
Mean-Field Dynamics of the Bose-Hubbard Model in High Dimension

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Abstract

The Bose-Hubbard model effectively describes bosons on a lattice with on-site interactions and nearest-neighbour hopping, serving as a foundational framework for understanding strong particle interactions and the superfluid to Mott insulator transition. We present a result establishing the validity of a mean-field approximation for the dynamics of quantum systems in high dimension, using the Bose-Hubbard model on a square lattice as a case study. Our result is a trace norm estimate between the one-lattice-site reduced density of the Schrödinger dynamics and the mean-field dynamics in the limit of large dimension.

Motivations

Goal: rigorously derives large dimensional mean field limits since claim from physics literature: mean field theory exact in $d = +\infty$.

Usual many-body $N \rightarrow \infty$ mean field:

$$H_N := \sum_{i=1}^N (-\Delta_i) + \frac{1}{N} \sum_{1 \leq i < j \leq N} w(X_i - X_j)$$

Bose-Hubbard model: interacting bosons on a lattice

- Simple mathematical description: finite lattice model
- Great success in physics: description of Mott-insulator \ Superfluid phase transition experimental observation [2] and theoretical description of mean field theory [1]
- Numerics shows mean field already effective in $d = 3$

Result: [3]

- Convergence of the many-body dynamics to the mean field dynamics when $d \rightarrow \infty$
- Describe a phase transition
- Strong particle interactions

Bose-Hubbard model

Lattice: $\Lambda := (\mathbb{Z}/L\mathbb{Z})^d$ with $d, L \in \mathbb{N}$ such that $d, L \geq 2$ of volume $|\Lambda| = L^d$

One-lattice-site Hilbert space: $\ell^2(\mathbb{C})$ of canonical basis $|n\rangle := (0, \dots, 0, \underbrace{1}_{n^{th} \text{ index}}, 0, \dots), n \in \mathbb{N}$

2^{nd} **quantization:** creation and annihilation operators:

$$\begin{aligned} a|0\rangle &:= 0 \quad \forall n \in \mathbb{N}^*, \quad a|n\rangle := \sqrt{n}|n-1\rangle, \\ \forall n \in \mathbb{N}, \quad a^\dagger|n\rangle &:= \sqrt{n+1}|n+1\rangle \\ [a, a^\dagger] &= 1 \end{aligned} \tag{CCR}$$

Particle number: $\mathcal{N} := a^\dagger a$

Fock space:

$$\mathcal{F} := \ell^2(\mathbb{C})^{\otimes |\Lambda|} \cong \mathcal{F}_+ (L^2(\Lambda, \mathbb{C})) := \bigoplus_{n \in \mathbb{N}} L^2(\Lambda, \mathbb{C})^{\otimes n}$$

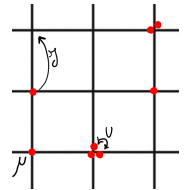
Indeed:

$$\mathcal{F}_+ (L^2(\Lambda, \mathbb{C})) = \mathcal{F}_+ \left(\bigoplus_{x \in \Lambda} \mathbb{C} \right) \cong \bigotimes_{x \in \Lambda} \mathcal{F}_+(\mathbb{C}) = \ell^2(\mathbb{C})^{\otimes |\Lambda|}$$

If A is an operator on $\ell^2(\mathbb{C})$ and $x \in \Lambda$ denote A_x the operator on \mathcal{F} acting on site x as A and as identity on other sites.

Bose-Hubbard hamiltonian of parameters $J, \mu, U \in \mathbb{R}$:

$$H_d := -\frac{J}{2d} \sum_{\substack{x, y \in \Lambda \\ x \sim y}} \overbrace{a_x^\dagger a_y}^{\mathcal{O}(2d|\Lambda|)} + (J - \mu) \sum_{x \in \Lambda} \mathcal{N}_x + \frac{U}{2} \sum_{x \in \Lambda} \mathcal{N}_x (\mathcal{N}_x - 1)$$



Mean field with respect to sites interactions and not particle interactions due to large coordination number.

Dynamics for $\gamma_d \in L^\infty(\mathbb{R}_+, \mathcal{L}^1(\mathcal{F}))$:

$$i\partial_t \gamma_d(t) = [H_d, \gamma_d(t)] \tag{B-H}$$

First one-lattice-site reduced density matrix:

$$\gamma_d^{(1)} := \frac{1}{|\Lambda|} \sum_{x \in \Lambda} \text{Tr}_{\Lambda \setminus \{x\}} (\gamma_d)$$

Mean field theory

Mean field hamiltonian for $\varphi \in \ell^2(\mathbb{C})$:

$$h^\varphi := -J(\overline{\alpha_\varphi}a + \alpha_\varphi a^\dagger - |\alpha_\varphi|^2) + (J - \mu)\mathcal{N} + \frac{U}{2}\mathcal{N}(\mathcal{N} - 1)$$

with the order parameter

$$\alpha_\varphi := \langle \varphi | a \varphi \rangle$$

Phase transition: Decompose

$$\varphi =: \sum_{n \in \mathbb{N}} \lambda_n |n\rangle \implies \alpha_\varphi = \sum_{n \in \mathbb{N}} \sqrt{n+1} \overline{\lambda_n} \lambda_{n+1}$$

- Mott Insulator (MI): $\alpha_\varphi = 0$
- Superfluid (SF): $\alpha_\varphi > 0$

Dynamics

For $\varphi \in L^\infty(\mathbb{R}_+, \ell^2(\mathbb{C}))$,

$$i\partial_t \varphi(t) = h^{\varphi(t)} \varphi(t)$$

Corresponding projection

$$p_\varphi := |\varphi\rangle \langle \varphi| \quad q_\varphi := 1 - p_\varphi$$

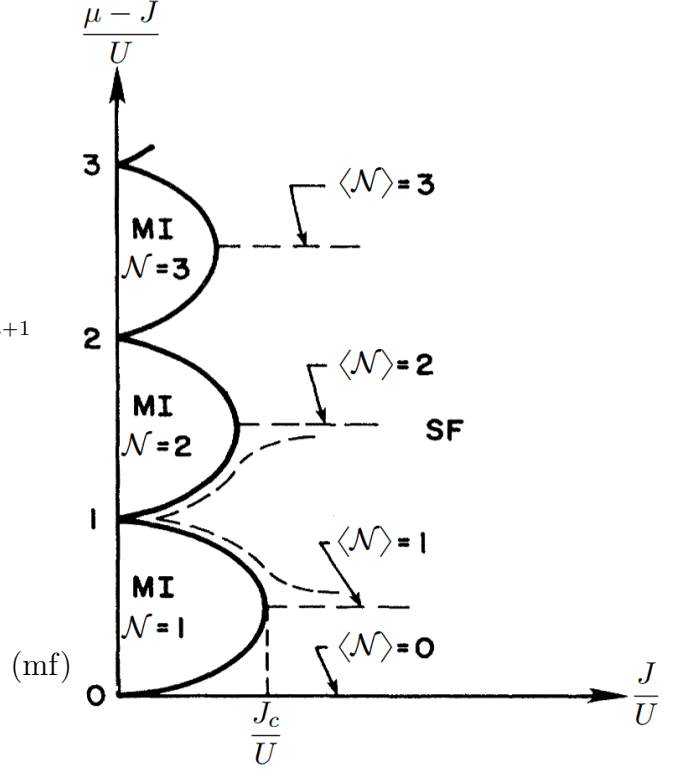


Figure 1: Mott insulator \ Superfluid phase diagram obtained by minimizing $\varphi \mapsto \langle \varphi | h^\varphi \varphi \rangle$ [1]

Main result

Theorem .1: *S.Farhat D.P S.Petrat 2025*

Assume

- γ_d solves (B-H) with $\gamma_d(0) \in \mathcal{L}^1(\mathcal{F})$ such that $\text{Tr}(\gamma_d(0)) = 1$
- φ solves (mf) with $\varphi(0) \in \ell^2(\mathbb{C})$ such that $\|\varphi\|_{\ell^2} = 1$
- $\exists c_1, c_2 > 0$ such that $\forall n \in \mathbb{N}$,

$$\text{Tr}(p_\varphi(0) \mathbf{1}_{\mathcal{N}=n}) \leq c_1 e^{-\frac{n}{c_2}} \quad \text{Tr}\left(\gamma_d^{(1)}(0) \mathbf{1}_{\mathcal{N}=n}\right) \leq c_1 e^{-\frac{n}{c_2}}.$$

Then $\exists C := C(J, c_1, c_2, \text{Tr}(p_\varphi(0)\mathcal{N})) > 0$ such that $\forall t \in \mathbb{R}_+$,

$$\left\| \gamma_d^{(1)}(t) - p_\varphi(t) \right\|_{\mathcal{L}^1} \leq e^{te^{C(t+1)}\sqrt{\ln(d)}} \left(\left\| \gamma_d^{(1)}(0) - p_\varphi(0) \right\|_{\mathcal{L}^1} + \frac{1}{d\sqrt{\ln(d)}} \right)$$

- If $\left\| \gamma_d^{(1)}(0) - p_\varphi(0) \right\|_{\mathcal{L}^1} = \mathcal{O}\left(\frac{1}{d}\right)$, then $\forall t \in \mathbb{R}_+$,

$$\left\| \gamma_d^{(1)}(t) - p_\varphi(t) \right\|_{\mathcal{L}^1} \leq 2e^{te^{C(t+1)}\sqrt{\ln(d)} - \ln(d)} \xrightarrow{d \rightarrow \infty} 0$$

- Proof relies on propagation of moments of \mathcal{N}
- Article has another result without the double exponential in t working with less assumptions on initial moments but requiring $U > 0$
- Well-posedness of the mean field equation treated
- Further works: improve error with corrections to the dynamics to get something small when $d = 3$

Convergence of the order parameter: since $a \leq \mathcal{N} + 1$ Insert a cut-off

$$\begin{aligned} & \left| \text{Tr} \left(\gamma_d^{(1)} a \right) - \text{Tr} (p_\varphi a) \right| \\ & \leq \left\| \left(\gamma_d^{(1)} - p_\varphi \right) a \right\|_{\mathcal{L}^1} \\ & \leq \left\| \left(\gamma_d^{(1)} - p_\varphi \right) a (\mathcal{N} + 1)^{-1} (\mathcal{N} + 1) \mathbb{1}_{\mathcal{N} < M} \right\|_{\mathcal{L}^1} + \left\| \left(\gamma_d^{(1)} - p_\varphi \right) a (\mathcal{N} + 1)^{-1} (\mathcal{N} + 1) \mathbb{1}_{\mathcal{N} \geq M} \right\|_{\mathcal{L}^1} \\ & \leq M \left\| \gamma_d^{(1)} - p_\varphi \right\|_{\mathcal{L}^1} + \underbrace{\text{Tr} \left(\gamma_d^{(1)} (\mathcal{N} + 1) \mathbb{1}_{\mathcal{N} \geq M} \right) + \text{Tr} (p_\varphi (\mathcal{N} + 1) \mathbb{1}_{\mathcal{N} \geq M})}_{\rightarrow 0 \text{ when } M \rightarrow \infty \text{ since the particle numbers are conserved}} \end{aligned}$$

Any choice of $M \gg 1$ such that $M \left\| \gamma_d^{(1)} - p_\varphi \right\|_{\mathcal{L}^1} \ll 1$ as $d \rightarrow \infty$ is sufficient to prove that

$$\left\| \left(\gamma_d^{(1)} - p_\varphi \right) a \right\|_{\mathcal{L}^1} \xrightarrow{d \rightarrow \infty} 0$$

Sketch of the proof

- Propagation of moments of \mathcal{N} :

$$\text{Tr} (p_\varphi(t) \mathcal{N}^k) \leq (\text{Tr} (p_\varphi(0) \mathcal{N}^k) + k^k) e^{C(t+1)},$$

and same for $\text{Tr} \left(\gamma_d^{(1)}(t) \mathcal{N}^k \right)$

- Gronwall estimate tentative

$$\left| \partial_t \text{Tr} \left(\gamma_d^{(1)} q_\varphi \right) \right| \leq C \left(\text{Tr} \left(\gamma_d^{(1)} q_\varphi \right) + \text{Tr} \left(\gamma_d^{(1)} q_\varphi \right)^{\frac{1}{2}} \underbrace{\text{Tr} \left(\gamma_d^{(1)} q_\varphi (\mathcal{N} + 1) q_\varphi \right)^{\frac{1}{2}}}_{\text{Insert cut-off } \mathbb{1}_{\mathcal{N} < M} + \mathbb{1}_{\mathcal{N} \geq M}} + d^{-1} \right).$$

since

$$\left\| \gamma_d^{(1)} - p_\varphi \right\|_{\mathcal{L}^1} \lesssim \sqrt{\text{Tr} \left(\gamma_d^{(1)} q_\varphi \right)}$$

- Controlling large \mathcal{N} terms

$$\text{Tr} \left(\gamma_d^{(1)} q_\varphi (\mathcal{N} + 1) \mathbb{1}_{\mathcal{N} \geq M} q_\varphi \right) \leq e^{C(t+1) - M e^{-C(t+1)}} \xrightarrow{M \rightarrow \infty} 0$$

- Close Gronwall and optimize in M .

Bibliography



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