Задача 8. Намерете неопределения интеграл $I = \int \dfrac{x-2}{x(x^2+2)}\,\mathrm{d}\,x$.

Решение.

$$\frac{x-2}{x(x^2+2)} = \frac{Ax+B}{x^2+2} + \frac{C}{x};$$

$$x - 2 = Ax^2 + Bx + C(x^2+2)$$

$$(A+C)x^2 + (B-1)x + 2(C+1) = 0$$

$$\begin{aligned}
x = 1 &: A + C + B - 1 + 2C + 2 = 0 \\
& A + B + 3C + 1 = 0 && (1) \\
x = -1 &: A + C - B + 1 + 2C + 2 = 0 \\
& A - B + 3C + 3 = 0 && (2) \\
x = 2 &: 4A + 4C + 2B - 2 + 2C + 2 = 0 \\
& 4A + 2B + 6C = 0 \\
& 2A + B + 3C = 0 && (3) \\
(1) + (2) &\Rightarrow 2A + 6C + 4 = 0 \\
& A + 3C + 2 = 0 && (4) \\
(2) + (3) &\Rightarrow 3A + 6C + 3 = 0 \\
& A + 2C + 1 = 0 && (5) \\
(4) - (5) &\Rightarrow C + 1 = 0 \Rightarrow C = -1 \\
\text{От } (4) &\Rightarrow A = 1 \\
\text{От } (3) &\Rightarrow B = 1 \\
&\Rightarrow \frac{x-2}{x(x^2+2)} = \frac{x+1}{x^2+2} - \frac{1}{x}
\end{aligned}$$

$$\int \frac{x-2}{x(x^2+2)}\,\mathrm{d}\,x = \int \left(\frac{x+1}{x^2+2} - \frac{1}{x}\right)\mathrm{d}\,x = \underbrace{\int \frac{x+1}{x^2+2}\,\mathrm{d}\,x}_{J} - \underbrace{\int \frac{1}{x}\,\mathrm{d}\,x}_{K}.$$

$$J = \int \frac{x+1}{x^2+2}\,\mathrm{d}\,x = \int \left(\frac{x}{x^2+2} + \frac{1}{x^2+2}\right)\mathrm{d}\,x =$$
$$= \underbrace{\int \frac{x}{x^2+2}\,\mathrm{d}\,x}_{J_1} + \underbrace{\int \frac{1}{x^2+2}{}_{J_2}\,\mathrm{d}\,x}_{J_2}.$$

$$J_1 = \int \frac{x}{x^2+2}\,\mathrm{d}\,x \underset{\frac{\mathrm{d}\,u}{\mathrm{d}\,x}=2x \to \mathrm{d}\,x = \frac{\mathrm{d}\,u}{2x}}{\overset{u=x^2+2}{=}} \frac{1}{2}\int \frac{x}{u} \times \frac{1}{x}\,\mathrm{d}\,u =$$

$$= \frac{1}{2}\int \frac{1}{u}\,\mathrm{d}\,u = \frac{\ln(u)}{2} = \frac{\ln(x^2+2)}{2}.$$

$$J_2 = \int \frac{1}{x^2+2}\,\mathrm{d}\,x = \int \frac{\frac{1}{2}}{\left(\frac{x}{\sqrt{2}}\right)^2+1} \underset{\frac{\mathrm{d}\,u}{\mathrm{d}\,x}=\frac{1}{\sqrt{2}} \to \mathrm{d}\,x = \sqrt{2}\,\mathrm{d}\,u}{\overset{u=\frac{x}{\sqrt{2}}}{=}} \int \frac{1}{2u^2+2} \times \sqrt{2}\,\mathrm{d}\,u =$$

$$= \frac{1}{\sqrt{2}}\int \frac{1}{u^2+1}\,\mathrm{d}\,u = \frac{\arctan(u)}{\sqrt{2}} = \frac{\arctan\left(\frac{x}{\sqrt{2}}\right)}{\sqrt{2}}.$$

$$K = \int \frac{1}{x}\,\mathrm{d}\,x = \ln(x)$$

$$\Rightarrow I = J - K = J_1 + J_2 - K = \frac{\ln(x^2+2)}{2} + \frac{\arctan\left(\frac{x}{\sqrt{2}}\right)}{\sqrt{2}} - \ln(|x|) + C.$$

$\square$