

Задача 8. Пресметнете определения интеграл:

$$\int_0^{\frac{1}{2}} \arcsin x \, dx.$$

Решение.

$$\arcsin x = y, x \in \left[0, \frac{1}{2}\right] \subset [-1, 1] \Rightarrow y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right].$$

I-ви начин: Интегриране по части.

$$\begin{aligned} I &= \int_0^{\frac{1}{2}} \arcsin x \, dx = x \arcsin x \Big|_0^{\frac{1}{2}} - \int_0^{\frac{1}{2}} x \, d \arcsin x = \\ &= \frac{1}{2} \times \frac{\pi}{6} - 0 - \int_0^{\frac{1}{2}} x (\arcsin x)' \, dx = \frac{\pi}{12} - \int_0^{\frac{1}{2}} \frac{x}{\sqrt{1-x^2}} \times x' \, dx \end{aligned}$$

Нека

$$\begin{aligned} J &= \int \frac{x}{\sqrt{1-x^2}} \, dx \stackrel{u=1-x^2}{=} \int \frac{x}{\sqrt{u}} \left(-\frac{1}{2x}\right) \, du = -\frac{1}{2} \int \frac{1}{\sqrt{u}} \, du = \\ &= -\frac{1}{2} \int u^{-\frac{1}{2}} \, du = -\frac{1}{2} \frac{u^{\frac{1}{2}}}{\frac{1}{2}} = -\sqrt{u} = -\sqrt{1-x^2} \end{aligned}$$

$$\begin{aligned} \Rightarrow I &= \frac{\pi}{12} - \left(-\sqrt{1-x^2}\right) \Big|_0^{\frac{1}{2}} = \frac{\pi}{12} + \sqrt{1-\frac{1}{4}} - 1 = \\ &= \frac{\pi}{12} + \frac{\sqrt{3}}{2} - 1 = \frac{\pi + 6\sqrt{3} - 12}{12} \approx 0.12782479... \end{aligned}$$

II-ри начин: Тригонометрична субституция.

Полагаме $x = \sin t$.

1. Ако $x = 0 \Rightarrow \sin t = 0$, за $t = 0$.

2. Ако $x = \frac{1}{2} \Rightarrow \sin t = \frac{1}{2}$, за $t = \frac{\pi}{6}$.

3.

Следователно,

$$\begin{aligned} I &= \int_0^{\frac{1}{2}} \arcsin x \, dx = \int_0^{\frac{\pi}{6}} \arcsin(\sin t) \, d \sin t = \int_0^{\frac{\pi}{6}} t \, d \sin t \stackrel{\text{И.Ч.}}{=} \\ &= t \sin t \Big|_0^{\frac{\pi}{6}} - \int_0^{\frac{\pi}{6}} \sin t \, dt = \frac{\pi}{6} \times \sin \frac{\pi}{6} - 0 - \int_0^{\frac{\pi}{6}} \sin t \, dt = \\ &= \frac{\pi}{6} \times \frac{1}{2} + \cos t \Big|_0^{\frac{\pi}{6}} = \frac{\pi}{12} + \cos \frac{\pi}{6} - \cos 0 = \frac{\pi}{12} + \frac{\sqrt{3}}{2} - 1. \end{aligned}$$

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