right] \\
\leqslant & \epsilon_n \left\|\left[\Pi_{n,z}, \mathcal{V}_z\right]\right\|_{\mathcal{L}^2}^2 + \frac{2\pi l_b^2}{\epsilon_n}\mathrm{Tr}\left[\gamma_b\Pi_{n,z}\right]
\end{aligned} \tag{III.5.16}$$

We estimate the first term with the changes of variables $x \coloneqq \dfrac{x-z}{\sqrt{2}l_b}$, $y \coloneqq \dfrac{y-z}{\sqrt{2}l_b}$

$$\begin{aligned}
\left\|\left[\Pi_{n,z}, \mathcal{V}_z\right]\right\|_{\mathcal{L}^2}^2 = & \iint\limits_{\mathbb{R}^2\times\mathbb{R}^2} \left|\left[\Pi_{n,z}, \mathcal{V}_z\right](x,y)\right|^2 dxdy = \iint\limits_{\mathbb{R}^2\times\mathbb{R}^2} \left(\mathcal{V}_z(x)-\mathcal{V}_z(y)\right)^2 \left|\Pi_{n,z}(x,y)\right|^2 dxdy \\
= & \frac{1}{(2\pi n! l_b^2)^2}\int\limits_{\mathbb{R}^2\times\mathbb{R}^2} \left(\mathcal{V}_z(x)-\mathcal{V}_z(y)\right)^2 \left|\frac{(x-z)(y-z)}{2l_b^2}\right|^{2n} e^{-\frac{|x-z|^2+|y-z|^2}{2l_b^2}} dxdy \\
= & \frac{1}{(\pi n!)^2}\int\limits_{\mathbb{R}^2\times\mathbb{R}^2} \left(\mathcal{V}_z(z+\sqrt{2}l_b x)-\mathcal{V}_z(z+\sqrt{2}l_b y)\right)^2 |xy|^{2n} e^{-|x|^2-|y|^2} dxdy
\end{aligned} \tag{III.5.17}$$

Let $2 \leqslant \alpha \leqslant 3$, using the expansion (III.5.8),

$$\left|\mathcal{V}_z(z+\sqrt{2}l_b x)\right| = \left|W(z+\sqrt{2}l_b x) - \sqrt{2}l_b dW_z(x) - l_b^2 d^2W_z(x,x)\right| \leqslant C\left\|W\right\|_{W^{\alpha,\infty}}|l_b x|^\alpha$$

and similarly with y instead of x. With Lemma III.5.2,

$$\begin{aligned}
\left\|\left[\Pi_{n,z}, \mathcal{V}_z\right]\right\|_{\mathcal{L}^2}^2 \leqslant & C\frac{\|W\|_{W^{\alpha,\infty}}^2}{n!^2}\int\limits_{\mathbb{R}^2\times\mathbb{R}^2} l_b^{2\alpha}\left(|x|^{2\alpha}+|y|^{2\alpha}\right)|xy|^{2n} e^{-|x|^2-|y|^2} dxdy \\
\leqslant & C\left\|W\right\|_{W^{3,\infty}}^2 \left((n+1)l_b^2\right)^\alpha
\end{aligned}$$

Inserting this along with (III.5.16) in (III.5.15) we get

$$\left|\mathcal{E}_{1,b}(z)\right| \leqslant \frac{C}{l_b^4}\left(\|W\|_{W^{3,\infty}}^2\sum_{n\in\mathbb{N}}\epsilon_n \min_{2\leqslant\alpha\leqslant3}\left(\left((n+1)l_b^2\right)^\alpha\right) + \sum_{n\in\mathbb{N}}\frac{l_b^2}{\epsilon_n}\mathrm{Tr}\left[\gamma_b\Pi_{n,z}\right]\right) \tag{III.5.18}$$

Integrating against $\varphi \in C_c^\infty(\mathbb{R}^2)$, choosing

$$\epsilon_n := \frac{\epsilon}{\sqrt{|\mathrm{supp}(\varphi)|}\,\|W\|_{W^{3,\infty}}(n+1)^k}$$

and using (III.4.9) (III.2.8) (III.5.1), we obtain

$$\begin{aligned}
&\left|\int_{\mathbb{R}^2}\varphi(z)\mathcal{E}_{1,b}(z)dz\right| \\
\leqslant &\frac{C}{l_b^4}\left(\|W\|_{W^{3,\infty}}^2\|\varphi\|_{L^1}\sum_{n\in\mathbb{N}}\epsilon_n\min_{2\leqslant\alpha\leqslant3}\left(\left((n+1)l_b^2\right)^\alpha\right)+\|\varphi\|_{L^\infty}\sum_{n\in\mathbb{N}}\frac{2\pi l_b^4}{\epsilon_n}\mathrm{Tr}\left[\gamma_b\Pi_n\right]\right) \\
\leqslant &C\|\varphi\|_{L^\infty}\left(|\mathrm{supp}(\varphi)|\,\|W\|_{W^{3,\infty}}^2\sum_{n\in\mathbb{N}}\frac{\epsilon_n}{l_b^4}\min_{2\leqslant\alpha\leqslant3}\left(\left((n+1)l_b^2\right)^\alpha\right)+\sum_{n\in\mathbb{N}}\frac{1}{\epsilon_n}\mathrm{Tr}\left[\gamma_b\Pi_n\right]\right) \\
\leqslant &C\sqrt{|\mathrm{supp}(\varphi)|}\,\|\varphi\|_{L^\infty}\|W\|_{W^{3,\infty}}\left(\epsilon\sum_{n\in\mathbb{N}}\frac{\min_{2\leqslant\alpha\leqslant3}\left(\left((n+1)l_b^2\right)^\alpha\right)}{l_b^4(n+1)^k}+\frac{1}{\epsilon}\sum_{n\in\mathbb{N}}(n+1)^k\mathrm{Tr}\left[\gamma_b\Pi_n\right]\right) \\
\leqslant &C\sqrt{|\mathrm{supp}(\varphi)|}\,\|\varphi\|_{L^\infty}\|W\|_{W^{3,\infty}}\left(\epsilon\sum_{n\in\mathbb{N}}\frac{\min_{2\leqslant\alpha\leqslant3}\left(\left((n+1)l_b^2\right)^\alpha\right)}{l_b^4(n+1)^k}+\frac{1}{\epsilon}\mathrm{Tr}\left[\gamma_b\mathscr{L}_b^k\right]\right)
\end{aligned}$$

We split the sum in two part depending on weather $(n+1)l_b^2 \leqslant 1$ or not, for the first part take $\alpha = 3$, and for the second part $\alpha = 2$:

$$\begin{aligned}
\left|\int_{\mathbb{R}^2}\varphi(z)\mathcal{E}_{1,b}(z)dz\right| \leqslant &C\sqrt{|\mathrm{supp}(\varphi)|}\,\|\varphi\|_{L^\infty}\|W\|_{W^{3,\infty}}\left(\epsilon l_b^2\sum_{(n+1)l_b^2\leqslant1}\frac{1}{(n+1)^{k-3}}\right. \\
&\left.+\epsilon\sum_{(n+1)l_b^2>1}\frac{1}{(n+1)^{k-2}}+\frac{1}{\epsilon}\mathrm{Tr}\left[\gamma_b\mathscr{L}_b^k\right]\right)
\end{aligned} \tag{III.5.19}$$

With the asymptotics (III.4.12) and (III.4.13),

$$\left|\int_{\mathbb{R}^2}\varphi(z)\mathcal{E}_{1,b}(z)dz\right| \leqslant C\sqrt{|\mathrm{supp}(\varphi)|}\,\|\varphi\|_{L^\infty}\|W\|_{W^{3,\infty}}\left(\epsilon l_b^2 p_{k-4}\left(l_b^{-2}\right)+\epsilon l_b^{2(k-3)}+\frac{1}{\epsilon}\mathrm{Tr}\left[\gamma_b\mathscr{L}_b^k\right]\right)$$

Similarly as in (III.4.14),

$$l_b^{2(k-4)} \leqslant p_{k-4}$$

so

$$l_b^{2(k-3)} \leqslant l_b^2 p_{k-4}$$

Hence choosing $\epsilon \coloneqq \frac{1}{l_b}\sqrt{\frac{\operatorname{Tr}\left[\gamma_b \mathscr{L}_b^k\right]}{p_{k-4}\left(l_b^{-2}\right)}}$ we conclude that

$$\begin{aligned}\left|\int_{\mathbb{R}^2} \varphi(z)\mathcal{E}_{1,b}(z)dz\right| \leqslant & C\sqrt{|\operatorname{supp}(\varphi)|}\, \|\varphi\|_{L^\infty} \|W\|_{W^{3,\infty}} \left(\epsilon l_b^2 p_{k-4}\left(l_b^{-2}\right) + \frac{1}{\epsilon}\operatorname{Tr}\left[\gamma_b \mathscr{L}_b^k\right]\right) \\ = & C\sqrt{|\operatorname{supp}(\varphi)|}\, \|\varphi\|_{L^\infty} \|W\|_{W^{3,\infty}} \sqrt{\operatorname{Tr}\left[\gamma_b \mathscr{L}_b^k\right]} l_b \sqrt{p_{k-4}\left(l_b^{-2}\right)} \qquad \text{(III.5.20)}\end{aligned}$$

Step 4: Estimate of $\mathcal{E}_{2,b}(z)$

We start from (III.5.12):

$$\mathcal{E}_{2,b}(z) \coloneqq \frac{1}{4\pi l_b^2} \iint_{\mathbb{R}^2\times\mathbb{R}^2} \gamma_b(x,y)\nabla_z^\perp \Pi_z(y,x)\cdot d^2W(z)(x-z)dxdy$$

Let $\odot$ denote the tensor contraction defined for $n, m \geqslant k$ by

$$u_1 \otimes \cdots \otimes u_n \odot^k v_1 \otimes \cdots \otimes v_m \coloneqq \langle u_n|v_1\rangle \ldots \langle u_{n-k+1}|v_k\rangle\, u_1 \otimes \cdots \otimes u_{n-k} \otimes v_{k+1} \otimes \cdots \otimes v_m$$

Identifying $d^nW(z)$ with the associated rank n tensor, we notice that

$$\begin{aligned}\nabla_z^\perp \cdot d^2W(z)(x-z) =& \left(\nabla \odot \nabla \otimes \nabla W(z)\right)(x-z) + \nabla \otimes \nabla W(z) \odot^2 \nabla_z^\perp \otimes (x-z) \\ =& d^2W(z) \odot^2 \begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix} = 0\end{aligned}$$

because $d^2W(z)$ is symmetric and $\begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix}$ anti-symmetric. An integration by parts yields

$$\begin{aligned}\int_{\mathbb{R}^2} \varphi(z)\mathcal{E}_{2,b}(z)dz =& \frac{-1}{4\pi l_b^2}\int_{\mathbb{R}^2} \nabla^\perp\varphi(z)\cdot d^2W(z)\left(\iint_{\mathbb{R}^2\times\mathbb{R}^2} \gamma_b(x,y)\Pi_z(y,x)(x-z)dxdy\right)dz \\ =& \frac{-1}{4\pi l_b^2}\int_{\mathbb{R}^2} \nabla^\perp\varphi(z)\cdot d^2W(z)\operatorname{Tr}\left[\gamma_b\Pi_z(X-z)\right]dz \qquad \text{(III.5.21)}\end{aligned}$$

With Young's inequality and an estimate similar to (III.4.6),

$$\begin{aligned}\left|\operatorname{Tr}\left[\gamma_b\Pi_z(X-z)\right]\right| \leqslant& \sum_{n\in\mathbb{N}} \left|\operatorname{Tr}\left[\gamma_b\Pi_{n,z}(X-z)\right]\right| \\ \leqslant& \sum_{n\in\mathbb{N}} \epsilon_n \operatorname{Tr}\left[\Pi_{n,z}\left|X-z\right|^2\right] + \sum_{n\in\mathbb{N}} \frac{1}{\epsilon_n}\operatorname{Tr}\left[\gamma_b^2\Pi_{n,z}\right] \\ \leqslant& 2l_b^2 \sum_{n\in\mathbb{N}} \epsilon_n(n+1) + 2\pi l_b^2 \sum_{n\in\mathbb{N}} \frac{1}{\epsilon_n}\operatorname{Tr}\left[\gamma_b\Pi_{n,z}\right] \qquad \text{(III.5.22)}\end{aligned}$$

Inserting this in (III.5.21), taking $\epsilon_n \coloneqq \dfrac{\epsilon}{\sqrt{|\mathrm{supp}(\varphi)|(n+1)^k}}$ and using (III.4.9) we obtain

$$\begin{aligned}
&\left|\int_{\mathbb{R}^2} \varphi(z)\mathcal{E}_{2,b}(z)dz\right| \\
\leqslant& C\left(\left\|d^2W\nabla^\perp\varphi\right\|_{L^1}\sum_{n\in\mathbb{N}}\epsilon_n(n+1) + \left\|d^2W\nabla^\perp\varphi\right\|_{L^\infty}\sum_{n\in\mathbb{N}}\frac{l_b^2}{\epsilon_n}\mathrm{Tr}\left[\gamma_b\Pi_n\right]\right) \\
\leqslant& C\sqrt{|\mathrm{supp}(\varphi)|}\left\|d^2W\nabla^\perp\varphi\right\|_{L^\infty}\left(\epsilon\sum_{n\in\mathbb{N}}\frac{1}{(n+1)^{k-1}} + \frac{l_b^2}{\epsilon}\sum_{n\in\mathbb{N}}(n+1)^k\mathrm{Tr}\left[\gamma_b\Pi_n\right]\right) \\
\leqslant& C\sqrt{|\mathrm{supp}(\varphi)|}\left\|\nabla\varphi\right\|_{L^\infty}\left\|d^2W\right\|_{L^\infty}\left(\epsilon + \frac{l_b^2}{\epsilon}\mathrm{Tr}\left[\gamma_b\mathscr{L}_b^k\right]\right) \qquad \text{(III.5.23)}
\end{aligned}$$

Choosing $\epsilon \coloneqq \dfrac{l_b}{\sqrt{\mathrm{Tr}\left[\gamma_b\mathscr{L}_b^k\right]}}$ we conclude that

$$\left|\int_{\mathbb{R}^2} \varphi(z)\mathcal{E}_{2,b}(z)dz\right| \leqslant C\sqrt{|\mathrm{supp}(\varphi)|}\left\|\nabla\varphi\right\|_{L^\infty}\left\|d^2W\right\|_{L^\infty}\sqrt{\mathrm{Tr}\left[\gamma_b\mathscr{L}_b^k\right]}l_b \qquad \text{(III.5.24)}$$

Step 5: Conclusion

We can control (III.5.13) with an estimate similar to (III.5.12) by exchanging x and y, obtaining

$$\left|\int_{\mathbb{R}^2} \varphi(z)\widetilde{\mathcal{E}}_{2,b}(z)dz\right| \leqslant C\sqrt{|\mathrm{supp}(\varphi)|}\left\|\nabla\varphi\right\|_{L^\infty}\left\|d^2W\right\|_{L^\infty}\sqrt{\mathrm{Tr}\left[\gamma_b\mathscr{L}_b^k\right]}l_b \qquad \text{(III.5.25)}$$

Finally, using the notation (III.4.11),

$$1 \leqslant p_{k-4}\left(l_b^{-2}\right) = \begin{cases} l_b^{2(k-4)} & \text{if } k<4 \\ \ln\left(l_b^{-2}\right) & \text{if } k=4 \\ 1 & \text{if } k>4 \end{cases}$$

so with (III.5.20) (III.5.24) (III.5.25) and (III.5.14) we obtain (III.5.3).

This proof can be adapted for the estimate to work with $k>2$ if we expand W to the second order in (III.5.8) for low Landau levels only, precisely when $(n+1)l_b^2 \leqslant 1$. But this idea works with $k=1$ when we consider the dynamics of the truncated semi-classical density (III.4.3) instead of (III.4.2). This is the goal of Proposition III.4.2 and Proposition III.4.3.

III.5.2 dynamics of the truncated semi-classical density

Proposition III.5.3: *Gyrokinetic equation for the truncated semi-classical density*

Let $t \in \mathbb{R}_+, k \geqslant 0, \gamma_b(t) \in \mathcal{L}^1\left(L^2\left(\mathbb{R}^2\right)\right), W \in W^{3,\infty}(\mathbb{R}^2)$ and assume

$$\begin{aligned}
&\operatorname{Tr}\left[\gamma_b(t)\right] = 1, 0 \leqslant \gamma_b(t) \leqslant 2\pi l_b^2 \\
&\partial_t \gamma_b(t) = \frac{1}{i l_b^2}\left[\mathscr{L}_b + W, \gamma_b(t)\right]
\end{aligned}$$

then $\forall \varphi \in C_c^\infty\left(\mathbb{R}^2\right)$,

$$\begin{aligned}
&\left|\int_{\mathbb{R}^2} \varphi(z)\left(\partial_t \rho_{\gamma_b}^{\leqslant N}(t,z) + \nabla^\perp W(z) \cdot \nabla \rho_{\gamma_b}^{\leqslant N}(t,z)\right) dz\right| \\
\leqslant & C\|\varphi\|_{L^\infty}\|\nabla W\|_{L^\infty} \frac{1}{l_b N^{k-\frac{1}{2}}} \operatorname{Tr}\left[\gamma_b \mathscr{L}_b^k\left(\Pi_N + \Pi_{N+1}\right)\right] + C\sqrt{|\operatorname{supp}(\varphi)|}\,\|\varphi\|_{W^{1,\infty}}\|W\|_{W^{3,\infty}} \Bigg(\\
& N^{1-\frac{k}{2}}\sqrt{\operatorname{Tr}\left[\gamma_b \mathscr{L}_b^k\left(\Pi_N + \Pi_{N+1}\right)\right]} + \sqrt{\operatorname{Tr}\left[\gamma_b(t) \mathscr{L}_b^k\right]} \cdot \begin{cases} l_b N^{2-\frac{k}{2}} & \text{if } k < 4 \\ l_b \sqrt{\ln(N)} & \text{if } k = 4 \\ l_b & \text{if } k > 4 \end{cases} \Bigg)
\end{aligned} \tag{III.5.26}$$

Proof:

Step 1: a direct computation

Using (III.2.14) in (III.5.6),

$$\begin{aligned}
&\nabla^\perp W(z) \cdot \nabla_z \rho_{\gamma_b}^{\leqslant N}(z) = \frac{-1}{2\pi l_b^2} \nabla W(z) \cdot \operatorname{Tr}\left[\gamma_b \nabla_z^\perp \Pi_{\leqslant N,z}\right] = \frac{-1}{2i\pi l_b^4} \operatorname{Tr}\left[\gamma_b\left[\Pi_{\leqslant N,z}, X \cdot \nabla W(z)\right]\right] \\
&+ \frac{\sqrt{N+1}}{2\sqrt{2}\pi l_b^3} \nabla W(z) \cdot \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix} \begin{pmatrix} \operatorname{Tr}\left[\gamma_b \left|\psi_{N,z}\right\rangle\left\langle\psi_{N+1,z}\right|\right] \\ \operatorname{Tr}\left[\gamma_b \left|\psi_{N+1,z}\right\rangle\left\langle\psi_{N,z}\right|\right] \end{pmatrix}
\end{aligned}$$

so with the same computation as for (III.5.7),

$$\begin{aligned}
&\partial_t \rho_{\gamma_b}^{\leqslant N}(z) + \nabla^\perp W(z) \cdot \nabla_z \rho_{\gamma_b}^{\leqslant N}(z) = \frac{1}{2i\pi l_b^4} \operatorname{Tr}\left[\gamma_b\left[\Pi_{\leqslant N,z}, W - X \cdot \nabla W(z)\right]\right] \\
&+ \frac{\sqrt{N+1}}{2\sqrt{2}\pi l_b^3} \nabla W(z) \cdot \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix} \begin{pmatrix} \operatorname{Tr}\left[\gamma_b \left|\psi_{N,z}\right\rangle\left\langle\psi_{N+1,z}\right|\right] \\ \operatorname{Tr}\left[\gamma_b \left|\psi_{N+1,z}\right\rangle\left\langle\psi_{N,z}\right|\right] \end{pmatrix}
\end{aligned} \tag{III.5.27}$$

Using (III.5.10) for $\Pi_{\leqslant N,z}$ instead of Π_z and then inserting (III.2.14),

$$\begin{aligned}
&\left[\Pi_{\leqslant N,z}, W - X \cdot \nabla W(z)\right] \\
&= \left[\Pi_{\leqslant N,z}, \mathcal{V}_z\right] + \frac{1}{2} d^2 W(z)(\left[\Pi_{\leqslant N,z}, X\right], X - z) + \frac{1}{2} d^2 W(z)(X - z, \left[\Pi_{\leqslant N,z}, X\right]) \\
&= \left[\Pi_{\leqslant N,z}, \mathcal{V}_z\right] + \frac{i l_b^2}{2} d^2 W(z)(\nabla_z^\perp \Pi_{\leqslant N,z}, X - z) + \frac{i l_b^2}{2} d^2 W(z)(X - z, \nabla_z^\perp \Pi_{\leqslant N,z})
\end{aligned}$$

$$+\frac{i\sqrt{N+1}l_b}{2\sqrt{2}}d^2W(z)\left(\begin{pmatrix}1 & 1\\ -i & i\end{pmatrix}\begin{pmatrix}|\psi_{N,z}\rangle\langle\psi_{N+1,z}|\\ |\psi_{N+1,z}\rangle\langle\psi_{N,z}|\end{pmatrix}, X-z\right)$$
$$+\frac{i\sqrt{N+1}l_b}{2\sqrt{2}}d^2W(z)\left(X-z,\begin{pmatrix}1 & 1\\ -i & i\end{pmatrix}\begin{pmatrix}|\psi_{N,z}\rangle\langle\psi_{N+1,z}|\\ |\psi_{N+1,z}\rangle\langle\psi_{N,z}|\end{pmatrix}\right) \tag{III.5.28}$$

Defining the error terms

$$\mathcal{E}_{0,b}(z) \coloneqq \frac{\sqrt{N+1}}{2\sqrt{2}\pi l_b^3}\nabla W(z)\cdot\begin{pmatrix}1 & 1\\ -i & i\end{pmatrix}\begin{pmatrix}\mathrm{Tr}\left[\gamma_b|\psi_{N,z}\rangle\langle\psi_{N+1,z}|\right]\\ \mathrm{Tr}\left[\gamma_b|\psi_{N+1,z}\rangle\langle\psi_{N,z}|\right]\end{pmatrix} \tag{III.5.29}$$
$$\mathcal{E}_{1,b}(z) \coloneqq \frac{1}{2i\pi l_b^4}\mathrm{Tr}\left[\gamma_b\left[\Pi_{\leqslant N,z},\mathcal{V}_z\right]\right] \tag{III.5.30}$$
$$\mathcal{E}_{2,b}(z) \coloneqq \frac{1}{4\pi l_b^2}\mathrm{Tr}\left[\gamma_b d^2W(z)\left(\nabla_z^\perp\Pi_{\leqslant N,z}, X-z\right)\right] \tag{III.5.31}$$
$$\widetilde{\mathcal{E}}_{2,b}(z) \coloneqq \frac{1}{4\pi l_b^2}\mathrm{Tr}\left[\gamma_b d^2W(z)\left(X-z,\nabla_z^\perp\Pi_{\leqslant N,z}\right)\right] \tag{III.5.32}$$
$$\mathcal{E}_{3,b}(z) \coloneqq \frac{\sqrt{N+1}}{4\sqrt{2}\pi l_b^3}\mathrm{Tr}\left[\gamma_b d^2W(z)\left(\begin{pmatrix}1 & 1\\ -i & i\end{pmatrix}\begin{pmatrix}|\psi_{N,z}\rangle\langle\psi_{N+1,z}|\\ |\psi_{N+1,z}\rangle\langle\psi_{N,z}|\end{pmatrix}, X-z\right)\right] \tag{III.5.33}$$
$$\widetilde{\mathcal{E}}_{3,b}(z) \coloneqq \frac{\sqrt{N+1}}{2\sqrt{4}\pi l_b^3}\mathrm{Tr}\left[\gamma_b d^2W(z)\left(X-z,\begin{pmatrix}1 & 1\\ -i & i\end{pmatrix}\begin{pmatrix}|\psi_{N,z}\rangle\langle\psi_{N+1,z}|\\ |\psi_{N+1,z}\rangle\langle\psi_{N,z}|\end{pmatrix}\right)\right] \tag{III.5.34}$$

and inserting (III.5.28) in (III.5.27) we have

$$\partial_t\rho_{\gamma_b}^{\leqslant N}+\nabla^\perp W\cdot\nabla\rho_{\gamma_b}^{\leqslant N}=\mathcal{E}_{0,b}+\mathcal{E}_{1,b}+\mathcal{E}_{2,b}+\widetilde{\mathcal{E}}_{2,b}+\mathcal{E}_{3,b}+\widetilde{\mathcal{E}}_{3,b} \tag{III.5.35}$$

Step 2: Estimate of $\mathcal{E}_{0,b}$

As operators, with Young's inequality

$$2\left||\psi_{N+1,z}\rangle\langle\psi_{N,z}|\right| \leqslant \Pi_{N,z}+\Pi_{N+1,z}$$
$$2\left||\psi_{N,z}\rangle\langle\psi_{N+1,z}|\right| \leqslant \Pi_{N,z}+\Pi_{N+1,z}$$

so

$$|\mathcal{E}_{0,b}(z)| \leqslant C\left\|\nabla W\right\|_{L^\infty}\frac{\sqrt{N}}{l_b^3}\left(\mathrm{Tr}\left[\gamma_b\Pi_{N+1,z}\right]+\mathrm{Tr}\left[\gamma_b\Pi_{N,z}\right]\right)$$

and

$$\left|\int_{\mathbb{R}^2}\varphi(z)\mathcal{E}_{0,b}(z)dz\right| \leqslant C\left\|\varphi\right\|_{L^\infty}\left\|\nabla W\right\|_{L^\infty}\frac{\sqrt{N}}{l_b}\left(\mathrm{Tr}\left[\gamma_b\Pi_N\right]+\mathrm{Tr}\left[\gamma_b\Pi_{N+1}\right]\right) \tag{III.5.36}$$

Note that

$$\mathrm{Tr}\left[\gamma_b\mathscr{L}_b^k\Pi_N\right]=\left(2\hbar b\left(N+\frac{1}{2}\right)\right)^k\mathrm{Tr}\left[\gamma_b\Pi_N\right]\underset{N\to\infty}{\sim}2^kN^k\mathrm{Tr}\left[\gamma_b\Pi_N\right]$$

$$\mathrm{Tr}\left[\gamma_b \mathscr{L}_b^2 \Pi_{N+1}\right] \underset{N\to\infty}{\sim} 2^k N^k \mathrm{Tr}\left[\gamma_b \Pi_{N+1}\right]$$

Inserting this in (III.5.36) we have

$$\left|\int_{\mathbb{R}^2} \varphi(z)\mathcal{E}_{0,b}(z)dz\right| \leqslant C\left\|\varphi\right\|_{L^\infty}\left\|\nabla W\right\|_{L^\infty}\frac{1}{l_b N^{k-\frac{1}{2}}}\mathrm{Tr}\left[\gamma_b \mathscr{L}_b^k\left(\Pi_N + \Pi_{N+1}\right)\right] \tag{III.5.37}$$

Step 3: Estimate of $\mathcal{E}_{1,b}$

Following the same proof as for (III.5.11) we obtain instead of (III.5.19):

$$\left|\int_{\mathbb{R}^2} \varphi(z)\mathcal{E}_{1,b}(z)dz\right| \leqslant C\sqrt{|\mathrm{supp}(\varphi)|}\left\|\varphi\right\|_{L^\infty}\left\|W\right\|_{W^{3,\infty}}\left(\epsilon l_b^2 \sum_{n=0}^{N}\frac{1}{(n+1)^{k-3}} + \frac{1}{\epsilon}\mathrm{Tr}\left[\gamma_b \mathscr{L}_b^k\right]\right)$$

With the asymptotic (III.4.13), taking $\epsilon := \frac{1}{l_b}\sqrt{\frac{\mathrm{Tr}\left[\gamma_b \mathscr{L}_b^k\right]}{p_{k-4}(N)}}$ we have

$$\begin{aligned}\left|\int_{\mathbb{R}^2} \varphi(z)\mathcal{E}_{1,b}(z)dz\right| &\leqslant C\sqrt{|\mathrm{supp}(\varphi)|}\left\|\varphi\right\|_{L^\infty}\left\|W\right\|_{W^{3,\infty}}\left(\epsilon l_b^2 p_{k-4}(N) + \frac{1}{\epsilon}\mathrm{Tr}\left[\gamma_b \mathscr{L}_b^k\right]\right)\\ &= C\sqrt{|\mathrm{supp}(\varphi)|}\left\|\varphi\right\|_{L^\infty}\left\|W\right\|_{W^{3,\infty}}\sqrt{\mathrm{Tr}\left[\gamma_b \mathscr{L}_b^k\right]}l_b\sqrt{p_{k-4}(N)}\end{aligned} \tag{III.5.38}$$

Step 4: Estimate of $\mathcal{E}_{2,b}$

Following the same proof as for (III.5.12) we obtain instead of (III.5.23)

$$\left|\int_{\mathbb{R}^2} \varphi(z)\mathcal{E}_{2,b}(z)dz\right| \leqslant C\sqrt{|\mathrm{supp}(\varphi)|}\left\|\nabla\varphi\right\|_{L^\infty}\left\|d^2 W\right\|_{L^\infty}\left(\epsilon \sum_{n=0}^{N}\frac{1}{(n+1)^{k-1}} + \frac{l_b^2}{\epsilon}\mathrm{Tr}\left[\gamma_b \mathscr{L}_b^k\right]\right)$$

With the asymptotic (III.4.13), taking $\epsilon := l_b\sqrt{\frac{\mathrm{Tr}\left[\gamma_b \mathscr{L}_b^k\right]}{p_{k-2}(N)}}$ we have

$$\begin{aligned}\left|\int_{\mathbb{R}^2} \varphi(z)\mathcal{E}_{2,b}(z)dz\right| &\leqslant C\sqrt{|\mathrm{supp}(\varphi)|}\left\|\nabla\varphi\right\|_{L^\infty}\left\|d^2 W\right\|_{L^\infty}\left(\epsilon p_{k-2}(N) + \frac{l_b^2}{\epsilon}\mathrm{Tr}\left[\gamma_b \mathscr{L}_b^k\right]\right)\\ &= C\sqrt{|\mathrm{supp}(\varphi)|}\left\|\nabla\varphi\right\|_{L^\infty}\left\|d^2 W\right\|_{L^\infty}\sqrt{\mathrm{Tr}\left[\gamma_b \mathscr{L}_b^k\right]}l_b\sqrt{p_{k-2}(N)}\end{aligned} \tag{III.5.39}$$

Step 5: Estimate of $\mathcal{E}_{3,b}$

We only need to estimate terms of the form

$$\frac{\sqrt{N}}{l_b^3}\mathrm{Tr}\left[\gamma_b \left|\psi_{N,z}\right\rangle\left\langle\psi_{N+1,z}\right|(X_i - z_i)\right]$$

or

$$\frac{\sqrt{N}}{l_b^3}\mathrm{Tr}\left[\gamma_b \left|\psi_{N+1,z}\right\rangle \left\langle\psi_{N,z}\right| (X_i - z_i)\right]$$

with $i \in \{1,2\}$. With Young's inequality and the same inequality as in (III.4.6),

$$\begin{aligned}\left|\mathrm{Tr}\left[\gamma_b \left|\psi_{N+1,z}\right\rangle \left\langle\psi_{N,z}\right| (X_i - z_i)\right]\right| \leqslant& \epsilon \mathrm{Tr}\left[(X_i - z_i)^2 \Pi_{N,z}\right] + \frac{1}{\epsilon}\mathrm{Tr}\left[\gamma_b^2 \Pi_{N+1,z}\right] \\ \leqslant& 2\epsilon(N+1)l_b^2 + \frac{2\pi l_b^2}{\epsilon}\mathrm{Tr}\left[\gamma_b \Pi_{N+1,z}\right]\end{aligned} \tag{III.5.40}$$

and similarly

$$\mathrm{Tr}\left[\gamma_b \left|\psi_{N,z}\right\rangle \left\langle\psi_{N+1,z}\right| (X_i - z_i)\right] \leqslant 2\epsilon(N+2)l_b^2 + \frac{2\pi l_b^2}{\epsilon}\mathrm{Tr}\left[\gamma_b \Pi_{N,z}\right]$$

Since $N+1 \sim N+2 \sim N$, integrating against $\varphi \in C_c^\infty(\mathbb{R}^2)$, we obtain

$$\begin{aligned}\left|\int_{\mathbb{R}^2} \varphi \mathcal{E}_{3,b}(z)dz\right| \leqslant& C \left\|d^2 W\right\|_{L^\infty} \frac{\sqrt{N}}{l_b^3}\left(\|\varphi\|_{L^1} \epsilon N l_b^2 + \|\varphi\|_{L^\infty} \frac{l_b^4}{\epsilon}\left(\mathrm{Tr}\left[\gamma_b \Pi_{N+1}\right] + \mathrm{Tr}\left[\gamma_b \Pi_N\right]\right)\right) \\ \leqslant& C \|\varphi\|_{L^\infty} \left\|d^2 W\right\|_{L^\infty}\left(\left|\mathrm{supp}(\varphi)\right| \epsilon \frac{N^{\frac{3}{2}}}{l_b} + \frac{l_b\sqrt{N}}{\epsilon N^k} N^k \mathrm{Tr}\left[\gamma_b\left(\Pi_N + \Pi_{N+1}\right)\right]\right) \\ \leqslant& C \|\varphi\|_{L^\infty} \left\|d^2 W\right\|_{L^\infty}\left(\left|\mathrm{supp}(\varphi)\right| \epsilon \frac{N^{\frac{3}{2}}}{l_b} + \frac{l_b}{\epsilon N^{k-\frac{1}{2}}} \mathrm{Tr}\left[\gamma_b \mathscr{L}_b^k\left(\Pi_N + \Pi_{N+1}\right)\right]\right)\end{aligned}$$

Taking $\epsilon := \dfrac{l_b}{\sqrt{\left|\mathrm{supp}(\varphi)\right|}N^{\frac{k+1}{2}}}\sqrt{\mathrm{Tr}\left[\gamma_b \mathscr{L}_b^k\left(\Pi_N + \Pi_{N+1}\right)\right]}$, we conclude that

$$\left|\int_{\mathbb{R}^2} \varphi \mathcal{E}_{3,b}(z)dz\right| \leqslant C\sqrt{\left|\mathrm{supp}(\varphi)\right|}\,\|\varphi\|_{L^\infty} \left\|d^2 W\right\|_{L^\infty} N^{1-\frac{k}{2}}\sqrt{\mathrm{Tr}\left[\gamma_b \mathscr{L}_b^k\left(\Pi_N + \Pi_{N+1}\right)\right]} \tag{III.5.41}$$

Step 6: Conclusion

Exchanging x and y in (III.5.32) and (III.5.34) we obtain with the same arguments as for (III.5.39) and (III.5.41):

$$\left|\int_{\mathbb{R}^2} \varphi \widetilde{\mathcal{E}}_{2,b}(z)dz\right| \leqslant C\sqrt{\left|\mathrm{supp}(\varphi)\right|}\,\|\nabla\varphi\|_{L^\infty} \left\|d^2 W\right\|_{L^\infty} \sqrt{\mathrm{Tr}\left[\gamma_b \mathscr{L}_b^k\right]} l_b \sqrt{p_{k-2}(N)} \tag{III.5.42}$$

$$\left|\int_{\mathbb{R}^2} \varphi \widetilde{\mathcal{E}}_{3,b}(z)dz\right| \leqslant C\sqrt{\left|\mathrm{supp}(\varphi)\right|}\,\|\varphi\|_{L^\infty} \left\|d^2 W\right\|_{L^\infty} N^{1-\frac{k}{2}}\sqrt{\mathrm{Tr}\left[\gamma_b \mathscr{L}_b^k\left(\Pi_N + \Pi_{N+1}\right)\right]} \tag{III.5.43}$$

Putting together (III.5.37) (III.5.38) (III.5.39) (III.5.41) (III.5.42) (III.5.43) with (III.5.35) we get

$$\left| \int_{\mathbb{R}^2} \varphi(z) \left(\partial_t \rho_{\gamma_b}^{sc,\leqslant N}(z) + \nabla^\perp W(z) \cdot \nabla \rho_{\gamma_b}^{sc,\leqslant N}(z) \right) dz \right|$$

$$\leqslant C \|\varphi\|_{L^\infty} \|\nabla W\|_{L^\infty} \frac{1}{l_b N^{k-\frac{1}{2}}} \mathrm{Tr}\left[\gamma_b \mathscr{L}_b^k (\Pi_N + \Pi_{N+1})\right] + C\sqrt{|\mathrm{supp}(\varphi)|}\, \|\varphi\|_{W^{1,\infty}} \|W\|_{W^{3,\infty}} \Big(\sqrt{\mathrm{Tr}\left[\gamma_b \mathscr{L}_b^k\right]} l_b \sqrt{p_{k-4}(N)} + \sqrt{\mathrm{Tr}\left[\gamma_b \mathscr{L}_b^k\right]} l_b \sqrt{p_{k-2}(N)} + N^{1-\frac{k}{2}} \sqrt{\mathrm{Tr}\left[\gamma_b \mathscr{L}_b^k (\Pi_N + \Pi_{N+1})\right]}\Big)$$

Notice from (III.4.11) that if $x \geqslant e$,

$$\alpha < \beta \implies p_\alpha(x) \geqslant p_\beta(x) \tag{III.5.44}$$

so

$$p_{k-2}(N) \leqslant p_{k-4}(N)$$

and we conclude that

$$\left| \int_{\mathbb{R}^2} \varphi(z) \left(\partial_t \rho_{\gamma_b}^{sc,\leqslant N}(z) + \nabla^\perp W(z) \cdot \nabla \rho_{\gamma_b}^{sc,\leqslant N}(z) \right) dz \right|$$

$$\leqslant C \|\varphi\|_{L^\infty} \|\nabla W\|_{L^\infty} \frac{1}{l_b N^{k-\frac{1}{2}}} \mathrm{Tr}\left[\gamma_b \mathscr{L}_b^k (\Pi_N + \Pi_{N+1})\right] + C\sqrt{|\mathrm{supp}(\varphi)|}\, \|\varphi\|_{W^{1,\infty}} \|W\|_{W^{3,\infty}} \Big(\sqrt{\mathrm{Tr}\left[\gamma_b \mathscr{L}_b^k\right]} l_b \sqrt{p_{k-4}(N)} + N^{1-\frac{k}{2}} \sqrt{\mathrm{Tr}\left[\gamma_b \mathscr{L}_b^k (\Pi_N + \Pi_{N+1})\right]}\Big)$$

If we assume

$$\mathrm{Tr}\left[\gamma_b \mathscr{L}_b^k\right] = \sum_{n \in \mathbb{N}} \mathrm{Tr}\left[\gamma_b \mathscr{L}_b^k \Pi_n\right] < \infty$$

then up to a subsequence, whose choice will be the object to Lemma III.6.1, we can assume

$$\mathrm{Tr}\left[\gamma_b \mathscr{L}_b^k \Pi_N\right] = \mathcal{O}\left(\frac{1}{N}\right)$$

This time we have a small error in (III.5.26) if

$$l_b N^{k+\frac{1}{2}} \gg 1, \quad N^{\frac{1-k}{2}} \ll 1, \quad l_b N^{2-\frac{k}{2}} \ll 1$$

When $1 < k < 4$, this is equivalent to

$$\frac{1}{l_b^{\frac{2}{2k+1}}} \ll N \ll \frac{1}{l_b^{\frac{2}{4-k}}}$$

which is a possible choice of N. Hence we will be able to control the dynamics of the truncated semi-classical density for $k > 1$. One could conclude the proof of the main results with the conservation of the second moment of the kinetic energy. As we only obtained Proposition III.3.4, for the proof to really work with the kinetic energy we further expand W.

Proposition III.5.4: *Gyrokinetic equation for the truncated semi-classical density*

Let $t \in \mathbb{R}_+, k \geqslant 0, \gamma_b(t) \in \mathcal{L}^1\left(L^2\left(\mathbb{R}^2\right)\right), W \in W^{4,\infty}(\mathbb{R}^2)$ and assume

$$\begin{aligned}\operatorname{Tr}\left[\gamma_b(t)\right] &= 1, 0 \leqslant \gamma_b(t) \leqslant 2\pi l_b^2 \\ \partial_t \gamma_b(t) &= \frac{1}{i l_b^2}\left[\mathscr{L}_b + W, \gamma_b(t)\right]\end{aligned}$$

then $\forall \varphi \in C_c^\infty\left(\mathbb{R}^2\right)$,

$$\begin{aligned}&\left|\int_{\mathbb{R}^2} \varphi(z)\left(\partial_t \rho_{\gamma_b}^{\leqslant N}(t,z) + \nabla^\perp W(z)\cdot\nabla\rho_{\gamma_b}^{\leqslant N}(t,z)\right) dz\right| \\ \leqslant & c(\varphi, W)\left(\frac{1}{l_b N^{k-\frac{1}{2}}}\operatorname{Tr}\left[\gamma_b(t)\mathscr{L}_b^k \Pi_{N-1:N+1}\right] + \left(N^{1-\frac{k}{2}} + l_b N^{\frac{3-k}{2}}\right)\sqrt{\operatorname{Tr}\left[\gamma_b(t)\mathscr{L}_b^k \Pi_{N-1:N+1}\right]}\right. \\ & \left. + l_b^2 N + \sqrt{\operatorname{Tr}\left[\gamma_b(t)\mathscr{L}_b^k\right]}\cdot\begin{cases} l_b N^{1-\frac{k}{2}} + l_b^2 N^{\frac{5-k}{2}} & \text{if } k < 2 \\ l_b\sqrt{\ln(N)} + l_b^2 N^{\frac{3}{2}} & \text{if } k = 2 \\ l_b + l_b^2 N^{\frac{5-k}{2}} & \text{if } 2 < k < 5 \\ l_b + l_b^2\sqrt{\ln(N)} & \text{if } k = 5 \\ l_b & \text{if } k > 5\end{cases}\right)\end{aligned}$$

where

$$c(\varphi, W) \coloneqq (1 + |\operatorname{supp}(\varphi)|)\,\|\varphi\|_{W^{1,\infty}}\|W\|_{W^{4,\infty}}$$

Proof:

Step 1: a direct computation

Redefine

$$\mathcal{V}_z(x) \coloneqq W(x) - dW(z)(x-z) - \frac{1}{2}d^2W(z)(x-z,x-z) - \frac{1}{6}d^3W(z)(x-z,x-z,x-z)$$

Notice that

$$\begin{aligned}& d^3W(z)(y-z,y-z,y-z) \\ =& d^3W(z)(y-x,y-z,y-z) + d^3W(z)(x-z,y-z,y-z) \\ =& d^3W(z)(y-x,y-z,y-z) + d^3W(z)(x-z,y-x,y-z) + d^3W(z)(x-z,x-z,y-z) \\ =& d^3W(z)(y-x,y-z,y-z) + d^3W(z)(x-z,y-x,y-z) + d^3W(z)(x-z,x-z,y-x) \\ & + d^3W(z)(x-z,x-z,x-z)\end{aligned}$$

so (III.5.9) becomes

$$\left[\Pi_{\leqslant N,z}, W - X\cdot\nabla W(z)\right](x,y) = \Pi_{\leqslant N,z}(x,y)\left(\mathcal{V}_z(y) - \mathcal{V}_z(x) + \frac{d^2W(z)}{2}(y-x,y-z)\right.$$

$$+\frac{d^2W(z)}{2}(x-z,y-x)+\frac{d^3W(z)}{6}(y-x,y-z,y-z)+\frac{d^3W(z)}{6}(x-z,y-x,y-z)$$
$$+\frac{d^3W(z)}{6}(x-z,x-z,y-x)\Big)$$

Then inserting (III.2.14),

$$\begin{aligned}
&[\Pi_{\leqslant N,z}, W - X\cdot\nabla W(z)]\\
&=[\Pi_{\leqslant N,z},\mathcal{V}_z]+\frac{d^2W(z)}{2}([\Pi_{\leqslant N,z},X],X-z)+\frac{d^2W(z)}{2}(X-z,[\Pi_{\leqslant N,z},X])\\
&\quad+\frac{d^3W(z)}{6}([\Pi_{\leqslant N,z},X],X-z,X-z)+\frac{d^3W(z)}{6}(X-z,[\Pi_{\leqslant N,z},X],X-z)\\
&\quad+\frac{d^3W(z)}{6}(X-z,X-z,[\Pi_{\leqslant N,z},X])\\
&=[\Pi_{\leqslant N,z},\mathcal{V}_z]+\frac{il_b^2}{2}d^2W(z)(\nabla_z^\perp\Pi_{\leqslant N,z},X-z)+\frac{il_b^2}{2}d^2W(z)(X-z,\nabla_z^\perp\Pi_{\leqslant N,z})\\
&\quad+\frac{il_b^2}{6}d^3W(z)(\nabla_z^\perp\Pi_{\leqslant N,z},X-z,X-z)+\frac{il_b^2}{6}d^3W(z)(X-z,\nabla_z^\perp\Pi_{\leqslant N,z},X-z)\\
&\quad+\frac{il_b^2}{6}d^3W(z)(X-z,X-z,\nabla_z^\perp\Pi_{\leqslant N,z})\\
&\quad+\frac{i\sqrt{N+1}l_b}{2\sqrt{2}}d^2W(z)\left(\begin{pmatrix}1&1\\-i&i\end{pmatrix}\begin{pmatrix}|\psi_{N,z}\rangle\langle\psi_{N+1,z}|\\|\psi_{N+1,z}\rangle\langle\psi_{N,z}|\end{pmatrix},X-z\right)\\
&\quad+\frac{i\sqrt{N+1}l_b}{2\sqrt{2}}d^2W(z)\left(X-z,\begin{pmatrix}1&1\\-i&i\end{pmatrix}\begin{pmatrix}|\psi_{N,z}\rangle\langle\psi_{N+1,z}|\\|\psi_{N+1,z}\rangle\langle\psi_{N,z}|\end{pmatrix}\right)\\
&\quad+\frac{i\sqrt{N+1}l_b}{6\sqrt{2}}d^3W(z)\left(\begin{pmatrix}1&1\\-i&i\end{pmatrix}\begin{pmatrix}|\psi_{N,z}\rangle\langle\psi_{N+1,z}|\\|\psi_{N+1,z}\rangle\langle\psi_{N,z}|\end{pmatrix},X-z,X-z\right)\\
&\quad+\frac{i\sqrt{N+1}l_b}{6\sqrt{2}}d^3W(z)\left(X-z,\begin{pmatrix}1&1\\-i&i\end{pmatrix}\begin{pmatrix}|\psi_{N,z}\rangle\langle\psi_{N+1,z}|\\|\psi_{N+1,z}\rangle\langle\psi_{N,z}|\end{pmatrix},X-z\right)\\
&\quad+\frac{i\sqrt{N+1}l_b}{6\sqrt{2}}d^3W(z)\left(X-z,X-z,\begin{pmatrix}1&1\\-i&i\end{pmatrix}\begin{pmatrix}|\psi_{N,z}\rangle\langle\psi_{N+1,z}|\\|\psi_{N+1,z}\rangle\langle\psi_{N,z}|\end{pmatrix}\right)
\end{aligned}\tag{III.5.45}$$

Defining the errors terms

$$\begin{aligned}
\mathcal{E}_{0,b}(z)&\coloneqq\frac{\sqrt{N+1}}{2\sqrt{2}\pi l_b^3}\nabla W(z)\cdot\begin{pmatrix}1&1\\-i&i\end{pmatrix}\begin{pmatrix}\mathrm{Tr}\left[\gamma_b\,|\psi_{N,z}\rangle\langle\psi_{N+1,z}|\right]\\\mathrm{Tr}\left[\gamma_b\,|\psi_{N+1,z}\rangle\langle\psi_{N,z}|\right]\end{pmatrix}\\
\mathcal{E}_{1,b}(z)&\coloneqq\frac{1}{2i\pi l_b^4}\mathrm{Tr}\left[\gamma_b\left[\Pi_{\leqslant N,z},\mathcal{V}_z\right]\right]\\
\mathcal{E}_{2,b}(z)&\coloneqq\frac{1}{4\pi l_b^2}\mathrm{Tr}\left[\gamma_b d^2W(z)\left(\nabla_z^\perp\Pi_{\leqslant N,z},X-z\right)\right]\\
\widetilde{\mathcal{E}}_{2,b}(z)&\coloneqq\frac{1}{4\pi l_b^2}\mathrm{Tr}\left[\gamma_b d^2W(z)\left(X-z,\nabla_z^\perp\Pi_{\leqslant N,z}\right)\right]\\
\mathcal{E}_{3,b}(z)&\coloneqq\frac{\sqrt{N+1}}{4\sqrt{2}\pi l_b^3}\mathrm{Tr}\left[\gamma_b d^2W(z)\left(\begin{pmatrix}1&1\\-i&i\end{pmatrix}\begin{pmatrix}|\psi_{N,z}\rangle\langle\psi_{N+1,z}|\\|\psi_{N+1,z}\rangle\langle\psi_{N,z}|\end{pmatrix},X-z\right)\right]
\end{aligned}$$

$$
\begin{aligned}
\widetilde{\mathcal{E}}_{3,b}(z) &\coloneqq \frac{\sqrt{N+1}}{4\sqrt{2}\pi l_b^3}\mathrm{Tr}\left[\gamma_b d^2W(z)\left(X-z,\begin{pmatrix}1 & 1\\ -i & i\end{pmatrix}\begin{pmatrix}\left|\psi_{N,z}\right\rangle\left\langle\psi_{N+1,z}\right|\\ \left|\psi_{N+1,z}\right\rangle\left\langle\psi_{N,z}\right|\end{pmatrix}\right)\right]\\
\mathcal{E}_{4,b}(z) &\coloneqq \frac{1}{12\pi l_b^2}\mathrm{Tr}\left[\gamma_b d^3W(z)\left(\nabla_z^\perp\Pi_{\leqslant N,z},X-z,X-z\right)\right]\\
\widetilde{\mathcal{E}}_{4,b}(z) &\coloneqq \frac{1}{12\pi l_b^2}\mathrm{Tr}\left[\gamma_b d^3W(z)\left(X-z,\nabla_z^\perp\Pi_{\leqslant N,z},X-z\right)\right]\\
\widetilde{\widetilde{\mathcal{E}}}_{4,b}(z) &\coloneqq \frac{1}{12\pi l_b^2}\mathrm{Tr}\left[\gamma_b d^3W(z)\left(X-z,X-z,\nabla_z^\perp\Pi_{\leqslant N,z}\right)\right]\\
\mathcal{E}_{5,b}(z) &\coloneqq \frac{\sqrt{N+1}}{12\sqrt{2}\pi l_b^3}\mathrm{Tr}\left[\gamma_b d^3W(z)\left(\begin{pmatrix}1 & 1\\ -i & i\end{pmatrix}\begin{pmatrix}\left|\psi_{N,z}\right\rangle\left\langle\psi_{N+1,z}\right|\\ \left|\psi_{N+1,z}\right\rangle\left\langle\psi_{N,z}\right|\end{pmatrix},X-z,X-z\right)\right] \qquad \text{(III.5.46)}\\
\widetilde{\mathcal{E}}_{5,b}(z) &\coloneqq \frac{\sqrt{N+1}}{12\sqrt{2}\pi l_b^3}\mathrm{Tr}\left[\gamma_b d^3W(z)\left(X-z,\begin{pmatrix}1 & 1\\ -i & i\end{pmatrix}\begin{pmatrix}\left|\psi_{N,z}\right\rangle\left\langle\psi_{N+1,z}\right|\\ \left|\psi_{N+1,z}\right\rangle\left\langle\psi_{N,z}\right|\end{pmatrix},X-z\right)\right]\\
\widetilde{\widetilde{\mathcal{E}}}_{5,b}(z) &\coloneqq \frac{\sqrt{N+1}}{12\sqrt{2}\pi l_b^3}\mathrm{Tr}\left[\gamma_b d^3W(z)\left(X-z,X-z,\begin{pmatrix}1 & 1\\ -i & i\end{pmatrix}\begin{pmatrix}\left|\psi_{N,z}\right\rangle\left\langle\psi_{N+1,z}\right|\\ \left|\psi_{N+1,z}\right\rangle\left\langle\psi_{N,z}\right|\end{pmatrix}\right)\right]
\end{aligned}
$$

and inserting (III.5.45) in (III.5.27) we have

$$
\begin{aligned}
&\partial_t\rho_{\gamma_b}^{\leqslant N}+\nabla^\perp W\cdot\nabla\rho_{\gamma_b}^{\leqslant N}\\
=&\mathcal{E}_{0,b}+\mathcal{E}_{1,b}+\mathcal{E}_{2,b}+\widetilde{\mathcal{E}}_{2,b}+\mathcal{E}_{3,b}+\widetilde{\mathcal{E}}_{3,b}+\mathcal{E}_{4,b}+\widetilde{\mathcal{E}}_{4,b}+\widetilde{\widetilde{\mathcal{E}}}_{4,b}+\mathcal{E}_{5,b}+\widetilde{\mathcal{E}}_{5,b}+\widetilde{\widetilde{\mathcal{E}}}_{5,b} \qquad \text{(III.5.47)}
\end{aligned}
$$

Step 2: Estimate of $\mathcal{E}_{1,b}$

Using that the fourth derivative of W is bounded, we get

$$
\left|\mathcal{V}_z(x+\sqrt{2}l_bx)\right|\leqslant C\left\|W\right\|_{W^{4,\infty}}\left|l_bx\right|^4
$$

so following the same proof as for (III.5.11) we obtain instead of (III.5.19)

$$
\left|\int_{\mathbb{R}^2}\varphi(z)\mathcal{E}_{1,b}(z)dz\right|\leqslant C\sqrt{\left|\mathrm{supp}(\varphi)\right|}\left\|\varphi\right\|_{L^\infty}\left\|W\right\|_{W^{4,\infty}}\left(\epsilon l_b^4\sum_{n=0}^{N}\frac{1}{(n+1)^{k-4}}+\frac{1}{\epsilon}\mathrm{Tr}\left[\gamma_b\mathscr{L}_b^k\right]\right)
$$

With the asymptotic (III.4.13), taking $\epsilon\coloneqq\frac{1}{l_b^2}\sqrt{\frac{\mathrm{Tr}\left[\gamma_b\mathscr{L}_b^k\right]}{p_{k-5}(N)}}$ we have

$$
\begin{aligned}
\left|\int_{\mathbb{R}^2}\varphi(z)\mathcal{E}_{1,b}(z)dz\right|\leqslant&C\sqrt{\left|\mathrm{supp}(\varphi)\right|}\left\|\varphi\right\|_{L^\infty}\left\|W\right\|_{W^{4,\infty}}\left(\epsilon l_b^4p_{k-5}(N)+\frac{1}{\epsilon}\mathrm{Tr}\left[\gamma_b\mathscr{L}_b^k\right]\right)\\
=&C\sqrt{\left|\mathrm{supp}(\varphi)\right|}\left\|\varphi\right\|_{L^\infty}\left\|W\right\|_{W^{4,\infty}}\sqrt{\mathrm{Tr}\left[\gamma_b\mathscr{L}_b^k\right]}l_b^2\sqrt{p_{k-5}(N)} \qquad \text{(III.5.48)}
\end{aligned}
$$

Step 3: Estimate of $\mathcal{E}_{4,b}$

By symmetry of the derivatives of W,

$$\mathcal{E}_{4,b}(z)=\frac{1}{12\pi l_b^2}\operatorname{Tr}\left[\gamma_b\nabla_z^\perp\Pi_{\leqslant N,z}\odot\nabla^{\otimes 3}W(z)\odot^2(X-z)^{\otimes 2}\right]$$

Next we notice that

$$\begin{aligned}
&\nabla_z^\perp\odot\nabla^{\otimes 3}W(z)\odot^2(X-z)^{\otimes 2}\\
=&\left(\nabla_z^\perp\odot\nabla^{\otimes 3}W(z)\right)\odot^2(X-z)^{\otimes 2}+\nabla^{\otimes 3}W(z)\odot^3\nabla_z^\perp\otimes(X-z)^{\otimes 2}\\
=&\nabla^{\otimes 3}W(z)\odot^3\begin{pmatrix}0&1\\-1&0\end{pmatrix}\otimes(X-z)+\nabla^{\otimes 3}W(z)\odot^3(X-z)\otimes\begin{pmatrix}0&1\\-1&0\end{pmatrix}\\
=&\left(\nabla^{\otimes 3}W(z)\odot^2\begin{pmatrix}0&1\\-1&0\end{pmatrix}\right)\odot(X-z)+\left(\nabla^{\otimes 3}W(z)\odot(X-z)\right)\odot\begin{pmatrix}0&1\\-1&0\end{pmatrix}=0
\end{aligned}$$

since $\nabla^{\otimes 3}W(z)$, $\nabla^{\otimes 3}W(z)\odot(X-z)$ are symmetric and $\begin{pmatrix}0&1\\-1&0\end{pmatrix}$ anti-symmetric. Hence with an integration by parts,

$$\int_{\mathbb{R}^2}\varphi(z)\mathcal{E}_{4,b}(z)dz=\frac{-1}{12\pi l_b^2}\int_{\mathbb{R}^2}\nabla^\perp\varphi(z)\odot d^3W(z)\odot^2\operatorname{Tr}\left[\gamma_b\Pi_{\leqslant N,z}(X-z)^{\otimes 2}\right]\tag{III.5.49}$$

From this point on, we follow the same proof as for (III.5.12) with one difference in (III.5.22) where we use

$$\operatorname{Tr}\left[\Pi_{n,z}|X-z|^4\right]\leqslant C(n+1)^2l_b^4$$

and obtain instead of (III.5.23),

$$\begin{aligned}
\left|\int_{\mathbb{R}^2}\varphi(z)\mathcal{E}_{2,b}(z)dz\right|\leqslant&C\sqrt{|\operatorname{supp}(\varphi)|}\,\|\nabla\varphi\|_{L^\infty}\left\|d^3W\right\|_{L^\infty}\left(\epsilon l_b^2\sum_{n=0}^N\frac{1}{(n+1)^{k-2}}+\frac{l_b^2}{\epsilon}\operatorname{Tr}\left[\gamma_b\mathscr{L}_b^k\right]\right)\\
\leqslant&C\sqrt{|\operatorname{supp}(\varphi)|}\,\|\nabla\varphi\|_{L^\infty}\left\|d^3W\right\|_{L^\infty}l_b^2\left(\epsilon p_{k-3}(N)+\frac{1}{\epsilon}\operatorname{Tr}\left[\gamma_b\mathscr{L}_b^k\right]\right)
\end{aligned}$$

and with $\epsilon:=\sqrt{\frac{\operatorname{Tr}\left[\gamma_b\mathscr{L}_b^k\right]}{p_{k-3}(N)}}$ we conclude that

$$\left|\int_{\mathbb{R}^2}\varphi(z)\mathcal{E}_{2,b}(z)dz\right|\leqslant C\sqrt{|\operatorname{supp}(\varphi)|}\,\|\nabla\varphi\|_{L^\infty}\left\|d^3W\right\|_{L^\infty}\sqrt{\operatorname{Tr}\left[\gamma_b\mathscr{L}_b^k\right]}\sqrt{p_{k-3}(N)}l_b^2\tag{III.5.50}$$

Step 4: Estimate of $\widetilde{\mathcal{E}}_{4,b}$

Similarly as for $\mathcal{E}_{4,b}$ we proceed to an integration by parts and instead of (III.5.49) we have

$$\int_{\mathbb{R}^2}\varphi(z)\mathcal{E}_{4,b}(z)dz=\frac{-1}{12\pi l_b^2}\int_{\mathbb{R}^2}\nabla^\perp\varphi(z)\odot d^3W(z)\odot^2\operatorname{Tr}\left[\gamma_b(X-z)\otimes\Pi_{\leqslant N,z}(X-z)\right]$$

But

$$|\mathrm{Tr}\left[\gamma_b(X-z)\otimes\Pi_{\leqslant N,z}(X-z)\right]|\leqslant Cl_b^2\mathrm{Tr}\left[\Pi_{n,Z}\left|X-z\right|^2\right]\leqslant CNl_b^4$$

and

$$\left|\int_{\mathbb{R}^2}\varphi(z)\mathcal{E}_{4,b}(z)dz\right|\leqslant C\left|\mathrm{supp}(\varphi)\right|\left\|\nabla\varphi\right\|_{L^\infty}\left\|d^3W\right\|_{L^\infty}l_b^2N \tag{III.5.51}$$

Step 5: Estimate of $\mathcal{E}_{5,b}$

We follow the same proof as for (III.5.33), the only difference being in (III.5.40) that we replace with

$$\begin{aligned}\left|\mathrm{Tr}\left[\gamma_b\left|\psi_{N+1,z}\right\rangle\left\langle\psi_{N,z}\right|(X_i-z_i)^2\right]\right|\leqslant&\epsilon\mathrm{Tr}\left[(X_i-z_i)^4\Pi_{N,z}\right]+\frac{1}{\epsilon}\mathrm{Tr}\left[\gamma_b^2\Pi_{N+1,z}\right]\\ \leqslant&2\epsilon(N+1)^2l_b^4+\frac{2\pi l_b^2}{\epsilon}\mathrm{Tr}\left[\gamma_b\Pi_{N+1,z}\right]\end{aligned} \tag{III.5.52}$$

So we gain a factor $\sqrt{N}l_b$ and instead of (III.5.41) we obtain

$$\left|\int_{\mathbb{R}^2}\varphi\mathcal{E}_{5,b}(z)dz\right|\leqslant C\sqrt{\left|\mathrm{supp}(\varphi)\right|}\left\|\varphi\right\|_{L^\infty}\left\|d^3W\right\|_{L^\infty}l_bN^{\frac{3-k}{2}}\sqrt{\mathrm{Tr}\left[\gamma_b\mathscr{L}_b^k\left(\Pi_N+\Pi_{N+1}\right)\right]} \tag{III.5.53}$$

Step 6: Estimate of $\widetilde{\mathcal{E}}_{5,b}$

$$\widetilde{\mathcal{E}}_{5,b}(z)=\frac{\sqrt{N+1}}{12\sqrt{2}\pi l_b^3}\mathrm{Tr}\left[\gamma_bd^3W(z)\left(X-z,\begin{pmatrix}1&1\\-i&i\end{pmatrix}\begin{pmatrix}\left|\psi_{N,z}\right\rangle\left\langle\psi_{N+1,z}\right|\\\left|\psi_{N+1,z}\right\rangle\left\langle\psi_{N,z}\right|\end{pmatrix},X-z\right)\right]$$

We only need to estimate terms of the form

$$\mathrm{Tr}\left[\gamma_b(X_i-z_i)\left|\psi_{N+1,z}\right\rangle\left\langle\psi_{N,z}\right|(X_j-z_j)\right]$$

with $i,j\in\{1,2\}$. But with Young's inequality,

$$\begin{aligned}2\left|\mathrm{Tr}\left[\gamma_b(X_i-z_i)\left|\psi_{N+1,z}\right\rangle\left\langle\psi_{N,z}\right|(X_j-z_j)\right]\right|\leqslant&\mathrm{Tr}\left[\gamma_b(X_i-z_i)\Pi_{N+1,z}(X_i-z_i)\right]\\&+\mathrm{Tr}\left[\gamma_b(X_j-z_j)\Pi_{N,z}(X_j-z_j)\right]\end{aligned}$$

We are going to control the second term with $j=1$, all the others terms being estimated the same way. With (III.2.15),

$$\begin{aligned}&\mathrm{Tr}\left[\gamma_b(X_1-z_1)\Pi_{N,z}(X_1-z_1)\right]\\&\leqslant\mathrm{Tr}\left[\gamma_b\Pi_{N,z}(X_1-z_1)^2\right]+\mathrm{Tr}\left[\gamma_b\left[X_1,\Pi_{N,z}\right](X_1-z_1)\right]\\&\leqslant\mathrm{Tr}\left[\gamma_b\Pi_{N,z}(X_1-z_1)^2\right]+il_b^2\mathrm{Tr}\left[\gamma_b\partial_{z_2}\Pi_{N,z}(X_1-z_1)\right]\end{aligned}$$

$$
\begin{aligned}
&- i\frac{\sqrt{N+1}l_b}{\sqrt{2}}\mathrm{Tr}\left[\gamma_b\left(\left|\psi_{N,z}\right\rangle\left\langle\psi_{N+1,z}\right| + \left|\psi_{N+1,z}\right\rangle\left\langle\psi_{N,z}\right|\right)(X_1 - z_1)\right] \\
&+ i\frac{\sqrt{N}l_b}{\sqrt{2}}\mathrm{Tr}\left[\gamma_b\left(\left|\psi_{N-1,z}\right\rangle\left\langle\psi_{N,z}\right| + \left|\psi_{N,z}\right\rangle\left\langle\psi_{N-1,z}\right|\right)(X_1 - z_1)\right]
\end{aligned}
$$

The first term is of the same form as (III.5.46), for the second we proceed to an integration by parts and obtain two terms of the form (III.5.33). The last two terms are also of the form (III.5.33). So using (III.5.53) and (III.5.41) we obtain

$$
\begin{aligned}
&\left|\int_{\mathbb{R}^2}\varphi(z)\widetilde{\mathcal{E}}_{5,b}(z)dz\right| \\
\leqslant& C\sqrt{|\mathrm{supp}(\varphi)|}\left\|\varphi\right\|_{L^\infty}\left\|d^3W\right\|_{L^\infty} l_b N^{\frac{3-k}{2}}\sqrt{\mathrm{Tr}\left[\gamma_b\mathscr{L}_b^k\Pi_N\right]} \\
&+C\sqrt{|\mathrm{supp}(\varphi)|}\left\|\varphi\right\|_{W^{1,\infty}}\left\|W\right\|_{W^{4,\infty}} l_b^2 N^{1-\frac{k}{2}}\sqrt{\mathrm{Tr}\left[\gamma_b\mathscr{L}_b^k\Pi_N\right]} \\
&+C\sqrt{|\mathrm{supp}(\varphi)|}\left\|\varphi\right\|_{L^\infty}\left\|d^3W\right\|_{L^\infty} l_b N^{\frac{3-k}{2}}\sqrt{\mathrm{Tr}\left[\gamma_b\mathscr{L}_b^k\left(\Pi_{N-1}+\Pi_N+\Pi_{N+1}\right)\right]} \\
\leqslant& C\sqrt{|\mathrm{supp}(\varphi)|}\left\|\varphi\right\|_{W^{1,\infty}}\left\|W\right\|_{W^{4,\infty}} l_b N^{\frac{3-k}{2}}\sqrt{\mathrm{Tr}\left[\gamma_b\mathscr{L}_b^k\left(\Pi_{N-1}+\Pi_N+\Pi_{N+1}\right)\right]} \qquad \text{(III.5.54)}
\end{aligned}
$$

Step 7: Conclusion

$\mathcal{E}_{0,b}, \mathcal{E}_{2,b}, \widetilde{\mathcal{E}}_{2,b}, \mathcal{E}_{3,b}, \widetilde{\mathcal{E}}_{3,b}$ are the same than in Proposition III.5.3. We control $\widetilde{\widetilde{\mathcal{E}}}_{4,b}, \widetilde{\widetilde{\mathcal{E}}}_{5,b}$ with respectively the same arguments as for $\mathcal{E}_{4,b}, \mathcal{E}_{5,b}$ by exchanging x and y. Therefore, putting together (III.5.37), (III.5.39), (III.5.41), (III.5.48), (III.5.50), (III.5.51), (III.5.53), (III.5.54) with (III.5.47) we obtain

$$
\begin{aligned}
&\left|\int_{\mathbb{R}^2}\varphi(z)\left(\partial_t\rho_{\gamma_b}^{sc,\leqslant N}(z) + \nabla^\perp W(z)\cdot\nabla\rho_{\gamma_b}^{sc,\leqslant N}(z)\right)dz\right| \\
\leqslant& C\left\|\varphi\right\|_{L^\infty}\left\|\nabla W\right\|_{L^\infty}\frac{1}{l_b N^{k-\frac{1}{2}}}\mathrm{Tr}\left[\gamma_b\mathscr{L}_b^k\left(\Pi_N+\Pi_{N+1}\right)\right] + C\sqrt{|\mathrm{supp}(\varphi)|}\left\|\varphi\right\|_{W^{1,\infty}}\left\|W\right\|_{W^{4,\infty}}\Big(\\
&\sqrt{\mathrm{Tr}\left[\gamma_b\mathscr{L}_b^k\right]}l_b\sqrt{p_{k-2}(N)} + N^{1-\frac{k}{2}}\sqrt{\mathrm{Tr}\left[\gamma_b\mathscr{L}_b^k\left(\Pi_N+\Pi_{N+1}\right)\right]} + \sqrt{\mathrm{Tr}\left[\gamma_b\mathscr{L}_b^k\right]}l_b^2\sqrt{p_{k-5}(N)} \\
&+\sqrt{\mathrm{Tr}\left[\gamma_b\mathscr{L}_b^k\right]}l_b^2\sqrt{p_{k-3}(N)} + l_b N^{\frac{3-k}{2}}\sqrt{\mathrm{Tr}\left[\gamma_b\mathscr{L}_b^k\left(\Pi_N+\Pi_{N+1}\right)\right]} \\
&+l_b N^{\frac{3-k}{2}}\sqrt{\mathrm{Tr}\left[\gamma_b\mathscr{L}_b^k\left(\Pi_{N-1}+\Pi_N+\Pi_{N+1}\right)\right]}\Big) + C\left|\mathrm{supp}(\varphi)\right|\left\|\nabla\varphi\right\|_{L^\infty}\left\|d^3W\right\|_{L^\infty} l_b^2 N
\end{aligned}
$$

With (III.5.44), we get

$$
\begin{aligned}
&\left|\int_{\mathbb{R}^2}\varphi(z)\left(\partial_t\rho_{\gamma_b}^{sc,\leqslant N}(z) + \nabla^\perp W(z)\cdot\nabla\rho_{\gamma_b}^{sc,\leqslant N}(z)\right)dz\right| \leqslant c(\varphi,W)\left(\frac{1}{l_b N^{k-\frac{1}{2}}}\mathrm{Tr}\left[\gamma_b\mathscr{L}_b^k\Pi_{N-1:N+1}\right]\right. \\
&+\sqrt{\mathrm{Tr}\left[\gamma_b\mathscr{L}_b^k\right]}\left(l_b\sqrt{p_{k-2}(N)} + l_b^2\sqrt{p_{k-5}(N)}\right)
\end{aligned}
$$

$$+ \left(N^{1-\frac{k}{2}} + l_b N^{\frac{3-k}{2}}\right) \sqrt{\mathrm{Tr}\left[\gamma_b \mathscr{L}_b^k \Pi_{N-1:N+1}\right] + l_b^2 N}\right)$$

We conclude with

$$\begin{aligned}
& l_b \sqrt{p_{k-2}(N)} + l_b^2 \sqrt{p_{k-5}(N)} \\
=& l_b \left(N^{1-\frac{k}{2}} \mathbb{1}_{k<2} + \sqrt{\ln(N)} \mathbb{1}_{k=2} + \mathbb{1}_{k>2}\right) + l_b^2 \left(N^{\frac{5-k}{2}} \mathbb{1}_{k<5} + \sqrt{\ln(N)} \mathbb{1}_{k=5} + \mathbb{1}_{k>5}\right) \\
\leqslant & \left(l_b N^{1-\frac{k}{2}} + l_b^2 N^{\frac{5-k}{2}}\right) \mathbb{1}_{k<2} + \left(l_b \sqrt{\ln(N)} + l_b^2 N^{\frac{3}{2}}\right) \mathbb{1}_{k=2} + \left(l_b + l_b^2 N^{\frac{5-k}{2}}\right) \mathbb{1}_{2<k<5} \\
& + \left(l_b + l_b^2 \sqrt{\ln(N)}\right) \mathbb{1}_{k=5} + l_b \mathbb{1}_{k>5}
\end{aligned}$$

III.6 Proofs of the main results

Lemma III.6.1: *Choice of a convenient N*

Let $N_- := (N_{-,b})_{b>0}, N_+ := (N_{+,b})_{b>0} \subset \mathbb{N}^*$ such that

$$1 \ll N_- \ll N_+$$

Let $k \geqslant 0, T > 0, \gamma_b \in L^\infty\left([0,T], \mathcal{L}^1\left(L^2\left(\mathbb{R}^2\right)\right)\right)$ and assume

$$\forall t \in [0,T], \gamma_b(t) \geqslant 0, \mathrm{Tr}\left[\gamma_b(t)\right] = 1$$
$$\int_0^T \mathrm{Tr}\left[\gamma_b(t)\mathscr{L}_b^k\right] dt < \infty$$

then $\exists N \in [\![N_-, N_+]\!]$ such that

$$\int_0^T \mathrm{Tr}\left[\gamma_b(t)\mathscr{L}_b^k \Pi_{N-1:N+1}\right] dt \leqslant \frac{6}{N \ln\left(\dfrac{N_+}{N_-}\right)} \int_0^T \mathrm{Tr}\left[\gamma_b(t)\mathscr{L}_b^k\right] dt$$

Proof:

Assume for contradiction that

$$\forall N \in [\![N_-, N_+]\!], \int_0^T \mathrm{Tr}\left[\gamma_b(t)\mathscr{L}_b^k \Pi_{N-1:N+1}\right] dt > \frac{6}{N \ln\left(\dfrac{N_+}{N_-}\right)} \int_0^T \mathrm{Tr}\left[\gamma_b(t)\mathscr{L}_b^k\right] dt$$

then

$$\begin{aligned}
\int_0^T \mathrm{Tr}\left[\gamma_b(t)\mathscr{L}_b^k\right] dt \geqslant& \frac{1}{3} \sum_{N=N_-}^{N_+} \int_0^T \mathrm{Tr}\left[\gamma_b(t)\mathscr{L}_b^k \Pi_{N-1:N+1}\right] dt \\
>& \frac{2}{\ln\left(\dfrac{N_+}{N_-}\right)} \int_0^T \mathrm{Tr}\left[\gamma_b(t)\mathscr{L}_b^k\right] dt \sum_{N=N_-}^{N_+} \frac{1}{N} \underset{b\to\infty}{\sim} 2 \int_0^T \mathrm{Tr}\left[\gamma_b(t)\mathscr{L}_b^k\right] dt
\end{aligned}$$

which yields the desired contradiction.

We may now conclude the proof of our main result.

Proof of Theorem III.1.3:

With an integration by part,

$$\int_{\mathbb{R}^2} \varphi(0,z)\rho_{\gamma_b}(0,z)dz - \int_{\mathbb{R}_+\times\mathbb{R}^2} \rho_{\gamma_b}(t,z)\mathrm{GYRO}_{\rho_{\gamma_b}}(\varphi)(t,z)dtdz$$
$$= \int_{\mathbb{R}^2} \varphi(0,z)\rho_{\gamma_b}^{sc,\leqslant N}(0,z)dz - \int_{\mathbb{R}_+\times\mathbb{R}^2} \rho_{\gamma_b}^{sc,\leqslant N}(t,z)\mathrm{GYRO}_{\rho_{\gamma_b}}(\varphi)(t,z)dtdz$$
$$+ \int_{\mathbb{R}^2} \varphi(0,z)\left(\rho_{\gamma_b} - \rho_{\gamma_b}^{sc,\leqslant N}\right)(0,z)dz - \int_{\mathbb{R}_+\times\mathbb{R}^2} \left(\rho_{\gamma_b} - \rho_{\gamma_b}^{sc,\leqslant N}\right)(t,z)\mathrm{GYRO}_{\rho_{\gamma_b}}(\varphi)(t,z)dtdz$$
$$= \int_{\mathbb{R}_+\times\mathbb{R}^2} \varphi\mathrm{GYRO}_{\rho_{\gamma_b}}(\rho_{\gamma_b}^{sc,\leqslant N})dtdz$$
$$+ \int_{\mathbb{R}^2} \varphi(0,z)\left(\rho_{\gamma_b} - \rho_{\gamma_b}^{sc,\leqslant N}\right)(0,z)dz - \int_{\mathbb{R}_+\times\mathbb{R}^2} \left(\rho_{\gamma_b} - \rho_{\gamma_b}^{sc,\leqslant N}\right)\mathrm{GYRO}_{\rho_{\gamma_b}}(\varphi)dtdz \tag{III.6.1}$$

Let $T \geqslant 0$ be such that $\mathrm{supp}(\varphi) \subset [0,T]\times\mathbb{R}^2$. By Proposition III.3.1,

$$\forall t \in [0,T]\,, \mathrm{Tr}\left[\gamma_b(t)\right] = 1 \text{ and } 0 \leqslant \gamma_b(t) \leqslant 2\pi l_b^2$$

Let $t \in [0,T]$, then by Proposition III.3.3 applied to $W = V + \frac{1}{2}w \star \rho_{\gamma_b(t)}$ and Proposition III.3.2,

$$\begin{aligned}\mathrm{Tr}\left[\gamma_b(t)\mathscr{L}_b\right] &\leqslant \left|\mathrm{Tr}\left[\gamma_b(t)H_b(t)\right]\right| + \left\|V + \frac{1}{2}w\star\rho_{\gamma_b(t)}\right\|_{L^\infty}\\ &= \left|\mathrm{Tr}\left[\gamma_b(0)H_b(0)\right]\right| + \left\|V + \frac{1}{2}w\star\rho_{\gamma_b(t)}\right\|_{L^\infty}\\ &\leqslant \left|\mathrm{Tr}\left[\gamma_b(0)H_b(0)\right]\right| + \|V\|_{L^\infty} + \frac{1}{2}\|w\|_{L^\infty} \leqslant C(V,w)\end{aligned} \tag{III.6.2}$$

and

$$\|V + w\star\rho_{\gamma_b}(t)\|_{W^{\infty,4}} \leqslant \|V\|_{W^{\infty,4}} + \|w\|_{W^{\infty,4}}$$

so from Proposition III.5.4 for $k = 1, W = V + w\star\rho_{\gamma_b}(t)$ we get

$$\left|\int_{\mathbb{R}^2}\varphi(t,z)\mathrm{GYRO}_{\gamma_b}\left(\rho_{\gamma_b}^{\leqslant N}\right)(t,z)dz\right| \leqslant C(\varphi,V,w)\left(\frac{1}{l_b\sqrt{N}}\mathrm{Tr}\left[\gamma_b(t)\mathscr{L}_b\Pi_{N-1:N+1}\right]\right.$$
$$\left.+\left(\sqrt{N}+l_bN\right)\sqrt{\mathrm{Tr}\left[\gamma_b(t)\mathscr{L}_b\Pi_{N-1:N+1}\right]} + l_b^2N + \sqrt{\mathrm{Tr}\left[\gamma_b(t)\mathscr{L}_b\right]}\left(l_b\sqrt{N}+l_b^2N^2\right)\right)$$

Integrating over time,

$$\left|\int_{\mathbb{R}_+\times\mathbb{R}^2}\varphi\mathrm{GYRO}_{\rho_{\gamma_b}}(\rho_{\gamma_b}^{sc,\leqslant N})dtdz\right| \leqslant C(\varphi,V,w)\left(\frac{1}{l_b\sqrt{N}}\int_0^T\mathrm{Tr}\left[\gamma_b(t)\mathscr{L}_b\Pi_{N-1:N+1}\right]dt\right.$$

$$
\begin{aligned}
&+ \left(\sqrt{N} + l_b N\right) \int_0^T \sqrt{\mathrm{Tr}\left[\gamma_b(t)\mathscr{L}_b \Pi_{N-1:N+1}\right]} dt + l_b^2 NT \\
&+ \int_0^T \sqrt{\mathrm{Tr}\left[\gamma_b(t)\mathscr{L}_b\right]} \left(l_b\sqrt{N} + l_b^2 N^2\right) dt \Bigg) \\
&\leqslant C(\varphi, V, w) \left(\frac{1}{l_b\sqrt{N}} \int_0^T \mathrm{Tr}\left[\gamma_b(t)\mathscr{L}_b \Pi_{N-1:N+1}\right] dt \right. \\
&+ \left(\sqrt{N} + l_b N\right) \sqrt{T} \left(\int_0^T \mathrm{Tr}\left[\gamma_b(t)\mathscr{L}_b \Pi_{N-1:N+1}\right] dt \right)^{\frac{1}{2}} + l_b^2 NT \\
&\left. + \sqrt{T} \left(\int_0^T \mathrm{Tr}\left[\gamma_b(t)\mathscr{L}_b\right] dt \right)^{\frac{1}{2}} \left(l_b\sqrt{N} + l_b^2 N^2\right) \right)
\end{aligned}
$$

Choosing N as in Lemma III.6.1 and using (III.6.2), we have

$$
\begin{aligned}
&\left| \int_{\mathbb{R}_+ \times \mathbb{R}^2} \varphi \mathrm{GYRO}_{\rho_{\gamma_b}}(\rho_{\gamma_b}^{sc,\leqslant N}) dt dz \right| \\
&\leqslant C(\varphi, V, w) \left(\frac{1}{l_b N^{\frac{3}{2}} \ln\left(\frac{N_+}{N_-}\right)} + \frac{1 + l_b\sqrt{N}}{\sqrt{\ln\left(\frac{N_+}{N_-}\right)}} + l_b^2 N + l_b\sqrt{N} + l_b^2 N^2 \right) \\
&\leqslant C(\varphi, V, w) \left(\frac{1}{l_b N_-^{\frac{3}{2}} \ln\left(\frac{N_+}{N_-}\right)} + \frac{1 + l_b\sqrt{N_+}}{\sqrt{\ln\left(\frac{N_+}{N_-}\right)}} + l_b\sqrt{N_+} + l_b^2 N_+^2 \right)
\end{aligned}
$$

We start by imposing the constrains

$$
\frac{1}{l_b^{\frac{2}{3}}} \leqslant N_-, \quad N_+ \leqslant \frac{1}{l_b^2}
$$

so that

$$
l_b\sqrt{N_+} \leqslant 1, \quad 1 \leqslant l_b N_-^{\frac{3}{2}} \leqslant l_b N_+^{\frac{3}{2}}, \quad l_b\sqrt{N_+} \leqslant l_b^2 N_+^2
$$

and

$$\left|\int\limits_{\mathbb{R}_+\times\mathbb{R}^2} \varphi \mathrm{GYRO}_{\rho_{\gamma_b}}(\rho_{\gamma_b}^{sc,\leqslant N})dtdz\right| \leqslant C(\varphi,V,w)\left(\frac{1}{l_b N_-^{\frac{3}{2}}\ln\left(\frac{N_+}{N_-}\right)} + \frac{1}{\sqrt{\ln\left(\frac{N_+}{N_-}\right)}} + l_b^2 N_+^2\right) \tag{III.6.3}$$

With Proposition III.4.3 for $k = 1$ and (III.6.2),

$$\begin{aligned}\left|\int\limits_{\mathbb{R}^2} \varphi(0,z)\left(\rho_{\gamma_b} - \rho_{\gamma_b}^{sc,\leqslant N}\right)(0,z)dz\right| \leqslant& C(\varphi)\left(\frac{1}{\sqrt{N}}\sqrt{\mathrm{Tr}\left[\gamma_b(0)\Pi_{>N}\mathscr{L}_b\right]} + \sqrt{\mathrm{Tr}\left[\gamma_b(0)\mathscr{L}_b\right]}l_b\sqrt{N}\right)\\ \leqslant& C(\varphi,V,w)\left(\frac{1}{\sqrt{N}} + l_b\sqrt{N}\right)\\ \leqslant& C(\varphi,V,w)\left(\frac{1}{\sqrt{N_-}} + l_b^2 N_+^2\right)\end{aligned} \tag{III.6.4}$$

Finally, since $\mathrm{GYRO}_{\rho_{\gamma_b}}(\varphi) \in C_c^\infty(\mathbb{R}^2)$, $\mathrm{supp}\left(\mathrm{GYRO}_{\rho_{\gamma_b}}(\varphi)\right) \subset \mathrm{supp}(\varphi)$, and $\forall t \in [0,T]$,

$$\begin{aligned}\left\|\mathrm{GYRO}_{\rho_{\gamma_b}}(\varphi)(t,\bullet)\right\|_{W^{1,\infty}} &= \left\|\partial_t\varphi(t,\bullet) + \nabla^\perp(V + w\star\rho_{\gamma_b(t)})\cdot\nabla\varphi(t)\right\|_{W^{1,\infty}}\\ &\leqslant \|\varphi\|_{W^{1,\infty}}\left(1 + \left\|V + w\star\rho_{\gamma_b(t)}\right\|_{W^{1,\infty}}\right)\\ &\leqslant \|\varphi\|_{W^{1,\infty}}\left(1 + \|V\|_{W^{1,\infty}} + \|w\|_{W^{1,\infty}}\right)\end{aligned}$$

thus similarly to (III.6.4) we obtain

$$\left|\int\limits_{\mathbb{R}_+\times\mathbb{R}^2}\left(\rho_{\gamma_b} - \rho_{\gamma_b}^{sc,\leqslant N}\right)\mathrm{GYRO}_{\rho_{\gamma_b}}(\varphi)dtdz\right| \leqslant C(\varphi,V,w)\left(\frac{1}{\sqrt{N_-}} + l_b^2 N_+^2\right) \tag{III.6.5}$$

Inserting (III.6.3), (III.6.4), (III.6.5) in (III.6.1), and then taking

$$N_- := l_b^{-\alpha}, \quad N_+ := l_b^{-\beta}, \quad \text{with} \quad \frac{2}{3} < \alpha, \quad \alpha < \beta < 1 \tag{III.6.6}$$

we conclude that

$$\begin{aligned}&\left|\int\limits_{\mathbb{R}^2}\varphi(0,z)\rho_{\gamma_b}(0,z)dz - \int\limits_{\mathbb{R}_+\times\mathbb{R}^2}\rho_{\gamma_b}(t,z)\mathrm{GYRO}_{\rho_{\gamma_b}}(\varphi)(t,z)dtdz\right|\\ \leqslant& C(\varphi,V,w)\left(\frac{1}{l_b N_-^{\frac{3}{2}}\ln\left(\frac{N_+}{N_-}\right)} + \frac{1}{\sqrt{\ln\left(\frac{N_+}{N_-}\right)}} + l_b^2N_+^2 + \frac{1}{\sqrt{N_-}}\right)\\ =& C(\varphi,V,w)\left(\frac{l_b^{\frac{3}{2}\alpha-1}}{(\beta-\alpha)\ln\left(l_b^{-1}\right)} + \frac{1}{\sqrt{(\beta-\alpha)\ln\left(l_b^{-1}\right)}} + l_b^{2(1-\beta)} + l_b^{\frac{\alpha}{2}}\right) \leqslant C(\varphi,V,w)\frac{1}{\sqrt{\ln\left(l_b^{-1}\right)}}\end{aligned}$$

Proof of Theorem III.1.4:

Step 1: Weak limit

With Proposition III.3.1,

$$\forall t \in \mathbb{R}_+, \rho_{\gamma_b}^{sc,\leqslant N}(t) \geqslant 0 \quad \left\|\rho_{\gamma_b}^{sc,\leqslant N}(t)\right\|_{L^1} = 1,$$

Let $T > 0$, then

$$\left\|\rho_{\gamma_b}^{sc,\leqslant N}{}_{|[0,T]}\right\|_{L^1} = T$$

The truncated semi-classical densities are bounded in $\mathcal{M}\left([0,T] \times \mathbb{R}^2\right)$ so up to a subsequence we have weak star convergence in the sense of measures:

$$\rho_{\gamma_b}^{sc,\leqslant N} \underset{b\to\infty}{\overset{*}{\rightharpoonup}} \rho_T \in \mathcal{M}\left([0,T] \times \mathbb{R}^2\right)$$

By uniqueness of the limit

$$T_2 \geqslant T_1 \implies \rho_{T_2|[0,T_1]} = \rho_{T_1}$$

hence we constructed a limit $\rho \in \mathcal{M}\left([0,T] \times \mathbb{R}^2\right)$ such that $\forall \varphi \in C_c^0\left(\mathbb{R}_+ \times \mathbb{R}^2\right)$,

$$\int_{\mathbb{R}_+\times\mathbb{R}^2} \rho_{\gamma_b}^{sc,\leqslant N}\varphi \underset{b\to\infty}{\rightarrow} \int_{\mathbb{R}_+\times\mathbb{R}^2} \rho\varphi \tag{III.6.7}$$

Let $(\Omega_n)_{n\in\mathbb{N}}$ be an increasing sequence of open bounded subsets of $\mathbb{R}_+^* \times \mathbb{R}^2$, such that

$$\bigcup_{n\in\mathbb{N}} \Omega_n = \mathbb{R}_+^* \times \mathbb{R}^2$$

By [50, 1. theorem 6], the embedding $\mathcal{M}(\Omega_n) \subset W^{-1,1}(\Omega_n)$ is compact. So after a diagonal extraction we can have

$$\forall n \in \mathbb{N}, \rho_{\gamma_b}^{sc,\leqslant N} \underset{b\to\infty}{\rightarrow} \rho_n \in W^{-1,1}(\Omega_n)$$

but $W^{1,\infty}(\Omega_n) \subset C^0(\Omega_n)$, so $\rho_n = \rho_{|\Omega_n}$. Hence, $\forall n \in \mathbb{N}$,

$$\left\|\rho - \rho_{\gamma_b}^{sc,\leqslant N}\right\|_{W^{-1,1}(\Omega_n)} \underset{b\to\infty}{\rightarrow} 0 \tag{III.6.8}$$

$\left(\rho_{\gamma_b}^{sc,\leqslant N}(0)\right)_{b>0}$ is also bounded in $\mathcal{M}\left(\mathbb{R}^2\right)$, so after another extraction one has

$$\rho_{\gamma_b}^{sc,\leqslant N}(0) \underset{b\to\infty}{\overset{*}{\rightharpoonup}} \rho_0 \in \mathcal{M}\left(\mathbb{R}^2\right) \tag{III.6.9}$$

in the sense of measures.

Step 2: Error decomposition

Let $\varphi \in C_c^\infty\left(\mathbb{R}_+^\times \mathbb{R}^2\right)$,

$$\begin{aligned}
&\int_{\mathbb{R}^2} \varphi(0)\rho_0 - \int_{\mathbb{R}_+\times\mathbb{R}^2} \rho \mathrm{GYRO}_\rho(\varphi) \\
&= \int_{\mathbb{R}^2} \varphi(0)\rho_0 - \int_{\mathbb{R}_+\times\mathbb{R}^2} \rho_{\gamma_b}^{sc,\leqslant N}\mathrm{GYRO}_\rho(\varphi) + \int_{\mathbb{R}_+\times\mathbb{R}^2} \left(\rho_{\gamma_b}^{sc,\leqslant N} - \rho\right)\mathrm{GYRO}_\rho(\varphi) \\
&= \int_{\mathbb{R}^2} \varphi(0)\rho_0 - \int_{\mathbb{R}_+\times\mathbb{R}^2} \rho_{\gamma_b}^{sc,\leqslant N}\mathrm{GYRO}_{\rho_{\gamma_b}}(\varphi) + \int_{\mathbb{R}_+\times\mathbb{R}^2} \rho_{\gamma_b}^{sc,\leqslant N}\left(\mathrm{GYRO}_{\rho_{\gamma_b}}(\varphi) - \mathrm{GYRO}_\rho(\varphi)\right) \\
&\quad + \int_{\mathbb{R}_+\times\mathbb{R}^2} \left(\rho_{\gamma_b}^{sc,\leqslant N} - \rho\right)\mathrm{GYRO}_\rho(\varphi) \\
&= \int_{\mathbb{R}_+\times\mathbb{R}^2} \varphi\mathrm{GYRO}_{\rho_{\gamma_b}}\left(\rho_{\gamma_b}^{sc,\leqslant N}\right) + \int_{\mathbb{R}^2} \varphi(0)(\rho_0 - \rho_{\gamma_b}^{sc,\leqslant N}(0)) + \int_{\mathbb{R}_+\times\mathbb{R}^2} \rho_{\gamma_b}^{sc,\leqslant N}\nabla_z\varphi\cdot(\rho_{\gamma_b} - \rho)\star\nabla^\perp w \\
&\quad + \int_{\mathbb{R}_+\times\mathbb{R}^2} \left(\rho_{\gamma_b}^{sc,\leqslant N} - \rho\right)\mathrm{GYRO}_\rho(\varphi) \\
&= \int_{\mathbb{R}_+\times\mathbb{R}^2} \varphi\mathrm{GYRO}_{\rho_{\gamma_b}}\left(\rho_{\gamma_b}^{sc,\leqslant N}\right) + \int_{\mathbb{R}^2} \varphi(0)(\rho_0 - \rho_{\gamma_b}^{sc,\leqslant N}(0)) \\
&\quad + \int_{\mathbb{R}_+\times\mathbb{R}^2} (\rho_{\gamma_b} - \rho)\left(\nabla_z\varphi\cdot\rho_{\gamma_b}^{sc,\leqslant N}\right)\star\nabla^\perp w + \int_{\mathbb{R}_+\times\mathbb{R}^2} \left(\rho_{\gamma_b}^{sc,\leqslant N} - \rho\right)\mathrm{GYRO}_\rho(\varphi) \\
&= \int_{\mathbb{R}_+\times\mathbb{R}^2} \varphi\mathrm{GYRO}_{\rho_{\gamma_b}}\left(\rho_{\gamma_b}^{sc,\leqslant N}\right) + \int_{\mathbb{R}^2} \varphi(0)(\rho_0 - \rho_{\gamma_b}^{sc,\leqslant N}(0)) \\
&\quad + \int_{\mathbb{R}_+\times\mathbb{R}^2} \left(\rho_{\gamma_b} - \rho_{\gamma_b}^{sc,\leqslant N}\right)\left(\nabla_z\varphi\cdot\rho_{\gamma_b}^{sc,\leqslant N}\right)\star\nabla^\perp w + \int_{\mathbb{R}_+\times\mathbb{R}^2} \left(\rho_{\gamma_b}^{sc,\leqslant N} - \rho\right)\left(\nabla_z\varphi\cdot\rho_{\gamma_b}^{sc,\leqslant N}\right)\star\nabla^\perp w \\
&+ \int_{\mathbb{R}_+\times\mathbb{R}^2} \left(\rho_{\gamma_b}^{sc,\leqslant N} - \rho\right)\mathrm{GYRO}_\rho(\varphi)
\end{aligned} \tag{III.6.10}$$

Step 3: $\left(\nabla_z\varphi\cdot\rho_{\gamma_b}^{sc,\leqslant N}\right)\star\nabla^\perp w \in W^{1,\infty}\left(\mathbb{R}_+\times\mathbb{R}^2\right)$

The goal of this part is to prove that $\left(\nabla_z\varphi\cdot\rho_{\gamma_b}^{sc,\leqslant N}\right)\star\nabla^\perp$ is bounded in $W^{1,\infty}\left(\mathbb{R}_+\times\mathbb{R}^2\right)$ uniformly in t and b. Let $\varphi \in C_c^\infty\left(\mathbb{R}_+^*\times\mathbb{R}^2\right)$, then $\forall t \in \mathbb{R}_+$,

$$\left\|\left(\nabla\varphi(t)\cdot\rho_{\gamma_b}^{sc,\leqslant N}(t)\right)\star\nabla^\perp w\right\|_{W^{1;\infty}} \leqslant \left\|\nabla\varphi(t)\rho_{\gamma_b}^{sc,\leqslant N}(t)\right\|_{L^1}\|w\|_{W^{1;\infty}} \leqslant \|\varphi\|_{W^{1,\infty}}\|w\|_{W^{1;\infty}} \tag{III.6.11}$$

and with $\partial_t\varphi$ instead of φ

$$\left\|\left(\nabla\partial_t\varphi(t)\cdot\rho_{\gamma_b}^{sc,\leqslant N}(t)\right)\star\nabla^\perp w\right\|_{L^\infty} \leqslant \left\|\nabla\partial_t\varphi(t)\rho_{\gamma_b}^{sc,\leqslant N}(t)\right\|_{L^1}\|w\|_{L^\infty} \leqslant \|\varphi\|_{W^{1,\infty}}\|w\|_{L^\infty} \tag{III.6.12}$$

Let $x \in \mathbb{R}^2$, with an integration by parts,

$$
\begin{aligned}
&\left(\nabla\varphi(t) \cdot \partial_t \rho_{\gamma_b}^{sc,\leqslant N}(t)\right) \star \nabla^\perp w(x) \\
&= \int_{\mathbb{R}^2} \partial_t \rho_{\gamma_b}^{sc,\leqslant N}(t,z) \nabla_z \varphi(t,z) \cdot \nabla^\perp w(x-z) dz \\
&= - \int_{\mathbb{R}^2} \nabla^\perp \left(V + w \star \rho_{\gamma_b(t)}\right)(z) \cdot \nabla_z \rho_{\gamma_b}(t,z) \nabla_z \varphi(t,z) \cdot \nabla^\perp w(x-z) dz \\
&\quad + \int_{\mathbb{R}^2} \mathrm{GYRO}_{\rho_{\gamma_b}}(\rho_{\gamma_b}^{sc,\leqslant N})(t,z) \nabla_z \varphi(t,z) \cdot \nabla^\perp w(x-z) dz \\
&= \int_{\mathbb{R}^2} \rho_{\gamma_b}(t,z) \nabla^\perp \left(V + w \star \rho_{\gamma_b(t)}\right)(z) \cdot \nabla_z \left(\nabla_z \varphi(t,z) \cdot \nabla^\perp w(x-z)\right) dz \\
&\quad + \int_{\mathbb{R}^2} \mathrm{GYRO}_{\rho_{\gamma_b}}(\rho_{\gamma_b}^{sc,\leqslant N})(t,z) \nabla_z \varphi(t,z) \cdot \nabla^\perp w(x-z) dz \\
&= \int_{\mathbb{R}^2} \rho_{\gamma_b}(t,z) \nabla^\perp \left(V + w \star \rho_{\gamma_b(t)}\right)(z) \cdot \nabla_z^{\otimes 2} \varphi(t,z) \nabla^\perp w(x-z) dz \\
&\quad + \int_{\mathbb{R}^2} \rho_{\gamma_b}(t,z) \nabla^\perp \left(V + w \star \rho_{\gamma_b(t)}\right)(z) \cdot \nabla_z \otimes \nabla_z^\perp w(x-z) \nabla_z \varphi(t,z) dz \\
&\quad + \int_{\mathbb{R}^2} \mathrm{GYRO}_{\rho_{\gamma_b}}(\rho_{\gamma_b}^{sc,\leqslant N})(t,z) \nabla_z \varphi(t,z) \cdot \nabla^\perp w(x-z) dz
\end{aligned}
\tag{III.6.13}
$$

For the two first terms in (III.6.13) we use Hölder inequality, and for the last term Proposition III.5.4 with $\nabla_z \varphi(t) \cdot \nabla^\perp w(x - \bullet)$ as test function. Hence with the choices (III.6.6),

$$
\begin{aligned}
&\left\|\left(\nabla\varphi(t) \cdot \partial_t \rho_{\gamma_b}^{sc,\leqslant N}(t)\right) \star \nabla^\perp w\right\|_{L^\infty} \leqslant \left\|\nabla V + \nabla w \star \rho_{\gamma_b(t)}\right\|_{L^\infty} \left\|d^2\varphi(t)\right\|_{L^\infty} \|\nabla w\|_{L^\infty} \\
&+ \left\|\nabla V + \nabla w \star \rho_{\gamma_b(t)}\right\|_{L^\infty} \left\|d^2 w\right\|_{L^\infty} \|\nabla\varphi(t)\|_{L^\infty} \\
&+ C\left(1 + |\mathrm{supp}(\varphi)|\right) \left\|\nabla\varphi(t) \cdot \nabla^\perp w(x - \bullet)\right\|_{W^{1,\infty}} \left\|V + w \star \rho_{\gamma_b(t)}\right\|_{W^{4,\infty}} \frac{1}{\sqrt{\ln\left(l_b^{-1}\right)}}
\end{aligned}
$$

and $\forall t \in \mathbb{R}_+$,

$$
\begin{aligned}
&\left\|\nabla\varphi(t) \cdot \nabla^\perp w(x - \bullet)\right\|_{W^{1,\infty}} \leqslant \|\varphi(t)\|_{W^{2,\infty}} \|w\|_{W^{2,\infty}} \\
&\|\varphi(t)\|_{W^{k,\infty}} \leqslant \|\varphi\|_{W^{k,\infty}}
\end{aligned}
$$

hence

$$
\left\|\left(\nabla\varphi(t) \cdot \partial_t \rho_{\gamma_b}^{sc,\leqslant N}(t)\right) \star \nabla^\perp w\right\|_{L^\infty} \leqslant C(\varphi, V, w) \tag{III.6.14}
$$

With (III.6.11), (III.6.12), (III.6.14) we conclude that

$$
\left\|\left(\nabla_z \varphi \cdot \rho_{\gamma_b}^{sc,\leqslant N}\right) \star \nabla^\perp w\right\|_{W^{1,\infty}} \leqslant C(\varphi, V, w) \tag{III.6.15}
$$

Step 4: Conclusion

With the choices (III.6.6), by (III.6.3),

$$\left| \int\limits_{\mathbb{R}_+\times\mathbb{R}^2} \varphi \mathrm{GYRO}_{\rho_{\gamma_b}}(\rho_{\gamma_b}^{sc,\leqslant N}) dt dz \right| \leqslant C(\varphi, V, w) \frac{1}{\sqrt{\ln\left(l_b^{-1}\right)}} \tag{III.6.16}$$

With (III.6.9), since $\varphi(0) \in C_c^0(\mathbb{R}^2)$,

$$\int\limits_{\mathbb{R}^2} \varphi(0)(\rho_0 - \rho_{\gamma_b}^{sc,\leqslant N}(0)) \underset{b\to\infty}{\to} 0 \tag{III.6.17}$$

Using (III.6.11), similarly as for (III.6.4) we have

$$\left| \int\limits_{\mathbb{R}_+\times\mathbb{R}^2} \left(\rho_{\gamma_b} - \rho_{\gamma_b}^{sc,\leqslant N}\right) \left(\nabla_z \varphi \cdot \rho_{\gamma_b}^{sc,\leqslant N}\right) \star \nabla^\perp w \right| \leqslant C(\varphi, V, w) \frac{1}{\sqrt{\ln\left(l_b^{-1}\right)}} \tag{III.6.18}$$

Using (III.6.15) and (III.6.8) for n large enough so $\mathrm{supp}(\varphi) \subset \Omega_n$,

$$\begin{aligned} &\left| \int\limits_{\mathbb{R}_+\times\mathbb{R}^2} \left(\rho_{\gamma_b}^{sc,\leqslant N} - \rho\right) \left(\nabla_z \varphi \cdot \rho_{\gamma_b}^{sc,\leqslant N}\right) \star \nabla^\perp w \right| \\ &\leqslant \left\| \rho - \rho_{\gamma_b}^{sc,\leqslant N} \right\|_{W^{-1,1}(\Omega_n)} \left\| \left(\nabla_z \varphi \cdot \rho_{\gamma_b}^{sc,\leqslant N}\right) \star \nabla^\perp w \right\|_{W^{1,\infty}} \underset{b\to\infty}{\to} 0 \end{aligned} \tag{III.6.19}$$

And since $\mathrm{GYRO}_\rho(\varphi) \in C_c^0(\mathbb{R}\times\mathbb{R}^2)$, (III.6.7) gives

$$\int\limits_{\mathbb{R}_+\times\mathbb{R}^2} \left(\rho_{\gamma_b}^{sc,\leqslant N} - \rho\right) \mathrm{GYRO}_\rho(\varphi) \underset{b\to\infty}{\to} 0 \tag{III.6.20}$$

Hence the five terms in (III.6.10) converge to 0 respectively because of (III.6.16), (III.6.17), (III.6.18), (III.6.19), (III.6.20) hence the conclusion

$$\int\limits_{\mathbb{R}^2} \varphi(0)\rho_0 - \int\limits_{\mathbb{R}_+\times\mathbb{R}^2} \rho \mathrm{GYRO}_\rho(\varphi) = 0$$

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