Lock-in amplifier equations

Amplitude units and factors

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Abstract

This document discusses the derivation of the equations that make up a dual-phase lock-in amplifier. Special attention is given to the physical units of the signal amplitudes and to the origin of factors like $\sqrt{2}$ in those equations. It serves as a supplement to the following sources that have been used as a background: section §2.4 from *Meade* [1], the white paper by *Zurich Instruments* [2] and the research note by *Stanford Research Systems* [3]. While these sources nicely explain the mathematical operations of a lock-in amplifier in varying degrees of detail, none of them are very clear on the aforementioned units and factors.

Chapter 1 starts with a short overview on trigonometric identities used within this document and on the relation between peak and root-mean-square amplitudes of standard waveform types. Chapter 2 proceeds with the derivation of the signal demodulation for sinusoidal waveforms.

1 Background

1.1 Trigonometric identities

$$\cos(a)\cos(b) = \frac{1}{2}\Big[\cos(a+b) + \cos(a-b)\Big] \tag{1}$$

$$\cos(a)\sin(b) = \frac{1}{2}\Big[\sin(a+b) - \sin(a-b)\Big] \tag{2}$$

1.2 Root-mean-square amplitude

The root-mean-square (rms) amplitude is useful as it relates to the average power contained in a waveform regardless of its type, something the peak amplitude does not. The following rms factors σ exist that relate the peak amplitude V_{peak} of a standard periodic waveform type to its rms amplitude V_{rms},

$$V_{\mathrm{peak}} = \sigma V_{\mathrm{rms}}\,, \tag{3}$$

where, depending on the waveform type,

$$\begin{aligned} \text{sine} : \sigma &= \sqrt{2}\,, \\ \text{square} : \sigma &= 1\,, \\ \text{triangle} : \sigma &= \sqrt{3}\,. \end{aligned}$$

2 Derivation for sinusoidal waveforms

This section derives the equations for the signal demodulation of a dual-phase lock-in amplifier. The demodulation consists of heterodyne mixing of the input and reference signals, followed by low-pass filtering.

2.1 Input signal

Let's assume we have a sinusoidal input signal whose time-dependent amplitude V_s is given by

$$V_s = A_s \cos(\omega_s t + \theta_s)\,, \tag{4}$$

where t is the time, A_s is the peak amplitude of the signal, ω_s is the signal frequency and θ_s is the phase offset of the signal. Furthermore for this discussion, we'll assume the input signal is measured as a voltage. Hence, the units of V_s and A_s are in volts, denoted V. Commonly and conveniently, lock-in amplifiers use the rms amplitude instead because it directly relates to the energy content in a waveform, in contrast to the peak amplitude. We substitute the peak amplitude with the rms amplitude using eq. 3, resulting in

$$V_s = \sqrt{2} R_s \cos(\omega_s t + \theta_s)\,, \tag{5}$$

where R_s is the rms amplitude of the signal measured in volts rms, denoted $\mathrm{V_{rms}}$.

2.2 Reference signal

Internal to a dual-phase lock-in amplifier are the time-dependent in-phase component V_X and the quadrature component V_Y of the reference signal given by

$$V_X = A_r \cos(\omega_r t + \theta_r)\,, \tag{6}$$

$$V_Y = A_r \sin(\omega_r t + \theta_r)\,, \tag{7}$$

where A_r is the peak amplitude of the reference signal, ω_r is the reference frequency and θ_r is the phase offset of the reference signal. Note that V_X, V_Y and A_r do not carry any physical units as they represent a mathematical construct. It is true that lock-in amplifiers can output V_X as a voltage – be it amplified and/or offset – to drive the device under test, but mathematically speaking V_X and V_Y must be non-dimensional for the upcoming heterodyne mixing.

Similar to the input signal it is preferable to use the rms amplitude instead. By definition, the rms amplitude of the in-phase and quadrature components of the reference signal are set to 1 as it represents the norm for the upcoming heterodyne mixing, resulting in

$$V_X = \sqrt{2} \cos(\omega_r t + \theta_r)\,, \tag{8}$$

$$V_Y = \sqrt{2} \sin(\omega_r t + \theta_r)\,. \tag{9}$$

2.3 Heterodyne mixing

With the input and reference signals now defined we follow with their heterodyne mixing, making use of the trigonometric identities from section 1.1:

$$\begin{aligned} M_X &= V_s V_X \\ &= 2R_s \cos(\omega_s t + \theta_s) \cos(\omega_r t + \theta_r) \\ &= R_s \Big[\cos\big((\omega_s + \omega_r)t + \theta_s + \theta_r\big) + \cos\big((\omega_s - \omega_r)t + \theta_s - \theta_r\big) \Big]\,, \end{aligned} \tag{10}$$

$$\begin{aligned} M_Y &= V_s V_Y \\ &= 2R_s \cos(\omega_s t + \theta_s) \sin(\omega_r t + \theta_r) \\ &= R_s \Big[\sin\big((\omega_s + \omega_r)t + \theta_s + \theta_r\big) - \sin\big((\omega_s - \omega_r)t + \theta_s - \theta_r\big) \Big]\,. \end{aligned} \tag{11}$$

Observe that the units of M_X and M_Y are both in V and not in V^2. To better understand the principles of the lock-in amplifier we simplify above equations **by assuming** $\omega_s = \omega_r$, resulting in

$$M_X = R_s\Big[\cos\big(2\omega_r t + \theta_s + \theta_r\big) + \cos\big(\theta_s - \theta_r\big)\Big]\,, \tag{12}$$

$$M_Y = R_s\Big[\sin\big(2\omega_r t + \theta_s + \theta_r\big) - \sin\big(\theta_s - \theta_r\big)\Big]\,. \tag{13}$$

2.4 Low-pass filtering and signal reconstruction

The final step of the signal demodulation involves filtering out the $2\omega_r$ component. With an appropriately chosen low-pass filter applied to M_X and M_Y we recover the in-phase signal X and quadrature signal Y:

$$X = R_s \cos\big(\theta_s - \theta_r\big)\,, \tag{14}$$

$$Y = -R_s \sin\big(\theta_s - \theta_r\big)\,. \tag{15}$$

Lastly, we can transform the Cartesian coordinates X and Y into the polar coordinates

$$\begin{aligned} R &= \sqrt{X^2 + Y^2} \\ &= R_s\sqrt{\cos^2(\theta_s - \theta_r) + \sin^2(\theta_s - \theta_r)} \\ &= R_s\,, \end{aligned} \tag{16}$$

$$\begin{aligned} \Theta &= \mathrm{atan2}(Y, X) \\ &= [\theta_s - \theta_r]_{\mathrm{wrapped}(-180^\circ, +180^\circ]}\,, \end{aligned} \tag{17}$$

where R is the signal amplitude measured in V, and Θ is the phase difference between the input and reference signals, commonly reported in degrees. From eq. 16 is becomes clear why it is convenient to work with the rms amplitude of the input signal, instead of its peak amplitude. Note that R reads as V, whereas R_s is given by $\mathrm{V_{rms}}$.

References

[1] Meade, M. L., *Lock-in amplifiers: principles and applications*. Printed book: P. Peregrinus Ltd (London), 1983. Free e-edition:
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