Classic Filters

There are 4 classic analogue filter types: Butterworth, Chebyshev, Elliptic and Bessel. There is no ideal filter; each filter is good in some areas but poor in others.

• **Butterworth**: Flattest pass-band but a poor roll-off rate.

• **Chebyshev:** Some pass-band ripple but a better (steeper) roll-off rate.

• **Elliptic:** Some pass- and stop-band ripple but with the steepest roll-off rate.

• **Bessel:** Worst roll-off rate of all four filters but the best phase response. Filters with a poor phase response will react poorly to a change in signal level.

Butterworth

The first, and probably best-known filter approximation is the **Butterworth** or **maximally-flat** response. It exhibits a nearly flat passband with no ripple. The rolloff is smooth and monotonic, with a low-pass or high-pass rolloff rate of 20 dB/decade (6 dB/octave) for every pole. Thus, a 5th-order Butterworth low-pass filter would have an attenuation rate of 100 dB for every factor of ten increase in frequency beyond the cutoff frequency. It has a reasonably good phase response.

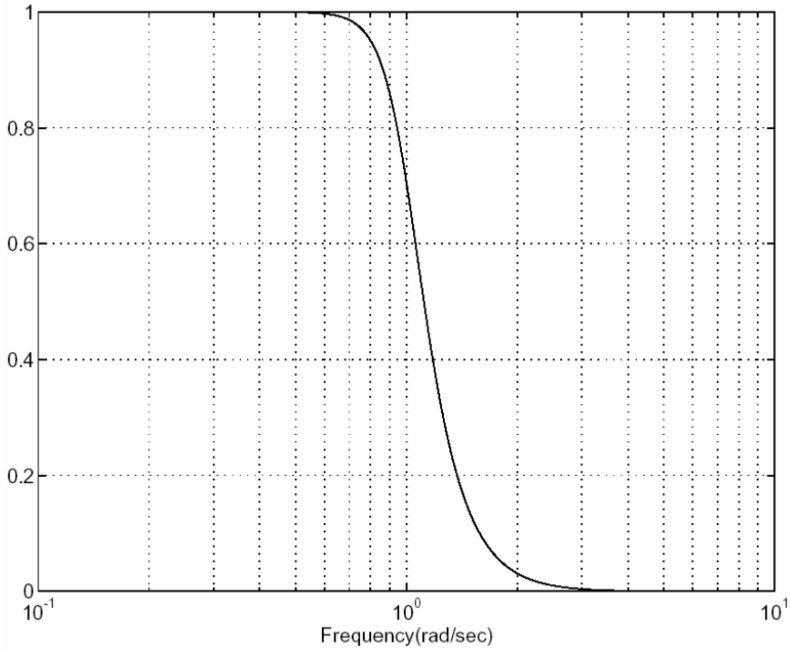


Figure 1 Butterworth Filter

Chebyshev

The Chebyshev response is a mathematical strategy for achieving a faster *roll-off* by allowing *ripple* in the frequency response. As the ripple increases (bad), the roll-off becomes sharper (good). The Chebyshev response is an optimal trade-off between these two parameters. Chebyshev filters where the ripple is only allowed in the *passband* are called **type 1** filters. Chebyshev filters that have ripple only in the *stopband* are called type 2 filters , but are are seldom used. Chebyshev filters have a poor phase response.

It can be shown that for a passband flatness within 0.1dB and a stopband attenuation of 20dB an 8^{th} order Chebyshev filter will be required against a 19^{th} order Butterworth filter. This may be important if you are using a lower specification processor.

The following figure shows the frequency response of a lowpass Chebyshev filter.

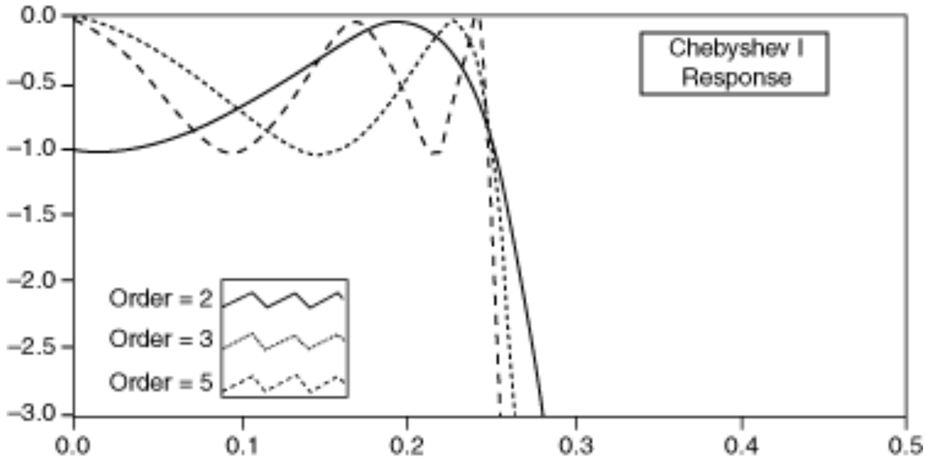


Figure 2

Compared to a Butterworth filter, a Chebyshev filter can achieve a sharper transition between the passband and the stopband with a lower order filter. The sharp transition between the passband and the stopband of a Chebyshev filter produces smaller absolute errors and faster execution speeds than a Butterworth filter.

The following figure shows the frequency response of a lowpass Chebyshev II filter.

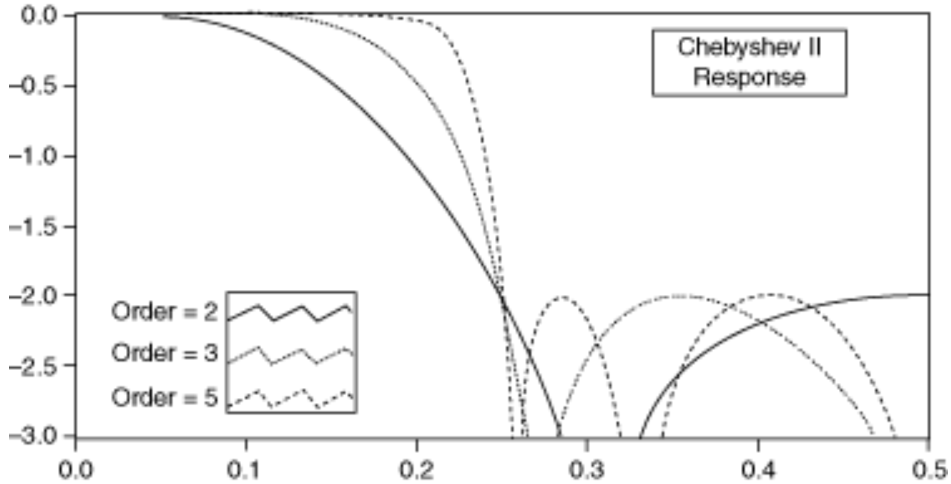


Figure 3

Chebyshev II filters have the same advantage over Butterworth filters that Chebyshev filters have—a sharper transition between the passband and the stopband with a lower order filter, resulting in a smaller absolute error and faster execution speed.

Elliptic

The cut-off slope of an **elliptic** filter is steeper than that of a Butterworth, Chebyshev, or Bessel, but the amplitude response has ripple in both the passband and the stopband, and the phase response is very non-linear. However, if the primary concern is to pass frequencies falling within a certain frequency band and reject frequencies outside that band, regardless of phase shifts or ringing, the elliptic response will perform that function with the lowest-order filter.

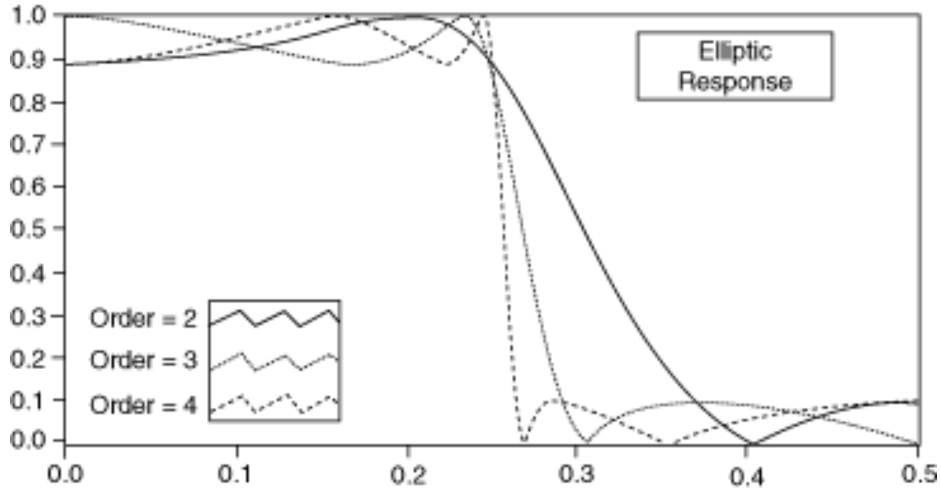


Figure 4

Compared with the same order Butterworth or Chebyshev filters, the elliptic filters provide the sharpest transition between the passband and the stopband, which accounts for their widespread use.

Bessell

- Maximally flat response in both magnitude and phase
- Nearly linear-phase response in the passband

You can use Bessel filters to reduce nonlinear-phase distortion inherent in all IIR filters. High-order IIR filters and IIR filters with a steep roll-off have a pronounced nonlinear-phase distortion, especially in the transition regions of the filters. You also can obtain linear-phase response with FIR filters.

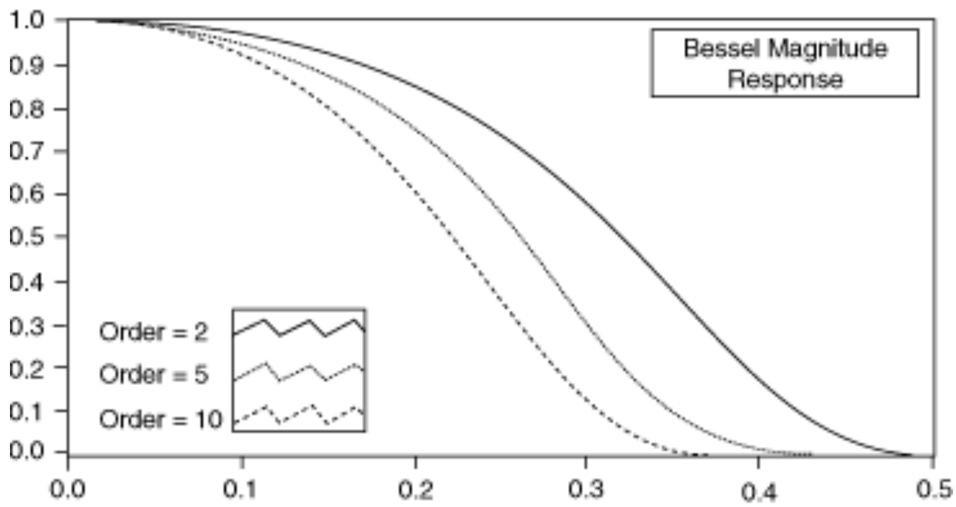


Figure 5

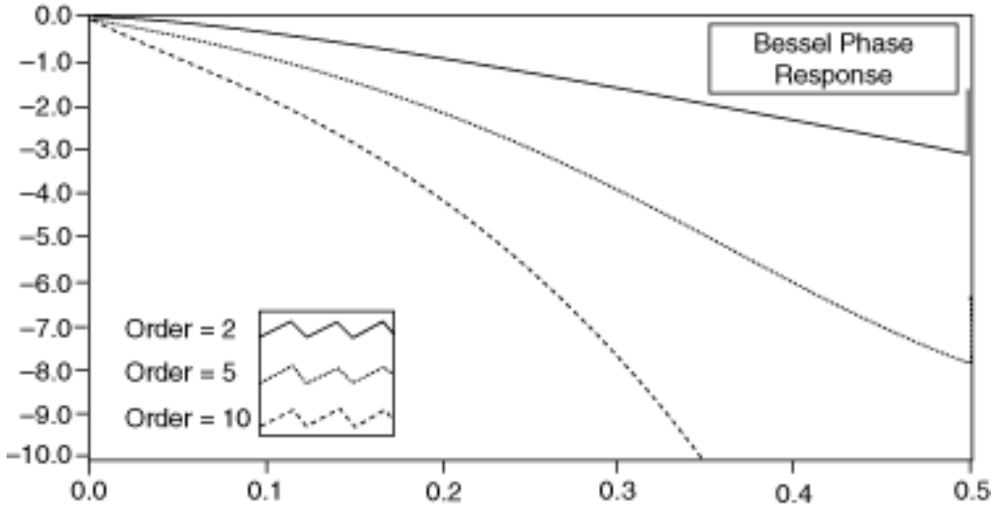


Figure 6

You can use Bessel filters to reduce nonlinear-phase distortion inherent in all IIR filters. High-order IIR filters and IIR filters with a steep roll-off have a pronounced nonlinear-phase distortion, especially in the transition regions of the filters. You also can obtain linear-phase response with FIR filters.

All the filters described above may be analogue or digital. However there is a lot of recorded data about the analogue varieties, so it is often the case that designers use the analogue equations and parameters used and convert them to their digital equivalents. There are two main methods for this, namely the **Impulse Invariant** method and the **Bilinear Transform** method.

Bilinear Transform

Analogue filters are designed using the Laplace transform (s domain) which is the analogue equivalent of the Z transform for digital filters.

Filters designed in the s domain have a transfer function like:

$$T(s) = \frac{1}{1 + \frac{s}{10}}$$

If we have a filter where 10 rads/sec = w_c. Then multiply top and bottom by 10

$$T(s) = \frac{10}{s + 10}$$

To apply the Bilinear transform we just need to replace the s by:

$$s = \frac{2(z-1)}{T(z+1)}$$

Where T is the sampling period. So for a sampling frequency of 16Hz (T= 0.065 s)

$$t(z) = \frac{10}{\frac{2(z-1)}{0.0625(z+1} + 10}$$

And then just work it out!

Near zero frequency, the relation between the analogue and digital frequency response is essentially linear. However as we near the Nyqist frequency it tends to become non-linear . This nonlinear compression is called *frequency warping*.

In the design of a digital filter, the effects of the frequency warping must be taken into account. The prototype filter frequency scale must be prewarped so that after the bilinear transform, the critical frequencies are in the correct places.

Impulse Invariant method

The approach here is to produce a digital filter that has the same impulse response as the analogue filter. It requires the following steps:

1. Compute the Inverse Laplace transform to get impulse response of the analogue filter
2. Sample the impulse response
3. Compute z-transform of resulting sequence

Sampling the impulse response has the advantage of preserving resonant frequencies but its big disadvantage is *aliasing* of the frequency response. Before a continuous impulse response is sampled, a *lowpass filter* should be used to eliminate all frequency components at half the sampling rate and above.

Using the low pass filter transfer function from the previous example:

$$T(s) = \frac{10}{s+10}$$

Now find the inverse Laplace transform from the Laplace transform tables, gives is:

$$y(t) = 10e^{-10t}$$

The final step is to find the z transform, Y(z) of this time variation. Once again from the Laplace/z transform tables, e^{at} has a z transformation of $z/(z - z^{-aT})$. With a sampling frequency of 16Hz:

$$Y(z) = \frac{10z}{z - e^{-0.625}} = \frac{10z}{z - 0.535}$$

As Y(z) = T(z) x 1 for an impulse then:

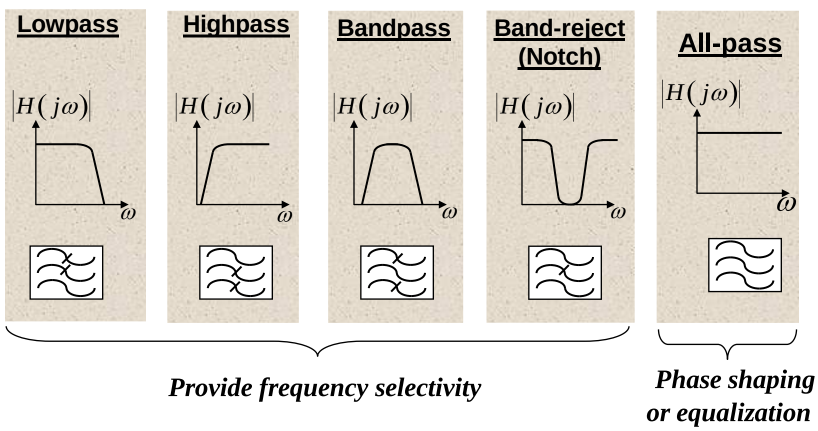
$$T(z) = \frac{10z}{z - 0.535}$$

https://inst.eecs.berkeley.edu/~ee247/fa10/files07/lectures/L2_2_f10n.pdf

EE247 - Lecture 2
Filters

- Filters:
  - Nomenclature
  - Specifications
    - Quality factor
    - Magnitude/phase response versus frequency characteristics
    - Group delay
  - Filter types
    - Butterworth
    - Chebyshev I & II
    - Elliptic
    - Bessel
  - Group delay comparison example
  - Biquads

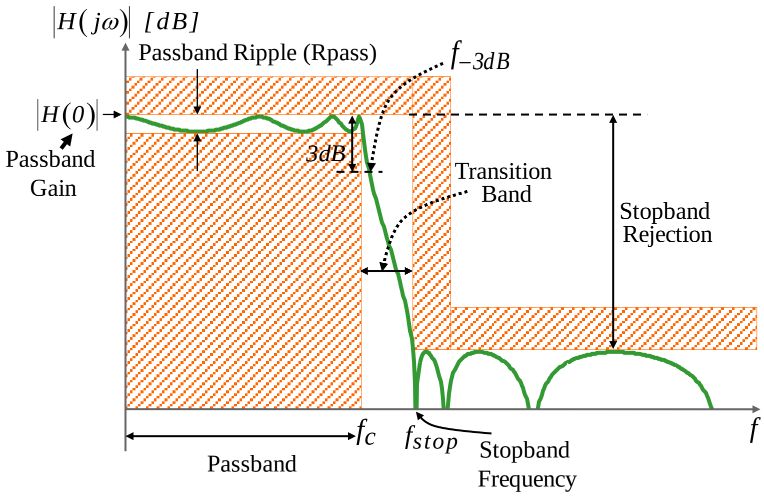
Nomenclature
Filter Types wrt Frequency Range Selectivity



Filter Specifications

- Magnitude response versus frequency characteristics:
  - Passband ripple (Rpass)
  - Cutoff frequency or *-3dB* frequency
  - Stopband rejection
  - Passband gain
- Phase characteristics:
  - Group delay
- SNR (Dynamic range)
- SNDR (Signal to Noise+Distortion ratio)
- Linearity measures: IM3 (intermodulation distortion), HD3 (harmonic distortion), IIP3 or OIP3 (Input-referred or output-referred third order intercept point)
- Area/pole & Power/pole

Filter Magnitude versus Frequency Characteristics
Example: Lowpass



Filters

- Filters:
  - Nomenclature
  - Specifications
    - Magnitude/phase response versus frequency characteristics
    - Quality factor
    - Group delay
  - Filter types
    - Butterworth
    - Chebyshev I & II
    - Elliptic
    - Bessel
  - Group delay comparison example
  - Biquads

Quality Factor (Q)

- The term quality factor (Q) has different definitions in different contexts:
  - Component quality factor (inductor & capacitor Q)
  - Pole quality factor
  - Bandpass filter quality factor
- Next 3 slides clarifies each

Component Quality Factor (Q)

- For any component with a transfer function:

$$H(j\omega)=\frac{1}{R(\omega)+jX(\omega)}$$

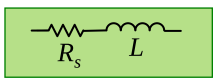
- Quality factor is defined as:

$$Q=\frac{X(\omega)}{R(\omega)} \rightarrow \frac{\text{Energy Stored}}{\text{Average Power Dissipation}} \text{ per unit time}$$

Component Quality Factor (Q)
Inductor & Capacitor Quality Factor

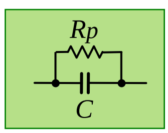
- Inductor Q :
  - ❖ R_s → series parasitic resistance

$$Y_L=\frac{1}{R_s+j\omega L} \qquad Q_L=\frac{\omega L}{R_s}$$



- Capacitor Q :
  - ❖ R_p → parallel parasitic resistance

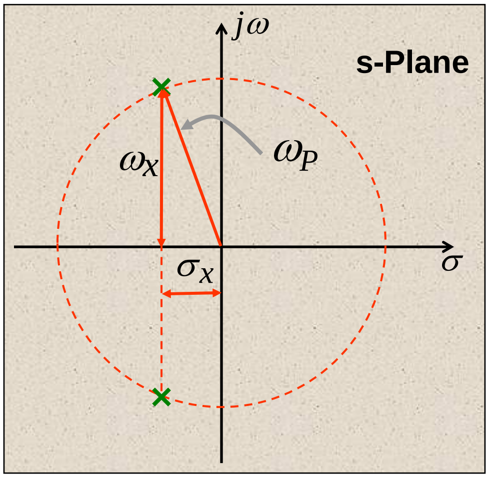
$$Z_C=\frac{1}{\frac{1}{Rp}+j\omega C} \qquad Q_C=\omega C Rp$$



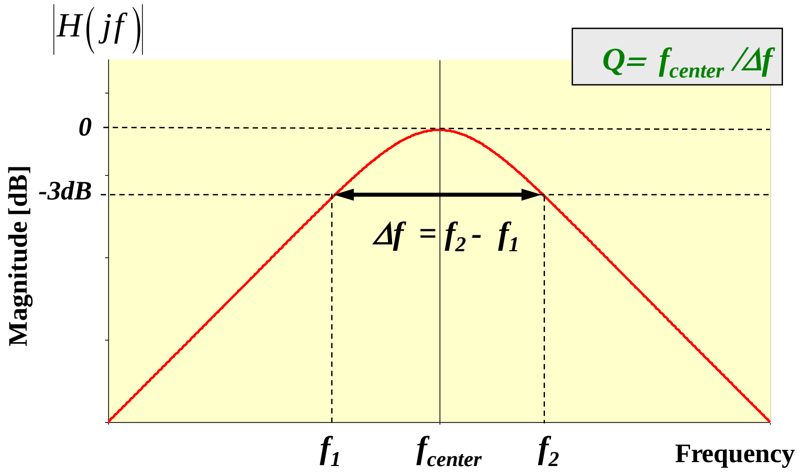
Pole Quality Factor

- Typically filter singularities include pairs of complex conjugate poles.
- Quality factor of complex conjugate poles are defined as:

$$Q_{Pole} = \frac{\omega_p}{2\sigma_x}$$



Bandpass Filter Quality Factor (Q)



Filters

- Filters:
  - Nomenclature
  - Specifications
    - Magnitude/phase response versus frequency characteristics
    - Quality factor
    - Group delay
  - Filter types
    - Butterworth
    - Chebyshev I & II
    - Elliptic
    - Bessel
  - Group delay comparison example
  - Biquads

What is Group Delay?

- Consider a continuous-time filter with s-domain transfer function G(s):

$$G(j\omega) \equiv |G(j\omega)| e^{j\theta(\omega)}$$

- Let us apply a signal to the filter input composed of sum of two sine waves at slightly different frequencies ($\Delta\omega \ll \omega$):

$$v_{IN}(t) = A_1 \sin(\omega t) + A_2 \sin[(\omega+\Delta\omega)\, t]$$

- The filter output is:

$$v_{OUT}(t) = A_1 \, |G(j\omega)| \, \sin[\omega t+\theta(\omega)] +$$

$$A_2 \, |G[\, j(\omega+\Delta\omega)]| \, \sin[(\omega+\Delta\omega)t+ \theta(\omega+\Delta\omega)]$$

What is Group Delay?

$$v_{OUT}(t) = A_1 \left|G(j\omega)\right| \sin\left\{\omega\left[t + \frac{\theta(\omega)}{\omega}\right]\right\} +$$

$$+ A_2 \left|G[j(\omega+\Delta\omega)]\right| \sin\left\{(\omega+\Delta\omega)\left[t + \frac{\theta(\omega+\Delta\omega)}{\omega+\Delta\omega}\right]\right\}$$

Since $\frac{\Delta\omega}{\omega} \ll 1$ then $\left[\frac{\Delta\omega}{\omega}\right]^2 \rightarrow 0$

$$\frac{\theta(\omega+\Delta\omega)}{\omega+\Delta\omega} \cong \left[\theta(\omega) + \frac{d\theta(\omega)}{d\omega}\Delta\omega\right]\left[\frac{1}{\omega}\left(1 - \frac{\Delta\omega}{\omega}\right)\right]$$

$$\cong \frac{\theta(\omega)}{\omega} + \left(\frac{d\theta(\omega)}{d\omega} - \frac{\theta(\omega)}{\omega}\right)\frac{\Delta\omega}{\omega}$$

What is Group Delay?
Signal Magnitude and Phase Impairment

$$v_{OUT}(t) = A_1 \left|G(j\omega)\right| \sin\left\{\omega\left[t + \frac{\theta(\omega)}{\omega}\right]\right\} +$$

$$+ A_2 \left|G[j(\omega+\Delta\omega)]\right| \sin\left\{(\omega+\Delta\omega)\left[t + \frac{\theta(\omega)}{\omega} + \underbrace{\left(\frac{d\theta(\omega)}{d\omega} - \frac{\theta(\omega)}{\omega}\right)\frac{\Delta\omega}{\omega}}_{\delta}\right]\right\}$$

- $\tau_{PD} \equiv -\theta(\omega)/\omega$ is called the "phase delay" and has units of time
- If the delay term δ is zero → the filter's output at frequency $\omega+\Delta\omega$ and the output at frequency ω are each delayed in time by $-\theta(\omega)/\omega$
- If the term δ is non-zero → the filter's output at frequency $\omega+\Delta\omega$ is time-shifted differently than the filter's output at frequency ω

→ "Phase distortion"

What is Group Delay?
Signal Magnitude and Phase Impairment

- Phase distortion is avoided only if:

$$\frac{d\theta(\omega)}{d\omega} - \frac{\theta(\omega)}{\omega} = 0$$

- Clearly, if $\theta(\omega)=k\omega$, k a constant, → no phase distortion
- This type of filter phase response is called "linear phase"
  →Phase shift varies linearly with frequency
- $\tau_{GR} \equiv -d\theta(\omega)/d\omega$ is called the "group delay" and also has units of time. For a linear phase filter $\tau_{GR} \equiv \tau_{PD} = -k$
  → $\tau_{GR} = \tau_{PD}$ implies linear phase
- Note: Filters with $\theta(\omega)=k\omega+c$ are also called linear phase filters, but they're not free of phase distortion

What is Group Delay?
Signal Magnitude and Phase Impairment

- If $\tau_{GR} = \tau_{PD}$ → No phase distortion

$$v_{OUT}(t) = A_1 \left|G(j\omega)\right| \sin\left[\omega\left(t - \tau_{GR}\right)\right] +$$
$$+ A_2 \left|G[j(\omega+\Delta\omega)]\right| \sin\left[(\omega+\Delta\omega)\left(t - \tau_{GR}\right)\right]$$

- If also $\left|G(j\omega)\right| = \left|G[j(\omega+\Delta\omega)]\right|$ for all input frequencies within the signal-band, v_{OUT} is a scaled, time-shifted replica of the input, with no "signal magnitude distortion"
- In most cases neither of these conditions are exactly realizable

Summary
Group Delay

- Phase delay is defined as:

  $\tau_{PD} \equiv -\theta(\omega)/\omega$ [time]

- Group delay is defined as :

  $\tau_{GR} \equiv -d\theta(\omega)/d\omega$ [time]

- If $\theta(\omega)=k\omega$, k a constant, → no phase distortion

- For a linear phase filter $\tau_{GR} \equiv \tau_{PD} = -k$

Filters

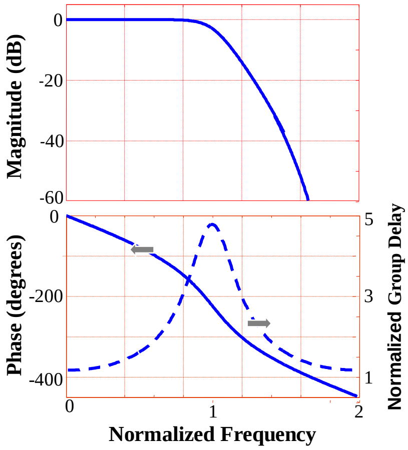
- Filters:
  - Nomenclature
  - Specifications
    - Magnitude/phase response versus frequency characteristics
    - Quality factor
    - Group delay
  - Filter types (examples considered all lowpass, the highpass and bandpass versions similar characteristics)
    - Butterworth
    - Chebyshev I & II
    - Elliptic
    - Bessel
  - Group delay comparison example
  - Biquads

Filter Types wrt Frequency Response
Lowpass Butterworth Filter

- Maximally flat amplitude within the filter passband

$$\left.\frac{d^N |H(j\omega)|}{d\omega}\right|_{\omega=0} = 0$$

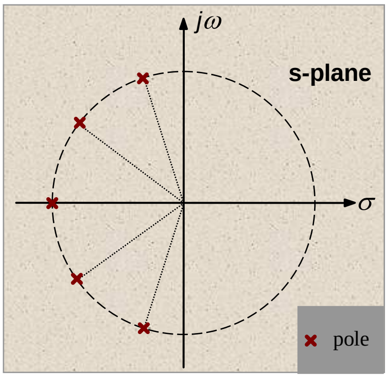
- Moderate phase distortion



Example: 5th Order Butterworth filter

Lowpass Butterworth Filter

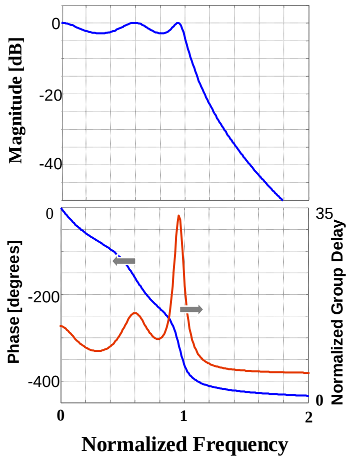
- All poles
- Number of poles equal to filter order
- Poles located on the unit circle with equal angles



Example: 5th Order Butterworth Filter

Filter Types
Chebyshev I Lowpass Filter

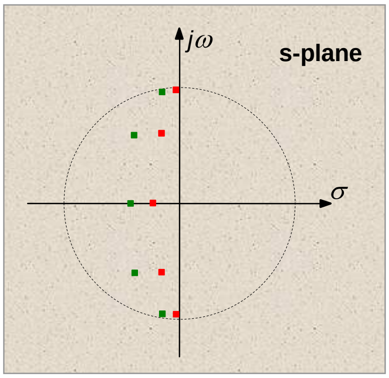
- Chebyshev I filter
  - Ripple in the passband
  - Sharper transition band compared to Butterworth (for the same number of poles)
  - Poorer group delay compared to Butterworth
  - More ripple in passband → poorer phase response



Example: 5th Order Chebyshev filter

Chebyshev I Lowpass Filter Characteristics

- All poles
- Poles located on an ellipse inside the unit circle
- Allowing more ripple in the passband:
  - ⇨Narrower transition band
  - ⇨Sharper cut-off
  - ⇨Higher pole Q
  - ⇨Poorer phase response

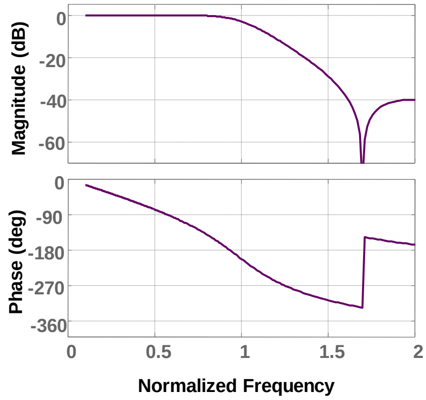


- Chebyshev I LPF 3dB passband ripple
- Chebyshev I LPF 0.1dB passband ripple

Example: 5th Order Chebyshev I Filter

Filter Types
Chebyshev II Lowpass

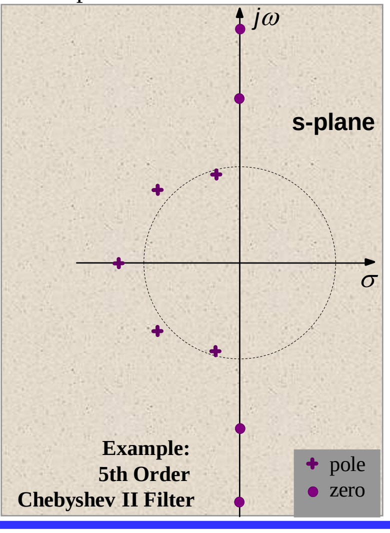
- Chebyshev II filter
  - No ripple in passband
  - Nulls or notches in stopband
  - Sharper transition band compared to Butterworth
  - Passband phase more linear compared to Chebyshev I



Example: 5th Order Chebyshev II filter

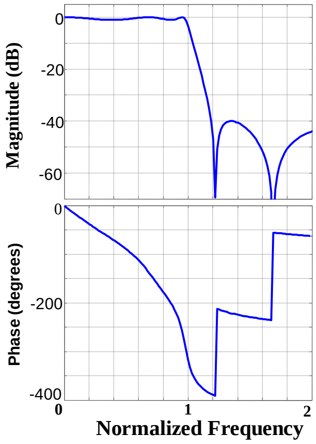
Filter Types
Chebyshev II Lowpass

- Poles & finite zeros
  - No. of poles *n* (*n* –> filter order)
  - No. of finite zeros: *n-1*
- Poles located both inside & outside of the unit circle
- Complex conjugate zeros located on $j\omega$ axis
- Zeros create nulls in stopband



Filter Types
Elliptic Lowpass Filter

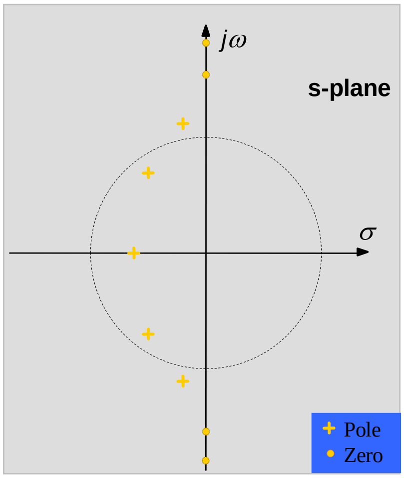
- Elliptic filter
  - Ripple in passband
  - Nulls in the stopband
  - Sharper transition band compared to Butterworth & both Chebyshevs
  - Poorest phase response



Example: 5th Order Elliptic filter

Filter Types
Elliptic Lowpass Filter

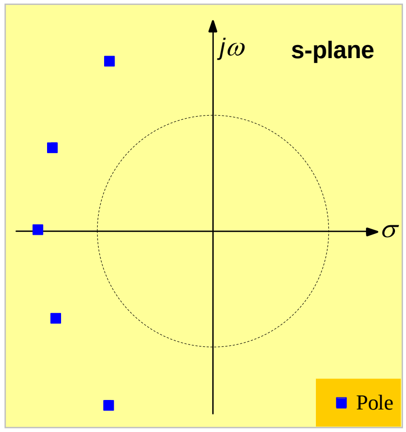
- Poles & finite zeros
  - No. of poles: *n*
  - No. of finite zeros: *n-1*
- Zeros located on $j\omega$ axis
- Sharp cut-off
  - ⇨Narrower transition band
  - ⇨Pole Q higher compared to the previous filter types



Example: 5th Order Elliptic Filter

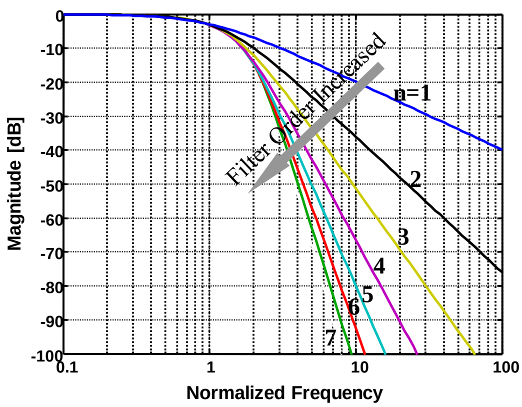
Filter Types
Bessel Lowpass Filter

- Bessel
  - All poles
  - Poles outside unit circle
  - Relatively low Q poles
  - **Maximally flat group delay**
  - Poor out-of-band attenuation



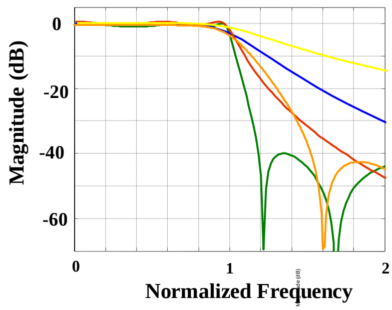
Example: 5th Order Bessel filter

Magnitude Response Behavior as a Function of Filter Order Example: Bessel Filter



n→ Filter order

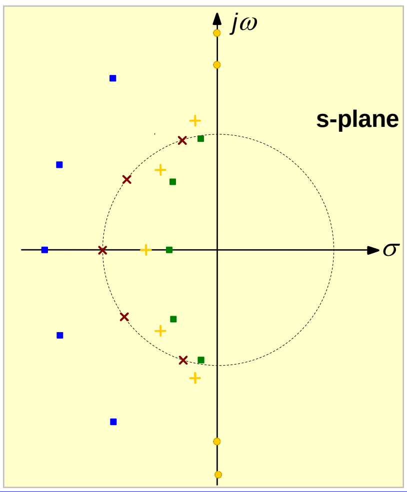
Filter Types
Comparison of Various Type LPF Magnitude Response



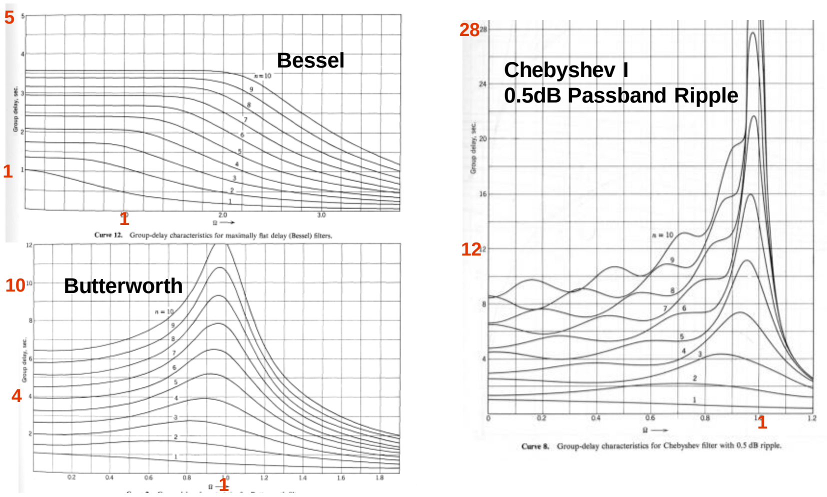
All 5th order filters with same corner freq.

Filter Types
Comparison of Various LPF Singularities

- Poles Bessel
- Poles Butterworth
- Poles Elliptic
- Zeros Elliptic
- Poles Chebyshev I 0.1dB



Comparison of Various LPF Groupdelay



Ref: A. Zverev, *Handbook of filter synthesis*, Wiley, 1967.

Filters

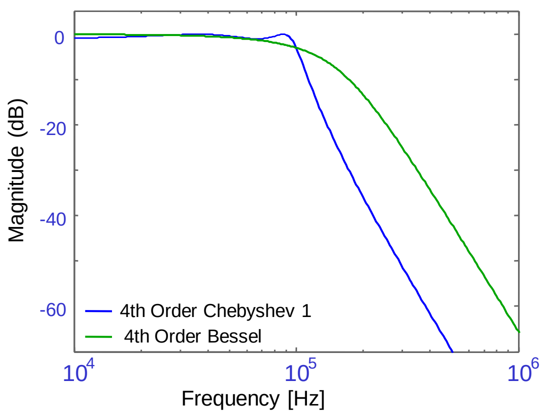
- Filters:
  - Nomenclature
  - Specifications
    - Magnitude/phase response versus frequency characteristics
    - Quality factor
    - Group delay
  - Filter types
    - Butterworth
    - Chebyshev I & II
    - Elliptic
    - Bessel
  - Group delay comparison example
  - Biquads

Group Delay Comparison Example

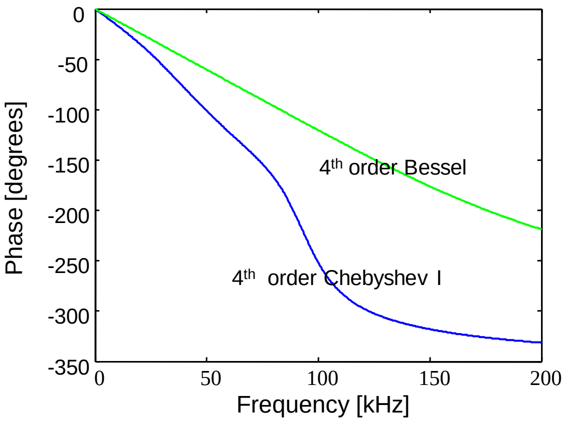
- Lowpass filter with 100kHz corner frequency
- Chebyshev I versus Bessel
  - Both filters 4th order- same ***-3dB*** point
  - Passband ripple of ***1dB*** allowed for Chebyshev I

Magnitude Response

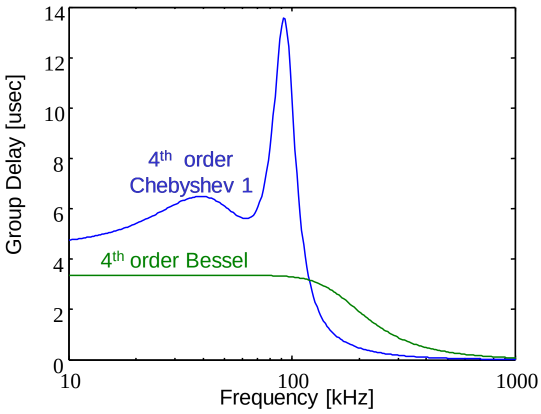
4th Order Chebyshev I versus Bessel



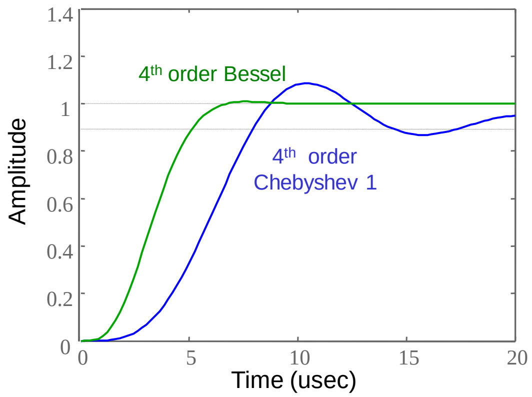
Phase Response
4^{th} Order Chebyshev I versus Bessel



Group Delay
4^{th} Order Chebyshev I versus Bessel



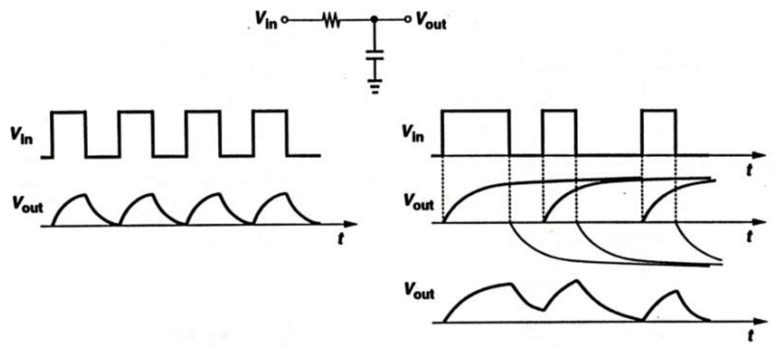
Step Response
4^{th} Order Chebyshev I versus Bessel



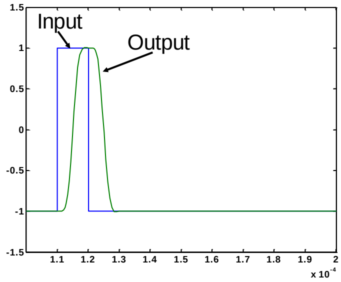
Intersymbol Interference (ISI)

ISI→ Broadening of pulses resulting in interference between successive transmitted pulses

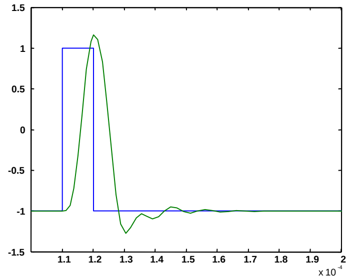
Example: Simple RC filter



Pulse Impairment
Bessel versus Chebyshev



4th order Bessel



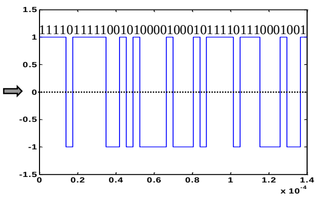
4th order Chebyshev I

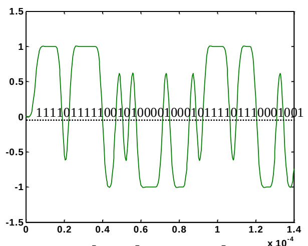
Note that in the case of the Chebyshev filter not only the pulse has broadened but it also has a long tail

→ More ISI for Chebyshev compared to Bessel

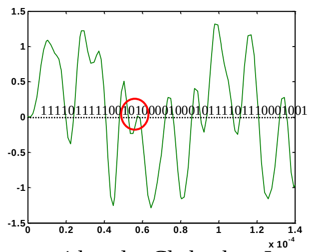
Response to Pseudo-Random Data
Chebyshev versus Bessel

Input Signal:
Symbol rate 1/130kHz ⇨





4th order Bessel



4th order Chebyshev I

Summary
Filter Types

– Filter types with high signal attenuation per pole ⇨ poor phase response

– For a given signal attenuation, requirement of preserving constant groupdelay → Higher order filter
  - In the case of passive filters ⇨ higher component count
  - For integrated active filters ⇨ higher chip area & power dissipation

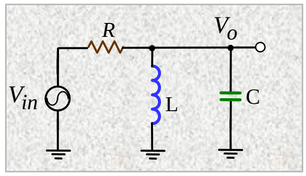
– In cases where filter is followed by ADC and DSP
  - In some cases possible to digitally correct for phase impairments incurred by the analog circuitry by using digital phase equalizers & thus possible to reduce the required analog filter order

Filters

- Filters:
  - Nomenclature
  - Specifications
    - Magnitude/phase response versus frequency characteristics
    - Quality factor
    - Group delay
  - Filter types
    - Butterworth
    - Chebyshev I & II
    - Elliptic
    - Bessel
  - Group delay comparison example
  - Biquads

RLC Filters

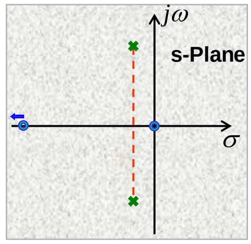
- Bandpass filter (2nd order):



$$\frac{V_o}{V_{in}} = \frac{\frac{s}{RC}}{s^2 + \frac{\omega_0}{Q}s + \omega_0^2}$$

$$\omega_0 = 1 \big/ \sqrt{LC}$$

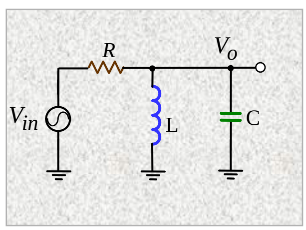
$$Q = \omega_0 RC = \frac{R}{L\omega_0}$$



Singularities: Pair of complex conjugate poles
Zeros @ $f=0$ & $f=inf$.

RLC Filters
Example

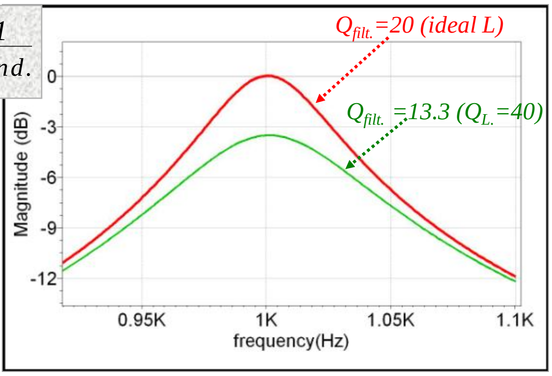
- Design a bandpass filter with:
  - Center frequency of 1kHz
  - Filter quality factor of 20



- First assume the inductor is ideal
- Next consider the case where the inductor has series R resulting in a finite inductor Q of 40
- What is the effect of finite inductor Q on the overall filter Q?

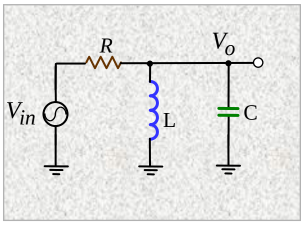
RLC Filters
Effect of Finite Component Q

$$\frac{1}{Q_{filt}} \approx \frac{1}{Q_{filt}^{ideal}} + \frac{1}{Q_{ind.}}$$



⇨Need to have component Q much higher compared to desired filter Q

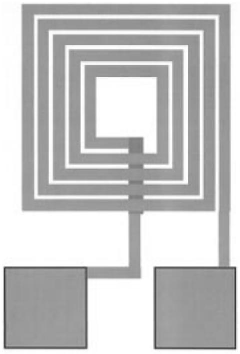
RLC Filters



Question:

Can RLC filters be integrated on-chip?

Monolithic Spiral Inductors



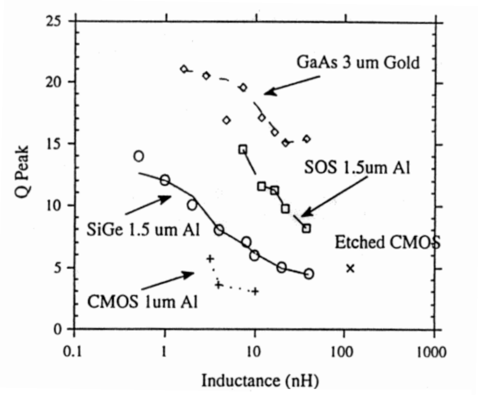
Top View

Monolithic Inductors
Feasible Quality Factor & Value

Typically, on-chip inductors built as spiral structures out of metal/s layers

$Q_L = (\omega L/R)$

Q_L measured at frequencies of operation ($>1GHz$)



➪ Feasible monolithic inductor in CMOS tech. <10nH with Q <7

❖Ref: "Radio Frequency Filters", Lawrence Larson; Mead workshop presentation 1999

Integrated Filters

- Implementation of RLC filters in CMOS technologies requires on-chip inductors
  - Integrated L<10nH with Q<10
  - Combined with max. cap. 20pF
  - → *LC filters in the monolithic form feasible: freq>350MHz*
  - → *(Learn more in EE242 & RF circuit courses)*

- Analog/Digital interface circuitry require fully integrated filters with critical frequencies << 350MHz

- Hence:

⇨ Need to build active filters without using inductors

Filters

2^{nd} Order Transfer Functions (Biquads)

- Biquadratic (2^{nd} order) transfer function:

$$H(s) = \frac{1}{1 + \frac{s}{\omega_P Q_P} + \frac{s^2}{\omega_P^2}}$$

$$|H(j\omega)| = \frac{1}{\sqrt{\left(1 - \frac{\omega^2}{\omega_P^2}\right)^2 + \left(\frac{\omega}{\omega_P Q_P}\right)^2}} \Rightarrow \begin{cases} \left|H(j\omega)\right|_{\omega=0} = 1 \\ \left|H(j\omega)\right|_{\omega\to\infty} = 0 \\ \left|H(j\omega)\right|_{\omega=\omega_P} = Q_P \end{cases}$$

$$\text{Biquad poles @: } s = -\frac{\omega_P}{2Q_P}\left(1 \pm \sqrt{1 - 4Q_P^2}\right)$$

Note: for $Q_P \le \frac{1}{2}$ *poles are real, complex otherwise*

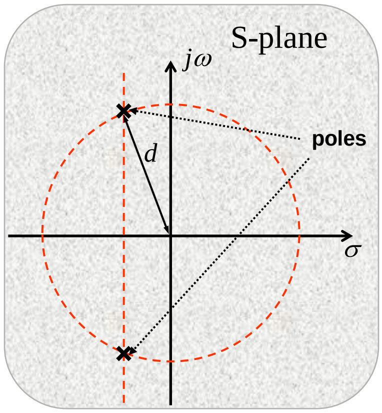
Biquad Complex Poles

$Q_P > \frac{1}{2} \quad \rightarrow$ Complex conjugate poles:

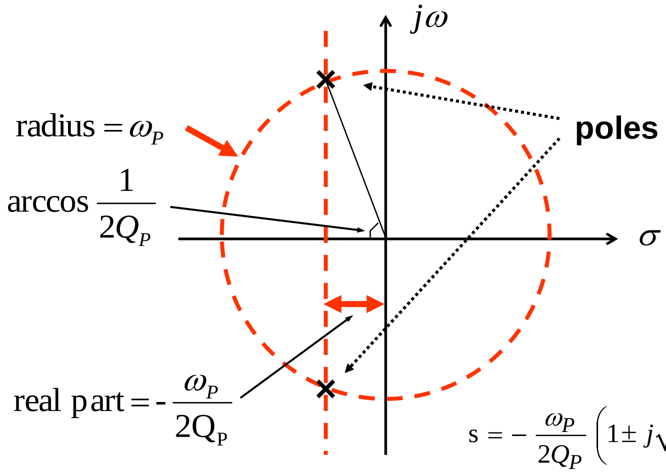
$$s = -\frac{\omega_P}{2Q_P}\left(1 \pm j\sqrt{4Q_P^2 - 1}\right)$$

Distance from origin in s-plane:

$$d^2 = \left(\frac{\omega_P}{2Q_P}\right)^2 \left(1 + 4Q_P^2 - 1\right)$$
$$= \omega_P^2$$



s-Plane



$$s = -\frac{\omega_P}{2Q_P}\left(1 \pm j\sqrt{4Q_P^2 - 1}\right)$$