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Parametric bootstrap simultaneous confidence intervals for differences of means from several two-parameter exponential distributions

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ABSTRACT

A parametric bootstrap method is proposed for constructing simultaneous confidence intervals (SCIs) for all pairwise differences of means from several two-parameter exponential distributions. The proposed SCIs are shown to have correct coverage probability asymptotically. Simulation studies show that, comparing with some existing methods, the proposed SCIs are generally closer to the nominal level and possess smaller volumes.

1. Introduction

The probability density function of the two-parameter exponential distribution $\mathrm{Exp}(\mu, \theta)$ is defined as

$$f(x; \mu, \theta) = \frac{1}{\theta}\exp\left(-\frac{x-\mu}{\theta}\right) I_{[\mu,\infty)}(x),$$

where μ is the location parameter, θ is the scale parameter, and $I_A(.)$ is the indicator function of A. The two-parameter exponential distribution family is widely used in the mechanical reliability, life testing, insurance and actuarial science fields, among others. Roy and Mathew (2005) proposed a method based on the concept of generalized confidence intervals to find a generalized confidence limit for the reliability function $e^{-\frac{(x-\mu)}{\theta}}$. Li and Zhang (2010) considered the problem of construct asymptotic confidence interval for the ratio of means of two two-parameter exponential distributions. Kharrati-Kopaei et al. (2013) consider simultaneous fiducial generalized confidence intervals for differences of the location parameters of several exponential distributions under heteroscedasticity. In the quality control study and the experimental design, a more important parameter of interest is the mean lifespan of certain products. For example, it is known that the product quality directly affects the competitive advantage of an enterprise in the market. The quality of the product and its lifespan are

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closely linked. If we assume that several component's life of a mechanical system are all follow life distribution, it is necessary for us to compare the mean life of these parts, timing to replacement and maintenance of these components to ensure the reliability of the product; In experimental design we often consider comparing the life of one or more reference products or one of more test products. Therefore, all pairwise differences of mean life of two or three products have become the urgent problem to address. It is typically assumed that the product life follows a two-parameter exponential distribution $\mathrm{Exp}(\mu, \theta)$, thus its mean life is $\delta = \mu + \theta$, and the question of interest is to compare the differences δ's from several such distributions. Surprisingly, to the best of our knowledge, the literature seems scant in this area. In this paper, we will try to fill this void by constructing simultaneous confidence intervals (SCIs) for differences of two-parameter exponential means using a parametric bootstrap (PB) method. For more about the two-parameter exponential distribution family, see Lawless (1982), Maurya et al. (2011) and the references therein.

The bootstrap approach is a computer method frequently used in applied statistics, which is a type of Monte Carlo method applied on observed data, see Efron and Tibshirani (1993) for more information on this important topic. The bootstrap method can be carried out in either parametric or nonparametric setting. However, the question addressed in this paper is in a strict parametric setting, therefore a parametric bootstrap approach will be constructed accordingly.

The paper is organized as follows. The proposed parametric bootstrap method for constructing SCIs for differences of means of several two-parameter exponential distributions is presented in Section 2, and a theorem on the asymptotic correct coverage probability of the PB SCIs is given in Section 3. In Section 4 simulation results are present to evaluate the empirical coverage probabilities and average volume of the proposed method in comparison to the fiducial generalized simultaneous confidence intervals (FG SCIs) and a nonparametric bootstrap simultaneous confidence intervals (NPB SCIs) procedures. Some concluding remarks are made in Section 5 and the FG SCIs and NPB SCIs methods are briefly described in the Appendix.

2. The parametric bootstrap approach

Let $X_{i1}, \ldots, X_{in_i}$ be random samples from $\mathrm{Exp}(\mu_i, \theta_i)$, $i = 1, \ldots, k$. Suppose that all X_{ij} are independent, $i = 1, \ldots, k$, $j = 1, \ldots, n_i$. Then for $i = 1, \ldots, k$, $\{X_{ij},\ j = 1, \ldots, n_i\}$ is an i.i.d. sample from a two-parameter exponential distribution with location parameter μ_i and scale parameter θ_i. The parameter of interest are $\delta_{il} = \delta_i - \delta_l$ for all $i \neq l$, $i, l = 1, \ldots, k$. In the following, we will construct simultaneous confidence intervals for δ_{il} using a parametric bootstrap algorithm.

For any fixed $i = 1, \ldots, k$, let $X_{(1)i}$ be the smallest order statistic of $X_{i1}, \ldots, X_{in_i}$, and $S_i = \frac{1}{n_i-1}\sum_{j=1}^{n_i}(X_{ij} - X_{(1)i})$. $(X_{i(1)}, S_i)$ are complete sufficient statistics for (μ_i, θ_i). Furthermore, it is well know that $X_{(1)i}$ and S_i are independently distributed with

$$X_{(1)i} \sim \mathrm{Exp}\left(\mu_i, \frac{\theta_i}{n_i}\right), \qquad \frac{(2n_i - 2)S_i}{\theta_i} \sim \chi^2(2n_i - 2) \tag{2.1}$$

where $\chi^2(m)$ denotes a central chi-square distribution with degrees of freedom m. As $n_i \to \infty$, it is easy to verify that $X_{i(1)}, S_i$ converge to μ_i, θ_i in probability, respectively. Therefore, a reasonable estimate of δ_i is

$$\widehat{\delta_i} = X_{(1)i} + S_i, \quad i = 1, \ldots, k.$$

Note that

$$\mathrm{Var}(\widehat{\delta_i} - \widehat{\delta_l}) = \frac{\theta_i^2}{n_i^2} + \frac{\theta_i^2}{n_i - 1} + \frac{\theta_l^2}{n_l^2} + \frac{\theta_l^2}{n_l - 1}, \quad i \neq l \tag{2.2}$$

and an unbiased estimator of (2.2) is

$$\mathrm{V}_{il} = \frac{n_i - 1}{n_i^3}S_i^2 + \frac{1}{n_i}S_i^2 + \frac{n_l - 1}{n_l^3}S_l^2 + \frac{1}{n_l}S_l^2. \tag{2.3}$$

Define

$$D_n = \max_{i \neq l}\left|\frac{(\widehat{\delta_i} - \widehat{\delta_l}) - (\delta_i - \delta_l)}{\sqrt{\mathrm{V}_{il}}}\right|. \tag{2.4}$$

Then the approximate $100(1-\alpha)\%$ two-sided SCIs for $\delta_i - \delta_l (i \neq l)$ can be constructed as

$$(\widehat{\delta_i} - \widehat{\delta_l}) \pm q_\alpha\sqrt{\mathrm{V}_{ij}}, \quad i,\ l = 1, \ldots, k\ (i \neq l) \tag{2.5}$$

where q_α denotes the approximate $(1-\alpha)$th quantile of the distribution of D_n. Unfortunately, the exact distribution of D_n is very hard to find, if not impossible. This motivates us to find an alternative way for constructing the SCIs for δ_{il} based on the characteristic of D_n.

To begin with, note that the distribution of D_n does not depend on the values of μ_i's, so without loss of generality, all μ_i's are assumed to be zeros. Based on the D_n in (2.4), and the fact (2.1) we can define the PB analogues of D_n as follows. Let

$$X_{(1)i}^{PB} \sim \mathrm{Exp}\left(0, \frac{s_i}{n_i}\right), \qquad S_i^{PB} \sim \frac{s_i\chi^2(2n_i - 2)}{2n_i - 2}$$

where s_i is the observed value of S_i, $i = 1, \ldots, k$. Thus, the PB version of D_n in (2.4) is defined as

$$D_n^{PB} = \max_{i \neq l} \left| \frac{X_{(1)i}{}^{PB} - X_{(1)l}{}^{PB} + S_i{}^{PB} - S_l{}^{PB} - (s_i - s_l)}{\sqrt{\mathrm{V}_{il}{}^{PB}}} \right|, \tag{2.6}$$

where

$$\mathrm{V}_{il}^{PB} = \frac{n_i - 1}{n_i^3}(S_i^{PB})^2 + \frac{1}{n_i}(S_i^{PB})^2 + \frac{n_l - 1}{n_l^3}(S_l^{PB})^2 + \frac{1}{n_l}(S_l^{PB})^2.$$

Then, the $100(1-\alpha)\%$ two-sided PB SCIs for all pairwise differences $\delta_i - \delta_l$ $(i \neq l)$ of means of more than two independent two-parameter exponential distributions are

$$(\widehat{\delta_i} - \widehat{\delta_l}) \pm q_\alpha^{n,PB}\sqrt{\mathrm{V}_{il}} \quad i,\ l = 1, \ldots, k\ (i \neq l), \tag{2.7}$$

where $q_\alpha^{n,PB}$ denotes the $(1-\alpha)$th quantile of the distribution of D_n^{PB}, and it can be obtained from the following algorithm.

Algorithm 1. For a given k independent samples from the corresponding two-parameter exponential distributions,

(i) Calculate the values of s_i, the observed value of S_i, $i = 1, \ldots, k$, from the data.
(ii) Generate $X_{(1)i}^{PB} \sim \mathrm{Exp}(0, \frac{s_i}{n_i})$ and $S_i^{PB} \sim \frac{s_i \chi^2(2n_i - 2)}{2n_i - 2}$, $i = 1, \ldots, k$, and calculate the value of D_n^{PB}, as given by (2.6).
(iii) Repeat step (ii) a large number of times, say M, and from these M values, obtain its $(1-\alpha)$th quantile as an estimate of $q_\alpha^{n,PB}$.

Remarks. An alternative inference problem would be to simultaneously test the hypotheses

$$H_{il}: \delta_i - \delta_l = \Delta_{0,il} \quad \text{vs.} \quad H'_{il}: \delta_i - \delta_l \neq \Delta_{0,il} \quad (\forall\, i \neq l),$$

where $\Delta_{0,il}$ is a known constant. Controlling the family-wise error rate, this problem usually can be addressed by simply inverting the simultaneous confidence intervals obtained from (2.7). That is, H_{il} is rejected if and only if the confidence interval for $\delta_i - \delta_l$ does not contain zero. But as pointed out by the associate editor, a more powerful stepwise multiple testing method proposed in Romano and Wolf (2005) can be used. More discussion on the multiple testing for the current set up deserves a further study, but we will not pursue this in this paper.

3. Asymptotic behaviour of PB SCIs

In this section, we shall show that the PB SCIs proposed in Section 2 has correct coverage probability asymptotically. This result, together with its accompanying conditions, is summarized in the following theorem.

Theorem. *Let $X_{i1}, \ldots, X_{in_i}$, $i = 1, \ldots, k$ be random samples from k two-parameter exponential populations $Exp(\mu_i, \theta_i)$ and be mutually independent. Assume that the ratios $\frac{n_i}{n} \to r_i \in (0, 1)$ as $n \to \infty$ for each i, where $n = n_1 + \cdots + n_k$. Then we have*

$$P\left(\delta_i - \delta_l \in \left(\widehat{\delta_i} - \widehat{\delta_l} \pm q_\alpha^{n,PB}\sqrt{\mathrm{V}_{il}}\right) \quad \forall\, i \neq l\right) \to 1 - \alpha.$$

Proof. Note that

$$P\left(\delta_i - \delta_l \in \left(\widehat{\delta_i} - \widehat{\delta_l} \pm q_\alpha^{n,PB}\sqrt{\mathrm{V}_{il}}\right) \quad \forall\, i \neq l\right) = P\left(D_n \leq q_\alpha^{n,PB}\right),$$

where D_n is defined in (2.4). To show the above SCIs has correct coverage probability asymptotically, it is sufficient to show that D_n has the same limiting distribution as D_n^{PB} when $n \to \infty$, where D_n^{PB} is given by (2.6). This can be justified, according to continuous mapping theorem if we can show that $d_{n,il} = [(\widehat{\delta_i} - \widehat{\delta_l}) - (\delta_i - \delta_l)]/\sqrt{\mathrm{V}_{il}}$, $i \neq l$, $i, l = 1, \ldots, k$ and $d_{n,il}^{PB} = [X_{(1)i}{}^{PB} - X_{(1)l}{}^{PB} + S_i{}^{PB} - S_l{}^{PB} - (s_i - s_l)]/\sqrt{\mathrm{V}_{il}{}^{PB}}$, $i \neq l$, $i, l = 1, \ldots, k$ have the same limiting joint distribution. For the sake of conciseness, take $k = 3$ as an example, the extension to the general k cases is straightforward. Note that

$$\begin{pmatrix} d_{n,12} \\ d_{n,13} \\ d_{n,23} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{nV_{12}}} & -\frac{1}{\sqrt{nV_{12}}} & 0 \\ \frac{1}{\sqrt{nV_{13}}} & 0 & -\frac{1}{\sqrt{nV_{13}}} \\ 0 & \frac{1}{\sqrt{nV_{23}}} & -\frac{1}{\sqrt{nV_{23}}} \end{pmatrix} \begin{pmatrix} \sqrt{n}(\widehat{\delta_1} - \delta_1) \\ \sqrt{n}(\widehat{\delta_2} - \delta_2) \\ \sqrt{n}(\widehat{\delta_3} - \delta_3) \end{pmatrix}.$$

From condition $n_i/n \to r_i$, $i = 1, \ldots, k$, and the fact (2.1), it is easy to show that $\sqrt{n}X_{i(1)} \to 0$ for $i = 1, 2, 3$ and $\sqrt{n}(S_i - \theta_i) \Longrightarrow N(0, \theta_i^2/r_i)$ for $i = 1, 2, 3$. This implies that $\sqrt{n}(\widehat{\delta_i} - \delta_i) \Longrightarrow N(0, \theta_i^2/r_i)$ for $i = 1, 2, 3$. Here and in the sequel, $\Longrightarrow$

denotes the convergence in distribution. On the other hand, by the law of large numbers, we can show that

$$nV_{il} \to \frac{\theta_i^2}{r_i} + \frac{\theta_l^2}{r_l}$$

for $i \neq l, i, l = 1, 2, 3$ in probability. Therefore, by Slutsky's theorem (see Ferguson, 1996), it follows that

$$\begin{pmatrix} d_{n,12} \\ d_{n,13} \\ d_{n,23} \end{pmatrix} \Longrightarrow \begin{pmatrix} \frac{1}{\sqrt{\theta_1^2/r_1 + \theta_2^2/r_2}} & -\frac{1}{\sqrt{\theta_1^2/r_1 + \theta_2^2/r_2}} & 0 \\ \frac{1}{\sqrt{\theta_1^2/r_1 + \theta_3^2/r_3}} & 0 & -\frac{1}{\sqrt{\theta_1^2/r_1 + \theta_3^2/r_3}} \\ 0 & \frac{1}{\sqrt{\theta_2^2/r_2 + \theta_3^2/r_3}} & -\frac{1}{\sqrt{\theta_2^2/r_2 + \theta_3^2/r_3}} \end{pmatrix} \begin{pmatrix} Z_1 \\ Z_2 \\ Z_3 \end{pmatrix}, \tag{3.1}$$

where Z_1, Z_2, Z_3 are i.i.d. standard normal random variables. Note that the independence is implied by the independence of the three samples.

To show that $d_{n,il}^{PB}, i \neq l, i, l = 1, 2, 3$ also weakly converges to the same joint distribution, we first note that $X_{i(1)}^{PB} \sim \frac{s_i}{n_i} Y_i$, $X_{l(1)}^{PB} \sim \frac{s_l}{n_l} Y_l$, $S_i^{PB} \sim s_i W_i/(2n_i - 2)$, and $S_l^{PB} \sim s_l W_l/(2n_l - 2)$, where $Y_i, Y_l \sim \text{Exp}(0, 1)$, $W_i \sim \chi^2_{2n_i-2}$, $W_l \sim \chi^2_{2n_l-2}$, and they are independent. Therefore, the numerator of $d_{n,il}^{PB}$ equals, in distribution

$$s_i Y_i/n_i - s_l Y_l/n_l + s_i[W_i/(2n_i - 2) - 1] - s_l[W_l/(2n_l - 2) - 1],$$

and V_{il}^{PB} equals, in distribution

$$\frac{s_i^2 W_i^2}{4n_i^3(n_i - 1)} + \frac{s_i^2 W_i^2}{4n_i(n_i - 1)^2} + \frac{s_l^2 W_l^2}{4n_l^3(n_l - 1)} + \frac{s_l^2 W_l^2}{4n_l(n_l - 1)^2}.$$

From (2.1), after multiplying $\sqrt{n}$ to the numerator and the denominator of $d_{n,il}^{PB}$, we can also show that $d_{n,il}^{PB}, i \neq l, i, l = 1, 2, 3$ converges in distribution to the right hand side of (3.1) using the similar approach.

Finally, because the distribution of D_n^{PB} and its limiting distribution are both continuous, so $q_\alpha^{n,PB} \to q_\alpha$ as $n, B \to \infty$. This completes the proof.

4. Simulation studies

In this section, simulation studies are carried out to evaluate the performance of the proposed PB SCIs for differences of means of several two-parameter exponential distributions, comparison studies are also conducted using the FG SCIs and the NPB SCIs. The FG SCIs approach was proposed by Hannig et al. (2006a) based on the concept of fiducial generalized pivotal quantities. The NPB SCIs is a pure nonparametric version of D_n based on direct resamples from the joint empirical distributions, which, together with the FG SCIs, is briefly discussed in the Appendix. The performance of these three procedures was evaluated through the empirical coverage probabilities, as well as the average volumes of the confidence intervals. More specifically, the following algorithm was used to estimate the empirical coverage probabilities of the PB SCIs:

Algorithm 2. (I) Generate $X_{i1}, \ldots, X_{in_i}$ from $\text{Exp}(\mu_i, \theta_i)$, $i = 1, \ldots, k$, and calculate the observed value of $X_{(1)i}$, S_i.

(II) Calculate the "critical value" (the value of $q_\alpha^{n,PB}$), using Algorithm 1 at the end of Section 2 when $M = 10{,}000$.

(III) Using the data from step (I) and the "critical value" ($q_\alpha^{n,PB}$) in step (II). Construct SCIs according the (2.7) and record whether or not all the values of $\delta_i - \delta_l$ $(i \neq l)$ fall in their corresponding confidence intervals.

(IV) Repeat steps (I)–(III) $N = 10{,}000$ times. Then the empirical coverage probabilities are calculated as the fraction of times that all $\delta_i - \delta_l$ $(i \neq l)$ fall into their corresponding SCIs.

Similar procedures can be used to estimate the empirical coverage probabilities for the FG SCIs and the NPB SCIs, except for the different resampling schemes. The average volume (AV) of the SCIs from each procedure is defined as the average of products of the lengths from all individual confidence intervals. In the case of $k = 2$, the AV is simply the average lengths of the confidence intervals.

In the simulation, four configuration factors are considered to evaluate the performance of the three SCI procedures: the number of two-parameter exponential populations; the sample sizes, the values of μ_i and θ_i, $i = 1, \ldots, k$. See the following tables for specific choices of these configurations. The nominal confidence level was chosen to be 95%. The simulation results from $k = 2, 3, 4$ are presented in the following three tables. For the sake of brevity, CP is used to denote the coverage probability in all three cases.

From Table 1, we can see that in the case of $k = 2$, both the PB SCIs and FG SCIs procedures perform quite satisfactorily in terms of the coverage probability, and the length of the confidence interval. Although the coverage probabilities of the

Table 1
The empirical coverage probability (CP), average lengths (AL) of 95% SCI for three methods when $k = 2$.

(n_1, n_2)	PB SCIs		FG SCIs		NPB SCIs	
	CP	AL	CP	AL	CP	AL
$(\mu_1, \mu_2) = (0, 0)$ $(\theta_1, \theta_2) = (5, 5)$						
(15, 15)	0.9643	7.2997	0.9840	8.6028	0.9575	7.6340
(30, 30)	0.9532	5.0889	0.9675	5.5408	0.9498	5.1275
(60, 60)	0.9503	3.5785	0.9601	3.7492	0.9516	3.5822
(15, 30)	0.9551	6.3665	0.9732	4.2806	0.9517	6.5666
(15, 60)	0.9502	5.9296	0.9611	3.8248	0.9427	6.1647
(30, 60)	0.9507	4.4236	0.9605	4.6962	0.9448	4.4545
(120, 120)	0.9509	2.5281	0.9544	2.5875	0.9485	2.5300
$(\mu_1, \mu_2) = (0.25, 0)$ $(\theta_1, \theta_2) = (0.75, 1)$						
(15, 15)	0.9613	1.2972	0.9791	1.5204	0.9781	1.7914
(30, 30)	0.9504	0.9012	0.9689	0.9771	0.9925	1.3580
(60, 60)	0.9495	0.6340	0.9579	0.6601	0.9992	1.0902
(15, 30)	0.9562	1.0551	0.9755	1.1972	0.9887	1.3980
(15, 60)	0.9547	0.9348	0.9684	1.0304	0.9918	1.1948
(30, 60)	0.9518	0.7405	0.9624	0.7903	0.9947	1.1118
(120, 120)	0.9510	0.4488	0.9548	0.4573	1.0000	0.9121

Table 2
The empirical coverage probability (CP), average volumes (AV) of 95% SCI for three methods when $k = 3$.

(n_1, n_2, n_3)	PB SCIs		FG SCIs		NPB SCIs	
	CP	AV	CP	AV	CP	AV
$(\mu_1, \mu_2, \mu_3) = (0, 0, 0)$ $(\theta_1, \theta_2, \theta_3) = (0.01, 0.02, 0.08)$						
(15, 15, 15)	0.9453	4.0616e−4	0.9618	4.3951e−4	0.8685	2.6082e−4
(30, 30, 30)	0.9501	1.0911e−4	0.9590	1.1990e−4	0.8729	6.5603e−5
(60, 60, 60)	0.9504	3.3958e−5	0.9493	3.5621e−5	0.8803	2.0814e−5
(15, 15, 30)	0.9453	1.7322e−4	0.9658	2.1323e−4	0.8741	1.2804e−4
(15, 15, 60)	0.9479	8.5166e−5	0.9683	1.0530e−4	0.8778	6.7142e−5
(15, 30, 30)	0.9499	1.1516e−4	0.9672	1.4312e−4	0.8794	7.0920e−5
(15, 60, 60)	0.9522	4.3374e−5	0.9690	5.3413e−5	0.8832	2.7749e−5
(30, 30, 120)	0.9493	2.7156e−5	0.9585	3.0084e−5	0.8755	1.7378e−5
(60, 60, 120)	0.9508	1.1719e−5	0.9525	1.7976e−5	0.8788	1.0598e−5
(15, 15, 120)	0.9491	4.7207e−5	0.9660	5.8924e−5	0.8815	3.8332e−5
(15, 30, 120)	0.9502	2.9705e−5	0.9712	3.6880e−5	0.8771	1.9517e−5
(15, 60, 120)	0.9542	2.2559e−5	0.9681	2.7792e−5	0.8811	1.4519e−5
(15, 120, 120)	0.9512	1.9678e−5	0.9631	2.3420e−5	0.8758	1.3465e−5
(30, 120, 120)	0.9519	1.4593e−5	0.9570	1.6226e−5	0.8771	9.0425e−6
(60, 120, 120)	0.9497	1.2422e−5	0.9536	1.3057e−5	0.8826	7.6807e−6
$(\mu_1, \mu_2, \mu_3) = (0, 0, 0)$ $(\theta_1, \theta_2, \theta_3) = (1, 2, 2)$						
(15, 15, 15)	0.9455	32.2401	0.9793	48.1005	0.8916	26.3308
(30, 30, 30)	0.9463	9.7383	0.9697	12.0390	0.8757	6.3564
(60, 60, 60)	0.9505	3.2173	0.9617	3.6086	0.8770	1.9999
(15, 15, 30)	0.9496	19.5056	0.9797	28.8949	0.8973	17.7674
(15, 15, 60)	0.9492	14.8493	0.9771	21.2184	0.9078	14.5353
(15, 30, 30)	0.9511	10.9192	0.9817	16.0095	0.8892	7.8383
(15, 60, 60)	0.9530	5.0645	0.9753	6.9637	0.8945	3.6989
(30, 30, 120)	0.9506	4.6793	0.9646	5.6512	0.8974	3.5672
(60, 60, 120)	0.9493	2.1313	0.9598	2.3632	0.8813	1.3752
(15, 15, 120)	0.9490	13.4650	0.9684	13.3447	0.9162	14.8628
(15, 30, 120)	0.9491	6.5325	0.9707	8.7573	0.9109	5.7872
(15, 60, 120)	0.9511	3.9761	0.9714	5.1986	0.9032	3.2490
(15, 120, 120)	0.9500	2.9100	0.9704	3.6283	0.8962	2.2693
(30, 120, 120)	0.9491	1.7573	0.9676	2.0844	0.8843	1.1571
(60, 120, 120)	0.9502	1.3182	0.9674	1.3794	0.8784	0.8164

PB SCIs procedure are smaller than those of the FG SCIs approach, they are closer to the claimed nominal confidence level 0.95. Moreover, we can see that, when the two samples have the same sizes, the PB SCIs are much shorter than the FG SCIs when the samples sizes are small, and they become close when sample size gets larger. The results are mixed when the two samples have different sizes. In some cases, for example, when $\mu_i = 0, \theta_i = 5, (n_1, n_2) = (15, 15), (n_1, n_2) = (30, 30), (n_1, n_2) = (15, 30)$, the NPB procedure performs better than the other two methods. For PB SCIs and FG SCIs procedures, similar pattern can be found in Table 2 for the case of $k = 3$. However, the performance NPB SCIs are very poor, as evidenced by the very low empirical coverage probabilities. For the case of $k = 4$, we note that the PB SCIs procedure performs consistently better than the other two methods for all configurations (see Table 3) .

Table 3
The empirical coverage probability (CP), average volumes (AV) of 95% SCI for three methods when $k = 4$.

(n_1, n_2, n_3, n_4)	PB SCIs		FG SCIs		NPB SCIs	
	CP	AV	CP	AV	CP	AV
$(\mu_1, \mu_2, \mu_3, \mu_4) = (2, 2, 2, 2)$ $(\theta_1, \theta_2, \theta_3, \theta_4) = (3, 3, 3, 3)$						
(15, 15, 15, 15)	0.9848	9.9442e4	0.9987	3.7502e5	0.8863	3.5319e4
(30, 30, 30, 30)	0.9591	3.9386e3	0.9852	8.4891e3	0.8430	1.5894e3
(60, 60, 60, 60)	0.9523	477.1935	0.9685	723.4636	0.8237	142.8578
(15, 30, 30, 30)	0.9546	8.6759e3	0.9889	1.9924e3	0.9970	1.2236e6
(15, 60, 60, 60)	0.9502	2.8025e3	0.9687	4.2352e3	1.0000	2.4186e6
$(\mu_1, \mu_2, \mu_3, \mu_4) = (3.25, 3.25, 3.5, 3.5)$ $(\theta_1, \theta_2, \theta_3, \theta_4) = (0.75, 0.75, 0.5, 0.5)$						
(15, 15, 15, 15)	0.9663	4.0572	0.9943	12.6595	0.9925	72.9898
(30, 30, 30, 30)	0.9529	0.3626	0.9775	0.7077	0.9998	6.7397
(60, 60, 60, 60)	0.9513	0.0425	0.9674	0.0600	0.9999	1.3626
(15, 30, 30, 30)	0.9510	1.0201	0.9765	1.9485	1.0000	4.1221e3
(15, 60, 60, 60)	0.9520	0.6245	0.9761	1.2167	1.0000	1.5980e4
$(\mu_1, \mu_2, \mu_3, \mu_4) = (4, 4, 4, 4)$ $(\theta_1, \theta_2, \theta_3, \theta_4) = (0.25, 0.25, 0.5, 0.5)$						
(15, 15, 15, 15)	0.9525	0.2536	0.9862	0.6550	0.8212	0.0901
(30, 30, 30, 30)	0.9506	0.0204	0.9743	0.0348	0.8008	0.0049
(60, 60, 60, 60)	0.9497	0.0022	0.9616	0.0029	0.8003	4.8783e−4
(15, 30, 30, 30)	0.9508	0.0313	0.9793	0.0638	0.9923	1.4920
(15, 60, 60, 60)	0.9482	0.0069	0.9702	0.0107	0.9997	1.1371

5. Concluding remarks

In this paper, we introduced a parametric bootstrap method to construct SCIs for all pairwise differences of means from several two-parameter exponential distributions. We prove that the PB SCIs have asymptotic correct coverage probabilities. Also, our simulation studies show that our proposed method consistently performs better than the FG SCIs and NPB SCIs methods from CP and AV perspective. Therefore, our proposed method is a very competitive candidate procedure for constructing SCIs for all pairwise differences of means from several two-parameter exponential distributions.

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Appendix. The FG SCIs and NPB SCIs approaches

In this section we briefly review the FG SCIs and NPB SCIs procedures for the sake of completeness. The same notations will be adopted here without further notice.

Let $X^*_{(1)i}$, S^*_i denote independent copies of $X_{(1)i}$, S_i, respectively. It is well known that $X_{(1)i}$ and S_i are independently distributed with

$$U_i = \frac{2n_i(X^*_{(1)i} - \mu_i)}{\theta_i} \sim \chi^2(2), \qquad V_i = \frac{(2n_i - 2)S^*_i}{\theta_i} \sim \chi^2(2n_i - 2)$$

where $\chi^2(m)$ denotes a central chi-square distribution with m degree.

In FG SCIs procedure, using the method provided in Hannig et al. (2006b), the fiducial generalized pivot quantities (FGPQ) for μ_i and θ_i, $i = 1, \ldots, k$ are defined as

$$R_{\mu_i} = X_{(1)i} - \frac{S_i}{S^*_i}(X^*_{(1)i} - \mu_i), \qquad R_{\theta_i} = \frac{S_i}{S^*_i}\theta_i.$$

Then

$$\begin{aligned} R_{\delta_{il}}(X, X^*, \delta) &= R_{\mu_i} + R_{\theta_i} - (R_{\mu_l} + R_{\theta_l}) \\ &= X_{(1)i} - \frac{(n_i - 1)}{n_i}\frac{U_i}{V_i}S_i + \frac{(2n_i - 2)S_i}{V_i} - \left(X_{(1)l} - \frac{(n_l - 1)}{n_l}\frac{U_l}{V_l}S_l + \frac{(2n_l - 2)S_l}{V_l}\right) \end{aligned}$$

are the FGPQs for $\delta_i - \delta_l$. Now, let

$$D(X, X^*, \delta)_n = \max_{i \neq l}\left|\frac{\widehat{\delta}_i - \widehat{\delta}_l - R_{\delta_{il}}(X, X^*, \delta)}{\sqrt{V_{il}}}\right|,$$

where

$$V_{il} = \frac{S_i^2(n_i-1)}{n_i^3} + \frac{S_i^2}{n_i} + \frac{S_l^2(n_l-1)}{n_l^3} + \frac{S_l^2}{n_l}.$$

Then, the $100(1-\alpha)\%$ two-sided FG SCIs for all pairwise differences $\delta_i - \delta_l$ $(i \neq l)$ of means of more than two independent two-parameter exponential distributions are

$$(\widehat{\delta_i} - \widehat{\delta_l}) \pm d_\alpha \sqrt{V_{il}},$$

where d_α is the $(1-\alpha)$th percentile of the conditional distribution of $D(X, X^*, \delta)_n$ given the observed data.

To introduce the NPB method, define

$$D_n^{NPB} = \max_{i \neq l} \left| \frac{X_{(1)i}{}^{NPB} - X_{(1)l}{}^{NPB} + S_i{}^{NPB} - S_l{}^{NPB} - (s_i - s_l)}{\sqrt{V_{il}{}^{NPB}}} \right| \tag{A.1}$$

where

$$V_{il}^{NPB} = \frac{n_i - 1}{n_i^3}(S_i^{NPB})^2 + \frac{1}{n_i}(S_i^{NPB})^2 + \frac{n_l - 1}{n_l^3}(S_l^{NPB})^2 + \frac{1}{n_l}(S_l^{NPB})^2.$$

Then, the $100(1-\alpha)\%$ two-side NPB SCIs for all pairwise differences $\delta_i - \delta_l$ $(i \neq l)$ of means of more than two independent two-parameter exponential distributions can be constructed as

$$(\widehat{\delta_i} - \widehat{\delta_l}) \pm q_\alpha^{n,NPB} \sqrt{V_{il}} \quad i,\ l = 1, \ldots, k\ (i \neq l), \tag{A.2}$$

where $q_\alpha^{n,NPB}$ denotes the $(1-\alpha)$th quantile of the distribution of D_n^{NPB}.

To facilitate the implementation of the NPB SCIs procedure, the following algorithm is designed. As we mentioned in Section 4, the following algorithm is similar to Algorithm 2 except for the resampling scheme.

Algorithm 3. (I) Generate $X_{i1}, \ldots, X_{in_i}$ from $Exp(\mu_i, \theta_i)$, $i = 1, \ldots, k$, and calculate the observed value of $X_{(1)i}$, S_i.

(II) Generate $X_{i1}^*, \ldots, X_{in_i}^*$ from $X_{i1}, \ldots, X_{in_i}$ and calculate the value of $X_{(1)i}^{NPB}$, S_i^{NPB}. Then calculate the value of D_n^{NPB}, as given by (A.1).

(III) Repeat step (II) a large number of times, $M = 10{,}000$, and from these M values, obtain the empirical distribution of D_n^{NPB} and its $(1-\alpha)$th quantile as an estimate of $q_\alpha^{n,NPB}$.

(IV) Using the data from step (I) and $q_\alpha^{n,NPB}$ in step (III). Construct SCIs according (A.2) and record whether or not all the values of $\delta_i - \delta_l$ $(i \neq l)$ fall in their corresponding confidence intervals.

(V) Repeat steps (I)–(IV) $N = 10{,}000$ times. Then, the fraction of times that all $\delta_i - \delta_l$ $(i \neq l)$ fall into their corresponding SCIs provides an estimate of the coverage probability.

As the associate editor pointed out that the validity of this nonparametric method follows immediately from the general results in Romano and Wolf (2005), and the advantage of this method is that it is robust to the violation of the assumption of the data indeed coming from the underlying distributions.

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