A closest number game solution

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The game description:

Three players A, B, C play the following game.

First, A picks a real number between 0 and 1 (both inclusive), then B picks a number in the same range (different from A's choice) and finally C picks a number, also in the same range, (different from the two chosen numbers).

We then pick a number in the range uniformly randomly. Whoever's number is closest to this random number wins the game.

Assume that A, B and C all play optimally and their sole goal is to maximise their chances of winning. Also assume that if one of them has several optimal choices, then that player will randomly pick one of the optimal choices.

The problems:

1. If A chooses 0, then what is the best choice for B?
2. What is the best choice for A?
3. What is the best choice for the first player when the game is played among n players?

Answers:

1. The optimal choice for B if A chooses 0 is $\boldsymbol{x_2 = \frac{2}{3}}$

2. The optimal choice for A is $\boldsymbol{x_1 = \frac{1}{4}}$ *or* $\boldsymbol{x_1 = \frac{3}{4}}$

3. The best choice for the A when the game is played among n players you can get running the programm.

Solutions:

Let's agree that the players choose numbers x_i, where i - tne number of a player.

1. The best choice for B if A chooses 0.

We have $x_1 = 0$. Let's assume the player B has already chosen his number x_2.

Now the player C is making decision: he can choose x_3 either $x_3 < x_2$ or $x_3 > x_2$.

In case of $x_3 < x_2$ probability of winning is $\frac{x_2}{2}$ regardless of the third player's choice.

In case of $x_3 > x_2$ optimally to choose $x_2 + \epsilon$ i.e. probability of winning is $1 - x_3 = 1 - x_2 - \epsilon$ where ϵ is infinitely small and can be omitted.

For the player B the optimal strategy is equalizing the chances of the third player: $\frac{x_2}{2} = 1 - x_2$

So the optimal choice for B if A chooses 0 is $\boldsymbol{x_2 = \frac{2}{3}}$

In this case if (A, B, C) choose $(0, \frac{2}{3}, \frac{1}{3})$, probabilities to win are $(\frac{1}{6}, \frac{1}{2}, \frac{1}{3})$ respectively.

2. The optimal choice for A.

Let's assume the player A and B have already chosen x_1 and x_2 respectively and $x_1 < x_2$.

Then for the third player the optimal strategy x_3 is among
$x_1 - \epsilon, \forall x_3 \in (x_1, x_2), x_2 + \epsilon$, where ϵ is infinitely small and can be omitted.

The probabilities to win are $x_1, \frac{(x_2 - x_1)}{2}, 1 - x_2$

Optimal strategies we will get resolving the system of equations:

$$\begin{cases} x_1 = \frac{(x_2 - x_1)}{2} \\ x_1 = 1 - x_2 \end{cases} => \begin{cases} 3x_1 - x_2 = 0 \\ x_1 + x_2 = 1 \end{cases} => \begin{cases} x_1 = \frac{1}{4} \\ x_2 = \frac{3}{4} \end{cases}$$

Thus considering symmetry (could be $x_1 > x_2$) both answers $\boldsymbol{\frac{1}{4}}$ and $\boldsymbol{\frac{3}{4}}$ are optimal for the first player.

In this case if (A, B, C) choose $(\frac{1}{4}, \frac{3}{4}, \frac{1}{2})$, probabilities to win are $(\frac{3}{8}, \frac{3}{8}, \frac{1}{4})$ respectively.

3. The best choice for the first player when the game is played among n players.

Now let's expand the previous solution to n players:

Let's assume the first $n-1$ players have already chosen $x_1, ..., x_{n-1}$ respectively and $x_1 < ... < x_{n-1}$.

Then for the third player the optimal strategy x_3 is among $x_1 - \epsilon, \forall x_n \in (x_1, x_2), ..., \forall x_n \in (x_{n-2}, x_{n-1}), x_{n-1} + \epsilon$, where ϵ is infinitely small and can be omitted.

The probabilities to win are $x_1, \frac{(x_2 - x_1)}{2}, ..., \frac{(x_{n-1} - x_{n-2})}{2}, 1 - x_{n-1}$

Optimal strategies we will get resolving the system of equations:

$$\begin{cases} x_1 = \frac{(x_2 - x_1)}{2} \\ x_1 = \frac{(x_3 - x_2)}{2} \\ ... \\ x_1 = \frac{(x_{n-1} - x_{n-2})}{2} \\ x_1 = 1 - x_{n-1} \end{cases} => \begin{cases} 3x_1 - x_2 = 0 \\ 2x_1 + x_2 - x_3 = 0 \\ ... \\ 2x_1 + x_{n-2} - x_{n-1} = 0 \\ x_1 + x_{n-1} = 1 \end{cases}$$

We can resolve this system using matrix method converting to matrix form $Ax = b$ where:

$$A = \begin{pmatrix} 3 & -1 & 0 & 0 & \cdots & 0 & 0 \\ 2 & 1 & -1 & 0 & \cdots & 0 & 0 \\ 2 & 0 & 1 & -1 & \cdots & 0 & 0 \\ \vdots & & & \ddots & & & \vdots \\ 2 & 0 & 0 & 0 & \cdots & -1 & 0 \\ 2 & 0 & 0 & 0 & \cdots & 1 & -1 \\ 1 & 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}, x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-3} \\ x_{n-2} \\ x_{n-1} \end{pmatrix}, b = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

The solution of this system is implemented in the CLI app.