Global-local shrinkage prior for variable selection in graph-structured models.

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Context

Dependence between variables may be induced by various factors in different applications:

- Disease mapping: structure in space and time for variables measured over time at adjacent locations,
- Functional MRI: spatial dependence between voxels within anatomical regions of the brain,
- Genomic studies: dependence structure between genes obtained from biological pathways or inferred computationally (e.g., based on co-expression),

$\looparrowright$ Most of the dependence between variables may be encoded by an undirected graph $\mathcal{G}$.

Objective:

To develop a unified Bayesian variable selection for graph-structured variables providing flexibility in the amount of shrinkage and smoothness over the graph.

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From a statistical point of view

Why incorporate the dependence structure into statistical models?

- ↬ helps the model building process by reducing the complexity of models and by circumventing the problem of high collinearity through identifiability constraints,
- ↬ increases power to detect associations,
- ↬ improves the predictive power.

Why incorporate the dependence structure into variable selection methods?

- ↬ It encourages the identification of groups of dependent variables acting jointly on the response, especially those with subtle individual effects.

How to incorporate the dependence structure into variable selection methods?

- ↬ Penalized likelihood approaches
- ↬ Bayesian regularization

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Bayesian regularization

In Bayesian context regularization is done by specifying shrinkage priors on the regression coefficients:

$\Rightarrow$ Natural framework for integrating knowledge about the covariates' structures into the analysis

Two classes of shrinkage priors:

- **Spike-and-slab prior** (Mitchell and Beauchamp, 1988; George and McCulloch, 1993): a discrete mixture of two distributions
- **Continuous shrinkage priors**: unimodal continuous distributions

$\rightarrowtail$ the class of global-local priors (Carvalho et al., 2010; Polson and Scott, 2010): a scale mixture of Gaussian distributions with the mixing density depending on two hyperparameters to control the **global** shrinkage and the **local** deviations

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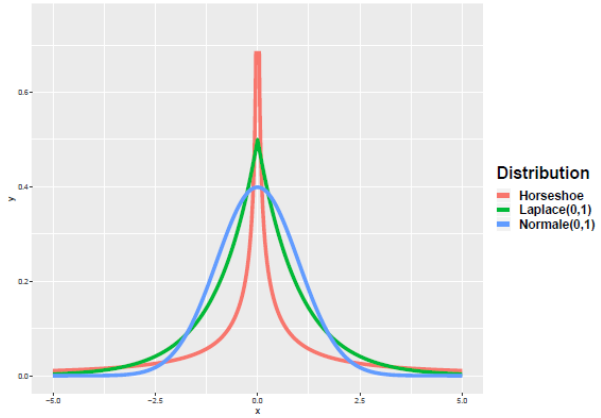
The proposed approach

We propose to combine:

- A global-local prior, the horseshoe (HS) prior (Carvalho et al., 2010), for its **efficiency and flexibility** in terms of selection and estimation with performances comparable to the spike-and-slab prior

↬ allows to shrink towards zero small coefficients while allowing large signals to escape from the overall shrinkage



- With a Gaussian Markov random field (GMRF) for its **appealing connection** with undirected graphs (Rue and Held, 2005)

↬ allows to impose the dependence structure between the parameters via the precision matrix of a conditionally Gaussian prior thus leading to sparse matrices and to smooth coefficients over the graph with possible abrupt changes

↬ An extension of the approach by Faulkner and Minin (2018); Faulkner (2019) to the more general context of graph-structured variable selection.

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Bayesian hierarchical model

We assume that $\mathcal{G} = \bigcup_{i=1}^{I} \mathcal{G}_i = \bigcup_{i=1}^{I} (V_i, E_i)$ a disjoint union of I subgraphs and $\mathcal{S}$ the set of indices associated to one representative of each of the I subgraphs.

HS-GMRF model

$$
\begin{aligned}
\boldsymbol{y}|\boldsymbol{\beta}, \sigma^2 &\sim \mathcal{N}_n(\boldsymbol{X}\boldsymbol{\beta}, \sigma^2 \mathbf{I}_n) \\
\beta_j - s_{jj'}\beta_{j'}|\tau^2_{jj'},\, \lambda^2 &\sim \mathcal{N}(0, \lambda^2\tau^2_{jj'}) \text{ for } (j,j') \in \bigcup_{i=1}^{I} E_i \\
\beta_j|\tau^2_j,\, \lambda^2 &\sim \mathcal{N}(0, \lambda^2\tau^2_j) \text{ for } j \in \mathcal{S} \\
\tau_{jj'} &\sim \mathcal{C}^+(0,1) \text{ for } (j,j') \in \bigcup_{i=1}^{I} E_i;\ \tau_j \sim \mathcal{C}^+(0,1) \text{ for } j \in \mathcal{S} \\
\lambda|\sigma &\sim \mathcal{C}^+(0,\sigma);\ \sigma^2 \sim \mathcal{IG}(a_0, b_0)
\end{aligned}
$$

with $s_{jj'} = \text{sign}\{\text{cor}(X_j, X_{j'})\}$ to encourage regression coefficients of negatively correlated variables to take opposite signs.

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Bayesian hierarchical model

Joint distribution of $\beta|\lambda^2, \boldsymbol{\tau^2}$

$$\boldsymbol{\beta}|\boldsymbol{\tau^2}, \lambda^2 \sim \mathcal{N}_p(0, \lambda^2 \boldsymbol{Q}^{-1}),$$

- A GMRF distribution where $\boldsymbol{Q}$ is the precision matrix of full rank with diagonal elements

$$Q_{jj} = \begin{cases} \frac{1}{\tau_j^2} + \sum_{j' \in \mathcal{N}(j)} s_{jj'} \frac{1}{\tau_{jj'}^2} & \text{if } j \in \mathcal{S} \\ \sum_{j' \in \mathcal{N}(j)} s_{jj'} \frac{1}{\tau_{jj'}^2} & \text{otherwise} \end{cases}$$

and off-diagonal elements

$$Q_{jj'} = \begin{cases} -s_{jj'} \frac{1}{\tau_{jj'}^2} & \text{if } (j, j') \in \bigcup_{i=1}^{I} E_i \\ 0 & \text{otherwise} \end{cases}.$$

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MCMC implementation

MCMC implementation

A Gibbs sampling algorithm is straightforward to fit the hierarchical models:

- by using the parametrization of a half-Cauchy as a mixture of inverse-gamma distributions (Makalic and Schmidt, 2016),
- by introducing a q-dimensional vector $\phi = (\phi_1, \ldots, \phi_q)' = C\beta$ (Martínez-Beneito and Botella-Rocamora, 2019) where $q = |E| + |\mathcal{S}|$ and C is a contrast matrix such that:

$$\phi \sim \mathcal{N}_q(0, \Sigma_\phi),$$

with $\Sigma_\phi = \text{diag}(\lambda^2 \boldsymbol{\tau^2})$.

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Simulation study

Objectives

- To evaluate the performances of the proposed approach with and without incorporating the sign of the sample correlation (HS-GMRF and HS-GMRF-nosign),
- To compare the results with two other approaches: the HS and the spike-and-slab with Ising prior (SS-Ising) (Smith and Fahrmeir, 2007; Li and Zhang, 2010) and when the true graph is known and unknown.

$Y = \sum_{g=1}^{G} \boldsymbol{X_g} \beta_g + \varepsilon$ with $X_{i,g} = (X_{i;g1}, \ldots, X_{i;gk})' \sim \mathcal{N}_k(0, \Sigma_g)$ and $\varepsilon \sim \mathcal{N}_n(0, \sigma^2 I_n)$

12 simulated scenarios

- Two covariance structures
- Two levels of correlation ($\rho = 0.5, 0.9$)
- Three regression coefficients

$\hookrightarrow$ Focus on the scenario where half of groups with Σ_g and

$$\beta_g = (5, -\frac{5}{\sqrt{10}}, -\frac{5}{\sqrt{10}}, \underbrace{\frac{5}{\sqrt{10}}, \ldots, \frac{5}{\sqrt{10}}}_{k-3})$$

Simulations

- $G = 14$ groups of $k = 10$ predictors,
- Only groups $g = 1, 3, 5, 8, 10$ have non-zero effects,
- $\sigma^2 = \sum_{g=1}^{G} \beta_g^2 / 5$
- Repetitions: 50

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Simulation study

Performance criteria

- Variable selection criteria:

↪ For HS-based: variable selected if 95% HPD interval does not contain 0,

↪ For SS-Ising: variable selected if marginal inclusion posterior probabilitie greater than 0.5.

- Matthews correlation coefficient (MCC),
- Mean squared error (MSE) of the regression coefficients,
- Mean squared prediction error (MSPE).

MCMC settings:

- iterations : 6000,
- burn-in: 1000.

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Results using the true graph

Table 1: Average MCC, MSE and MSPE (with SE) over 50 simulated replications.

		MCC	MSE	MSPE
$\Sigma_{g,\text{half}}$ $\rho = 0.5$	HS-GMRF	**0.708** ($\pm$ 0.018)	**0.513** ($\pm$ 0.067)	**94.871** ($\pm$ 13.632)
	HS-GMRF-nosign	0.624 ($\pm$ 0.034)	0.728 ($\pm$ 0.155)	122.188 ($\pm$ 21.609)
	HS	0.240 ($\pm$ 0.041)	1.009 ($\pm$ 0.200)	126.252 ($\pm$ 19.657)
	SS-Ising	0.323 ($\pm$ 0.054)	1.386 ($\pm$ 0.204)	149.294 ($\pm$ 27.384)
$\Sigma_{g,\text{half}}$ $\rho = 0.9$	HS-GMRF	**0.668** ($\pm$ 0.046)	**0.541** ($\pm$ 0.089)	**84.954** ($\pm$ 14.485)
	HS-GMRF-nosign	0.444 ($\pm$ 0.117)	1.038 ($\pm$ 0.259)	99.123 ($\pm$ 17.694)
	HS	0.219 ($\pm$ 0.038)	2.243 ($\pm$ 0.551)	95.219 ($\pm$ 19.279)
	SS-Ising	0.312 ($\pm$ 0.048)	2.359 ($\pm$ 0.437)	109.387 ($\pm$ 23.713)

- HS-GMRF-based approaches lead to the best results in terms of MCCs, MSEs, and MSPEs,
- HS-GMRF outperforms HS-GMRF-nosign especially when $\rho = 0.9$

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Results using the true graph

Table 2: Average MCC and MSE for connected and non-connected covariates over 50 simulated replications.

	MCC		MSE	
	Connected	Non-connected	Connected	Non-connected
	$\Sigma_{g,\text{half}}$ $\rho = 0.5$			
HS-GMRF	**0.956** ($\pm$ 0.033)	0.277 ($\pm$ 0.039)	**0.558** ($\pm$ 0.061)	**0.469** ($\pm$ 0.111)
HS-GMRF-nosign	0.810 ($\pm$ 0.053)	0.264 ($\pm$ 0.057)	0.913 ($\pm$ 0.202)	0.542 ($\pm$ 0.151)
HS	0.237 ($\pm$ 0.038)	0.244 ($\pm$ 0.054)	1.464 ($\pm$ 0.374)	0.553 ($\pm$ 0.139)
SS-Ising	0.332 ($\pm$ 0.062)	**0.295** ($\pm$ 0.096)	2.028 ($\pm$ 0.372)	0.744 ($\pm$ 0.208)
	$\Sigma_{g,\text{half}}$ $\rho = 0.9$			
HS-GMRF	**0.883** ($\pm$ 0.078)	0.278 ($\pm$ 0.049)	**0.611** ($\pm$ 0.138)	**0.470** ($\pm$ 0.091)
HS-GMRF-nosign	0.526 ($\pm$ 0.177)	0.265 ($\pm$ 0.053)	1.582 ($\pm$ 0.465)	0.495 ($\pm$ 0.112)
HS	0.188 ($\pm$ 0.043)	0.271 ($\pm$ 0.046)	3.998($\pm$ 1.105)	0.488 ($\pm$ 0.103)
SS-Ising	0.310 ($\pm$ 0.047)	**0.304** ($\pm$ 0.081)	4.055 ($\pm$ 0.855)	0.662 ($\pm$ 0.135)

- Performances for non-connected predictors are similar for HS and HS-GMRF-based approaches.
- For connected variables the integration of the dependence structure helps to select variables with small effects

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Results using the true graph

Table 3: Coverage probability (CP) and width of 95% HPD intervals averaged over the 50 simulated replications.

			CP of 95% HPD	Width of 95% HPD
$\Sigma_{g,\text{half}}$	$\rho = 0.5$	HS-GMRF	0.923 ($\pm$0.026)	2.047 ($\pm$0.188)
		HS-GMRF nosign	0.931 ($\pm$0.027)	2.712 ($\pm$0.231)
		HS	0.894 ($\pm$0.037)	2.871 ($\pm$0.278)
		SS-Ising	0.751 ($\pm$0.026)	0.656 ($\pm$0.117)
	$\rho = 0.9$	HS-GMRF	0.928 ($\pm$0.019)	2.415 ($\pm$0.248)
		HS-GMRF nosign	0.922 ($\pm$0.031)	3.212 ($\pm$0.284)
		HS	0.908 ($\pm$0.05)	3.255 ($\pm$0.419)
		SS-Ising	0.773 ($\pm$0.029)	0.927 ($\pm$0.181)

- CPs similar for HS-based approaches but wider intervals for HS

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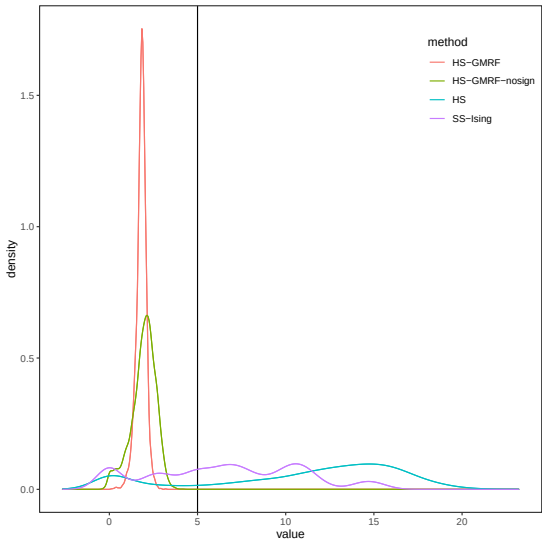
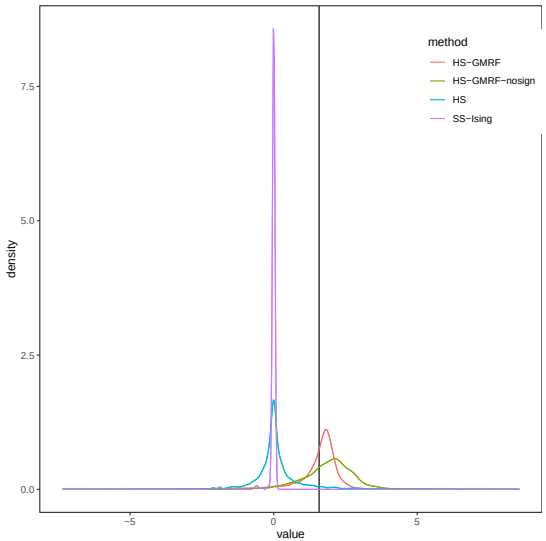
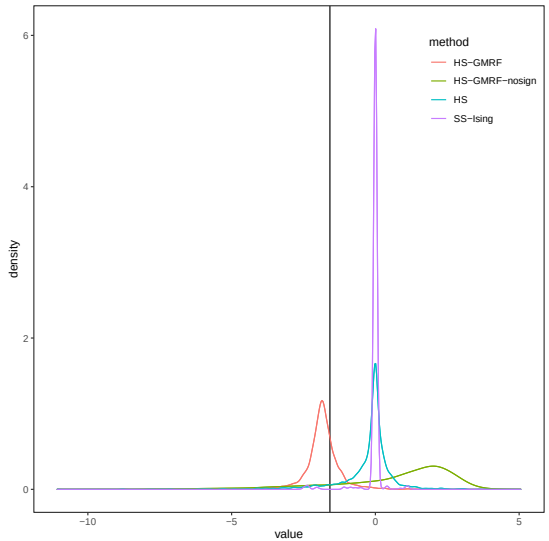
Results using the true graph







- HS leads to a bimodal posterior density or a distribution concentrated around 0 with large tails
- HS-GMRF-based approaches give narrower posterior densities away from 0
- For $\beta = -5/\sqrt{(10)}$ HS-GMRF-nosign spreads out posterior density around the average of β's

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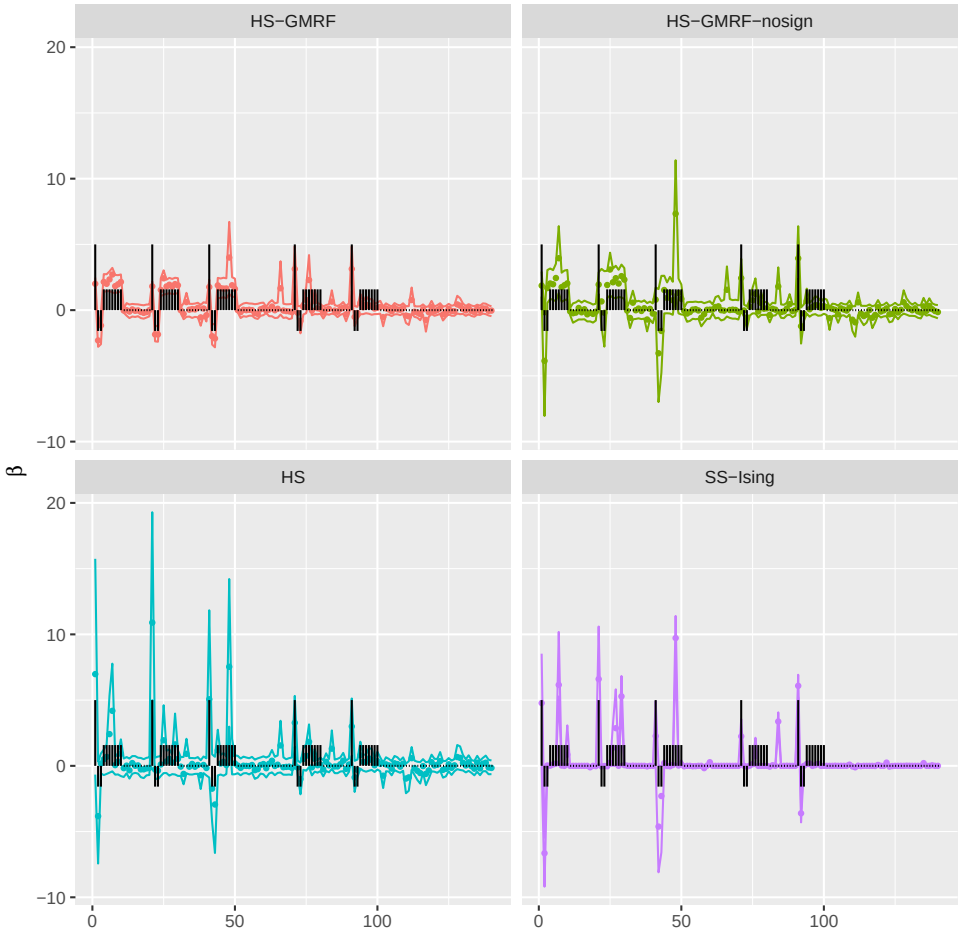
Results using the true graph



- HS and SS-Ising tend to select one representative of a group of correlated variables,
- HS gives wide HPD intervals,
- HS-GMRF-based approaches give similar estimates for highly correlated covariates,
- HS-GMRF yields narrower HPD intervals with good coverage and fairly accurate estimates for regression coefficients with opposite signs.

Estimated coefficients along with 80% HPD intervals in one simulated replication ($\Sigma_{g,\text{half}}$, $\rho = 0.9$).

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Results using an estimated graphs

Graph structure may not be known and may need to be estimated. Graphical Lasso approach used to estimate the graph (Friedman et al., 2008)

- HS-GMRF-based approaches outperform the other approaches,
- For moderate correlation: graph is underestimated $\Rightarrow$ slightly poorer selection and estimation for the HS-GMRF-based approaches than with the true graph,
- For high correlation: graph is overestimated $\Rightarrow$ improved selection for the HS-GMRF-based approaches compared to the true graph but with an overesmoothing of the regressions coefficients

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Application

Objective

To identify gene expressions involved in the variability of riboflavin production using data on 71 samples

- A total of 142 gene expressions considered
- Estimation of an undirected graph with 157 edges
- 5-fold cross-validation procedure

Methods	CV-MSPE	Selected genes
HS-GMRF	**0.29**	4 (90% HPD) 8 (80% HPD)
HS-GMRF-nosign	0.31	4(90% HPD) 6 (80% HPD)
HS	0.33	0(90% HPD) 0 (80% HPD)
SS-Ising	0.37	21 ($PPI > 0.5$) 16 ($PPI > 0.8$)
Lasso	0.41	16

↬ HS-GMRF yields the smallest CV-MSPE

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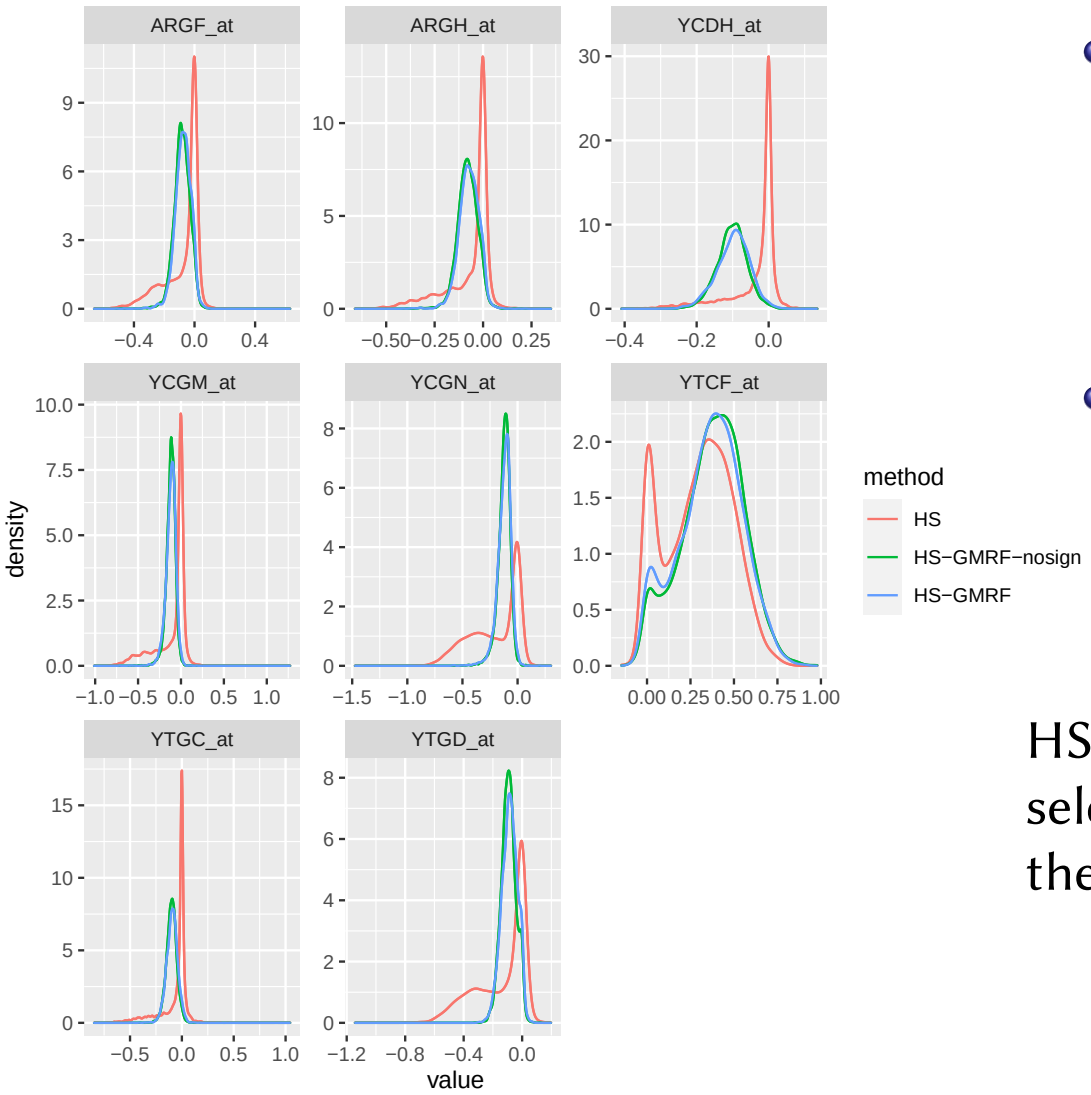
Application



For moderate non-zero effects:

- HS estimates densities concentrated around 0 with long tails or bimodal densities with one of the modes around 0,
- HS-GMRF-based methods estimate unimodal densities or bimodal densities with the mode around 0 less than with HS.

HS-GMRF-based approaches select groups of genes involved in the same biological pathway.

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Conclusion/Perspective

The proposed approaches allow to:

- consider a broad type of dependence structures,
- achieve flexibility in the estimation and the selection due to the local and global shrinkage hyperparameters,
- need to consider the sign of the sample correlation,
- give better predictive performances notably by selecting groups of connected variables,
- give good results even when true graph is unknown and needs to be estimated.

Limitation:

- tend to encourage similar values fo connected variables, especially for highly correlated variables or overestimated graphs.

For future research:

- Extension to non-Gaussian distributions,
- Integration of prior knowledge on strengths of connections between variables.

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Outline

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