Mean field limit for 2D fermions under large magnetic field with magnetic periodic boundary conditions

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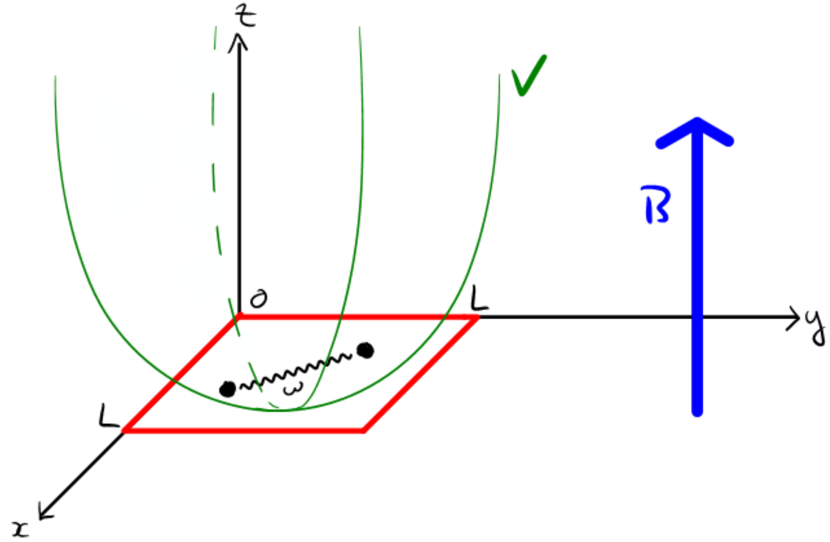
Part I
Model

I.1 Physical model

Our particles:

- spinless fermions
- 2D compact domain $\Omega := [0, L]^2$
- perpendicular uniform magnetic field B
- magnetic periodic boundary conditions



Mean field Hamiltonian

$$H_N := \sum_{j=1}^{N} \left(\left[-i\nabla_j - A_j \right]^2 + V_j \right) + \frac{1}{N} \sum_{1 \leqslant i < j \leqslant N} w_{ij} \quad \text{on} \quad L^2_{asym}(\Omega^N) := \bigwedge^N L^2(\Omega) \quad \text{(I.1)}$$

Interested in

$$E_N^0 := \inf \left\{ \langle \Psi_N | H_N | \Psi_N \rangle , \Psi_N \in Dom(H_N), \langle \Psi_N | \Psi_N \rangle = 1 \right\} \quad \text{(I.2)}$$

and the reduced densities ρ_N when $N \to \infty$ and $B \to \infty$

I.2 Magnetic periodic boundary conditions

Assume $\exists\phi \in C^\infty(\Omega, \mathbb{R})$ st

$$A = \nabla^\perp \phi = \begin{pmatrix} -\partial_y \phi \\ \partial_x \phi \end{pmatrix} \text{ and } B = \nabla \wedge A \tag{I.3}$$

in Landau gauge

$$\phi_{Lan}(x+iy) = \frac{B}{2} y^2 \text{ and } A_{Lan} := B \begin{pmatrix} -y \\ 0 \end{pmatrix} \tag{I.4}$$

Translation operator $T_R \psi(z) := \psi(z - R)$, problem:

$$[T_R, i\nabla + A] = [T_R, A] \neq 0 \tag{I.5}$$

$B = \nabla \wedge A \implies T_R A - A = \nabla \varphi_R$, we define

$$\tau_R := e^{i\varphi_R} T_R \tag{I.6}$$

We recover

$$[\tau_R, i\nabla + A] = 0 \tag{I.7}$$

I.3 Landau level quantization

$$H := (i\nabla + A)^2 \quad \text{with} \quad Dom(H) := \left\{\psi \in H^2(\Omega) \text{ st } \tau_L\psi = \psi, \tau_{iL}\psi = \psi\right\} \tag{I.8}$$

Proposition I.1: *spectral analysis of H*

$Dom(H)$ is dense in $L^2(\Omega)$, H is a closed positive self-adjoint operator, its spectrum is completely punctual and $L^2(\Omega)$ is a Hilbertian direct sum of H eigen-spaces.

Annihilation and creation operators

Magnetic momentum:

$$i\nabla + A := \begin{pmatrix} \pi_x \\ \pi_y \end{pmatrix} \tag{I.9}$$

Annihilation and creation operators:

$$a := \frac{\pi_y - i\pi_x}{\sqrt{2B}} \qquad a^\dagger := \frac{\pi_y + i\pi_x}{\sqrt{2B}} \tag{I.10}$$

Commutation relations:

$$[\pi_x, \pi_y] = iB \implies [a, a^\dagger] = \mathbb{1} \tag{I.11}$$

We obtain

$$H = 2B\left(\hat{n} + \frac{1}{2}\right) \quad \text{with} \quad \hat{n} := a^\dagger a \tag{I.12}$$

Let $n \in \mathbb{N}$, the n^{th} Landau level is

$$nLL := \{\psi \in Dom(H) \text{ st } \hat{n}\psi = n\psi\} \quad \text{and has energy} \quad E_n := 2B\left(n + \frac{1}{2}\right) \tag{I.13}$$

Proposition I.2: *Landau levels properties*

$\frac{a^\dagger}{\sqrt{n+1}}$ is a unitary mapping from nLL to $(n+1)LL$ of inverse $\frac{a}{\sqrt{n+1}}$

The lowest Landau level is

$$LLL := 0LL = \left\{\psi \in Dom(H) \text{ st } \exists f \in \mathcal{O}(\Omega_L) \text{ and } \psi = fe^{-\phi}\right\} \tag{I.14}$$

I.4 Landau level degeneracy

Properties I.3: *Magnetic translation properties*

Assume there exist a wave function in LLL, let d be its number of zeros in Ω, then

$$\int_{\partial\Omega} A \cdot dl = 2\pi d = BL^2 \text{ and therefore } d_N := \frac{B_N L^2}{2\pi} \in \mathbb{N}^* \tag{I.15}$$

and d is also the LLL degeneracy

Sketch of the proof:

Stockes theorem:

$$\int_{\partial\Omega_L} A.dl = \int_{\Omega_L} BdS = BL^2 \tag{I.16}$$

traduce boundary conditions on $\partial_z ln(f)$ and compute

$$d = \frac{1}{2i\pi} \int_{\partial\Omega_L} \partial_z ln(f) dz \tag{I.17}$$

Fourier transform f, use holomorphy and pseudo-periodicity

I.5 Mean field scaling

Characteristic lengths

- $N^{-\frac{1}{2}}$ for particle density
- $l_B := \frac{1}{\sqrt{B}}$, the magnetic length

The square ratio is $\frac{\frac{1}{N}}{l_B^2} = \frac{B}{N}$

Large magnetic field limit :

- $q \in \mathbb{N}$ first Landau level fully filled
- qLL partially filled with filling ratio $r \in [0, 1)$

Fix the limit

$$N \to \infty, \frac{N}{d_N} \underset{N\to\infty}{\to} q + r \tag{I.18}$$

so

$$\frac{N}{d_N} = \frac{2\pi N}{BL^2} \underset{N\to\infty}{\to} q + r \text{ therefore } B \underset{N\to\infty}{\sim} \frac{2\pi N}{(q+r)L^2} \tag{I.19}$$

I.6 Limit energy functional

$$\mathcal{E}_{class}[\rho] = \int_{\Omega} V\rho + \frac{1}{2}\int_{\Omega}\int_{\Omega} w(x-y)\rho(x)\rho(y)dxdy \tag{I.20}$$

with $\rho \in L^1(\Omega, \mathbb{R}_+)$ st

$$\int \rho = \lim_{N\to\infty} \frac{rd_N}{N} = \frac{r}{q+r} \quad \text{and} \quad \rho \leqslant \frac{1}{(q+r)L^2} \tag{I.21}$$

Fundamental energies:

$$\mathcal{E}^0_{class} := \inf\left\{\mathcal{E}_{class}[\rho], \rho \in L^1(\Omega, \mathbb{R}_+), \quad \text{satisfying} \quad (I.21)\right\} \tag{I.22}$$

I.7 Main result

We assume

$$w(x,y) := \tilde{w}[d(x,y)] \ \text{ with } \ d(x,y) := \min_{r\in L\mathbb{Z}^2} \|x-y+r\| \tag{I.23}$$

Let E_q be the energy of the q lowest Landau levels (fully filled) and their interactions with qLL (partially filled)

Theorem I.1: *Convergence with magnetic periodic conditions*

If $V \in L^1([0,L]^2)$ and $u \to u\tilde{w}(u) \in L^1\left(\left[0, \frac{L}{\sqrt{2}}\right]\right)$

$$E(N) := \frac{E_N^0 - E_q}{N} \to \mathcal{E}_{class}^0 \tag{I.24}$$

and up to a sub-sequence ρ_N converges in $\sigma\big(L^\infty(\Omega), L^1(\Omega)\big)$ to a convex combination of $\mathcal{E}_{class}$ minimizers

Part II
Tools

II.1 nLL basis and projector

Proposition II.1: *nLL wave functions*

We have the commutation

$$[a^\dagger, \tau_{-il\frac{L}{d}}] = [a, \tau_{-il\frac{L}{d}}] = 0 \tag{II.1}$$

The following family indexed by $(n, l) \in \mathbb{N} \times [\![0, d-1]\!]$ is an Hilbert basis of eigen functions of H in Landau gauge:

$$\psi_{nl} := \frac{{a^\dagger}^n}{\sqrt{n!}} \tau^l_{-i\frac{L}{d}} \psi_{00} \tag{II.2}$$

with

$$\psi_{00} = \frac{B^{\frac{1}{4}}}{\pi^{\frac{1}{4}}\sqrt{L}} e^{-\frac{B}{2}y^2} \theta\left(\frac{d}{L}z, id\right) \quad \text{and} \quad \theta(z, \tau) = \sum_{k\in\mathbb{Z}} e^{i\pi\tau k^2 + 2i\pi kz} \tag{II.3}$$

nLL projector

Define

$$\Pi_n := \sum_{l=0}^{d-1} |\psi_{nl}\rangle\langle\psi_{nl}| \tag{II.4}$$

$$\Pi_{n,R}(x,y) := g(x-R)\Pi_n(x-y)g(y-R) \tag{II.5}$$

They satisfy the resolution of identity

$$\sum_{n=0}^{\infty} \Pi_n = \mathbb{1} \tag{II.6}$$

$$\sum_{n=0}^{\infty} \int_{\Omega} \Pi_{n,R} = \mathbb{1} \tag{II.7}$$

II.2 Useful tools

Lemma II.1: *convergence of the projector*

$$\Pi_n(x,y) \underset{N\to\infty}{\sim} \frac{1}{2\pi l_B^2} e^{-\frac{|x-y|}{4l_B^2}+\frac{i}{2l_B^2}Im[\overline{x}y]} \tag{II.8}$$

uniformly with the convergence rate

$$l_B^2 \left\| \Pi_{n,L}(x,y) - \frac{1}{2\pi} e^{-\frac{|x-y|}{4l_B^2}+\frac{i}{2l_B^2}Im[\overline{x}y]} \right\|_\infty \leqslant C(n) l_B \tag{II.9}$$

Theorem II.1: *De Finetti or Hewitt-Savage*

Let $\mu \in \mathcal{P}_s(\Omega^{\mathbb{N}})$ be a symmetric probability measure, $\exists P_\mu \in \mathcal{P}(\mathcal{P}(\Omega))$ such that :

$$\forall n \in \mathbb{N}, \mu^{(n)} = \int_{\mathcal{P}(\Omega)} \rho^{\otimes n} dP_\mu(\rho) \tag{II.10}$$

where $\mu^{(n)}$ is the n^{th} marginal of μ

II.3 N-body ground state and densities

Let Γ_N be a density matrix on $L^2_{asym}(\Omega^N)$, with $\gamma_N^{(1)}$ and $\gamma_N^{(2)}$ its first and second reduced densities.

Quantum energy

$$\mathcal{E}_N[\Gamma_N] := \mathrm{Tr}(h\gamma_N^{(1)}) + \frac{1}{2}\mathrm{Tr}(w\gamma_N^{(2)}) = \mathrm{Tr}(H_N\Gamma_N) \text{ if } \Gamma_N = |\psi_N\rangle\langle\psi_N| \tag{II.11}$$

Husimi functions

- $m^{(1)}(n, R) := \mathrm{Tr}(\Pi_{n,R}\gamma_N^{(1)})$
- $m^{(2)}(n_1, n_2; R_1, R_2) := \mathrm{Tr}\left((\Pi_{n_1,R_1} \otimes \Pi_{n_2,R_2})\gamma_N^{(2)}\right)$

Total density $\rho_{\gamma_N}(x) := \gamma_N^{(1)}(x, x)$ approximated by

$$\rho_{\gamma_N} = \sum_{n=0}^{\infty} m^{(1)}(n, .) + \text{error term} \tag{II.12}$$

II.4 Energy computation

$$
\begin{aligned}
\mathcal{E}_N[\Gamma_N] = \sum_{n=0}^{\infty} E_n \int_\Omega m^{(1)}(n,x)dx \\
+ \sum_{n=0}^{\infty} \int_\Omega V(x) m^{(1)}(n,x)dx \\
+ \frac{1}{2N} \sum_{n_1,n_2} \int_\Omega \int_\Omega w(x-y) m^{(2)}(n_1,x;n_2,y)dxdy + \text{error terms}
\end{aligned}
\tag{II.13}
$$

- Mean field approximation : $m^{(2)} = m^{(1)} \otimes m^{(1)}$
- Pauli principle : $0 \leqslant \Gamma_N \leqslant \mathbb{1} \implies 0 \leqslant m^{(1)}(q,R) \leqslant \frac{N}{(q+r)L^2}$
- subtract E_q, set $\rho_N = \frac{m^{(1)}(q,.)}{N}$

$$
\mathcal{E}_{class}[\rho] = \int_\Omega V\rho + \frac{1}{2} \int_\Omega \int_\Omega w(x-y)\rho(x)\rho(y)dxdy \tag{II.14}
$$

Part III
Sketch of the proof

III.1 Lower bound

Let $(\Gamma_N)_{N\in\mathbb{N}}$ be a minimizing sequence of $\lim E(N)$

- Extract a weakly* convergent sequence from $\rho_N^{(2)} := \frac{m_N^{(2)}(q,q;\cdot,\cdot)}{N^2}$ of limit $\rho^{(2)}$

$$\frac{E_N(\Gamma_N) - E_q}{N} \geqslant \mathcal{E}_{class}[\rho_N] + errors \tag{III.1}$$

$$\int_\Omega \int_\Omega [w(x-y) + V(x) + V(y)]\, d\rho_N^{(2)}(x,y) \underset{N\to\infty}{\longrightarrow} \tag{III.2}$$

$$\int_\Omega \int_\Omega [w(x-y) + V(x) + V(y)]\, d\rho^{(2)}(x,y) \tag{III.3}$$

- Then, with De Finetti theorem $\rho^{(2)} = \int_{\mathcal{P}(\Omega)} \rho^{\otimes 2} dP_\mu(\rho)$:

$$\lim_{N\to\infty} E(N) \geqslant \frac{1}{2} \int_{\mathcal{P}(\Omega)} \int_\Omega \int_\Omega [w(x-y) + V(x) + V(y)]\, d\rho^{\otimes 2}(x,y) dP_\mu(\rho) \geqslant \mathcal{E}_{class}^0$$

III.2 Upper bound

Let ρ be an argument of $\mathcal{E}_{class}$

Construct Slater determinant with density that approximate ρ to apply variational principle.

- $H_\rho = (i\nabla + A)^2 + r(\rho)$ with negatives eigen values $(\lambda_j)_j$
- Weyl asymptotic to approximate

$$\sum_{\lambda_j \leqslant 0} \lambda_j = \mathrm{Tr}(H_\rho \mathbb{1}_{H_\rho \leqslant 0}) = \lim_{N \to \infty} \mathrm{Tr}(H_\rho \gamma_N) \tag{III.4}$$

- Feynman Hallmann theorem to show ρ_{γ_N} converges to ρ in $\sigma\big(L^\infty(\Omega), L^1(\Omega)\big)$

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