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Centralizers of spherical-transitive elements in the group of finite-state automorphisms of binary rooted tree

In this work the centralizer of spherical-transitive finite-state automorphisms is investigated.

Key Words: rooted tree, automorphism group, state, centralizer.

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Централізатори шарово-транзитивних елементів в групі скінченно-станових автоморфізмів бінарного кореневого дерева

В роботі досліджено централізатори шарово-транзитивних скінченно-станових автоморфізмів.

Ключові слова: кореневе дерево, група автоморфізмів, стан, централізатор.

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1 Introduction

In this paper, we shall use the following definitions:

Definition 1.1. T_2 - binary rooted tree,
$AutT_2$ - group of automorphisms of T_2,
$FAutT_2$ - group of finite-state automorphisms T_2,
$x * a$ - effect of the automorphism a onto the end x of a tree T_2,
$a \circ b$ - superposition of the automorphisms a and b of a tree T_2,
Z_2 - ring of integer 2-adic numbers,
we shall state, that an automorphism χ_0 is a 0- solution of an equation $a^{\chi_0} = b$, if $0 * \chi_0 = 0$

Let's identify an automorphism a of a tree T_2 as a function $f_a : Z_2 \to Z_2$ in the following way:

$$x * a = f_a(x)$$

In study [1] the author has proved the following theorem:

Theorem 1.1. *Let x be a spherical-transitive automorphism. Then*

$$C_{AutT_2}(x) = \{x^p | p \in Z_2\}$$

The aim of this study is to research the centralizers of the spherical-transitive elements in $FAutT_2$, as there is no result for $FAutT_2$, similar to the theorem 1.1.

2 Centralizers of the spherical-transitive elements in $FAutT_2$

Let ε be an adding machine, that is $x * \varepsilon = x + 1$. Then, we have the following lemma:

Lemma 1. *For $p \in Z_2$ we have an equality:*

$$0 * \varepsilon^p = p$$

Proof. *Since $t * \varepsilon^p = t + p$, therefore $0 * \varepsilon^p = 0 + p = p$.*

Theorem 2.1. *Let χ_x be a 0-solution of the conjugacy equation $\varepsilon^t = x$ with respect to the automorphism t. Then we have an equality:*

$$0 * x^p = p * \chi_x$$

Proof. *Since $\varepsilon^{\chi_x} = x$, therefore we have a correspondence:*

$$x^p = (\chi_x^{-1} \circ \varepsilon \circ \chi_x)^p = \chi_x^{-1} \circ \varepsilon^p \circ \chi_x$$

That is, according to the lemma 1 *and the equation $0 * \chi_x = 0$ we have:*

$$0 * x^p = 0 * (\chi_x^{-1} \circ \varepsilon^p \circ \chi_x) =$$

$$= ((0 * \chi_x^{-1}) * \varepsilon^p) * \chi_x) = (0 * \varepsilon^p) * \chi_x = p * \chi_x$$

q.e.d.

We have the following lemma:

Lemma 2. *Let x be a spherical-transitive automorphism. Then*

$$0 * C_{AutT_2}(x) = Z_2$$

Proof. *According to the theorem* 1.1

$$C_{AutT_2}(x) = \{x^p | p \in Z_2\}$$

Then, using the theorem 2.1, *we obtain:*

$$0 * x^{Z_2} = Z_2 * \chi_x$$

where χ_x is a 0-solution of the conjugacy equation $\varepsilon^t = x$ with respect to the automorphism t.

Since χ_x is an automorphism, therefore

$$Z_2 * \chi_x = Z_2$$

q.e.d.

Definition 2.1. Let us define a set $F_p (p \in Z_2)$ in the following way:

$p \in F_p$,

if $2t + 1 \in F_p$, then $t \in F_p, t + 1 \in F_p$,

if $2t \in F_p$, then $t \in F_p$.

We shall state, that t_k belongs to the k-th level in F_p, if obtained from p in k steps.

Definition 2.2. Let us define a set $P_{m,n} (m \in \mathbb{Z}, n \in \mathbb{Z}^+ \cup 0)$ in the following way:

$m \in P_{m,n}$,

if $2t+1 \in P_{m,n}$, then $t-n \in P_{m,n}$, $t+n+1 \in P_{m,n}$,

if $2t \in P_{m,n}$, then $t \in P_{m,n}$.

We shall state, that t_k belongs to the k-th level in $P_{m,n}$, if obtained from m in k steps.

Lemma 3. *Let a 2-adic quasiperiodic number p equals to $\frac{m}{2n+1}$, where $m \in \mathbb{Z}, n \in \mathbb{Z}^+ \cup 0$. Then the sets $P_{m,n}$ and F_p are both finite or infinite.*

Proof. *Since we have the following equalities:*

$$\frac{2m+1}{2n+1} = 2\frac{m-n}{2n+1} + 1$$

$$\frac{2m}{2n+1} = 2\frac{m}{2n+1}$$

therefore in F_p $\frac{2m+1}{2n+1}$ generates $\frac{m-n}{2n+1}$ and $\frac{m+n+1}{2n+1}$, and $\frac{2m}{2n+1}$ generates $\frac{m}{2n+1}$.

Therefore, if t_k belongs to the k-th level in F_p, then $t_k(2n+1)$ belongs to the k-th level in $P_{m,n}$, and vice versa, if t'_k belongs to the k-th level in $P_{m,n}$, then $\frac{t'_k}{2n+1}$ belongs to the k-th level in F_p. Therefore, we have an equality:

$$|P_{m,n}| = |F_p|$$

q.e.d.

Lemma 4. *A set $P_{m,n} (m \in \mathbb{Z}, n \in \mathbb{Z}^+ \cup 0)$ is finite.*

Proof. *According to the definition, if $t \in P_{m,n}$, then either $\frac{t}{2}$ or $\frac{t-1}{2} - n$ and $\frac{t-1}{2} + n + 1$.Let t_k belong to the k-th level in $P_{m,n}$, then we have an equality:*

$$t_k = \frac{t_{k-1} + a * (2n+1)}{2}, a = 0, 1, -1$$

After applying this equality k times, we obtain:

$$t_k = \frac{m}{2^k} + (2n+1)(\frac{a_0}{2^k} + \dots + \frac{a_{k-1}}{2})$$

Since $|a_i| \leqslant 1$, we have the following estimation:

$$|t_k| = \left|\frac{m}{2^k} + (2n+1)(\frac{a_0}{2^k} + \dots + \frac{a_{k-1}}{2})\right| \leqslant$$

$$\leqslant \left|\frac{m}{2^k}\right| + |2n+1| \leqslant |m| + 2n + 1$$

Thus, the number of elements of the set $P_{m,n}$ is limited by an inequality:

$$|P_{m,n}| \leqslant 2(|m| + 2n + 1)$$

so the set $P_{m,n}$ is finite, q.e.d.

Lemma 5. *A set F_p is finite if and only if p is a quasiperiodic number.*

Proof. $\Rightarrow$ *For $2t+1$ and $2t$ a number t is obtained by skipping the last digit of the binary representation, therefore F_p contains all numbers, obtained from p by skipping several last digits. If p is not quasiperiodic, then we have an infinite amount of such numbers, thus F_p is not finite.*

$\Leftarrow$ *p is a quasiperiodic number if and only if $p = \frac{m}{2n+1} (m \in \mathbb{Z}, n \in \mathbb{Z}^+ \cup 0)$. Therefore, according to the lemmas* 3 *and* 4 *the set F_p is finite.*

Theorem 2.2. *Let ε be an adding machine. Then*

$$C_{FAutT_2}(\varepsilon) = \{\varepsilon^p | p \in Z_2 \cap \mathbb{Q}\}$$

Proof. *Since we have an equality*

$$C_{FAutT_2}(\varepsilon) = C_{AutT_2}(\varepsilon) \cap FAutT_2$$

therefore, according to the theorem 1.1, *the elements of the centralizer* $C_{FAutT_2}(\varepsilon)$ *are described as* $\{\varepsilon^p | \varepsilon^p \in FAutT_2\}$. *Obviously, if* p *is not a quasiperiodic number, then* ε^p *is infinite-state, as it translates a quasiperiodic number 0 into a non-quasiperiodic number* p. *Then, let* $p \in Z_2 \cap \mathbb{Q}\}$, *or be quasiperiodic. According to the lemma* 5 *the set* F_p *is finite. On the other hand, we have the equalities:*

$$\varepsilon^{2t+1} = (\varepsilon^t, \varepsilon^{t+1}) \circ \sigma$$
$$\varepsilon^{2t} = (\varepsilon^t, \varepsilon^t)$$

Thus, the states of the automorphism ε^p *are limited to the automorphisms, described as*

$$\varepsilon^t, t \in F_p$$

Since F_p *is finite, therefore* ε^p *is a finite-state automorphism, q.e.d.*

Theorem 2.3. *Let* ε *be an adding machine. Then*

$$0 * C_{FAutT_2}(\varepsilon) = (Z_2 \cap \mathbb{Q})$$

Proof. *According to the theorem* 2.2

$$C_{FAutT_2}(\varepsilon) = \{\varepsilon^p | p \in Z_2 \cap \mathbb{Q}\}$$

Then, using the lemma 1, *we obtain:*

$$0 * \varepsilon^{Z_2 \cap \mathbb{Q}} = Z_2 \cap \mathbb{Q}$$

q.e.d.

The theorems 2.2 and 2.3 can be applied to research of finite-state conjugacy with the automorphism ε (an adding machine). The following theorem illustrates that:

Theorem 2.4. *If a 0-solution* t_0 *of a conjugacy equation with respect to* t

$$\varepsilon^t = a$$

is not finite-state, then this equation has no finite-state solutions.

Proof. *Let us assume, that* t_0 *is infinite-state, and the equation* $\varepsilon^t = a$ *has a finite-state solution* $t' : p \to 0$, *where* p *is a quasiperiodic number. Since each solution can be uniquely represented as*

$$t' = x \circ t_0, x \in C_{FAutT_2}(\varepsilon)$$

ma $p * \varepsilon^{-p} = 0$, *then, according to the theorem* 2.2 $t' = \varepsilon^{-p} \circ t_0$. *Since* t_0 *is infinite-state, and* ε^{-p} *is finite-state, therefore* t' *is infinite-state. We've come to a contradiction.*

Theorem 2.5. *Let* a *be a spherical-transitive automorphism. Then*

$$C_{FAutT_2}(a) \subseteq \{a^{(p*\chi_a^{-1})} | p \in Z_2 \cap \mathbb{Q}\}$$

where χ_a *is a 0-solution of the conjugacy equation* $\varepsilon^t = a$ *with respect to* t.

Proof. *We have the following equality:*

$$0 * a^{(p*\chi_a^{-1})} = p$$

Indeed, according to the theorem 2.1 *we obtain:*

$$0 * a^{(p*\chi_a^{-1})} = (p * \chi_a^{-1}) * \chi_a = p * (\chi_a^{-1} \circ \chi_a) = p$$

Thus, $a^{(p*\chi_a^{-1})}$ *can be finite-state only if* $p \in Z_2 \cap \mathbb{Q}$. *On the other hand, according to the theorem* 1.1 *all the elements of a centralizer* $C_{AutT_2}(a)$ *can be represented as* $a^{(p*\chi_a^{-1})}$, *since* χ_a^{-1} *is an automorphism* Z_2. *Taking into account that*

$$C_{FAutT_2}(a) = C_{AutT_2}(a) \cap FAutT_2$$

we obtain an inclusion

$$C_{FAutT_2}(a) \subseteq \{a^{(p*\chi_a^{-1})} | p \in Z_2 \cap \mathbb{Q}\}$$

q.e.d.

Theorem 2.6. *Let* x *be a spherical-transitive finite-state automorphism. Then*

$$0 * C_{FAutT_2}(x) \subseteq (Z_2 \cap \mathbb{Q})$$

Proof. *According to the theorem* 2.5 *we have an inclusion:*

$$0 * C_{FAutT_2}(x) \subseteq \{0 * a^{(p*\chi_a^{-1})} | p \in Z_2 \cap \mathbb{Q}\} = Z_2 \cap \mathbb{Q}$$

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