Eigenvalue Computation in the 20th Century

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A clear statement of the topic

The main task of numerical methods is to create an algorithm that would optimally calculate eigenvalues and eigenvectors. Calculating eigenvalues is a more difficult task than solving linear equations, so the necessary research on algorithms began in the mid-20th century when computers appeared. A lot of big tasks from different areas of science are based on numerical methods. A number of problems in physics, engineering, and also text- and image-processing tasks also use numerical methods for finding matrix eigendecomposition. From the beginning of the history of applied mathematics, various eigenproblem solutions have been developed, but in our work, we want to examine modern methods, implement and compare some modern approaches developed in the middle-to-late 20th century.

In order to find eigenvalues we need to find non-trivial solutions for the equation Ax = Lx. Such an equation can be solved explicitly by construction of the characteristic equation, but this approach has a significant disadvantage: its accuracy. For most of the matrices, finding the roots of the characteristic equation does not give exact(same) results that affect eigenvalues and eigenvectors. And also, the Abel – Ruffini theorem, which states that any such algorithm for measurements greater than 4 must either be infinite or include functions of greater complexity than elementary arithmetic operations and fractional degrees.
Therefore, the direct method works only with a limited number of matrices. In other cases, it is optimal to use iterative algorithms.

Some of the iterative algorithms are:
Power iteration, Inverse iteration, QR algorithm, Jacobi eigenvalue algorithm, Folded spectrum method and others. These algorithms use different types, properties of matrices and give different convergence.

Description of the problem you will discuss

In our work, we briefly consider some iterative algorithms that were developed in the 20th century and are used now, for example, the Power method (even though it finds one of the eigenvalues) that Google uses to build PageRank and Twitter to create a recommended model. We will also take two modern algorithms: Arnoldi iteration (1951) and the Jacobi-Davidson algorithm (based on Jacobi’s work from 1846 and Davidson’s algorithm from mid-1990s). We

describe the work of these algorithms, implement them, conduct tests and write the results of the comparison of these algorithms.

Sketch of the possible approaches

There is a number of possible approaches to the numeric solving of eigenvalue problem. Among them:
- QR decomposition
- Divide and conquer methods
- Power methods
- Iteration methods;

The former two are very elegant solutions but they require the matrix to symmetric which is quite a serious limitation. So in our work we will concentrate on the power and iterative methods.

Short explanations of the pros and cons of your chosen method

One of the most popular approaches is Arnoldi algorithm which is a power iterative method, which uses the Gram-Schmidt process to produce a sequence of orthonormal vectors q_1, q_2, … called the Arnoldi vectors, such that for every n, the vectors q_1, … , q_n span the Krylov subspace K_n.

Another popular method is Jacobi-Davidson algorithm, another iterative method, which searches approximate eigenvectors in a subspace which is extracted and expanded on each iteration.

Arnoldi method shows excellent performance in finding extreme eigenvalues which are well separated from the rest of the spectrum but can fail in finding interior eigenvalues and Jacobi-Davidson algorithm shows better results for such case.

A short discussion of the planned numerical realization

We are going to implement both Arnoldi and Jacobi-Davidson algorithms, test them on matrices of different types (e.g. sparse/dense) and sizes and provide the analysis of cases where each of them fails/succeeds and provide results of the benchmarking.