

## ПРОИЗВОДНИ

Function $f(x)$	First derivative $f'(x), f^{(1)}(x), \frac{\partial f(x)}{\partial x}$
$x^n$	$n x^{n-1}$
$a^x$	$a^x \ln(a)$
$e^x$	$e^x$
$\ln x$	$\frac{1}{x}$
$\log_a x$	$\left(\frac{\ln x}{\ln a}\right)' = \frac{1}{x \ln(a)}$
$\sin(x)$	$\cos(x)$
$\cos(x)$	$-\sin(x)$
$\tan(x)$	$\frac{1}{\cos^2(x)}$
$\cotan(x)$	$-\frac{1}{\sin^2(x)}$
$\arcsin(x)$	$\frac{1}{\sqrt{1-x^2}}$
$\arccos(x)$	$-\frac{1}{\sqrt{1-x^2}}$
$\arctan(x)$	$\frac{1}{1+x^2}$
$\operatorname{arccotan}(x)$	$-\frac{1}{1+x^2}$
$\sec(x)$	$\left(\frac{1}{\cos(x)}\right)' = \sec(x) \cdot \tan(x) = \frac{\tan(x)}{\cos(x)} = \frac{\sin(x)}{\cos^2(x)}$
$\csc(x)$	$\left(\frac{1}{\sin(x)}\right)' = -\csc(x) \cdot \cotan(x) = -\frac{\cotan(x)}{\sin(x)} = -\frac{\cos(x)}{\sin^2(x)}$
$\operatorname{arcsec}(x)$	$\frac{1}{ x  \sqrt{x^2-1}}$
$\operatorname{arccsc}(x)$	$-\frac{1}{ x  \sqrt{x^2-1}}$

## ПРАВИЛА И ТЪЖДЕСТВА

- $(fg)' = f'g + fg'$
- $\frac{f}{g} = \frac{f'g - fg'}{g^2}$
- $\arcsin(-x) = -\arcsin(x)$
- $\arccos(-x) = \pi - \arccos(x)$
- $\arctan(x) = -\arctan(x)$
- $\operatorname{arccotan}(-x) = \pi - \operatorname{arccotan}(x)$   
 $\arcsin(x) = \frac{\pi}{2} - \arccos(x) \Leftrightarrow$
- $\arcsin(x) + \arccos(x) = \frac{\pi}{2}$
- $\arctan(x) + \operatorname{arccotan}(x) = \frac{\pi}{2}$
- $\arcsin(x) = y, x \in [-1, 1],$   
 $y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right], x = \sin(y)$
- $\arccos(x) = y, x \in [-1, 1],$   
 $y \in [0, \pi], x = \cos(y)$
- $\arctan(x) = y, x \in (-\infty, +\infty),$   
 $y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), x = \tan(y)$
- $\operatorname{arccotan}(x) = y, x \in (-\infty, +\infty),$   
 $y \in (0, \pi), x = \cotan(y)$
- $\arcsin(x) = \arccos \sqrt{1-x^2},$   
 $0 \leq x \leq 1$
- $\arcsin(x) = -\arccos \sqrt{1-x^2},$   
 $-1 \leq x \leq 0$
- $\arccos(x) = \arcsin \sqrt{1-x^2},$   
 $0 \leq x \leq 1$
- $\arccos(x) = \pi - \arcsin \sqrt{1-x^2},$   
 $-1 \leq x \leq 0$
- $\arcsin(x) = \arctan \left( \frac{x}{\sqrt{1-x^2}} \right),$   
 $x^2 < 1$
- $\arctan(x) = \arctan \left( \frac{1}{x} \right), x > 0$
- $\arctan(x) = \arctan \left( \frac{1}{x} \right) - \pi,$   
 $x < 0$

## ИНТЕГРАЛИ

Function $f(x)$	Integral $\int f(x) \, dx$
$x^n$	$\frac{x^{n+1}}{n+1} + C, n \neq -1$
$\frac{1}{x}$	$\ln x  + C$
$a^x$	$\frac{a^x}{\ln a} + C$
$e^x$	$dx = e^x + C$
$\sin(x)$	$-\cos(x) + C$
$\cos(x)$	$\sin(x) + C$
$\tan(x)$	$\int \frac{\sin(x)}{\cos(x)} dx = -\int \frac{d \cos(x)}{\cos(x)}$ $= -\ln \cos(x)  + C$
$\cotan(x)$	$\int \frac{\cos(x)}{\sin(x)} dx = \int \frac{d \sin(x)}{\sin(x)}$ $= \ln \sin(x)  + C$
$\frac{1}{\sin^2(x)}$	$-\cotan(x) + C$
$\frac{1}{\cos^2(x)}$	$\tan(x) + C$
$\frac{1}{a^2 + x^2}$	$\frac{1}{a} \arctan\left(\frac{x}{a}\right) + C$
$\frac{1}{\sqrt{x^2 \pm a^2}}$	$\ln x + \sqrt{x^2 \pm a^2}  + C$
$\frac{1}{\sqrt{a^2 - x^2}}$	$\frac{1}{\sin \frac{x}{a}}$

## ФОРМУЛИ ЗА УНИВЕРСАЛНА СУБСТИТУЦИЯ

$t = \tan \frac{x}{2}, -\frac{\pi}{2} < t < \frac{\pi}{2}$
$\sin \frac{x}{2} = \frac{t}{\sqrt{1+t^2}}$
$\cos \frac{x}{2} = \frac{1}{\sqrt{1+t^2}}$
$\sin x = \frac{2t}{1+t^2}$
$\cos x = \frac{1-t^2}{1+t^2}$
$d \frac{x}{2} = d \arctan t = \frac{1}{1+t^2} dt \Rightarrow$ $dx = \frac{2}{1+t^2} dt$

## НЕПРАВИЛНИ ИНТЕГРАЛИ

Определен в интервала  $(a, b]$  или  $[a, b)$  интеграл.

$$\text{Ако } \lim_{x \rightarrow b^-} f(x) = \infty, \text{ то } \int_a^b f(x) dx = \lim_{t \rightarrow b^-} \int_a^t f(x) dx$$

$$\text{Ако } \lim_{x \rightarrow a^+} f(x) = \infty, \text{ то } \int_a^b f(x) dx = \lim_{t \rightarrow a^+} \int_t^b f(x) dx$$

Примери:

$$\int_1^2 \frac{x}{\sqrt{x^2-1}} dx = \lim_{t \rightarrow 1^+} \int_t^2 \frac{x}{\sqrt{x^2-1}} dx$$

$$\int_1^6 \frac{1}{x-4} dx = \lim_{t \rightarrow 4^-} \int_1^t \frac{1}{x-4} dx + \lim_{t \rightarrow 4^+} \int_t^6 \frac{1}{x-4} dx$$

### ДЪЛЖИНА НА ГРАФИКА НА ФУНКЦИЯ

$$\text{arclen}(f(x)) = \int_a^b \sqrt{1 + (f'(x))^2} dx$$

$f \in [a, b]$

### ТРИГОНОМЕТРИЧНИ ЗАВИСИМОСТИ

- 1)  $\sin^2 \theta + \cos^2 \theta = 1$
- 2)  $\cos(-\theta) = \cos(\theta)$ ,  $\cos$  е четна функция  
 $\sin(-\theta) = -\sin(\theta)$ ,  $\sin$  е нечетна функция
- 3)  $\sin(A + B) = \sin A \cdot \cos B + \cos A \cdot \sin B$   
 $\cos(A + B) = \cos A \cdot \cos B - \sin A \cdot \sin B$   
 $\tan(\alpha + \beta) = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta}$   
 $\tan(\alpha - \beta) = \frac{\tan \alpha - \tan \beta}{1 + \tan \alpha \tan \beta}$   
 $\tan \frac{\theta}{2} = \frac{\sin \theta}{1 + \cos \theta} = \frac{1 - \cos \theta}{\sin \theta}$
- 4)  $\tan^2 \theta + 1 = \frac{1}{\cos^2 \theta}$ ,  $1 + \cotan^2 \theta = \frac{1}{\sin^2 \theta}$
- 5)  $\sin(2\theta) = \sin(\theta + \theta) = 2 \sin \theta \cdot \cos \theta$   
 $\cos(2\theta) = \cos(\theta + \theta) = \cos^2 \theta - \sin^2 \theta = 1 - 2 \sin^2 \theta = 2 \cos^2 \theta - 1$
- 6)  $\sin^2 \theta = \frac{1 - \cos 2\theta}{2}$   
 $\cos^2 \theta = \frac{1 + \cos 2\theta}{2}$

Други:

- 7)  $\sec \theta = \frac{1}{\sin \theta}$ ,  $\csc \theta = \frac{1}{\cos \theta}$
- 8)  $\cos \theta = \sin \left( \frac{\pi}{2} - \theta \right)$ ,  $\sin \theta = \cos \left( \frac{\pi}{2} - \theta \right)$ ,  $\tan \theta = \cotan \left( \frac{\pi}{2} - \theta \right)$ ,  
 $\cotan = \tan \theta \left( \frac{\pi}{2} - \theta \right)$
- 9)  $\sin(\theta \pm 2\pi) = \sin \theta$ ,  $\cos(\theta \pm 2\pi) = \cos(\theta)$ ,  $\tan(\theta \pm \pi) = \tan \theta$
- 10)  $\sin \alpha + \cos \alpha = \sqrt{2} \cos \left( \frac{\pi}{4} - \alpha \right) = \sqrt{2} \sin \left( \frac{\pi}{2} + \alpha \right)$

## ЛОГАРИТМИЧНИ ЗАВИСИМОСТИ

$$1) a^x = b \Leftrightarrow \log_a a^x = \log_a b \Leftrightarrow x = \log_a b$$

$$2) \log_a x^b = b \log_a x$$

$$3) \log_a x^{2b} = 2b \log_a |x|$$

$$4) \log_a b = \frac{\log_c b}{\log_c a}$$

$$5) \log_a b = \log_{a^k} b^k$$

$$6) \log_{a^k} b^m = \frac{m}{k} \log_a b, b > 0, k \neq 0$$

## ТЪЖДЕСТВА

$$1) a^3 \pm b^3 = (a + b)(a^2 \mp ab + b^2)$$

$$2) (a + b)^n = a^n + \binom{n}{1} a^{n-1}b + \binom{n}{2} a^{n-2}b^2 + \dots + \binom{n}{k} a^{n-k}b^k + \dots + b^n$$

$$3) \binom{n}{k} = C_n^k = \frac{n(n-1)\dots(n-k+1)}{k!}$$

## ИНТЕГРАЛИ ОТ ЧЕТНИ/НЕЧЕТНИ ФУНКЦИИ

Ако  $f(-x) = f(x)$ , то  $f(x)$  е четна функция и тогава  $\int_{-a}^a f(x) dx = 2 \int_0^a f(x) dx$ .

Ако  $f(-x) = -f(x)$ , то  $f(x)$  е нечетна. Тогава  $\int_{-a}^a f(x) dx = 0$ .

Някои интересни интеграли и как се решават:

$$\left\{ \begin{array}{l} \int x^k \arcsin(x) dx \\ \int x^k \arctan(x) dx, \text{ чрез интегриране по части и внасяне на } x^k \text{ под диференциала.} \\ \int x^k \ln x dx \end{array} \right.$$

$$\left\{ \begin{array}{l} \int x^k \sin(ax + c) dx \\ \int x^k \cos(ax + c) dx, \text{ чрез интегриране по части и внасяне на } \sin, \cos, e^x \text{ под} \\ \int x^k e^{ax} dx \end{array} \right.$$

диференциала толкова пъти, колкото е степента на  $x$ , т.е.  $k$  пъти.

## ПРАВИЛО НА ХОРНЕР

**Пример 1)**  $x^3 - 2x^2 - 5x + 6 = 0$ . Потенциални корени:  $\pm 1, \pm 2, \pm 3, \pm 6$ .

	1	-2	-5	6
1	1	-1	-6	0

На първия ред на таблицата стоят коефициентите на полинома. На втория ред от таблицата, на първата колона поставяме потенциалният корен, след което на втората колона преписваме първият коефициент от горния ред. Прилагаме процедурата на Хорнер.

$$1 \times 1 - 2 = -1, 1 \times (-1) - 5 = -6, 1 \times (-6) + 6 = 0$$

Следователно  $x = 1$  е корен и  $x^3 - 2x^2 - 5x + 6 = (x - 1)(x^2 - x - 6)$ .

**Пример 2)**  $3x^3 + 3x^2 - 2x - 1 = 0$

Потенциални корени:  $\pm 1, \pm \frac{1}{3}$ .

	3	4	-2	-1
-4	3	-8	30	-121 $\neq 0$

Следователно  $x = -4$  не е корен.  $-121$  е остатък при деление на  $3x^3 + 4x^2 - 2x - 1$  с  $x + 4$ . Тоест  $\frac{3x^3 + 4x^2 - 2x - 1}{x - 4} = 3x^2 - 8x + 30 - \frac{121}{2x^3 + 4x^2 - 2x - 1}$ .

## ХИЛЯДНИК

$$\int \frac{dx}{(x^2 + a^2)^2}, a \neq 0$$

$$\begin{aligned} I &= \int \frac{dx}{(x^2 + a^2)^2} = \int \frac{a^2 + x^2 - x^2}{a^2(x^2 + a^2)^2} dx = \frac{1}{a^2} \int \frac{(a^2 + x^2) - x^2}{(x^2 + a^2)^2} dx = \\ &= \frac{1}{a^2} \int \frac{1}{x^2 + a^2} - \frac{x^2}{(x^2 + a^2)^2} dx = \frac{1}{a^2} \underbrace{\int \frac{1}{x^2 + a^2} dx}_{\text{табличен}} - \frac{1}{a^2} \underbrace{\int \frac{x^2}{(x^2 + a^2)^2} dx}_J \end{aligned}$$

$$\begin{aligned} J &= \frac{1}{2} \int \frac{2x \cdot x}{(x^2 + a^2)^2} dx = \frac{1}{2} \int \frac{x}{(x^2 + a^2)} dx^2 = \frac{1}{2} \int \frac{x}{(x^2 + a^2)^2} d(x^2 + a^2) \stackrel{(*)}{=} \\ &= \frac{1}{2} \int x d\left(-\frac{1}{x^2 + a^2}\right) = -\frac{1}{2} \int x d\left(\frac{1}{x^2 + a^2}\right) \stackrel{\text{и.ч.}}{=} -\frac{1}{2} \cdot x \cdot \frac{1}{x^2 + a^2} + \frac{1}{2} \int \frac{1}{x^2 + a^2} dx. \end{aligned}$$

$$(*) \quad \left(-\frac{1}{u}\right)' = \frac{1}{u^2}, \text{ т.е. } \frac{f(u)du}{u^2} = f(u)d\left(-\frac{1}{u}\right)$$

Следователно,

$$I = \frac{1}{a^2} \int \frac{1}{x^2 + a^2} dx - \frac{1}{a^2} \left( -\frac{1}{2}x \cdot \frac{1}{x^2 + a^2} + \frac{1}{2} \int \frac{1}{x^2 + a^2} dx \right) =$$

$$= \frac{1}{a^2} \int \frac{1}{x^2 + a^2} dx \left( 1 - \frac{1}{2} \right) + \frac{x}{2a^2(x^2 + a^2)} = \frac{1}{2a^2} \left( \frac{1}{a} \arctan \left( \frac{x}{a} \right) + \frac{x}{x^2 + a^2} \right) + C.$$

### ТРИГОНОМЕТРИ

1)  $\sin(2x)\cos(x) = ?$

$$+ \begin{cases} \sin(2x + x) = \sin(2x)\cos(x) + \cos(2x)\sin(x) \\ \sin(2x - x) = \sin(2x)\cos(x) - \cos(2x)\sin(x) \end{cases}$$

$$\Rightarrow \sin(3x) + \sin(x) = 2 \sin(2x)\cos(x)$$

2)  $\cos(x)\cos(2x) = ?$

$$+ \begin{cases} \cos(x + 2x) = \cos(x)\cos(2x) - \sin(x)\sin(2x) \\ \cos(x - 2x) = \cos(x)\cos(2x) + \sin(x)\sin(2x) \end{cases}$$

$$\Rightarrow \cos(3x) + \cos(-x) = \cos(3x) + \cos(x) = 2 \cos(x)\cos(2x)$$

3)  $\sin(x)\sin(2x) = ?$

$$- \begin{cases} \cos(x + 2x) = \cos(x)\cos(2x) - \sin(x)\sin(2x) \\ \cos(x - 2x) = \cos(x)\cos(2x) + \sin(x)\sin(2x) \end{cases}$$

$$\Rightarrow \cos(3x) - \cos(x) = -2 \sin(x)\sin(2x)$$

### ИНТЕРЕСНИ ИНТЕГРАЛИ

1)  $\int_0^{\frac{1}{2}} \arcsin x dx$

**I-ви начин:** Интегриране по части.

**II-ри начин:** Полагаме  $x = \sin(t)$ ,  $dx = d \sin(t) = \cos(t)dt$

Ако  $x = 0 \Rightarrow \sin(t) = 0$ , за  $t = 0$

Ако  $x = \frac{1}{2} \Rightarrow \sin(t) = \frac{1}{2}$ , за  $t = \frac{\pi}{6}$

Следователно,

$$\begin{aligned}
I_1 &= \int_0^{\frac{1}{2}} \arcsin x \, dx = \int_0^{\frac{\pi}{6}} \arcsin(\sin(t)) \, d\sin(t) = \int_0^{\frac{\pi}{6}} t \, d\sin(t) \stackrel{\text{И.Ч.}}{=} \\
&= t \sin(t) \Big|_0^{\frac{\pi}{6}} - \int_0^{\frac{\pi}{6}} \sin(t) \, dt = \frac{\pi}{6} \cdot \sin \frac{\pi}{6} - 0 - \int_0^{\frac{\pi}{6}} \sin(t) \, dt = \\
&= \frac{\pi}{6} \times \frac{1}{2} + \cos(t) \Big|_0^{\frac{\pi}{6}} = \frac{\pi}{12} + \cos \frac{\pi}{6} - \cos 0 = \frac{\pi}{12} + \frac{\sqrt{3}}{2} - 1.
\end{aligned}$$

$$2) \int x \cos x \sin x \, dx$$

$$\begin{aligned}
I_2 &= \int x \sin(x) \cos(x) \, dx = \frac{1}{2} \int x \cdot 2 \sin(x) \cos(x) \, dx = \frac{1}{2} \int x \sin(2x) \, dx = \\
&= \frac{1}{4} \int x \sin(2x) \, d(2x) = -\frac{1}{4} \int x \, d\cos(2x) \stackrel{\text{И.Ч.}}{=} -\frac{1}{4} x \cos(2x) + \frac{1}{4} \int \cos(2x) \, dx = \\
&= -\frac{1}{4} x \cos(2x) + \frac{1}{8} \int \cos(2x) \, d2x = -\frac{x}{4} \cos(2x) + \frac{1}{8} \sin(2x) + C.
\end{aligned}$$

$$3) \int x^2 \cos(x) \sin(x) \, dx$$

$$\begin{aligned}
I_3 &= \int x^2 \cos(x) \sin(x) \, dx = \frac{1}{2} \int x^2 \sin(2x) \, dx = \frac{1}{4} \int x^2 \sin(2x) \, d(2x) = -\frac{1}{4} \int x^2 \, d\cos(2x) \stackrel{\text{И.Ч.}}{=} \\
&= -\frac{1}{4} \left( x^2 \cos(2x) - \int \cos(2x) \, dx^2 \right) = -\frac{1}{4} \left( x^2 \cos(2x) - \int 2x \cos(2x) \, dx \right) = \\
&= -\frac{1}{4} \left( x^2 \cos(2x) - \underbrace{\int x \, d\sin(2x)}_J \right) \\
&\stackrel{\text{И.Ч.}}{=} x \sin(2x) - \int \sin(2x) \, dx = x \sin(2x) - \frac{1}{2} \int \sin(2x) \, d(2x) = \\
&= x \sin(2x) + \frac{1}{2} \cos(2x) + C_1
\end{aligned}$$

Следовательно,

$$\begin{aligned}
I_3 &= \int x^2 \cos(x) \sin(x) \, dx = -\frac{1}{4} \left( x^2 \cos(2x) - x \sin(2x) - \frac{1}{2} \cos(2x) - C_1 \right) = \\
&= -\frac{1}{4} \left( \left( x^2 - \frac{1}{2} \right) \cos(2x) - x \sin(2x) - C_1 \right) = -\frac{1}{8} [(2x^2 - 1)\cos(2x) - 2x \sin(2x)] + C = \\
&= \frac{2x \sin(2x) - (2x^2 - 1)\cos(2x)}{8} + C = \frac{2x \sin(2x) + (1 - 2x^2)\cos(2x)}{8} + C.
\end{aligned}$$

$$4) \int x \sin x \cos 2x \, dx$$

Тъй като  $\sin(x \pm 2x) = \sin(x)\cos(2x) \pm \cos(x)\sin(2x)$ , то  
 $\sin(x + 2x) + \sin(x - 2x) = 2 \sin(x)\cos(2x)$ ,  $\frac{\sin(3x) - \sin(x)}{2} = \sin(x)\cos(2x)$ .

Следователно,

$$I_4 = \int x \sin(x)\cos(2x) \, dx = \frac{1}{2} \int x (\sin(3x) - \sin(x)) \, dx =$$

$$= \frac{1}{2} \underbrace{\int x \sin(3x) \, dx}_{J_1} - \frac{1}{2} \underbrace{\int x \sin(x) \, dx}_{J_2}$$

$$J_1 = \frac{1}{3} \int x \sin(3x) \, d3x = -\frac{1}{3} \int x \, d\cos(3x) \stackrel{\text{И.Ч.}}{=} -\frac{1}{3} x \cos(3x) + \frac{1}{3} \int \cos(3x) \, d3x =$$

$$= -\frac{1}{3} x \cos(3x) + \frac{1}{9} \int \cos(3x) \, d3x = -\frac{1}{3} x \cos(3x) + \frac{1}{9} \sin(3x) + C_1.$$

$$J_2 = \int x \sin(x) \, dx = - \int x \, d\cos(x) \stackrel{\text{И.Ч.}}{=} -x \cos(x) + \int \cos(x) \, dx =$$

$$= -x \cos(x) + \sin(x) + C_2.$$

$$\text{Окончательно, } I_4 = \frac{1}{2} \left( -\frac{1}{3} x \cos(3x) + \frac{1}{9} \sin(3x) \right) - \frac{1}{2} (-x \cos(x) + \sin(x)) + C.$$

5)

$$I_5 = \int \underbrace{\frac{3x+2}{x^2+2x+4}}_{\substack{\text{неразложим} \\ \text{полином}}} \, dx \stackrel{\substack{\text{отделяне} \\ \text{на точен} \\ \text{квадрат}}}{=} \int \frac{3x+2}{x^2+2x+1+3} \, dx = \int \frac{3x+2}{(x+1)^2+3} \, dx =$$

$$= \int \frac{3x+3-1}{(x+1)^2+3} \, dx = \int \frac{3(x+1)-1}{(x+1)^2+3} \, dx = \int \frac{3(x+1)}{(x+1)^2+3} \, dx - \int \frac{1}{(x+1)^2+3} \, dx =$$

$$= \frac{3}{2} \int \frac{2(x+1)}{(x+1)^2+3} \, d(x+1) - \int \frac{1}{(x+1)^2+3} \, d(x+1) =$$

$$= \frac{3}{2} \underbrace{\int \frac{d(x+1)^2}{(x+1)^2+3}}_I - \underbrace{\int \frac{1}{(x+1)^2+3} \, d(x+1)}_J.$$

$$J_1 \stackrel{u=(x+1)^2}{=} \int \frac{du}{u+3} = \int \frac{du+3}{u+3} = \ln|u+3| + C_1 =$$

$$= \ln((x+1)^2+3) + C_2 = \ln(x^2+2x+4) + C_1$$



$$J_2 \stackrel{u=(x+1)}{=} \frac{1}{\sqrt{3}} \arctan \frac{u}{\sqrt{3}} + C_2 = \frac{1}{\sqrt{3}} \arctan \frac{x+1}{\sqrt{3}} + C_2$$

$$\Rightarrow I_5 = \frac{3}{2} \ln(x^2 + 2x + 4) - \frac{\sqrt{3}}{3} \arctan \frac{x+1}{\sqrt{3}} + C.$$

$$6) \int \frac{dx}{\sqrt[3]{x} + \sqrt{x}} = ?$$

$$\text{Полагаме } x = t^6 \Rightarrow \sqrt[3]{x} = t^2, \sqrt{x} = t^3, dx = dt^6 = 6t^5 dt$$

$$I_6 = \int \frac{dx}{\sqrt[3]{x} + \sqrt{x}} = \int \frac{1}{t^2 + t^3} \times 6t^5 dt = \int \frac{1}{t^2(1+t)} \times 6t^5 dt = 6 \int \frac{t^3}{t+1} dt$$

Делим полиноми:

$$t^3 : t + 1 = t^2 - t + 1$$

$$\begin{array}{r} t^3 + t^2 \\ -t^2 \\ \hline -t^2 - t \\ t \\ \hline t + 1 \\ -1 \leftarrow \text{остатък} \end{array}$$

$$\text{Следователно, } \frac{t^3}{t+1} = t^2 - t + 1 - \frac{1}{t+1}.$$

$$I_6 = 6 \int t^2 dt - 6 \int t dt + 6 \int 1 dt - 6 \int \frac{d(t+1)}{t+1} =$$

$$= 6 \frac{t^3}{3} - 6 \frac{t^2}{2} + 6t - 6 \ln|t+1| + C =$$

$$= 2t^3 = 3t^2 + 6t - 6 \ln|t+1| + C = 2\sqrt{x} - 3\sqrt[3]{x} + 6\sqrt[6]{x} - 6 \ln|\sqrt[6]{x} + 1| + C.$$

$$7) \int_0^{2\pi} \frac{1}{2 + \sin x} dx = ?$$

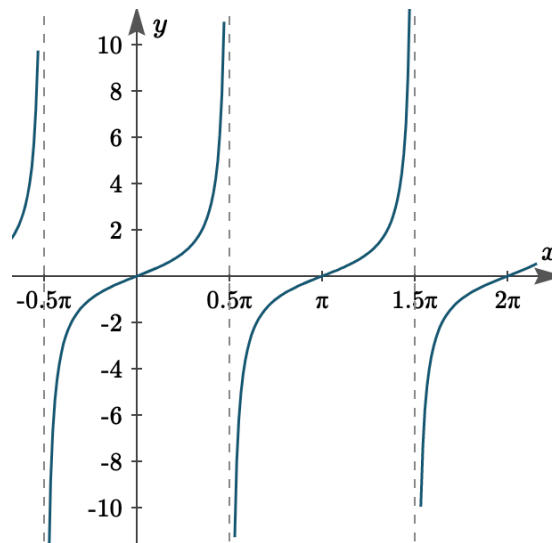
$$\text{Универсална субституция: } t = \tan \frac{x}{2}, t \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \sin x = \frac{2t}{1+t^2}, \quad x = 2 \arctan(t)$$

$$x = 2 \arctan(t), dx = d 2 \arctan(t) = \frac{2}{1+t^2} dt.$$

Пресмятаме интеграла като неопределен.

$$\begin{aligned}
I_7 &= \int \frac{1}{2 + \sin x} dx = \int \frac{1}{2 + \frac{2t}{1+t^2}} \times \frac{2}{1+t^2} dt = \int \frac{1}{\frac{2+2t^2+2t}{1+t^2}} \times \frac{2}{1+t^2} dt = \\
&= \int \frac{1}{1+t+t^2} dt = \int \frac{1}{t^2 + 2 \times \frac{1}{2} \times t + \frac{1}{4} + \frac{3}{4}} dt = \int \frac{1}{\left(t + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} dt = \\
&= \frac{1}{\frac{\sqrt{3}}{2}} \arctan \frac{t + \frac{1}{2}}{\frac{\sqrt{3}}{2}} + C = \frac{2}{\sqrt{3}} \arctan \frac{2t + 1}{\sqrt{3}} + C = \frac{2}{\sqrt{3}} \arctan \frac{2 \tan \frac{x}{2} + 1}{\sqrt{3}} + C.
\end{aligned}$$

Ако  $x \in [0, 2\pi]$ , то  $\frac{x}{2} \in [0, \pi]$ . Да разгледаме графиката на  $\tan \frac{x}{2}$  за  $\frac{x}{2} \in [0, \pi]$ .



Разбиваме интеграла на  $\frac{x}{2} \in \left[0, \frac{\pi}{2}\right]$  и  $\frac{x}{2} \in \left(\frac{\pi}{2}, \pi\right]$ , което е еквивалентно на  $x \in [0, \pi]$  и  $x \in (\pi, 2\pi]$ .  
Следователно,

$$\begin{aligned}
I &= \int_0^{2\pi} \frac{1}{2 + \sin x} dx = \int_0^{\pi} \frac{1}{2 + \sin x} dx + \int_{\pi}^{2\pi} \frac{1}{2 + \sin x} dx = \\
&= \lim_{b \rightarrow \pi^-} \int_0^b \frac{1}{2 + \sin x} dx + \lim_{b \rightarrow \pi^+} \int_b^{2\pi} \frac{1}{2 + \sin x} dx = \\
&= \lim_{b \rightarrow \pi^-} \frac{2}{\sqrt{3}} \arctan \frac{2 \tan \frac{x}{2}}{\sqrt{3}} \Big|_0^b + \lim_{b \rightarrow \pi^+} \frac{2}{\sqrt{3}} \arctan \frac{2 \tan \frac{x}{2} + 1}{\sqrt{3}} \Big|_b^{2\pi} = \\
&= \lim_{c \rightarrow \infty} \frac{2}{\sqrt{3}} \arctan c - \frac{2}{\sqrt{3}} \arctan \frac{2 \tan 0 + 1}{\sqrt{3}} + \frac{2}{\sqrt{3}} \arctan \frac{2 \tan \frac{2\pi}{2} + 1}{\sqrt{3}} - \lim_{c \rightarrow -\infty} \frac{2}{\sqrt{3}} \arctan c = \\
&= \frac{2}{\sqrt{3}} \times \frac{\pi}{2} - \frac{2}{\sqrt{3}} \arctan \frac{1}{\sqrt{3}} + \frac{2}{\sqrt{3}} \arctan \frac{2 \tan \pi + 1}{\sqrt{3}} - \frac{2}{\sqrt{3}} \left( -\frac{\pi}{2} \right) = \\
&= \frac{2}{\sqrt{3}} \times \frac{\pi}{2} - \cancel{\frac{2}{\sqrt{3}} \arctan \frac{1}{\sqrt{3}}} + \cancel{\frac{2}{\sqrt{3}} \arctan \frac{1}{\sqrt{3}}} + \frac{2\pi}{\sqrt{3} \times 2} = \\
&= \frac{\pi}{\sqrt{3}} + \frac{\pi}{\sqrt{3}} = \frac{2\pi}{\sqrt{3}} = \frac{2\pi\sqrt{3}}{3}.
\end{aligned}$$

Отговор:  $\int_0^{2\pi} \frac{1}{2 \sin x} dx = \frac{2\pi\sqrt{3}}{3}.$

8)  $\int e^{-x}(x^2 + 1) dx = ?$

$$\begin{aligned}
I_8 &= \int e^{-x}(x^2 + 1) dx = - \int x^2 + 1 d e^{-x} \stackrel{\text{И.Ч.}}{=} - e^{-x}(x^2 + 1) + \int e^{-x} d x^2 + 1 = \\
&= - e^{-x}(x^2 + 1) + \int 2x e^{-x} dx = - e^{-x}(x^2 + 1) - 2 \int x d e^{-x} \stackrel{\text{И.Ч.}}{=} \\
&= - e^{-x}(x^2 + 1) - 2x e^{-x} + 2 \int e^{-x} dx = - e^{-x}(x^2 + 1) - 2x e^{-x} - 2 \int e^{-x} d(-x) = \\
&= - e^{-x}(x^2 + 1) - 2x e^{-x} - 2e^{-x} + C = - e^{-x}(x^2 + 2x + 3) + C.
\end{aligned}$$

9)  $\int \sqrt{1+x^2} dx = ?$

$$\begin{aligned}
I_9 &= \int \sqrt{1+x^2} dx \stackrel{\text{И.Ч.}}{=} x \sqrt{1+x^2} - \int x d \sqrt{1+x^2} = \\
&= x \sqrt{1+x^2} - \int x \left( (1+x^2)^{\frac{1}{2}} \right) dx = x \sqrt{1+x^2} - \int x \times \frac{1}{2} \times \frac{1}{\sqrt{1+x^2}} \times 2 dx = \\
&= x \sqrt{1+x^2} - \int \frac{x^2}{\sqrt{1+x^2}} dx = x \sqrt{1+x^2} - \int \frac{1+x^2-1}{\sqrt{1+x^2}} dx = \\
&= x \sqrt{1+x^2} - \int \sqrt{1+x^2} dx + \int \frac{1}{\sqrt{1+x^2}} dx = x \sqrt{1+x^2} - I_9 + \ln |x + \sqrt{1+x^2}|.
\end{aligned}$$

Следователно,

$$2I_9 = x\sqrt{1+x^2} + \ln|x + \sqrt{1+x^2}| + C$$

$$I_9 = \int \sqrt{1+x^2} dx = \frac{x\sqrt{1+x^2} + \ln|x + \sqrt{1+x^2}|}{2} + C$$

Забележка, интегралът  $\int \sqrt{1-x^2} dx$  се решава аналогично, с разликата, че получаваме  $\arcsin$  от табличния интеграл.

$$10) \int \sqrt{4-x^2} dx = ?$$

$$\begin{aligned} I_{10} &= \int \sqrt{4-x^2} dx \stackrel{\text{И.Ч}}{=} x\sqrt{4-x^2} - \int x d\sqrt{4-x^2} = \\ &= x\sqrt{4-x^2} - \int x(\sqrt{4-x^2})' dx = x\sqrt{4-x^2} - \int x \left( \frac{1}{2} \times \frac{1}{(4-x^2)} \times (-2x) \right) dx = \\ &= x\sqrt{4-x^2} + \int \frac{x^2}{\sqrt{4-x^2}} = x\sqrt{4-x^2} - \int \frac{4-x^2-4}{\sqrt{4-x^2}} dx = \\ &= x\sqrt{4-x^2} - I + 4 \arcsin\left(\frac{x}{2}\right) + C. \end{aligned}$$

$$\text{Следователно, } I = \frac{x\sqrt{4-x^2} + 4 \arcsin\left(\frac{x}{2}\right) + C}{2}.$$

$$11) \int_1^2 \frac{x+1}{\sqrt{x}(1+\sqrt{x})} dx = ?$$

Полагаме  $x = t^2 \Rightarrow dx = dt^2 = 2t dt$ .

Ако  $x = 1$ , то  $t = 1$ . Ако  $x = 2$ , то  $t = \sqrt{2}$ .

$$I_{11} = \int_1^2 \frac{x+1}{\sqrt{x}(1+\sqrt{x})} dx = \int_1^{\sqrt{2}} \frac{t^2+1}{t(1+t)} \times 2t dt = 2 \int_1^{\sqrt{2}} \frac{t^2+1}{t+1} dt$$

Делим полиноми

$$t^2 + 1 : t + 1 = t$$

$$\frac{t^2+t}{t^2+t}$$

$$\frac{-t+1}{-t+1}$$

$$\frac{-t-1}{-t-1}$$

$$2 \leftarrow \text{остатък}$$

$$\Rightarrow \frac{t^2+1}{t+1} = t - 1 + \frac{2}{t+1}$$

Следователно,

$$\begin{aligned}
I_{11} &= 2 \int_1^{\sqrt{2}} (t-1)dt + 2 \int_1^{\sqrt{2}} \frac{2}{t+1} dt = \\
&= 2 \frac{(t-1)^2}{2} \Big|_1^{\sqrt{2}} + 4 \int_1^{\sqrt{2}} \frac{d(t+1)}{t+1} = \\
&= (\sqrt{2}-1)^2 - 0 + 4 \ln|t+1| \Big|_1^{\sqrt{2}} = (\sqrt{2}-1)^2 + 4 \ln(\sqrt{2}+1) - 4 \ln 2 = \\
&= 4 \ln \frac{\sqrt{2}+1}{2} + 3 - 2\sqrt{2}.
\end{aligned}$$

$$12) \int_1^3 \frac{dx}{\sqrt{x}(x+1)^2}.$$

Полагаме  $x = t^2 \Rightarrow dx = dt^2 = 2t dt$ . Ако  $x = 1$ , то  $t = 1$ . Ако  $x = 3$ , то  $t = \sqrt{3}$ .

Следователно

$$\begin{aligned}
I_{12} &= \int_1^{\sqrt{3}} \frac{2t dt}{t(t^2+1)^2} = 2 \int_1^{\sqrt{3}} \frac{dt}{(t^2+1)^2} = 2 \int_1^{\sqrt{3}} \frac{1+t^2-t^2}{(t^2+1)^2} dt = \\
&= 2 \int_1^{\sqrt{3}} \frac{1}{t^2+1} dt - \underbrace{2 \int_1^{\sqrt{3}} \frac{t^2}{(t^2+1)^2} dt}_J.
\end{aligned}$$

$$\frac{f(u)du}{u^2} = f(u)d\left(-\frac{1}{u}\right), \text{ тъй като } \left(-\frac{1}{u}\right)' = \frac{1}{u^2}$$

$$\begin{aligned}
J &= \int_1^{\sqrt{3}} \frac{2t \times t}{(t^2+1)^2} dt = \int_1^{\sqrt{3}} \frac{t d(t^2+1)}{(t^2+1)^2} = \int_1^{\sqrt{3}} t d\left(-\frac{1}{t^2+1}\right) = \\
&= - \int_1^{\sqrt{3}} t d\frac{1}{t^2+1} \stackrel{\text{и.ч.}}{=} - \frac{t}{t^2+1} \Big|_1^{\sqrt{3}} + \int_1^{\sqrt{3}} \frac{1}{t^2+1} dt = -\frac{\sqrt{3}}{4} + \frac{1}{2} + \int_1^{\sqrt{3}} \frac{1}{t^2+1} dt.
\end{aligned}$$

Следователно,

$$\begin{aligned}
I_{12} &= 2 \arctan t \Big|_1^{\sqrt{3}} + \frac{\sqrt{3}}{4} - \frac{1}{2} - \arctan t \Big|_1^{\sqrt{3}} = \\
&= \arctan \sqrt{3} - \arctan 1 + \frac{\sqrt{3}}{4} - \frac{1}{2} = \left(\frac{\pi}{3} - \frac{\pi}{4}\right) + \frac{\sqrt{3}}{4} - \frac{1}{2} = \frac{\pi + 3\sqrt{3} - 6}{12}.
\end{aligned}$$

Коментар: Задачата е от държавния изпит на специалност „Приложна математика“ проведен на 2014-07.