Задача 7. Намерете определения интеграл $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{|x|}(\sin x + \cos x)\mathrm{d}\, x$.

Решение.

$$I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{|x|}(\sin x + \cos x)\mathrm{d}\, x = \underbrace{\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{|x|} \sin x \,\mathrm{d}\, x}_{A} + \underbrace{\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{|x|} \cos x \,\mathrm{d}\, x}_{B}.$$

Нека $f(x) = e^{|x|} \sin x$ и $g(x) = e^{|x|} \cos x$.

Проверяваме, че

$f(x) = e^{|x|} \sin x = e^{|-x|} \sin x = -\left(e^{|-x|} \sin(-x)\right) = -f(-x)$ и
$g(x) = e^{|x|} \cos x = e^{|-x|} \cos(-x) = g(-x)$.

Следователно f е нечетна функция, а g е четна функция. Това означава, че $A = 0$, а $B = 2\int_0^{\frac{\pi}{2}} e^{|x|} \cos x \,\mathrm{d}\, x = 2\int_0^{\frac{\pi}{2}} e^x \cos x \,\mathrm{d}\, x$, тъй като $e^{|x|} = e^x$ за $x \in \left[0, \frac{\pi}{2}\right]$.

Нека $J = \int_0^{\frac{\pi}{2}} e^x \cos x \,\mathrm{d}\, x$. Тогава $I = B = 2J$.

$$J = \int_0^{\frac{\pi}{s}} e^x \cos x \,\mathrm{d}\, x = \int_0^{\frac{\pi}{2}} \cos x \,\mathrm{d}\, e^x \overset{\text{И.Ч.}}{=} e^x \cos x \Big|_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} e^x \,\mathrm{d} \cos x =$$

$$= e^{\frac{\pi}{2}} \cos \frac{\pi}{2} - e^0 \cos 0 + \int_0^{\frac{\pi}{2}} e^x \sin x \,\mathrm{d}\, x = e^{\frac{\pi}{2}} \times 0 - 1 \times 1 + \int_0^{\frac{\pi}{2}} \sin x \,\mathrm{d}\, e^x \overset{\text{И.Ч.}}{=}$$

$$= -1 + e^x \sin x \Big|_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} e^x \,\mathrm{d} \sin x = -1 + e^{\frac{\pi}{2}} \sin \frac{\pi}{2} - e^0 \sin 0 - \int_0^{\frac{\pi}{2}} e^x \cos x \,\mathrm{d}\, x =$$

$$= -1 + e^{\frac{\pi}{2}} - 0 - J$$

Следователно, $J = -1 + e^{\frac{\pi}{2}} - J \Rightarrow \underline{\underline{e^{\frac{\pi}{2}} - 1}} = 2J = B = I$, което търсехме.

$\square$