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Answer 1

a.
Let's start from the right hand-side. It is stated that $f_k^{-1}(A_k) = \{x \in E | f_k(x) \in A_k\}$. Therefore we can write right hand-side as follows,

$$\bigcap_{k=1}^{n} f_k^{-1}(A_k)$$
$$= f_1^{-1}(A_1) \cap f_2^{-1}(A_2) \cap ... \cap f_n^{-1}(A_n)$$
$$= \{x \in E | f_1(x) \in A_1 \wedge f_2(x) \in A_2 \wedge ... \wedge f_n(x) \in A_n\}$$
$$= \{x \in E | x_1 \in A_1 \wedge x_2 \in A_2 \wedge ... \wedge x_n \in A_n\}$$
$$\text{for } x = (x_1, x_2, x_3, ..., x_n)$$

This means that any $i'th$ element of any tuple $x \in E$ should also be an element of A_i. The set we formulate in such a way is obviously the definition of cartesian product

$$A_1 \times A_2 \times ... \times A_n = \prod_{k=1}^{n} A_k$$

Which forms a set of tuples s.t. $x = (x_1, x_2, ..., x_n)$ and $x_i \in A_i$.
b.
Function f_2 is not 1-to-1 if $\exists i \, |E_i| \geq 2, \; i = 1, 3, 4, 5, ..., n$

The reason is, in such a case we can easily find two different tuples $x_a, x_b \in E$ s.t. their second elements are both $x_2 \in E_2$ but their $i'th$ elements are different therefore we would have such a case that

$$f_2(x_a) = x_2 \text{ and } f_2(x_b) = x_2 \text{ and } x_a \neq x_b$$

Therefore f_2 is not for the mentioned case 1-to-1.
On the other hand, if $\forall i \, |E_i| < 2$, in cartesian product there will only be one tuple s.t. $x \in E and |E| = 1$ which in that case obviously

$$f_2(x) = x_2$$

And the function is 1-to-1.

The case for all sets E_i are empty the proof is trivial and f_2 is an empty function.
c.
For f_1 to be on to, it's co-domain should equal to it's range. Range of f_1 is given as E_1. Let's choose an arbitrary $x_1 \in E_1$ and see if we can find an $x \in E$ s.t. $f(x) = x_1$.

Since the definition of E is the cartesian product of all tuples $E_i, i = 1, 2, ..., n$, if an element x_i is in E_i we can find at least one tuple $x \in E$ s.t. $i'th$ element of x is x_i and we can say for that tuple $f(x) = x_i$. Applying the same steps for our x_1 show that we indeed can find such an x.

Since our x_1 was chosen arbitrarily from $f's$ range, it's range should be equal to it's co-domain, ergo function f_1 is on to.

d.

$$\begin{aligned}\overline{f_K^{-1}(A_k)} &= \{x \in E | \neg(f(k) \in A_k)\} \\ &= \{x \in E | f(k) \notin A_k\} \\ &= \left\{x \in E | f(k) \in \overline{A_k}\right\} \\ &= f_k^{-1}(\overline{A_k})\end{aligned}$$

e.

The cartesian product

$$A_1 \times E_2 \times E_3 \times ... \times E_n$$

will include all tuples from E where first element of the tuple is an element of A_1 such that

$$\{x \in E | x_1 \in A_1 \land x_i \in E_i\}$$
$$x = (x_1, x_2, ..., x_n) \text{ and } i = 2, 3, 4...n$$

We can inverse this set as follows, since by definition $x_i \in E_i$

$$\{x \in E | x_1 \notin A_1 \lor x_i \notin E_i\}$$
$$\{x \in E | x_1 \notin A_1 \lor (false)\}$$
$$\{x \in E | x_1 \notin A_1\}$$

In other words we are looking for the set of $x \in E$ tuples s.t. $x_1 \notin A_1$. Since it is given that E_k is the universal set of A_k we can formulate this as follows

$$(E_1 \setminus A_1) \times \prod_{i=2}^{n} E_i$$
$$\overline{A_1} \times \prod_{i=2}^{n} E_i$$

This concludes our proof.

Answer 2

a.

To show that f has inverse, we will show it is both 1-to-1 and on to.

1-TO-1

Let's choose two arbitrary $x_1, x_2 \in Z$ s.t.

$$f(x_1) = f(x_2)$$

To prove then $x_1 = x_2$ is also true, we need to consider three cases.
1)$x_1, x_2 < 0$ Then by the definition of the function f we have

$$2|x_1| = 2|x_2|$$
$$-x_1 = -x_2$$
$$x_1 = x_2$$

2)$x_1, x_2 \geq 0$ Then by the definition of function f we have

$$2x_1 + 1 = 2x_2 + 1$$
$$x_1 = x_2$$

3)$x_1 < 0, x_2 \geq 0$ Then by the definition of the function f we have

$$2|x_1| = 2x_2 + 1$$
$$-x_1 - x_2 = \frac{1}{2}$$
$$x_1 + x_2 = -\frac{1}{2}$$

Which can't happen because we chose x_1, x_2 to be whole numbers so in that case $f(x_1) \neq f(x_2)$, ergo in all cases we conclude $f(x_1) = f(x_2) \rightarrow x_1 = x_2$ holds true and function f is 1-to-1.
<u>On To</u>
For a function to be on to, it's co-domain should be equal to it's range. Let's try to find the co-domain of each partial function f_1 and f_2.
For f_1 let's choose an arbitrary $y \in N^+$ and assume an $x < 0$ and $x \in Z$ s.t. $f(x) = y$. Then we have

$$y = -2x$$
$$x = \frac{y}{-2}$$

We see that as long as we choose our y to be divisible by 2, we indeed have such an x. We can say co-domain of f_1 contains

$$C_1 = \{y \in N^+ | \exists k y = 2k\}$$

Inversely for f_2 we do the same steps with an arbitrary y and assumed x to find

$$y = 2x + 1$$
$$x = \frac{y - 1}{2}$$

We see that as long as we choose our y so that $y - 1$ is divisible by 2,in other words for y to be odd, we indeed have such an x. We can say co-domain of f_2 contains

$$C_2 = \{y \in N^+ | \exists k y = 2k + 1\}$$

Finally, if we take $C_1 \cup C_2$ which is the subset of the co-domain of function f, it is clear that co-domain is equal to range N^+. This concludes our proof.
b.
It is easy to show from part (a) that $f^{-1} : N^+ \to Z$ and

$$f^{-1}(x) = \begin{cases} \dfrac{x}{-2} & x \text{ } is \text{ } even \\ \dfrac{x-1}{2} & x \text{ } is \text{ } odd \end{cases}$$

So we find

$$f(26) = \frac{26}{-2} = -13$$

Answer 3

First let's set these to hold true for $\forall n \geq 2$

$$n \geq 2 \qquad \log_2 n \geq 1 \qquad n \log_2 n \geq 1 \qquad n^2 \geq 1$$

From these with help of Lemma1 we can conclude the below inequalities

$$\log_2 n \leq n^2 \log_2 n \qquad n \leq n^2 \leq n^2 \log_2 n \qquad n \log_2 n \leq n^2 \log_2 n$$

We can also conclude $n \log_2^2 n \leq n^2 \log_2 n$ from $\log_2 n \leq n$ with the help of Lemma2.
Now by the definition of big-oh let us try to find k, c s.t. $\forall n \geq k \, \exists c \big(f(n) \leq cg(n)\big)$

$$\begin{aligned} f(n) = 12(\log_2 n + n)(n + 3n \log_2 n) + 6n^2 &= 12\big[n \log_2 n + 3n \log_2^2 n + n^2 + 3n^2 \log_2 n\big] + 6n^2 \\ \leq 12\big[n^2 log_2 n + 3n^2 \log_2 n + n^2 \log_2 n + 3n^2 \log_2 n\big] &+ 6n^2 \log_2 n \leq 102 n^2 \log_2 n \end{aligned}$$

We see for $k = 2$ and $c = 102$

$$f(n) = O\big(n^2 \log_2 n\big) = O\big(g(n)\big)$$

Lemma1
Let's choose two $x, y \in N, x > 0, y > 1$.
Let's assume $x > xy$. Since $x > 0$, we can divide both sides by x to get $1 > y$ which contradicts with our premise. Therefore our assumption is wrong, therefore $x \leq xy$.
In other words, a positive number multiplied with another number greater than one will always be less than or equal to himself.

Lemma2
By refering to Chapter 3 Figure 3(Rosem& Kenneth, Discrete Mathematics and It's Applications,p. 211) we can see for $n \geq 2$, $\log_2 n \leq n$. Than from this we can multiple both sides with $n \log_2 n$ to get

$$n \log_2^2 n \leq n^2 \log_2 n$$

Answer 4

Let's assume $E \setminus S$ is countable.

It is given that S is countable, therefore by Lemma1 $(E \setminus S) \cup S$ is also countable. But then by Lemma2 we see $E \subseteq (E \setminus S) \cup S$ so E is contained in a countable set so it must itself be countable. This contradicts with our premise that E is uncountable, which means our assumption is wrong, ergo $E \setminus S$ is uncountable.

$\underline{Lemma1}$

Let A and B be two countable sets. That means both can be written as
$A = \{a_1, a_2, a_3...\}$
$B = \{b_1, b_2, b_3...\}$
We can take union of these sets such a way that
$A \cup B = \{a_1, b_1, a_2, b_2...\}$
Which can be listed and is clearly countable.

$\underline{Lemma2}$

We will try to prove $A \subseteq ((A \setminus B) \cup B)$

$(A \setminus B) \cup B = \{x | x \in (A \setminus B) \vee x \in B\} = \{x | (x \in A \wedge x \notin B) \vee x \in B\} = \{x | x \in A \wedge (x \notin B \vee x \in B)\} = \{x | x \in A \wedge (TRUE)\} = \{x | x \in A\} = A$

Answer 5

a.
Assume $n \equiv 1(mod3)$. Keeping in mind our Lemma1, then $n + 1 = 1 + 1 = 2(mod3)$. By multiplying the formulas with each other we conclude the following,

$$n(n + 1) = 1 \cdot 2 = 2(mod3)$$

This concludes the first part of our proof. Otherwise, if $n \not\equiv 1(mod3)$ then either $n \equiv 2(mod3)$ or $n \equiv 0(mod3)$ must be true.
For the first case we have $n + 1 \equiv 2 + 1 \equiv 3 \equiv 0(mod3)$ and multiplying we get

$$n(n + 1) = 2 \cdot 0 \equiv 0(mod3)$$

For the second case, $n + 1 \equiv 0 + 1 \equiv 1(mod3)$ and again multiplying with each other

$$n(n + 1) \equiv 0 \cdot 1 \equiv 0(mod3)$$

And this concludes our proof.
Lemma1
Refer to Chapter 4 Theorem 5(Rosen& Kenneth,Discrete Mathematics and It's Applications,p.

242).

b.
$gcd\big(123, 277\big) = gcd\big(277, 123\big) = gcd\big(123, 277(mod\,123)\big) = gcd\big(123, 31\big) = gcd\big(31, 123(mod\,31)\big)$
$= gcd\big(31, 30\big) = gcd\big(30, 31(mod\,30)\big) = gcd\big(30, 1\big) = gcd\big(1, 30(mod\,1)\big) = gcd\big(1, 0\big) = 1$

c.
Let's first see if the first part of the implication holds true. For that, we will try to find the set of possible p values. It is given $p > 2$, p is even and p is a prime, so let's write all three sets as follows and take their intersection.

$$P_1 = \{x \in N | x > 2\} = \{3, 4, 5...\}$$

$$P_2 = \{x \in N | x\ is\ even\} = \{2, 4, 6, 8, ...\}$$

$$P_3 = \{x \in N | x\ is\ prime\} = \{2, 3, 5, 7, 9, ...\}$$

Let's first try to intersect P_2 and P_3. P_2 implies

$$x \in P_2 \rightarrow \exists k\ x = 2k$$

And P_3 implies

$$x \in P_3 \rightarrow \forall x_1, x_2\ \big(x = x_1 x_2 \rightarrow ((x_1 = x \wedge x_2 = 1) \vee (x_2 = x \wedge x_1 = 1))\big)$$

To find an element of intersection $P_2 \cap P_3$ we need to find x such that satisfies both equations. Since $x = 2k$ that means either $2 = 1\ k = x$ or $k = 1\ 2 = x$, obviously only the later is possible. So we can conclude

$$P_2 \cap P_3 = \{x \in N | x = 2\} = \{2\}$$

And finally it is obvious that when we intersect the resulting set with P_1 we get $\emptyset$ which means we can't find such p, ergo the first part of the implication always holds false.
We conclude that since $false \rightarrow P$ is always true, the given implication is also true.