Задача 8. Да се пресметнете интеграла $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\mathrm{d}\,x}{2+\sin x}$.

Решение.

Прилагаме универсалната субституция: $\tan\dfrac{x}{2} = t,\ t \in \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right) \Rightarrow \sin x = \dfrac{2t}{1+t^2}$.

$\mathrm{d}\,x = 2\,\mathrm{d}\arctan t = 2 \times \dfrac{1}{1+t^2}\,\mathrm{d}\,t = \dfrac{2}{1+t^2}\,\mathrm{d}\,t$. Граници: $x \in \left[0, \dfrac{\pi}{2}\right] \Rightarrow t \in [0, 1]$.

$$I = \int_0^1 \frac{1}{2+\frac{2t}{1+t^2}} \times \frac{2}{1+t^2}\,\mathrm{d}\,t = \int_0^1 \frac{\cancel{1+t^2}}{2+2t^2+2t} \times \frac{2}{\cancel{1+t^2}}\,\mathrm{d}\,t = \int_0^1 \frac{1}{t^2+t+1}\,\mathrm{d}\,t =$$

$$= \int_0^1 \frac{1}{t^2+t+\frac{1}{4}+\frac{3}{4}}\,\mathrm{d}\,t = \int_0^1 \frac{1}{\left(t+\frac{1}{2}\right)^2+\frac{3}{4}}\,\mathrm{d}\,t = \frac{1}{\left(t+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\,\mathrm{d}\left(t+\frac{1}{2}\right) =$$

$$= \frac{1}{\frac{\sqrt{3}}{2}} \arctan \frac{t+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\Bigg|_0^1 = \frac{2}{\sqrt{3}}\left(\frac{\pi}{3}-\frac{\pi}{6}\right) = \frac{\pi\sqrt{3}}{9}.$$

$\square$