Задача 8. (10 т.). Пресметнете определения интеграл:

$$\int_0^{\frac{1}{2}} \arcsin x \, \mathrm{d}\, x.$$

Решение.

$\arcsin x = y, x \in \left[0, \frac{1}{2}\right] \subset [-1,1] \Rightarrow y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right].$

<u>I-ви наин: Интегриране по части.</u>

$$I = \int_0^{\frac{1}{2}} \arcsin x \, \mathrm{d}\, x = x \arcsin x \, \mathrm{d}\, x = x \arcsin x \Big|_0^{\frac{1}{2}} - \int_0^{\frac{1}{2}} x \, \mathrm{d} \arcsin x =$$

$$= \frac{1}{2} \times \frac{\pi}{6} - 0 - \int_0^{\frac{1}{2}} x (\arcsin x)' \mathrm{d}\, x = \frac{\pi}{12} - \int_0^{\frac{1}{2}} \frac{x}{\sqrt{1-x^2}} \times x' \, \mathrm{d}\, x$$

Нека

$$J = \int \frac{x}{\sqrt{1-x^2}} \mathrm{d}\, x \underset{\frac{\mathrm{d}\, u}{\mathrm{d}\, x} = -2x \Rightarrow \mathrm{d}\, x = -\frac{1}{2x} \mathrm{d}\, u}{\overset{u = 1 - x^2}{=}} \int \frac{x}{\sqrt{u}} \left(-\frac{1}{2x}\right) \mathrm{d}\, u = -\frac{1}{2} \int \frac{1}{\sqrt{u}} \mathrm{d}\, u =$$

$$= -\frac{1}{2} \int u^{-\frac{1}{2}} \mathrm{d}\, x = -\frac{\cancel{1}}{\cancel{2}} \frac{u^{\frac{1}{2}}}{\frac{\cancel{1}}{\cancel{2}}} = -\sqrt{u} = -\sqrt{1-x^2}$$

$$\Rightarrow I = \frac{\pi}{12} - \left(-\sqrt{1-x^2}\right) \Big|_0^{\frac{1}{2}} = \frac{\pi}{12} + \sqrt{1 - \frac{1}{4}} - 1 =$$

$$= \frac{\pi}{12} + \frac{\sqrt{3}}{2} - 1 = \frac{\pi + 6\sqrt{3} - 12}{12} \approx 0.12782479...$$

<u>II-ри начин: Тригонометрична субституция.</u>

Полагаме $x = \sin t$.

1. Ако $x = 0 \Rightarrow \sin t = 0$, за $t = 0$.
2. Ако $x = \frac{1}{2} \Rightarrow \sin t = \frac{1}{2}$, за $t = \frac{\pi}{6}$.
3.

Следователно,

$$I = \int_0^{\frac{1}{2}} \arcsin x \, \mathrm{d}\, x = \int_0^{\frac{\pi}{6}} \arcsin\left(\sin t\right) \mathrm{d} \sin t = \int_0^{\frac{\pi}{6}} t \, \mathrm{d} \sin t \overset{\text{И.Ч.}}{=}$$

$$= t \sin t \Big|_0^{\frac{\pi}{6}} - \int_0^{\frac{\pi}{6}} \sin t \, \mathrm{d}\, t = \frac{\pi}{6} \times \sin \frac{\pi}{6} - 0 - \int_0^{\frac{\pi}{6}} \sin t \, \mathrm{d}\, t =$$

$$= \frac{\pi}{6} \times \frac{1}{2} + \cos t \Big|_0^{\frac{\pi}{6}} = \frac{\pi}{12} + \cos \frac{\pi}{6} - \cos 0 = \frac{\pi}{12} + \frac{\sqrt{3}}{2} - 1.$$

□