Pascal's triangle divisibility problem

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January 14, 2020

The problem description:

We can easily verify that none of the entries in the first seven rows of Pascal's triangle are divisible by 7:

$$\begin{array}{lccccccccccccc}
m=0: & & & & & & & 1 & & & & & & \\
m=1: & & & & & & 1 & & 1 & & & & & \\
m=2: & & & & & 1 & & 2 & & 1 & & & & \\
m=3: & & & & 1 & & 3 & & 3 & & 1 & & & \\
m=4: & & & 1 & & 4 & & 6 & & 4 & & 1 & & \\
m=5: & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\
m=6: & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1
\end{array}$$

However, if we check the first one hundred rows, we will find that only 2361 of the 5050 entries are *not* divisible by 7. Find the number of entries which are not divisible by 7 in the first one billion (10^9) rows of Pascal's triangle.

Let's reformulate the problem in general form:

Given a non-negative integer n and prime p, count the number of binomial coefficients $\binom{m}{k}$ for $m \le n$ that are not divisible by p.

Solution:

Let's address to one a consequence of Lucas' theorem:

A binomial coefficient $\binom{m}{k}$ is divisible by a prime p (p divides $\binom{m}{k}$) if and only if at least one of the base p digits of k is greater than the corresponding digit of m.

Now let's agree on the overline notation used to express the base p expansion:

$$n = n_i p^i + n_{i-1} p^{i-1} + \cdots + n_1 p + n_0 = \overline{n_i n_{i-1} \cdots n_0},$$
$$m = m_i p^i + m_{i-1} p^{i-1} + \cdots + m_1 p + m_0 = \overline{m_i m_{i-1} \cdots m_0},$$
$$k = k_i p^i + k_{i-1} p^{i-1} + \cdots + k_1 p + k_0 = \overline{k_i k_{i-1} \cdots k_0}.$$

So, in other words, $p \mid \binom{m}{k} <=> \exists\, j : k_j > m_j$

Let's denote:

The amount of $\binom{m}{k}$ not divisible by p in a row with index m as a function $d(m)$.

And the target function D(n) - the number of binomial coefficients $\binom{m}{k}$ for $m \leq n$ that are not divisible by p:

$$D(n) = \sum_{i=1}^{n} d(i)$$

First we'll find d(m). As an example, let's take a look on Pascal's triangle's row with index 6 and consider $p = 3$: $m = 6 = 2 * 3 + 0 = \overline{20}$.

decimal k	0	1	2	3	4	5	6
p expansion of k	$\overline{00}$	$\overline{01}$	$\overline{02}$	$\overline{10}$	$\overline{11}$	$\overline{12}$	$\overline{20}$
$j : k_j > m_j$	$\nexists$	0	0	$\nexists$	0	0	$\nexists$
$p \mid \binom{m}{k}$	0	1	1	0	1	1	0
$\binom{m}{k}$	1	6	15	20	15	6	1

So, in the row with index 6 we have $d(6) = 3$. Now let's count this number without checking each coefficient in the row: for $m = \overline{20}$, $\binom{m}{k}$ is not divisible if $k_1 \leq 2$ and $k_0 \leq 0$ Consequently, $d(6) = (2+1) * (0+1)$

In general case for the $m = \overline{m_i m_{i-1} \cdots m_0}$

$$d(m) = \prod_{j=0}^{i} (m_j + 1)$$

Now, we can notice the fractal structure of non-divisible by p numbers in the triangle, if we mark non-divisible as * and space for divisible ones. Example for $p = 3$:

```
m = 0:                  *
m = 1:                *   *
m = 2:              *   *   *
m = 3:            *           *
m = 4:          *   *       *   *
m = 5:        *   *   *   *   *   *
m = 6:      *           *           *
m = 7:    *   *       *   *       *   *
m = 8:  *   *   *   *   *   *   *   *   *
```

So for the first p^i rows we'll have a triangle constructed by $\dfrac{p(p+1)}{2}$ (sum of arithmetic progression) triangles of size p^{i-1}

Thus,

$$D(p^i - 1) = \frac{p(p+1)}{2} D(p^{i-1} - 1) = \left(\frac{p(p+1)}{2}\right)^i$$

Now we'll extend this result for a multiple of p^i:

$$n = n_i p^i - 1 = \overline{n_i 0 \cdots 0} - 1$$

$$D(n_i p^i - 1) = \frac{n_i(n_i+1)}{2} D(p^i - 1) = \frac{n_i(n_i+1)}{2} \left(\frac{p(p+1)}{2}\right)^i$$

In turn, we can now expand this formula to arbitrary n if we notice that

$$n = n_i p^i + \overline{n_{i-1} \cdots n_0}$$

$$D(\overline{n_i n_{i-1} \cdots n_0}) = D(\overline{n_i 0 \cdots 0} - 1) + d(\overline{n_i 0 \cdots 0}) + \cdots + d(\overline{n_i n_{i-1} \cdots n_0}) =$$

$$D(n_i p^i - 1) + (n_i - 1)(d(\overline{0 \cdots 0}) + \cdots + d(\overline{n_{i-1} \cdots n_0})) =$$

$$\frac{n_i(n_i+1)}{2} \left(\frac{p(p+1)}{2}\right)^i + (n_i - 1) D(\overline{n_{i-1} \cdots n_0})$$

This formula is implemented in the CLI app.