Multiples of 3 and 5 problem

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The problem description:

If we list all the natural numbers below 10 that are multiples of 3 or 5, we get 3, 5, 6 and 9. The sum of these multiples is 23.
Find the sum of all the multiples of 3 or 5 below 1000.

Let's reformulate the problem in general form for two dividers:

Given a non-negative integer n and two integers d_1 and d_2, find the sum of all the multiples of d_1 or d_2 below n

Solution:

We can notice that multiples of d_1 and d_2 form two Arithmetic Progression(AP):

$$d_1,\ 2d_1,\ 3d_1,\ \ldots$$
$$d_2,\ 2d_2,\ 3d_2,\ \ldots$$

Let's denote:

$T(n, d_1, d_2)$ - the sum of all the multiples of d_1 or d_2 below n

$S_d(m)$ - sum of the first m multiples of d, thus it is the sum of the first m terms of the AP: $d,\ 2d,\ 3d,\ \ldots$

$$S_d(n) = \frac{m(d + md)}{2} = \frac{m(m + 1)d}{2}$$

In both sequences there will be common terms:

$$d_1d_2,\ 2d_1d_2,\ 3d_1d_2,\ \ldots$$

So if we will sum terms: $S_{d_1}(m_1) + S_{d_2}(m_2)$ these common terms will be repeated. Hence we have to subtract them:

$$T(n, d_1, d_2) = S_{d_1}(m(d_1, n)) + S_{d_2}(m(d_1, n)) - S_{d_1 d_2}(m(d_1d_2, n)),$$

where $m(d, n) = \left\lfloor \frac{n}{d} \right\rfloor$ - amount of the multiples of d below n.