1 Overview

In *Ray Tracing in One Weekend* and *Ray Tracing: the Next Week*, you built a "real" ray tracer.

If you are motivated, you can take the source and information contained in those books to implement any visual effect you want. The source provides a meaningful and robust foundation upon which to build out a raytracer for a small hobby project. Most of the visual effects found in commercial ray tracers rely on the techniques described in these first two books. However, your capacity to add increasingly complicated visual effects like subsurface scattering or nested dielectrics will be severely limited by a missing mathematical foundation. In this volume, I assume that you are either a highly interested student, or are someone who is pursuing a career related to ray tracing. We will be diving into the math of creating a very serious ray tracer. When you are done, you should be well equipped to use and modify the various commercial ray tracers found in many popular domains, such as the movie, television, product design, and architecture industries.

There are many many things I do not cover in this short volume. For example, there are many ways of writing Monte Carlo rendering programs—I dive into only one of them. I don't cover shadow rays (deciding instead to make rays more likely to go toward lights), nor do I cover bidirectional methods, Metropolis methods, or photon mapping. You'll find many of these techniques in the so-called "serious ray tracers", but they are not covered here because it is more important to cover the concepts, math, and terms of the field. I think of this book as a deep exposure that should be your first of many, and it will equip you with some of the concepts, math, and terms that you'll need in order to study these and other interesting techniques.

I hope that you find the math as fascinating as I do.

See the project README file for information about this project, the repository on GitHub, directory structure, building & running, and how to make or reference corrections and contributions.

As before, see our Further Reading wiki page for additional project related resources.

These books have been formatted to print well directly from your browser. We also include PDFs of each book with each release, in the "Assets" section.

Thanks to everyone who lent a hand on this project. You can find them in the acknowledgments section at the end of this book.

```
            return p;
        }
    }
```

Listing 1: `[vec3.h]` *A Las Vegas algorithm*

This code will always eventually arrive at a random point in the unit sphere, but we can't say beforehand how long it'll take. It may take only 1 iteration, it may take 2, 3, 4, or even longer. Whereas, an MC program will give a statistical estimate of an answer, and this estimate will get more and more accurate the longer you run it. Which means that at a certain point, we can just decide that the answer is accurate *enough* and call it quits. This basic characteristic of simple programs producing noisy but ever-better answers is what MC is all about, and is especially good for applications like graphics where great accuracy is not needed.

2.1 Estimating Pi

The canonical example of a Monte Carlo algorithm is estimating π, so let's do that. There are many ways to estimate π with the Buffon Needle problem being a classic case study. We'll do a variation inspired by this method. Suppose you have a circle inscribed inside a square:

Figure 1: *Estimating π with a circle inside a square*

Now, suppose you pick random points inside the square. The fraction of those random points that end up inside the circle should be proportional to the area of the circle. The exact fraction should in fact be the ratio of the circle area to the square area:

$$\frac{\pi r^2}{(2r)^2} = \frac{\pi}{4}$$

Since the r cancels out, we can pick whatever is computationally convenient. Let's go with $r = 1$, centered at the origin:

```
#include "rtweekend.h"
```

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```
#include "rtweekend.h"

#include <iostream>
#include <iomanip>
#include <math.h>
#include <stdlib.h>

int main() {
    int inside_circle = 0;
    int runs = 0;
    std::cout << std::fixed << std::setprecision(12);
    while (true) {
        runs++;
        auto x = random_double(-1,1);
        auto y = random_double(-1,1);
        if (x*x + y*y < 1)
            inside_circle++;

        if (runs % 100000 == 0)
            std::cout << "Estimate of Pi = "
                      << (4.0 * inside_circle) / runs
                      << '\n';
    }
}
```

Listing 3: `[pi.cc]` *Estimating π, v2*

2.3 Stratified Samples (Jittering)

We get very quickly near π, and then more slowly zero in on it. This is an example of the *Law of Diminishing Returns*, where each sample helps less than the last. This is the worst part of Monte Carlo. We can mitigate this diminishing return by *stratifying* the samples (often called *jittering*), where instead of taking random samples, we take a grid and take one sample within each:

Figure 2: *Sampling areas with jittered points*

This changes the sample generation, but we need to know how many samples we are taking in advance because we need to know the grid. Let's take a million and try it both ways:

```
#include "rtweekend.h"
```

On my computer, I get:

```
Regular    Estimate of Pi = 3.141184000000
Stratified Estimate of Pi = 3.141460000000
```

Where the first 12 decimal places of pi are:

```
3.141592653589
```

Interestingly, the stratified method is not only better, it converges with a better asymptotic rate! Unfortunately, this advantage decreases with the dimension of the problem (so for example, with the 3D sphere volume version the gap would be less). This is called the *Curse of Dimensionality*. Ray tracing is a very high-dimensional algorithm, where each reflection adds two new dimensions: ϕ_o and θ_o. We won't be stratifying the output reflection angle in this book, simply because it is a little bit too complicated to cover, but there is a lot of interesting research currently happening in this space.

As an intermediary, we'll be stratifying the locations of the sampling positions around each pixel location.

```
#include "rtweekend.h"
```

```
    // Box 2
    shared_ptr<hittable> box2 = box(point3(0,0,0), point3(165,165,165), white);
    box2 = make_shared<rotate_y>(box2, -18);
    box2 = make_shared<translate>(box2, vec3(130,0,65));
    world.add(box2);

    camera cam;

    cam.aspect_ratio      = 1.0;
    cam.image_width       = 600;
    cam.samples_per_pixel = 64;
    cam.max_depth         = 50;
    cam.background        = color(0,0,0);

    cam.vfov     = 40;
    cam.lookfrom = point3(278, 278, -800);
    cam.lookat   = point3(278, 278, 0);
    cam.vup      = vec3(0, 1, 0);

    cam.defocus_angle = 0;

    cam.render(world);
}
```

Listing 5: *[main.cc] Stratifying the samples inside pixels*

```
class camera {
  public:
```

```
    void initialize() {
        image_height = int(image_width / aspect_ratio);
        image_height = (image_height < 1) ? 1 : image_height;

        sqrt_spp = int(sqrt(samples_per_pixel));
        pixel_samples_scale = 1.0 / (sqrt_spp * sqrt_spp);
        recip_sqrt_spp = 1.0 / sqrt_spp;

        center = lookfrom;
        ...
    }
    ...
    ray get_ray(int i, int j, int s_i, int s_j) const {
        // Construct a camera ray originating from the defocus disk and directed at a randomly
        // sampled point around the pixel location i, j for stratified sample square s_i, s_j.

        auto offset = sample_square_stratified(s_i, s_j);
        auto pixel_sample = pixel00_loc
                          + ((i + offset.x()) * pixel_delta_u)
                          + ((j + offset.y()) * pixel_delta_v);

        auto ray_origin = (defocus_angle <= 0) ? center : defocus_disk_sample();
        auto ray_direction = pixel_sample - ray_origin;
        auto ray_time = random_double();

        return ray(ray_origin, ray_direction, ray_time);
    }

    vec3 sample_square_stratified(int s_i, int s_j) const {
        // Returns the vector to a random point in the square sub-pixel specified by grid
        // indices s_i and s_j, for an idealized unit square pixel [-.5,-.5] to [+.5,+.5].

        auto px = ((s_i + random_double()) * recip_sqrt_spp) - 0.5;
        auto py = ((s_j + random_double()) * recip_sqrt_spp) - 0.5;

        return vec3(px, py, 0);
    }

    vec3 sample_square() const {
        ...
    }

    ...
};
```

Listing 6: *[camera.h] Stratifying the samples inside pixels (render)*

If we compare the results from without stratification:

$$\frac{\text{area(circle)}}{\text{area(square)}} = \frac{\pi}{4}$$

We picked a bunch of random points in the circumscribed square and counted the fraction of them that were also in the unit circle. This fraction was an estimate that tended toward $\frac{\pi}{4}$ as more points were added. If we didn't know the area of a circle, we could still solve for it using the above ratio. We know that the ratio of areas of the unit circle and the circumscribed square is $\frac{\pi}{4}$, and we know that the area of a circumscribed square is $4r^2$, so we could then use those two quantities to get the area of a circle:

$$\frac{\text{area(circle)}}{\text{area(square)}} = \frac{\pi}{4}$$

$$\frac{\text{area(circle)}}{(2r)^2} = \frac{\pi}{4}$$

$$\text{area(circle)} = \frac{\pi}{4} 4r^2$$

$$\text{area(circle)} = \pi r^2$$

We choose a circle with radius $r = 1$ and get:

$$\text{area(circle)} = \pi$$

Our work above is equally valid as a means to solve for pi as it is a means to solve for the area of a circle:

```
#include "rtweekend.h"
```

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1. A list of values X that contains members x_i:

$$X = (x_0, x_1, \ldots, x_{N-1})$$

2. A continuous function $f(x)$ that takes members from the list:

$$y_i = f(x_i)$$

3. A function $F(X)$ that takes the list X as input and produces the list Y as output:

$$Y = F(X)$$

4. Where output list Y has members y_i:

$$Y = (y_0, y_1, \ldots, y_{N-1}) = (f(x_0), f(x_1), \ldots, f(x_{N-1}))$$

If we assume all of the above, then we could solve for the arithmetic mean—the average—of the list Y with the following:

$$\text{average}(Y) = E[Y] = \frac{1}{N} \sum_{i=0}^{N-1} y_i$$

$$= \frac{1}{N} \sum_{i=0}^{N-1} f(x_i)$$

$$= E[F(X)]$$

Where $E[Y]$ is referred to as the *expected value of* Y. If the values of x_i are chosen randomly from a continuous interval $[a, b]$ such that $a \leq x_i \leq b$ for all values of i, then $E[F(X)]$ will approximate the average of the continuous function *[Math Processing Error]* over the the same interval *[Math Processing Error]*.

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$$\approx E[Y = \{y_i = f(x_i) | a \leq x_i \leq b\}]$$

$$\approx \frac{1}{} \sum^{N-1} f(x_i)$$

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Take the limit as N approaches ∞

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This is, of course, just a regular integral:

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If you recall your introductory calculus class, the integral of a function is the area under the curve over that interval:

$$\text{area}(f(x), a, b) = \int_a^b f(x)dx$$

Therefore, the average over an interval is intrinsically linked with the area under the curve in that interval.

$$E[f(x)|a \le x \le b] = \frac{1}{b-a} \cdot \text{area}(f(x), a, b)$$

Both the integral of a function and a Monte Carlo sampling of that function can be used to solve for the average over a specific interval. While integration solves for the average with the sum of infinitely many infinitesimally small slices of the interval, a Monte Carlo algorithm will approximate the same average by solving the sum of ever increasing random sample points within the interval. Counting the number of points that fall inside of an object isn't the only way to measure its average or area. Integration is also a common mathematical tool for this purpose. If a closed form exists for a problem, integration is frequently the most natural and clean way to formulate things.

I think a couple of examples will help.

3.2 Integrating x^2

Let's look at a classic integral:

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$$\text{average}(x^2, 0, 2) = \frac{1}{2-0} \cdot \text{area}(x^2, 0, 2)$$

$$\text{average}(x^2, 0, 2) = \frac{1}{2-0} \cdot I$$

$$I = 2 \cdot \text{average}(x^2, 0, 2)$$

The Monte Carlo approach:

```
#include "rtweekend.h"

#include <iostream>
#include <iomanip>
#include <math.h>
#include <stdlib.h>

int main() {
    int a = 0;
    int b = 2;
    int N = 1000000;
    auto sum = 0.0;
    for (int i = 0; i < N; i++) {
        auto x = random_double(a, b);
        sum += x*x;
    }
    std::cout << std::fixed << std::setprecision(12);
    std::cout << "I = " << (b - a) * (sum / N) << '\n';
}
```

Listing 8: `[integrate_x_sq.cc]` *Integrating* x^2

This, as expected, produces approximately the exact answer we get with integration, *i.e.* $I = 8/3$. You could rightly point to this example and say that the integration is actually a lot less work than the Monte Carlo. That might be true in the case where the function is $f(x) = x^2$, but there exist many functions where it might be simpler to solve for the Monte Carlo than for the integration, like $f(x) = \sin^5(x)$.

```
        for (int i = 0; i < N; i++) {
            auto x = random_double(a, b);
```

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3.3 Density Functions

The `ray_color` function that we wrote in the first two books, while elegant in its simplicity, has a fairly *major* problem. Small light sources create too much noise. This is because our uniform sampling doesn't sample these light sources often enough. Light sources are only sampled if a ray scatters toward them, but this can be unlikely for a small light, or a light that is far away. If the background color is black, then the only real sources of light in the scene are from the lights that are actually placed about the scene. There might be two rays that intersect at nearby points on a surface, one that is randomly reflected toward the light and one that is not. The ray that is reflected toward the light will appear a very bright color. The ray that is reflected to somewhere else will appear a very dark color. The two intensities should really be somewhere in the middle. We could lessen this problem if we steered both of these rays toward the light, but this would cause the scene to be inaccurately bright.

For any given ray, we usually trace from the camera, through the scene, and terminate at a light. But imagine if we traced this same ray from the light source, through the scene, and terminated at the camera. This ray would start with a bright intensity and would lose energy with each successive bounce around the scene. It would ultimately arrive at the camera, having been dimmed and colored by its reflections off various surfaces. Now, imagine if this ray was forced to bounce toward the camera as soon as it could. It would appear inaccurately bright because it hadn't been dimmed by successive bounces. This is analogous to sending more random samples toward the light. It would go a long way toward solving our problem of having a bright pixel next to a dark pixel, but it would then just make *all* of our pixels bright.

We can remove this inaccuracy by downweighting those samples to adjust for the over-sampling. How do we do this adjustment? Well, we'll first need to understand the concept of a *probability density function*. But to understand the concept of a *probability density function*, we'll first need to know what a *density function* is.

A *density function* is just the continuous version of a histogram. Here's an example of a histogram from the histogram Wikipedia page:

Figure 3: *Histogram example*

If we had more items in our data source, the number of bins would stay the same, but each bin would have a higher frequency of each item. If we divided the data into more bins, we'd have

more bins, but each bin would have a lower frequency of each item. If we took the number of bins and raised it to infinity, we'd have an infinite number of zero-frequency bins. To solve for th
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Indeed, with this continuous probability function, we can now say the likelihood that any given tree has a height that places it within any arbitrary span of multiple bins. This is a *probability density function* (henceforth *PDF*). In short, a PDF is a continuous function that can be integrated over to determine how likely a result is over an integral.

3.4 Constructing a PDF

Let's make a PDF and play around with it to build up an intuition. We'll use the following function:

Figure 4: *A linear PDF*

What does this function do? Well, we know that a PDF is just a continuous function that defines the likelihood of an arbitrary range of values. This function $p(r)$ is constrained between 0 and 2 and linearly increases along that interval. So, if we used this function as a PDF to generate a random number then the *probability* of getting a number near zero would be less than the probability of getting a number near two.

The PDF $p(r)$ is a linear function that starts with 0 at $r = 0$ and monotonically increases to its highest point at $p(2)$ for $r = 2$. What is the value of $p(2)$? What is the value of $p(r)$? Maybe $p(2)$ is 2? The PDF increases linearly from 0 to 2, so guessing that the value of $p(2)$ is 2 seems reasonable. At least it looks like it can't be 0.

Remember that the PDF is a probability function. We are constraining the PDF so that it lies in the range [0,2]. The PDF represents the continuous density function for a probabilistic list. If we know that everything in that list is contained within 0 and 2, we can say that the probability of getting a value between 0 and 2 is 100%. Therefore, the area under the curve must sum to 1:

$$\text{area}(p(r), 0, 2) = 1$$

All linear functions can be represented as a constant term multiplied by a variable.

$$p(r) = C \cdot r$$

We need to solve for the value of C. We can use integration to work backwards.

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To confirm your understanding, you should integrate over the region r = 0 to r = 2, you should get a probability of 1.

After spending enough time with PDFs you might start referring to a PDF as the probability that a variable r is value x, *i.e.* p(r = x). Don't do this. For a continuous function, the probability that a variable is a specific value is always zero. A PDF can only tell you the probability that a variable will fall within a given interval. If the interval you're checking against is a single value, then the PDF will always return a zero probability because its "bin" is infinitely thin (has zero width). Here's a simple mathematical proof of this fact:

$$\text{Probability}(r = x) = \int_x^x p(r)dr$$

$$= P(r)\Big|_x^x$$

$$= P(x) - P(x)$$

$$= 0$$

Finding the probability of a region surrounding x may not be zero:

$$\text{Probability}(r|x - \Delta x < r < x + \Delta x) = \text{area}(p(r), x - \Delta x, x + \Delta x)$$

$$= P(x + \Delta x) - P(x - \Delta x)$$

3.5 Choosing our Samples

If we have a PDF for the function that we care about, then we have the probability that the function will return a value within an arbitrary interval. We can use this to determine where we should sample. Remember that this started as a quest to determine the best way to sample a scene so that we wouldn't get very bright pixels next to very dark pixels. If we have a PDF for the scene, then we can probabilistically steer our samples toward the light without making the image inaccurately bright. We already said that if we steer our samples toward the light then we *will* make the image inaccurately bright. We need to figure out how to steer our samples without introducing this inaccuracy, this will be explained a little bit later, but for now we'll

focus on generating samples if we have a PDF. How do we generate a random number with a PDF? For that we will need some more machinery. Don't worry — this doesn't go on forever!

$$50\% = \int_0^x \frac{r}{2} dr = \int_x^2 \frac{r}{2} dr$$

Solving gives us:

$$0.5 = \frac{r^2}{4}\Big|_0^x$$

$$0.5 = \frac{x^2}{4}$$

$$x^2 = 2$$

$$x = \sqrt{2}$$

As a crude approximation we could create a function `f(d)` that takes as input `double d = random_double()`. If `d` is less than (or equal to) 0.5, it produces a uniform number in $[0, \sqrt{2}]$, if `d` is greater than 0.5, it produces a uniform number in $[\sqrt{2}, 2]$.

```
double f(double d)
{
    if (d <= 0.5)
        return sqrt(2.0) * random_double();
    else
        return sqrt(2.0) + (2 - sqrt(2.0)) * random_double();
}
```

Listing 11: *A crude, first-order approximation to nonuniform PDF*

While our initial random number generator was uniform from 0 to 1:

Figure 5: *A uniform distribution*

Our, new, crude approximation for $\frac{r}{2}$ is nonuniform (but only just):

Figure 6: *A nonuniform distribution for r/2*

We had the analytical solution to the integration above, so we could very easily solve for the 50% value. But we could also solve for this 50% value experimentally. There will be functions th
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```
#include "rtweekend.h"
```

```
        sample this_sample = {x, p_x};
        samples.push_back(this_sample);
    }

    // Sort the samples by x
    std::sort(samples.begin(), samples.end(), compare_by_x);

    // Find out the sample at which we have half of our area
    double half_sum = sum / 2.0;
    double halfway_point = 0.0;
    double accum = 0.0;
    for (unsigned int i = 0; i < N; i++){
        accum += samples[i].p_x;
        if (accum >= half_sum) {
            halfway_point = samples[i].x;
            break;
        }
    }

    std::cout << std::fixed << std::setprecision(12);
    std::cout << "Average = " << sum / N << '\n';
    std::cout << "Area under curve = " << 2 * pi * sum / N << '\n';
    std::cout << "Halfway = " << halfway_point << '\n';
}
```

Listing 12: *[estimate_halfway.cc] Estimating the 50% point of a function*

This code snippet isn't too different from what we had before. We're still solving for the sum over an interval (0 to 2π). Only this time, we're also storing and sorting all of our samples by their input and output. We use this to determine the point at which they subtotal half of the sum across the entire interval. Once we know that our first j samples sum up to half of the total sum, we know that the jth x roughly corresponds to our halfway point:

```
Average = 0.314686555791
Area under curve = 1.977233943713
Halfway = 2.016002314977
```

If you solve for the integral from 0 to 2.016 and from 2.016 to 2π you should get almost exactly the same result for both.

We have a method of solving for the halfway point that splits a PDF in half. If we wanted to, we could use this to create a nested binary partition of the PDF:

1. Solve for halfway point of a PDF

2. Recurse into lower half, repeat step 1
3. Recurse into upper half, repeat step 1

$$p(r) = \begin{cases} \frac{r}{2} & 0 \le r \le 2 \\ 0 & 2 < r \end{cases}$$

If you consider what we were trying to do in the previous section, a lot of math revolved around the *accumulated* area (or *accumulated* probability) from zero. In the case of the function

$$f(x) = e^{\frac{-x}{2\pi}} \sin^2(x)$$

we cared about the accumulated probability from 0 to 2π (100%) and the accumulated probability from 0 to 2.016 (50%). We can generalize this to an important term, the *Cumulative Distribution Function* $P(x)$ is defined as:

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Or,

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Which is the amount of *cumulative* probability from $-\infty$. We rewrote the integral in terms of *[Math Processing Error]* instead of x because of calculus rules, if you're not sure what it means, don't worry about it, you can just treat it like it's the same. If we take the integration outlined above, we get the piecewise $P(r)$:

$$P(r) = \begin{cases} 0 & r < 0 \\ \frac{r^2}{4} & 0 \le r \le 2 \\ 1 & 2 < r \end{cases}$$

The *Probability Density Function* (PDF) is the probability function that explains how likely an interval of numbers is to be chosen. The *Cumulative Distribution Function* (CDF) is the distribution function that explains how likely all numbers smaller than its input is to be chosen. To go from the PDF to the CDF, you need to integrate from $-\infty$ to x, but to go from the CDF to the PDF, all you need to do is take the derivative:

$$p(x) = \frac{d}{dx} P(x)$$

If we evaluate the CDF, P(r), at r = 1.0, we get:

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$$P(1.0) = \frac{\cdot}{4}$$

$$P(1.5) = \frac{9}{16}$$

$$P(2.0) = 1$$

so, the function f() has values

$$f(P(0.0)) = f(0) = 0$$

$$f(P(0.5)) = f(\frac{1}{16}) = 0.5$$

$$f(P(1.0)) = f(\frac{1}{4}) = 1.0$$

$$f(P(1.5)) = f(\frac{9}{16}) = 1.5$$

$$f(P(2.0)) = f(1) = 2.0$$

We could use these intermediate values and interpolate between them to approximate f(d):

Figure 8: *Approximating the nonuniform f()*

If you can't solve for the PDF analytically, then you can't solve for the CDF analytically. After all, the CDF is just the integral of the PDF. However, you can still create a distribution that approximates the PDF. If you take a bunch of samples from the random function you want the PDF from, you can approximate the PDF by getting a histogram of the samples and then converting to a PDF. Alternatively, you can do as we did above and sort all of your samples.

Looking closer at the equality:

$$f(P(x)) = x$$

That just means that f() just undoes whatever P() does. So, f() is the inverse function:

$f(d) = P^{-1}(x)$

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Note that this ranges from 0 to 2 as we hoped, and if we check our work, we replace `random_double()` with $1/4$ to get 1, and also replace with $1/2$ to get $\sqrt{2}$, just as expected.

3.7 Importance Sampling

You should now have a decent understanding of how to take an analytical PDF and generate a function that produces random numbers with that distribution. We return to our original integral and try it with a few different PDFs to get a better understanding:

$$I = \int_0^2 x^2 dx$$

The last time that we tried to solve for the integral we used a Monte Carlo approach, uniformly sampling from the interval $[0, 2]$. We didn't know it at the time, but we were implicitly using a uniform PDF between 0 and 2. This means that we're using a PDF $= 1/2$ over the range $[0, 2]$, which means the CDF is $P(x) = x/2$, so $f(d) = 2d$. Knowing this, we can make this uniform PDF explicit:

```
#include "rtweekend.h"
```

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from within this distribution. We were previously multiplying the average over the interval (`sum / N`) times the length of the interval (`b - a`) to arrive at the final answer. Here, we don't need to multiply by the interval length—that is, we no longer need to multiply the average by 2.

We need to account for the nonuniformity of the PDF of x. Failing to account for this nonuniformity will introduce bias in our scene. Indeed, this bias is the source of our inaccurately bright image—if we account for nonuniformity, we will get accurate results. The PDF will "steer" samples toward specific parts of the distribution, which will cause us to converge faster, but at the cost of introducing bias. To remove this bias, we need to down-weight where we sample more frequently, and to up-weight where we sample less frequently. For our new nonuniform random number generator, the PDF defines how much or how little we sample a specific portion. So the weighting function should be proportional to 1/pdf. In fact it is *exactly* 1/pdf. This is why we divide `x*x` by `pdf(x)`.

We can try to solve for the integral using the linear PDF $p(r) = \frac{r}{2}$, for which we were able to solve for the CDF and its inverse. To do that, all we need to do is replace the functions $f = \sqrt{4d}$ and pdf = x/2.

```
double f(double d) {
    return sqrt(4.0 * d);
}

double pdf(double x) {
    return x / 2.0;
}

int main() {
    int N = 1000000;
    auto sum = 0.0;
    for (int i = 0; i < N; i++) {
        auto x = f(random_double());
        sum += x*x / pdf(x);
    }
    std::cout << std::fixed << std::setprecision(12);
    std::cout << "I = " << sum / N << '\n';
}
```

Listing 14: *[integrate_x_sq.cc] Integrating x^2 with linear PDF*

If you compared the runs from the uniform PDF and the linear PDF, you would have probably

found that the linear PDF converged faster. If you think about it, a linear PDF is probably a better approximation for a quadratic function than a uniform PDF, so you would expect it to c
b

L

Which gives us:

$$p(r) = \begin{cases} 0 & r < 0 \\ \frac{3}{8}r^2 & 0 \le r \le 2 \\ 0 & 2 < r \end{cases}$$

And we get the corresponding CDF:

$$P(r) = \frac{r^3}{8}$$

and

$$P^{-1}(x) = f(d) = 8d^{\frac{1}{3}}$$

For just one sample we get:

```
double f(double d) {
    return 8.0 * pow(d, 1.0/3.0);
}

double pdf(double x) {
    return (3.0/8.0) * x*x;
}

int main() {
    int N = 1;
    auto sum = 0.0;
    for (int i = 0; i < N; i++) {
        auto x = f(random_double());
        sum += x*x / pdf(x);
    }
    std::cout << std::fixed << std::setprecision(12);
    std::cout << "I = " << sum / N << '\n';
}
```

Listing 15: *[integrate_x_sq.cc] Integrating x^2, final version*

This always returns the exact answer. Which, honestly, feels a bit like magic.

A nonuniform PDF "steers" more samples to where the PDF is big, and fewer samples to where

the PDF is small. By this sampling, we would expect less noise in the places where the PDF is bi
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by integrating p analytically), but it's a good exercise to make sure our code works.

Let's review the main concepts that underlie Monte Carlo ray tracers:

1. You have an integral of f(x) over some domain [a, b]
2. You pick a PDF p that is non-zero and non-negative over [a, b]
3. You average a whole ton of $\frac{f(r)}{p(r)}$ where r is a random number with PDF p.

Any choice of PDF p will always converge to the right answer, but the closer that p approximates f, the faster that it will converge.

4 Monte Carlo Integration on the Sphere of Directions

In chapter One Dimensional Monte Carlo Integration we started with uniform random numbers and slowly, over the course of a chapter, built up more and more complicated ways of producing random numbers, before ultimately arriving at the intuition of PDFs, and how to use them to generate random numbers of arbitrary distribution.

All of the concepts covered in that chapter continue to work as we extend beyond a single dimension. Moving forward, we might need to be able to select a point from a two, three, or even higher dimensional space and then weight that selection by an arbitrary PDF. An important case of this—at least for ray tracing—is producing a random direction. In the first two books we generated a random direction by creating a random vector and rejecting it if it fell outside of the unit sphere. We repeated this process until we found a random vector that fell inside the unit sphere. Normalizing this vector produced points that lay exactly on the unit sphere and thereby represent a random direction. This process of generating samples and rejecting them if they are not inside a desired space is called *the rejection method*, and is found all over the literature. The method covered last chapter is referred to as *the inversion method* because we invert a PDF.

Every direction in 3D space has an associated point on the unit sphere and can be generated by solving for the vector that travels from the origin to that associated point. You can think of choosing a random direction as choosing a random point in a constrained two dimensional plane: the plane created by mapping the unit sphere to Cartesian coordinates. The same methodology as before applies, but now we might have a PDF defined over two dimensions.

Suppose we want to integrate this function over the surface of the unit sphere:

```
        vec3 d = random_unit_vector();
        auto f_d = f(d);
        sum += f_d / pdf(d);
    }
    std::cout << std::fixed << std::setprecision(12);
    std::cout << "I = " << sum / N << '\n';
}
```

Listing 16: *[sphere_importance.cc] Generating importance-sampled points on the unit sphere*

The analytic answer is $\frac{4}{3}\pi$ — if you remember enough advanced calc, check me! And the code above produces that. The key point here is that all of the integrals and the probability and everything else is over the unit sphere. The way to represent a single direction in 3D is its associated point on the unit sphere. The way to represent a range of directions in 3D is the amount of area on the unit sphere that those directions travel through. Call it direction, area, or *solid angle* — it's all the same thing. Solid angle is the term that you'll usually find in the literature. You have radians (r) in θ over one dimension, and you have *steradians* (sr) in θ *and* ϕ over two dimensions (the unit sphere is a three dimensional object, but its surface is only two dimensional). Solid Angle is just the two dimensional extension of angles. If you are comfortable with a two dimensional angle, great! If not, do what I do and imagine the area on the unit sphere that a set of directions goes through. The solid angle ω and the projected area A on the unit sphere are the same thing.

Figure 9: *Solid angle / projected area of a sphere*

Now let's go on to the light transport equation we are solving.

5 Light Scattering

In this chapter we won't actually program anything. We'll just be setting up for a big lighting change in the next chapter. Our ray tracing program from the first two books scatters a ray when it interacts with a surface or a volume. Ray scattering is the most commonly used model for simulating light propagation through a scene. This can naturally be modeled probabilistically. There are many things to consider when modeling the probabilistic scattering of rays.

5.1 Albedo

the light color rather than RGB. As an example, we would replace our *tristimulus* RGB renderer with something that specifically samples at 300nm, 350nm, 400nm, ..., 700nm. We can extend our intuition by thinking of R, G, and B as specific algebraic mixtures of wavelengths where R is *mostly* red wavelengths, G is *mostly* green wavelengths, and B is *mostly* blue wavelengths. This is an approximation of the human visual system which has 3 unique sets of color receptors, called *cones*, that are each sensitive to different algebraic mixtures of wavelengths, roughly RGB, but are referred to as long, medium, and short cones (the names are in reference to the wavelengths that each cone is sensitive to, not the length of the cone). Just as colors can be represented by their strength in the RGB color space, colors can also be represented by how excited each set of cones is in the *LMS color space* (long, medium, short).

5.2 Scattering

If the light does scatter, it will have a directional distribution that we can describe as a PDF over solid angle. I will refer to this as its *scattering PDF*: $\text{pScatter}()$ The scattering PDF will vary with outgoing direction: $\text{pScatter}(\omega_o)$ The scattering PDF can also vary with *incident direction*: $\text{pScatter}(\omega_i, \omega_o)$ You can see this varying with incident direction when you look at reflections off a road — they become mirror-like as your viewing angle (incident angle) approaches grazing. The scattering PDF can vary with the wavelength of the light: $\text{pScatter}(\omega_i, \omega_o, \lambda)$ A good example of this is a prism refracting white light into a rainbow. Lastly, the scattering PDF can also depend on the scattering position: $\text{pScatter}(x, \omega_i, \omega_o, \lambda)$ The x is just math notation for the scattering position: $x = (x, y, z)$. The albedo of an object can also depend on these quantities: $A(x, \omega_i, \omega_o, \lambda)$.

The color of a surface is found by integrating these terms over the unit hemisphere by the incident direction:

$$\text{Color}_o(x, \omega_o, \lambda) = \int_{\omega_i} A(x, \omega_i, \omega_o, \lambda) \cdot \text{pScatter}(x, \omega_i, \omega_o, \lambda) \cdot \text{Color}_i(x, \omega_i, \lambda)$$

We've added a Color_i term. The scattering PDF and the albedo at the surface of an object are acting as filters to the light that is shining on that point. So we need to solve for the light that is shining on that point. This is a recursive algorithm, and is the reason our `ray_color` function returns the color of the current object multiplied by the color of the next ray.

5.2 The Scattering PDF

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So:

$$1 = C \cdot \int_0^{2\pi} \int_0^{\pi/2} \cos(\theta)\sin(\theta) d\theta d\phi$$

$$1 = C \cdot 2\pi \frac{1}{2}$$

$$1 = C \cdot \pi$$

$$C = \frac{1}{\pi}$$

The integral of $\cos(\theta_o)$over the hemisphere is π, so we need to normalize by $\frac{1}{\pi}$. The PDF pScatter is only dependent on outgoing direction (ω_o), so we'll simplify its representation to just $\text{pScatter}(\omega_o)$.Put all of this together and you get the scattering PDF for a Lambertian surface:

$$\text{pScatter}(\omega_o) = \frac{\cos(\theta_o)}{\pi}$$

We'll assume that the $p(x, \omega_i, \omega_o, \lambda)$ is equal to the scattering PDF:

$$p(\omega_o) = \text{pScatter}(\omega_o) = \frac{\cos(\theta_o)}{\pi}$$

The numerator and denominator cancel out, and we get:

$$\text{Color}_o(x, \omega_o, \lambda) \approx \sum A(\dots) \cdot \text{Color}_i(\dots)$$

This is exactly what we had in our original `ray_color()` function!

```
        return attenuation * ray_color(scattered, depth-1, world);
```

The treatment above is slightly non-standard because I want the same math to work for surfaces and volumes. If you read the literature, you'll see reflection defined by the *Bidirectional Reflectance Distribution Function* (BRDF). It relates pretty simply to our terms:

$$\text{BRDF}(\omega_i, \omega_o, \lambda) = \frac{A(x, \omega_i, \omega_o, \lambda) \cdot \text{pScatter}(x, \omega_i, \omega_o, \lambda)}{}$$

As long as the weights are positive and add up to one, any such mixture of PDFs is a PDF. Remember, we can use any PDF: *all PDFs eventually converge to the correct answer*. So, the game is to figure out how to make the PDF larger where the product

$$\text{pScatter}(x, \omega_i, \omega_o) \cdot \text{Color}_i(x, \omega_i)$$

is largest. For diffuse surfaces, this is mainly a matter of guessing where $\text{Color}_i(x, \omega_i)$ is largest. Which is equivalent to guessing where the most light is coming from.

For a mirror, pScatter()is huge only near one direction, so pScatter()matters a lot more. In fact, most renderers just make mirrors a special case, and make the pScatter()/p()implicit — our code currently does that.

6.1 Returning to the Cornell Box

Let's adjust some parameters for the Cornell box:

```
int main() {
    ...
    cam.samples_per_pixel = 100;
    ...
}
```

Listing 17: *[main.cc] Cornell box, refactored*

At 600×600 my code produces this image in 15min on 1 core of my Macbook:

Image 3: Cornell box, refactored

Reducing that noise is our goal. We'll do that by constructing a PDF that sends more rays to the light.

First, let's instrument the code so that it explicitly samples some PDF and then normalizes for that. Remember Monte Carlo basics: $\int f(x) \approx \sum f(r)/p(r)$. For the Lambertian material, let's sample like we do now: $p(\omega_o) = \cos(\theta_o)/\pi$

We modify the base-class `material` to enable this importance sampling:

A

```
        attenuation = tex->value(rec.u, rec.v, rec.p);
        return true;
    }

    double scattering_pdf(const ray& r_in, const hit_record& rec, const ray& scattered) const
        auto cos_theta = dot(rec.normal, unit_vector(scattered.direction()));
        return cos_theta < 0 ? 0 : cos_theta/pi;
    }

  private:
    shared_ptr<texture> tex;
};
```

Listing 19: *[material.h] Lambertian material, modified for importance sampling*

And the `camera::ray_color` function gets a minor modification:

```
class camera {
  ...
```

Listing 20: *[camera.h] The ray_color function, modified for importance sampling*

You should get exactly the same picture. Which *should make sense*, as the scattered part of `ray_color` is getting multiplied by `scattering_pdf / pdf`, and as `pdf` is equal to `scattering_pdf` is just the same as multiplying by one.

6.2 Using a Uniform PDF Instead of a Perfect Match

Now, just for the experience, let's try using a different sampling PDF. We'll continue to have our reflected rays weighted by Lambertian, so $\cos(\theta_o)$ and we'll keep the scattering PDF as is, but we'll use a different PDF in the denominator. We will sample using a uniform PDF about the hemisphere, so we'll set the denominator to $1/2\pi$ This will still converge on the correct answer, as all we've done is change the PDF, but since the PDF is now less of a perfect match for the real distribution, it will take longer to converge. Which, for the same number of samples means a noisier image:

```
class camera {
  ...
```

Listing 21: *[camera.h] The ray_color function, now with a uniform PDF in the denominator*

You should get a very similar result to before, only with slightly more noise, it may be hard to see.

Image 4: Cornell box, with imperfect PDF

Make sure to return the PDF to the scattering PDF.

```
        ...

        double scattering_pdf = rec.mat->scattering_pdf(r, rec, scattered);
        double pdf = scattering_pdf;

        ...
```

Listing 22: *[camera.h] Return the PDF to the same as scattering PDF*

6.3 Random Hemispherical Sampling

To confirm our understanding, let's try a different scattering distribution. For this one, we'll attempt to repeat the uniform hemispherical scattering from the first book. There's nothing wrong with this technique, but we are no longer treating our objects as Lambertian. Lambertian is a specific type of diffuse material that requires a $\cos(\theta_o)$ scattering distribution. Uniform hemispherical scattering is a different diffuse material. If we keep the material the same but change the PDF, as we did in last section, we will still converge on the same answer, but our convergence may take more or less samples. However, if we change the material, we will have fundamentally changed the render and the algorithm will converge on a different answer. So when we replace Lambertian diffuse with uniform hemispherical diffuse we should expect the outcome of our render to be *materially* different. We're going to adjust our scattering direction and scattering PDF:

```
class lambertian : public material {
  public:
```

T
th
hemispherical diffuse. When rendering, we should get a slightly different image.

Image 5: Cornell box, with uniform hemispherical sampling

It's pretty close to our old picture, but there are differences that are not just noise. The front of the tall box is much more uniform in color. If you aren't sure what the best sampling pattern for your material is, it's pretty reasonable to just go ahead and assume a uniform PDF, and while that might converge slowly, it's not going to ruin your render. That said, if you're not sure what the correct sampling pattern for your material is, your choice of PDF is not going to be your biggest concern, as incorrectly choosing your scattering function *will* ruin your render. At the very least it will produce an incorrect result. You may find yourself with the most difficult kind of bug to find in a Monte Carlo program — a bug that produces a reasonable looking image! You won't know if the bug is in the first version of the program, or the second, or both!

Let's build some infrastructure to address this.

7 Generating Random Directions

In this and the next two chapters, we'll harden our understanding and our tools.

7.1 Random Directions Relative to the Z Axis

Let's first figure out how to generate random directions. We already have a method to generate random directions using the rejection method, so let's create one using the inversion method. To simplify things, assume the z axis is the surface normal, and θ is the angle from the normal. We'll set everything up in terms of the z axis this chapter. Next chapter we'll get them oriented to the surface normal vector. We will only deal with distributions that are rotationally symmetric about z. So $p(\omega) = f(\theta)$.

Given a directional PDF on the sphere (where $p(\omega) = f(\theta)$), the one dimensional PDFs on θ and ϕ are:

$$a(\phi) = \frac{1}{}$$

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[Math Processing Error]

[Math Processing Error]

Let's try some different functions for $f()$. Let's first try a uniform density on the sphere. The area of the unit sphere is 4π, so a uniform $p(\omega) = \frac{1}{4\pi}$ on the unit sphere.

[Math Processing Error]

[Math Processing Error]

$$= \frac{-\cos(\theta)}{2} - \frac{-\cos(0)}{2}$$

$$= \frac{1 - \cos(\theta)}{2}$$

Solving for $\cos(\theta)$gives:

$$\cos(\theta) = 1 - 2r_2$$

We don't solve for theta because we probably only need to know $\cos(\theta)$anyway, and don't want needless $\arccos()$calls running around.

To generate a unit vector direction toward (θ, ϕ) we convert to Cartesian coordinates:

$$x = \cos(\phi) \cdot \sin(\theta)$$

$$y = \sin(\phi) \cdot \sin(\theta)$$

$$z = \cos(\theta)$$

And using the identity $\cos^2 + \sin^2 = 1$, we get the following in terms of random (r_1, r_2):

$$x = \cos(2\pi \cdot r_1)\sqrt{1 - (1 - 2r_2)^2}$$

$$y = \sin(2\pi \cdot r_1)\sqrt{1 - (1 - 2r_2)^2}$$

$$z = 1 - 2r_2$$

Listing 24: `[sphere_plot.cc]` *Random points on the unit sphere*

And plot them for free on plot.ly (a great site with 3D scatterplot support):

Figure 10: *Random points on the unit sphere*

On the plot.ly website you can rotate that around and see that it appears uniform.

7.2 Uniform Sampling a Hemisphere

Now let's derive uniform on the hemisphere. The density being uniform on the hemisphere means $p(\omega) = f(\theta) = \frac{1}{2\pi}$. Just changing the constant in the theta equations yields:

[Math Processing Error]

[Math Processing Error]

[Math Processing Error]

…

$$\cos(\theta) = 1 - r_2$$

This means that $\cos(\theta)$ will vary from 1 to 0, so θ will vary from 0 to $\pi/2$ which means that nothing will go below the horizon. Rather than plot it, we'll solve for a 2D integral with a known solution. Let's integrate cosine cubed over the hemisphere (just picking something arbitrary with a known solution). First we'll solve the integral by hand:

$$\int_\omega \cos^3(\theta) dA$$

$$= \int_0^{2\pi} \int_0^{\pi/2} \cos^3(\theta) \sin(\theta) d\theta d\phi$$

$$= 2\pi \int_0^{\pi/2} \cos^3(\theta) \sin(\theta) d\theta = \frac{\pi}{2}$$

Now for integration with importance sampling. $p(\omega) = \frac{1}{2\pi}$, so we average $f()/p() = \cos^3(\theta)/\frac{1}{2\pi}$, a

```
    std::cout << "PI/2 = " << pi / 2.0 << '\n';
    std::cout << "Estimate = " << sum / N << '\n';
}
```

Listing 25: *[cos_cubed.cc] Integration using* $\cos^3(x)$

7.3 Cosine Sampling a Hemisphere

We'll now continue trying to solve for cosine cubed over the horizon, but we'll change our PDF to generate directions with $p(\omega) = f(\theta) = \cos(\theta)/\pi$

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[Math Processing Error]

[Math Processing Error]

$$= 1 - \cos^2(\theta)$$

So,

$$\cos(\theta) = \sqrt{1 - r_2}$$

We can save a little algebra on specific cases by noting

$$z = \cos(\theta) = \sqrt{1 - r_2}$$

$$x = \cos(\phi)\sin(\theta) = \cos(2\pi r_1)\sqrt{1 - z^2} = \cos(2\pi r_1)\sqrt{r_2}$$

$$y = \sin(\phi)\sin(\theta) = \sin(2\pi r_1)\sqrt{1 - z^2} = \sin(2\pi r_1)\sqrt{r_2}$$

Here's a function that generates random vectors weighted by this PDF:

```
inline vec3 random_cosine_direction() {
    auto r1 = random_double();
```

```
    auto sum = 0.0;
    for (int i = 0; i < N; i++) {
        vec3 d = random_cosine_direction();
        sum += f(d) / pdf(d);
    }

    std::cout << std::fixed << std::setprecision(12);
    std::cout << "PI/2 = " << pi / 2.0 << '\n';
    std::cout << "Estimate = " << sum / N << '\n';
}
```

Listing 27: *[cos_density.cc] Integration with cosine density function*

We can generate other densities later as we need them. This `random_cosine_direction()` function produces a random direction weighted by $\cos(\theta)$where θ is the angle from the z axis.

8 Orthonormal Bases

In the last chapter we developed methods to generate random directions relative to the z axis. If we want to be able to produce reflections off of any surface, we are going to need to make this more general: Not all normals are going to be perfectly aligned with the z axis. So in this chapter we are going to generalize our methods so that they support arbitrary surface normal vectors.

8.1 Relative Coordinates

An *orthonormal basis* (ONB) is a collection of three mutually orthogonal unit vectors. It is a strict subtype of coordinate system. The Cartesian xyz axes are one example of an orthonormal basis. All of our renders are the result of the relative positions and orientations of the objects in a scene projected onto the image plane of the camera. The camera and objects must be described in the same coordinate system, so that the projection onto the image plane is logically defined, otherwise the camera has no definitive means of correctly rendering the objects. Either the camera must be redefined in the objects' coordinate system, or the objects must be redefined in the camera's coordinate system. It's best to start with both in the same coordinate system, so no redefinition is necessary. So long as the camera and scene are described in the same coordinate system, all is well. The orthonormal basis defines how distances and orientations are represented in the space, but an orthonormal basis alone is not enough. The objects and the

camera need to described by their displacement from a mutually defined location. This is just the origin O of the scene; it represents the center of the universe for everything to displace from.

model has only one cotangent vector, then the process of making an ONB is a nontrivial one. Suppose we have any vector a that is of nonzero length and nonparallel with n. We can get vectors s and t perpendicular to n by using the property of the cross product that $n \times a$ is perpendicular to both n and a:

$$s = \text{unit\_vector}(n \times a)$$

$$t = n \times s$$

This is all well and good, but the catch is that we may not be given an a when we load a model, and our current program doesn't have a way to generate one. If we went ahead and picked an arbitrary a to use as an initial vector we may get an a that is parallel to n. So a common method is to pick an arbitrary axis and check to see if it's parallel to n (which we assume to be of unit length), if it is, just use another axis:

```
if (fabs(n.x()) > 0.9)
    a = vec3(0, 1, 0)
else
    a = vec3(1, 0, 0)
```

We then take the cross product to get s and t

```
vec3 s = unit_vector(cross(n, a));
vec3 t = cross(n, s);
```

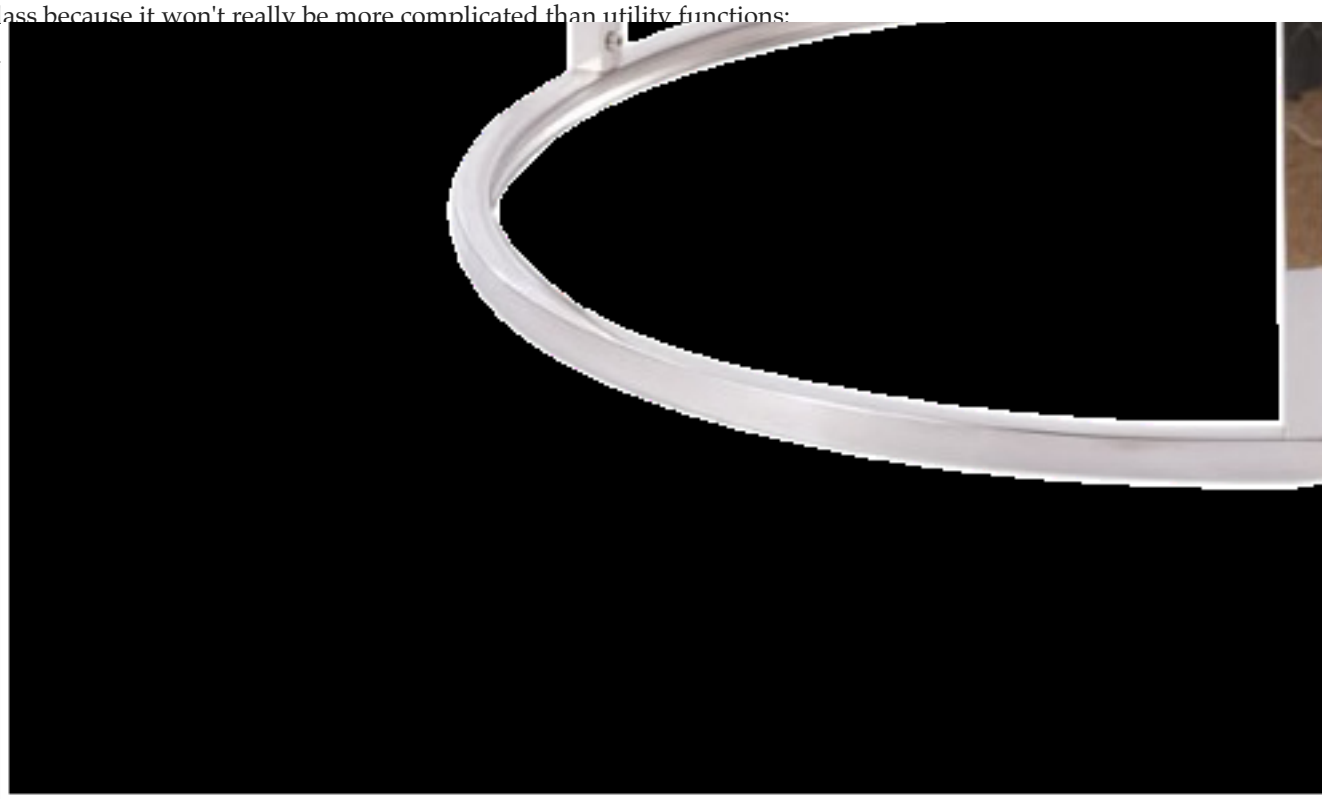
Note that we don't need to take the unit vector for t. Since n and s are both unit vectors, their cross product t will be also. Once we have an ONB of s, t, and n, and we have a random (x, y, z) relative to the z axis, we can get the vector relative to n with:

$$\text{Randomvector} = xs + yt + zn$$

If you remember, we used similar math to produce rays from a camera. You can think of that as a change to the camera's natural coordinate system.

8.3 The ONB Class

Should we make a class for ONBs, or are utility functions enough? I'm not sure, but let's make a class because it won't really be more complicated than utility functions:



```
        axis[0] = u;
        axis[1] = v;
        axis[2] = unit_w;
    }

  public:
    vec3 axis[3];
};


#endif
```

Listing 28: *[onb.h] Orthonormal basis class*

We can rewrite our Lambertian material using this to get:

```
class lambertian : public material {
  public:
    ...
    bool scatter(
        const ray& r_in, const hit_record& rec, color& attenuation, ray& scattered, double& pd
    ) const override {
        onb uvw;
        uvw.build_from_w(rec.normal);
        auto scatter_direction = uvw.local(random_cosine_direction());

        scattered = ray(rec.p, unit_vector(scatter_direction), r_in.time());
        attenuation = tex->value(rec.u, rec.v, rec.p);
        pdf = dot(uvw.w(), scattered.direction()) / pi;
        return true;
    }

    ...
};
```

Listing 29: *[material.h] Scatter function, with orthonormal basis*

Which produces:

Image 6: Cornell box, with orthonormal basis scatter function

Let's get rid of some of that noise.

But first, let's quickly update the `isotropic` material:

The problem with sampling uniformly over all directions is that lights are no more likely to be sampled than any arbirary or unimportant direction. We could use shadow rays to solve for the direct lighting at any given point. Instead, I'll just use a PDF that sends more rays to the light. We can then turn around and change that PDF to send more rays in whatever direction we want.

It's really easy to pick a random direction toward the light; just pick a random point on the light and send a ray in that direction. But we'll need to know the PDF, $p(\omega)$ so that we're not biasing our render. But what is that?

9.1 Getting the PDF of a Light

For a light with a surface area of A, if we sample uniformly on that light, the PDF on the surface is just $\frac{1}{A}$. How much area does the entire surface of the light take up if its projected back onto the unit sphere? Fortunately, there is a simple correspondence, as outlined in this diagram:

Figure 11: *Projection of light shape onto PDF*

If we look at a small area dA on the light, the probability of sampling it is $p_q(q) \cdot dA$. On the sphere, the probability of sampling the small area $d\omega$ on the sphere is $p(\omega) \cdot d\omega$ There is a geometric relationship between $d\omega$and dA:

$$d\omega = \frac{dA \cdot \cos(\theta)}{\text{distance}^2(p, q)}$$

Since the probability of sampling $d\omega$and dA must be the same, then

$$p(\omega) \cdot d\omega = p_q(q) \cdot dA$$

$$p(\omega) \cdot \frac{dA \cdot \cos(\theta)}{\text{distance}^2(p, q)} = p_q(q) \cdot dA$$

We know that if we sample uniformly on the light the PDF on the surface is $\frac{1}{A}$:

```
            if (depth <= 0)
                return color(0,0,0);

            hit_record rec;

            // If the ray hits nothing, return the background color.
            if (!world.hit(r, interval(0.001, infinity), rec))
                return background;

            ray scattered;
            color attenuation;
            double pdf;
            color color_from_emission = rec.mat->emitted(rec.u, rec.v, rec.p);

            if (!rec.mat->scatter(r, rec, attenuation, scattered, pdf))
                return color_from_emission;

            auto on_light = point3(random_double(213,343), 554, random_double(227,332));
            auto to_light = on_light - rec.p;
            auto distance_squared = to_light.length_squared();
            to_light = unit_vector(to_light);

            if (dot(to_light, rec.normal) < 0)
                return color_from_emission;

            double light_area = (343-213)*(332-227);
            auto light_cosine = fabs(to_light.y());
            if (light_cosine < 0.000001)
                return color_from_emission;

            pdf = distance_squared / (light_cosine * light_area);
            scattered = ray(rec.p, to_light, r.time());

            double scattering_pdf = rec.mat->scattering_pdf(r, rec, scattered);

            color color_from_scatter =
                (attenuation * scattering_pdf * ray_color(scattered, depth-1, world)) / pdf;

            return color_from_emission + color_from_scatter;
        }
    };
```

Listing 31: *[camera.h] Ray color with light sampling*

With 10 samples per pixel this yields:

Image 7: Cornell box, sampling only the light, 10 samples per pixel

This is about what we would expect from something that samples only the light sources, so this appears to work.

```
            return color(0,0,0);
        return emit->value(u, v, p);
    }

    ...
};
```

Listing 32: *[material.h] Material emission, directional*

```
class camera {
  ...
  private:
    color ray_color(const ray& r, int depth, const hittable& world) const {
        ...

        color color_from_emission = rec.mat->emitted(r, rec, rec.u, rec.v, rec.p);

        ...
    }
};
```

Listing 33: *[camera.h] Material emission, camera::ray_color() changes*

This gives us:

Image 8: Cornell box, light emitted only in the downward direction

10 Mixture Densities

We have used a PDF related to $\cos(\theta)$ and a PDF related to sampling the light. We would like a PDF that combines these.

10.1 The PDF Class

We've worked with PDFs in quite a lot of code already. I think that now is a good time to figure out how we want to standardize our usage of PDFs. We already know that we are going to have a PDF for the surface and a PDF for the light, so let's create a `pdf` base class. So far, we've had a `pdf()` function that took a direction and returned the PDF's distribution value for that direction. This value has so far been one of $1/4\pi$ $1/2\pi$ and $\cos(\theta)/\pi$ In a couple of our examples we

generated the random direction using a different distribution than the distribution of the PDF. We covered this quite a lot in the chapter Playing with Importance Sampling. In general, if we k d th

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```
    virtual vec3 generate() const = 0;
};

#endif
```

Listing 34: *[pdf.h] The abstract pdf class*

We'll see if we need to add anything else to `pdf` by fleshing out the subclasses. First, we'll create a uniform density over the unit sphere:

```
class sphere_pdf : public pdf {
  public:
    sphere_pdf() { }

    double value(const vec3& direction) const override {
        return 1/ (4 * pi);
    }

    vec3 generate() const override {
        return random_unit_vector();
    }
};
```

Listing 35: *[pdf.h] The uniform_pdf class*

Next, let's try a cosine density:

```
class cosine_pdf : public pdf {
  public:
    cosine_pdf(const vec3& w) { uvw.build_from_w(w); }

    double value(const vec3& direction) const override {
        auto cosine_theta = dot(unit_vector(direction), uvw.w());
        return fmax(0, cosine_theta/pi);
    }

    vec3 generate() const override {
        return uvw.local(random_cosine_direction());
    }

  private:
    onb uvw;
};
```

Listing 36: *[pdf.h] The cosine_pdf class*

We can try this cosine PDF in the `ray_color()` function:

```
            (attenuation * scattering_pdf * ray_color(scattered, depth-1, world)) / pdf_val;

        return color_from_emission + color_from_scatter;
    }
};
```

Listing 37: *[camera.h] The ray_color function, using cosine pdf*

This yields an exactly matching result so all we've done so far is move some computation up into the `cosine_pdf` class:

Image 9: Cornell box with a cosine density PDF

10.2 Sampling Directions towards a Hittable

Now we can try sampling directions toward a `hittable`, like the light.

```
...
#include "hittable_list.h"
...
class hittable_pdf : public pdf {
  public:
    hittable_pdf(const hittable& objects, const point3& origin)
      : objects(objects), origin(origin)
    {}

    double value(const vec3& direction) const override {
        return objects.pdf_value(origin, direction);
    }

    vec3 generate() const override {
        return objects.random(origin);
    }

  private:
    const hittable& objects;
    point3 origin;
};
```

Listing 38: *[pdf.h] The hittable_pdf class*

If we want to sample the light, we will need `hittable` to answer some queries that it doesn't yet

have an interface for. The above code assumes the existence of two as-of-yet unimplemented functions in the `hittable` class: `pdf_value()` and `random()`. We need to add these functions for th
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```
    : Q(Q), u(u), v(v), mat(mat)
    {
        auto n = cross(u, v);
        normal = unit_vector(n);
        D = dot(normal, Q);
        w = n / dot(n,n);

        area = n.length();

        set_bounding_box();
    }
    ...

    double pdf_value(const point3& origin, const vec3& direction) const override {
        hit_record rec;
        if (!this->hit(ray(origin, direction), interval(0.001, infinity), rec))
            return 0;

        auto distance_squared = rec.t * rec.t * direction.length_squared();
        auto cosine = fabs(dot(direction, rec.normal) / direction.length());

        return distance_squared / (cosine * area);
    }

    vec3 random(const point3& origin) const override {
        auto p = Q + (random_double() * u) + (random_double() * v);
        return p - origin;
    }

  private:
    point3 Q;
    vec3 u, v;
    vec3 w;
    shared_ptr<material> mat;
    aabb bbox;
    vec3 normal;
    double D;
    double area;
};
```

Listing 40: *[quad.h] quad with pdf*

We only need to add `pdf_value()` and `random()` to `quad` because we're using this to importance sample the light, and the only light we have in our scene is a `quad`. if you want other light geometries, or want to use a PDF with other objects, you'll need to implement the above functions for the corresponding classes.

Add a `lights` parameter to the camera `render()` function:

```
        ...

        ray scattered;
        color attenuation;
        double pdf_val;
        color color_from_emission = rec.mat->emitted(r, rec, rec.u, rec.v, rec.p);

        if (!rec.mat->scatter(r, rec, attenuation, scattered, pdf_val))
            return color_from_emission;

        hittable_pdf light_pdf(light_ptr, rec.p);
        scattered = ray(rec.p, light_pdf.generate(), r.time());
        pdf_val = light_pdf.value(scattered.direction());

        double scattering_pdf = rec.mat->scattering_pdf(r, rec, scattered);

        color sample_color = ray_color(scattered, depth-1, world, lights);
        color color_from_scatter = (attenuation * scattering_pdf * sample_color) / pdf_val;

        return color_from_emission + color_from_scatter;
    }
};
```

Listing 41: *`[camera.h]` ray_color function with light PDF*

Create a light in the middle of the ceiling:

```
int main() {
    ...

    // Box 2
    shared_ptr<hittable> box2 = box(point3(0,0,0), point3(165,165,165), white);
    box2 = make_shared<rotate_y>(box2, -18);
    box2 = make_shared<translate>(box2, vec3(130,0,65));
    world.add(box2);

    // Light Sources
    hittable_list lights;
    auto m = shared_ptr<material>();
    lights.add(make_shared<quad>(point3(343,554,332), vec3(-130,0,0), vec3(0,0,-105), m));

    camera cam;
    ...

    cam.render(world, lights);
}
```

Listing 42: [main.cc] *Adding a light to the Cornell box*

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not quite as easy as one might expect. Generating the random direction for a mixture PDF is simple:

```
if (random_double() < 0.5)
    pick direction according to pSurface
else
    pick direction according to pLight
```

But solving for the PDF value of pMixture is slightly more subtle. We can't just

```
if (direction is from pSurface)
    get PDF value of pSurface
else
    get PDF value of pLight
```

For one, figuring out which PDF the random direction came from is probably not trivial. We don't have any plumbing for `generate()` to tell `value()` what the original `random_double()` was, so we can't trivially say which PDF the random direction comes from. If we thought that the above was correct, we would have to solve backwards to figure which PDF the direction could come from. Which honestly sounds like a nightmare, but fortunately we don't need to do that. There are some directions that both PDFs could have generated. For example, a direction toward the light could have been generated by either pLight *or* pSurface It is sufficient for us to solve for the pdf value of pSurfaceand of pLight for a random direction and then take the PDF mixture weights to solve for the total PDF value for that direction. The mixture density class is actually pretty straightforward:

```
class mixture_pdf : public pdf {
  public:
```

```
    color ray_color(const ray& r, int depth, const hittable& world, const hittable& lights)
    const {
        ...

        if (!rec.mat->scatter(r, rec, attenuation, scattered, pdf_val))
            return color_from_emission;

        auto p0 = make_shared<hittable_pdf>(light_ptr, rec.p);
        auto p1 = make_shared<cosine_pdf>(rec.normal);
        mixture_pdf mixed_pdf(p0, p1);

        scattered = ray(rec.p, mixed_pdf.generate(), r.time());
        pdf_val = mixed_pdf.value(scattered.direction());

        double scattering_pdf = rec.mat->scattering_pdf(r, rec, scattered);

        color sample_color = ray_color(scattered, depth-1, world, lights);
        color color_from_scatter = (attenuation * scattering_pdf * sample_color) / pdf_val;

        return color_from_emission + color_from_scatter;
    }
```

Listing 44: *[camera.h] The ray_color function, using mixture PDF*

1000 samples per pixel yields:

Image 11: Cornell box, mixture density of cosine and light sampling

11 Some Architectural Decisions

We won't write any code in this chapter. We're at a crossroads and we need to make some architectural decisions.

The mixture-density approach is an alternative to having more traditional shadow rays. These are rays that check for an unobstructed path from an intersection point to a given light source. Rays that intersect an object between a point and a given light source indicate that the intersection point is in the shadow of that particular light source. The mixture-density approach is something that I personally prefer, because in addition to lights, you can sample windows or bright cracks under doors or whatever else you think might be bright — or important. But you'll still see shadow rays in most professional path tracers. Typically they'll have a predefined

number of shadow rays (*e.g* 1, 4, 8, 16) where over the course of rendering, at each place where the path tracing ray intersects, they'll send these terminal shadow rays to random lights in the sc
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don't generate NaNs. I don't have an opinion on which way to do it (I have tried both and they both have their advantages), but we have smooth metal and glass code anyway, so we'll add perfect specular surfaces that just skip over explicit f()/p() calculations.

We also lack a real background function infrastructure in case we want to add an environment map or a more interesting functional background. Some environment maps are HDR (the RGB components are normalized floats rather than 0–255 bytes). Our output has been HDR all along; we've just been truncating it.

Finally, our renderer is RGB. A more physically based one — like an automobile manufacturer might use — would probably need to use spectral colors and maybe even polarization. For a movie renderer, most studios still get away with RGB. You can make a hybrid renderer that has both modes, but that is of course harder. I'm going to stick to RGB for now, but I will touch on this at the end of the book.

12 Cleaning Up PDF Management

So far I have the `ray_color()` function create two hard-coded PDFs:

1. `p0()` related to the shape of the light
2. `p1()` related to the normal vector and type of surface

We can pass information about the light (or whatever `hittable` we want to sample) into the `ray_color()` function, and we can ask the `material` function for a PDF (we would have to add instrumentation to do that). We also need to know if the scattered ray is specular, and we can do this either by asking the `hit()` function or the `material` class.

12.1 Diffuse Versus Specular

One thing we would like to allow for is a material — like varnished wood — that is partially ideal specular (the polish) and partially diffuse (the wood). Some renderers have the material generate two rays: one specular and one diffuse. I am not fond of branching, so I would rather have the material randomly decide whether it is diffuse or specular. The catch with that approach is that we need to be careful when we ask for the PDF value, and `ray_color()` needs to be aware of whether this ray is diffuse or specular. Fortunately, we have decided that we

should only call the `pdf_value()` if it is diffuse, so we can handle that implicitly.

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```
    bool scatter(const ray& r_in, const hit_record& rec, scatter_record& srec) const override
        srec.attenuation = tex->value(rec.u, rec.v, rec.p);
        srec.pdf_ptr = make_shared<cosine_pdf>(rec.normal);
        srec.skip_pdf = false;
        return true;
    }

    double scattering_pdf(const ray& r_in, const hit_record& rec, const ray& scattered) const
        auto cosine = dot(rec.normal, unit_vector(scattered.direction()));
        return cosine < 0 ? 0 : cosine/pi;
    }

  private:
    shared_ptr<texture> tex;
};
```

Listing 46: *[material.h] New lambertian scatter() method*

As does the `isotropic` material:

```
class isotropic : public material {
  public:
    isotropic(const color& albedo) : tex(make_shared<solid_color>(albedo)) {}
    isotropic(shared_ptr<texture> tex) : tex(tex) {}

    bool scatter(const ray& r_in, const hit_record& rec, scatter_record& srec) const override
        srec.attenuation = tex->value(rec.u, rec.v, rec.p);
        srec.pdf_ptr = make_shared<sphere_pdf>();
        srec.skip_pdf = false;
        return true;
    }

    double scattering_pdf(const ray& r_in, const hit_record& rec, const ray& scattered)
    const override {
        return 1 / (4 * pi);
    }

  private:
    shared_ptr<texture> tex;
};
```

Listing 47: *[material.h] New isotropic scatter() method*

And `ray_color()` changes are small:

```
class camera {
  ...
```

```
        return color_from_emission + color_from_scatter;
    }
};
```

Listing 48: *[camera.h] The ray_color function, using mixture PDF*

12.2 Handling Specular

We have not yet dealt with specular surfaces, nor instances that mess with the surface normal. But this design is clean overall, and those are all fixable. For now, I will just fix `specular`. Metal and dielectric materials are easy to fix.

```
class metal : public material {
  public:
```

```
        vec3 unit_direction = unit_vector(r_in.direction());
        double cos_theta = fmin(dot(-unit_direction, rec.normal), 1.0);
        double sin_theta = sqrt(1.0 - cos_theta*cos_theta);

        bool cannot_refract = ri * sin_theta > 1.0;
        vec3 direction;

        if (cannot_refract || reflectance(cos_theta, ri) > random_double())
            direction = reflect(unit_direction, rec.normal);
        else
            direction = refract(unit_direction, rec.normal, ri);

        srec.skip_pdf_ray = ray(rec.p, direction, r_in.time());
        return true;
    }

  ...
};
```

Listing 49: *[material.h] The metal and dielectric scatter methods*

Note that if the fuzziness is nonzero, this surface isn't really ideally specular, but the implicit sampling works just like it did before. We're effectively skipping all of our PDF work for the materials that we're treating specularly.

`ray_color()` just needs a new case to generate an implicitly sampled ray:

```
class camera {
  ...
```

```
    ...
    // Light
    world.add(make_shared<quad>(point3(213,554,227), vec3(130,0,0), vec3(0,0,105), light));

    // Box 1
    shared_ptr<material> aluminum = make_shared<metal>(color(0.8, 0.85, 0.88), 0.0);
    shared_ptr<hittable> box1 = box(point3(0,0,0), point3(165,330,165), aluminum);
    box1 = make_shared<rotate_y>(box1, 15);
    box1 = make_shared<translate>(box1, vec3(265,0,295));
    world.add(box1);

    // Box 2
    shared_ptr<hittable> box2 = box(point3(0,0,0), point3(165,165,165), white);
    box2 = make_shared<rotate_y>(box2, -18);
    box2 = make_shared<translate>(box2, vec3(130,0,65));
    world.add(box2);

    // Light Sources
    hittable_list lights;
    auto m = shared_ptr<material>();
    lights.add(make_shared<quad>(point3(343,554,332), vec3(-130,0,0), vec3(0,0,-105), m));

    ...
}
```

Listing 51: *[main.cc] Cornell box scene with aluminum material*

The resulting image has a noisy reflection on the ceiling because the directions toward the box are not sampled with more density.

Image 12: Cornell box with arbitrary PDF functions

12.3 Sampling a Sphere Object

The noisiness on the ceiling could be reduced by making a PDF of the metal block. We would also want a PDF for the block if we made it glass. But making a PDF for a block is quite a bit of work and isn't terribly interesting, so let's create a PDF for a glass sphere instead. It's quicker and makes for a more interesting render. We need to figure out how to sample a sphere to determine an appropriate PDF distribution. If we want to sample a sphere from a point outside of the sphere, we can't just pick a random point on its surface and be done. If we did that, we would frequently pick a point on the far side of the sphere, which would be occluded by the

front side of the sphere. We need a way to uniformly sample the side of the sphere that is visible from an arbitrary point. When we sample a sphere's solid angle uniformly from a point outside th
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$$r_2 = 2\pi \cdot C \cdot (1 - \cos(\theta))$$

$$1 = 2\pi \cdot C \cdot (1 - \cos(\theta_{max}))$$

$$C = \frac{1}{2\pi \cdot (1 - \cos(\theta_{max}))}$$

Which gives us an equality between θ, θ_{max}, and r_2:

$$\cos(\theta) = 1 + r_2 \cdot (\cos(\theta_{max}) - 1)$$

We sample ϕ like before, so:

$$z = \cos(\theta) = 1 + r_2 \cdot (\cos(\theta_{max}) - 1)$$

$$x = \cos(\phi) \cdot \sin(\theta) = \cos(2\pi \cdot r_1) \cdot \sqrt{1 - z^2}$$

$$y = \sin(\phi) \cdot \sin(\theta) = \sin(2\pi \cdot r_1) \cdot \sqrt{1 - z^2}$$

Now what is θ_{max}?

Figure 12: *A sphere-enclosing cone*

We can see from the figure that $\sin(\theta_{max}) = R/\text{length}(c - p)$ So:

$$\cos(\theta_{max}) = \sqrt{1 - \frac{R^2}{\text{length}^2(c - p)}}$$

We also need to evaluate the PDF of directions. For a uniform distribution toward the sphere the PDF is $1/\text{solid_angle}$What is the solid angle of the sphere? It has something to do with the C above. It is — by definition — the area on the unit sphere, so the integral is

$$\text{solid\_angle} = \int_0^{2\pi} \int_0^{\theta_{max}} \sin(\theta) = 2\pi \cdot (1 - \cos(\theta_{max}))$$

It's good to check the math on all such calculations. I usually plug in the extreme cases (thank you for that concept, Mr. Horton — my high school physics teacher). For a zero radius sphere

```
        auto distance_squared = direction.length_squared();
        onb uvw;
        uvw.build_from_w(direction);
        return uvw.local(random_to_sphere(radius, distance_squared));
    }

  private:
    ...
    static vec3 random_to_sphere(double radius, double distance_squared) {
        auto r1 = random_double();
        auto r2 = random_double();
        auto z = 1 + r2*(sqrt(1-radius*radius/distance_squared) - 1);

        auto phi = 2*pi*r1;
        auto x = cos(phi)*sqrt(1-z*z);
        auto y = sin(phi)*sqrt(1-z*z);

        return vec3(x, y, z);
    }
};
```

Listing 52: *[sphere.h] Sphere with PDF*

We can first try just sampling the sphere rather than the light:

```
int main() {
    ...
```

Image 13: Cornell box with glass sphere, using new PDF functions

12.5 Adding PDF Functions to Hittable Lists

We should probably just sample both the sphere and the light. We can do that by creating a mixture density of their two distributions. We could do that in the `ray_color()` function by passing a list of hittables in and building a mixture PDF, or we could add PDF functions to `hittable_list`. I think both tactics would work fine, but I will go with instrumenting `hittable_list`.

```
class hittable_list : public hittable {
  public:
    ...

    double pdf_value(const point3& origin, const vec3& direction) const override {
        auto weight = 1.0 / objects.size();
        auto sum = 0.0;

        for (const auto& object : objects)
            sum += weight * object->pdf_value(origin, direction);

        return sum;
    }

    vec3 random(const point3& origin) const override {
        auto int_size = int(objects.size());
        return objects[random_int(0, int_size-1)]->random(origin);
    }

    ...
};
```

Listing 54: *[hittable_list.h] Creating a mixture of densities*

We assemble a list to pass to `render()` from `main()`:

```
int main() {
    ...
```

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If you find yourself getting some form of acne in your renders, and this acne is white or black where one "bad" sample seems to kill the whole pixel — then that sample is probably a huge number or a `NaN` (Not A Number). This particular acne is probably a `NaN`. Mine seems to come up once in every 10–100 million rays or so.

So big decision: sweep this bug under the rug and check for `NaN`s, or just kill `NaN`s and hope this doesn't come back to bite us later. I will always opt for the lazy strategy, especially when I know that working with floating point is hard. First, how do we check for a `NaN`? The one thing I always remember for `NaN`s is that a `NaN` does not equal itself. Using this trick, we update the `write_color()` function to replace any `NaN` components with zero:

```
void write_color(std::ostream& out, const color& pixel_color) {
    auto r = pixel_color.x();
    auto g = pixel_color.y();
    auto b = pixel_color.z();

    // Replace NaN components with zero.
    if (r != r) r = 0.0;
    if (g != g) g = 0.0;
    if (b != b) b = 0.0;

    // Apply a linear to gamma transform for gamma 2
    r = linear_to_gamma(r);
    g = linear_to_gamma(g);
    b = linear_to_gamma(b);

    // Translate the [0,1] component values to the byte range [0,255].
    static const interval intensity(0.000, 0.999);
    int rbyte = int(256 * intensity.clamp(r));
    int gbyte = int(256 * intensity.clamp(g));
    int bbyte = int(256 * intensity.clamp(b));

    // Write out the pixel color components.
    out << rbyte << ' ' << gbyte << ' ' << bbyte << '\n';
}
```

Listing 56: *`[color.h]` NaN-tolerant write_color function*

Happily, the black specks are gone:

Image 15: Cornell box with anti-acne color function

Regardless of what direction you take, add a glossy BRDF model. There are many to choose from, and each has its advantages.

Have fun!

Peter Shirley
Salt Lake City, March, 2016

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15 Citing This Book

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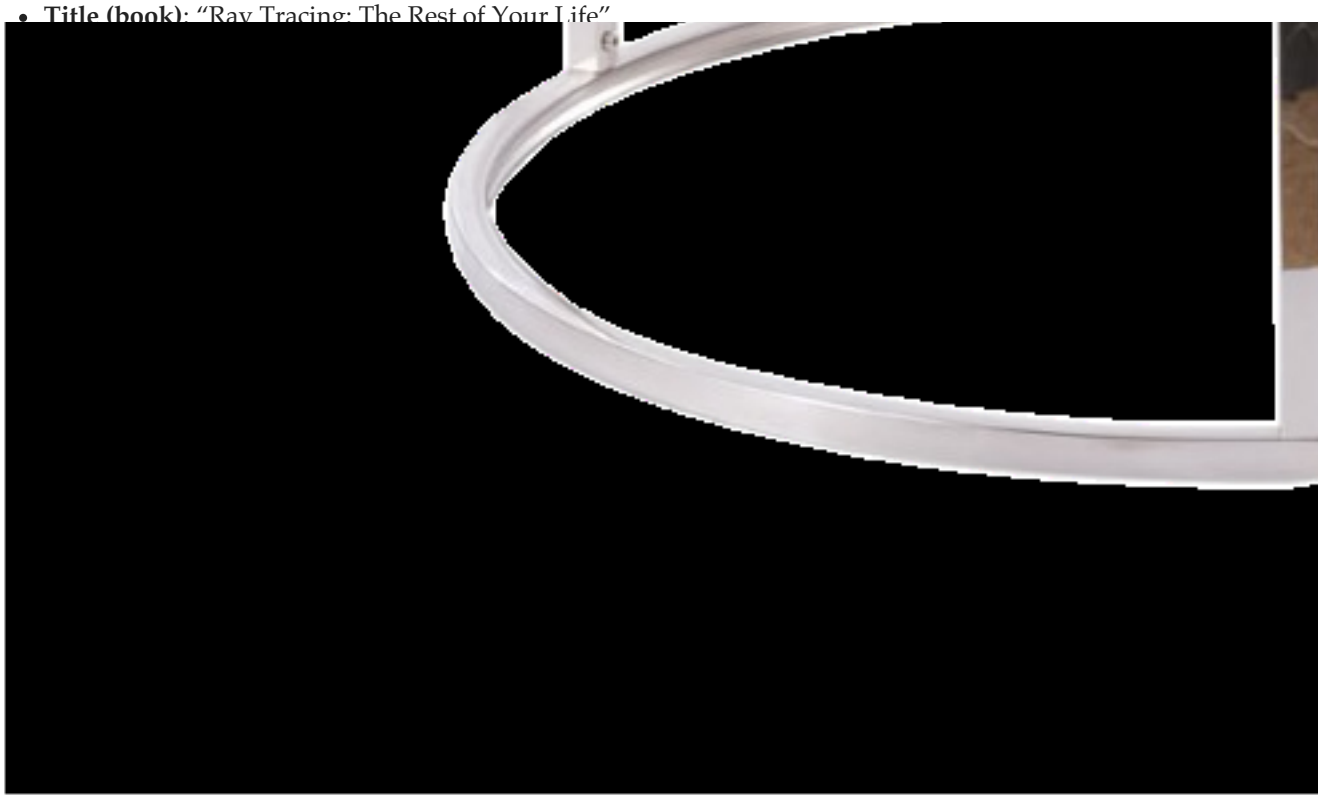
- **Title (series)**: “Ray Tracing in One Weekend Series”
- **Title (book)**: “Ray Tracing: The Rest of Your Life”



```
~\cite{Shirley2024RTW3}

@misc{Shirley2024RTW3,
   title = {Ray Tracing: The Rest of Your Life},
   author = {Peter Shirley, Trevor David Black, Steve Hollasch},
   year = {2024},
   month = {April},
   note = {\small \texttt{https://raytracing.github.io/books/RayTracingTheRestOfYourLife.html}
   url = {https://raytracing.github.io/books/RayTracingTheRestOfYourLife.html}
}
```

15.2.4 BibLaTeX

```
\usepackage{biblatex}

~\cite{Shirley2024RTW3}

@online{Shirley2024RTW3,
   title = {Ray Tracing: The Rest of Your Life},
   author = {Peter Shirley, Trevor David Black, Steve Hollasch},
   year = {2024},
   month = {April},
   url = {https://raytracing.github.io/books/RayTracingTheRestOfYourLife.html}
}
```

15.2.5 IEEE

```
“Ray Tracing: The Rest of Your Life.”
raytracing.github.io/books/RayTracingTheRestOfYourLife.html
(accessed MMM. DD, YYYY)
```

15.2.6 MLA:

```
Ray Tracing: The Rest of Your Life. raytracing.github.io/books/RayTracingTheRestOfYourLife.htm
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