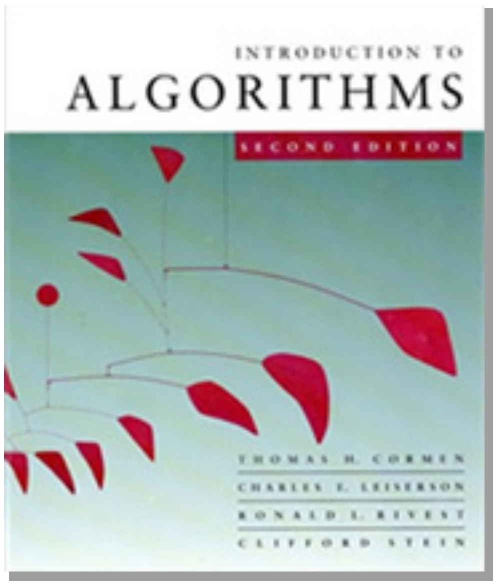
# *Introduction to Algorithms*

## 6.046J/18.401J




## LECTURE 2

**Asymptotic Notation**

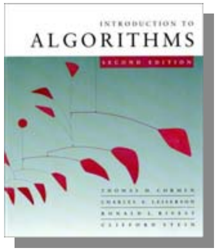
- $O$-, $\Omega$-, and $\Theta$-notation

**Recurrences**

- Substitution method
- Iterating the recurrence
- Recursion tree
- Master method

**Prof. Erik Demaine**

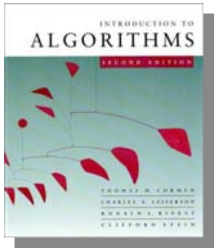
  




# Asymptotic notation

$O$-notation (upper bounds):

> We write $f(n) = O(g(n))$ if there exist constants $c > 0$, $n_0 > 0$ such that $0 \le f(n) \le cg(n)$ for all $n \ge n_0$.

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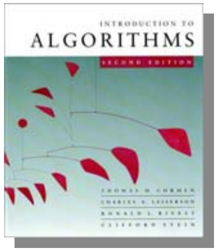
**Example:** $2n^2 = O(n^3)$ $\quad (c = 1, n_0 = 2)$

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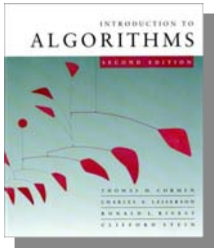
*functions, not values*

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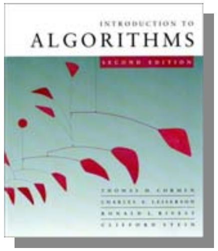
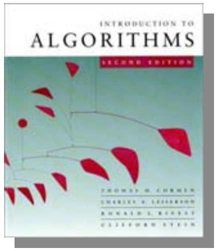
*functions, not values*

*funny, "one-way" equality*




# Set definition of O-notation

$O(g(n)) = \{ f(n)$ : there exist constants $c > 0$, $n_0 > 0$ such that $0 \leq f(n) \leq cg(n)$ for all $n \geq n_0$ $\}$

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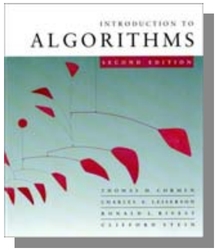


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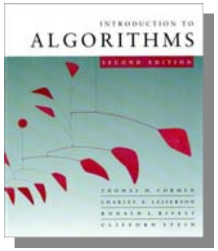
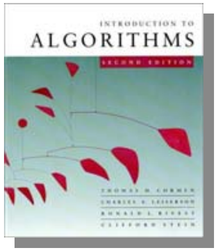
**EXAMPLE:** $2n^2 \in O(n^3)$

(*Logicians:* $\lambda n.2n^2 \in O(\lambda n.n^3)$, but it's convenient to be sloppy, as long as we understand what's *really* going on.)




# Macro substitution

***Convention:*** A set in a formula represents an anonymous function in the set.

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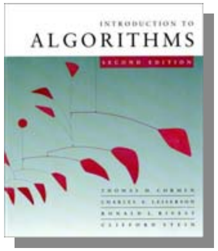
**EXAMPLE:** $f(n) = n^3 + O(n^2)$

means

$f(n) = n^3 + h(n)$

for some $h(n) \in O(n^2)$ .

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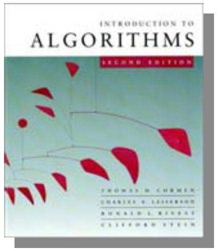
means

for any $f(n) \in O(n)$:

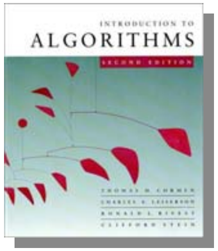
$$n^2 + f(n) = h(n)$$

for some $h(n) \in O(n^2)$ .




# Ω-notation (lower bounds)

$O$-notation is an *upper-bound* notation. It makes no sense to say $f(n)$ is at least $O(n^2)$.

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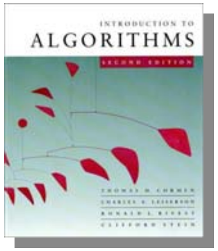
$\Omega(g(n)) = \{ f(n) :$ there exist constants $c > 0, n_0 > 0$ such that $0 \le cg(n) \le f(n)$ for all $n \ge n_0 \}$

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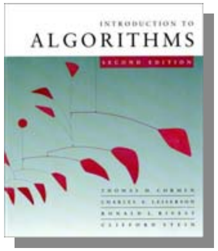
**EXAMPLE:** $\sqrt{n} = \Omega(\lg n)$ $(c = 1,\ n_0 = 16)$

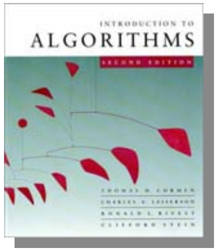
  




# Θ-notation (tight bounds)

$$\Theta(g(n)) = O(g(n)) \cap \Omega(g(n))$$

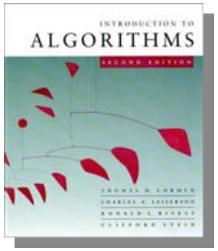



# Θ-notation (tight bounds)

$$\Theta(g(n)) = O(g(n)) \cap \Omega(g(n))$$

**EXAMPLE:** $\frac{1}{2}n^2 - 2n = \Theta(n^2)$




# $o$-notation and $\omega$-notation

$O$-notation and $\Omega$-notation are like $\leq$ and $\geq$.
$o$-notation and $\omega$-notation are like $<$ and $>$.

$o(g(n)) = \{\, f(n)$ : for any constant $c > 0$,
there is a constant $n_0 > 0$
such that $0 \leq f(n) < cg(n)$
for all $n \geq n_0 \,\}$

**EXAMPLE:** $2n^2 = o(n^3)$ $\quad (n_0 = 2/c)$

# o-notation and ω-notation

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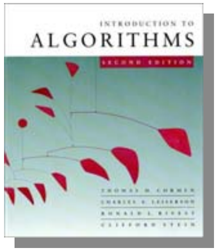
$\omega(g(n)) = \{ f(n)$ : for any constant $c > 0$, there is a constant $n_0 > 0$ such that $0 \leq cg(n) < f(n)$ for all $n \geq n_0$ $\}$

**EXAMPLE:** $\sqrt{n} = \omega(\lg n)$ $\quad$ $(n_0 = 1 + 1/c)$



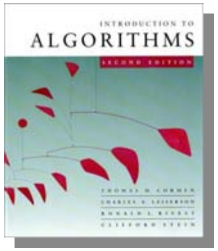

# Solving recurrences

- The analysis of merge sort from ***Lecture 1*** required us to solve a recurrence.
- Recurrences are like solving integrals, differential equations, etc.
  - Learn a few tricks.
- ***Lecture 3***: Applications of recurrences to divide-and-conquer algorithms.

# Substitution method

*The most general method:*

1. ***Guess*** the form of the solution.
2. ***Verify*** by induction.
3. ***Solve*** for constants.

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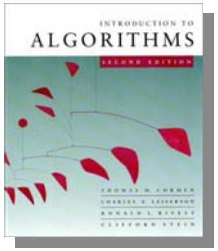
**EXAMPLE:** $T(n) = 4T(n/2) + n$

- [Assume that $T(1) = \Theta(1)$.]
- Guess $O(n^3)$ . (Prove $O$ and $\Omega$ separately.)
- Assume that $T(k) \le ck^3$ for $k < n$ .
- Prove $T(n) \le cn^3$ by induction.




# Example of substitution

$$
\begin{aligned}
T(n) &= 4T(n/2) + n \\
&\le 4c(n/2)^3 + n \\
&= (c/2)n^3 + n \\
&= cn^3 - ((c/2)n^3 - n) \longleftarrow \textit{desired} - \textit{residual} \\
&\le cn^3 \longleftarrow \textit{desired}
\end{aligned}
$$

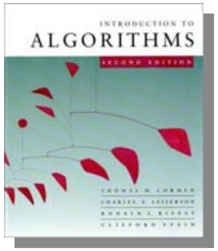
whenever $(c/2)n^3 - n \ge 0$, for example, if $c \ge 2$ and $n \ge 1$.

*residual*

# Example (continued)

- We must also handle the initial conditions, that is, ground the induction with base cases.
- ***Base:*** $T(n) = \Theta(1)$ for all $n < n_0$, where $n_0$ is a suitable constant.
- For $1 \le n < n_0$, we have "$\Theta(1)$" $\le cn^3$, if we pick $c$ big enough.

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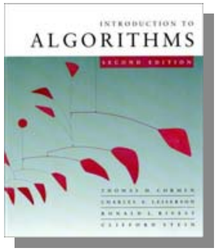
***This bound is not tight!***

# A tighter upper bound?

We shall prove that $T(n) = O(n^2)$.

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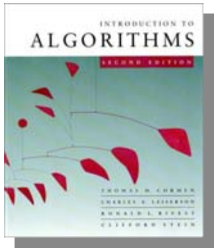
Assume that $T(k) \le ck^2$ for $k < n$:

$$\begin{aligned} T(n) &= 4T(n/2) + n \\ &\le 4c(n/2)^2 + n \\ &= cn^2 + n \\ &= O(n^2) \end{aligned}$$

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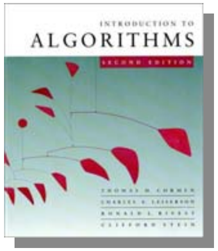
***Wrong!*** We must prove the I.H.

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$$= cn^2 - (-n)$$ [ desired – residual ]

$$\le cn^2$$ for ***no*** choice of $c > 0$. Lose!

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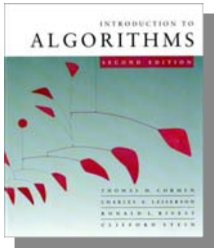
**IDEA:** Strengthen the inductive hypothesis.

- ***Subtract*** a low-order term.

*Inductive hypothesis*: $T(k) \leq c_1k^2 - c_2k$ for $k < n$.

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T(n) &= 4T(n/2) + n \\
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&= c_1 n^2 - 2c_2 n + n \\
&= c_1 n^2 - c_2 n - (c_2 n - n) \\
&\le c_1 n^2 - c_2 n \ \text{ if } c_2 \ge 1.
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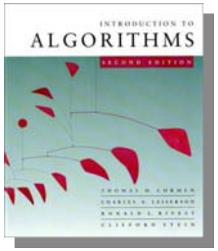


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\end{aligned}
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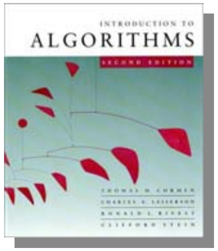
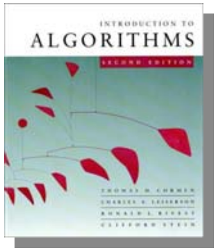
Pick $c_1$ big enough to handle the initial conditions.




# Recursion-tree method

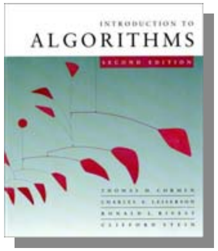
- A recursion tree models the costs (time) of a recursive execution of an algorithm.
- The recursion-tree method can be unreliable, just like any method that uses ellipses (…).
- The recursion-tree method promotes intuition, however.
- The recursion tree method is good for generating guesses for the substitution method.




# Example of recursion tree

Solve $T(n) = T(n/4) + T(n/2) + n^2$:

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$$T(n)$$

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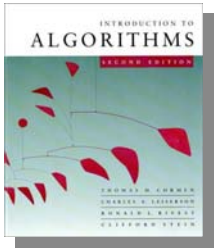
Solve $T(n) = T(n/4) + T(n/2) + n^2$:

$$n^2$$

$$T(n/4) \qquad\qquad\qquad T(n/2)$$




# Example of recursion tree

Solve $T(n) = T(n/4) + T(n/2) + n^2$:

$$n^2$$

$$(n/4)^2 \qquad\qquad (n/2)^2$$

$$T(n/16) \quad T(n/8) \quad T(n/8) \quad T(n/4)$$

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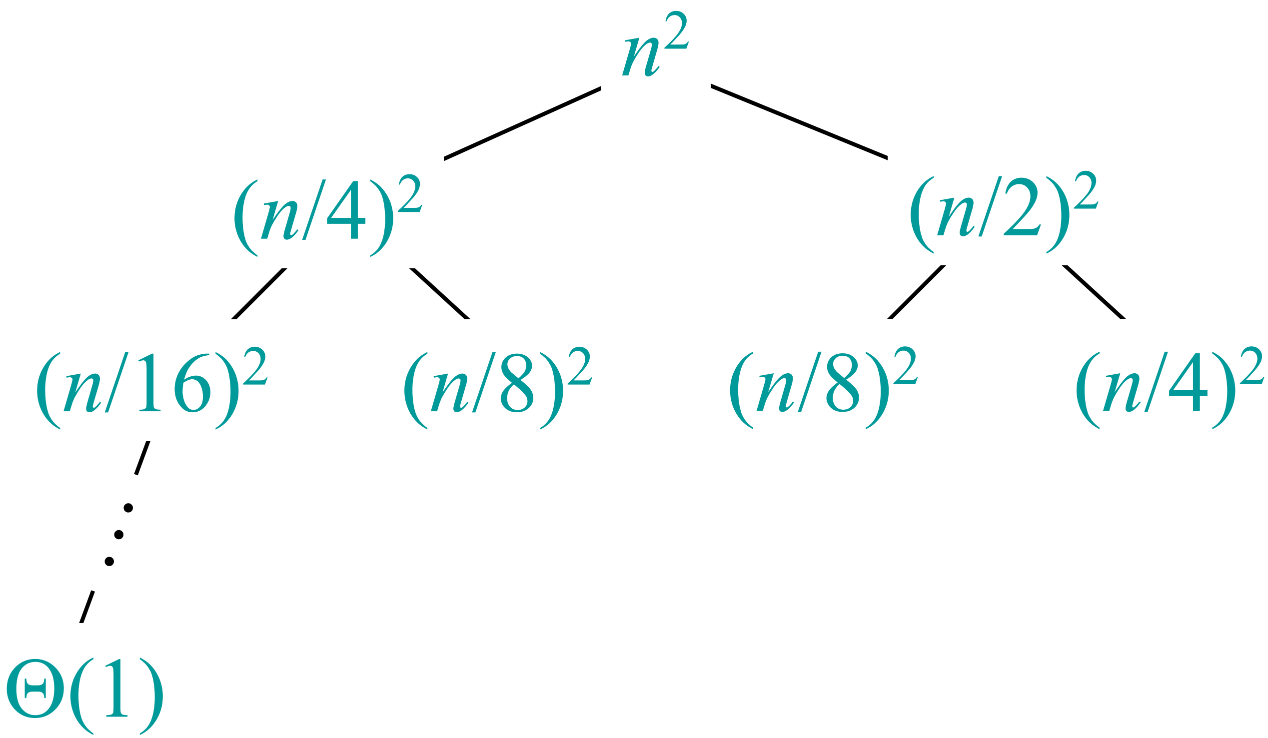


  

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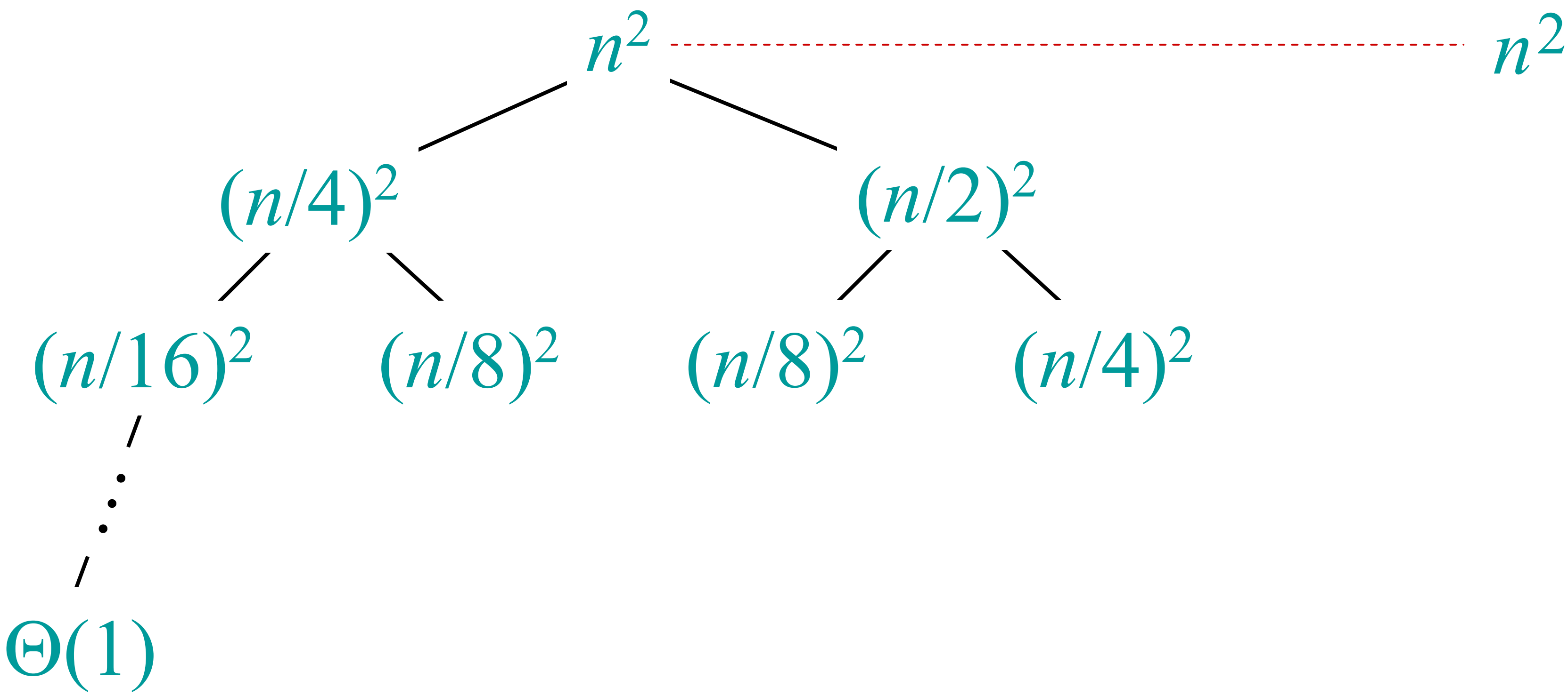

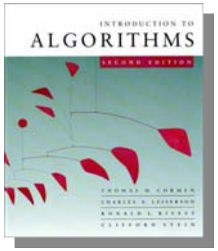


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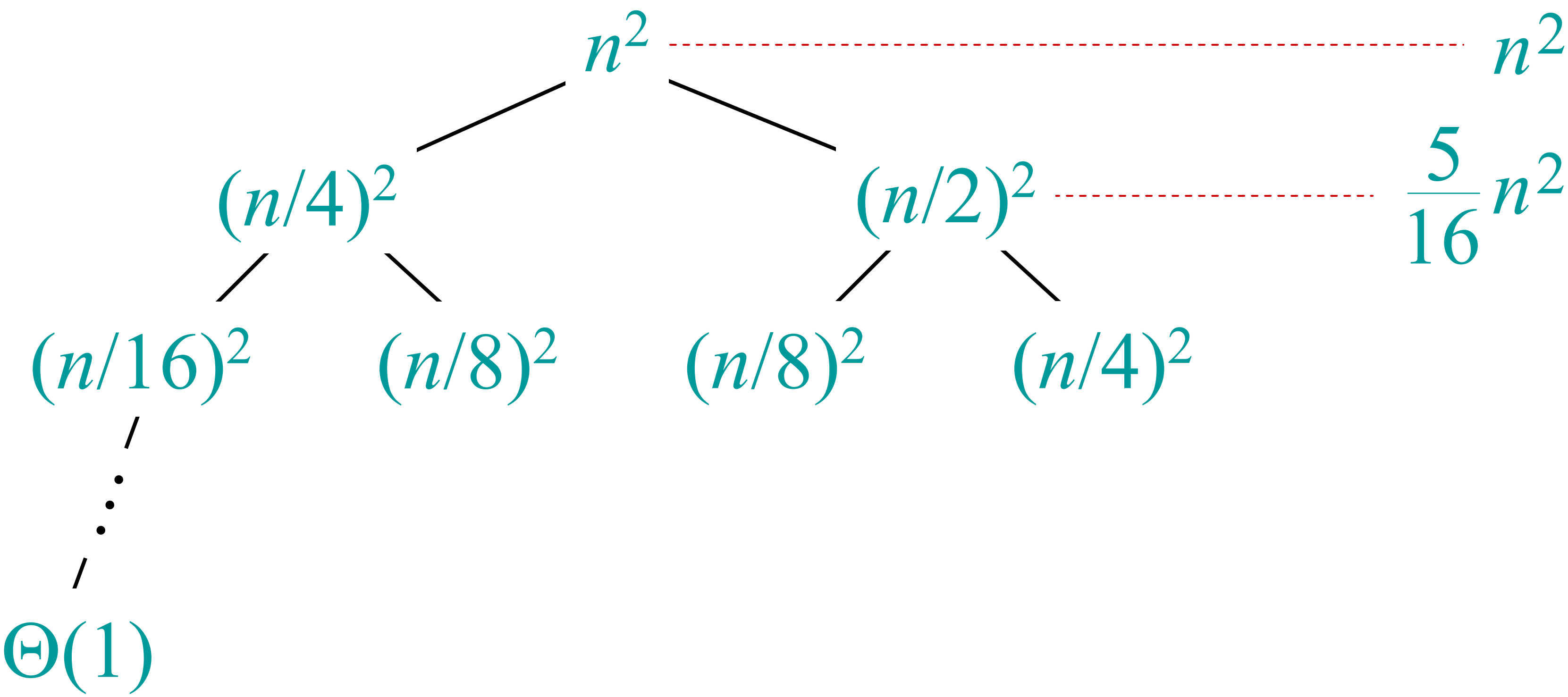

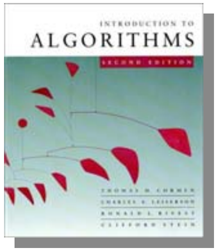


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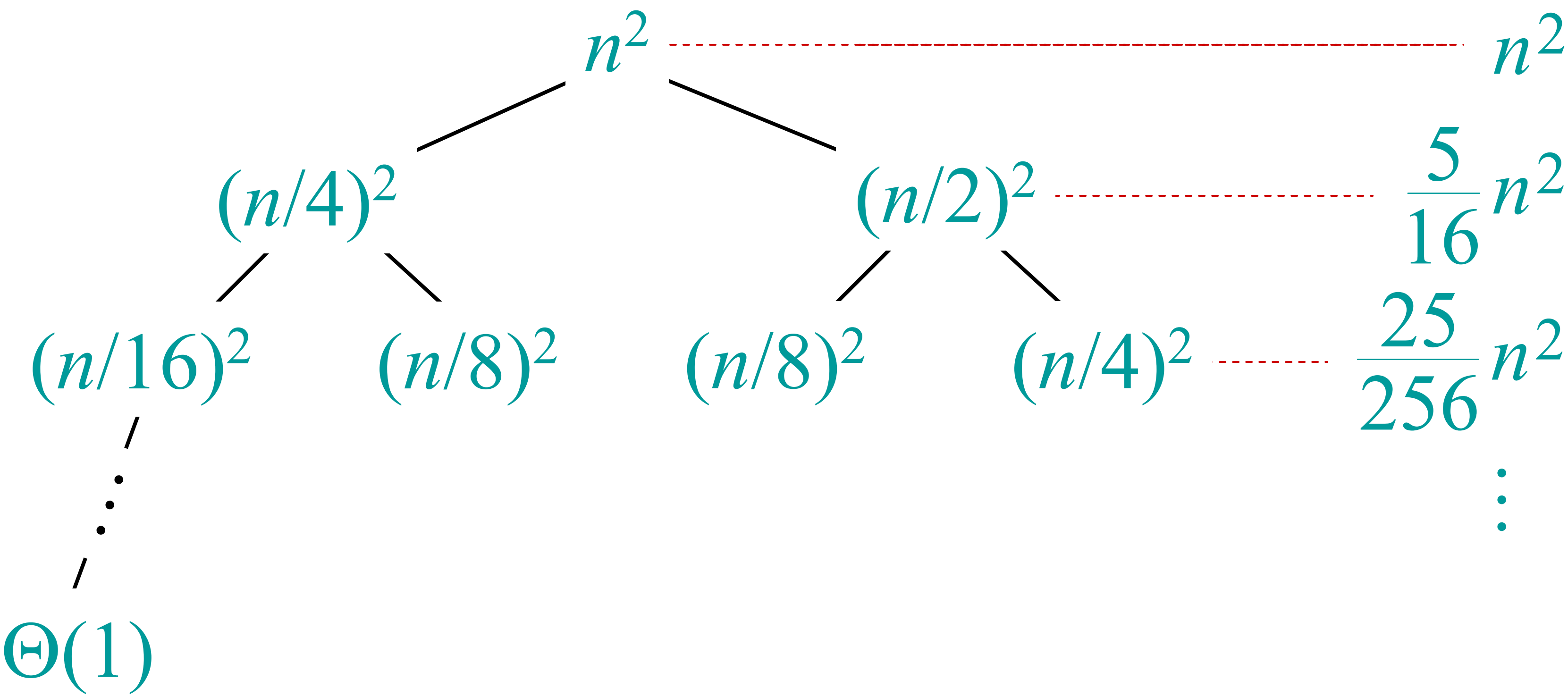
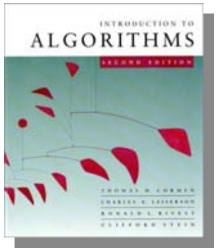





# Example of recursion tree

Solve $T(n) = T(n/4) + T(n/2) + n^2$:

$$n^2 \quad\cdots\quad n^2$$

$$(n/4)^2 \qquad (n/2)^2 \quad\cdots\quad \frac{5}{16}n^2$$

$$(n/16)^2 \qquad (n/8)^2 \qquad (n/8)^2 \qquad (n/4)^2 \quad\cdots\quad \frac{25}{256}n^2$$

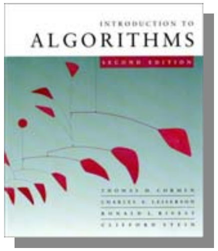
$$\vdots \qquad\qquad \vdots$$

$$\Theta(1)$$

$$\text{Total} = n^2\left(1 + \frac{5}{16} + \left(\frac{5}{16}\right)^2 + \left(\frac{5}{16}\right)^3 + \cdots\right)$$

$$= \Theta(n^2)$$

*geometric series*

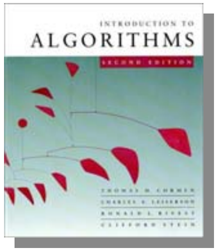



# The master method

The master method applies to recurrences of the form

$$T(n) = a\,T(n/b) + f(n)\ ,$$

where $a \geq 1$, $b > 1$, and $f$ is asymptotically positive.




# Three common cases

Compare $f(n)$ with $n^{\log_b a}$:

1. $f(n) = O(n^{\log_b a - \varepsilon})$ for some constant $\varepsilon > 0$.
   - $f(n)$ grows polynomially slower than $n^{\log_b a}$ (by an $n^{\varepsilon}$ factor).

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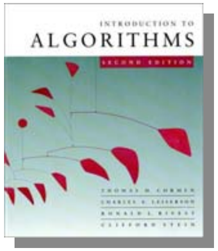
2. $f(n) = \Theta(n^{\log_b a} \lg^k n)$ for some constant $k \geq 0$.
   - $f(n)$ and $n^{\log_b a}$ grow at similar rates.

   ***Solution:*** $T(n) = \Theta(n^{\log_b a} \lg^{k+1} n)$ .




# Three common cases (cont.)

Compare $f(n)$ with $n^{\log_b a}$:

3. $f(n) = \Omega(n^{\log_b a + \varepsilon})$ for some constant $\varepsilon > 0$.
   - $f(n)$ grows polynomially faster than $n^{\log_b a}$ (by an $n^{\varepsilon}$ factor),

   ***and*** $f(n)$ satisfies the ***regularity condition*** that $a f(n/b) \le c f(n)$ for some constant $c < 1$.

   ***Solution:*** $T(n) = \Theta(f(n))$ .

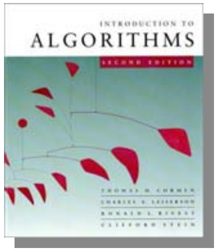
# Examples

**Ex.** $T(n) = 4T(n/2) + n$

$a = 4,\ b = 2 \Rightarrow n^{\log_b a} = n^2;\ f(n) = n.$

**Case 1**: $f(n) = O(n^{2-\varepsilon})$ for $\varepsilon = 1$.

$\therefore\ T(n) = \Theta(n^2).$

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$a = 4,\ b = 2 \Rightarrow n^{\log_b a} = n^2;\ f(n) = n^2.$

**CASE 2**: $f(n) = \Theta(n^2 \lg^0 n)$, that is, $k = 0$.

$\therefore\ T(n) = \Theta(n^2 \lg n).$

# Examples

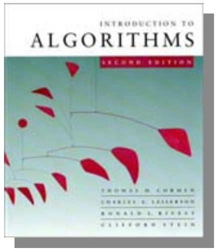
**Ex.** $T(n) = 4T(n/2) + n^3$

$a = 4,\ b = 2 \Rightarrow n^{\log_b a} = n^2;\ f(n) = n^3.$

**CASE 3**: $f(n) = \Omega(n^{2+\varepsilon})$ for $\varepsilon = 1$

***and*** $4(n/2)^3 \leq cn^3$ (reg. cond.) for $c = 1/2$.

$\therefore\ T(n) = \Theta(n^3)$.

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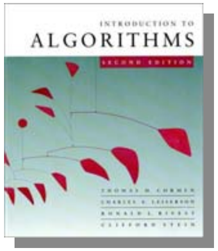
**Ex.** $T(n) = 4T(n/2) + n^2/\lg n$

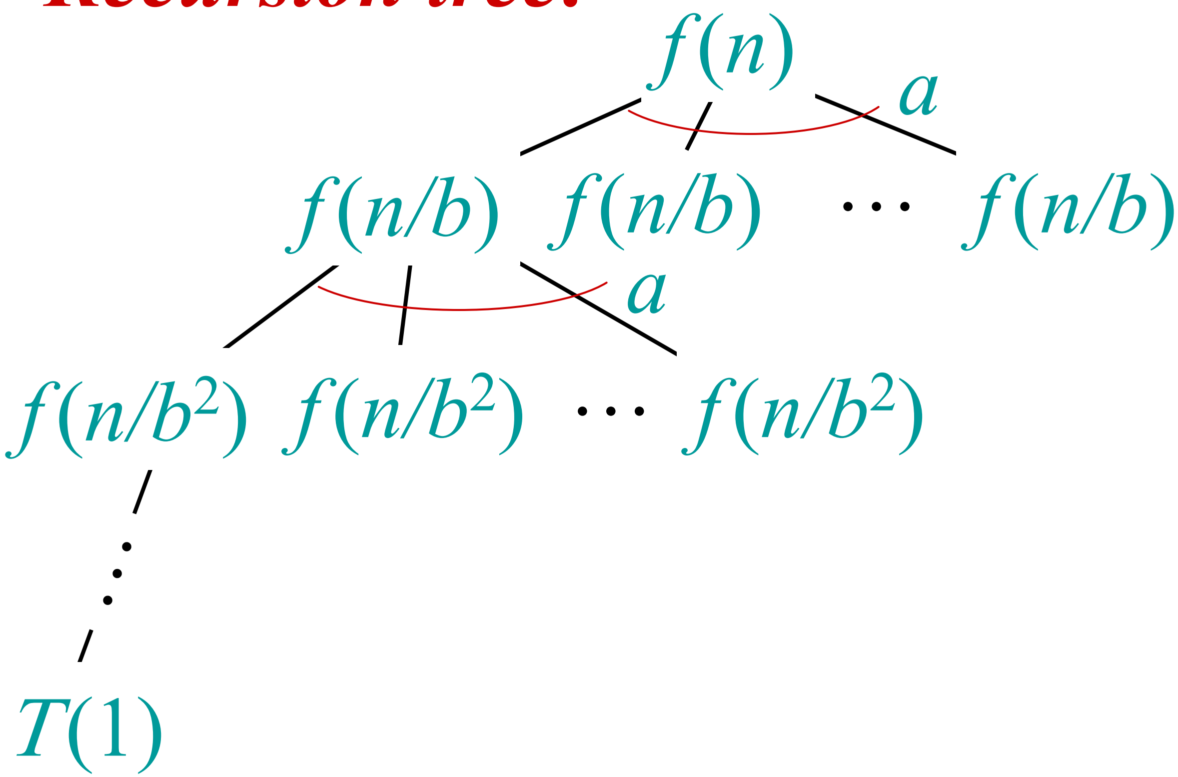
$a = 4,\ b = 2 \Rightarrow n^{\log_b a} = n^2;\ f(n) = n^2/\lg n.$

Master method does not apply. In particular, for every constant $\varepsilon > 0$, we have $n^{\varepsilon} = \omega(\lg n)$.

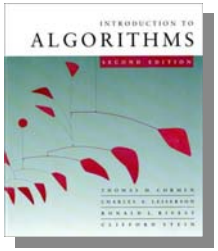


# Idea of master theorem

***Recursion tree:***



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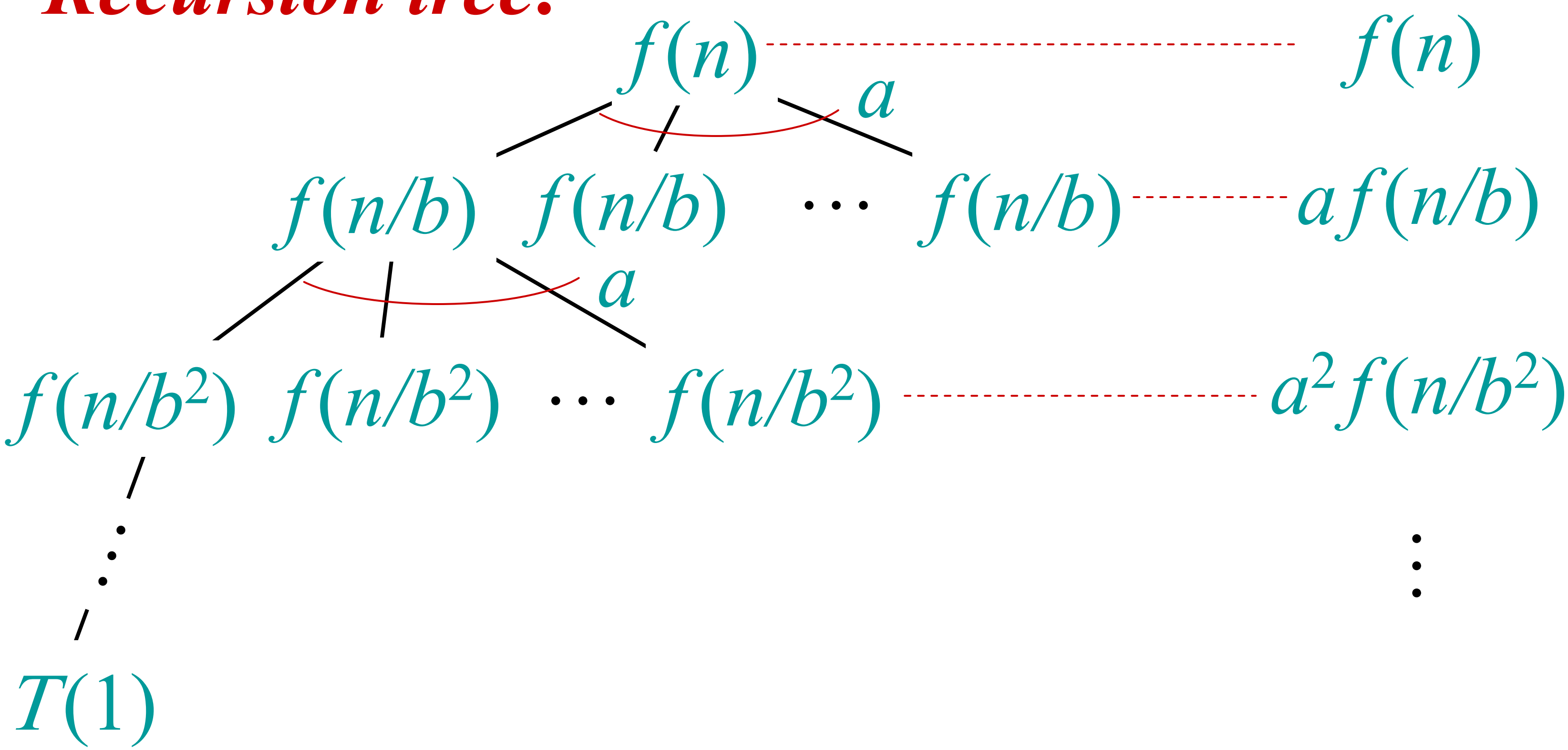

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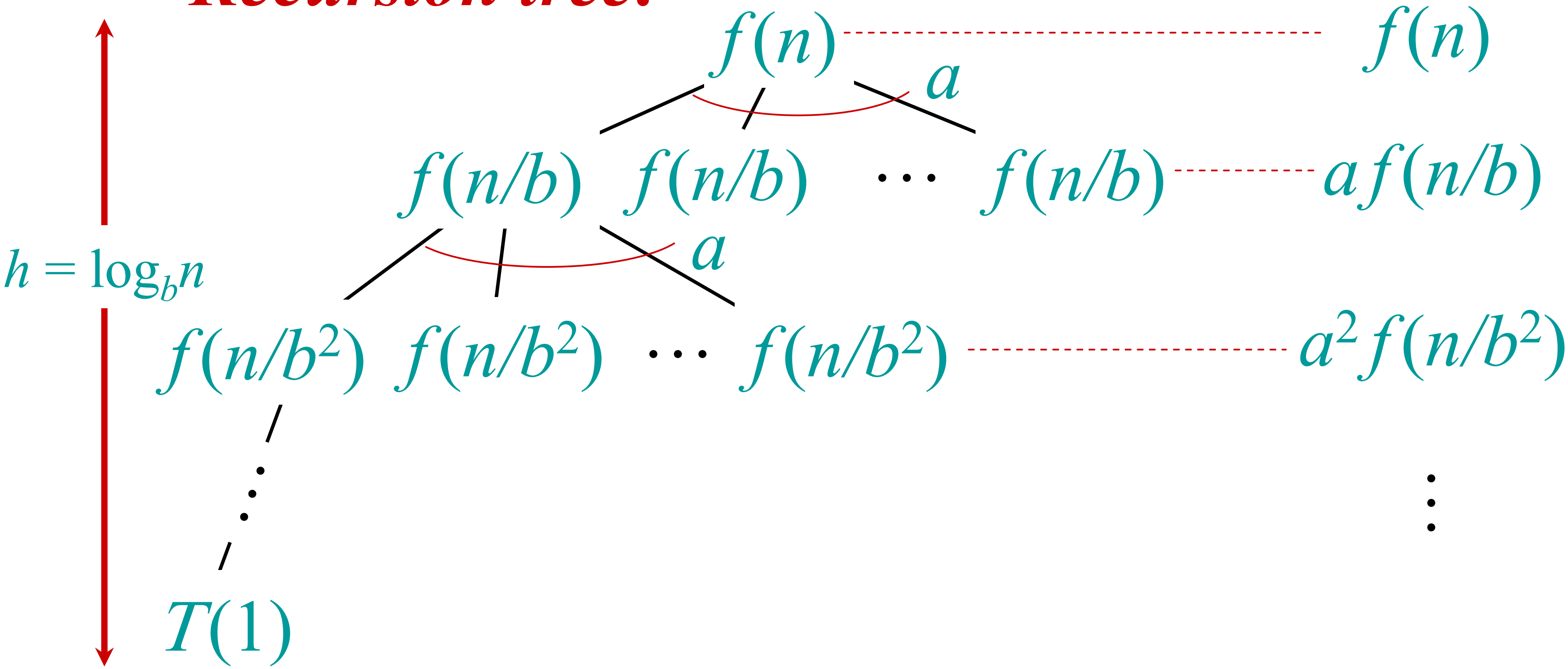
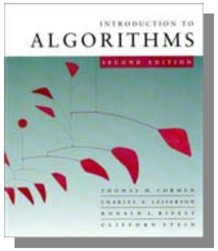





# Idea of master theorem

***Recursion tree:***

$$f(n) \quad \cdots\cdots \quad f(n)$$

$a$

$$f(n/b) \quad f(n/b) \quad \cdots \quad f(n/b) \quad \cdots\cdots \quad a\,f(n/b)$$

$a$

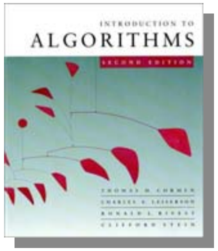
$$h = \log_b n$$

$$f(n/b^2) \quad f(n/b^2) \quad \cdots \quad f(n/b^2) \quad \cdots\cdots \quad a^2 f(n/b^2)$$

$$\vdots$$

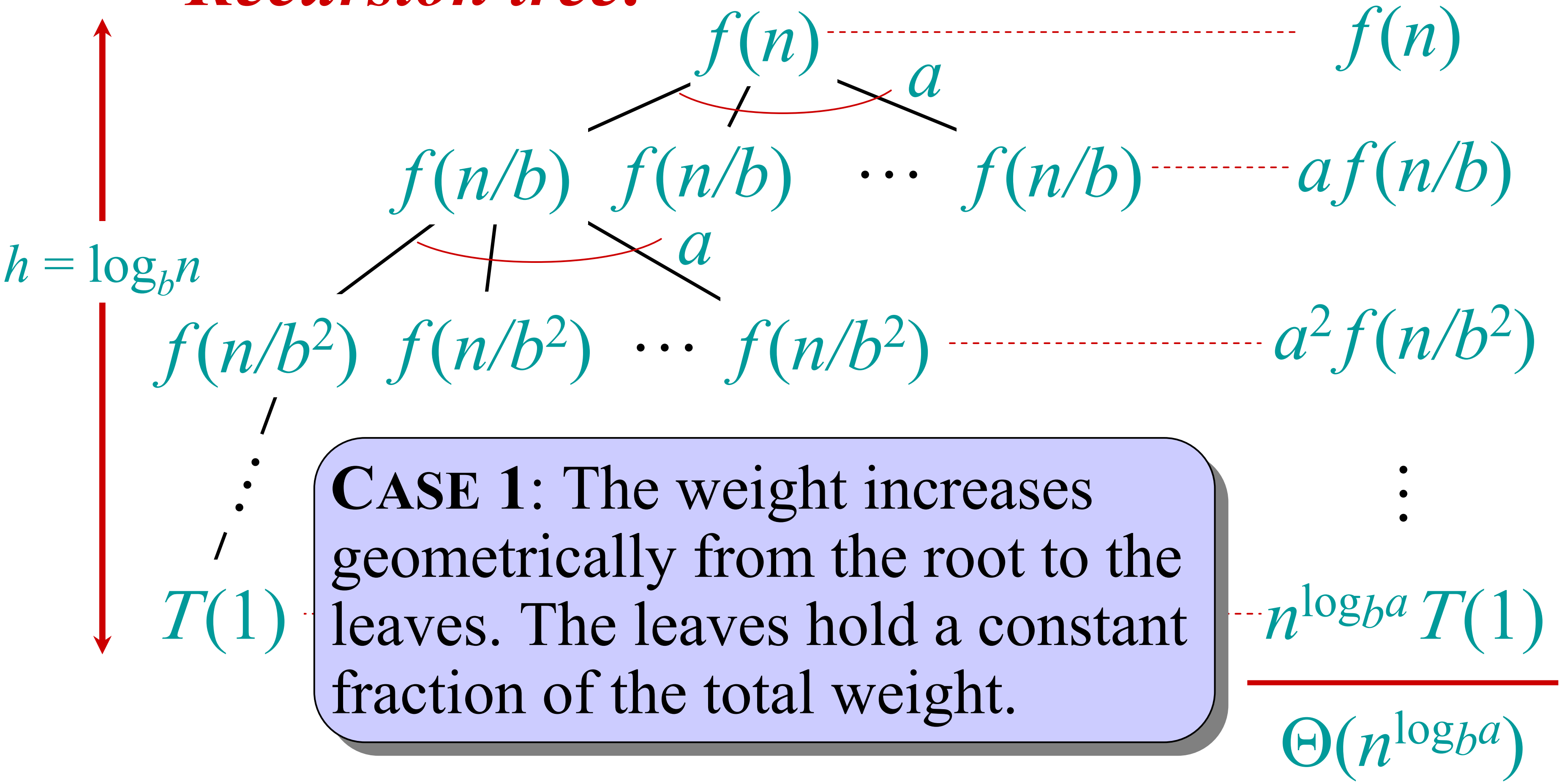
$$T(1) \quad \cdots\cdots \quad n^{\log_b a}\,T(1)$$

$$\begin{aligned} \#\text{leaves} &= a^h \\ &= a^{\log_b n} \\ &= n^{\log_b a} \end{aligned}$$

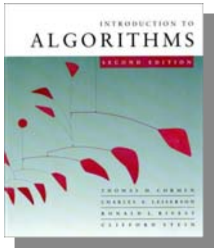


# Idea of master theorem

## *Recursion tree:*




**CASE 1**: The weight increases geometrically from the root to the leaves. The leaves hold a constant fraction of the total weight.




# Idea of master theorem

***Recursion tree:***

$f(n)$ ........ $f(n)$

$a$

$f(n/b)$ $f(n/b)$ $\cdots$ $f(n/b)$ ........ $a\,f(n/b)$

$a$

$h = \log_b n$

$f(n/b^2)$ $f(n/b^2)$ $\cdots$ $f(n/b^2)$ ........ $a^2 f(n/b^2)$

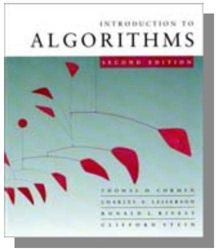
$\vdots$ $\vdots$

$T(1)$ ........ $n^{\log_b a}\,T(1)$

**CASE 2**: ($k = 0$) The weight is approximately the same on each of the $\log_b n$ levels.

$\Theta(n^{\log_b a} \lg n)$




# Idea of master theorem

***Recursion tree:***

$f(n)$ ——— $f(n)$

$a$

$f(n/b) \quad f(n/b) \quad \cdots \quad f(n/b)$ ——— $a f(n/b)$

$a$

$h = \log_b n$

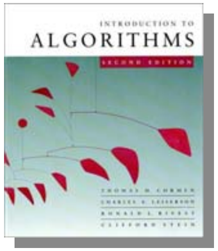
$f(n/b^2) \quad f(n/b^2) \quad \cdots \quad f(n/b^2)$ ——— $a^2 f(n/b^2)$

$\vdots$

$T(1)$ ——— $n^{\log_b a} T(1)$

$\Theta(f(n))$

**CASE 3**: The weight decreases geometrically from the root to the leaves. The root holds a constant fraction of the total weight.




# Appendix: geometric series

$$1 + x + x^2 + \cdots + x^n = \frac{1 - x^{n+1}}{1 - x} \quad \text{for } x \neq 1$$

$$1 + x + x^2 + \cdots = \frac{1}{1 - x} \quad \text{for } |x| < 1$$

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