ПРОИЗВОДНИ

Function $f(x)$	First derivative $f'(x)$, $f^{(1)}(x)$, $\dfrac{\partial f(x)}{\partial x}$
x^n	$n\,x^{n-1}$
a^x	$a^x \ln(a)$
e^x	e^x
$\ln x$	$\dfrac{1}{x}$
$\log_a x$	$\left(\dfrac{\ln x}{\ln a}\right)' = \dfrac{1}{x\ln(a)}$
$\sin(x)$	$\cos(x)$
$\cos(x)$	$-\sin(x)$
$\tan(x)$	$\dfrac{1}{\cos^2(x)}$
$\text{cotan}(x)$	$-\dfrac{1}{\sin^2(x)}$
$\arcsin(x)$	$\dfrac{1}{\sqrt{1-x^2}}$
$\arccos(x)$	$-\dfrac{1}{\sqrt{1-x^2}}$
$\arctan(x)$	$\dfrac{1}{1+x^2}$
$\text{arccotan}(x)$	$-\dfrac{1}{1+x^2}$
$\sec(x)$	$\left(\dfrac{1}{\cos(x)}\right)' = \sec(x)\,.\,\tan(x) = \dfrac{\tan(x)}{\cos(x)} = \dfrac{\sin(x)}{\cos^2(x)}$
$\csc(x)$	$\left(\dfrac{1}{\sin(x)}\right)' = -\csc(x)\,.\,\text{cotan(x)} = -\dfrac{\text{cotan(x)}}{\sin(x)} = -\dfrac{\cos(x)}{\sin^2(x)}$
$\text{arcsec}(x)$	$\dfrac{1}{\lvert x\rvert\sqrt{x^2-1}}$
$\text{arccsc}(x)$	$-\dfrac{1}{\lvert x\rvert\sqrt{x^2-1}}$

ПРАВИЛА И ТЪЖДЕСТВА

1. $(fg)' = f'g + fg'$
2. $\dfrac{f}{g} = \dfrac{f'g - fg'}{g^2}$
3. $\arcsin(-x) = -\arcsin(x)$
4. $\arccos(-x) = \pi - \arccos(x)$
5. $\arctan(x) = -\arctan(x)$
6. $\text{arccotan}(-x) = \pi - \text{arccotan}(x)$
7. $\arcsin(x) = \dfrac{\pi}{2} - \arccos(x) \Leftrightarrow$
   $\arcsin(x) + \arccos(x) = \dfrac{\pi}{2}$
8. $\arctan(x) + \text{arccotan}(x) = \dfrac{\pi}{2}$
9. $\arcsin(x) = y,\ x \in [-1, 1],$
   $y \in \left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right],\ x = \sin(y)$
10. $\arccos(x) = y,\ x \in [-1, 1],$
    $y \in [0, \pi],\ x = \cos(y)$
11. $\arctan(x) = y,\ x \in (-\infty, +\infty),$
    $y \in \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right),\ x = \tan(y)$
12. $\text{arccotan}(x) = y,\ x \in (-\infty, +\infty),$
    $y \in (0, \pi),\ x = \text{cotan}(y)$
13. $\arcsin(x) = \arccos\sqrt{1-x^2},$
    $0 \le x \le 1$
14. $\arcsin(x) = -\arccos\sqrt{1-x^2},$
    $-1 \le x \le 0$
15. $\arccos(x) = \arcsin\sqrt{1-x^2},$
    $0 \le x \le 1$
16. $\arccos(x) = \pi - \arcsin\sqrt{1-x^2},$
    $-1 \le x \le 0$
17. $\arcsin(x) = \arctan\left(\dfrac{x}{\sqrt{1-x^2}}\right),$
    $x^2 < 1$
18. $\arctan(x) = \arctan\left(\dfrac{1}{x}\right),\ x > 0$
19. $\arctan(x) = \arctan\left(\dfrac{1}{x}\right) - \pi,$
    $x < 0$

ИНТЕГРАЛИ

Function $f(x)$	Integral $\int f(x)\ \mathrm{d}\,x$
x^n	$\frac{x^{n+1}}{n+1} + C, n \neq -1$
$\frac{1}{x}$	$\ln\lvert x \rvert + C$
a^x	$\frac{a^x}{\ln a} + C$
e^x	$\mathrm{d}\,x = e^x + C$
$\sin(x)$	$-\cos(x) + C$
$\cos(x)$	$\sin(x) + C$
$\tan(x)$	$\int \frac{\sin(x)}{\cos(x)}\mathrm{d}\,x = -\int \frac{\mathrm{d}\cos(x)}{\cos(x)} = -\ln\lvert\cos(x)\rvert + C$
$\mathrm{cotan}(x)$	$\int \frac{\cos(x)}{\sin(x)}\mathrm{d}\,x = \int \frac{\mathrm{d}\sin(x)}{\sin(x)} = \ln\lvert\sin(x)\rvert + C$
$\frac{1}{\sin^2(x)}$	$-\mathrm{cotan}(x) + C$
$\frac{1}{\cos^2(x)}$	$\tan(x) + C$
$\frac{1}{a^2+x^2}$	$\frac{1}{a}\arctan\left(\frac{x}{a}\right) + C$
$\frac{1}{\sqrt{x^2 \pm a^2}}$	$\ln\lvert x + \sqrt{x^2 \pm a^2}\rvert + C$
$\frac{1}{\sqrt{a^2 - x^2}}$	$\frac{1}{\sin\frac{x}{a}}$

ФОРМУЛИ ЗА УНИВЕРСАЛНА СУБСТИТУЦИЯ

$t = \tan\frac{x}{2},\ -\frac{\pi}{2} < t < \frac{\pi}{2}$
$\sin\frac{x}{2} = \frac{t}{\sqrt{1+t^2}}$
$\cos\frac{x}{2} = \frac{1}{\sqrt{1+t^2}}$
$\sin x = \frac{2t}{1+t^2}$
$\cos x = \frac{1-t^2}{1+t^2}$
$\mathrm{d}\frac{x}{2} = \mathrm{d}\arctan t = \frac{1}{1+t^2}\mathrm{d}\,t \Rightarrow \mathrm{d}\,x = \frac{2}{1+t^2}\mathrm{d}\,t$

НЕПРАВИЛНИ ИНТЕГРАЛИ

Определен в интервала $(a, b]$ или $[a, b)$ интеграл.

Ако $\lim_{x\to b^-} f(x) = \infty$, то $\int_a^b f(x)\mathrm{d}\,x = \lim_{t\to b^-}\int_a^t f(x)\mathrm{d}\,x$

Ако $\lim_{x\to a^+} f(x) = \infty$, то $\int_a^b f(x)\mathrm{d}\,x = \lim_{t\to a^+}\int_t^b f(x)\mathrm{d}\,x$

Примери:

$$\int_1^2 \frac{x}{\sqrt{x^2-1}}\mathrm{d}\,x = \lim_{t\to 1^+}\int_t^2 \frac{x}{\sqrt{x^2-1}}\mathrm{d}\,x$$

$$\int_1^6 \frac{1}{x-4}\,\mathrm{d}\,x = \lim_{t\to 4^-}\int_1^t \frac{1}{x-4}\,\mathrm{d}\,x + \lim_{t\to 4^+}\int_t^6 \frac{1}{x-4}\,\mathrm{d}\,x$$

ДЪЛЖИНА НА ГРАФИКА НА ФУНКЦИЯ

$$\underset{f\in[a,b]}{\mathrm{arclen}}\big(f(x)\big) = \int_a^b \sqrt{1+\big(f'(x)\big)^2\mathrm{d}\,x}$$

ТРИГОНОМЕТРИЧНИ ЗАВИСИМОСТИ

1) $\sin^2\theta + \cos^2\theta = 1$
2) $\cos(-\theta) = \cos(\theta)$, cos е четна функция
   $\sin(-\theta) = -\sin(\theta)$, sin е нечетна функция
3) $\sin(A+B) = \sin A \,.\cos B + \cos A\,.\sin B$
   $\cos(A+B) = \cos A\,.\cos B - \sin A\,.\sin B$
   $\tan(\alpha+\beta) = \dfrac{\tan\alpha + \tan\beta}{1-\tan\alpha\,\tan\beta}$
   $\tan(\alpha-\beta) = \dfrac{\tan\alpha - \tan\beta}{1+\tan\alpha\,\tan\beta}$
   $\tan\dfrac{\theta}{2} = \dfrac{\sin\theta}{1+\cos\theta} = \dfrac{1-\cos\theta}{\sin\theta}$
4) $\tan^2\theta + 1 = \dfrac{1}{\cos^2\theta}$, $1+\mathrm{cotan}^2\theta = \dfrac{1}{\sin^2\theta}$
5) $\sin(2\theta) = \sin(\theta+\theta) = 2\sin\theta\,.\cos\theta$
   $\cos(2\theta) = \cos(\theta+\theta) = \cos^2\theta - \sin^2\theta = 1-2\sin^2\theta = 2\cos^2\theta - 1$
6) $\sin^2\theta = \dfrac{1-\cos 2\theta}{2}$
   $\cos^2\theta = \dfrac{1+\cos 2\theta}{2}$

Други:

7) $\sec\theta = \dfrac{1}{\sin\theta}$, $\csc\theta = \dfrac{1}{\sin\theta}$
8) $\cos\theta = \sin\left(\dfrac{\pi}{2}-\theta\right)$, $\sin\theta = \cos\left(\dfrac{\pi}{2}-\theta\right)$, $\tan\theta = \mathrm{cotan}\left(\dfrac{\pi}{2}-\theta\right)$,
   $\mathrm{cotan} = \tan\theta\left(\dfrac{\pi}{2}-\theta\right)$
9) $\sin(\theta\pm 2\pi) = \sin\theta$, $\cos(\theta\pm 2\pi) = cos(\theta)$, $\tan(\theta\pm\pi) = \tan\theta$
10) $\sin\alpha + \cos\alpha = \sqrt{2}\cos\left(\dfrac{\pi}{4}-\alpha\right) = \sqrt{2}\sin\left(\dfrac{\pi}{2}+\alpha\right)$

ЛОГАРИТМИЧНИ ЗАВИСИМОСТИ

1) $a^x = b \Leftrightarrow \log_a a^x = \log_a b \Leftrightarrow x = \log_a b$

2) $\log_a x^b = b \log_a x$

3) $\log_a x^{2b} = 2b \log_a |x|$

4) $\log_a b = \dfrac{\log_c b}{\log_c a}$

5) $\log_a b = \log_{a^k} b^k$

6) $\log_{a^k} b^m = \dfrac{m}{k} \log_a b, b > 0, k \neq 0$

ТЪЖДЕСТВА

1) $a^3 \pm b^3 = (a+b)(a^2 \mp ab + b^2)$

2) $(a+b)^n = a^n + \binom{n}{1} a^{n-1}b + \binom{n}{2} a^{n-2}b^2 + \ldots + \binom{n}{k} a^{n-k}b^k + \ldots + b^n$

3) $\binom{n}{k} = C_n^k = \dfrac{n(n-1)\ldots(n-k+1)}{k!}$

ИНТЕГРАЛИ НА ЧЕТНИ/НЕЧЕТНИ ФУНКЦИИ

Ако $f(-x) = f(x)$, то $f(x)$ е четна функция и тогава $\displaystyle\int_{-a}^{a} f(x)\mathrm{d}\,x = 2\int_0^a f(x)\mathrm{d}\,x$.

Ако $f(-x) = -f(x)$, то $f(x)$ е нечетна. Тогава $\displaystyle\int_{-a}^{a} f(x)\mathrm{d}\,x = 0$.

Някои интересни интеграли и как се решават:

$$\begin{cases} \int x^k \arcsin(x)\mathrm{d}\,x \\ \int x^k \arctan(x)\mathrm{d}\,x, \text{ чрез интегриране по части и внасяне на } x^k \text{ под диференциала.} \\ \int x^k \ln x \,\mathrm{d}\,x \end{cases}$$

$$\begin{cases} \int x^k \sin(ax+c)\mathrm{d}\,x \\ \int x^k \cos(ax+c)\mathrm{d}\,x, \text{ чрез интегриране по части и внасяне на } \sin, \cos, e^x \text{ под} \\ \int x^k e^{ax}\,\mathrm{d}\,x \end{cases}$$

диференциала толкова пъти, колкото е степента на x, т.е. k пъти.

ПРАВИЛО НА ХОРНЕР

Пример 1) $x^3 - 2x^2 - 5x + 6 = 0$. Потенциални корени: $\pm 1, \pm 2, \pm 3, \pm 6$.

	1	-2	-5	6
1	1	-1	-6	0

На първия ред на таблицата стоят коефициентите на полинома. На втория ред от таблицата, на първата колона поставяме потенциалният корен, след което на втората колона преписваме първият коефициент от горния ред. Прилагаме процедурата на Хорнер.

$$1 \times 1 - 2 = -1,\ 1 \times (-1) - 5 = -6,\ 1 \times (-6) + 6 = 0$$

Следователно $x = 1$ е корен и $x^3 - 2x^2 - 5x + 6 = (x-1)(x^2 - x - 6)$.

Пример 2) $3x^3 + 3x^2 - 2x - 1 = 0$

Потенциални корени: $\pm 1, \pm \dfrac{1}{3}$.

	3	4	-2	-1
-4	3	-8	30	-121 $\neq$ 0

Следователно $x = -4$ не е корен. -121 е остатък при деление на $3x^3 + 4x^2 - 2x - 1$ с $x + 4$. Тоест $\dfrac{3x^3 + 4x^2 - 2x - 1}{x - 4} = 3x^2 - 8x + 30 - \dfrac{121}{2x^3 + 4x^2 - 2x - 1}$.

ХИЛЯДНИК

$$\int \frac{\mathrm{d}\,x}{(x^2+a^2)^2},\ a \neq 0$$

$$I = \int \frac{\mathrm{d}\,x}{(x^2+a^2)^2} = \int \frac{a^2 + x^2 - x^2}{a^2(x^2+a^2)}\,\mathrm{d}\,x = \frac{1}{a^2}\int \frac{(a^2+x^2) - x^2}{(x^2+a^2)^2}\,\mathrm{d}\,x =$$

$$= \frac{1}{a^2}\int \frac{1}{x^2+a^2} - \frac{x^2}{(x^2+a^2)^2}\,\mathrm{d}\,x = \frac{1}{a^2}\underbrace{\int \frac{1}{x^2+a^2}\,\mathrm{d}\,x}_{\text{табличен}} - \frac{1}{a^2}\underbrace{\int \frac{x^2}{(x^2+a^2)^2}\,\mathrm{d}\,x}_{J}.$$

$$J = \frac{1}{2}\int \frac{2x\,.\,x}{(x^2+a^2)^2}\,\mathrm{d}\,x = \frac{1}{2}\int \frac{x}{(x^2+a^2)}\,\mathrm{d}\,x^2 = \frac{1}{2}\int \frac{x}{(x^2+a^2)^2}\,\mathrm{d}(x^2+a^2) \overset{(*)}{=}$$

$$= \frac{1}{2}\int x\,\mathrm{d}\left(-\frac{1}{x^2+a^2}\right) = -\frac{1}{2}\int x\,\mathrm{d}\left(\frac{1}{x^2+a^2}\right) \overset{\text{и.ч.}}{=} -\frac{1}{2}\,.\,x\,.\,\frac{1}{x^2+a^2} + \frac{1}{2}\int \frac{1}{x^2+a^2}\,\mathrm{d}\,x\,.$$

$(*)$ $\left(-\dfrac{1}{u}\right)' = \dfrac{1}{u^2}$, т.е. $\dfrac{f(u)\mathrm{d}\,u}{u^2} = f(u)\mathrm{d}\left(-\dfrac{1}{u}\right)$

Следователно,

$$I = \frac{1}{a^2}\int \frac{1}{x^2+a^2}\,\mathrm{d}\,x - \frac{1}{a^2}\left(-\frac{1}{2}x\,.\,\frac{1}{x^2+a^2} + \frac{1}{2}\int \frac{1}{x^2+a^2}\,\mathrm{d}\,x\right) =$$

$$= \frac{1}{a^2}\int \frac{1}{x^2+a^2}\,\mathrm{d}\,x\left(1-\frac{1}{2}\right) + \frac{x}{2a^2(x^2+a^2)} = \frac{1}{2a^2}\left(\frac{1}{a}\arctan\left(\frac{x}{a}\right) + \frac{x}{x^2+a^2}\right) + C\,.$$

ТРИГОНОМЕТРИЯ

1) $\sin(2x)\cos(x) = ?$

$$+\begin{cases}\sin(2x+x) = \sin(2x)\cos(x) + \cos(2x)\sin(x)\\ \sin(2x-x) = \sin(2x)\cos(x) - \cos(2x)\sin(x)\end{cases}$$

$$\Rightarrow \sin(3x) + \sin(x) = 2\sin(2x)\cos(x)$$

2) $\cos(x)\cos(2x) = ?$

$$+\begin{cases}\cos(x+2x) = \cos(x)\cos(2x) - \sin(x)\sin(2x)\\ \cos(x-2x) = \cos(x)\cos(2x) + \sin(x)\sin(2x)\end{cases}$$

$$\Rightarrow \cos(3x) + \cos(-x) = \cos(3x) + \cos(x) = 2\cos(x)\cos(2x)$$

3) $\sin(x)\sin(2x) = ?$

$$-\begin{cases}\cos(x+2x) = \cos(x)\cos(2x) - \sin(x)\sin(2x)\\ \cos(x-2x) = \cos(x)\cos(2x) + \sin(x)\sin(2x)\end{cases}$$

$$\Rightarrow \cos(3x) - \cos(x) = -\,2\sin(x)\sin(2x)$$

ИНТЕРЕСНИ ИНТЕГРАЛИ

1) $\displaystyle\int_0^{\frac{1}{2}} \arcsin x\,\mathrm{d}\,x$

I-ви начин: Интегриране по части.
II-ри начин: Полагаме $x = \sin(t)$, $\mathrm{d}\,x = \mathrm{d}\sin(t) = \cos(t)\mathrm{d}\,t$
Ако $x = 0 \Rightarrow \sin(t) = 0$, за $t = 0$
Ако $x = \dfrac{1}{2} \Rightarrow \sin(t) = \dfrac{1}{2}$, за $t = \dfrac{\pi}{6}$
Следователно,

$$I_1 = \int_0^{\frac{1}{2}} \arcsin x\,\mathrm{d}\,x = \int_0^{\frac{\pi}{6}} \arcsin\big(\sin(t)\big)\,\mathrm{d}\sin(t) = \int_0^{\frac{\pi}{6}} t\,\mathrm{d}\sin(t) \overset{\text{И.Ч.}}{=}$$

$$= t\sin(t)\Big|_0^{\frac{\pi}{6}} - \int_0^{\frac{\pi}{6}}\sin(t)\mathrm{d}\,t = \frac{\pi}{6}\,.\,\sin\frac{\pi}{6} - 0 - \int_0^{\frac{\pi}{6}}\sin(t)\mathrm{d}\,t =$$

$$= \frac{\pi}{6}\times\frac{1}{2} + \cos(t)\Big|_0^{\frac{\pi}{6}} = \frac{\pi}{12} + \cos\frac{\pi}{6} - \cos 0 = \frac{\pi}{12} + \frac{\sqrt{3}}{2} - 1.$$

2) $\displaystyle\int x\cos x\sin x\,\mathrm{d}\,x$

$$I_2=\int x\sin(x)\cos(x)\mathrm{d}\,x=\frac{1}{2}\int x.2\sin(x)\cos(x)\mathrm{d}\,x=\frac{1}{2}\int x\sin(2x)\mathrm{d}\,x=$$
$$=\frac{1}{4}\int x\sin(2x)\mathrm{d}(2x)=-\frac{1}{4}\int x\,\mathrm{d}\cos(2x)\overset{\text{И.Ч.}}{=}-\frac{1}{4}x\cos(2x)+\frac{1}{4}\int\cos(2x)\mathrm{d}\,x=$$
$$=-\frac{1}{4}x\cos(2x)+\frac{1}{8}\int\cos(2x)\mathrm{d}\,2x=-\frac{x}{4}\cos(2x)+\frac{1}{8}\sin(2x)+C\,.$$

3) $\displaystyle\int x^2\cos(x)\sin(x)\mathrm{d}\,x$

$$I_3=\int x^2\cos(x)\sin(x)\mathrm{d}\,x=\frac{1}{2}\int x^2\sin(2x)\mathrm{d}\,x=\frac{1}{4}\int x^2\sin(2x)\mathrm{d}(2x)=-\frac{1}{4}\int x^2\,\mathrm{d}\cos(2x)\overset{\text{И.Ч.}}{=}$$
$$=-\frac{1}{4}\left(x^2\cos(2x)-\int\cos(2x)\mathrm{d}\,x^2\right)=-\frac{1}{4}\left(x^2\cos(2x)-\int 2x\cos(2x)\mathrm{d}\,x\right)=$$
$$=-\frac{1}{4}\left(x^2\cos(2x)-\underbrace{\int x\,\mathrm{d}\sin(2x)}_{J}\right)$$
$$J\overset{\text{И.Ч.}}{=}x\sin(2x)-\int\sin(2x)\mathrm{d}\,x=x\sin(2x)-\frac{1}{2}\int\sin(2x)\mathrm{d}(2x)=$$
$$=x\sin(2x)+\frac{1}{2}\cos(2x)+C_1$$

Следователно,
$$I_3=\int x^2\cos(x)\sin(x)\mathrm{d}\,x=-\frac{1}{4}\left(x^2\cos(2x)-x\sin(2x)-\frac{1}{2}\cos(2x)-C_1\right)=$$
$$=-\frac{1}{4}\left(\left(x^2-\frac{1}{2}\right)\cos(2x)-x\sin(2x)-C_1\right)=-\frac{1}{8}\left[(2x^2-1)\cos(2x)-2x\sin(2x)\right]+C=$$
$$=\frac{2x\sin(2x)-(2x^2-1)\cos(2x)}{8}+C=\frac{2x\sin(2x)+(1-2x^2)\cos(2x)}{8}+C\,.$$

4) $\displaystyle\int x\sin x\cos 2x\,\mathrm{d}\,x$

Тъй като $\sin(x\pm 2x)=\sin(x)\cos(2x)\pm\cos(x)\sin(2x)$, то
$\sin(x+2x)+\sin(x-2x)=2\sin(x)\cos(2x)$, $\dfrac{\sin(3x)-\sin(x)}{2}=\sin(x)\cos(2x)$.

Следователно,

$$I_4 = \int x \sin(x)\cos(2x)\mathrm{d}\, x \;=\; \frac{1}{2}\int x\left(\sin(3x) - \sin(x)\right)\mathrm{d}\, x \;=$$

$$= \frac{1}{2}\underbrace{\int x \sin(3x)\mathrm{d}\, x}_{J_1} - \frac{1}{2}\underbrace{\int x \sin(x)\mathrm{d}\, x}_{J_2}$$

$$J_1 = \frac{1}{3}\int x \sin(3x)\mathrm{d}\, 3x \;=\; -\frac{1}{3}\int x\, \mathrm{d}\cos(3x) \overset{\text{И.Ч.}}{=} -\frac{1}{3}x\cos(3x) + \frac{1}{3}\int \cos(3)\mathrm{d}\, x \;=$$

$$= -\frac{1}{3}x\cos(3x) + \frac{1}{9}\int \cos(3x)\mathrm{d}\, 3x \;=\; -\frac{1}{3}x\cos(3x) + \frac{1}{9}\sin(3x) + C_1\,.$$

$$J_2 = \int x \sin(x)\mathrm{d}\, x \;=\; -\int x\, \mathrm{d}\cos(x) \overset{\text{И.Ч.}}{=} -x\cos(x) + \int \cos(x)\mathrm{d}\, x \;=$$

$$= -x\cos(x) + \sin(x) + C_2\,.$$

Окончателно, $I_4 = \dfrac{1}{2}\left(-\dfrac{1}{3}x\cos(3x) + \dfrac{1}{9}\sin(3x)\right) - \dfrac{1}{2}\left(-x\cos(x) + \sin(x)\right) + C.$

5)

$$I_5 = \int \frac{3x+2}{\underbrace{x^2+2x+4}_{\text{неразложим полином}}}\mathrm{d}\, x \overset{\substack{\text{отделяне}\\ \text{на точен}\\ \text{квадрат}}}{=} \int \frac{3x+2}{x^2+2x+1+3}\mathrm{d}\, x = \int \frac{3x+2}{(x+1)^2+3}\mathrm{d}\, x =$$

$$= \int \frac{3x+3-1}{(x+1)^2+3}\mathrm{d}\, x = \int \frac{3(x+1)-1}{(x+1)^2+3}\mathrm{d}\, x = \int \frac{3(x+1)}{(x+1)^2+3}\mathrm{d}\, x - \int \frac{1}{(x+1)^2+3}\mathrm{d}\, x =$$

$$= \frac{3}{2}\int \frac{2(x+1)}{(x+1)^2+3}\mathrm{d}(x+1) - \int \frac{1}{(x+1)^2+3}\mathrm{d}(x+1) =$$

$$= \frac{3}{2}\underbrace{\int \frac{\mathrm{d}(x+1)^2}{(x+1)^2+3}}_{I} - \underbrace{\int \frac{1}{(x+1)^2+3}\mathrm{d}(x+1)}_{J}.$$

$$J_1 \overset{u=(x+1)^2}{=} \int \frac{\mathrm{d}\, u}{u+3} = \int \frac{\mathrm{d}\, u+3}{u+3} = \ln|\, u+3\,| + C_1 =$$

$$= \ln\left((x+1)^2+3\right) + C_2 = \ln(x^2+2x+4) + C_1$$

$$J_2 \overset{u=(x+1)}{=} \frac{1}{\sqrt{3}}\arctan\frac{u}{\sqrt{3}} + C_2 = \frac{1}{\sqrt{3}}\arctan\frac{x+1}{\sqrt{3}} + C_2$$

$$\Rightarrow I_5 = \frac{3}{2}\ln(x^2+2x+4) - \frac{\sqrt{3}}{3}\arctan\frac{x+1}{\sqrt{3}} + C.$$

6) $\displaystyle\int \frac{\mathrm{d}\,x}{\sqrt[3]{x}+\sqrt{x}} = ?$

Полагаме $x = t^6 \Rightarrow \sqrt[3]{x} = t^2, \sqrt{x} = t^3, \mathrm{d}\,x = \mathrm{d}\,t^6 = 6t^5\,\mathrm{d}\,t$

$$I_6 = \int \frac{\mathrm{d}\,x}{\sqrt[3]{x}+\sqrt{x}} = \int \frac{1}{t^2+t^3} \times 6t^5\,\mathrm{d}\,t = \int \frac{1}{\cancel{t^2}(1+t)} \times 6t^{\cancel{5}3}\,\mathrm{d}\,t = 6\int \frac{t^3}{t+1}\,\mathrm{d}\,t$$

Делим полиноми:

$$\begin{array}{l}
t^3 : t+1 = t^2 - t + 1 \\
\underline{t^3 + t^2} \\
\quad -t^2 \\
\quad \underline{-t^2 - t} \\
\qquad\quad t \\
\qquad\quad \underline{t+1} \\
\qquad\quad\ -1 \leftarrow \text{остатък}
\end{array}$$

Следователно, $\dfrac{t^3}{t+1} = t^2 - t + 1 - \dfrac{1}{t+1}$.

$$I_6 = 6\int t^2\,\mathrm{d}\,t - 6\int t\,\mathrm{d}\,t + 6\int 1\,\mathrm{d}\,t - 6\int \frac{\mathrm{d}(t+1)}{t+1} =$$

$$= 6\frac{t^3}{3} - 6\frac{t^2}{2} + 6t - 6\ln|t+1| + C =$$

$$= 2t^3 = 3t^2 + 6t - 6\ln|t+1| + C = 2\sqrt{x} - 3\sqrt[3]{x} + 6\sqrt[6]{x} - 6\ln|\sqrt[6]{x}+1| + C.$$

7) $\displaystyle\int_0^{2\pi} \frac{1}{2+\sin x}\,\mathrm{d}\,x = ?$

Универсална субституция: $t = \tan\dfrac{x}{2}$, $t \in \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$, $\sin x = \dfrac{2t}{1+t^2}$, $\quad x = 2\arctan(t)$

$x = 2\arctan(t)$, $\mathrm{d}\,x = \mathrm{d}\,2\arctan(t) = \dfrac{2}{1+t^2}\,\mathrm{d}\,t$.

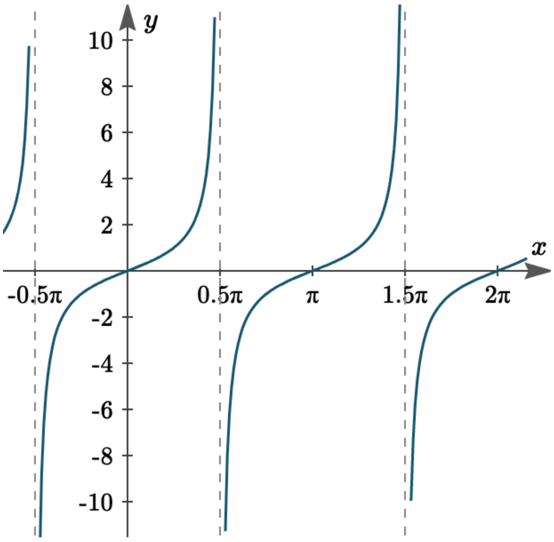
Пресмятаме интеграла като неопределен.

$$I_7 = \int \frac{1}{2+\sin x}\,\mathrm{d}\,x = \int \frac{1}{2+\frac{2t}{1+t^2}} \times \frac{2}{1+t^2}\,\mathrm{d}\,t = \int \frac{1}{\frac{2+2t^2+2t}{\cancel{1+t^2}}} \times \frac{2}{\cancel{1+t^2}}\,\mathrm{d}\,t =$$

$$= \int \frac{1}{1+t+t^2}\,\mathrm{d}\,t = \int \frac{1}{t^2+2\times\frac{1}{2}\times t+\frac{1}{4}+\frac{3}{4}}\,\mathrm{d}\,t = \int \frac{1}{\left(t+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\,\mathrm{d}\left(t+\frac{1}{2}\right) =$$

$$= \frac{1}{\frac{\sqrt{3}}{2}}\arctan\frac{t+\frac{1}{2}}{\frac{\sqrt{2}}{2}} + C = \frac{2}{\sqrt{3}}\arctan\frac{2t+1}{\sqrt{3}} + C = \frac{2}{\sqrt{3}}\arctan\frac{2\tan\frac{x}{2}+1}{\sqrt{3}} + C.$$

Ако $x \in [0,\, 2\pi]$, то $\dfrac{x}{2} \in [0,\, \pi]$. Да разгледаме графиката на tan $\dfrac{x}{2}$ за $\dfrac{x}{2} \in [0,\, \pi]$.



Разбиваме интеграла на $\dfrac{x}{2} \in \left[0,\, \dfrac{\pi}{2}\right]$ и $\dfrac{x}{2} \in \left(\dfrac{\pi}{2},\, \pi\right]$, което е еквивалентно на $x \in [0,\, \pi]$ и $x \in (\pi,\, 2\pi]$.

Следователно,

$$I = \int_0^{2\pi} \frac{1}{2+\sin x}\,\mathrm{d}\,x = \int_0^{\pi} \frac{1}{2+\sin x}\,\mathrm{d}\,x + \int_{\pi}^{2\pi} \frac{1}{2+\sin x}\,\mathrm{d}\,x =$$

$$= \lim_{b\to\pi^-} \int_0^{b} \frac{1}{2+\sin x}\,\mathrm{d}\,x + \lim_{b\to\pi^+} \int_b^{2\pi} \int \frac{1}{2+\sin x}\,\mathrm{d}\,x =$$

$$= \lim_{b\to\pi^-} \frac{2}{\sqrt{3}} \arctan \frac{2\tan\frac{x}{2}}{\sqrt{3}}\Bigg|_0^{b} + \lim_{b\to\pi^+} \frac{2}{\sqrt{3}} \arctan \frac{2\tan\frac{x}{2}+1}{\sqrt{3}}\Bigg|_b^{2\pi} =$$

$$= \lim_{c\to\infty} \frac{2}{\sqrt{3}} \arctan c - \frac{2}{\sqrt{3}} \arctan \frac{2\tan 0 + 1}{\sqrt{3}} + \frac{2}{\sqrt{3}} \arctan \frac{2\tan\frac{2\pi}{2}+1}{\sqrt{3}} - \lim_{c\to-\infty} \frac{2}{\sqrt{3}} \arctan c =$$

$$= \frac{2}{\sqrt{3}} \times \frac{\pi}{2} - \frac{2}{\sqrt{3}} \arctan \frac{1}{\sqrt{3}} + \frac{2}{\sqrt{3}} \arctan \frac{2\tan\pi + 1}{\sqrt{3}} - \frac{2}{\sqrt{3}}\left(-\frac{\pi}{2}\right) =$$

$$= \frac{2}{\sqrt{3}} \times \frac{\pi}{2} - \cancel{\frac{2}{\sqrt{3}} \arctan \frac{1}{\sqrt{3}}} + \cancel{\frac{2}{\sqrt{3}} \arctan \frac{1}{\sqrt{3}}} + \frac{2\pi}{\sqrt{3}\times 2} =$$

$$= \frac{\pi}{\sqrt{3}} + \frac{\pi}{\sqrt{3}} = \frac{2\pi}{\sqrt{3}} = \frac{2\pi\sqrt{3}}{3}.$$

Отговор: $\displaystyle\int_0^{2\pi} \frac{1}{2\sin x}\,\mathrm{d}\,x = \frac{2\pi\sqrt{3}}{3}$.

8) $\int e^{-x}(x^2+1)\mathrm{d}\,x = ?$

$$I_8 = \int e^{-x}(x^2+1)\mathrm{d}\,x = -\int x^2+1\,\mathrm{d}\,e^{-x} \overset{\text{И.Ч.}}{=} -e^{-x}(x^2+1) + \int e^{-x}\,\mathrm{d}\,x^2+1 =$$

$$= -e^{-x}(x^2+1) + \int 2xe^{-x}\,\mathrm{d}\,x = -e^{-x}(x^2+1) - 2\int x\,\mathrm{d}\,e^{-x} \overset{\text{И.Ч.}}{=}$$

$$= -e^{-x}(x^2+1) - 2xe^{-x} + 2\int e^{-x}\,\mathrm{d}\,x = -e^{-x}(x^2+1) - 2xe^{-x} - 2\int e^{-x}\,\mathrm{d}(-x) =$$

$$= -e^{-x}(x^2+1) - 2xe^{-x} - 2e^{-x} + C = -e^{-x}(x^2+2x+3) + C.$$

9) $\int \sqrt{1+x^2}\,\mathrm{d}\,x = ?$

$$I_9 = \int \sqrt{1+x^2}\,\mathrm{d}\,x \overset{\text{И.Ч.}}{=} x\sqrt{1+x^2} - \int x\,\mathrm{d}\sqrt{1+x^2} =$$

$$= x\sqrt{1+x^2} - \int x\left((1+x^2)^{\frac{1}{2}}\right)\mathrm{d}\,x = x\sqrt{1+x^2} - \int x \times \frac{1}{\cancel{2}} \times \frac{1}{\sqrt{1+x^2}} \times \cancel{2}\,\mathrm{d}\,x =$$

$$= x\sqrt{1-x^2} - \int \frac{x^2}{\sqrt{1+x^2}}\,\mathrm{d}\,x = x\sqrt{1+x^2} - \int \frac{1+x^2-1}{\sqrt{1+x^2}}\,\mathrm{d}\,x =$$

$$= x\sqrt{1+x^2} - \int \sqrt{1+x^2}\,\mathrm{d}\,x + \int \frac{1}{\sqrt{1+x^2}}\,\mathrm{d}\,x = x\sqrt{1+x^2} - I_9 + \ln|x+\sqrt{1+x^2}|.$$

Следователно,

$$2I_9 = x\sqrt{1+x^2} + \ln|x+\sqrt{1+x^2}| + C$$

$$I_9 = \int \sqrt{1+x^2}\,\mathrm{d}\,x = \frac{x\sqrt{1+x^2} + \ln|x+\sqrt{1+x^2}|}{2} + C$$

Забележка, интегралът $\int \sqrt{1-x^2}\,\mathrm{d}\,x$ се решава аналогично, с разликата, че получаваме $\arcsin$ от табличния интеграл.

10) $\int \sqrt{4-x^2}\,\mathrm{d}\,x = ?$

$$I_{10} = \int \sqrt{4-x^2}\,\mathrm{d}\,x \overset{\text{И.Ч}}{=} x\sqrt{4-x^2} - \int x\,\mathrm{d}\sqrt{4-x^2} =$$

$$= x\sqrt{4-x^2} - \int x\left(\sqrt{4-x^2}\right)'\mathrm{d}\,x = x\sqrt{4-x^2} - \int x\left(\frac{1}{2} \times \frac{1}{(4-x^2)} \times (-2x)\right)\mathrm{d}\,x =$$

$$= x\sqrt{4-x^2} + \int \frac{x^2}{\sqrt{4-x^2}} = x\sqrt{4-x^2} - \int \frac{4-x^2-4}{\sqrt{4-x^2}}\,\mathrm{d}\,x =$$

$$= x\sqrt{4-x^2} - I + 4\arcsin\left(\frac{x}{2}\right) + C.$$

Следователно, $I = \dfrac{x\sqrt{4-x^2} + 4\arcsin\left(\frac{x}{2}\right) + C}{2}$.

11) $\displaystyle\int_1^2 \frac{x+1}{\sqrt{x}(1+\sqrt{x})}\,\mathrm{d}\,x = ?$

Полагаме $x = t^2 \Rightarrow \mathrm{d}\,x = \mathrm{d}\,t^2 = 2t\,\mathrm{d}\,t$.
Ако $x = 1$, то $t = 1$. Ако $x = 2$, то $t = \sqrt{2}$.

$$I_{11} = \int_1^2 \frac{x+1}{\sqrt{x}(1+\sqrt{x})}\,\mathrm{d}\,x = \int_1^{\sqrt{2}} \frac{t^2+1}{\cancel{t}(1+t)} \times 2\cancel{t}\,\mathrm{d}\,t = 2\int_1^{\sqrt{2}} \frac{t^2+1}{t+1}\,\mathrm{d}\,t$$

Делим полиноми

$$\begin{array}{l} t^2+1 : t+1 = t \\ \underline{t^2+t} \\ \quad -t+1 \\ \quad \underline{-t-1} \\ \qquad\quad 2 \leftarrow \text{остатък} \end{array}$$

$$\Rightarrow \frac{t^2+1}{t+1} = t - 1 + \frac{2}{t+1}$$

Следователно,

$$I_{11} = 2\int_1^{\sqrt{2}} (t-1)\mathrm{d}(\mathrm{t}-1) + 2\int_1^{\sqrt{2}} \frac{2}{t+1}\,\mathrm{d}\,t =$$

$$= 2\frac{(t-1)^2}{2}\Bigg|_1^{\sqrt{2}} + 4\int_1^{\sqrt{2}} \frac{\mathrm{d}(t+1)}{t+1} =$$

$$= (\sqrt{2}-1)^2 - 0 + 4\ln|\,t+1\,|\Bigg|_1^{\sqrt{2}} = (\sqrt{2}-1)^2 + 4\ln(\sqrt{2}+1) - 4\ln 2 =$$

$$= 4\ln\frac{\sqrt{2}+1}{2} + 3 - 2\sqrt{2}\,.$$

12) $\displaystyle\int_1^3 \frac{\mathrm{d}\,x}{\sqrt{x}(x+1)^2}$.

Полагаме $x = t^2 \Rightarrow \mathrm{d}\,x = \mathrm{d}\,t^2 = 2t\,\mathrm{d}\,t$. Ако $x = 1$, то $t = 1$. Ако $x = 3$, то $t = \sqrt{3}$.

Следователно

$$I_{12} = \int_1^{\sqrt{3}} \frac{2\cancel{t}\,\mathrm{d}\,t}{\cancel{t}(t^2+1)^2} = 2\int_1^{\sqrt{3}} \frac{\mathrm{d}\,t}{(t^2+1)^2} = 2\int_1^{\sqrt{3}} \frac{1+t^2-t^2}{(t^2+1)^2}\,\mathrm{d}\,t =$$

$$= 2\int_1^{\sqrt{3}} \frac{1}{t^2+1}\,\mathrm{d}\,t - \underbrace{2\int_1^{\sqrt{3}} \frac{t^2}{(t^2+1)^2}\,\mathrm{d}\,t}_{J}.$$

$\dfrac{f(u)\mathrm{d}\,u}{u^2} = f(u)\mathrm{d}\left(-\dfrac{1}{u}\right)$, тъй като $\left(-\dfrac{1}{u}\right)' = \dfrac{1}{u^2}$

$$J = \int_1^{\sqrt{3}} \frac{2t \times t}{(t^2+1)^2}\,\mathrm{d}\,t = \int_1^{\sqrt{3}} \frac{t\,\mathrm{d}(t^2+1)}{(t^2+1)^2} = \int_1^{\sqrt{3}} t\,\mathrm{d}\left(-\frac{1}{t^2+1}\right) =$$

$$= -\int_1^{\sqrt{3}} t\,\mathrm{d}\,\frac{1}{t^2+1} \overset{\text{И.Ч.}}{=} -\frac{t}{t^2+1}\Bigg|_1^{\sqrt{3}} + \int_1^{\sqrt{3}} \frac{1}{t^2+1}\,\mathrm{d}\,t = -\frac{\sqrt{3}}{4} + \frac{1}{2} + \int_1^{\sqrt{3}} \frac{1}{t^2+1}\,\mathrm{d}\,t.$$

Следователно,

$$I_{12} = 2\arctan t\Bigg|_1^{\sqrt{3}} + \frac{\sqrt{3}}{4} - \frac{1}{2} - \arctan t\Bigg|_1^{\sqrt{3}} =$$

$$= \arctan\sqrt{3} - \arctan 1 + \frac{\sqrt{3}}{4} - \frac{1}{2} = \left(\frac{\pi}{3} - \frac{\pi}{4}\right) + \frac{\sqrt{3}}{4} - \frac{1}{2} = \frac{\pi + 3\sqrt{3} - 6}{12}.$$

Коментар: Задачата е от държавния изпит на специалност „Приложна математика“ провел се на 07/2014.