COMPUTING UNCERTAINTY REGIONS WITH SIMULTANEOUS CONFIDENCE DEGREE USING BOOTSTRAP

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Abstract: We discuss the importance of constructing confidence regions of simultaneous confidence degree for certain statistics, e.g., the frequency function. In this contribution we show how bootstrap can be used to obtain reliable confidence regions of simultaneous confidence degree, independently of how many confidence regions we calculate. The procedure is illustrated by comparison with "classical" methods and Monte Carlo simulations. We will also provide an evaluation of the quality of the obtained confidence regions. *Copyright ©2000 IFAC*

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1. INTRODUCTION

This paper discusses the problem of constructing reliable confidence regions of simultaneous confidence degree for some statistic associated with the system. (The example and the text will be oriented towards looking at the frequency function, but the procedure works equally well for other statistics.) The paper presents a bootstrap procedure to estimate uncertainty regions of simultaneous confidence degree for any smooth statistic associated with the system. Since the procedure is simulation based it will be computer intensive, but straightforward to implement and calculate. In the discussion we also try to illustrate the difference between these confidence regions and the regions obtained using more classical methods. The strength and the quality of the method is also shown by a simulation example.

First we will give a brief introduction to the bootstrap. Section 3 describes the system identification setup and gives some classical results that have been used to construct confidence regions for certain statistics. The extension of adopting bootstrap to identification will be given in Section 4. In Section 5 we show how to use bootstrap to construct confidence regions of predefined simultaneous confidence degree. In Section 6 an example is given to show how the regions will change using the new procedure described in Section 5.2 and to show how good degree of coverage these regions have. Finally, Section 7 will give some conclusions.

2. THE BOOTSTRAP IDEA?

Bootstrap was introduced in Efron (1979) as a method to calculate accuracy measures of a statistic by simulations. The development of bootstrap has together with computer speed improvements made it possible to compute uncertainty regions for a very wide class of problems. Bootstrap is still in focus for a lot of research in the statistical community, and much work is spent on improving the bootstrap method for constructing accurate confidence regions. A lot of work has also been spent on trying to extend the bootstrap to more complicated problems than it was originally designed for. Introductions to bootstrap can be found in Efron and Tibshirani (1993), Davison and Hinkley (1997), Politis (1998), and a survey of signal processing applications in Zoubir and Boashash (1998).

The main idea behind bootstrap is that we would like to get to a situation where we could perform Monte Carlo simulations to judge the uncertainty in the estimate that we have, instead of going through tedious or unfeasible calculations. The setup can be described as follows. Let $\mathbf{x} = (x_1, \ldots, x_N)$ be an independent, identically distributed (i.i.d.) sample from a stochastic variable X with distribution function F. We are interested in estimating some parameter, $\tau = \tau(F)$ associated with F. As an estimator for τ we will use the statistic T, i.e., $\hat{\tau} = T(\mathbf{x})$. This estimator will also be used to find the accuracy of $\hat{\tau}$.

It is not hard to realize that this problem would be easy to solve if F was known. Then B new "samples" from F, $\mathbf{x_1}, \ldots, \mathbf{x_B}$, could be generated and we could calculate $\hat{\tau}_1, \ldots, \hat{\tau}_B$ from each one of them. From this it is not hard to calculate the empirical probability density function for $\hat{\tau}$.

Since we do not know F, the simple and natural idea is to estimate F with the empirical distribution estimate

$$\hat{F}(x) = \frac{\#\{x_i \leq x\}}{N}, \tag{1}$$

where $\#\{x_i \leq x\}$ denotes the number of x_is less than or equal to x. From this we generate (bootstrap) resamples, $\mathbf{x}_j^* = (x_1^*, \ldots, x_N^*)$, $j = 1, \ldots, B$ and calculate $\tau_1^*, \ldots, \tau_B^*$. (Stars indicate that we are working with bootstrap samples.) The empirical distribution for τ is estimated with

$$\hat{F}_\tau(t) = \frac{\#\{\tau_j^* \leq t\}}{N}.$$

From this a level α confidence interval for $\tau(F)$ can easily be calculated. The key is to generate the bootstrap resamples from an i.i.d. sample. If we have such a sample and are interested in some smooth statistic, $\tau = \tau(F)$, bootstrap will generally work.

Extensions of these methods to dependent data structures, e.g., time-series, have been proposed by several authors, e.g., Freedman (1984), Bose (1988), and Nordgaard (1995). They propose different strategies to construct data sets associated with the measured data which can be bootstrapped. We will here illustrate one of the simplest ideas.

Assume that data, $y(1), \ldots, y(N)$, is generated by an n'th order AR-process

$$A(q)y(t) = e(t),$$
$$A(q) = 1 + a_1 q^{-1} + \cdots + a_n q^{-n},$$

where q^{-1} is the time delay operator, $q^{-1}y(t) = y(t-1)$ and $\{e(t)\}$ is white noise with zero mean. We estimate $A(q)$ by standard least-squares and compute the residuals

$$\varepsilon(t) = \hat{A}(q)y(t).$$

Now $\{\varepsilon(t)\}$ is approximately a white noise sequence from which we generate our resamples and then simulate "new" output signals according to

$$y_j^*(t) = \frac{1}{\hat{A}(q)}\varepsilon_j^*(t).$$

Here $\{\varepsilon_j^*(t)\}$, $j = 1, \ldots, B$ are drawn from the distribution $\hat{F}_\varepsilon$, constructed as in (1). We simulate B resampled sequences of $\{y(t)\}$. From the simulated outputs we construct the estimates $\hat{A}_1^*, \ldots, \hat{A}_B^*$ of $\hat{A}$. We compute our statistic of interest $\tau_j^* = \tau(\hat{A}_j^*), j = 1, \ldots, B$ and compute some kind of uncertainty measure for it.

Proofs of the validity of this method can be found in Freedman (1984) and Bose (1988). In the article Freedman (1984) the author also shows that the same method works in case of auto-regression with exogenous input (so called ARX-models). Here the data is assumed to be collected in open-loop and the control signal is assumed to white noise.

3. IDENTIFICATION SETUP

Let the data be generated from a linear time-invariant system represented as

$$y(t) = G_0(q)u(t) + H_0(q)e(t).$$

Here $\{y(t)\}$ is the output, $\{u(t)\}$ is the input and $\{e(t)\}$ is zero mean, white noise with variance λ_0. Furthermore, $H_0(q)$ is an inversely stable monic filter. This system will be modeled in a parameterized way by

$$y(t) = G(q,\theta)u(t) + H(q,\theta)e(t).$$

The aim is to estimate θ in such a way that $G(e^{i\omega}, \hat{\theta}) \approx G_0(e^{i\omega})$ and $H(e^{i\omega}, \hat{\theta}) \approx H_0(e^{i\omega})$ for all frequencies. This will be done using measurements $Z_N = \{y(1), u(1), \ldots, y(N), u(N)\}$ of the input-output data.

In obtaining this estimate, we will minimize sum of the squared prediction errors, $\varepsilon(t,\theta)$, as follows

$$\hat{\theta}_N = \arg\min_\theta \frac{1}{N}\sum_{t=1}^{N} \varepsilon^2(t,\theta)$$
$$\varepsilon(t,\theta) = H^{-1}(q,\theta)\left(y(t) - G(q,\theta)u(t)\right).$$

The estimated transfer functions will be denoted by $\hat{G}_N(q)$, $\hat{H}_N(q)$; $\hat{G}_N(q) = G(q,\hat{\theta}_N)$, $\hat{H}_N(q) = H(q,\hat{\theta}_N)$. We assume the existence of some parameter θ_0 such that $G(q,\theta_0) = G_0(q)$. This in order to get an asymptotically unbiased estimate, i.e., $\mathrm{E}\,\hat{G}_N(q) = G_0(q)$. In this case one can show the following asymptotic characterization of the distribution of $\hat{\theta}_N$ under weak conditions (Ljung 1999)

$$\sqrt{N}\left(\hat{\theta}_N - \theta_0\right) \in As N(0, P_\theta)$$
$$P_\theta = \lambda_0 \left[\lim_{N\to\infty}\frac{1}{N}\sum_{t=1}^{N}\mathrm{E}\,\psi(t,\theta_0)\psi^T(t,\theta_0)\right]^{-1} \tag{2}$$
$$\psi(t,\theta_0) = -\frac{d}{d\theta}\varepsilon(t,\theta)\Big|_{\theta=\theta_0}.$$

In many cases it is important to construct confidence regions for other statistics than the parameter vector θ. These regions can be approximated using Gauss' approximation formula. This formula is obtained using Taylor series expansion to first order around the $\theta_0 = \mathrm{E}\,\hat{\theta}_N$. The results is that if $\hat{\theta}_N$ is sufficiently close to θ_0, we can approximate

$$\mathrm{Cov}\, f(\hat{\theta}) \approx f'(\theta_0)P_\theta\left[f'(\theta_0)\right]^T, \tag{3}$$

where $f'(\bar{\theta})$ is a $1 \times d$ dimensional matrix being the derivative of $f(\theta)$ evaluated at $\theta = \bar{\theta}$. If the higher

order derivatives are small, this approximation will be good. An extension to this is that if the higher order derivatives are sufficiently smooth and if $\hat{\theta}_N$ is asymptotically Gaussian, then $f(\hat{\theta}_N)$ will also be asymptotically Gaussian. If we for instance are interested in uncertainty bounds for the frequency function, we get from (3)

$$\operatorname{Cov}\begin{bmatrix}\operatorname{Re}\hat{G}_N(e^{i\omega})\\ \operatorname{Im}\hat{G}_N(e^{i\omega})\end{bmatrix} \approx \begin{bmatrix}\operatorname{Re}\hat{G}'_\theta(e^{i\omega},\hat{\theta}_N)\\ \operatorname{Im}\hat{G}'_\theta(e^{i\omega},\hat{\theta}_N)\end{bmatrix} \times \times P_\theta \begin{bmatrix}\operatorname{Re}\hat{G}'_\theta(e^{i\omega},\hat{\theta}_N)\\ \operatorname{Im}\hat{G}'_\theta(e^{i\omega},\hat{\theta}_N)\end{bmatrix}^T. \quad (4)$$

This could be used to construct confidence regions for the frequency function in the Nyquist diagram.

4. BOOTSTRAP AND SYSTEM IDENTIFICATION

The idea on how use bootstrap in system identification is clearly inspired by the one described for autoregressive models for time-series data in Section 2.

We know that as $N \to \infty$ we will have that the estimates $\hat{G}_N \to G_0$ and $\hat{H}_N \to H_0$. This also means that the residuals we compute

$$\varepsilon(t,\hat{\theta}_N) = \hat{H}_N^{-1}(q)\big(y(t) - \hat{G}_N(q)u(t)\big)$$

will tend to the "true" noise sequence $\{e(t)\}_{t=1}^N$. These are by assumption i.i.d. and could therfore be used as the basis for our resampling procedure. In practice one should always perform some kind of whiteness test on the residuals to ensure that the residuals actually are not non-white.

Given the residuals, the empirical distribution function can easily be calculated

$$\hat{F}_\varepsilon(x) = \frac{\#\{\varepsilon \le x\}}{N}. \quad (5)$$

There are other and more sophisticated methods to estimate the distribution, see e.g., cite-WandJ:1995. These methods are however some what more involved to compute and we will not look into that here. Now the approximate distribution function (5) could be used to draw resampled residuals $\varepsilon_j^*(t)$, $j = 1,\ldots,B$ from. These bootstrapped residuals will thereafter act as the driving noise for the estimated system. This will generate B pseudo-output signals with statistical properties similar to those of the original output $y(t)$;

$$y_j^*(t) = \hat{G}_N(q)u(t) + \hat{H}_N(q)\varepsilon_j^*(t),\ j = 1,\ldots,B$$

From each of the B generated output sequences, reestimates of the system and noise models, $\hat{G}_N$ and $\hat{H}_N$, can be obtained. The B reestimated parameter vectors can now be used to approximate the distribution function for any statistic of interest.

The whole procedure above can be summarized in the following steps:

(1) Estimate $G(q)$ and $H(q)$ and compute the residuals:
$\varepsilon(t,\hat{\theta}_N) = \hat{H}_N^{-1}(q)(y(t) - \hat{G}_N(q)u(t))$
(2) Compute the empirical distribution of the residuals:
$\hat{F}_\varepsilon(x) = \dfrac{\#\{\varepsilon \le x\}}{N}$
(3) Generate B resampled noise sequences from $\hat{F}_\varepsilon(x)$:
$\left\{\varepsilon_j^*(t,\hat{\theta}_N)\right\}_{t=1}^N$, $j = 1,\ldots,B$
(4) Simulate the new outputs:
$y_j^*(t) = \hat{G}_N(q)u(t) + \hat{H}_N(q)\varepsilon_j^*(t,\hat{\theta}_N)$,
$j = 1,\ldots,B$
(5) Estimate $\hat{\theta}_{N,j}$ from our resampled data:
$Z_{N,j}^* = \{y_j^*(1),\ u(1),\ \ldots,\ y_j^*(N),\ u(N)\}$,
$j = 1,\ldots,B$.

5. CONSTRUCTING UNCERTAINTY REGIONS FOR FREQUENCY RESPONSE

In estimating a system from input-output data it is of great importance to come up with a tight and reliable uncertainty description. This uncertainty is preferably displayed in the frequency domain since it provides useful information to, e.g., a control designer. The simplest (and probably most used) way to create such description is to construct confidence regions with a certain confidence degree at a specific frequency for the frequency function. In order to make a confidence band out of this, the procedure is repeated at several frequencies and the regions are connected to produce a band for the entire frequency function. It should be noted that the different regions are dependent through the estimate of a parametric model and it is thus not possible to say much about the simultaneous confidence degree. This follows from the Bonferroni inequality (Manoukian 1986). This inequality gives a lower bound on the simultaneous confidence degree. The bound is $1 - d\cdot(1-\alpha)$, where d is the number of confidence regions and α is the confidence degree for one single region. In situations where a guaranteed simultaneous confidence degree is wanted, the parameters α and d should be chosen according to this. However, constructing too many intervals will force the designer to choose α very small and hence also make the bands (possibly) more conservative.

One exception to this basic procedure is given in Vuerinckx *et al.* (1998). Here the authors look at uncertainty descriptions for the poles and zeros of the system in a simultaneous manner. In the article there is no simulation study to evaluate the actual coverage probability of the intervals, but the approach looks interesting. In Politis *et al.* (1992) a method to construct simultaneous confidence bands for the spectra and cross-spectra of stationary weakly dependent time series is presented.

5.1 *Using estimated covariance matrices*

With estimated covariance matrices for the frequency response, approximate confidence regions in the Nyquist plot at frequency ω_k can be constructed. The regions will be in form of ellipsoids centered around the nominal estimate $\hat{G}_N(e^{i\omega_k})$:

$$\begin{bmatrix}\mathrm{Re}[G(e^{i\omega_k}) - \hat{G}_N(e^{i\omega_k})] \\ \mathrm{Im}[G(e^{i\omega_k}) - \hat{G}_N(e^{i\omega_k})]\end{bmatrix} \left(\frac{1}{N} \cdot P(e^{i\omega_k})\right)^{-1} \times \begin{bmatrix}\mathrm{Re}[G(e^{i\omega_k}) - \hat{G}_N(e^{i\omega_k})] \\ \mathrm{Im}[G(e^{i\omega_k}) - \hat{G}_N(e^{i\omega_k})]\end{bmatrix}^T \leq C_\alpha, \qquad (6)$$

where $P(e^{i\omega_k})$ is covariance matrix for $\hat{G}(e^{i\omega_k})$ and C_α is defined through $\mathrm{Prob}(X \leq C_\alpha) = \alpha$, $X \in \chi^2(2)$. $P(e^{i\omega_k})$ could be estimated using (2) and (4) or by using bootstrap methods. The results obtained using this method very closely resembles with the ones obtained using a bootstrap procedure described in Tjärnström and Forssell (1999), where also a comparison to Monte Carlo simulations is made, with excellent agreement. See also Tjärnström (2000).

5.2 *Obtaining confidence degree using bootstrap*

The discussion in the beginning of Section 5 points out the problem with the method described above. This method simply do not take the dependence between different frequency regions into account. Fortunately, this can be taken care of quite easily using bootstrap.

It is rather easily seen that the confidence region in a high dimensional space should be rectangularly shaped if we want to project the region to low dimensional spaces without using any approximations. This type of region can however be quite difficult in a parametric setting, but is actually a relatively easy task in a bootstrap situation. Since we aim at constructing a rectangular box in a d-dimensional space, we must choose a norm measuring the distance between points in a suitable way. After appropriate scaling and translation of the estimates the l_∞-norm is the natural choice since it measures the longest orthogonal distance to the sides of the rectangle. In this way we do not take any cross-correlation between estimates into account (just as we want to).

We will now formalize this idea. Assume that we estimate some d-dimensional parameter vector θ and aim at constructing d single confidence intervals with a simultaneous confidence degree α. B bootstrap estimates of θ are calculated and will be used to construct the intervals. Denote these estimates with

$$\hat{\theta}_1^*, \ldots, \hat{\theta}_B^*.$$

Let the elements of $\hat{\theta}_k^*$ be denoted

$$\begin{pmatrix}\hat{\theta}_{1,k}^* & \cdots & \hat{\theta}_{d,k}^*\end{pmatrix}^T, \quad k = 1, \ldots, B,$$

and the estimated means and standard deviations of the elements of $\hat{\theta}$ by

$$\bar{\theta}_{j,\cdot}^* = \frac{1}{B}\sum_{k=1}^{B} \hat{\theta}_{j,k}^*, \quad j = 1, \ldots, d,$$

$$\hat{\sigma}_j^* = \sqrt{\frac{1}{B-1}\sum_{k=1}^{B}\left(\hat{\theta}_{j,k}^* - \bar{\theta}_{j,\cdot}^*\right)}, \quad j = 1, \ldots, d.$$

Normalize and translate the estimates of the different elements such that they all have unit variance and zero mean

$$\tilde{\theta}_{j,k}^* = \frac{\hat{\theta}_{j,k}^* - \bar{\theta}_{j,\cdot}^*}{\hat{\sigma}_j^*}.$$

This will give all elements in θ equal influence. Define the distances from the origin to the bootstrap estimates as

$$Q_k = \|\tilde{\theta}_k^*\|_\infty = \max(|\tilde{\theta}_{1,k}^*|, \ldots, |\tilde{\theta}_{d,k}^*|)$$

Order the estimates in increasing distance to the mean

$$\hat{\theta}_{(1)}^*, \ldots, \hat{\theta}_{(B)}^*,$$

where the number between the parentheses denotes the order. From the ordered set of estimates we pick the α-fraction closest to the mean. That is, the confidence region should be built from

$$\hat{\theta}_{(1)}^*, \ldots, \hat{\theta}_{(\lceil B\cdot\alpha\rceil)}^*,$$

where $\lceil\cdot\rceil$ denotes ceiling. The boundaries

$$b_1^{low}, b_1^{high}, \ldots, b_d^{low}, b_d^{high}$$

of the rectangular region will be decided from

$$b_j^{low} = \min(\hat{\theta}_{j,(1)}^*, \ldots, \hat{\theta}_{j,(\lceil B\cdot\alpha\rceil)}^*), \quad j = 1, \ldots, d,$$
$$b_j^{high} = \max(\hat{\theta}_{j,(1)}^*, \ldots, \hat{\theta}_{j,(\lceil B\cdot\alpha\rceil)}^*), \quad j = 1, \ldots, d.$$

These boundaries also form the single confidence intervals (that should be interpreted simultaneously) according to

$$I_{\theta_j} = (b_j^{low}, b_j^{high}), \quad j = 1, \ldots, d.$$

The choice of the number of estimates that should be included in the confidence region actually gives the user some freedom of choice. With the choice $\lceil B\cdot\alpha\rceil$ we have $B\cdot(1-\alpha)$ of the estimates strictly outside the region and $B\cdot\alpha - 2\cdot d$ estimates strictly inside the region. This is because we have one estimate on each boundary of the region. It should also be clear that changing the number of estimates that decides the boundaries of region to $\lceil B\cdot\alpha\rceil + 2\cdot d$ gives $B\cdot\alpha$ estimates strictly inside the region and $B\cdot(1-\alpha) - 2\cdot d$ estimates lying strictly outside the region. We recommend the choice of using $\lceil B\cdot\alpha\rceil + d$ as a standard choice (lying between the other two), but all choices in the interval $[\,\lceil B\cdot\alpha\rceil, \lceil B\cdot\alpha\rceil + 2\cdot d\,]$ are valid. The difference between these choices will vanish as B tends to infinity, but can be significant when B is finite. When $d/B \ll 1$ the choice is more or less immaterial. One could however argue that

the regions are reliable only when all choices gives approximately the same regions.

Finally we remark that, if the dimension of the parameter vector is high, we will need a substantial amount of reestimates of θ to accurately describe the probability density function $f_{\hat{\theta}}$. This could mean that for d in the range of $30 - 50$ we would probably need B at least as high as 10000.

6. EXAMPLE

To illustrate the bootstrap procedure given in Section 5.2 we will look at the following system:

$$\begin{aligned} A(q)y(t) &= B(q)u(t) + e(t) \qquad (7)\\ A(q) &= 1 - 2.5q^{-1} + 3.3q^{-2} \\ &\quad - 2.5q^{-3} + 1.2q^{-4} - 0.3q^{-5} \\ B(q) &= 0.21q^{-1} + 0.35q^{-2} \\ &\quad - 0.12q^{-3} - 0.11q^{-4} + 0.23q^{-5} \end{aligned}$$

where $\{u(t)\}$ and $\{e(t)\}$ are mutually independent Gaussian white noise processes with zero mean and variances 1 and 0.04, respectively. This system was simulated with $N = 1000$ data points and an ARX(5,5,1)-model was fitted to the data.

The evaluation of the results are displayed along the Nyquist curve using confidence regions calculated along a grid defined by the following 32 frequencies (in MATLAB notation):

$$\begin{aligned} \mathcal{W}_1 = &\{10^{-5},\ 0.01,\ (0.1 : 0.1 : 0.6),\ 0.65,\\ &(0.68 : 0.015 : 0.9),\ 0.93,\ 0.96,\ 1,\ 1.1,\ 1.2,\\ &1.5,\ 2,\ 3.1\} \end{aligned}$$

(All frequencies in rad/s.) First of all we compare the size of the confidence regions along the Nyquist curve for the different "methods". The first method is based on the "classical" method described by (6), where the covariance matrix is estimated using (2) and (4). The result is depicted in Figure 1. Figures 2 and 3 shows the confidence regions obtained using the method in Section 5.2 applied to bootstrap and Monte Carlo simulations, respectively. In the bootstrap and Monte Carlo based confidence regions, $B = 3000$ resamples/samples were used to construct the regions.

We see that the method used to construct the confidence regions in Figure 1 give smaller confidence regions compared to the ones in Figures 2 and 3, as expected. We also see that the presented bootstrap based procedure described in Section 5.2 gives resulting regions in comparable size to the ones obtained using Monte Carlo. It is also worth noting that the regions in Figure 2 and 3 are not very "smooth". This has to do with the fact that the regions are only based upon 3000 sample/resamples. Increasing this number will of course increase the smoothness and correctness of these regions.

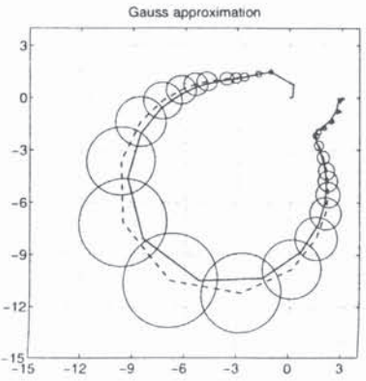

Fig. 1. Nyquist curve with 95 % confidence degree uncertainty regions using the classical method described in Section 5.1. True system (solid), estimated system (dashed).

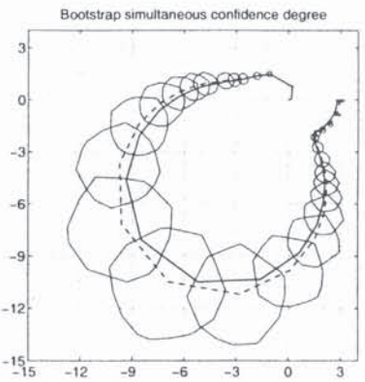

Fig. 2. Nyquist curve with 95 % simultaneous confidence degree uncertainty regions using bootstrap and the method described in Section 5.2. True system (solid), estimated system (dashed).

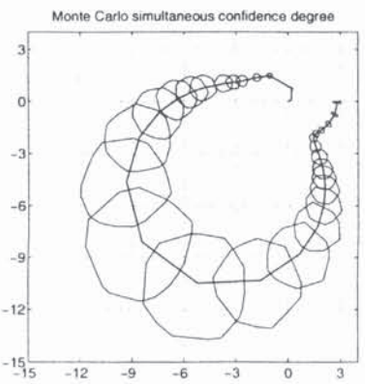

Fig. 3. Nyquist curve with 95 % simultaneous confidence degree uncertainty regions using Monte Carlo simulations and the method described in Section 5.2. True system (solid).

The figures shown in this example illustrate the fact that bootstrap achieves confidence regions which are very similar to the ones obtained using Monte Carlo simulations. These figures do not however say anything about the confidence degree actually achieved. Therefore, we evaluate

$N = 300$	$\alpha = 0.85$	$\alpha = 0.9$	$\alpha = 0.95$
$B = 1000$	0.834	0.912	0.942
$B = 3000$	0.842	0.904	0.944
Bonferroni	0.952	0.968	0.98
$N = 1000$	$\alpha = 0.85$	$\alpha = 0.9$	$\alpha = 0.95$
$B = 1000$	0.810	0.886	0.942
$B = 3000$	0.832	0.884	0.946
Bonferroni	0.948	0.958	0.974
$B = 3000$	$\lceil 0.9B \rceil$	$\lceil 0.9B \rceil + 19$	$\lceil 0.9B \rceil + 38$
$N = 300$	0.896	0.904	0.916
$N = 1000$	0.876	0.886	0.900

Table 1. Obtained simultaneous confidence degree for the amplitude estimate of (7) using the simultaneous bootstrap and the Bonferroni inequality. The explanation is given in the text.

the quality of the simultaneous bootstrap confidence regions. This is accomplished by constructing confidence bands for the amplitude curve and evaluating how many of these bands that actually cover the true system at *every* frequency. This is done by generating 500 realizations from the system (7). We evaluate the results on the following frequency grid

$$\mathcal{W}_2 = \{10^{-5},\ 0.01,\ (0.1 : 0.1 : 0.6),\ (0.65 : 0.05 : 0.9),\ 1,\ 1.2,\ 1.5,\ 2,\ 3.1\}.$$

We used $N = 300$ and $N = 1000$ data in each realization and constructed the confidence bands from each one of them using $B = 1000$ and $B = 3000$ resamples, respectively. The confidence degrees we wanted to achieve were $\alpha = 0.85$, 0.9, and 0.95. The obtained confidence degrees are depicted in Table 1. In this table we also included a comparison of three different choices of the number of samples that was included in the construction of the confidence band. Here we compared the three choices $\lceil B \cdot \alpha \rceil$, $\lceil B \cdot \alpha + d \rceil$, and $\lceil B \cdot \alpha + 2d \rceil$. It is clear that if we cannot use a high B (due to time limitations for example) we should use the conservative choice. If $d/B \ll 0.01$ the choice is more or less immaterial. The table shows that the simultaneous bootstrap seems to work really well in most cases. It is however a bit surprising that the method produces slightly worse results when the data length increases (in this example). To this we have no explanation. We also included an evaluation of the actual confidence degree obtained when choosing the confidence degrees to meet the guaranteed lower bound given by the Bonferroni inequality. The table shows that this bound is rather conservative in this case.

7. CONCLUSIONS

We have shown how bootstrap can be used to obtain confidence regions of a given simultaneous confidence degree for any smooth statistics of the system. The simulations on the example show very good results. The accuracy of the regions can certainly be improved by using more reestimates. This can however be very time consuming in some situations. The amount of time and computer power needed is also the drawback of this method. Computing the regions presented in paper takes approximately 1 minute on a standard PC using MATLAB. With increase in speed and memory of computers this time will drastically decrease in the near future, and thus make it an interesting alternative to more "classical" methods.

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