

MODEL PREDICTIVE CONTROL

HYBRID MPC EXAMPLES

Alberto Bemporad

imt.lu/ab

COURSE STRUCTURE

- ✓ Basic concepts of model predictive control (MPC) and linear MPC
- ✓ Linear time-varying and nonlinear MPC
- ✓ Quadratic programming (QP) and explicit MPC
 - **Hybrid** MPC
 - **Stochastic** MPC
 - **Learning-based** MPC

Course page:

http://cse.lab.imtlucca.it/~bemporad/mpc_course.html

HYBRID MPC EXAMPLES

HYBRID MPC FOR CRUISE CONTROL



Disclaimer: This is
an academic example

GOAL:

Command **gear ratio**, **gas pedal**, and **brakes** to **track** a
desired **speed** and minimize fuel **consumption**

HYBRID MPC FOR CRUISE CONTROL - MODEL

- Vehicle dynamics

$$m\ddot{x} = F_e - F_b - \beta\dot{x}$$

\dot{x} = vehicle speed

F_e = traction force

F_b = brake force

→ discretized with sampling time $T_s = 0.5$ s

- Transmission kinematics

$$\omega = \frac{R_g(i)}{k_s} \dot{x}$$

ω = engine speed

C = engine torque

power balance:

$$F_e = \frac{R_g(i)}{k_s} C$$

i = gear

$$F_e \dot{x} = C \omega$$

HYBRID MPC FOR CRUISE CONTROL - MODEL

- Gear selection: for each gear #i, define a binary input $g_i \in \{0, 1\}$
 $i = R, 1, 2, 3, 4, 5$

- Gear selection (traction force):

$$F_e = \frac{R_g(i)}{k_s} C \quad \text{depends on gear } \#i$$

define auxiliary continuous variables:



IF $g_i = 1$ THEN $F_{ei} = \frac{R_g(i)}{k_s} C$ ELSE 0

$$\rightarrow F_e = F_{eR} + F_{e1} + F_{e2} + F_{e3} + F_{e4} + F_{e5}$$

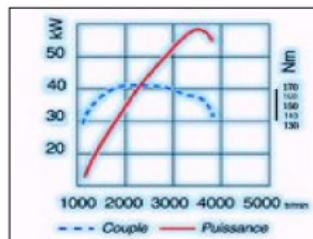
- Gear selection (engine/vehicle speed):

$$\omega = \frac{R_g(i)}{k_s} \dot{x} \quad \text{similarly, also requires 6 auxiliary continuous variables}$$

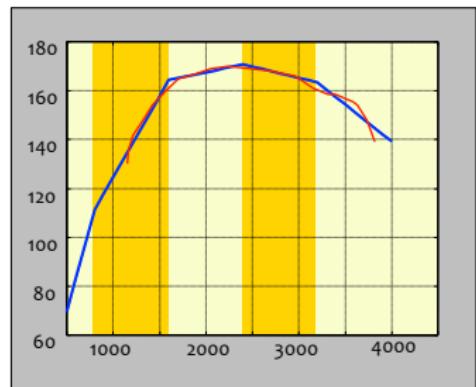
HYBRID MPC FOR CRUISE CONTROL - MODEL

- engine torque $-C_e^-(\omega) \leq C \leq C_e^+(\omega)$

- max engine torque $C_e^+(\omega)$



Piecewise-linearization
(PWL Toolbox, Julián, 2000)



requires: 4 binary aux variables
4 continuous aux variables

- Min engine torque $C_e^-(\omega) = \alpha_1\omega + \beta_1$

Note: in this case the PWL constraint $C \leq C_e^+(\omega)$ is convex, it could be handled by linear constraints without introducing any binary variable !

HYBRID MPC FOR CRUISE CONTROL - HYSDEL

```
SYSTEM cruisecontrolmodel {  
  
INTERFACE {  
    PARAMETER {  
        REAL mass = 1020; /* kg */  
        REAL beta_friction = 25; /* W/m*s */  
  
        [snip]  
    }  
  
    STATE { REAL position [0,10000];  
            REAL speed [vmin,vmax]; }  
  
    INPUT { REAL torque [Cmin,Cmax];  
            REAL F_brake [0,max_brake_force];  
            BOOL gear1, gear2, gear3, gear4, gear5, gearR; }  
}  
  
IMPLEMENTATION {  
    AUX (REAL F, Fe1, Fe2, Fe3, Fe4, Fe5, FeR;  
        REAL w, w1, w2, w3, w4, w5, wR;  
        BOOL dPWL1,dPWL2,dPWL3,dPWL4;  
        REAL DCe1,DCe2,DCe3,DCe4; )  
  
    LINEAR (F = Fe1+Fe2+Fe3+Fe4+Fe5+FeR;  
            w = w1+w2+w3+w4+w5+wR; )  
  
    AD { dPWL1 = wPWL1-w<=0;  
        dPWL2 = wPWL2-w<=0;  
        dPWL3 = wPWL3-w<=0;  
        dPWL4 = wPWL4-w<=0; }  
  
    DA { Fe1 = {IF gear1 THEN torque/speed_factor*Rgear1};  
        Fe2 = {IF gear2 THEN torque/speed_factor*Rgear2};  
        Fe3 = {IF gear3 THEN torque/speed_factor*Rgear3};  
        Fe4 = {IF gear4 THEN torque/speed_factor*Rgear4};  
        Fe5 = {IF gear5 THEN torque/speed_factor*Rgear5};  
        FeR = {IF gearR THEN torque/speed_factor*RgearR};  
    }  
  
    w1 = {IF gear1 THEN speed/speed_factor*Rgear1};  
    w2 = {IF gear2 THEN speed/speed_factor*Rgear2};  
    w3 = {IF gear3 THEN speed/speed_factor*Rgear3};  
    w4 = {IF gear4 THEN speed/speed_factor*Rgear4};  
    w5 = {IF gear5 THEN speed/speed_factor*Rgear5};  
    wR = {IF gearR THEN speed/speed_factor*RgearR};  
  
    DCe1 = {IF dPWL1 THEN (aPWL2-aPWL1)+(bPWL2-bPWL1)*w};  
    DCe2 = {IF dPWL2 THEN (aPWL3-aPWL2)+(bPWL3-bPWL2)*w};  
    DCe3 = {IF dPWL3 THEN (aPWL4-aPWL3)+(bPWL4-bPWL3)*w};  
    DCe4 = {IF dPWL4 THEN (aPWL5-aPWL4)+(bPWL5-bPWL4)*w};  
}  
  
    CONTINUOUS { position = position+Ts*speed;  
                 speed = speed+Ts/mass*(F-F_brake-beta_friction*speed); }  
  
    MUST { /* max engine speed */  
            /* wemin <= w1+w2+w3+w4+w5+wR <= wemax */  
  
            -w1 <= -wemin; w1 <= wemax;  
            -w2 <= -wemin; w2 <= wemax;  
            -w3 <= -wemin; w3 <= wemax;  
            -w4 <= -wemin; w4 <= wemax;  
            -w5 <= -wemin; w5 <= wemax;  
            -wR <= -wemin; wR <= wemax;  
  
            -F_brake <= 0;  
            F_brake <= max_brake_force;  
  
            -torque-(alpha1+beta1*w) <= 0;  
            torque-(aPWL1+bPWL1*w+DCe1+DCe2+DCe3+DCe4)-1<=0;  
  
            -((REAL gear1)+(REAL gear2)+(REAL gear3)+(REAL gear4)+  
                (REAL gear5)+(REAL gearR))<=-0.9999;  
            (REAL gear1)+(REAL gear2)+(REAL gear3)+(REAL gear4)+  
                (REAL gear5)+(REAL gearR)<=1.0001;  
  
            dPWL4 -> dPWL3; dPWL4 -> dPWL2;  
            dPWL4 -> dPWL1; dPWL3 -> dPWL2;  
            dPWL3 -> dPWL1; dPWL2 -> dPWL1;  
        }  
    }  
}
```

go to demo /demos/cruise/init.m

HYBRID MPC FOR CRUISE CONTROL - MLD MODEL

- MLD model

$$\begin{aligned}x(t+1) &= Ax(t) + B_1u(t) + B_2\delta(t) + B_3z(t) + B_5 \\y(t) &= Cx(t) + D_1u(t) + D_2\delta(t) + D_3z(t) + D_5 \\E_2\delta(t) + E_3z(t) &\leq E_4x(t) + E_1u(t) + E_5\end{aligned}$$

- 2 continuous states: x, v (vehicle position and speed)
- 2 continuous inputs: C, F_b (engine torque, brake force)
- 6 binary inputs: $g_R, g_1, g_2, g_3, g_4, g_5$ (gears)
- 1 continuous output: v (vehicle speed)
- 18 auxiliary continuous vars: (6+1 traction force, 6+1 engine speed, 4 PWL max engine torque)
- 4 auxiliary binary vars: (PWL max engine torque breakpoints)
- 100 mixed-integer inequalities

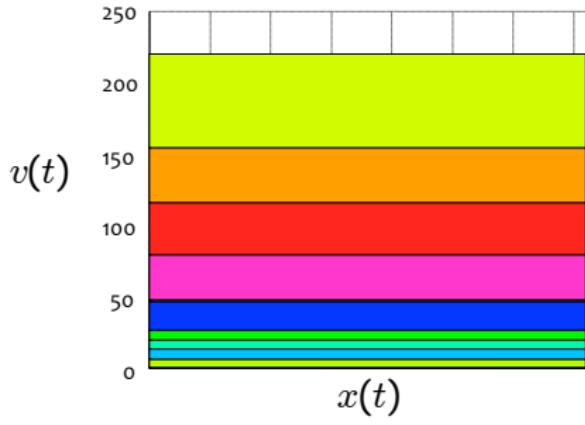
HYBRID MPC FOR CRUISE CONTROL - CONTROLLER

- Maximum-speed controller

$$\begin{aligned} \max_{u(t)} \quad & v(t+1|t) \\ \text{s.t.} \quad & \text{MLD model} \\ & v(t|t) = v(t) \\ & x(t|t) = x(t) \end{aligned}$$

Objective: maximize speed

(to reproduce max acceleration plots)



$(x(t)$ is irrelevant)

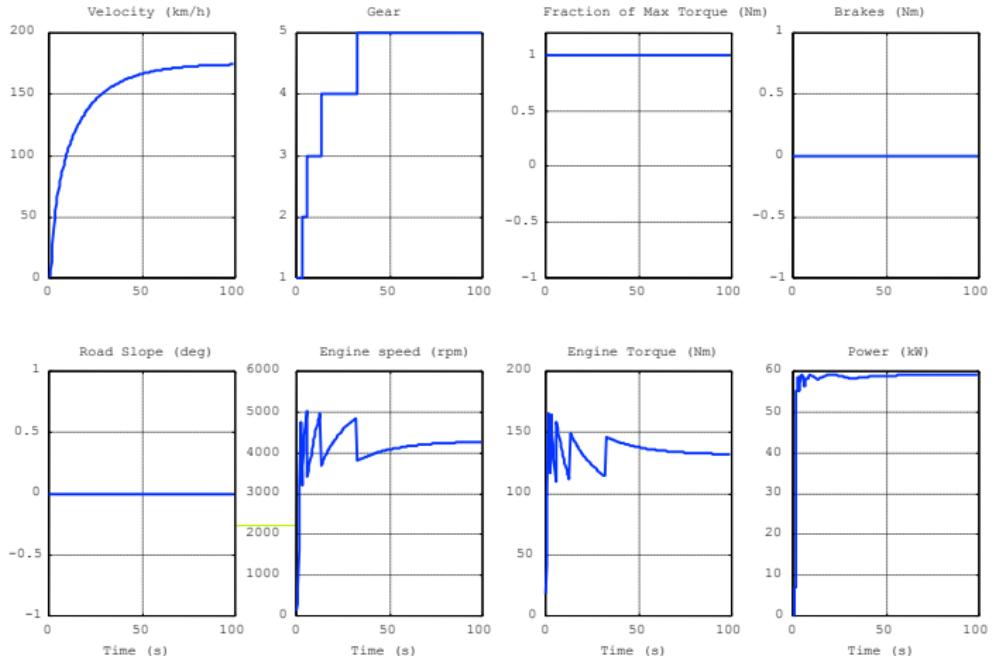
MILP optimization problem

Linear constraints	96
Continuous variables	18
Binary variables	10
Parameters	1
Time to solve mp-MILP (Sun Ultra 10)	45 s
Number of regions	11

(parameters: Renault Clio 1.9 DTI RXE)

HYBRID MPC FOR CRUISE CONTROL - RESULTS

- Maximum-speed controller



HYBRID MPC FOR CRUISE CONTROL - CONTROLLER

- **Cruise controller**

$$\min_{u(t)} |v(t+1|t) - v_d(t)| + \rho |\omega|$$

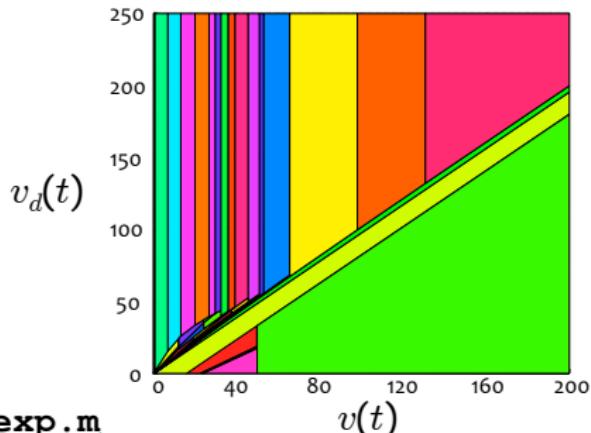
s.t. MLD model

$$v(t|t) = v(t)$$

$$x(t|t) = x(t)$$

MILP optimization problem

Linear constraints	98
Continuous variables	19
Binary variables	10
Parameters	2
Time to solve mp-MILP (PC 850Mhz)	43 s
Number of regions	49

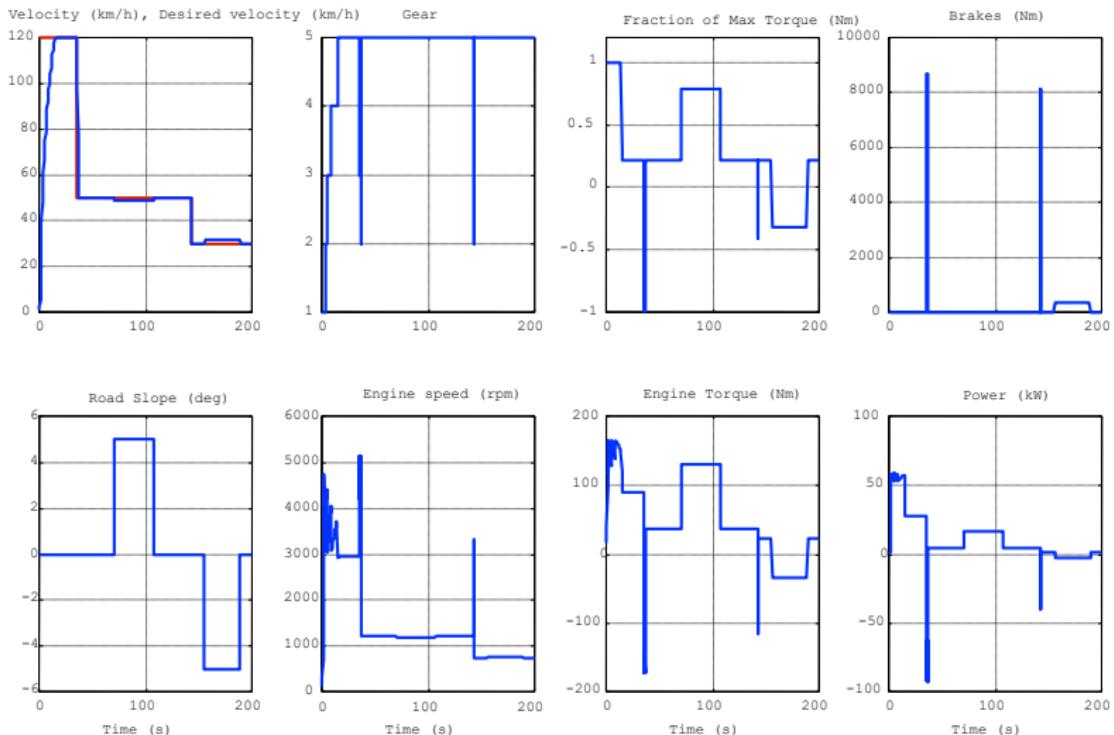


go to demo /**demos/cruise/init_exp.m**

HYBRID MPC FOR CRUISE CONTROL - RESULTS

- **Cruise controller**

$$\min_{u(t)} |v(t+1|t) - v_d(t)| + \rho|\omega|, \rho = 0.001$$



HYBRID MPC FOR CRUISE CONTROL - CONTROLLER

- Smoother cruise controller

$$\min_{u(t)} |v(t+1|t) - v_d(t)| + \rho|\omega|$$

s.t. MLD model

$$|v(t+1|t) - v(t)| \leq a_{\max} T_s$$

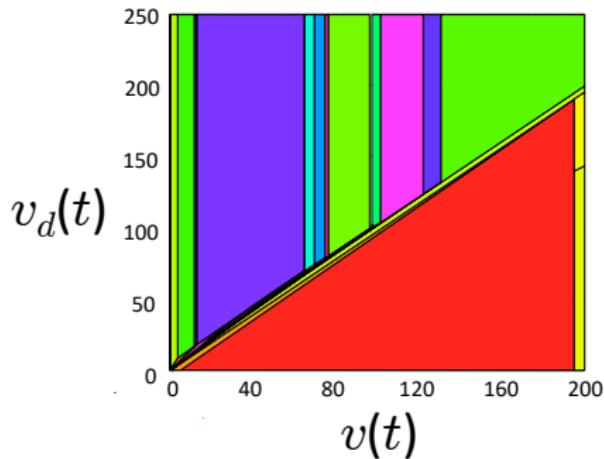
$$v(t|t) = v(t)$$

$$x(t|t) = x(t)$$



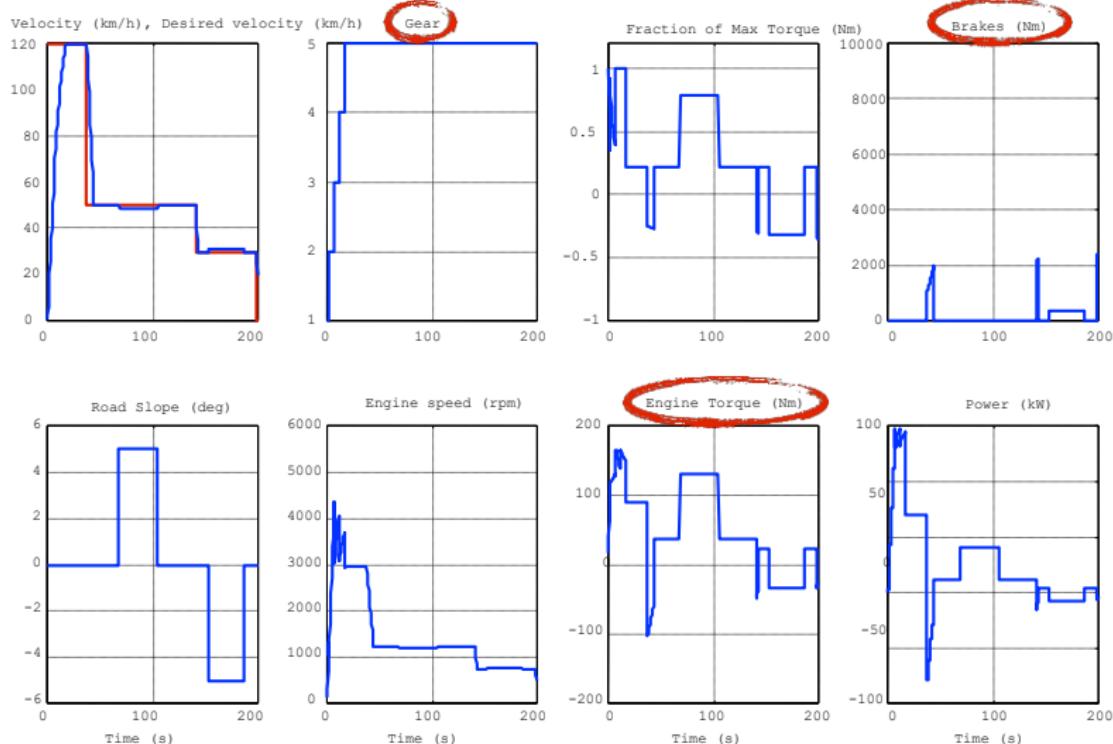
MILP optimization problem

Linear constraints	100
Continuous variables	19
Binary variables	10
Parameters	2
Time to solve mp-MILP (PC 850Mhz)	47 s
Number of regions	54



HYBRID MPC FOR CRUISE CONTROL - RESULTS

- Smoother cruise controller



HYBRID MPC FOR TRACTION CONTROL

VEHICLE TRACTION CONTROL PROBLEM

Improve driver's ability to control a vehicle under adverse external conditions (wet or icy roads)

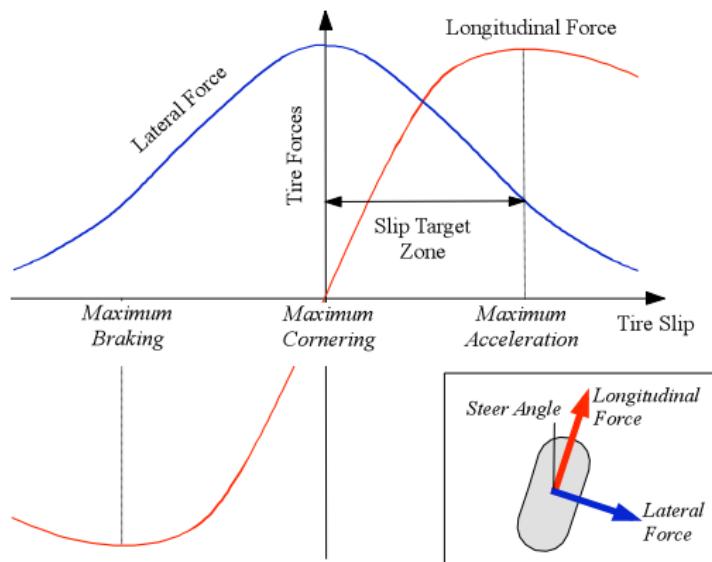


Model: nonlinear, uncertain, constraints

Controller: suitable for real-time implementation

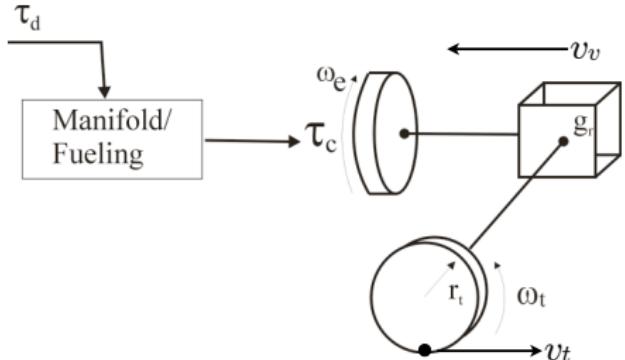
Solution: MLD hybrid framework + explicit hybrid MPC strategy

TIRE FORCE CHARACTERISTICS



SIMPLE TRACTION MODEL

(Borrelli, Bemporad, Fodor, Hrovat, 2006)



$$v_t = \omega_t r_t = \frac{\omega_e}{g_r} r_t$$

$$\Delta\omega = \frac{1}{r_t} (v_t - v_v) = \frac{\omega_e}{g_r} - \frac{v_v}{r_t} \quad \text{wheel slip}$$

- Mechanical system

$$\begin{aligned}\dot{\omega}_e &= \frac{1}{J_e} \left(\tau_c - b_e \omega_e - \frac{\tau_t}{g_r} \right) \\ \dot{v}_v &= \frac{\tau_t}{m_v r_t}\end{aligned}$$

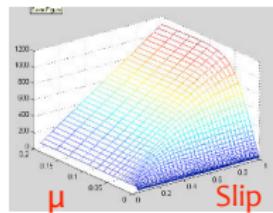
- Manifold/fueling dynamics

$$\tau_c = b_i \tau_d (t - \tau_f)$$

- Tire torque τ_t is a function of slip $\Delta\omega$ and road surface adhesion coefficient μ

HYBRIDIZATION OF TIRE CHARACTERISTICS

Torque



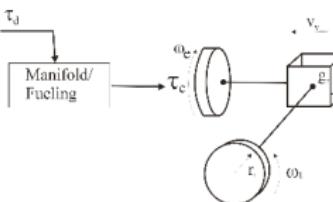
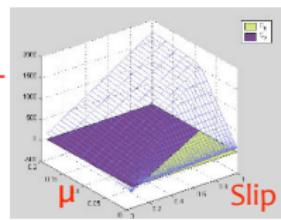
Nonlinear tire torque

$$\tau_t = f(\Delta\omega, \mu)$$

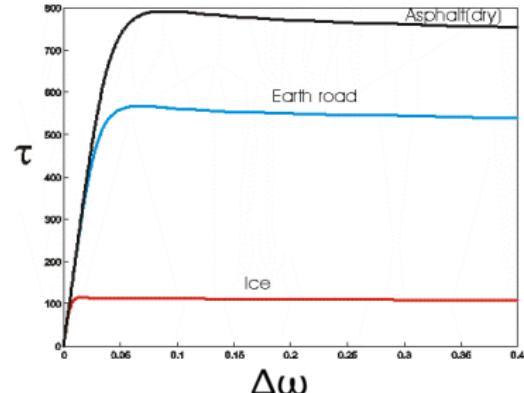


PWA approximation

Torque



HYSDEL



Mixed-Logical
Dynamical (MLD)
Hybrid Model
(discrete time)

MLD TRACTION MODEL

$$\begin{aligned}x(t+1) &= Ax(t) + B_1u(t) + B_2\delta(t) + B_3z(t) + B_5 \\y(t) &= Cx(t) + D_1u(t) + D_2\delta(t) + D_3z(t) + D_5 \\E_2\delta(t) + E_3z(t) &\leq E_4x(t) + E_1u(t) + E_5\end{aligned}$$

state	$x(t) \in \mathbb{R}^4$
input	$u(t) \in \mathbb{R}$
aux. binary	$\delta(t) \in \{0, 1\}$
aux. continuous	$z(t) \in \mathbb{R}^3$

number of mixed-integer inequalities = 14



The MLD matrices are automatically generated in MATLAB format by HYSDEL

PERFORMANCE AND CONSTRAINTS

- Control objective:

$$\begin{aligned} \min \quad & \sum_{k=0}^N |\Delta\omega(t+k|t) - \Delta\omega_{\text{des}}| \\ \text{s.t.} \quad & \text{MLD dynamics} \end{aligned}$$

- Constraints:

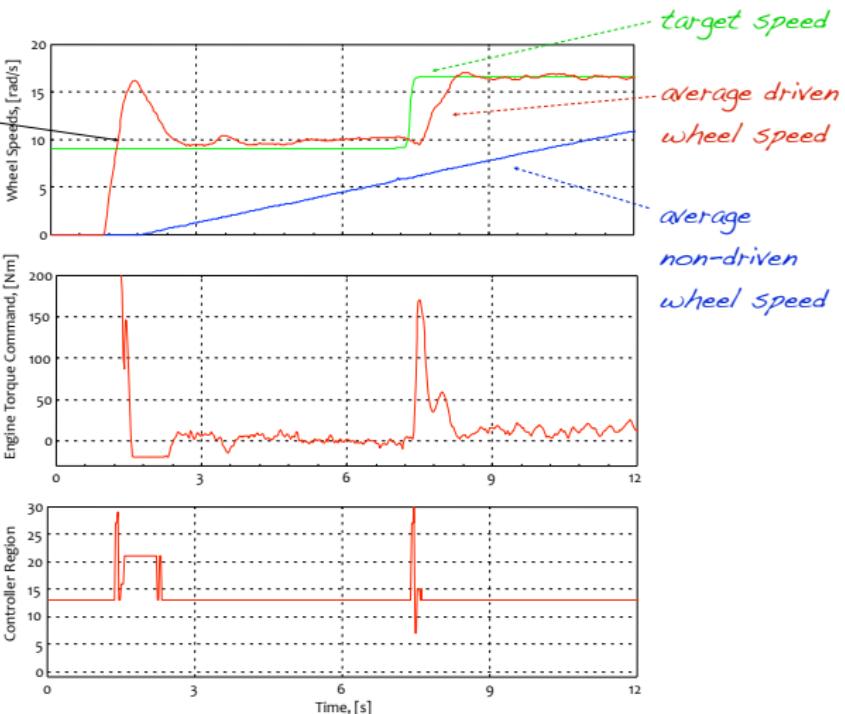
- Limits on the engine torque:

$$-20 \text{ Nm} \leq \tau_d \leq 176 \text{ Nm}$$

EXPERIMENTAL RESULTS

controller is triggered ON

(250 ms delay from commanded to actual engine torque → initial overspin)



Ford Motor Company

EXPERIMENTS

(Borrelli, Bemporad, Fodor, Hrovat, 2006)

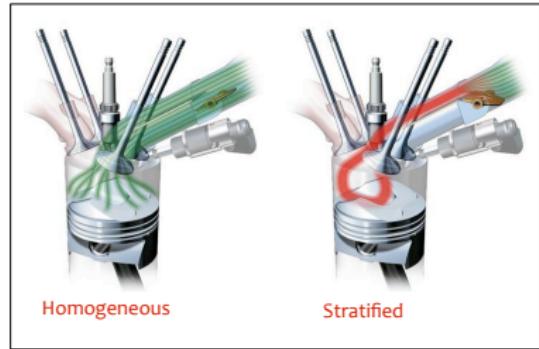


HYBRID CONTROL OF A DISC ENGINE

DISC ENGINE CONTROL PROBLEM

Objective: develop a controller for a **Direct-Injection Stratified Charge (DISC)** engine that:

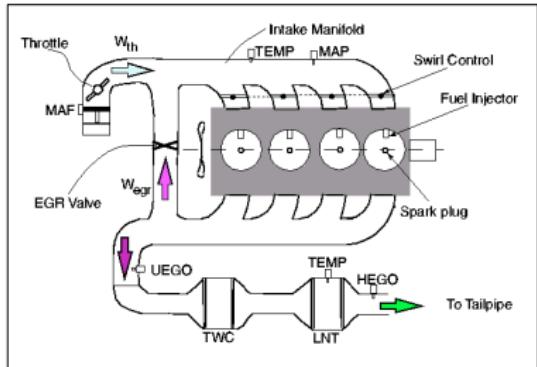
- automatically chooses operating **mode** (homogeneous/stratified)
- can cope with **nonlinear** dynamics
- handles **constraints** on A/F ratio, air-flow, spark
- achieves **optimal** performance (track desired torque and A/F ratio)



DISC ENGINE

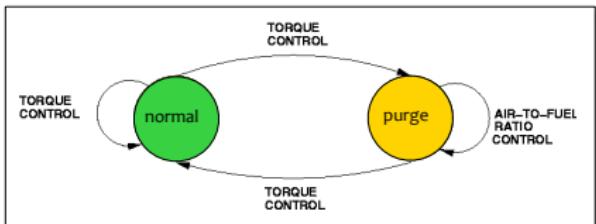
Two distinct regimes:

regime	fuel injection	air-to-fuel ratio
Homogeneous combustion	intake stroke	$\lambda=14.64$
Stratified combustion	compression stroke	$\lambda>14.64$



- Mode is **switched** by changing **fuel injection timing** (late / early)
- Better **fuel economy** during stratified mode

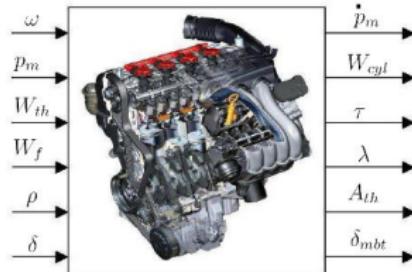
Periodical cleaning of the aftertreatment system needed ($\lambda=14.00$, homogeneous regime)



the stratified operation
can only be sustained in a restricted
part of the engine
operating range

DISC ENGINE

- **States:** intake manifold pressure (p_m)
- **Outputs:** Air-to-fuel ratio (λ), torque (τ), max-brake-torque spark timing (δ_{mbt})
- **Continuous inputs:** spark advance (δ), air flow (W_{th}), fuel flow (W_f)
- **Binary input:** spark **combustion regime** (ρ)
- **Disturbance:** engine speed (ω) [measured]
- **Constraints on:**
 - Air-to-fuel ratio (due to engine roughness, misfiring, smoke emiss.)
 - Spark timing (to avoid excessive engine roughness)
 - Mass flow rate on intake manifold (constraints on throttle)



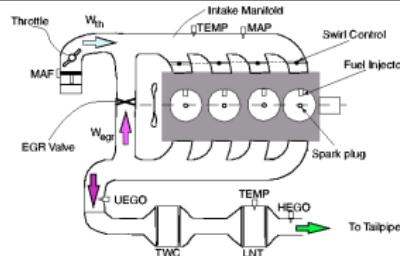
Dynamic equations are **nonlinear**, dynamics and constraints depend on regime ρ

DISC ENGINE DYNAMICS

Nonlinear model of the engine developed and validated at Ford
(Kolmanovsky, Sun, ...)

Assumptions:

- no EGR (exhaust gas recirculation) rate,
- engine speed=2000 rpm.



- Intake manifold pressure:

$$\dot{p}_m = c_m (W_{th} - W_{cyl}) = c_m (W_{th} - k_{cyl} p_m)$$

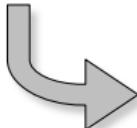
- In-cylinder Air-to-Fuel ratio:

$$\lambda = \frac{W_{cyl}}{W_f} = \frac{k_{cyl} p_m}{W_f}$$

- Engine torque:

$$\tau = \tau_{mfr} + \tau_{pump} + \tau_{ind} \quad \text{with } \tau_{mfr}, \tau_{pump} \text{ functions of } p_m$$

$$\tau_{ind} = (\theta_a + \theta_b(\delta - \delta_{mbt})^2) W_f \quad \text{where } \theta_a, \theta_b, \delta_{mbt} \text{ are functions of } \lambda, \delta \text{ and } \rho$$



✓ Good for simulation

✗ Not suitable for optimization-based controller synthesis

HYBRIDIZATION OF DISC MODEL

DYNAMICS (intake pressure, air-to-fuel ratio, torque):

- Definition of two operating points;
- Linearization of nonlinear dynamics;
- Time discretization of the linear models.



ρ -dependent dynamic equations

CONSTRAINTS on:

- Air-to-Fuel Ratio: $\lambda_{min}(\rho) \leq \lambda(t) \leq \lambda_{max}(\rho)$
- Mass of air through the throttle: $0 \leq W_{th} \leq K$
- Spark timing: $0 \leq \delta(t) \leq \delta_{mbt}(\lambda, \rho)$



ρ -dependent constraints



Hybrid system with 2 modes (switching affine system)

INTEGRAL ACTION

Integrators on torque error and air-to-fuel ratio error are added to obtain zero offsets in steady-state:

$$\begin{aligned}\epsilon_{\tau,k+1} &= \epsilon_{\tau,k} + T_s(\tau_{\text{ref}}(t) - \tau_k) \\ \epsilon_{\lambda,k+1} &= \epsilon_{\lambda,k} + T_s(\lambda_{\text{ref}}(t) - \lambda_k)\end{aligned}$$

T_s = sampling time

$\tau_{\text{ref}}, \lambda_{\text{ref}}$ = references on brake torque and air-to-fuel ratio



Simulation based on nonlinear model confirms zero offsets in steady-state
(despite the model mismatch)

MPC OF DISC ENGINE

$$\min_{\xi} J(\xi, x(t)) = \sum_{k=0}^{N-1} u_k' R u_k + y_k' Q y_k + x_{k+1}' S x_{k+1}$$

subj. to $\begin{cases} x_0 = x(t), \\ \text{hybrid model} \end{cases}$

$$\xi = [u_0', \gamma_0', z_0', \dots, u_{N-1}', \gamma_{N-1}', z_{N-1}']'$$

where: $u_k = [W_{th,k} - W_{th,ref}, W_{f,k} - W_{f,ref}, \delta_k - \delta_{ref}, \rho_k - \rho_{ref}]'$

$$y_k = [\tau_k - \tau_{ref}, \lambda_k - \lambda_{ref}, \delta_{mbt} - \delta_k - \Delta\delta_{ref}]'$$

$$x_k = [p_{m,k} - p_{m,ref}, \epsilon_{\tau,k}, \epsilon_{\lambda,k}]'$$

and: $R = \begin{pmatrix} r_{W_{th}} & 0 & 0 & 0 \\ 0 & r_{W_f} & 0 & 0 \\ 0 & 0 & r_{\delta} & 0 \\ 0 & 0 & 0 & r_{\rho} \end{pmatrix} \quad Q = \begin{pmatrix} q_{\tau} & 0 & 0 \\ 0 & q_{\lambda} & 0 \\ 0 & 0 & q_{\Delta\delta} \end{pmatrix} \quad S = \begin{pmatrix} s_{p_m} & 0 & 0 \\ 0 & s_{\epsilon_{\tau}} & 0 \\ 0 & 0 & s_{\epsilon_{\lambda}} \end{pmatrix}$

Reference values are automatically generated from τ_{ref} and λ_{ref} by numerical computations based on the nonlinear model

N = control horizon

$x(t)$ = current state

DISC ENGINE - HYSDEL

```
SYSTEM hysdisc{
  INTERFACE{
    STATE{
      REAL pm      [1, 101.325];
      REAL xtau    [-1e3, 1e3];
      REAL xlam    [-1e3, 1e3];
      REAL taud    [0, 100];
      REAL lamd    [10, 60];
    }
    OUTPUT{
      REAL lambda, tau, ddelta;
    }
    INPUT{
      REAL Wth      [0, 38.5218];
      REAL Wf       [0, 2];
      REAL delta    [0, 40];
      BOOL rho;
    }
    PARAMETER{
      REAL Ts, pml, pm2;
      ...
    }
  }

  IMPLEMENTATION{
    AUX{
      REAL lam,taul,dmbtl,lmin,lmax;
    }
    DA{
      lam={(IF rho THEN 111*pm+112*Wth...
            +113*Wf+114*delta+11c
            ELSE 101*pm+102*Wth+103*Wf...
            +104*delta+10c    );
    }
  }
  STATE{
    tau1={IF rho THEN taull*pm+...
           tau12*Wth+tau13*Wf+tau14*delta+tau1c
           ELSE tau01*pm+tau02*Wth...
           +tau03*Wf+tau04*delta+tau0c };
  }
  CONTINUOUS{
    pm=pm1*pm+pm2*Wth;
    xtau=xtau+Ts*(taud-tau);
    xlam=xlam+Ts*(lamd-lam);
    taud=taud; lamd=lamd;
  }
  OUTPUT{
    lambda=lam-lamd;
    tau=taul-taud;
    ddelta=dmbtl-delta;
  }
  MUST{
    lmin-lam      <=0;
    lam-lmax      <=0;
    delta-dmbtl  <=0;
  }
}
```

MPC – TORQUE CONTROL MODE

$$\min \sum_{k=0}^{N-1} (u_k - u_r)' R (u_k - u_r) + (y_k - y_r)' Q (y_k - y_r) + (x_{k+1} - x_r)' S (x_{k+1} - x_r)$$

subj. to $\begin{cases} x_0 = x(t), \\ \text{hybrid model} \end{cases}$



Solve
MIQP problem
to compute $u(t)$

$$u(t) = [W_{th}(t), W_f(t), \delta(t), \rho(t)]$$

Weights:

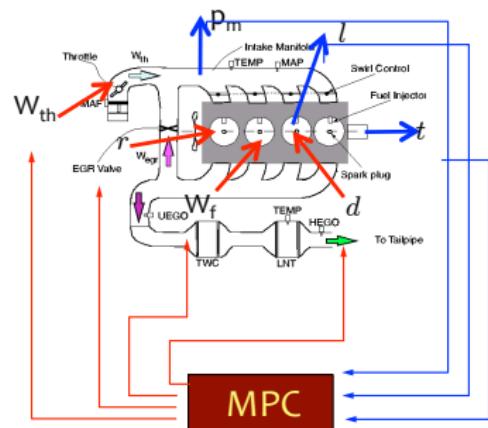
$$R = \begin{pmatrix} 0.01 & 0 & 0 & 0 \\ 0 & 0.001 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad r_p \quad \begin{matrix} \text{(prevents} \\ \text{unnecessary} \\ \text{chattering)} \end{matrix}$$

$$Q = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0.001 & 0 \\ 0 & 0 & 0.01 \end{pmatrix} \quad S = \begin{pmatrix} 0.04 & 0 & 0 \\ 0 & 1500 & 0 \\ 0 & 0 & 0.01 \end{pmatrix}$$

q_T q_λ

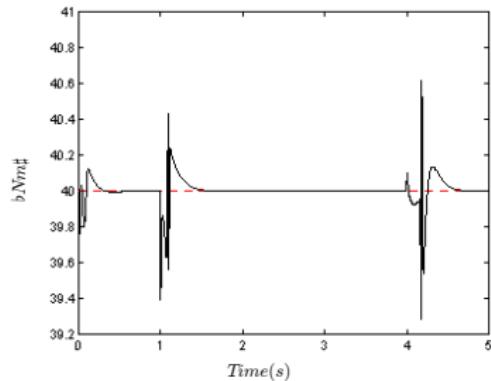
s_{ε_T} s_{ε_λ}

main emphasis on torque



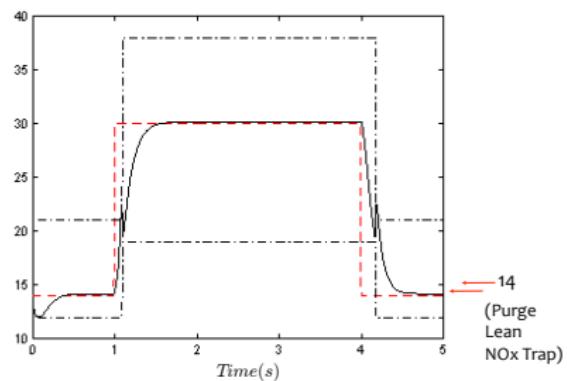
SIMULATION RESULTS (NOMINAL ENGINE SPEED)

Engine brake torque

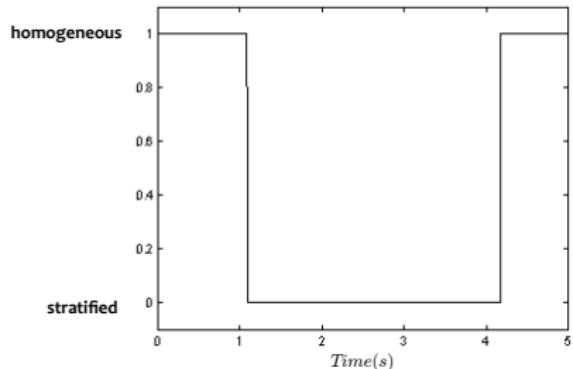


$\omega/2000 \text{ rpm}$

Air-to-fuel ratio



Combustion mode



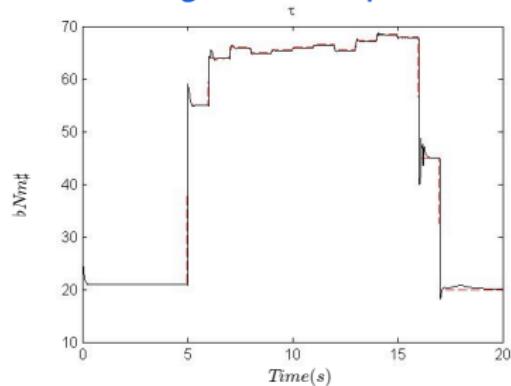
- Control horizon $N=1$;
- Sampling time $T_s=10 \text{ ms}$;
- PC Xeon 2.8 GHz + Cplex 9.1



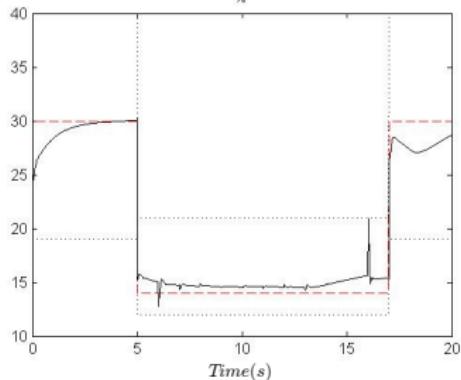
$\approx 3 \text{ ms per time step}$

SIMULATION RESULTS (VARYING ENGINE SPEED)

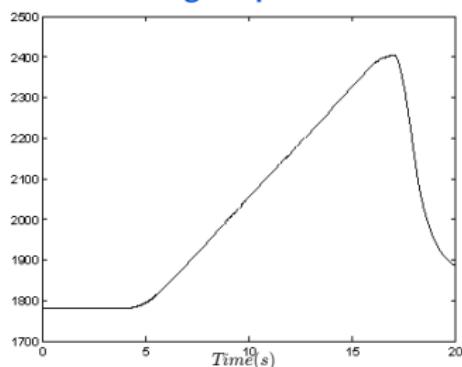
Engine Brake Torque



Air-to-Fuel Ratio



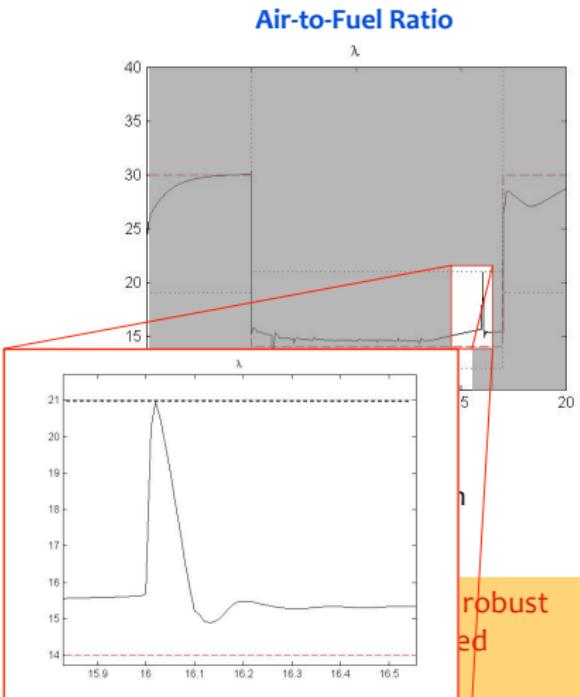
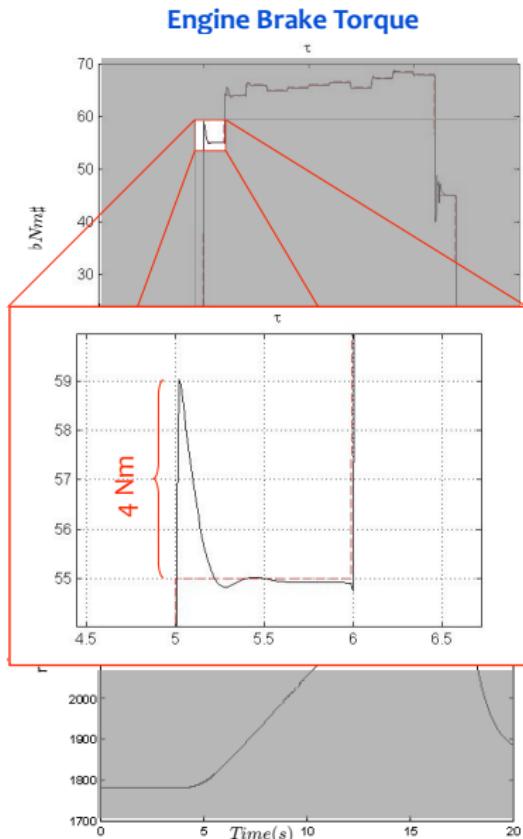
Engine speed



20 s segment of the European drive cycle (NEDC)

Hybrid MPC design is quite robust with respect to engine speed variations

SIMULATION RESULTS (VARYING ENGINE SPEED)



Control code too complex
(MIQP) \rightarrow not implementable!

EXPLICIT MPC

Explicit control law:

$$u(t) = f(\theta(t))$$

N=1 (control horizon)

where: $u = [W_{th} \ W_f \ \delta \ \rho]'$

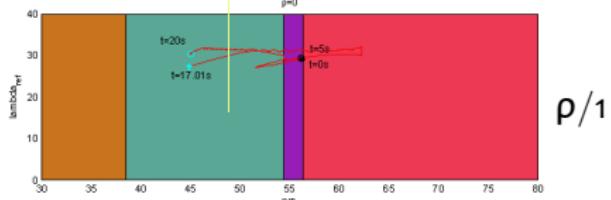
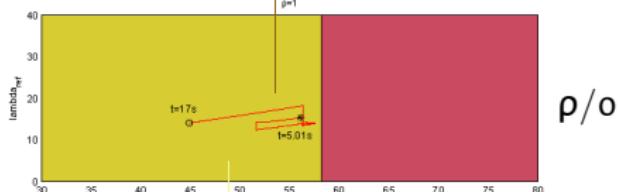


42 partitions

$$\theta = [p_m \ \epsilon_\tau \ \epsilon_\lambda \ \tau_{ref} \ \lambda_{ref}]$$

$$p_{m,ref} \ W_{th,ref} \ W_{f,ref} \ \delta_{ref}]'$$

Cross-section by the $\tau_{ref} \lambda_{ref}$ plane



- Time to compute explicit MPC:

$\approx 3s$;

- Sampling time $T_s=10$ ms;

- PC Xeon 2.8 GHz + Cplex 9.1

$\rightarrow 8 \mu\text{s}$ per time step

$\approx 3\text{ms}$ on

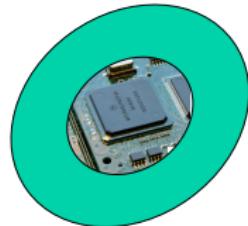
μ -controller

Motorola

MPC 555

43kb RAM

(custom made for Ford)



EXPLICIT MPC CONTROLLER (N=2)

Explicit control law:

$$u(t) = f(\theta(t))$$

N=2 (control horizon)

where: $u = [W_{th} \ W_f \ \delta \ \rho]'$

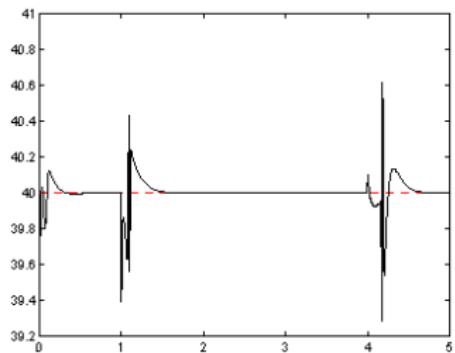


747 partitions

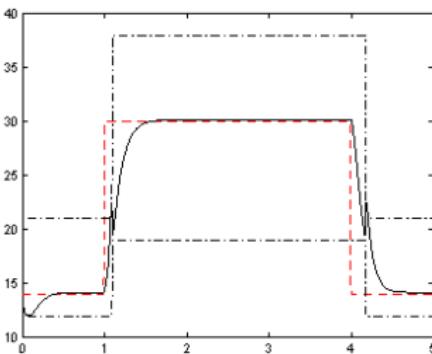
$$\theta = [p_m \ \epsilon_\tau \ \epsilon_\lambda \ \tau_{ref} \ \lambda_{ref}$$

$$p_{m,ref} \ W_{th,ref} \ W_{f,ref} \ \delta_{ref}]'$$

Engine Brake Torque



Air-to-Fuel Ratio



Closed-loop N=2

Closed-loop N=1

adequate !

EXPLICIT HYBRID MPC OF SEMIACTIVE SUSPENSIONS

ACTIVE SUSPENSIONS

(Giorgetti, Bemporad, Tseng, Hrovat, 2006)

Active Suspension System Ford Mercur XR 40i



active
suspensions

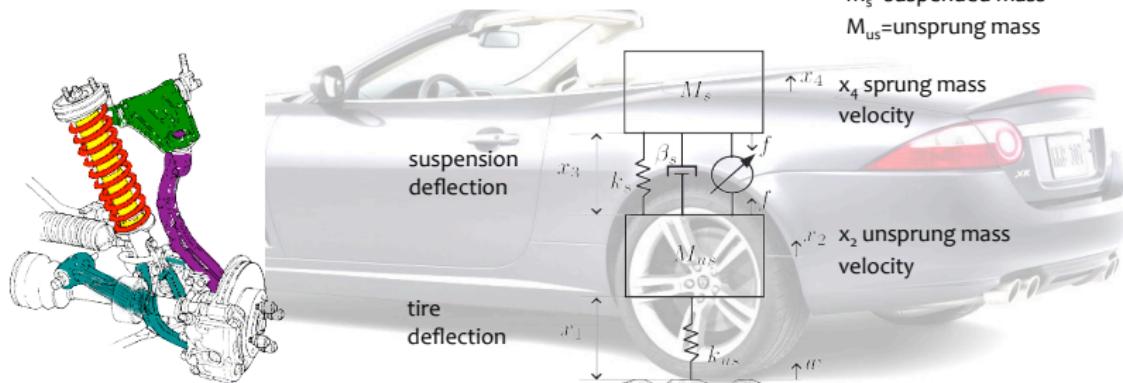


passive
suspensions

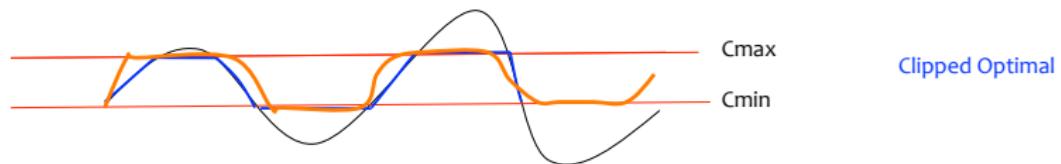


Ford Motor Company

QUEST OF OPTIMAL SEMI-ACTIVE SUSPENSIONS



For Semi-Active with Variable Damping, $f(x) = C^*(x_4 - x_2)$



— $C = f(x)/(x_4 - x_2)$, where $f(x)$ is the optimal active suspension force

— $C = \text{sat}[f(x)/(x_4 - x_2)]$

— Optimal

— ? — = —

SUSPENSION MODEL

- State-space model

$$\dot{x} = Ax + B\bar{f} + B_w w$$

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\omega_{us}^2 & -2\rho\zeta\omega_s & \rho\omega_s^2 & 2\rho\zeta\omega_s \\ 0 & -1 & 0 & 1 \\ 0 & 2\zeta\omega_s & -\omega_s^2 & -2\zeta\omega_s \end{bmatrix}$$

$$B = \begin{bmatrix} 0 \\ \rho \\ 0 \\ -1 \end{bmatrix} \quad B_w = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

- x_1 = tire deflection from equilibrium
 x_2 = unsprung mass velocity
 x_3 = suspension deflection from equilibrium
 x_4 = sprung mass velocity
 \bar{f} = normalized adjustable force
 w = road velocity disturbance

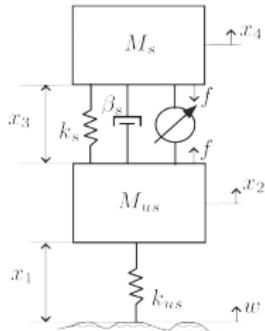
$$\rho = \frac{M_s}{M_{us}}, \omega_{us} = \sqrt{\frac{k_{us}}{M_{us}}}, \omega_s = \sqrt{\frac{k_s}{M_s}}, \zeta = \frac{\beta_s}{2\sqrt{M_s k_s}}, \bar{f} = \frac{f}{M_s}$$

- Output:

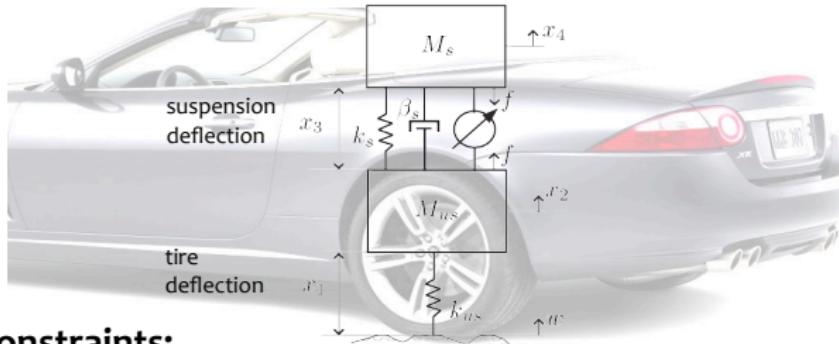
$$y = \frac{dx_4}{dt} = \begin{bmatrix} 0 & 2\zeta\omega_s & -\omega_s^2 & -2\zeta\omega_s \end{bmatrix} x - \bar{f}$$

$$\begin{aligned}
 \bullet \text{ Cost: } J &= \int (q_{x_1} x_1^2 + q_{x_3} x_3^2 + \dot{x}_4^2) dt \\
 &= \int (x' Q x + \dot{x}_4^2) dt
 \end{aligned}$$

$$\bullet \text{ Time-discretization: } T_s = 10 \text{ ms}$$



CONSTRAINTS ON SUSPENSION MODEL



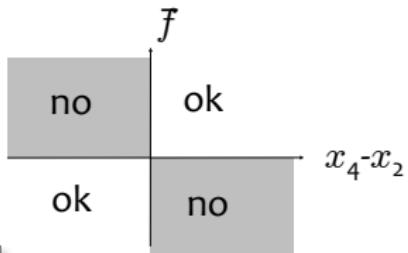
Quarter-car model

→ linear model

Constraints:

1) Passivity condition:

$$\bar{f}(x_4 - x_2) \geq 0$$



2) Max dissipation power:

$$\bar{f}(x_4 - x_2) \leq (2 \cdot 25.5 \cdot \omega_s)(x_4 - x_2)^2$$

3) Saturation:

$$|\bar{f}| \leq \sigma$$

(1), (2) are **nonlinear & nonconvex**
physical constraints

hybrid model

HYBRID MODEL OF SEMIACTIVE CONSTRAINTS

1) Passivity condition:

$$\bar{f}(x_4 - x_2) \geq 0$$



no	\bar{f}	$x_4 - x_2$
	ok	
ok		no

$$\begin{aligned} [\delta_v = 1] &\leftrightarrow [x_4 - x_2 \geq 0] \\ [\delta_{\bar{f}} = 1] &\leftrightarrow [\bar{f} \geq 0] \\ [\delta_v = 1] &\rightarrow [\delta_{\bar{f}} = 1] \\ [\delta_v = 0] &\rightarrow [\delta_{\bar{f}} = 0] \end{aligned}$$

2) Max dissipation power:

$$\bar{f}(x_4 - x_2) \leq (2 \cdot 25.5 \cdot \omega_s)(x_4 - x_2)^2$$



$$F \geq 0$$

where

$$F = \begin{cases} \bar{f} - (2 \cdot 25.5 \cdot \omega_s)(x_4 - x_2) & \text{if } (x_4 - x_2) \leq 0 \\ -\bar{f} + (2 \cdot 25.5 \cdot \omega_s)(x_4 - x_2) & \text{otherwise} \end{cases}$$

3) Saturation:

$$|\bar{f}| \leq \sigma$$



$$\begin{array}{c} \bar{f} \leq \sigma \\ \bar{f} \geq -\sigma \end{array}$$

SEMIACTIVE SUSPENSION MODEL - HYSDEL

```
/* Semiactive suspension system

(C) 2003-2005 by A.Bemporad, D.Hrovat,
      E.Tseng, N.Giorgetti
 */

SYSTEM suspension {

  INTERFACE {
    STATE {
      REAL x1 [-0.05,0.05];
      REAL x2 [-5,5];
      REAL x3 [-0.2,0.2];
      REAL x4 [-2,2];
    }
    INPUT{
      REAL u [-10,10]; /* m/s^2 */
    }
    OUTPUT {
      REAL y;
    }
  }
  PARAMETER {
    REAL A1dot,A2dot,A3dot,A4dot,B4dot,ws;
    REAL A11,A12,A13,A14,B1,A21,A22,A23,A24,B2;
    REAL A31,A32,A33,A34,B3,A41,A42,A43,A44,B4;
  }
}
```

```
IMPLEMENTATION {

  AUX {
    BOOL sign;
    BOOL usign;
    REAL F;
  }
  AD {
    sign = x4-x2<=0;
    usign = u<=0;
  }
  DA {
    F=( IF sign THEN u-(2*25.5*ws)*(x4-x2)
        ELSE -u+(2*25.5*ws)*(x4-x2) );
  }
  OUTPUT { y=A1dot*x1+A2dot*x2+A3dot*x3
            +A4dot*x4+B4dot*u;
  }
  CONTINUOUS {
    x1 = A11*x1+A12*x2+A13*x3+A14*x4+B1*u;
    x2 = A21*x1+A22*x2+A23*x3+A24*x4+B2*u;
    x3 = A31*x1+A32*x2+A33*x3+A34*x4+B3*u;
    x4 = A41*x1+A42*x2+A43*x3+A44*x4+B4*u;
  }
  MUST {
    sign -> usign;
    ~sign -> ~usign;
    F>=0;
  }
}
```

```
>>S=mld('semiact3',Ts)
```

get the MLD model in MATLAB

```
>>[X,T,D,Z,Y]=sim(S,x0,U);
```

simulate the MLD model

SEMIACTIVE SUSPENSION MODEL - PWA FORM

- PWA model

$$x(k+1) = A_{i(k)}x(k) + B_{i(k)}u(k) + f_{i(k)}$$

$$y(k) = C_{i(k)}x(k) + D_{i(k)}u(k) + g_{i(k)}$$

$$i(k) \text{ s.t. } H_{i(k)}x(k) + J_{i(k)}u(k) \leq K_{i(k)}$$

- 4 continuous states

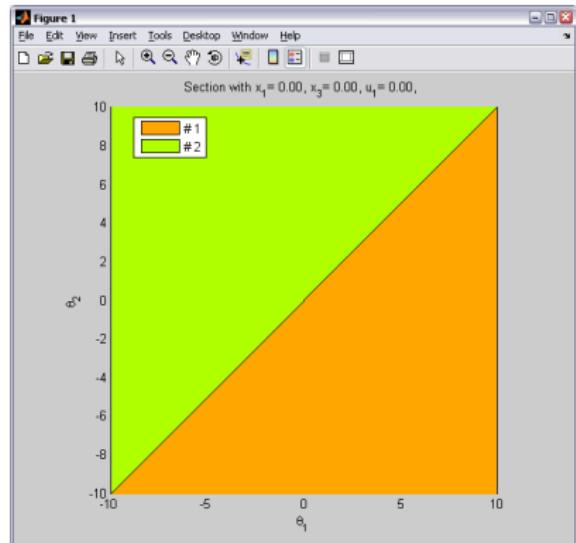
$$(x_1, x_2, x_3, x_4)$$

- 1 continuous input

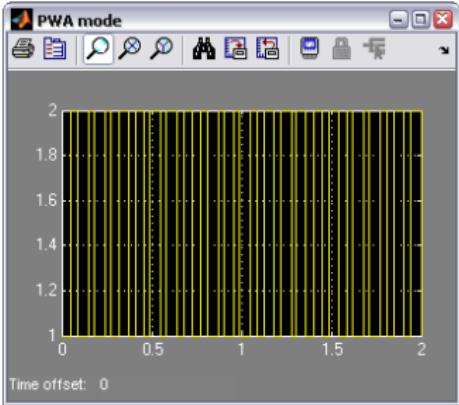
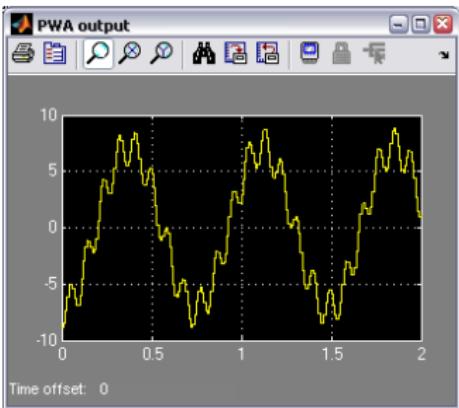
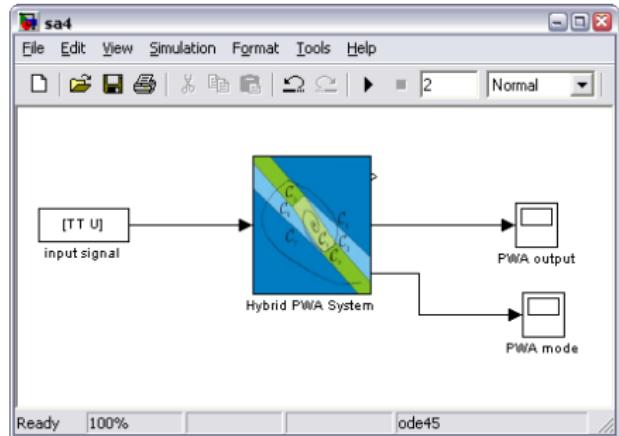
(normalized adjustable damping force \bar{f})

- 2 polyhedral regions

```
>>P=pwa(S);
```



OPEN-LOOP SIMULATION OF PWA SUSPENSION MODEL



MPC PERFORMANCE SPECIFICATIONS

tire deflection

suspension
deflection

vertical
acceleration

$$\min \left(\sum_{k=1}^{N-1} 1100x_{1,k}^2 + 100x_{3,k}^2 + x_{4,k}^2 \right) + x_N' P x_N$$

terminal weight
(Riccati matrix)

CLOSED-LOOP MPC RESULTS

$$J = \int (q_1 x_1^2 + q_3 x_3^2 + \dot{x}_4^2)$$

```
>>refs.y=1; % weights output #1
>>Q.y=Ts*rx4d;% output weight
...
>>Q.norm=2; % quadratic costs
>>N=1; % optimization horizon
>>limits.umin=umin;
>>limits.umax=umax;
```

```
>>C=hybcon(S,Q,N,limits,refs);
```

```
>> C
```

Hybrid controller based on MLD model S <semiact3.hys> [2-norm]

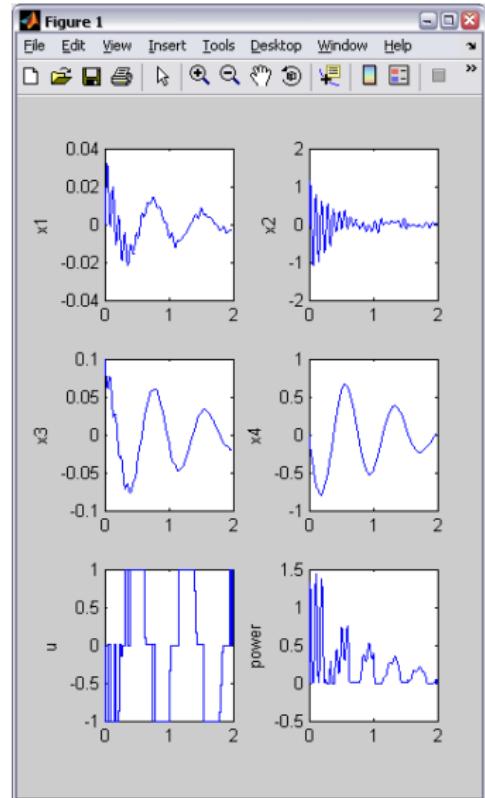
```
4 state measurement(s)
1 output reference(s)
1 input reference(s)
4 state reference(s)
0 reference(s) on auxiliary continuous z-variables
```

```
4 optimization variable(s) (2 continuous, 2 binary)
13 mixed-integer linear inequalities
sampling time = 0.01, MIQP solver = 'cplex'
```

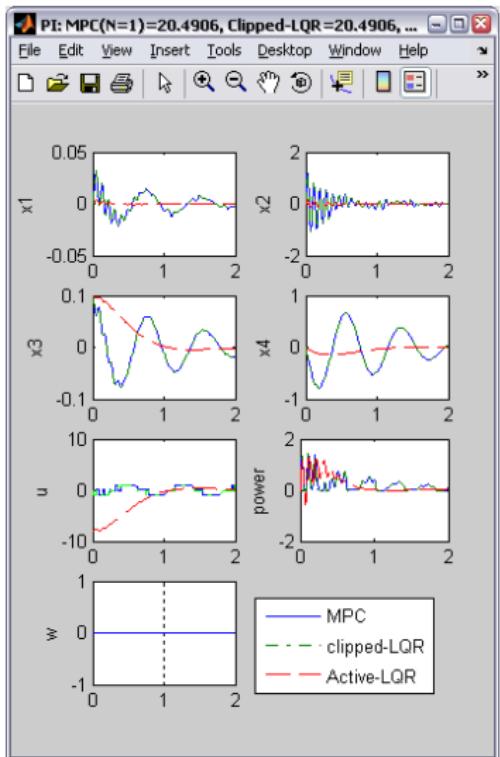
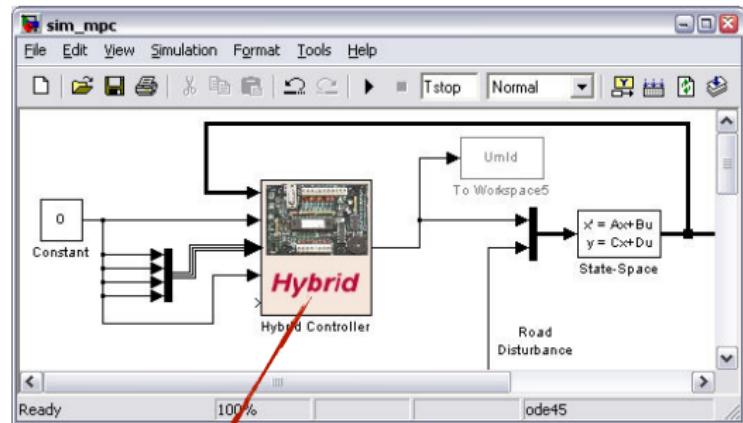
Type "struct(C)" for more details.

```
>>
```

```
>>[XX,UU,DD,ZZ,TT]=sim(C,S,r,x0,Tstop);
```



CLOSED-LOOP MPC RESULTS (SIMULINK)



EXPLICIT HYBRID MPC

```
>> E=expcon(C, range, options);
```

```
>> E

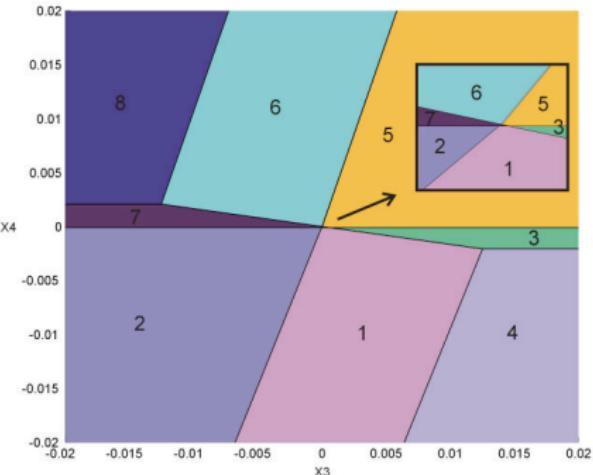
Explicit controller (based on hybrid controller C)
  4 parameter(s)
  1 input(s)
  8 partition(s)
sampling time = 0.01

The controller is for hybrid systems (tracking)
[2-norm]

This is a state-feedback controller.

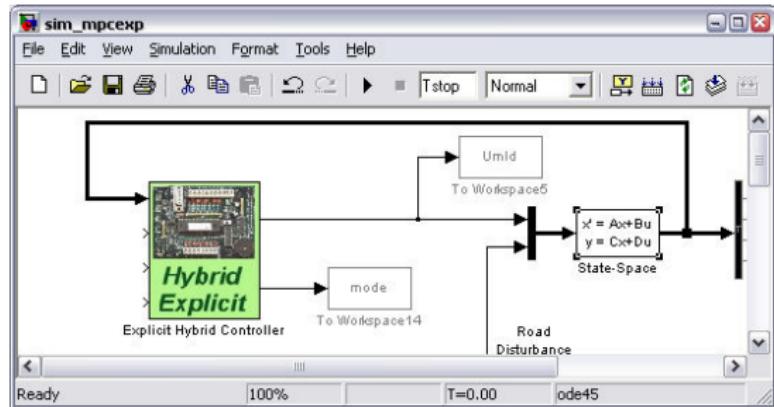
Type "struct(E)" for more details.
>>
```

Explicit solution ($N=1, x_1=x_2=0$):



$$u(x) = \begin{cases} 10.4748x_1 + 0.2446x_2 & +79.1519x_3 - 3.9235x_4 \\ (= K_{LQ}) & \\ 0 & \text{Regions } \#1, \#6 \\ (2 \cdot 25.5 \cdot \omega_s)(x_4 - x_2) & \text{Regions } \#2, \#5 \\ -1 & \text{Regions } \#3, \#7 \\ 1 & \text{Region } \#4 \\ & \text{Region } \#8 \end{cases}$$

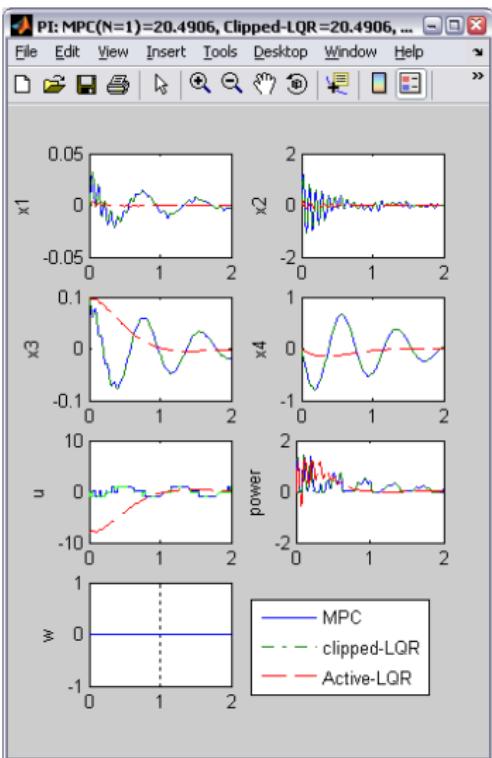
EXPLICIT HYBRID MPC



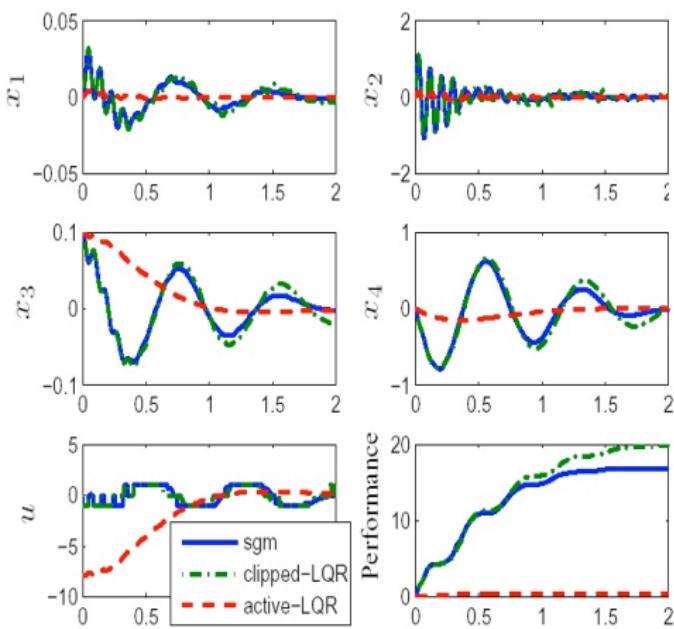
generated
C-code

```
#define EXPCON_NU 1
#define EXPCON_NZ 4
#define EXPCON_NY 1
#define EXPCON_TS 0.01000000
#define EXPCON_REG 8
#define EXPCON_NTH 4
#define EXPCON_NYM 4
#define EXPCON_NUC 1
#define EXPCON_NUB 0
#define EXPCON_NGAIN 1
#define EXPCON_NH 21
#define EXPCON_NF 8
static double EXPCON_F[] = {
    10.4748, 0, 0, 0, 10.4748, 0, 0, 0, -0.244594, 0,
    480.664, 0,
    3.92349, 0,
    480.664, 0
};

static double EXPCON_G[] = {
    0, 1e-006, -1e-006, -1, 0, 0, 1e-006, 1
}
```



QUEST OF OPTIMAL SEMIACTIVE SUSPENSIONS



PARAMETER VALUES USED IN SIMULATION

Parameter	Value	Description
T_s	10 ms	Sampling time
ω_s	1.5 Hz	Sprung mass natural frequency
ω_{us}	10 Hz	Wheel-hop natural frequency
ρ	10	Sprung-to-unprung mass ratio
ζ	0	Damping ratio
σ	1	Maximum force capacity
q_1	1100	Weight on tire deflection
q_3	100	Weight on suspension deflection

TABLE II
SHOCK TEST: MPC COST VALUE FOR DIFFERENT CONTROL HORIZONS SUBJECTED TO I.C. = [0 0 0 1 0].

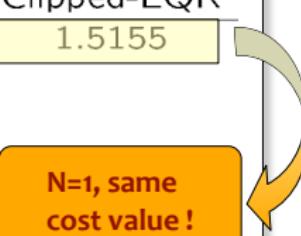
N	MPC	Clipped-LQR	SGM	LQR
1	20.4282	20.4282	17.4944	0.4446
2	20.4054	20.4282		
3	20.3290			
4	20.1100			
5	19.7380			
10	20.9840			
12	19.3084			
14	18.4842			
15	18.5996			
16	19.3212			
20	18.0764			
30	17.1494			
40	17.1304			

EXPLICIT HYBRID MPC - RESULTS

- Horizon N=1: same as Clipped-LQR !
- Better closed-loop performance for increasing N

Performance Index

N	MPC	Clipped-LQR
1	1.5155	1.5155
5	1.4416	
10	1.5238	
15	1.3083	
20	1.2204	
30	1.1456	
40	1.1462	



N=1, same cost value !

- Simulations with road noise.
- Initial condition $x(0)=[0 \ 0 \ 0]'$
- Simulation time $T=20$ s, sampling time $T_s=10$ ms

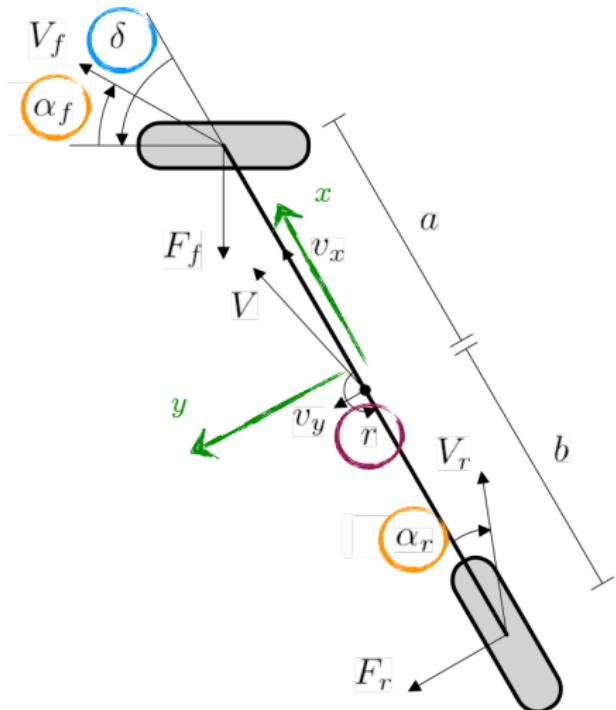
VEHICLE YAW STABILITY CONTROL

YAW STABILITY CONTROL PROBLEM

- **Problem:** Control **vehicle stability** while tracking driver's desired trajectory
 - Electronic Stability Control (ESC) (Koibuchi et al., 1996)
 - Active Front Steering (AFS) (Ackermann, 1997)
- **Main control objective:** Make the **yaw rate** of the vehicle track a time-varying reference computed from the driver's steering angle and current velocity
- **Approach:** Consider the steering command as a reference generator and actuate **steering** and **differential braking** (=coordinated AFS and ESC action)
(Bernardini, Di Cairano, Bemporad, Tseng, 2009) (Di Cairano, Tseng, Bernardini, Bemporad, 2012)

VEHICLE MODEL

- **Bicycle model** appropriate in high speed turns (Gillespie, 1992)



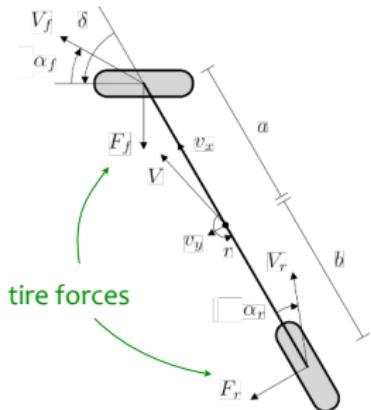
- **reference frame** (x,y,z) moving with the vehicle
- **front steering angle** δ [rad]
- **tire slip angles** α_f, α_r [rad]
- **yaw rate** r [rad/s]

$$\begin{aligned}\tan(\alpha_f + \delta) &= \frac{v_y + ar}{v_x} \\ \tan \alpha_r &= \frac{v_y - br}{v_x}\end{aligned}$$

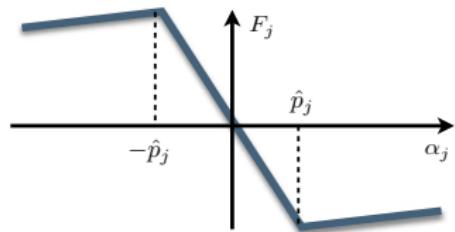
TIRE FORCE MODEL

- **Tire force characteristics:** are nonlinear functions of the **slip angles** and of the **longitudinal slip**
- For a constant longitudinal slip, we use a **piecewise affine model**

$$F_f(\alpha_f) = \begin{cases} -c_f \alpha_f & \text{if } -\hat{p}_f \leq \alpha_f \leq \hat{p}_f \\ -(d_f \alpha_f + e_f) & \text{if } \alpha_f > \hat{p}_f \end{cases}$$
$$F_r(\alpha_r) = \begin{cases} -c_r \alpha_r & \text{if } -\hat{p}_r \leq \alpha_r \leq \hat{p}_r \\ -(d_r \alpha_r + e_r) & \text{if } \alpha_r > \hat{p}_r \end{cases}$$



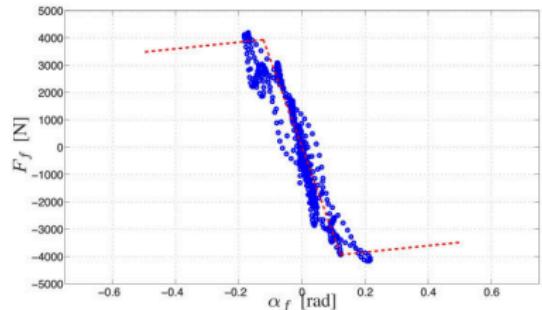
- **Critical slip angles** \hat{p}_f, \hat{p}_r are threshold values where dynamics switch
- For symmetry, we can restrict to analyze **clockwise turns** (counter-clockwise turns can be handled by opportunely inverting signs)



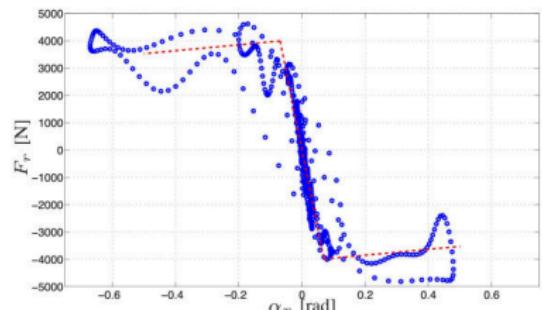
tire forces characteristics

TIRE FORCE MODEL

Sideslip angle-force characteristics



(a) Front tires



(b) Rear tires

Experimental tire data and piecewise linear approximation of the tire



Rear-wheel drive test vehicle equipped with active front steering and differential braking used for experimental validation

VEHICLE DYNAMICAL MODEL

slip angles

$$\begin{aligned}\dot{\alpha}_f &= \frac{\dot{v}_y + a\dot{r}}{v_x} - \dot{\delta} \\ \dot{\alpha}_r &= \frac{\dot{v}_y - b\dot{r}}{v_r}\end{aligned}$$

static yaw rate

$$r = \frac{v_x}{a+b}(\alpha_f - \alpha_r + \delta)$$

overall dynamical model

$$\begin{aligned}\dot{\alpha}_f &= \frac{F_f + F_r}{mv_x} - \frac{v_x}{a+b}(\alpha_f - \alpha_r + \delta) + \frac{a}{v_x I_z}(aF_f - bF_r + Y) \\ \dot{\alpha}_r &= \frac{F_f + F_r}{mv_x} - \frac{v_x}{a+b}(\alpha_f - \alpha_r + \delta) - \frac{b}{v_x I_z}(aF_f - bF_r + Y) \\ r &= \frac{v_x}{a+b}(\alpha_f - \alpha_r + \delta)\end{aligned}$$

$$\dot{v}_y = \frac{F_f \cos \delta + F_r}{m} - rv_x$$

lateral velocity

$$\dot{r} = \frac{aF_f \cos \delta - bF_r + Y}{I_z}$$

yaw rate derivative

VEHICLE DYNAMICAL MODEL - PWA FORM

- The overall dynamics model is recast as a **PWA system** by introducing the Boolean variables

$$\begin{aligned}\gamma_f = 0 &\leftrightarrow \alpha_f \leq \hat{p}_f \\ \gamma_r = 0 &\leftrightarrow \alpha_r \leq \hat{p}_r\end{aligned}$$

- By **discretizing** with sampling period $T_s = 0.1$ s we obtain

$$\begin{aligned}x(k+1) &= A_i x(k) + B_i u(k) + f_i \\ y(k) &= C x(k) + D u(k) \\ i \in \{1, \dots, 4\} &: H_i x(k) \leq K_i\end{aligned}$$

where

$$x = [\alpha_f \ \alpha_r]'$$

slip angles

$$u = [Y \ \delta]'$$

yaw moment
front steering

$$y = r$$

yaw rate

REFERENCE GENERATION

- **Control goal:**

stabilize the system at the equilibrium obtained with $\delta(k) = \hat{\delta}(k)$
while **minimizing** the use of the brake actuator ($\hat{Y}(k) = 0$) 

- **Equilibrium** condition in the linear region:

$$\begin{aligned}\dot{\alpha}_f^0 &= \frac{F_f + F_r}{mv_x} - \frac{v_x}{a+b}(\alpha_f - \alpha_r + \hat{\delta}) + \frac{a}{v_x I_z}(aF_f - bF_r + \hat{Y}^0) \\ \dot{\alpha}_r^0 &= \frac{F_f + F_r}{mv_x} - \frac{v_x}{a+b}(\alpha_f - \alpha_r + \hat{\delta}) - \frac{b}{v_x I_z}(aF_f - bF_r + \hat{Y}^0)\end{aligned}$$

driver's steering angle

- **Time-varying set-points** are defined using the overall dynamical model

$$\begin{aligned}\hat{\alpha}_f &= \frac{m\tilde{v}_x^2 b c_r \hat{\delta}}{m\tilde{v}_x^2 (a c_f - b c_r) - c_f c_r (a + b)^2} \\ \hat{\alpha}_r &= \hat{\alpha}_f \frac{a c_f}{b c_r} \\ \hat{r} &= \frac{\tilde{v}_x}{a + b}(\hat{\alpha}_f - \hat{\alpha}_r + \hat{\delta})\end{aligned}$$

- Current longitudinal velocity $v_x(k)$ is used to **update set-points**

HYBRID PREDICTION MODEL

- **Yaw rate tracking:**

zero tracking error in steady state is provided by **integral action**

$$\begin{aligned} \text{integral of} \\ \text{tracking error} &\rightarrow I_r(k+1) = I_r(k) + r(k) - r_s(k) \\ \text{yaw rate set-} \\ \text{point} &\rightarrow r_s(k+1) = r_s(k) \end{aligned}$$

- The **global hybrid dynamical model** of the vehicle is given by

$$\begin{aligned} z(k+1) &= \tilde{A}_i z(k) + \tilde{B}_i u(k) + \tilde{f}_i && \text{augmented state} \\ y(k) &= \tilde{C} z(k) + \tilde{D} u(k) \\ i &\in \{1, \dots, 4\} : \tilde{H}_i z(k) \leq \tilde{K}_i \\ \tilde{A}_i &= \begin{bmatrix} A_i & 0 & 0 \\ \frac{v_x}{a+b} & \frac{-v_x}{a+b} & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \tilde{B}_i = \begin{bmatrix} B_i \\ 0 & \frac{v_x}{a+b} \\ 0 & 0 \end{bmatrix}, \quad \tilde{f}_i = \begin{bmatrix} f_i \\ 0 \\ 0 \end{bmatrix}, \\ \tilde{C} &= \begin{bmatrix} \frac{v_x}{a+b} & \frac{-v_x}{a+b} & 0 & 0 \end{bmatrix}, \quad \tilde{D} = \begin{bmatrix} 0 & \frac{v_x}{a+b} \end{bmatrix}, \\ \tilde{H}_i &= \begin{bmatrix} H_i & 0 & 0 \end{bmatrix}, \quad \tilde{K}_i = K_i. \end{aligned}$$
$$z = \begin{bmatrix} \alpha_f \\ \alpha_r \\ I_r \\ r_s \end{bmatrix}$$

CONTROL PROBLEM FORMULATION

- The **optimal control problem** solved at every time step k is

$$\begin{aligned} \min_{\mathbf{u}_k} \quad & \sum_{j=0}^{N-1} \left\{ (z_{k+j|k} - \hat{z})' Q_z (z_{k+j|k} - \hat{z}) \right. \\ & + (y_{k+j|k} - \hat{y})' Q_y (y_{k+j|k} - \hat{y}) \left. + (u_{k+j|k} - \hat{u})' Q_u (u_{k+j|k} - \hat{u}) \right\} \\ \text{s.t.} \quad & z_{k|k} = z(k) \end{aligned}$$

hybrid dynamics

state and input constraints

slip angles tracking & int. action

yaw rate tracking

penalty on actuators action

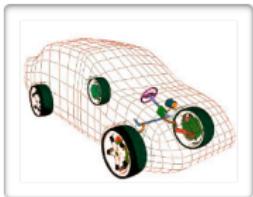


- State and input constraints:

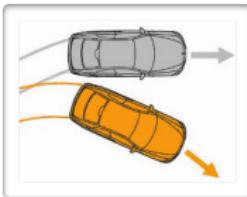
$$\begin{array}{rclcrcl} [z]_1(k) & \geq & -\hat{p}_f & \text{front slip angle} & |[u]_1(k)| & \leq & 1000 \text{ [Nm]} & \text{yaw moment} \\ [z]_2(k) & \geq & -\hat{p}_r & \text{rear slip angle} & |[u]_2(k)| & \leq & 0.35 \text{ [rad]} & \text{front steering} \end{array}$$

SIMULATION SETUP

- Simulations run on a **nonlinear vehicle model** including:



longitudinal and lateral
vehicle dynamics



yaw rate dynamics



steering actuation
dynamics

- Driver's steering command **constant** over the simulation interval
- Controller setup** after calibration:

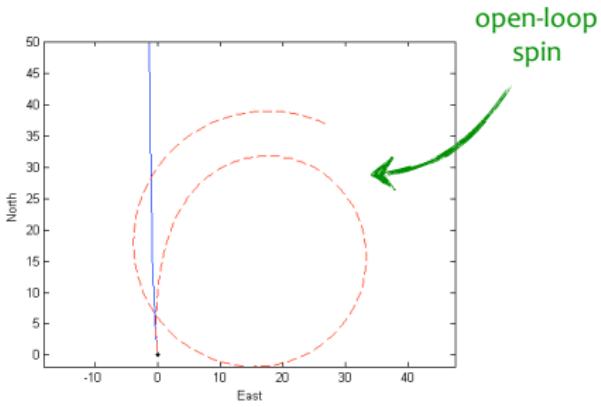
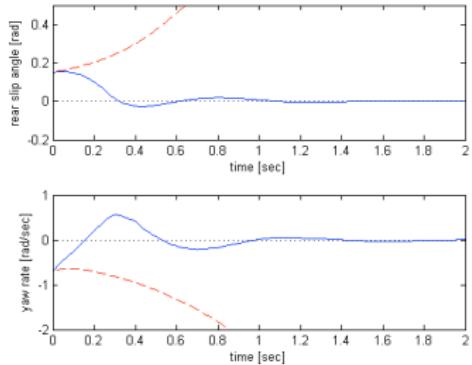
▶ Prediction horizon $N = 3$

▶ Weight matrices $Q_z = \begin{bmatrix} .1 & 0 & 0 & 0 \\ 0 & .1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, $Q_u = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $Q_y = 1$

CLOSED-LOOP SIMULATIONS RESULTS

- **Stability analysis** under nominal conditions (with $\hat{\delta} = 0$):

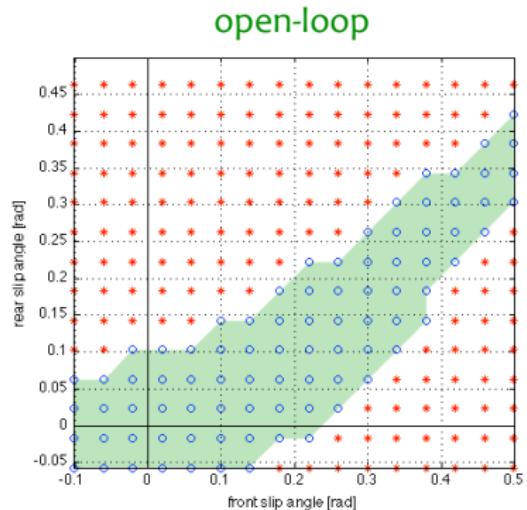
open-loop vs closed-loop



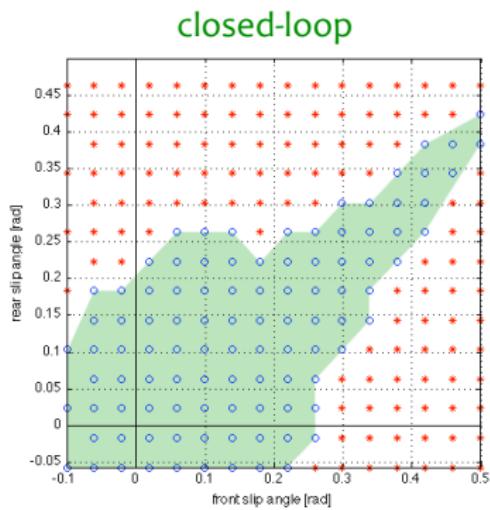
- Controller has to cope with **linearization errors**

CLOSED-LOOP SIMULATIONS RESULTS

- **Stability analysis** under nominal conditions (with $\hat{\delta} = 0$):



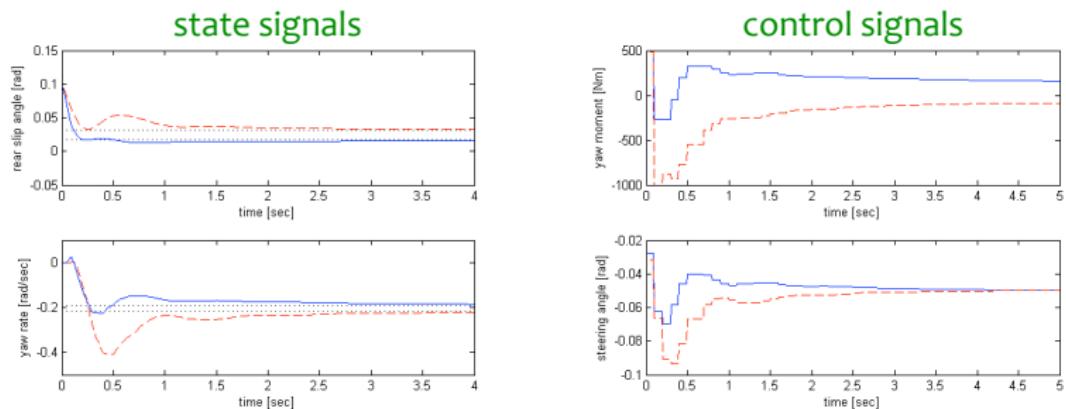
stable initial conditions
unstable initial conditions



closed-loop provides a
larger stability region

CLOSED-LOOP SIMULATIONS RESULTS

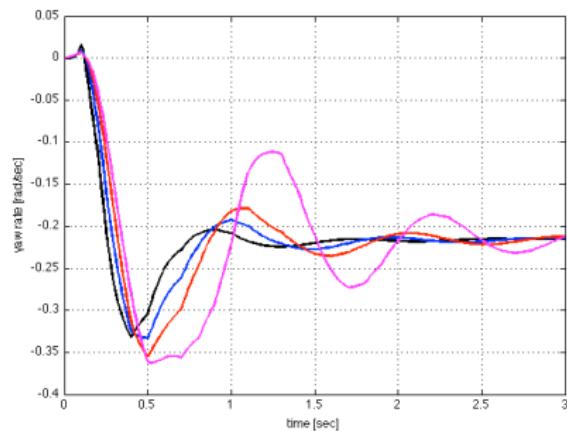
- Robustness analysis w.r.t. model mismatches (with $\hat{\delta} = -0.05$):
 - ▶ nominal longitudinal velocity $\hat{v}_x = 20$ m/s
 - ▶ real longitudinal velocity $v_x = 15$ m/s and $v_x = 25$ m/s



- Stability and **fast tracking response** are provided

CLOSED-LOOP SIMULATIONS RESULTS

- Turns on slippery road surface (with $\hat{\delta} = -0.05$):
 - ▶ several values tested: $\hat{s} = 0$, $s = 0.20$, $s = 0.30$, $s = 0.35$



- Good **degree of robustness** with respect to slip mismatches

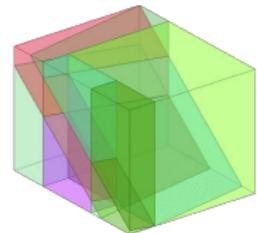
COMPUTATIONAL ISSUES

- **Computational issues:**

- MPC-based approach is **viable for experimental tests**
(average CPU time 17ms, worst-case CPU time 63 ms in MATLAB),
but requires a MIQP solver in the ECU

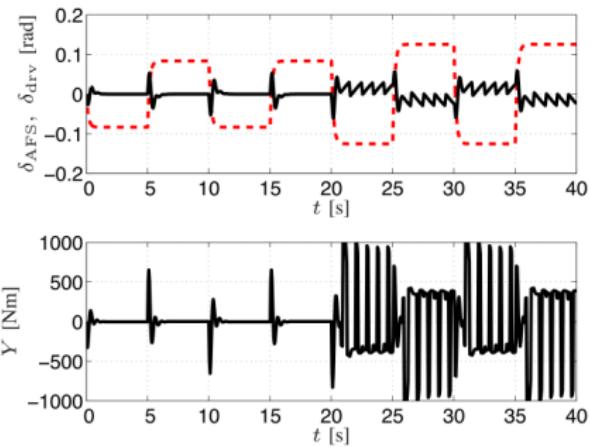
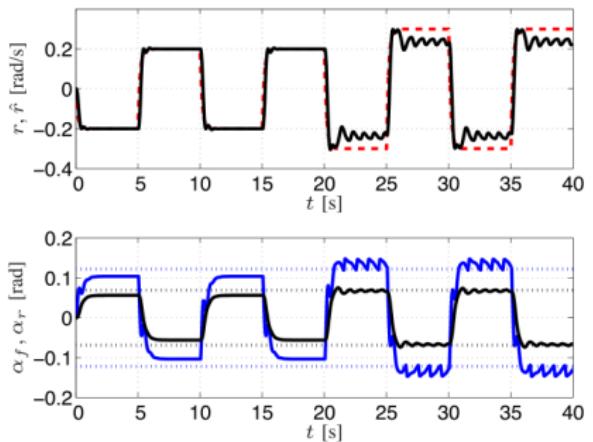
- **Explicit solution** of hybrid MPC control problem:

- Exploits **multiparametric programming** techniques to provide a description of the control law as an **explicit** function of the state
- All the computation is executed **off-line**, only simple set-membership tests and function evaluations are performed on-line to compute the control action
- However, the explicit solution requires **memory**
(around **5000 polytopes** to be stored, in this case!)
- An **approximation** of the solution is needed for real implementation



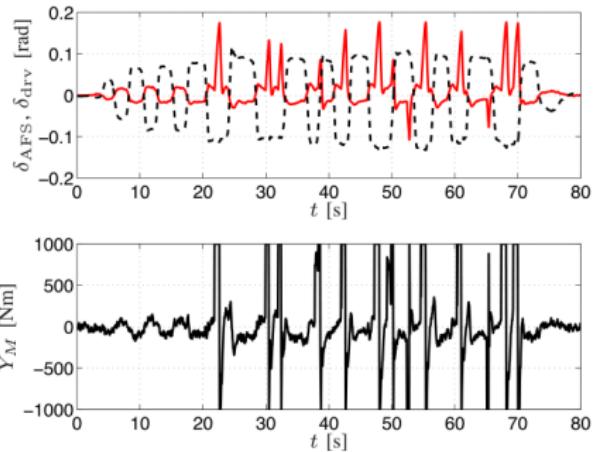
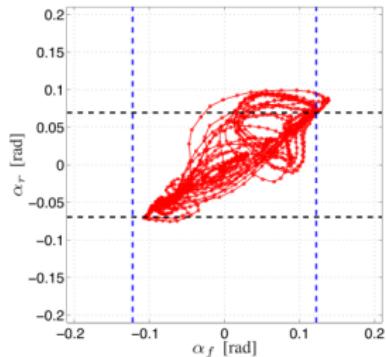
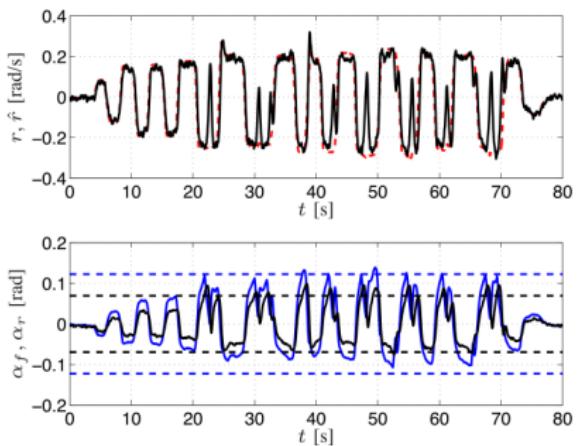
SWITCHED MPC SOLUTION

- Simpler solution: assume PWA mode remains constant in prediction
- Design a linear MPC controller for each mode, make it explicit
- 4 linear explicit MPC's are enough (linear/saturation \times front/rear)
- Simulation results:



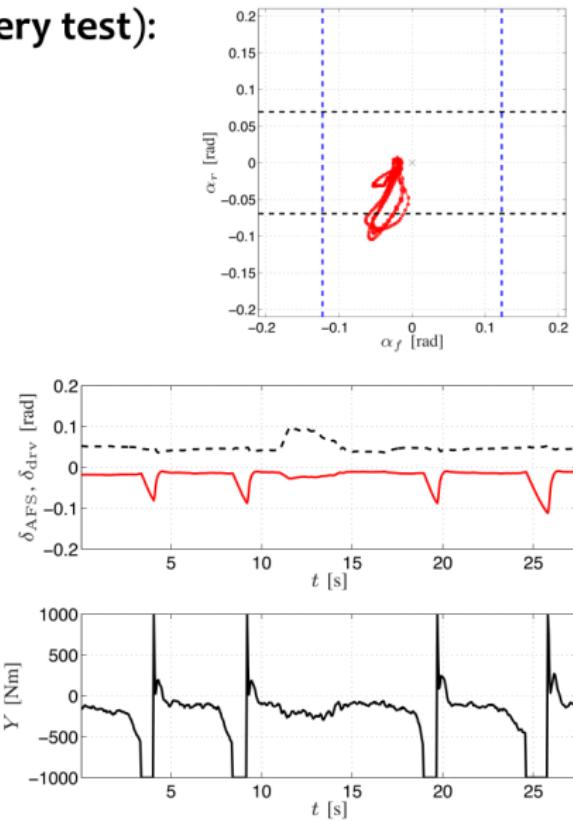
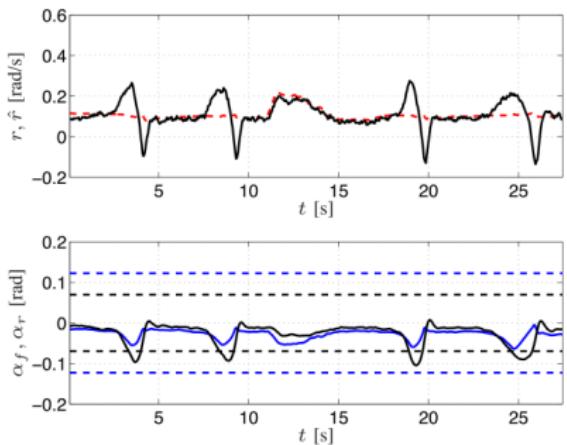
EXPERIMENTAL RESULTS

- Experimental results (slalom test):



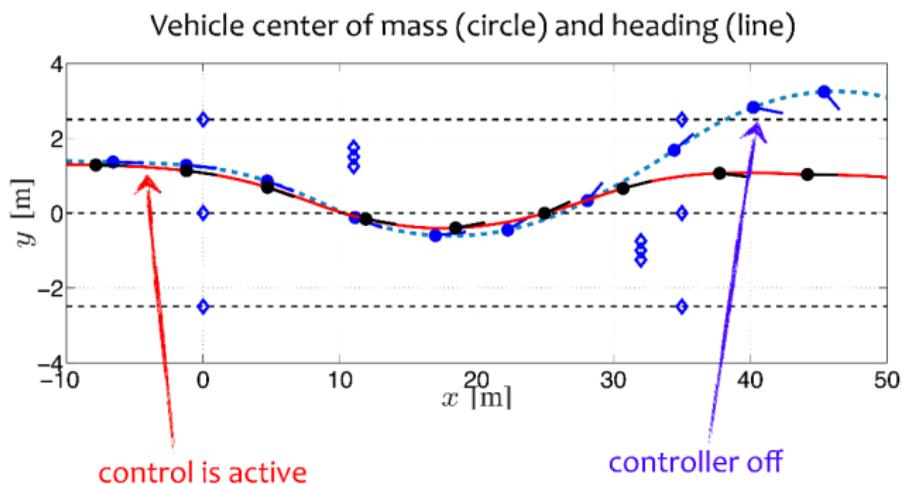
EXPERIMENTAL RESULTS

- Experimental results (stability recovery test):



EXPERIMENTAL RESULTS

- Experimental results (double-lane change):



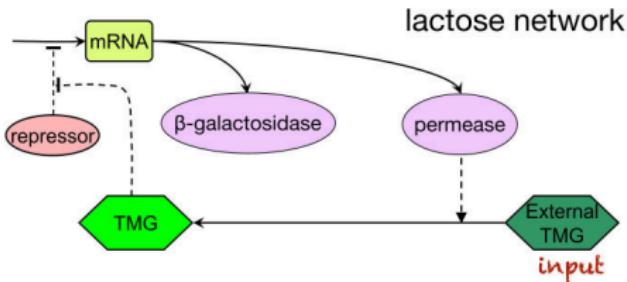
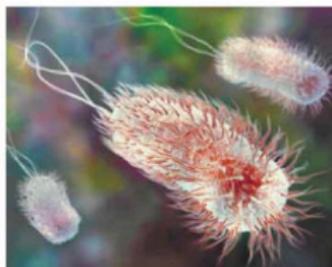
- Alternative: use switched linear MPC and mpQP (Di Cairano, Tseng, 2010)

HYBRID MPC OF INDUCTION OF ESCHERICHIA COLI

HYBRID MPC OF INDUCTION OF ESCHERICHIA COLI

(Julius, Sakar, Bemporad, Pappas, 2007)

- Goal: control the **lactose regulation system** of a colony of *E. coli*

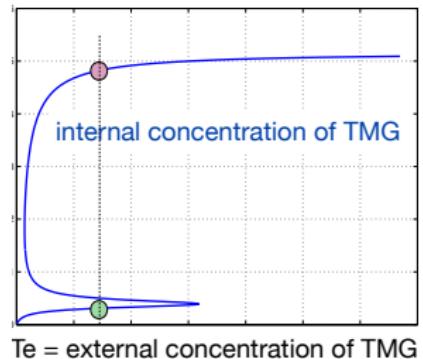
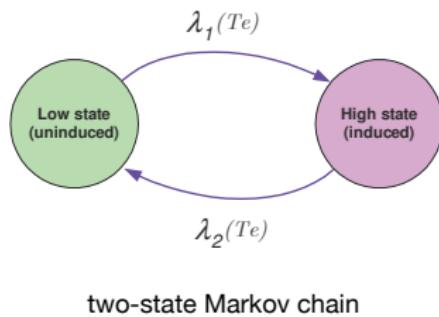


TMG = *thio-methyl galactosidase* concentration

- Model, measurements, and actuation are at the **entire colony** level

HYBRID MPC OF INDUCTION OF ESCHERICHIA COLI

- Bistable lactose regulation system of E. coli



- The probabilities x_{lo} , x_{hi} to be in low/high state satisfy the dynamics

$$\frac{d}{dt} \begin{bmatrix} x_{lo} \\ x_{hi} \end{bmatrix} = \begin{bmatrix} -\lambda_1(T_e) & \lambda_2(T_e) \\ \lambda_1(T_e) & -\lambda_2(T_e) \end{bmatrix} \begin{bmatrix} x_{lo} \\ x_{hi} \end{bmatrix}$$

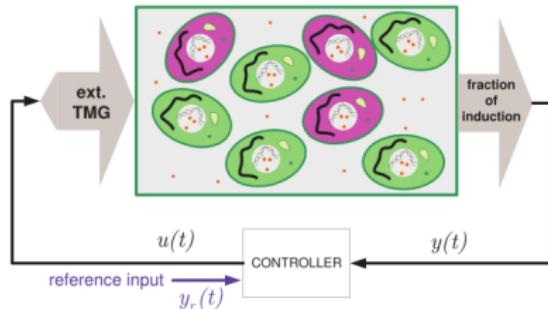
- Transition rates λ_1 , λ_2 modeled as **piecewise constant** functions of T_e

$T_e [10^{-3} \text{ mM}]$	$\lambda_1(T_e) [\text{min}^{-1}]$	$\lambda_2(T_e) [\text{min}^{-1}]$
[1.4, 1.5)	$8.68 \cdot 10^{-4}$	$5.91 \cdot 10^{-3}$
[1.5, 1.6)	$9.27 \cdot 10^{-4}$	$3.61 \cdot 10^{-3}$
[1.6, 1.7)	$1.13 \cdot 10^{-3}$	$2.36 \cdot 10^{-3}$
[1.7, 1.8)	$1.39 \cdot 10^{-3}$	$1.54 \cdot 10^{-3}$
[1.8, 1.9)	$1.67 \cdot 10^{-3}$	$9.53 \cdot 10^{-4}$
[1.9, 2.0)	$1.93 \cdot 10^{-3}$	$5.54 \cdot 10^{-4}$

HYBRID MPC OF INDUCTION OF ESCHERICHIA COLI

- Hybrid MPC problem

- switched linear system
- constraints on input T_e and dT_e/dt
- penalties on tracking error $y - y_r$ and input rate dT_e/dt



- Closed-loop results

- MPC controller developed with **Hybrid Toolbox** in MATLAB
- Mixed-Integer Linear Program solver GLPK
- solution time: **32 ms** (worst case=**280 ms**) on 1.2 GHz laptop
- **sampling time = 10 min**

