Université de Genève

Multimedia Security and Privacy

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TP 3 : Elements of Detection Theory

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1 Exercices

1.1

Let $X \sim \mathcal{N}(0,1)$ and $Y \sim \mathcal{N}(0,3)$:

$$P[-2 < Y \leq 1] : \tag{1}$$
$$=> \{(-2-(0))/3 < N \leq (1-0)/3\} : \tag{2}$$
$$=> P[-2/3 < N \leq 1/3] = \phi(1/3) - \phi(-2/3) = \phi(1/3) - (1-\phi(2/3)) = 0.63 - (1-0.74) = 0.37 \tag{3}$$

$$P[Y > 5.5] = 1 - P[Y \leq 5.5] : \tag{4}$$
$$P[Y \leq 5.5] = \phi(5.5/3) = 0.96 \tag{5}$$
$$=> 1 - 0.96 = 0.04 \tag{6}$$

$$P[-2 < X \leq 2] : \tag{7}$$
$$=> P[-2 < N \leq 2] = \phi(2) - (1-\phi(2)) = 0.97 - 0.03 = 0.94 \tag{8}$$

$$P[X > 1.5] = 1 - P[X \leq 1.5] : \tag{9}$$
$$=> P[X \leq 1.5] = \phi(1.5) = 0.93 \tag{10}$$
$$=> 1 - 0.93 = 0.07 \tag{11}$$

1.2

Let X denote the peak temperature in Geneva, in June, as measured in Celsius, for which holds: $X \sim N(27, 9)$. What is:

$$P[X > 35] = 1 - P[X \leq 35] : \tag{12}$$
$$P[X \leq 35] = \phi((35-27)/9) = 0.81 \tag{13}$$
$$=> 1 - 0.81 = 0.19 \tag{14}$$

$$P[X \leq 5] = \phi((5-27)/9) = \phi(-2.4) = 1 - \phi(2.4) = 0.01 \tag{15}$$

$$P[20 < X \leq 40] = P[((20-27)/9) < N \leq ((40-27)/9)] = P[-7/9 < N \leq 13/9] \tag{16}$$
$$= \phi(13/9) - \phi(-7/9) = \phi(13/9) - (1-\phi(7/9)) = 0.91 - (1-0.77) = 0.68 \tag{17}$$

1.3

Let X be a Gaussian random variable, for which E[X] = 0 and P[|X| $\leq$ 10] = 0.3. What is σ_X?

$$P[|X| \leq 10] = P[-10 \leq X \leq 10] = P[-10/\sigma \leq N \leq 10/\sigma] \tag{18}$$
$$= \phi(10/\sigma) - (1 - \phi(10/\sigma)) = 0.3 \tag{19}$$
$$=> \phi(10/\sigma) = 0.65 \tag{20}$$
$$=> 10/\sigma \cong 0.4 \tag{21}$$
$$=> \sigma = 25 \tag{22}$$

1.4

Prove that:

$$Q(n) = \frac{1}{2} erfc(\frac{n}{\sqrt{2}}) :$$

$$=> Q(n) = \frac{1}{\sqrt{2\pi}} \int_n^\infty e^{\frac{-x^2}{2}} dx = \frac{1}{2}(\frac{2}{\sqrt{\pi}} \int_{n/\sqrt{2}}^\infty exp(-x^2)dx) \tag{23}$$
$$= \frac{1}{2} - \frac{1}{2} erf(\frac{n}{\sqrt{2}}) = \frac{1}{2} erfc(\frac{n}{\sqrt{2}}) \tag{24}$$

1.5

Let there be two hypothesizes, H_0 and H_1:

H_0 : X = Z

H_1 : X = μ1 + Z

where Z $\sim$ N (0, 1) and μ1 = 1.

Determine the separation threshold τ following the MAP hypothesis, or likelihood ratio test:

We have that $X \sim \mathcal{N}(0, 1)$ with hypothesis 0 and X $\sim \mathcal{N}(1, 1)$ with hypothesis 1:

$$x \in A_0, if x \leq \frac{e^{\frac{-x^2}{2\sigma^2}}}{e^{\frac{-(x-1)^2}{2\sigma^2}}} \geq \frac{P[H_1]}{P[H_0]}$$

$$x \in A_0, if x \leq ln(\frac{P[H_1]}{P[H_0]}) \frac{2\sigma^2 + 1}{-2}$$

$$x \in A_0, if x \leq \tau$$

Determine the probability of correct detection $p_d = 1 - p_m$:

$$p_d = 1 - P(A_0|H_1) = 1 - \int_{x<\tau} f_{x|H_1}(x)dx = 1 - \phi(\tau - 1)$$

1.6

Derive the general formula's for the Probability of Miss, p_m, the Probability of correct detection, p_d, and the Probability of False Alarm, p_f :

$$p_d = 1 - P(A_0|H_1) = 1 - \int_{x<\tau} f_{x|H_1}(x)dx = 1 - \phi(\tau - \mu 1)$$

$$p_m = P(A_0|H_1) = \phi(\tau - \mu 1)$$

$$p_f = P(A_1|H_0) = 1 - \phi(\tau)$$

Receiver Operating Characteristic (ROC) curve for each different value of $\mu 1 = 0$, 1, 2:

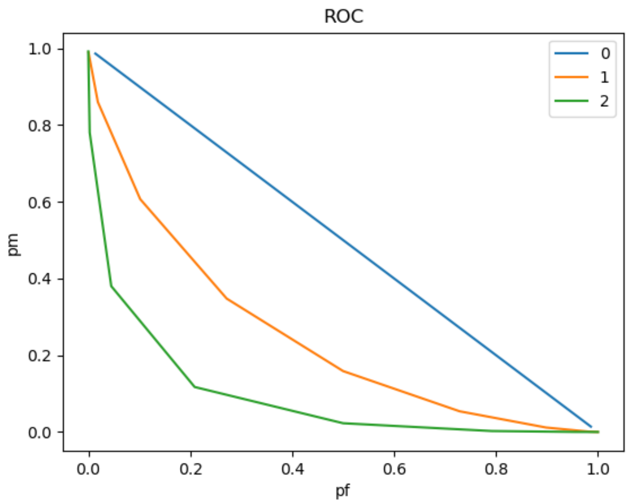

Figure 1: ROC

We clearly observe that for $\mu 1 = 0$, pf = 1 - pm. The more $\mu 1$ is big the more the pm decrease quickly over the pf. It can be explained by the fact that we centralize τ by $\mu 1$ so the obtained pm by using ϕ is much smaller. Furthermore, the bigger is $\mu 1$ the bigger is the p_d.

1.7

Formulate hypothesises H_0 and H_1:

H_0: Y = W

H_1: Y = V + W

Use the binary hypothesis likelihood ratio test to determine the rule that minimises p_{err}, the probability of a decoding error:

$$x \in A_0, if y \leq \frac{f_{Y|H_0}}{f_{Y|H_1}} \tag{25}$$

$$x \in A_0, if y \leq \frac{e^{-y}}{ye^{-y}} \tag{26}$$

$$=> y \leq \frac{1}{y} \tag{27}$$

$$=> \tau = 1 \tag{28}$$

Determine p_{err} for the optimum decision rule:

$$p_{err} = p_m p(H_1) + p_f p(H_0) = p_m 0.5 + p_f 0.5 \tag{29}$$

$$<=> p_f = \int_{y>\tau} f_{Y|H_0} = 0.36 <=> p_m = \int_{y<\tau} f_{Y|H_1} = 0.26 => p_{err} = 0.5 * 0.26 + 0.5 * 0.36 = 0.31 \tag{30}$$