Stochastic Control with Filtering

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September 10, 2025

Abstract:

Many financial models consider the dynamics of stock prices as stochastic processes, and stochastic control has thus been a key concept in optimising investment portfolio to maximise the investment return. Filtering theory, on the other hand, is widely used to estimate unobservable latent variables, such as the underlying trend in the stock price. Therefore, in this project, we consider a hypothetical market consisting of only one stock, and propose a novel scheme of combining both stochastic control with filtering techniques to apply portfolio control to a calibrated stock price model. The performance has been carefully examined by applying the proposed scheme to the S&P 500 data. Our results has empirically demonstrated that this novel approach improves the investment return over a tested 3-, 6- and 9-year period.

Authors' Contribution:

All authors have contributed equally to group meetings, discussions, presentations and report preparations.
All authors have seen and approved this statement.

Contents

1 Introduction

Suppose you are an investor in the financial markets but only have access to the price data and volatility data, for example the SPY and VIX indices, how can you invest in the stock in such a way that your wealth at the end of the investment period is maximised? The aim of this project is to investigate portfolio optimality in the setting of partial observations. Studies on optimal portfolios in this framework were mainly pioneered by Huyen Pham and Marie-Claire Quenez [1] and further developed by Dalia Ibrahim and Frédéric Abergel [2].

This project adopts a multi-technique approach as we assume that the evolution of the investor's wealth follows a set of stochastic differential equations that is dependent the evolution of the price, the volatility and the drift of the risky asset. Since the investor has no access to data on the evolution of the drift, it must be estimated using methods from stochastic filtering theory. Using the Girsanov's Theorem we then perform a series of transformations on the partially observed model to obtain a fully observable model that depends on the filtering estimates of the drift given the available information. Through the fully observable model, we then proceed to apply the Martingale Duality approach to obtain an explicit expression for the optimal portfolio under log utility. This approach is chosen over the famous dynamic programming method [3], which would have required extra constraints on the admissible set of controls [1, 2].

We furthermore present some results to validate the theory through application of these findings to the Log Ornstein-Uhlenbeck (O-U) model. Reasons for choosing the Log O-U model for our application aspect revolve around the points that for starters, this model is very well-known in finance literature and will render our results to be more relatable to members of this community. In addition, the Log O-U model is one of the few models for which a closed form solution of the optimisation problem we tackled exist and lastly, as will further be elaborated in the course of this reading, the Log O-U model satisfied the necessary assumptions to enable us operate with the classical Kalman-Bucy filter.

In the work of Pham and Abergel [1], the parameters of the model are assumed to be known which of course is of no use in practice. In this report we also examine the problem of parameter calibration under partial data and its limitations using the Maximum Likelihood estimation method.

The significance of this project lies in addressing the prevalent real-life challenge of uncertainty in dynamic systems. Just as is seen in this project, in the area of finance, optimising trading strategies requires accurate state estimation, however, this goes beyond just the finance and economics industry as in the engineering industry for example, for systems such as autonomous vehicles, the ability to navigate through uncertain environments relies also on precise state information.

2 Problem Formulation

For simplicity, we consider a market with one risk-free asset (bond) whose price remains constant at 1 throughout the time period, and one risky asset with price S_t which is generally governed by the following dynamics:

$$\frac{dS_t}{S_t} = \mu_t dt + g(V_t)dW_t^1 \tag{1}$$
$$dV_t = f(\beta_t, V_t)dt + k(V_t)(\rho dW_t^1 + \sqrt{1-\rho^2}dW_t^2) \tag{2}$$
$$d\mu_t = \zeta(\mu_t)dt + \vartheta(\mu_t)dW_t^3 \tag{3}$$

where $\mu = \{\mu_t \,|\, 0 \leq t \leq T\}$ is the unobservable drift dynamics of the risky asset, V_t represents the implied volatility of the stock price, and W_t^1, W_t^2 and W_t^3, are three independent Brownian motions defined on the filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, where $\mathbb{F} = \{\mathcal{F}_t \,|\, 0 \leq t \leq T\}$. We assume that $g(v), k(v) > 0$ to avoid the solution of (2) exploding [2]. Furthermore we assume that β_t is a function of the unobservable drift process μ_t.

In addition, we can reformulate the evolution of the risky asset as a factor model by assuming that one can replace μ_t with $\mu_t g(V_t)$. Through this assumption, we can further capture the dependency of the risky asset's drift evolution on its volatility [4]. Hence equation (1) becomes

$$\frac{dS_t}{S_t} = g(V_t)(\mu_t dt + dW_t^1) \tag{4}$$

The natural filtration of S is denoted by $\mathcal{F}_t^S = \sigma(S_s \,|\, 0 \leq s \leq t)$; similarly for the natural filtration of V. $\forall t \in [0, T]$, we define the natural $\mathbb{P}$-augmentation of $\mathcal{F}_t^S$ as

$$\mathcal{G}_t = \sigma\left(\mathcal{F}_t^S \cup \mathcal{N}_{\mathbb{P}}\right), \tag{5}$$

where $\mathcal{N}_{\mathbb{P}}$ is the set of null events under the measure $\mathbb{P}$, and . The augmentation then in essence *completes* the filtration to contain the sets of events that have a probability of zero under $\mathbb{P}$. Furthermore, we can define the *join* of two filtrations $\mathbb{F}^S \vee \mathbb{F}^V$ as

$$(\mathcal{F}^S \vee \mathcal{F}^V)_t = \sigma\left(\mathcal{F}_t^S \cup \mathcal{F}_t^V\right) \quad \forall t \in [0, T],$$

that is, as the smallest sigma-algebra that contains both filtrations at time $t \in [0, T]$. These notions will come crucial in our quest to reduce this partially observable model to a fully observable model.

Now suppose alongside the risky asset we have a riskless asset $\mathfrak{S}_t$ whose price grows with an interest rate of r and hence is governed by

$$\frac{d\mathfrak{S}_t}{\mathfrak{S}_t} = rdt.$$

We can model the portfolio composed of the risky and riskless assets through a *control* policy determing the proportion of wealth that should be invested in the risky and riskless asset

$$dR_t^\pi = R_t^\pi \left(\pi_t \frac{dS_t}{S_t} + (1 - \pi_t)\frac{d\mathfrak{S}_t}{\mathfrak{S}_t} \right)$$

with $R_0^\pi = x$. Thus the objective is to maximise the expected value of R_t^π at time T under some utility. However to make the optimisation feasible, we will assume that the price of the riskless asset remains constant, that is we have $r = 0$ and thus our portfolio becomes

$$dR_t^\pi = R_t^\pi g(V_t)\pi_t \Big(\mu_t dt + dW_t^1 \Big) \tag{6}$$

Then formally, the aim of this report is to investigate the following objective

$$J(x) = \sup_{\pi \in \mathcal{A}} E[U(R_T^\pi)] \qquad x > 0$$

with U being a utility function and $\mathcal{A}$ being the class of admissible portfolios such that

$$\int_0^T g^2(V_s)\pi_s^2 \, ds < \infty$$

2.1 Data

The primary datasets that were made use of during this project were publicly available, daily cumulated S&P 500 (SPY) and VIX index data. The daily S&P 500 data was used as a proxy for the stock price dynamics since, as an index that tracks the stock performance of the top 500 companies on the American stock market, it serves as the best indicator of the dynamics of real-world stock prices. The VIX index, which goes hand in hand with the S&P 500 index, measures the implied stock price volatility within a 30-day period and hence was used to represent the volatility dynamics in our model. These datasets were obtained from Yahoo Finance.

2.2 Preliminaries

2.2.1 Filtering theory

As a motivating example as was in [5], consider a scenario where we are tracking the position of a moving object which has the true dynamics given by

$$dX_t = f(X_t)\, dt + \xi_t \, dW_t,$$

Instead of observing X_t directly, we make observations represented by the process Y_t, which relates to the state process X_t through

$$dY_t = h(X_t)\, dt + \epsilon_t \, dB_t,$$

where $h(X_t)$ represents the relationship between the observed process and the actual position of the object at time t, not necessarily linear, ϵ_t, ξ_t represent the intensity of the independent Brownian motions W_t, B_t. The primary focus of filtering theory is on estimating the true state of a system, X_t, based on the imperfect, noisy observation process, Y_t. Given that by time t, we can only observe $\mathcal{Y}_t = \sigma(Y_s : s \leq t) \vee \mathcal{N}$, the state of the process X_t is then estimated in a mean square sense as

$$\alpha_t(\varphi) = \mathbb{E}[\varphi(X_t) \mid \mathcal{Y}_t]$$

where φ is a bounded function such that $\varphi : \mathbb{S} \to \mathbb{R}$, which will be taken to be the identity operator in our examples, and $(\mathbb{S}, \mathcal{S})$ is the state space of the signal process, X_t. The filtering problem is solved in theory. That is, the evolution of the filtering estimates $D\alpha_t(\varphi)$ are governed by the *Kushner-Stratonovich* equations [6]. Explicit solutions of these equations are usually intractable due to their complex nature and simulating it's solution still remains an active area of research in non-linear stochastic filtering.

In the special case of f, h being linear in X, the Kushner-Stratonovich equations simplify to obtain the celebrated *Kalman-Bucy* filter, which is derived and studied in Chapter 4.

2.2.2 Change of Probability Measure

Girsanov's theorem provides steps to transform a stochastic process under one measure to its corresponding process under another measure. In the case when the underlying process is a Brownian motion, it can also be viewed as a way to discover a new measure under which a stochastic process becomes a martingale with the property of zero drift. More formally, the theorem is given as follows:

Theorem 1 (Girsanov's Theorem) *Let W_t be a standard Brownian motion under the probability measure $\mathbb{P}$ and let $\mathbb{Q}$ be an equivalent measure such that $\mathcal{N}_{\mathbb{P}} = \mathcal{N}_{\mathbb{Q}}$. If the above conditions are met then for any adapted process X_t which is integrable under the measure $\mathbb{P}$, the process M_t defined by $M_t = \exp\left(-\int_0^t X_s dW_s - \frac{1}{2}\int_0^t X_s^2 ds\right)$ is a martingale under measure $\mathbb{Q}$.*

The applicability of Girsanov's theorem relies on specific assumptions. These include the existence of a well-defined probability space, the presence of a martingale under the original measure, and the existence of an equivalent measure under which the stochastic process retains its martingale property. Understanding these assumptions is crucial for the proper application of the theorem. The strongest findings from Girsanov's theorem, on the other hand, are obtained when the local martingale is actually a Brownian motion, such that when the measure is changed, the Brownian motion only acquires a stochastic drift term. As these results will be directly applied in the transformation of our partially observable model to a fully observable one, consider the following well-known theorem which generalizes this concept for a $d-$dimensional Brownian motion.

Theorem 2 *Let $W = (W^1, \ldots, W^d)$ be a standard d-dimensional Brownian motion on the underlying filtered probability space, and $\{\xi^i\}_{i=1,\ldots,d}$ be predictable processes satisfying $\int_0^\infty \sum_{i=1}^d (\xi_s^i)^2\, ds < \infty$ almost surely. If*

$$X_t \equiv \exp\left(-\sum_{i=1}^{d}\int_0^t \xi_s^i\, dW_s^i - \frac{1}{2}\sum_{i=1}^{d}\int_0^t (\xi_s^i)^2\, ds\right)$$

is a uniformly integrable martingale, then $\mathbb{E}[X_T] = 1$, and the measure $\mathbb{Q} = X_T \cdot \mathbb{P}$ is equivalent to $\mathbb{P}$. Then, W decomposes as

$$\tilde{W}_t^i = W_t^i + \int_0^t \xi_s^i\, ds$$

for a d-dimensional Brownian motion $\tilde{W}$ with respect to $\mathbb{Q}$ [7].

2.2.3 Martingale Duality Method

As described by Tchamga [7], martingale methods in portfolio optimisation aim to simplify the initial dynamic problem, involving optimisation over a control process, into a static optimisation problem. This static problem is focused on the state variable represented by the portfolio's terminal value, incorporating a linear constraint. The transformation involves using a Radon-Nikodým density, referred to as a dual variable, which describes the change in an equivalent probability measure. In other words, the optimisation problem is restructured to depend on a suitable family of martingales. In a general, but more detailed sense, suppose we seek to solve an optimisation problem

$$\sup_{\{\theta_t\}} E[X_T \theta_T]$$

subject to certain constraints on θ_t. To apply the martingale duality approach, we introduce a dual process, Z_t which is a martingale representing the Lagrange multiplier associated with the constraint. This enables the formulation of the dual problem

$$\inf_{\{Z_t\}} E[Z_T^2]$$

such that under the Martingale Duality theorem and under certain conditions, the solution to the primal problem is associated with that of the dual problem via a known duality relation.

3 Transformation of the Partial Observation Model to a Full Observable Model

The model as introduced by (2), (3) and (4) describes the market as a partially observable system, as the drift μ is adapted to the unobservable filtration $\mathcal{F}$, which one does not have access to. We need to restrict this filtration in an eloquent way using Girsanov's theorem to reconstruct the model such that all components will be defined on the observable filtration $\mathbb{G} = \{\mathcal{G}_t \,|\, 0 \leq t \leq T\}$, with $\mathcal{G}_t$ as defined in (5).
Consider, first, the price dynamics S_t,

$$\frac{dS_t}{S_t} = g(V_t)\left(\mu_t dt + dW_t^1\right)$$

Defining[1] $\tilde{\mu}_t = \mu_t$, by Girsanov's theorem, a new measure $\tilde{\mathbb{P}}$ can be constructed such that $\tilde{W}_t^1$ is a $\tilde{\mathbb{P}}$-Brownian motion satisfying

$$d\tilde{W}_t^1 = \tilde{\mu}_t dt + dW_t^1.$$

In that case,

$$\frac{dS_t}{S_t} = g(V_t) d\tilde{W}_t^1 \tag{7}$$

For the volatility dynamics,

[1]This is the point of having a factor model, the stochastic drift directly accounts for the market price of risk so we have $\tilde{\mu}_t = \mu_t$. If one doesn't wish to consider factor models, then the new process is defined as $\tilde{\mu}_t = \frac{\mu_t}{g(V_t)}$

$$dV_t = f(\beta_t, V_t)dt + k(V_t)(\rho dW_t^1 + \sqrt{1-\rho^2}dW_t^2)$$

We employ the fact from our initial transformation of S_t above that $d\tilde{W}_t^1 = \tilde{\mu}dt + dW_t^1$ so that

$$\begin{aligned}
dV_t &= f(\beta_t, V_t)dt + k(V_t)[\rho(d\tilde{W}_t^1 - \tilde{\mu}dt) + \sqrt{1-\rho^2}dW_t^2] \\
&= \big[f(\beta_t, V_t) - k(V_t)\rho\tilde{\mu}_t\big]dt + k(V_t)\rho d\tilde{W}_t^1 + k(V_t)\sqrt{1-\rho^2}dW_t^2 \\
&= k(V_t)\sqrt{1-\rho^2}\Big[dW_t^2 + (k(V_t)\sqrt{1-\rho^2})^{-1}\big(f(\beta_t, V_t) - k(V_t)\rho\tilde{\mu}_t\big)dt + (k(V_t)\sqrt{1-\rho^2})^{-1}k(V_t)\rho d\tilde{W}_t^1\Big] \\
&= k(V_t)\sqrt{1-\rho^2}[dW_t^2 + \tilde{\beta}_t dt + (k(V_t)\sqrt{1-\rho^2})^{-1}k(V_t)\rho d\tilde{W}_t^1]
\end{aligned}$$

where

$$\tilde{\beta}_t = (k(V_t)\sqrt{1-\rho^2})^{-1}\big(f(\beta_t, V_t) - k(V_t)\rho\tilde{\mu}_t\big). \tag{8}$$

Similarly by Girsanov's theorem, we have

$$d\tilde{W}_t^2 = dW_t^2 + \tilde{\beta}_t dt,$$

where $d\tilde{W}_t^2$ is a Brownian motion under measure $\tilde{\mathbb{P}}$, independent of $\tilde{W}_t^1$. Therefore the V_t dynamics becomes

$$dV_t = k(V_t)\sqrt{1-\rho^2}d\tilde{W}_t^2 + \rho k(V_t)d\tilde{W}_t^1 \tag{9}$$

alongside the transformed dS_t under the space $(\tilde{\mathbb{P}}, \mathbb{F})$

Since these dynamics now exist in the probability space $(\tilde{\mathbb{P}}, \mathbb{F})$, in essence we are in a more complicated position than we began with. The oracle is that we want the dynamics to exist on the original probability measure $\mathbb{P}$ and a fully observable filtration. Under certain assumptions, Lemma 3.1 in [2] shows that under the new measure $\tilde{\mathbb{P}}$ we have

$$\mathbb{G} = \mathbb{F}^S \vee \mathbb{F}^V = \mathbb{F}^{\tilde{W}^1} \vee \mathbb{F}^{\tilde{W}^2}$$

That is, the Brownian motions obtained from Girsanov's theorem generate the fully observable filtration $\mathbb{G}$. The final challenge is then to transform the new measure back to the original measure $\mathbb{P}$, which is where stochastic filtering theory comes into play.

4 Stochastic Filtering

As introduced above, the goal of stochastic filtering is to estimate the true position X_t, based on the noisy observations Y_t, with the *filtering estimates*:

$$\alpha_t(\varphi) = \mathbb{E}[\varphi(X_t) \mid \mathcal{G}_t],$$

where $\mathcal{G}_t$ is the *information given by the observation up to time* t, that is the filtration generated by the observation process Y_t. This translates directly into our study, with the truth given by

$$dX_t = A(X_t)dt + G(X_t)dM_t + B(X_t)dW_t$$

and the observations by

$$d\Theta_t = h(X_t)dt + dW_t$$

such that

$$X_t = \begin{bmatrix} \tilde{\mu}_t \\ \tilde{\beta}_t \end{bmatrix}, \ \Theta_t = \begin{bmatrix} \tilde{W}_t^1 \\ \tilde{W}_t^2 \end{bmatrix}$$

where the operators A, G, B are given by

$$A = \begin{pmatrix} a \\ \bar{a} \end{pmatrix}, \quad G = \begin{pmatrix} g_1 & g_2 \\ \bar{g}_1 & \bar{g}_2 \end{pmatrix}, \quad B = \begin{pmatrix} b_1 & b_2 \\ \bar{b}_1 & \bar{b}_2 \end{pmatrix} \quad h = \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}$$

where $a := a(\tilde{\mu}_t, \tilde{\beta}_t)$ and respectively for $\bar{a},\ g_1,\ g_2,\ \bar{g}_1,\ \bar{g}_2,\ b_1,\ b_2,\ \bar{b}_1,\ \bar{b}_2, h_1, h_2$. The Brownian motions are then given by

$$M_t = \begin{pmatrix} W_t^3 \\ W_t^4 \end{pmatrix}, \quad W_t = \begin{pmatrix} W_t^1 \\ W_t^2 \end{pmatrix}$$

We consider the evolution when $\varphi = Id.$ in (20), the filtering estimates are thus

$$\mathbb{E}^{\tilde{\mathbb{P}}}[\tilde{\mu}_t \mid \mathcal{G}_t] := \bar{\mu}_t \tag{10}$$

$$\mathbb{E}^{\tilde{\mathbb{P}}}[\tilde{\beta}_t \mid \mathcal{G}_t] := \bar{\beta}_t \tag{11}$$

where the expectation here is taken with respect to $\tilde{\mathbb{P}}$. Then using Girsanov's theorem again, we define the so-called *innovation processes* $\bar{W}_t^1$ and $\bar{W}_t^2$, which are independent $(\mathbb{P}, \mathbb{G})$-Brownian motions satisfying

$$\begin{aligned} d\bar{W}_t^1 &= dW_t^1 + \tilde{\mu}_t dt - \bar{\mu}_t dt \\ &= d\tilde{W}_t^1 - \bar{\mu}_t dt \end{aligned}$$

and similarly,

$$\begin{aligned} d\bar{W}_t^2 &= dW_t^2 + \tilde{\beta}_t dt - \bar{\beta}_t dt \\ &= d\tilde{W}_t^2 - \bar{\beta}_t dt \end{aligned}$$

This result is proved explicitly in [1]. Rearranging these equations for $d\tilde{W}_t^1$ and $d\tilde{W}_t^2$ and substituting into (7) and (9), we achieve the oracle. That is, we have a fully observable equivalent model in the space $(\mathbb{P}, \mathbb{G})$

$$\begin{aligned} \frac{dS_t}{S_t} &= g(V_t)\bar{\mu}_t dt + g(V_t) d\bar{W}_t^1 \\ dV_t &= (\rho k(V_t)\bar{\mu}_t + \sqrt{1-\rho^2} k(V_t)\bar{\beta}_t)dt + \rho k(V_t) d\bar{W}_t^1 + \sqrt{1-\rho^2} k(V_t) d\bar{W}_t^2 \end{aligned}$$

In these fully observable dynamics, we still have to calculate the filtering estimates $\bar{\mu}_t, \bar{\beta}_t$ for all time. As discussed in the preliminaries, the evolution of these filtering estimates satisfies the Kushner-Stratonovich equations which in this case with $\varphi = Id.$ takes the form:

$$d\bar{\mu}_t = \alpha_t(a)dt + \left(\alpha_t(h_1^2 + b_1) - \bar{\mu}_t^2\right) d\bar{W}_t^1 + \left(\alpha_t(h_2 h_1 + b_2) - \bar{\beta}_t \bar{\mu}_t\right) d\bar{W}_t^2$$

$$d\bar{\beta}_t = \alpha_t(\bar{a})dt + \left(\alpha_t(h_1 h_2 + \bar{b}_1) - \bar{\mu}_t \bar{\beta}_t\right) d\bar{W}_t^1 + \left(\alpha_t(h_2^2 + \bar{b}_2) - \bar{\beta}_t^2\right) d\bar{W}_t^2$$

Let's now apply this theory to the Log OU model.

4.1 Log Ornstein-Uhlenbeck model

Aligned with the general model presented in chapter 2, we have $g(V_t) = e^{V_t}$, $f(\beta_t, V_t) = \lambda_V(\theta - V_t)$, $k(V_t) = \sigma_V$, $\zeta(\mu_t) = \lambda_\mu(\theta_\mu - \mu_t)$, $\vartheta(\mu_t) = \sigma_\mu$. In this case, both V_t and the stochastic drift μ_t are mean reverting processes. The partially observable model is thus given by

$$\frac{dS_t}{S_t} = e^{V_t}(\mu_t dt + dW_t^1) \tag{12}$$

$$dV_t = \lambda_V(\theta - V_t)dt + \sigma_V \rho dW_t^1 + \sigma_V\sqrt{1-\rho^2}dW_t^2 \tag{13}$$

$$d\mu_t = \lambda_\mu(\theta_\mu - \mu_t)dt + \sigma_\mu dW_t^3 \tag{14}$$

In order to reduce to the fully observable model, we must first define a procedure to obtain the filtering estimates. This can be done in the following four steps:

Step 1: Describe the dynamics of the risks $\tilde{\mu}_t, \tilde{\beta}_t$. Since we only consider factor models, we have $\tilde{\mu}_t = \mu_t$ and hence the dynamics of $\tilde{\mu}_t$ is the same as that of μ_t. As for $\tilde{\beta}_t$, using (8) we have

$$\tilde{\beta}_t = \frac{\lambda_V(\theta - V_t)}{\sigma_V\sqrt{1-\rho^2}} - \frac{\rho}{\sqrt{1-\rho^2}}\tilde{\mu}_t$$

Step 2: Use Ito's formula to find $d\tilde{\mu}_t$, $d\tilde{\beta}_t$ on $\tilde{\beta}_t = f(t, V_t)$ (with it's application on $\tilde{\mu}_t = \mu_t$ trivial). We have

$$d\tilde{\mu}_t = \lambda_\mu(\theta_\mu - \tilde{\mu}_t)dt + \sigma_\mu dW_t^3$$

$$d\tilde{\beta}_t = \left(\frac{\rho(\lambda_\mu - \lambda_V)\tilde{\mu}_t}{\bar{\rho}} - \lambda_V\tilde{\beta}_t - \frac{\rho\lambda_\mu\theta_\mu}{\bar{\rho}}\right)dt - \frac{\rho\sigma_\mu}{\bar{\rho}}dW_t^4 - \left(\frac{\rho\lambda_V}{\bar{\rho}} + \lambda_V\right)dW_t^2$$

In matrix vector notation, this becomes

$$d\begin{pmatrix}\tilde{\mu}_t \\ \tilde{\beta}_t\end{pmatrix} = \left(A\begin{pmatrix}\tilde{\mu}_t \\ \tilde{\beta}_t\end{pmatrix} + b\right)dt + Gd\begin{pmatrix}W_t^3 \\ W_t^4\end{pmatrix} + B\begin{pmatrix}W_t^1 \\ W_t^2\end{pmatrix}$$

with the matrices given by

$$A = \begin{pmatrix}-\lambda_\mu & 0 \\ \frac{\rho(\lambda_\mu - \lambda_V)}{\bar{\rho}} & -\lambda_V\end{pmatrix},\ b = \begin{pmatrix}\lambda_\mu\theta_\mu \\ -\frac{\rho}{\bar{\rho}}\lambda_\mu\theta_\mu\end{pmatrix},\ G = \begin{pmatrix}\sigma_\mu & 0 \\ -\frac{\rho}{\bar{\rho}}\sigma_\mu & 0\end{pmatrix},\ B = \begin{pmatrix}0 & 0 \\ -\frac{\rho}{\bar{\rho}}\lambda_V & -\lambda_V\end{pmatrix}$$

Step 3: Recall the general frame work of the signal - observation process:

$$dX_t = A(X_t)dt + G(X_t)dM_t + B(X_t)dW_t$$

$$d\Theta_t = h(X_t)dt + dW_t.$$

with the operators defined above, which is consistent with the notation in the example. That is, we have

$$A(X_t) = AX_t + b,\ G(X_t) = G,\ B(X_t) = B$$

We can then discover dynamics of the filtering estimates by evaluating the Kushner-Stratonovich equation

$$d\bar{\mu}_t = \alpha_t(a)dt + \left[\alpha_t\left(h_1^2 + b_1\right) - \bar{\mu}^2\right]d\bar{W}_t^1 + \left[\alpha_t\left(h_2h_1 + b_2\right) - \bar{\mu}_t\bar{\beta}_t\right]d\bar{W}_t^2 \tag{15}$$

We have $\alpha_t(a) = \alpha_t(\lambda_\mu(\theta_\mu - \tilde{\mu}_t)) = \lambda_\mu(\theta_\mu - \bar{\mu}_t)$, $\alpha_t(h_1^2 + b_1) = \alpha_t(h_1^2)$ since $b_1 = 0$. Then since $h_1\left(\begin{bmatrix}\tilde{\mu}_t \\ \tilde{\beta}_t\end{bmatrix}\right) = \tilde{\mu}_t$, we have $\alpha_t(h_2 h_1) = \bar{\beta}_t \bar{\mu}_t$, thus

$$\begin{aligned} d\bar{\mu}_t &= \lambda_\mu(\theta_\mu - \bar{\mu}_t)dt + \left[\alpha_t\left(h_1^2\right) - \bar{\mu}^2\right] d\bar{W}_t^1 \\ &= \lambda_\mu(\theta_\mu - \bar{\mu}_t)dt + \left[\mathbb{E}[\tilde{\mu}^2 \mid \mathcal{G}_t] - \bar{\mu}^2\right] d\bar{W}_t^1 \\ &= \lambda_\mu(\theta_\mu - \bar{\mu}_t)dt + \left[\mathrm{Var}[\tilde{\mu} \mid \mathcal{G}_t]\right] d\bar{W}_t^1 \end{aligned}$$

Notice the variance term here. Similar calculations for $\bar{\beta}_t$ then give rise to stochastic differential equations of the filters

$$d\begin{pmatrix}\bar{\mu}_t \\ \bar{\beta}_t\end{pmatrix} = \left(A\begin{pmatrix}\bar{\mu}_t \\ \bar{\beta}_t\end{pmatrix} + b\right) dt + (B + \Sigma_t) d\begin{pmatrix}\bar{W}_t^1 \\ \bar{W}_t^2\end{pmatrix}$$

such that

$$d\Sigma_t = A\Sigma_t + \Sigma_t A^T + GG^T - \Sigma_t \Sigma_t^T - \Sigma_t B^T - B\Sigma_t^T$$

is the conditional covariance matrix satisfying a deterministic Riccati equation [2], which can be evaluated using a basic finite difference schemes. Figure 1 illustrates the evolution of the filtering estimates given some parameters using an elementary matrix finite difference scheme to solve the Riccati equation. Observe the dependency on the initial choice of covariance matrix Σ_0. For an in-depth analysis of this filter, see [5].

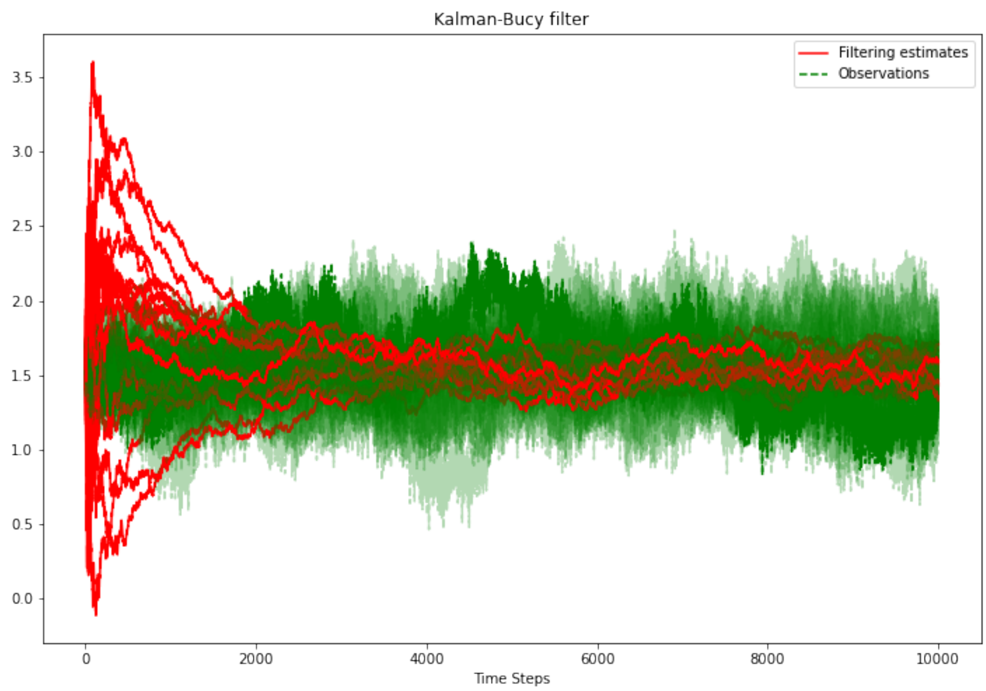

Figure 1: Illustration of the Kalman-Bucy Filter

5 Solving the Optimisation Problem with the Martingale Dual Approach

We recall the Martingale Representation Theorem and an important Lemma which backs it up,

Theorem 3 *Suppose that M_t is a $\mathbb{G}$-adapted martingale with respect to $\mathbb{P}$, such that $M_t \in L^2(\mathbb{P}), \forall t \geq 0$. Then there exists a unique $\mathbb{G}$-adapted process $\phi(s)$ such that*

$$M_t = E[M_0] + \int_0^t \phi(s) dW_s$$

a.s $\forall t \geq 0$.

Lemma 4 *The linear span of random variables of the type*

$$\exp\left(\int_0^T \psi(t)dW_t - \frac{1}{2}\int_0^T \psi^2(t)dt\right), \quad \psi \in L^2(0,T)$$

is dense in the space $L^2(\mathbb{P})$.

By applying the above information with the fact that there exists a $\mathbb{G}$-adapted process $\nu = \{\nu_t, 0 \leq t \leq T\}$ such that $\int_0^T \nu_t^2 dt < \infty$, we can define a $(\mathbb{P}, \mathbb{G})$-local martingale given as

$$Z_t^\nu = \exp\Bigg(-\int_0^t \bar{\mu}_s d\bar{W}_s^1 - \int_0^t \nu_s d\bar{W}_s^2 - \frac{1}{2}\int_0^t \bar{\mu}_s^2 ds - \frac{1}{2}\int_0^t \nu_s^2 ds\Bigg)$$

such that $\int_0^T \bar{\mu}_t^2 dt < \infty, \mathbb{P} - a.s.$
By the optional stopping theorem, when $E[Z_T^\nu] = E[Z_0^\nu] = 1$, the process Z^ν is a martingale and can therefore be used to introduce a new measure equivalent to $\mathbb{P}$. That is,

$$\frac{d\mathbb{Q}}{d\mathbb{P}}\bigg|_{\mathcal{G}_t} = Z_T^\nu$$

such that $E^{\mathbb{P}}[X_t] = E^{\mathbb{Q}}[\frac{d\mathbb{Q}}{d\mathbb{P}} X_t]$ for a process X_t.
In addition, by various standing assumptions, the utility function U is known to be strictly increasing, concave, continuous and continuously differentiable satisfying the following

$$U'(0) = \lim_{x \to 0} U'(x) = \infty, \qquad U'(\infty) = \lim_{x \to \infty} U'(x) = 0$$

Now let us introduce the function

$$\tilde{U}(y) = \max_{x>0}[U(x) - xy] = U(I(y)) - yI(y),$$

where $I = (U')^{-1}$. $\tilde{U}$, as the convex dual of U is strictly decreasing, convex and satisfies the following

$$U(x) = \min[\tilde{U}(y) + xy] = \tilde{U}(U'(x)) + xU'(x)$$

These properties and the above information all come together to show that seeking the solution to the dual problem

$$J_{\text{dual}}(z) = \inf_{\mathbb{Q}} E\Big[\tilde{U}\Big(z\frac{d\mathbb{Q}}{d\mathbb{P}}\Big)\Big] := \inf_{\nu \in \mathcal{H}} E[\tilde{U}(zZ_T^\nu)], \quad z > 0,$$

is equivalent to solving the primal optimisation problem. Here, $\mathcal{H}$ is the Hilbert space of $\mathbb{G}$-adapted processes satisfying an integrability condition. According to Theorem 5.2 in [2], under the following assumptions,

- For some $p \in (0,1), \gamma \in (1,\infty)$ we have $pU'(x) \geq U'(\gamma x)$
- $x \to xU'(x)$ is non-decreasing on $(0,\infty)$
- $\forall z \in (0,\infty)$, $\nu \in \mathcal{H}$ there is $J_{\text{dual}}(z) < \infty$

a minimising solution ν^* may be found, and the optimal wealth process will then be

$$R_t^* = E\Big[\frac{Z_T^{\nu^*}}{Z_t^{\nu^*}} I(z_x Z_T^{\nu^*})|\mathcal{G}_t\Big]$$

The corresponding optimal portfolio π^* then satisfies

$$dR_t^* = R_t^*\pi_t^* g(V_t)(d\bar{W}_t^1 + \bar{\mu}dt) = R_t^*\pi_t^* g(V_t)d\tilde{W}_t^1.$$

For simplicity, the property

$$\tilde{U}(\lambda x) = g_1(\lambda)\tilde{U}(x) + g_2(\lambda)$$

for $\lambda > 0$ and any functions g_1, g_2 is assumed to enable the value function to be evaluated without knowledge of z. In that case

$$J_{\text{dual}}(z) = \inf_{\nu\in\mathcal{H}} E[\tilde{U}(Z_T^\nu z)] = g_1(z)\inf_{\nu\in\mathcal{H}} E[\tilde{U}(Z_T^\nu)] + g_2(z)$$

In obtaining ν^*, on which the optimal wealth and the corresponding optimal portfolio values depend, we consider the case where the infimum of the dual value function is assumed to be attained when $\nu^* = 0$. For that case,

$$J_{\text{dual}}(z) = \inf_{\nu\in\mathcal{H}} E[\tilde{U}(zZ_T^\nu)] = \inf_{\nu\in\mathcal{H}} E[\tilde{U}(zZ_T^0)].$$

We also considered specifically the log utility function defined as

$$U(x) = \ln(x) \quad x > 0$$

with convex dual given by

$$\tilde{U}(z) = -(1+\ln(z)) \quad z > 0.$$

This particular utility function is preferred because it satisfies the necessary assumptions, and also, due to the homogeneity of the dual function and the linearity of ν, the dual value function J_{dual} can be characterized through a solution to a semi-linear PDE.
According to the verification theorem in [2], if we assume all standing assumptions hold, and that ϕ, the solution to the semi-linear PDE for which J_{dual} is characterized, is a $C^{1,2}([0,T)\times\mathbb{R}^2)\cap C^0([0,T]\times\mathbb{R}^2)$ function and satisfies a polynomial growth condition, then the dual value function devoid of z is given as

$$J_{\text{dual}} = -1 - \ln(Z_T^\nu) - \phi(t,y)$$

Now to obtain the explicit form of the optimal portfolio, consider the following steps as carried out in [2].

Recall

$$R_t^* = E\Big[\frac{Z_T^{\nu^*}}{Z_t^{\nu^*}} I(z_x Z_T^{\nu^*})|\mathcal{G}_t\Big]$$

Since under this setting, $I = z_x = \frac{1}{x}$, the following is obtained,

$$R_t^* = E\Big[\frac{Z_T^{\nu^*}}{Z_t^{\nu^*}} I(z_x Z_T^{\nu^*})|\mathcal{G}_t\Big] = E\Bigg[\frac{x}{Z_t^{\nu^*}}|\mathcal{G}_t\Bigg] = \frac{x}{Z_t^0}$$

To add, by applying Ito's formula to $R_t^* = \frac{x}{Z_t^0}$ while taking note that by assumption (Z_t^ν, Y_t) is a controlled process such that

$$\begin{aligned} dZ_t^\nu &= -Z_t^\nu \psi(Y_t) d\bar{W}_t^1 - Z_t^\nu \nu_t d\bar{W}_t^2 \quad \text{where } \psi(Y_t) = \bar{\mu}_t \\ dY_t &= \Gamma(Y_t)dt + \Sigma(Y_t)dW_t, \end{aligned}$$

we obtain

$$dR_t^* = R_t^* \psi(Y_t) d\tilde{W}_t^1,$$

However, from above it was clear that

$$dR_t^* = R_t^* \pi_t^* g(V_t)(d\tilde{W}_t^1)$$

and so by simple comparison, the optimal portfolio is given by

$$\psi(Y_t) = \pi_t^* g(V_t) \implies \pi_t^* = \frac{\psi(Y_t)}{g(V_t)} = \frac{\bar{\mu}_t}{g(V_t)} \tag{16}$$

It can be observed then that the value for the optimal portfolio depends on the filtering estimate, $\bar{\mu}_t$, however, in order to apply filtering, one must first have some ground truth dynamics. Recall that, all the investor has access to is the price and volatility data, for example the S&P 500 and VIX datasets. The investor also has absolutely no idea what the correct parameters of the model should be without analysis of the data.
To overcome this, we discuss the Maximum Likelihood Estimation technique for parameter calibration in the subsequent chapter.

6 Application

6.1 Model Calibration with Non-constant μ_t

In order to apply Kalman-Bucy filter to the Log O-U model, one needs to first calibrate the model by estimating all unknown parameters $\Theta = \{\rho, \sigma_V, \sigma_\mu, \lambda_V, \lambda_\mu, \theta, \theta_\mu\}$. Initially, we attempted to simplify the problem by setting μ_t constant. It was later realised that the assumption of constant μ_t would lead to an ill-posed control problem that violates the necessary hypothesis as stated in [2], specifically *Assumption H.* Details of the work on constant μ_t can be found in Appendix A, and the rest of the section only considers the non-constant μ_t case.

Assuming dS_t, dV_t and $d\mu_t$ are independent and identically distributed between each time increment [8]. the model can be calibrated by finding Θ that maximises the discretised likelihood

$$L(\Theta|x_0, x_1, \ldots, x_n) = f(x_0, x_1, \ldots, x_n|\Theta) = \prod_{t=0}^{n} f(x_t|\Theta), \tag{17}$$

where x_t represents the discretised changes of the observable variables.

The difficulty of calibrating models in the non-constant μ_t case lies in the unobservable nature of $d\mu_t$, which makes it not possible to directly optimise a μ_t-dependent likelihood function. To formulate a μ_t-independent likelihood function that also considers the presence of μ_t in the S dynamics, one needs to marginalise $f(x_t|\Theta)$ against μ_t and integrate using the law of total probability. More explicitly,

$$\begin{aligned} f(x_0, x_1 \dots, x_n|\Theta) &= \int_{\mathbb{R}^n} f(x_0, x_1, \dots, x_n, \mu_0, \mu_1, \dots, \mu_{n-1}|\Theta)\, d\mu \\ &= \int_{\mathbb{R}^n} f(x_0, x_1, \dots, x_n|\mu_0, \mu_1, \dots, \mu_{n-1}, \Theta) f(\mu_0, \mu_1, \dots, \mu_{n-1}|\Theta)\, d\mu \\ &= \int_{\mathbb{R}^n} \prod_{t=0}^{n} f(x_t|\mu_0, \mu_1, \dots, \mu_{n-1}, \Theta) f(\mu_0, \mu_1, \dots, \mu_{n-1}|\Theta)\, d\mu \end{aligned}$$

where $d\mu = d\mu_0\, d\mu_1 \dots d\mu_{n-1}$

The observable variables $x_t = \{d \log S_t, dV_t\}$ are assumed to follow a bivariate normal distribution [8], which can be derived by applying Ito's lemma to equation (12),

$$\begin{aligned} d \log S_t &= \frac{1}{S_t}(dS_t) - \frac{1}{2S_t^2}(dS_t)^2 \\ &= (e^{V_t}\mu - \frac{1}{2}e^{2V_t})dt + e^{V_t} dW_t \end{aligned}$$

Following Euler-Maruyama scheme to discretise the continuous time into Δt, the joint probability distribution function at time t can be written as

$$x_t \sim \mathcal{N}\Big(m_t, \Sigma_t\Big)$$

$$x_t = \begin{bmatrix} \Delta \log S_t \\ \Delta V_t \end{bmatrix}, \quad m_t = \begin{bmatrix} (e^{V_{t-1}}\mu_{t-1} - \frac{1}{2}e^{2V_{t-1}})\Delta t \\ \lambda_V(\theta - V_{t-1})\Delta t \end{bmatrix}, \quad \Sigma_t = \begin{bmatrix} e^{2V_{t-1}}\Delta t & \rho e^{V_{t-1}}\sigma_V \Delta t \\ \rho e^{V_{t-1}}\sigma_V \Delta t & \sigma_V^2 \Delta t \end{bmatrix}$$

$$f(\Delta \log S_t, \Delta V_t \mid \mu_0, \mu_1, \dots \mu_{t-1}, \Theta) = (2\pi e^{V_{t-1}}\sigma_V \Delta t\sqrt{1-\rho^2})^{-1} \exp\left(-\frac{1}{2}(x_t - m_t)^T \Sigma_i^{-1}(x_t - m_t)\right)$$

where $\Delta \log S_t := \log S_t - \log S_{t-1}$, $\Delta V_t := V_t - V_{t-1}$

Similarly, by assuming $\Delta \mu_t$ to be also independently and normally distributed

$$\Delta \mu_t := \mu_{t+1} - \mu_t \sim \mathcal{N}(\lambda_\mu(\theta_\mu - \mu_t)\Delta t, \sigma_\mu^2 \Delta t)$$

$$f(\mu_0, \mu_1, \dots, \mu_{n-1} \mid \Theta) = \prod_{t=1}^{n-1} \frac{1}{\sigma_\mu\sqrt{2\pi\Delta t}} \exp\left(-\frac{1}{2}\frac{(\mu_t - \mu_{t-1} - \Delta t(\lambda_\mu(\theta_\mu - \mu_{t-1})))^2}{\sigma_\mu^2 \Delta t}\right)$$

Hence, we have to solve the following integral

$$\begin{aligned} L(\Theta|x_1, x_2, \dots, x_{n-1}) = \int_{\mathbb{R}^n} &\prod_{t=1}^{n} \left[(2\pi e^{V_{t-1}}\sigma_V \Delta t\sqrt{1-\rho^2})^{-1} \exp\left(-\frac{1}{2}(x_t - m_t)^T \Sigma_i^{-1}(x_t - m_t)\right)\right] \\ &\cdot \prod_{t=1}^{n-1} \left[\frac{1}{\sigma_\mu\sqrt{2\pi\Delta t}} \exp\left(-\frac{1}{2}\frac{(\mu_t - \mu_{t-1} - \Delta t(\lambda_\mu(\theta_\mu - \mu_{t-1})))^2}{\sigma_\mu^2 \Delta t}\right)\right] d\mu \end{aligned}$$

Let $M_t = \begin{bmatrix} -e^{V_{t-1}}\Delta t & 0 \\ 0 & 0 \end{bmatrix}$, $r_t = x_t - \begin{bmatrix} -\frac{1}{2}e^{2V_{t-1}}\Delta t \\ \lambda_V(\theta - V_{t-1})\Delta t \end{bmatrix}$, then $x_t - m_t = M_t \begin{bmatrix} \mu_{t-1} \\ \mu_t \end{bmatrix} + r_t$.

Also, let $P = \frac{1}{\sigma_\mu\sqrt{2\pi\Delta t}}\begin{bmatrix}-1+(\Delta t)\lambda_\mu & 1\end{bmatrix}$, and $s = -\frac{1}{\sigma_\mu\sqrt{2\pi\Delta t}}(\Delta t)\lambda_\mu\theta_\mu$, then

$$\frac{1}{\sigma_\mu\sqrt{2\pi\Delta t}}(\mu_t - \mu_{t-1} - \Delta t(\lambda_\mu(\theta_\mu - \mu_{t-1}))) = P\begin{bmatrix}\mu_{t-1}\\ \mu_t\end{bmatrix} + s.$$

Therefore, the integral can be written as

$$L(\Theta|x_1,x_2,\ldots,x_{n-1}) = \frac{1}{\left(\sigma_\mu\sqrt{2\pi\Delta t}\right)^{n-1}}\prod_{t=1}^{n}\left(2\pi e^{V_{t-1}}\sigma_V\Delta t\sqrt{1-\rho^2}\right)^{-1}$$

$$\cdot\int_{\mathbb{R}^n}\exp\left(-\frac{1}{2}\sum_{t=1}^{n}\left\{\left(M_t\begin{bmatrix}\mu_{t-1}\\ \mu_t\end{bmatrix}+r_t\right)^T C_t\left(M_t\begin{bmatrix}\mu_{t-1}\\ \mu_t\end{bmatrix}+r_t\right)+\left(P\begin{bmatrix}\mu_{t-1}\\ \mu_t\end{bmatrix}+s\right)^T\left(P\begin{bmatrix}\mu_{t-1}\\ \mu_t\end{bmatrix}+s\right)\right\}\right)$$

$$\begin{aligned}
&= \frac{1}{\left(\sigma_\mu\sqrt{2\pi\Delta t}\right)^{n-1}}\left(\prod_{t=1}^{n}\left(2\pi e^{V_{t-1}}\sigma_V\Delta t\sqrt{1-\rho^2}\right)^{-1}\right)\cdot\exp\left(-\frac{1}{2}\sum_{t=1}^{n}\left\{r_t^TC_tr_t+s^2\right\}\right)\\
&\quad\cdot\int_{\mathbb{R}^n}\exp\left(-\frac{1}{2}\sum_{t=1}^{n}\left\{\begin{bmatrix}\mu_{t-1}\\ \mu_t\end{bmatrix}^T\left(M_t^TC_tM_t+P^TP\right)\begin{bmatrix}\mu_{t-1}\\ \mu_t\end{bmatrix}+2(r_t^TC_tM_t+sP)\begin{bmatrix}\mu_{t-1}\\ \mu_t\end{bmatrix}\right\}\right)
\end{aligned}\tag{18}$$

where

$$C_t := \Sigma_t^{-1} = \frac{1}{(1-\rho^2)(e^{2V_{t-1}}\sigma_V^2)(\Delta t)}\begin{bmatrix}\sigma_V^2 & -\rho e^{V_{t-1}}\sigma_V\\ -\rho e^{V_{t-1}}\sigma_V & e^{2V_{t-1}}\end{bmatrix}$$

Now let $\mu = \begin{bmatrix}\mu_0\\ \vdots\\ \mu_{n-1}\end{bmatrix}$, $E^{(t-1,t)}$ denote the $2\times n$ matrix with $E^{(t-1,t)}_{1,t} = 1$, $E^{(t-1,t)}_{2,t+1} = 1$, and all other elements zero. $E^{(n-1)}$ denote the $2\times n$ matrix with $E^{(n-1)}_{1,n} = 1$ and all other elements zero. Then $\begin{bmatrix}\mu_{t-1}\\ \mu_t\end{bmatrix} = E^{(t-1,t)}\mu$. Hence, we can re-write the integral from equation (18) as following

$$\begin{aligned}
&\int_{\mathbb{R}^n}\exp\left(-\frac{1}{2}\sum_{t=1}^{n}\left\{\begin{bmatrix}\mu_{t-1}\\ \mu_t\end{bmatrix}^T\left(M_t^TC_tM_t+P^TP\right)\begin{bmatrix}\mu_{t-1}\\ \mu_t\end{bmatrix}+2(r_t^TC_tM_t+sP)\begin{bmatrix}\mu_{t-1}\\ \mu_t\end{bmatrix}\right\}\right)\\
&= \int_{\mathbb{R}^n}\exp\Bigg(-\frac{1}{2}\mu^T\left((E^{(n-1)})^TM_{n-1}^TC_{n-1}M_{n-1}E^{(n-1)}+\sum_{t=1}^{n-1}(E^{(t-1,t)})^T\left(M_t^TC_tM_t+P^TP\right)E^{(t-1,t)}\right)\mu\\
&\qquad+\left(r_{n-1}^TC_{n-1}M_{n-1}E^{(n-1)}+\sum_{t=1}^{n-1}(r_t^TC_tM_t+sP)E^{(t-1,t)}\right)\mu\Bigg)\\
&= \int_{\mathbb{R}^n}\exp\left(-\frac{1}{2}\mu^TA\mu+B\mu\right)d\mu\\
&= \frac{(2\pi)^{n/2}}{(\det(A))^{1/2}}\cdot\exp\left(\frac{1}{2}B^TA^{-1}B\right)
\end{aligned}\tag{19}$$

where equation (19) is obtained using the standard Gaussian integral, with A and B defined as following

$$A = (E^{(n-1)})^T M_{n-1}^T C_{n-1} M_{n-1} E^{(n-1)} + \sum_{t=1}^{n-1} (E^{(t-1,t)})^T \left(M_t^T C_t M_t + P^T P \right) E^{(t-1,t)}$$

$$B = r_{n-1}^T C_{n-1} M_{n-1} E^{(n-1)} + \sum_{t=1}^{n-1} (r_t^T C_t M_t + sP) E^{(t-1,t)}$$

By substituting equation (19) back to (18), the final log likelihood function reads

$$\begin{aligned} l(\Theta|x_1, x_2, \ldots, x_{n-1}) &= \log(L(\Theta|x_1, x_2, \ldots, x_{n-1})) \\ &= (1-n)\log(\sigma_\mu \sqrt{2\pi\Delta t}) - n\log(2\pi\sigma_V \Delta t\sqrt{1-\rho^2}) - \sum_{t=1}^{n}(V_{t-1}) \\ &\quad - \frac{1}{2}\sum_{t=1}^{n}\{r_t^T C_t r_t + s^2\} + \frac{n}{2}\log(2\pi) - \frac{1}{2}\log(\det(A)) + \frac{1}{2}BA^{-1}B^T \end{aligned}$$

6.2 Results

Although many different optimisers, such as CG, Powell, BFGS, L-BFGS-B, were investigated to minimise the negative log likelihood, Nelder-Mead seems to achieve better and more consistent results. Furthermore, to improve the optimisation stability as well as to ensure the validity of the MLE parameters, we enforce a set of constraints to the MLE solution: $|\rho| \leq 1$ and $\sigma_V, \sigma_\mu, \lambda_V, \lambda_\mu \geq 0$. The results of the optimisation performed over the period from 01/01/2013 to 01/06/2014 is shown below in Figure 2.

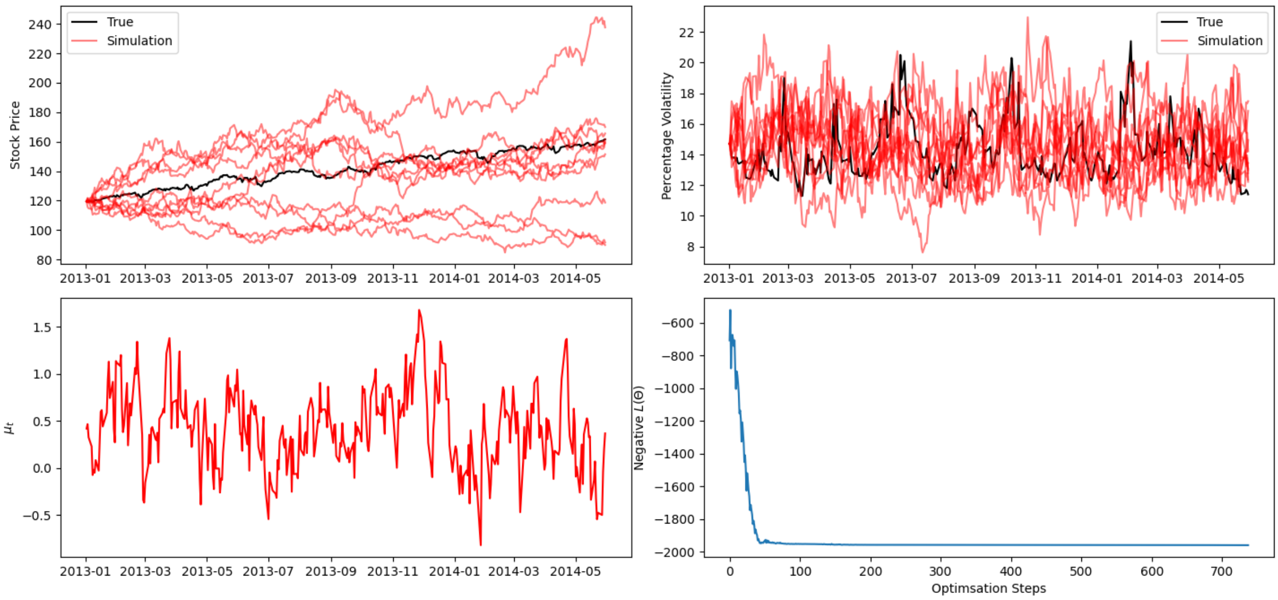

Figure 2: **Top**: A comparison between the true stock price (SPY, **top left**) and volatility (VIX, **top right**) with ten independently simulated dynamics using the calibrated parameters : $[\rho = -0.2,\ \sigma_V = 0.64,\ \sigma_\mu = 3.05,\ \lambda_V = 10.01,\ \lambda_\mu = 27.23,\ \theta = -1.97,\ \theta_\mu = 0.44]$. **Bottom left**: A single example of the simulated μ_t during the same time period. No comparison can be made as the true μ_t is unobservable. **Bottom right**: The plot shows the descending negative log likelihood over all optimisation steps until convergence.

Although the proper method to quantify the calibration accuracy is through inference, due to the time constraints, we adopted a simpler empirical approach to visualise the model calibration error by comparing the normalised distributions of the true ΔS_t and ΔV_t with their simulations. As shown in Figure 3, the distributions of our simulations may not be an entirely perfect fit to the true distributions of the ΔS_t and ΔV_t, however, they are in some sense manageable for the purposes of this time bound project as areas of fair similarity in the distributions are visible and suggest that given ample time, simulated values and the true values of ΔS_t and ΔV_t as given by the SPY and VIX datasets will be closer in distributions.

The optimal portfolio as defined in equation (16) can then be calculated as shown by Figure 4.

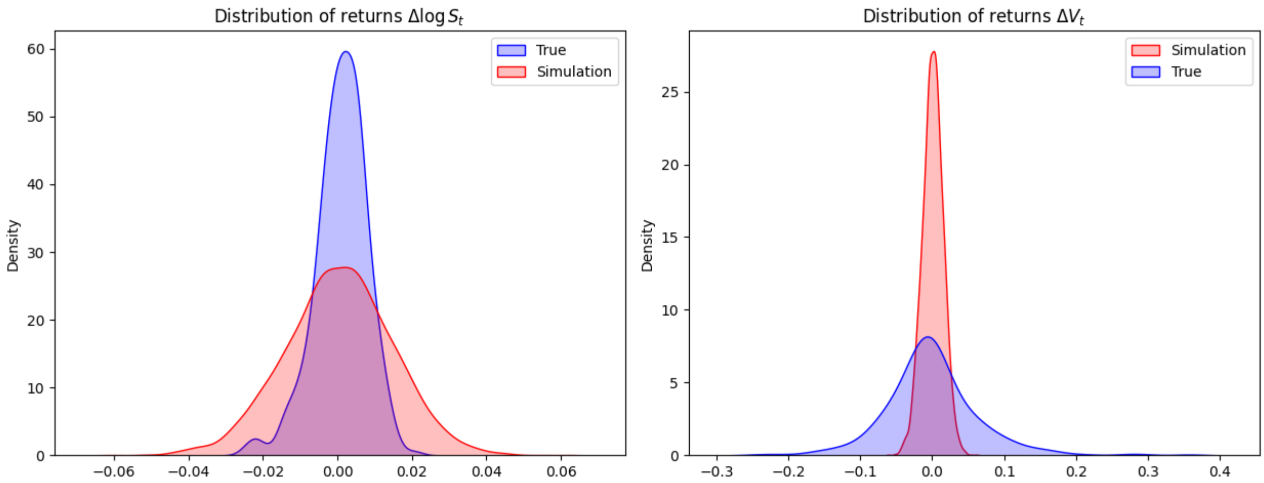

Figure 3: A comparison between the normalised distributions of the true and simulated ΔS_t (**left**) and ΔV_t (**right**). The distributions for the simulated case are obtained by independently simulating ΔS_t and ΔV_t ten times for roughly 1000 time steps before applying normalisation.

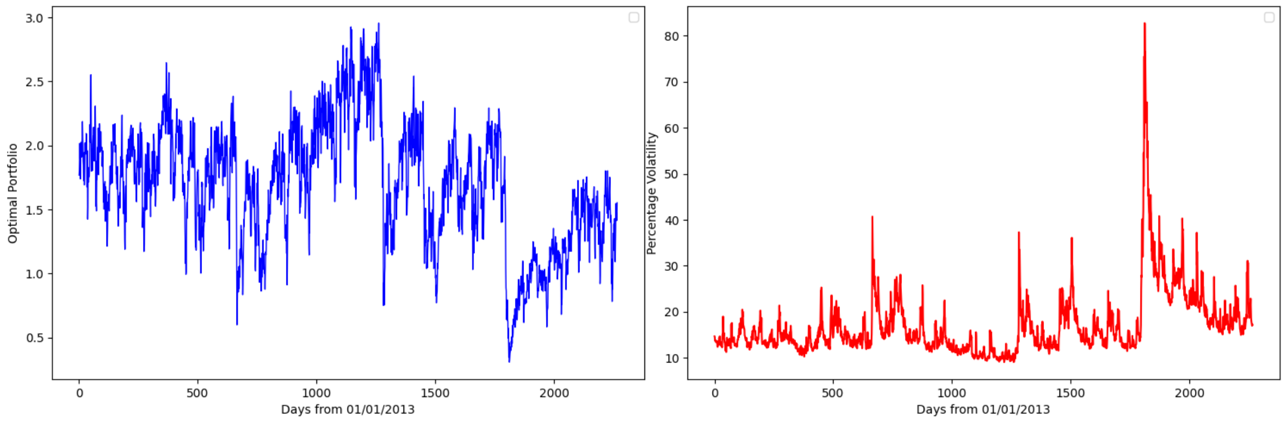

Figure 4: The two graphs illustrate the proposed optimal portfolio (**left**) compared with the percentage volatility (**right**) over the same time period. It is obvious that the higher the volatility, the riskier the asset is, the smaller the portfolio size becomes. A portfolio value above 1 indicates borrowing.

From equation (6) the wealth process can then be explicitly written as

$$dR_t^{\pi^*} = R_t^{\pi^*}(\bar{\mu}_t^2 dt + \bar{\mu}_t d\bar{W}_t^1)$$

And by applying Ito's Lemma,

$$d\log R_t^{\pi^*} = \frac{1}{2}\bar{\mu}_t^2 dt + \bar{\mu}_t d\bar{W}_t^1$$

We then compared expectation of the terminal wealth in two cases: **1.** The entire portfolio is invested according to the optimal control; **2.** Only half of the portfolio is invested according to the optimal control, while the other half is investigated in a riskless asset of zero return. The experiment is run over three to nine years, using real volatility data of the S&P 500 index. As shown in table 1, case **1** achieves better performance than case **2**, empirically demonstrating the impressive performance of the calculated control.

Period	$\mathbb{E}[\log(R_T^{\pi^*})]$ **:Optimal**	$\mathbb{E}[\log(R_T^{\pi})]$ **:Half**
2013-2016	10.27	9.96
2013-2019	10.48	10.21
2013-2022	10.79	10.51

Table 1: The table illustrates the expectation of the log wealth return at the end of a 3-, 6-, or 9-year period, with an initial log wealth of 10 units. Each simulation uses the real VIX data over the same period, while independently sampling $d\bar{W}$ to simulate $\bar{\mu}$ dynamics according to equation (15). The expectation of the log wealth is calculated by averaging 1000 simulations. Terminal log wealth less than 10 represents a loss to the initial investment.

7 Conclusion

In this project, we investigated in depth the novel approach of combining stochastic control derived through the martingale dual approach with filtering techniques to maximise the investment return in a hypothetical market consisting of a single stock. Although problems such as the highly non-convex and thus often unstable optimisation converging to heavily data- and initialisation-dependent solutions still remain as a challenge aspect, we expect these can be potentially solved in a future project by investigating different models [2] and more accurate time discretisation schemes. Another avenue for future work is an investigation into how model performance would change if we were to instead calibrate the parameters of the full observation Log O-U model (see (5.46-49) of [2]), so as to address the question of whether there is any practical purpose to filtering. Overall, the project has nevertheless seen some highly promising initial successes in model calibration involving latent variables and impressive results when applied to the real S&P 500 data.

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A Model Calibration in Constant μ_t Case

There had been an attempt early in the project to simplify the problem by setting $\mu_t = \mu_0 \,\forall t \in [0,T]$, but this approach is later abandoned after realising that assuming constant μ_t in the dynamics leads to an ill-posed optimal control problem and thus cannot be used as a simple case in the optimal portfolio theory. The likelihood formulation below is thus simply a motivation for the case of estimating parameters of non-constant μ_t.

Following the same assumptions and similar arguments as in Section 6.1, the parameter set $\Theta = \{\rho, \mu_0, \lambda_V, \theta, \sigma_V\}$ can be estimated by maximising the log likelihood of the bivariant normal distribution of the discretised $\{\Delta \log S_t, \Delta V_t\}$

$$x_t \sim \mathcal{N}\Big(m_t, \Sigma_t\Big)$$

$$x_t = \begin{bmatrix} \Delta \log S_t \\ \Delta V_t \end{bmatrix}, \quad m_t = \begin{bmatrix} (e^{V_{t-1}}\mu_{t-1} - \frac{1}{2}e^{2V_{t-1}})\Delta t \\ \lambda_V(\theta - V_{t-1})\Delta t \end{bmatrix}, \quad \Sigma_t = \begin{bmatrix} e^{2V_{t-1}}\Delta t & \rho e^{V_{t-1}}\sigma_V \Delta t \\ \rho e^{V_{t-1}}\sigma_V\Delta t & \sigma_V^2 \Delta t \end{bmatrix}$$

$$f(\Delta \log S_t, \Delta V_t \mid \mu_0, \mu_1, \ldots \mu_{t-1}, \Theta) = (2\pi e^{V_{t-1}}\sigma_V \Delta t\sqrt{1-\rho^2})^{-1} \exp\left(-\frac{1}{2}(x_t - m_t)^T \Sigma_i^{-1}(x_t - m_t)\right)$$

The probability density function at each time step is thus

$$\begin{aligned} f_\Theta(\Delta \log S_t, \Delta V_t) = {} & \frac{\alpha(\sigma_V, \rho)}{e^{2V_t}} \exp\Big(- \frac{1}{2(1-\rho^2)}\Big((\frac{\Delta \log S_t - (e^{V_t}\mu_0 - \frac{1}{2}e^{2V_t})\Delta t}{e^{2V_t}\sqrt{\Delta t}})^2 \\ & - 2\rho(\frac{(\Delta \log S_t - (e^{V_t}\mu_0 - \frac{1}{2}e^{2V_t})\Delta t)(\Delta V_t - \lambda_V(\theta - V_t)\Delta t)}{\sigma_V e^{2V_t}\Delta t}) + (\frac{\Delta V_t - \lambda_V(\theta - V_t)\Delta t}{\sigma_V\sqrt{\Delta t}})^2\Big)\Big) \\ \alpha(\sigma_V, \rho) = {} & \frac{1}{2\sigma_V\sqrt{1-\rho^2}\Delta t} \end{aligned}$$

Following equation (17), the log likelihood over the entire given data sequence $\{S_t, V_t\}_{i=0}^N$ can thus be written as

$$\begin{aligned} \log(L(\Theta)) &= \sum_{t=1}^{N} \log(f_\Theta(\Delta \log S_t, \Delta V_t)) \\ &= N\log(\alpha(\sigma_V, \rho)) + \sum_{t=1}^{N}\Big(- V_t - \frac{1}{2(1-\rho^2)}\Big((\frac{\Delta \log S_t - (e^{V_t}\mu_0 - \frac{1}{2}e^{2V_t})\Delta t}{e^{2V_t}\sqrt{\Delta t}})^2 \\ & - 2\rho(\frac{(\Delta \log S_t - (e^{V_t}\mu_0 - \frac{1}{2}e^{2V_t})\Delta t)(\Delta V_t - \lambda_V(\theta - V_t)\Delta t)}{\sigma_V e^{2V_t}\Delta t}) + (\frac{\Delta V_t - \lambda_V(\theta - V_t)\Delta t}{\sigma_V\sqrt{\Delta t}})^2\Big)\Big) \end{aligned}$$

This log likelihood can then be maximised following the same procedure as described in Section 6 above.