1. *The case of deterministic trends.* Let the regressor in equation (2) be the deterministic trend $x_t = t$. Find the limit distributions of estimates of β obtained by 2SLS on (1), OLS on (1)′, and full system maximum likelihood. Which econometrician is right or are both of them wrong?

Set $x_t = t$ in the structural model (1) and (2) of the previous problem. How do the results differ from before?

2. *The case of stochastic trends.* Set $\gamma = 1$ and $x_t = y_{2t-1}$ in equation (2) of the structural model, leading to the random walk

$$y_{2t} = y_{2t-1} + u_{2t}. \tag{2$'$}$$

Econometrician (A) persists in believing that 2SLS applied to equation (1) using y_{2t-1} as an instrument leads to optimal estimates of β. Econometrician B now asserts that his approach is even more compelling than before. Not only is Σ_0 known but so also is the coefficient γ in equation (2). This means that the residual u_{2t} is known and subtracting (2)′ from (1) we get

$$y_{1t} - \Delta y_{2t} = \beta y_{2t} + v_{1t}. \tag{1$''$}$$

Again he suggests that OLS will be asymptotically efficient and to use all the information available he recommends equation (1)″ for the regression.

Which econometrician is right this time?

3. Now suppose that both econometricians believe the model with deterministic trends to be correct, whereas in fact the model has a stochastic trend as in (2)′. Thus, the true model is (1) and (2)′ but the model used for estimation is (1) and (2). Again, econometrician A recommends 2SLS on equation (1) and econometrician B argues for the use of OLS on (1)′. In both cases, $x_t = t$.

Which estimation procedure is preferable this time?

SOLUTIONS

89.3.2. *Simultaneous Confidence Ellipsoids*—Solution, proposed by Ali S. Hadi and Martin T. Wells, Cornell University.

Since $\sigma^{-1}\hat{\beta} \sim N_p(\beta, V)$ and $VH(H'VH)^{-1}H'VH(H'VH)^{-1}H'V = VH(H'VH)^{-1}H'V$, then $Q \equiv \sigma^{-2}(\beta - \hat{\beta})'H(H'VH)^{-1}H'(\beta - \hat{\beta})$ has a χ^2 distribution with rank $(H(H'VH)^{-1}H'V)$ degrees of freedom (see, e.g., [1]). Note that the matrix $H(H'VH)^{-1}H'V$ is idempotent, thus

$$\begin{aligned}\text{rank}(H(H'VH)^{-1}H'V) &= \text{trace}(H(H'VH)^{-1}H'V)\\ &= \text{trace}(H'VH(H'VH)^{-1}) = \text{trace}(I_q) = q.\end{aligned}$$

Since $\hat{\sigma}^2$ is independent of $\hat{\beta}$, $\hat{\sigma}^2$ is independent of Q, therefore

$$\frac{Q/q}{m\hat{\sigma}^2/m\sigma^2} = \frac{(\beta - \hat{\beta})'H(H'VH)^{-1}H'(\beta - \hat{\beta})}{q\hat{\sigma}^2} \sim F_{q,m}.$$

Hence, $\Pr\{(\beta - \hat{\beta})'H(H'VH)^{-1}H'(\beta - \hat{\beta}) \leq c\hat{\sigma}^2\} = 1 - \alpha$, for all $p \times q$ matrices H of rank q, where c/q is the upper α critical value of the F distribution with q and m degrees of freedom. Q.E.D.

REFERENCE

1. Rao, C.R. *Linear statistical inference and its applications*. 2nd ed., New York: John Wiley & Sons, 1973.

NOTE

Very good solutions have been proposed independently by Badi H. Baltagi and R. W. Farebrother (the poser of the problem).

89.3.3. *The Equivalence of the Boothe-MacKinnon and the Hausman Specification Tests in the Context of Panel Data*—Solution, proposed by Ruud H. Koning.

Consider the model

$$y_{it} = \beta' x_{it} + \mu_i + \nu_{it} \qquad i = 1, \ldots, N$$
$$t = 1, \ldots, T, \tag{1}$$

or in matrix format

$$y = X\beta + \mu + \nu$$
$$\equiv X\beta + \epsilon, \tag{1'}$$

with the time index running fast and the individual index running slow. In order to test whether μ_i is correlated with x_{ii}, Hausman and Taylor [1] proposed three equivalent tests, based on $\hat{q}_1 \equiv \hat{\beta}_{\text{GLS}} - \hat{\beta}_w$, $\hat{q}_2 \equiv \hat{\beta}_{\text{GLS}} - \hat{\beta}_B$ and $\hat{q}_3 \equiv \hat{\beta}_w - \hat{\beta}_B$, respectively. All these vectors are nonsingular linear transformations of each other, and hence, they yield the same numerical value of the test statistic $\hat{q}_i' V_i^{-1} \hat{q}_i$, $i = 1, 2, 3$. A fourth equivalent test is given by

$$(\hat{\beta}_{\text{GLS}} - \hat{\beta}_{\text{OLS}})'[V(\hat{\beta}_{\text{GLS}} - \hat{\beta}_{\text{OLS}})]^{-1}(\hat{\beta}_{\text{GLS}} - \hat{\beta}_{\text{OLS}}).$$

This will be shown in the sequel. We define

$$P = 1/T\, I_N \otimes J_T,$$

$$Q = I_{NT} - P,$$

$$R = 1/T\, I_N \otimes \iota'_T,$$

with J_T a $T \times T$ matrix of ones and ι_T a T-vector of ones. Now we have