Задача 8. Пресметнете интеграла $\int_0^1 x^2 \arctan x \, \mathrm{d}\, x$.

Решение.

$$I = \int_0^1 x^2 \arctan x \, \mathrm{d}\, x = \frac{1}{3} \int_0^1 \arctan x \, \mathrm{d}\, x^3 \overset{\text{и.ч.}}{=} \frac{1}{3} x^3 \arctan x \Big|_0^1 - \frac{1}{3} \int_0^1 x^3 \, \mathrm{d} \arctan x =$$

$$= \frac{1}{3} \times 1 \times \frac{\pi}{4} - 0 - \frac{1}{3} \int_0^1 \frac{x^3}{1+x^2} \, \mathrm{d}\, x = \frac{\pi}{12} - \frac{1}{3} \times \frac{1}{2} \int_0^1 \frac{x^2 \times 2x}{1+x^2} \, \mathrm{d}\, x =$$

$$= \frac{\pi}{12} - \frac{1}{6} \underbrace{\int_0^1 \frac{x^2}{1+x^2} \, \mathrm{d}\, x^2}_{J}.$$

$$J = \int_0^1 \frac{x^2}{1+x^2} \, \mathrm{d}\, x^2 \overset{x^2 = u}{=} \int_0^1 \frac{u}{1+u} \, \mathrm{d}\, u = \int_0^1 \frac{u+1-1}{u+1} \, \mathrm{d}\, u = \int_0^1 1 \, \mathrm{d}\, u - \int_0^1 \frac{1}{u+1} \, \mathrm{d}\, u$$

$$= u \Big|_0^1 - \int_0^1 \frac{1}{u+1} \, \mathrm{d}(u+1) = 1 - 0 - \ln(u+1) \Big|_0^1 = 1 - \ln 2 + \ln 1 = 1 - \ln 2.$$

Следователно, $I = \dfrac{\pi}{12} - \dfrac{1}{6}(1 - \ln 2) \approx 0.210657251225807$.

$\square$