Mean-Field limit of the Bose-Hubbard model in high dimension

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Motivation

Study: large system of quantum bosons

Usually [3]: many-body $N \to \infty$ mean field:

$$H_N := \sum_{i=1}^{N}(-\Delta_i) + \frac{1}{N}\sum_{1\leqslant i<j\leqslant N} w(X_i - X_j) \quad \text{acting on } L^2(\mathbb{R}^d,\mathbb{C})^{\otimes_+ N}$$

Statistical description of the interaction for a mean particle $\varphi \in L^2(\mathbb{R}^d)$:

$$h^{\varphi}_{\text{Hartree}} = -\Delta + |\varphi|^2 \star w$$

Bose-Hubbard model: interacting bosons on a lattice

- Great success in physics:
  Mott-insulator \ Superfluid phase transition, experimental observation [2] & theoretical description of the mean field theory [1]
- Mean field justified when $d \to \infty$ and effective in $d = 3$
- Simple mathematical description

Goals:

- Mean field limit as $d \to \infty$ of the dynamics and the ground state energy
- Describe a phase transition
- Strong and local particle interactions

Bose-Hubbard model

Lattice: $\Lambda := (\mathbb{Z}/L\mathbb{Z})^d$ with $d, L \in \mathbb{N}$ such that $d, L \geqslant 2$ of volume $|\Lambda| = L^d$

One-lattice-site Hilbert space: $\ell^2(\mathbb{C})$ of canonical basis

$$|n\rangle := (0,\dots,0,\underbrace{1}_{n^{th}\text{index}},0,\dots),\ n \in \mathbb{N}$$

2^{nd} quantization: creation and annihilation operators:

$$a\,|0\rangle := 0,\quad \forall n \in \mathbb{N}^*,\ a\,|n\rangle := \sqrt{n}\,|n-1\rangle,$$
$$\forall n \in \mathbb{N},\ a^\dagger\,|n\rangle := \sqrt{n+1}\,|n+1\rangle$$
$$[a, a^\dagger] = \mathbb{1}_{\ell^2} \tag{CCR}$$

Particle number: $\mathcal{N} := a^\dagger a$

Fock space: $\ell^2(\mathbb{C})^{\otimes|\Lambda|} \cong \mathcal{F}_+\left(L^2(\Lambda,\mathbb{C})\right) := \bigoplus_{n\in\mathbb{N}} L^2(\Lambda,\mathbb{C})^{\otimes_+ n}$

Bose-Hubbard hamiltonian of parameters $J, \mu, U \in \mathbb{R}$:

$$H_d := -\frac{J}{2d}\overbrace{\sum_{\substack{x,y\in\Lambda\\ x\sim y}} a_x^\dagger a_y}^{\mathcal{O}(2d|\Lambda|)} + (J-\mu)\sum_{x\in\Lambda}\mathcal{N}_x + \frac{U}{2}\sum_{x\in\Lambda}\mathcal{N}_x(\mathcal{N}_x - 1)$$

Dynamics for $\gamma_d \in L^\infty\left(\mathbb{R}_+, S^1\left(\ell^2(\mathbb{C})^{\otimes|\Lambda|}\right)\right)$:

$$i\partial_t\gamma_d(t) = [H_d, \gamma_d(t)] \tag{B-H}$$

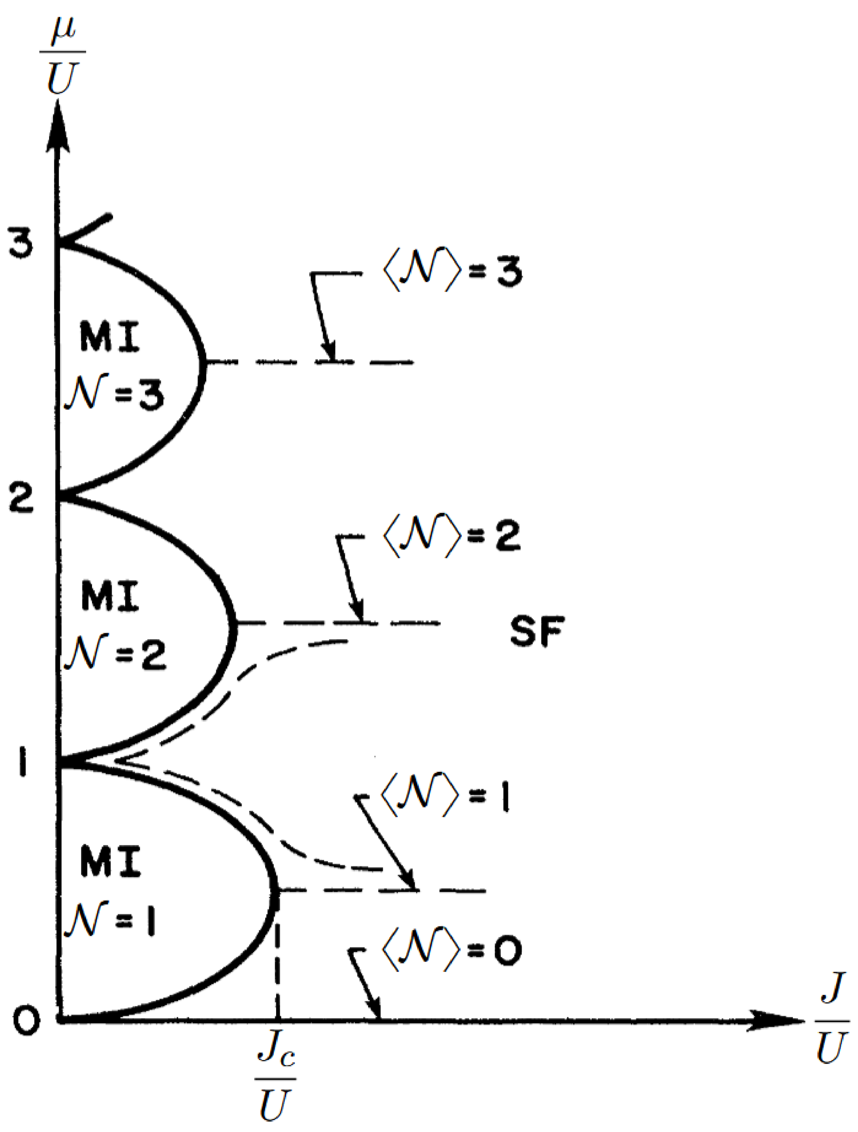
First reduced one-lattice-site density matrix:

$$\gamma_d^{(1)} := \frac{1}{|\Lambda|}\sum_{x\in\Lambda}\mathrm{Tr}_{\Lambda\setminus\{x\}}(\gamma_d)$$

Mean field theory

Mean field hamiltonian for $\varphi \in \ell^2(\mathbb{C})$:

$$h^\varphi := -J\left(\overline{\alpha_\varphi}a + \alpha_\varphi a^\dagger - |\alpha_\varphi|^2\right) + (J-\mu)\mathcal{N} + \frac{U}{2}\mathcal{N}(\mathcal{N}-1)$$



Phase diagram obtained [1] by minimizing $\varphi \mapsto \langle\varphi|h^\varphi\varphi\rangle$

with the order parameter

$$\alpha_\varphi := \langle\varphi|a\varphi\rangle$$

Phase transition: decompose

$$\varphi =: \sum_{n\in\mathbb{N}}\lambda_n\,|n\rangle$$
$$\implies \alpha_\varphi = \sum_{n\in N}\sqrt{n+1}\,\overline{\lambda_n}\lambda_{n+1}$$

- Mott Insulator (MI): $\alpha_\varphi = 0$
- Superfluid (SF): $\alpha_\varphi > 0$

Dynamics:

For $\varphi \in L^\infty\left(\mathbb{R}_+, \ell^2(\mathbb{C})\right)$,

$$i\partial_t\varphi(t) = h^{\varphi(t)}\varphi(t) \tag{mf}$$

Corresponding projection

$$p_\varphi := |\varphi\rangle\langle\varphi|$$

Results

Theorem: *S.Farhat D.P S.Petrat 2025*

Dynamics [4]: Assume

- γ_d solves (B-H) with $\gamma_d(0) \in S^1\left(\ell^2(\mathbb{C})^{\otimes|\Lambda|}\right)$ such that $\mathrm{Tr}(\gamma_d(0)) = 1$
- φ solves (mf) with $\varphi(0) \in \ell^2(\mathbb{C})$ such that $\|\varphi\|_{\ell^2} = 1$
- $\exists c_1, c_2 > 0$ such that $\forall n \in \mathbb{N}$,

$$\mathrm{Tr}\left(p_\varphi(0)\mathbb{1}_{\mathcal{N}=n}\right) \leqslant c_1 e^{-\frac{n}{c_2}},\quad \mathrm{Tr}\left(\gamma_d^{(1)}(0)\mathbb{1}_{\mathcal{N}=n}\right) \leqslant c_1 e^{-\frac{n}{c_2}}.$$

Then $\exists\, C := C\left(J, c_1, c_2, \mathrm{Tr}\left(p_\varphi(0)\mathcal{N}\right)\right) > 0$ such that $\forall t \in \mathbb{R}_+$,

$$\left\|\gamma_d^{(1)}(t) - p_\varphi(t)\right\|_{S^1} \leqslant C\left(\left\|\gamma_d^{(1)}(0) - p_\varphi(0)\right\|_{S^1} + \frac{1}{d\sqrt{\ln(d)}}\right)e^{Cte^{Ct}\sqrt{\ln(d)}}$$

Ground state (WIP): If $J, \mu \geqslant 0, U > 0$, then for d large enough,

$$-\frac{\ln(d)^3}{d} \lesssim \inf_{\substack{\psi_d\in\ell^2(\mathbb{C})^{\otimes|\Lambda|}\\ \|\psi_d\|=1}}\frac{\langle\psi_d|H_d\psi_d\rangle}{|\Lambda|} - \inf_{\substack{\varphi\in\ell^2(\mathbb{C})\\ \|\varphi\|=1}}\langle\varphi|h^\varphi\varphi\rangle \leqslant 0$$

If $\left\|\gamma_d^{(1)}(0) - p_\varphi(0)\right\|_{S^1} = \mathcal{O}\left(\frac{1}{d}\right)$, then $\forall t \in \mathbb{R}_+$, $\left\|\gamma_d^{(1)}(t) - p_\varphi(t)\right\|_{S^1} \lesssim e^{Cte^{Ct}\sqrt{\ln(d)}-\ln(d)} \underset{d\to\infty}{\to} 0$

Convergence of the order parameter: Use $a \leqslant \mathcal{N} + 1$ and insert a $\mathcal{N}$-cut-off:

$$\begin{aligned}\left|\mathrm{Tr}\left(\gamma_d^{(1)}a\right) - \mathrm{Tr}\left(p_\varphi a\right)\right| &\leqslant \left\|\left(\gamma_d^{(1)} - p_\varphi\right)a\right\|_{S^1} = \left\|\left(\gamma_d^{(1)} - p_\varphi\right)a\left(\mathcal{N}+1\right)^{-1}\left(\mathcal{N}+1\right)\right\|_{S^1}\\ &\leqslant \left\|\left(\gamma_d^{(1)} - p_\varphi\right)\underbrace{a\left(\mathcal{N}+1\right)^{-1}\left(\mathcal{N}+1\right)\mathbb{1}_{\mathcal{N}<M}}_{\leqslant M}\right\|_{S^1} + \left\|\left(\gamma_d^{(1)} - p_\varphi\right)\underbrace{a\left(\mathcal{N}+1\right)^{-1}}_{\leqslant 1}\left(\mathcal{N}+1\right)\mathbb{1}_{\mathcal{N}\geqslant M}\right\|_{S^1}\\ &\leqslant M\left\|\gamma_d^{(1)} - p_\varphi\right\|_{S^1} + \underbrace{\mathrm{Tr}\left(\gamma_d^{(1)}(\mathcal{N}+1)\mathbb{1}_{\mathcal{N}\geqslant M}\right) + \mathrm{Tr}\left(p_\varphi(\mathcal{N}+1)\mathbb{1}_{\mathcal{N}\geqslant M}\right)}_{\to 0 \text{ when } M\to\infty \text{ since the particle numbers are conserved}}\end{aligned}$$

Any choice of $M \gg 1$ such that $M\left\|\gamma_d^{(1)} - p_\varphi\right\|_{S^1} \ll 1$ as $d \to \infty$ is sufficient to prove that $\left\|\left(\gamma_d^{(1)} - p_\varphi\right)a\right\|_{S^1} \underset{d\to\infty}{\to} 0$

References

[1] M.P.A. Fisher P.B.Weichman G.Grinstein D.S.Fisher. “Boson localization and the superfluid-insulator transition”. In: *Phys. Rev. B* (1989). DOI: https://doi.org/10.1103/PhysRevB.40.546.

[2] M.Greiner O.Mandel T.Rom A.Altmeyer A.Widera T.W.Hänsch I.Bloch. “Quantum phase transition from a superfluid to a Mott insulator in an ultracold gas of atoms”. In: *Physica B: Condensed Matter* (2003). DOI: https://doi.org/10.1016/S0921-4526(02)01872-0.

[3] M.Lewin, P.T.Nam, and N.Rougerie. “Derivation of Hartree’s theory for generic mean-field Bose systems”. In: *Advances in Mathematics* 254 (2014). DOI: https://doi.org/10.1016/j.aim.2013.12.010.

[4] S.Farhat D.Périce S.Petrat. “Mean-Field Dynamics of the Bose-Hubbard Model in High Dimension”. In: (2025). DOI: https://doi.org/10.48550/arXiv.2501.05304.