Stats Exercise 1

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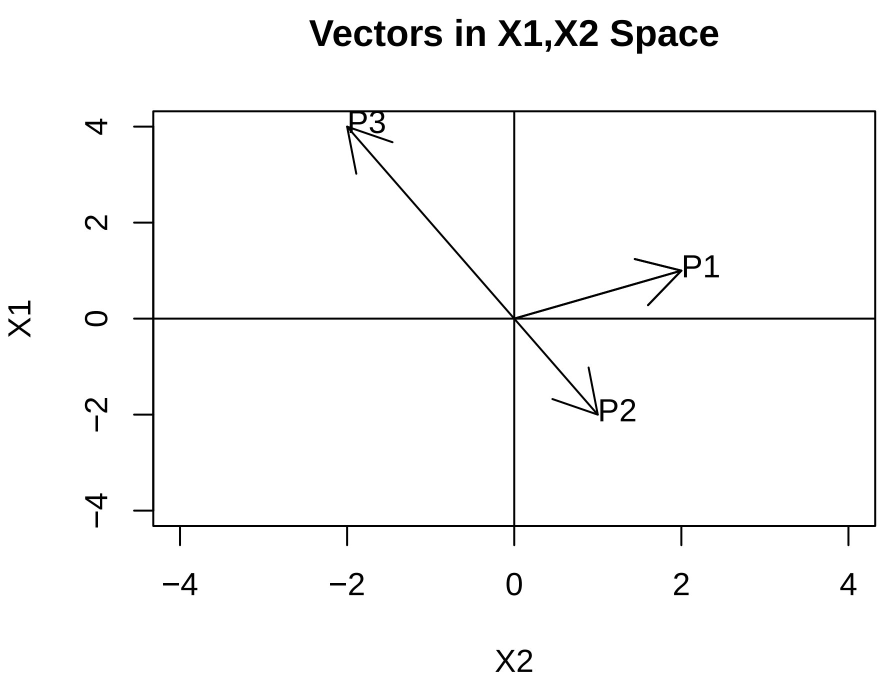
R Markdown

Task 1. The following table lists the standarized values that three persons P_1, P_2 and P_3 from a sample provided in tests X_1 and X_2:

	X_1	X_2
P_1	2	1
P_2	1	-2
P_3	-2	4

a. Enter the person specific values as column vectors $\vec{p_1}$, $\vec{p_2}$ and $\vec{p_3}$ and draw them as vectors in the variable space spanned by X_1 and X_2.



b. Is the system of vectors $\vec{p_1}$, $\vec{p_2}$ and $\vec{p_3}$ linearly independent? Justify your decision both graphically and geometrically, as well as formally via the definition of linear independence.

- Graphically: No. P_1 and P_2 form a straight line.
- Geometrically:
- Analytically: $c \cdot \vec{P_3} = \vec{P_2} \rightarrow -0.5 \cdot \vec{P_3} = \vec{P_2}$

Task 2

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 2 \\ 2 & 2 & 3 \end{pmatrix},\ B = \begin{pmatrix} 0 & 1 \\ 2 & -1 \\ 3 & 0 \end{pmatrix},\ C = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix},\ and\ D = \begin{pmatrix} 4 & 1 & 0 \\ 0 & 5 & 0 \end{pmatrix}$$

Identify the sums **A + B, A + C, A + D, B + D** and products **AB, AC, AD, BD** that are properly defined. Specify the number of rows and columns of the resulting matrices and calculate them by hand and using R.

- A+B, doesn't work. Non-conforming dimensions.
- A+C = $\begin{pmatrix} 2 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 5 \end{pmatrix}$
- A+D, doesn't work. Non-conforming dimensions.
- B+D, doesn't work. Non-conforming dimensions.
- AB: (3x3,3x2) -> 3x2: 0x1+2x2+3x0=0,1x1+0x-1+0x0 = 1… 0x1+2x1+3x2=8, 1x1+-1x1+0x2=0… $\begin{pmatrix} 0 & 1 \\ 8 & 0 \\ 13 & 0 \end{pmatrix}$
- AC (3x3,3x3)-> 3x3: 1x1 + 0x0 +0x0 = 1, 1x1 +0x1 + 0x1 = 1, 2x1+1x0+2x0=2… $\begin{pmatrix} 1 & 1 & 2 \\ 1 & 1 & 3 \\ 2 & 7 & 12 \end{pmatrix}$
- AD (3x3 , 2x3), doesn't work. Non-conforming dimensions.
- BD (3x2, 2x3) -> 3x3: 4x0+0x1=0, 1x0+5x1=1, 0x0+0x1 = 0… $\begin{pmatrix} 0 & 5 & 0 \\ 8 & -3 & 0 \\ 12 & 3 & 0 \end{pmatrix}$

```
a<-matrix(c(1,0,0,1,1,2,2,2,3),3,3,byrow=T)
b<-matrix(c(0,1,2,-1,3,0),3,2,byrow=T)
c <- matrix(c(1,1,2,0,1,1,0,1,2),3,3,byrow=T)
d <- matrix(c(4,1,0,0,5,0),2,3,byrow=T)
#a+b
a+c
```

```
##      [,1] [,2] [,3]
## [1,]    2    1    2
## [2,]    1    2    3
## [3,]    2    3    5
```

```
#a+d
#b+d
a%*%b
```

```
##      [,1] [,2]
## [1,]    0    1
## [2,]    8    0
## [3,]   13    0
```

```
a%*%c
```

```
##      [,1] [,2] [,3]
## [1,]    1    1    2
## [2,]    1    4    7
## [3,]    2    7   12
```

```
#a%*%d
b%*%d
```

```
##      [,1] [,2] [,3]
## [1,]    0    5    0
## [2,]    8   -3    0
## [3,]   12    3    0
```

Task 3. Let $\vec{x}$ be a n-vector and $\vec{1}$ be the n-vector consisting of ones exclusively.

a. Compute the products $\vec{1}'\vec{1}$, and $\vec{1}\vec{1}'$. $\vec{1}'\vec{1} = \sum_n 1 \cdot 1 = n$ $\vec{1}\vec{1}' = (n \times n) \cdot 1$ (n x n) matrix of ones.
b. What do we get by computing $\frac{1}{n}\vec{1}'\vec{x}$? $\frac{1}{n}\vec{1}'\vec{x} = \sum_n 1 \cdot x \cdot \frac{1}{n} = \bar{x}$

** Task 4.** Let the vectors **x,y** and the matrix **A** be given by

$$x = \begin{pmatrix} 7 \\ 4 \\ 1 \end{pmatrix}, y = \begin{pmatrix} 3 \\ 5 \end{pmatrix}, A = \begin{pmatrix} 2 & 7 & 1 \\ 3 & 0 & 4 \end{pmatrix}.$$

Solve the following problems by hand and using R.

a. Calculate (whenever defined) $\vec{x}'\vec{x}$, $\vec{x}\vec{x}'$, $\vec{x}\vec{y}$, $\vec{x}'\vec{y}$ and $\vec{x}\vec{y}'$

- $\vec{x}'\vec{x} = 7^2 + 4^2 + 1^2 = 66$
- $\vec{x}\vec{x}' = 7 * 7, 7 * 4, 7 * 1 ... = \begin{pmatrix} 49 & 28 & 7 \\ 28 & 16 & 4 \\ 7 & 4 & 1 \end{pmatrix}$
- $\vec{x}\vec{y}$ Doesn't work. Non-conforming dimensions (3x1, 2x1).
- $\vec{x}'\vec{y}$ Doesn't work. Non-conforming dimensions (1x3, 2x1).
- $\vec{x}\vec{y}' = 7 * 3, 7 * 5..4 * 3, 4 * 5..1 * 3, 1 * 5... = \begin{pmatrix} 21 & 35 \\ 12 & 20 \\ 3 & 5 \end{pmatrix}$

b. Specify **Ax** and **x'A'**.

- $A\vec{x} = \begin{pmatrix} 7 * 2 + 4 * 7 + 1 * 1 \\ 7 * 3 + 4 * 0 + 1 * 4 \end{pmatrix} = \begin{pmatrix} 43 \\ 25 \end{pmatrix}$
- $x'A' = (Ax)' = \begin{pmatrix} 70 & 40 & 10 \\ 49 & 28 & 7 \end{pmatrix}$

c. Calculate trace and determinant for **x'x** and **xx'**.

- $Trace(66) = 66$
- $Det(66) = 66$
- $Trace(\begin{pmatrix} 49 & 28 & 7 \\ 28 & 16 & 4 \\ 7 & 4 & 1 \end{pmatrix}) + 49 + 16 + 1 = 66$
- $Det(\begin{pmatrix} 49 & 28 & 7 \\ 28 & 16 & 4 \\ 7 & 4 & 1 \end{pmatrix}) = (49*16*1+28*4*7+7*28*4)-(7*16*7+4*4*49+1*28*28) = 2352-2352 = 0$

d. Are row or column vectors, respectively, in **xx'** linearly independent?

- Rows: No. $r_3 \cdot 4 = r_2$,$r_3 \cdot 7 = r_1$
- Columns: No. $c_3 \cdot 4 = c_2$,$c_3 \cdot 7 = c_1$

**Task 5*. ** The set of all real (2×2) matrices forms a vector space with respect to matrix addition and scalar multiplication. Specify a base for this vector space by showing:

a. that every arbitrary matrix

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

with $a_{11}, a_{12}, a_{21}, a_{22} \in R$ can be expressed as a linear combination of the elements of the basis;

- Bases $= e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, e_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, e_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. Therefor:$e1 \cdot A + e2 \cdot A + e3 \cdot A + e4 \cdot A = A$

b. that the elements of the basis are linearly independent.

- $a_1(a_{11}, a_{21}) + a_2(a_{21}, a_{22}) = 0$

Task 6. Given the matrix

$$\mathbf{A} = \begin{pmatrix} 4 & 5 \\ -2 & 1 \end{pmatrix},$$

solve the following problems by hand and using R. a. Is $\mathbf{A}$ invertible? Why/ Why not? * Yes. Is square and has a determinant. b. If yes, calculate $\mathbf{A}^{-1}$ $\mathbf{A}^{-1} = \frac{1}{det(\mathbf{A})}\begin{pmatrix} 1 & -5 \\ 2 & 4 \end{pmatrix}, det(\mathbf{A}) = 14 \rightarrow \frac{1}{14}\begin{pmatrix} 1 & -5 \\ 2 & 4 \end{pmatrix} = \begin{pmatrix} \frac{1}{14} & \frac{-5}{14} \\ \frac{2}{14} & \frac{4}{14} \end{pmatrix} = \begin{pmatrix} 0.07 & -0.357 \\ 0.143 & 0.286 \end{pmatrix}$

c. Check if $\mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$

d. Which vectors $\mathbf{x}$ fulfill the equation:

$$\mathbf{A}\mathbf{x} = \begin{pmatrix} 2 \\ 6 \end{pmatrix}?$$

- if$\mathbf{A}\mathbf{x} = \mathbf{b}, then \mathbf{A}^{-1}\mathbf{b} = \mathbf{x}$
- $\begin{pmatrix} 0.07 & -0.357 \\ 0.143 & 0.286 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 6 \end{pmatrix} = \begin{pmatrix} 0.07 * 2 & -0.357 * 6 \\ 0.143 * 2 & 0.286 * 6 \end{pmatrix} = \begin{pmatrix} 0.07 * 2 & -0.357 * 6 \\ 0.143 * 2 & 0.286 * 6 \end{pmatrix} = \begin{pmatrix} -2 \\ 2 \end{pmatrix}$

Task 7. Determine the number a such that the two vectors are orthogonal

$$\mathbf{x} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}, \mathbf{y} = \begin{pmatrix} -2 \\ a \\ 5 \end{pmatrix}$$

Orthogonal $= x \cdot y = 0$.

$$0 = 2 * -2 + 1 * a + 3 * 5 = -4 + a + 15 = 0 = a = -11$$

Task 8. The points in R^2 given by the vectors

$$\mathbf{x} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}, \mathbf{y} = \begin{pmatrix} 4 \\ -1 \end{pmatrix}, \mathbf{z} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

define a triangle in the coordinate plane. Calculate the result of multiplying these vectors with the matrices:

$$\mathbf{A} = \begin{pmatrix} -0.707 & -0.707 \\ 0.707 & -0.707 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 0.5 & 0 \\ 0 & 0.5 \end{pmatrix}$$

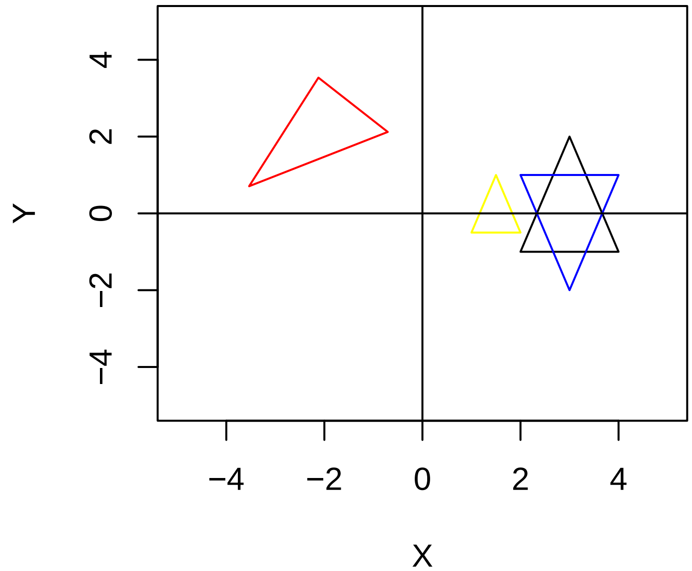
(multiply in the order matrix-vector).

Illustrate the respective transformations of the triangle graphically, and provide geometric interpretations of the associated linear mappings.

```
x <- matrix(c(2,-1),2,1)
y <- matrix(c(4,-1),2,1)
z <- matrix(c(3,2),2,1)
A <-matrix(c(-0.707,-0.707,0.707,-0.707),2,2,byrow=T)
B <-matrix(c(1,0,0,-1),2,2,byrow=T)
C <-matrix(c(0.5,0,0,0.5),2,2,byrow=T)

t1 <- rbind(t(x),t(y),t(z))
```

```r
t1a <- rbind(t(A%*%x),t(A%*%y),t(A%*%z))
t1b <- rbind(t(B%*%x),t(B%*%y),t(B%*%z))
t1c <- rbind(t(C%*%x),t(C%*%y),t(C%*%z))
plot(NULL,xlim=c(-5,5),ylim=c(-5,5),xlab="X",ylab="Y")
polygon(t1,border="black")
polygon(t1a,border="red")
polygon(t1b,border="blue")
polygon(t1c,border="yellow")
abline(h=0)
abline(v=0)
```



- A (red) rotates the triangle.
- B (blue) flips the triangle.
- C (yellow) scales the triangle.