Even Fibonacci numbers

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The problem description:

Each new term in the Fibonacci sequence is generated by adding the previous two terms. By starting with 0 and 1, the first 12 terms will be:

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \ldots$$

By considering the terms in the Fibonacci sequence whose values do not exceed four million, find the sum of the even-valued terms.

Let's reformulate the problem in general form:

Given a non-negative integer n, find the sum of the even-valued terms in the Fibonacci sequence whose values are below n

Solution:

We can notice notice and easily prove that every third Fibonacci number is even:

$$\textit{0},\ 1,\ 1,\ \textit{2},\ 3,\ 5,\ \textit{8},\ 13,\ 21,\ \textit{34},\ 55,\ 89,\ \textit{144},\ 233,\ 377,\ \ldots$$

The proof easily follows from the fact that the sum is even if both terms are even or odd:

$$2k + 2l = 2(k + l)$$
$$(2k + 1) + (2l + 1) = 2(k + l + 1)$$

If we only write the even numbers:

$$0,\ 2,\ 8,\ 34,\ 144,\ \ldots$$

we can prove that they obey the following recursive relation:

$$E(n) = 4E(n-1) + E(n-2)$$

If we can prove that for the Fibonacci numbers the formula

$$F(n) = 4F(n-3) + F(n-6)$$

holds we have proven this recursion and we can easily program it.

Proof:

$$\begin{aligned}
F(n) &= F(n-1) + F(n-2) \\
&= F(n-2) + F(n-3) + F(n-2) \\
&= 2F(n-2) + F(n-3) \\
&= 2(F(n-3) + F(n-4)) + F(n-3) \\
&= 3F(n-3) + 2F(n-4) \\
&= 3F(n-3) + F(n-4) + F(n-4) \\
&= 3F(n-3) + F(n-4) + F(n-5) + F(n-6) \\
&= 4F(n-3) + F(n-6)
\end{aligned}$$