APPENDIX A

Cumulant Calculations

In this book several chapters and some of the problems involve moment calculations, which are often simplified by using cumulants.

The cumulant-generating function of a random variable Y is

$$K(t) = \log \mathrm{E}\left(e^{tY}\right) = \sum_{s=1}^{\infty} \frac{1}{s!} t^s \kappa_s,$$

where κ_s is the sth cumulant, while the moment-generating function of Y is

$$M(t) = \mathrm{E}\left(e^{tY}\right) = \sum_{s=0}^{\infty} \frac{1}{s!} t^s \mu'_s,$$

where $\mu'_s = \mathrm{E}(Y^s)$ is the sth moment. A simple example is a $N(\mu, \sigma^2)$ random variable, for which $K(t) = t\mu + \frac{1}{2}t^2\sigma^2$; note the appealing fact that its cumulants of order higher than two are zero. By equating powers of t in the expansions of $K(t)$ and $\log M(t)$ we find that $\kappa_1 = \mu'_1$ and that

$$\begin{aligned}
\kappa_2 &= \mu'_2 - (\mu'_1)^2, \\
\kappa_3 &= \mu'_3 - 3\mu'_2\mu'_1 + 2(\mu'_1)^3, \\
\kappa_4 &= \mu'_4 - 4\mu'_3\mu'_1 - 3(\mu'_2)^2 + 12\mu'_2(\mu'_1)^2 - 6(\mu'_1)^4,
\end{aligned}$$

with inverse formulae

$$\begin{aligned}
\mu'_2 &= \kappa_2 + (\kappa_1)^2, \\
\mu'_3 &= \kappa_3 + 3\kappa_2\kappa_1 + (\kappa_1)^3, \\
\mu'_4 &= \kappa_4 + 4\kappa_3\kappa_1 + 3(\kappa_2)^2 + 6\kappa_2(\kappa_1)^2 + (\kappa_1)^4.
\end{aligned} \tag{A.1}$$

The cumulants κ_1, κ_2, κ_3 and κ_4 are the mean, variance, skewness and kurtosis of Y.

For vector Y it is better to drop the power notation used above and to adopt index notation and the summation convention. In this notation Y has components $Y^1,\dots,Y^n$ and we write Y^iY^i and $Y^iY^iY^i$ for the square and cube of Y^i. The joint cumulant-generating function $K(t)$ of $Y^1,\dots,Y^n$ is the logarithm of their joint moment-generating function,

$$\log \mathrm{E}\left(e^{t_1Y^1+\cdots+t_nY^n}\right) = t_i\kappa^i + \tfrac{1}{2!}t_it_j\kappa^{i,j} + \tfrac{1}{3!}t_it_jt_k\kappa^{i,j,k} + \tfrac{1}{4!}t_it_jt_kt_l\kappa^{i,j,k,l} + \cdots,$$

where summation is implied over repeated indices, so that, for example,

$$t_i\kappa^i = t_1\kappa^1 + \cdots + t_n\kappa^n, \quad t_it_j\kappa^{i,j} = t_1t_1\kappa^{1,1} + t_1t_2\kappa^{1,2} + \cdots + t_nt_n\kappa^{n,n}.$$

Thus the n-dimensional normal distribution with means κ^i and covariance matrix $\kappa^{i,j}$ has cumulant-generating function $t_i\kappa^i + \frac{1}{2}t_it_j\kappa^{i,j}$. We sometimes write $\kappa^{i,j} = \mathrm{cum}(Y^i, Y^j)$, $\kappa^{i,j,k} = \mathrm{cum}(Y^i, Y^j, Y^k)$ and so forth for the coefficients of t_it_j, $t_it_jt_k$ in $K(t)$. The cumulant arrays $\kappa^{i,j}$, $\kappa^{i,j,k}$ etc. are invariant to index permutation, so for example $\kappa^{1,2,3} = \kappa^{2,3,1}$.

A key feature that simplifies calculations with cumulants as opposed to moments is that cumulants involving two or more independent random variables are zero: for independent variables, $\kappa^{i,j} = \kappa^{i,j,k} = \cdots = 0$ unless all the indices are equal.

The above notation extends to generalized cumulants such as

$$\begin{aligned}\mathrm{cum}(Y^iY^jY^k) &= \mathrm{E}(Y^iY^jY^k) = \kappa^{ijk},\\ \mathrm{cum}(Y^i, Y^jY^k) &= \kappa^{i,jk}, \quad \mathrm{cum}(Y^iY^j, Y^k, Y^l) = \kappa^{ij,k,l},\end{aligned}$$

which can be obtained from the joint cumulant-generating functions of $Y^iY^jY^k$, of Y^i and Y^jY^k and of Y^iY^j, Y^k, and Y^l. Note that ordinary moments can be regarded as generalized cumulants.

Generalized cumulants can be expressed in terms of ordinary cumulants by means of complementary set partitions, the most useful of which are given in Table A.1. For example, we use its second column to see that $\kappa^{ij} = \kappa^{i,j} + \kappa^i\kappa^j$, or

$$\mathrm{E}(Y^iY^j) = \mathrm{cum}(Y^iY^j) = \mathrm{cum}(Y^i, Y^j) + \mathrm{cum}(Y^i)\mathrm{cum}(Y^j),$$

more familiarly written $\mathrm{cov}(Y^i, Y^j) + \mathrm{E}(Y^i)\mathrm{E}(Y^j)$. The boldface **12** represents κ^{12}, while the 12 **[1]** and 1|2 **[1]** immediately below it represent $\kappa^{1,2}$ and $\kappa^1\kappa^2$. With this understanding we use the third column to see that $\kappa^{ijk} = \kappa^{i,j,k} + \kappa^{i,j}\kappa^k[3] + \kappa^i\kappa^j\kappa^k$, where $\kappa^{i,j}\kappa^k[3]$ is shorthand for $\kappa^{i,j}\kappa^k + \kappa^{i,k}\kappa^j + \kappa^{j,k}\kappa^i$; this is the multivariate version of (A.1). Likewise $\kappa^{ij,k} = \kappa^{i,j,k} + \kappa^{i,k}\kappa^j[2]$, where the term $\kappa^{i,k}\kappa^j[2]$ on the right is understood in the context of the left-hand side to equal $\kappa^{i,k}\kappa^j + \kappa^{j,k}\kappa^i$: each index in the first block of the partition $ij \mid k$ appears once with the index in the second block. The expression 123|4 **[2][2]** in the fourth column of the table represents the partitions 123|4, 124|3, 134|2, 234|1.

To illustrate these ideas, we calculate $\mathrm{cov}\{\bar{Y}, (n-1)^{-1}\sum(Y_i - \bar{Y})^2\}$, where

$\bar{Y} = n^{-1}\sum Y_i$ is the average of the independent and identically distributed random variables $Y_1, \ldots, Y_n$. Note first that the covariance does not depend on the mean of the Y_i, so we can take $\kappa^i \equiv 0$. We then express $\bar{Y}$ and $(n-1)^{-1}\sum(Y_i - \bar{Y})^2$ in index notation as $a_i Y^i$ and $b_{ij} Y^i Y^j$, where $a_i = 1/n$ and $b_{ij} = (\delta_{ij} - 1/n)/(n-1)$, with

$$\delta_{ij} = \begin{cases} 1, & i = j, \\ 0, & \text{otherwise}, \end{cases}$$

the Kronecker delta symbol. The covariance is

$$\mathrm{cum}(a_i Y^i, b_{jk} Y^j Y^k) = a_i b_{jk} \kappa^{i,jk} = a_i b_{jk} \kappa^{i,j,k} = n a_1 b_{11} \kappa^{1,1,1},$$

the second equality following on use of Table A.1 because $\kappa^i \equiv 0$ and the third equality following because the observations are independent and identically distributed. In power notation $\kappa^{1,1,1}$ is κ_3, the third cumulant of Y_i, so $\mathrm{cov}\{\bar{Y}, (n-1)^{-1}\sum(Y_i - \bar{Y})^2\} = \kappa_3/n$. Similarly

$$\mathrm{var}\{(n-1)^{-1}\sum(Y_j - \bar{Y})^2\} = \mathrm{cum}(b_{ij} Y^i Y^j, b_{kl} Y^k Y^l) = b_{ij} b_{kl} \kappa^{ij,kl},$$

which Table A.1 shows to be equal to $b_{ij} b_{kl}(\kappa^{i,j,k,l} + \kappa^{i,k}\kappa^{j,l} + \kappa^{i,l}\kappa^{j,k})$. This reduces to

$$n b_{11} b_{11} \kappa^{1,1,1,1} + 2n b_{11} b_{11} \kappa^{1,1} \kappa^{1,1} + 2n(n-1) b_{12} b_{12} \kappa^{1,1} \kappa^{1,1},$$

which in turn is $\kappa_4/n + 2(\kappa_2)^2/(n-1)$ in power notation. To perform this calculation using moments and power notation will convince the reader of the elegance and relative simplicity of cumulants and index notation.

McCullagh (1987) makes a cogent more-extended case for these methods. His book includes more-extensive tables of complementary set partitions.

Table A.1
Complementary set partitions

<table>
<tr><th>1</th><th>2</th><th>3</th><th>4</th></tr>
<tr><td>1</td><td>12</td><td>123</td><td>1234</td></tr>
<tr><td>1 [1]</td><td>12 [1]</td><td>123 [1]</td><td>1234 [1]</td></tr>
<tr><td></td><td>1|2 [1]</td><td>12|3 [3]</td><td>123|4 [4]</td></tr>
<tr><td></td><td></td><td>1|2|3 [1]</td><td>12|34 [3]</td></tr>
<tr><td></td><td>1|2</td><td></td><td>12|3|4 [6]</td></tr>
<tr><td></td><td>12 [1]</td><td>12|3</td><td>1|2|3|4 [1]</td></tr>
<tr><td></td><td></td><td>123 [1]</td><td></td></tr>
<tr><td></td><td></td><td>13|2 [2]</td><td>123|4</td></tr>
<tr><td></td><td></td><td></td><td>1234 [1]</td></tr>
<tr><td></td><td></td><td>1|2|3|</td><td>124|3 [3]</td></tr>
<tr><td></td><td></td><td>123 [1]</td><td>12|34 [3]</td></tr>
<tr><td></td><td></td><td></td><td>14|2|3 [3]</td></tr>
<tr><td></td><td></td><td></td><td>12|34</td></tr>
<tr><td></td><td></td><td></td><td>1234 [1]</td></tr>
<tr><td></td><td></td><td></td><td>123|4 [2][2]
134|2</td></tr>
<tr><td></td><td></td><td></td><td>13|24 [2]</td></tr>
<tr><td></td><td></td><td></td><td>13|2|4 [4]</td></tr>
<tr><td></td><td></td><td></td><td>12|3|4</td></tr>
<tr><td></td><td></td><td></td><td>1234 [1]</td></tr>
<tr><td></td><td></td><td></td><td>134|2 [2]</td></tr>
<tr><td></td><td></td><td></td><td>13|24 [2]</td></tr>
<tr><td></td><td></td><td></td><td>1|2|3|4</td></tr>
<tr><td></td><td></td><td></td><td>1234 [1]</td></tr>
</table>