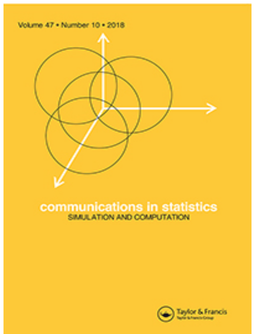

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Confidence Intervals of Performance Measures for an $M/G/1$ Queueing System

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In this article, we discuss constructing confidence intervals (CIs) of performance measures for an $M/G/1$ queueing system. Fiducial empirical distribution is applied to estimate the service time distribution. We construct fiducial empirical quantities (FEQs) for the performance measures. The relationship between generalized pivotal quantity and fiducial empirical quantity is illustrated. We also present numerical examples to show that the FEQs can yield new CIs dominate the bootstrap CIs in relative coverage *(defined as the ratio of coverage probability to average length of CI) for performance measures of an $M/G/1$ queueing system in most of the cases.*

Keywords Confidence intervals; Fiducial empirical quantity.

Mathematics Subject Classification 62G15; 60K25.

1. Introduction

In a queueing system, it is important to carry out a statistical analysis. When operating a queueing system, monitoring and control of the performance measures of the system are essential to ensure that the system performance is up to design standards. These problems often require statistical analysis of performance measures such as traffic intensity (ρ), mean system size (L), mean queue size (L_q), mean sojourn time (W), and mean waiting time (W_q), of which the statistical behaviors reflect the stability of the system. Interest in statistical inference of the performance measures has increased in recent years. Chu and Ke (2006, 2007a) examined the statistical behavior of the mean response time (W) for the $M/G/1$ and $G/G/1$ queueing system using bootstrap simulation. In addition, they propose a consistent and asymptotically normal estimator of W for a $G/M/1$ queueing system (see Chu and Ke, 2007b). Nonparametric inference of traffic intensity for a queueing system was also studied by Ke and Chu (2006). Each of the methods above construct confidence intervals (CIs) for one performance measure. This motivates us to develop more general method for five performance measures of an $M/G/1$ First Come First Service (FCFS) queueing system.

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Throughout this article, we consider an experiment in which we measure m interarrival times and n service times. We use $\{X_i, i = 1, \ldots, m\}$ and $\{Y_i, i = 1, \ldots, n\}$ to denote the positive random variables of interarrival and service time for the ith customer of an $M/G/1$ FCFS queueing system. Let $X_1, X_2, \ldots, X_m$ be independent and identically distributed (i.i.d.) with exponential distribution $F(t) = 1 - e^{-\lambda t}, x \geq 0$, and $Y_1, Y_2, \ldots, Y_n$ be i.i.d. variables with distribution function $H(t), t \geq 0$. All random variables $(X_1, \ldots, X_m)$, $(Y_1, \ldots, Y_n)$ are assumed to be mutually independent.

It is very frequent to assume steady-state in queueing theory. Steady-state implies traffic intensity $\rho = \lambda E(Y) < 1$. Expressions for the steady-state performance measures of the $M/G/1$ queue can be found in Gross and Harris (1998). The mean waiting time in queue is $W_q = \frac{E(Y^2)}{2(1/\lambda - E(Y))}$, the mean number of customers in queue is $L_q = \frac{\lambda^2 E(Y^2)}{2(1-\rho)}$, the mean waiting time in system (mean response time) is $W = E(Y) + \frac{E(Y^2)}{2(1/\lambda - E(Y))}$, and the mean number of customers in system is $L = \rho + \frac{\lambda^2 E(Y^2)}{2(1-\rho)}$.

We consider first the nonparametric estimation of the general service time distribution. There has been recent interest in nonparametric methods for queueing systems. Bingham and Pitts (1999) developed nonparametric methods for the $M/G/\infty$ queueing system based on the Laplace transform using incomplete information. Hall and Park (2004) constructed empirical approximations to the service time distribution for the $M/G/\infty$ queueing system with only the busy periods information. Ausín et al. (2004) suggested a Bayesian procedure to estimate the service time distribution for the $M/G/1$ queue by a class of Erlang mixtures which are phase-type distributions. An explicit evaluation of measures such as the stationary queue size, waiting time, and busy period distributions can be obtained from this phase-type approximation. In this context, we estimate the service time distribution using the service time data. The classical nonparametric estimator of a distribution function is the empirical distribution function (EDF) H_n. Xu et al. (2008) introduced a new randomized estimator, say $\widetilde{H}_n$, called fiducial EDF. This estimator can be applied to estimate H. Denote $Y_{(1)} \leq Y_{(2)} \leq \cdots \leq Y_{(n)}$ as the order statistics of $Y_1, \ldots, Y_n$. Similarly, $U_{(1)} \leq U_{(2)} \leq \cdots \leq U_{(n)}$ are the order statistics from $U(0, 1)$. Given observations $y_1, \ldots, y_n$ of the sample $Y_1, \ldots, Y_n$, $\widetilde{H}_n$ is defined by

$$\widetilde{H}_n(t; \underline{\mathrm{U}}, \underline{\mathrm{y}}) = \begin{cases} \dfrac{U_{(1)}}{y_{(1)}} t & 0 \leq t \leq y_{(1)}, \\[2ex] (U_{(i)} - U_{(i-1)}) \cdot \dfrac{t - y_{(i-1)}}{y_{(i)} - y_{(i-1)}} + U_{(i-1)} & y_{(i-1)} \leq t \leq y_{(i)},\ \ 2 \leq i \leq n, \\[2ex] 1 - (1 - U_{(n)}) \exp\left(-\dfrac{t - y_{(n)}}{k} \right) & y_{(n)} \leq t, \end{cases} \tag{1}$$

where $\underline{\mathrm{U}} = (U_1, \ldots, U_n)^T$, $\underline{\mathrm{y}} = (y_1, \ldots, y_n)^T$, k are constants or depend on $\underline{\mathrm{y}}$, $k \geq 0$. For $k = 0$, define $\widetilde{H}_n(t; \underline{\mathrm{U}}, \underline{\mathrm{y}}) = 1$, when $t \geq y_{(n)}$.

The purpose of this article is to develop a simple approach which can be used to obtain CIs for ρ, L, L_q, W, and W_q. Toward this, we develop methods based on the concept of generalized variable. Weeranhandi (1993) introduced the concept of a generalized pivotal quantity (GPQ) for deriving interval estimation. In this article, fiducial EDF is applied to construct new CIs for performance measures.

Our choice of this estimator is based on two considerations. First, it's a new distribution estimator, which has a good performance in constructing goodness-of-fit test statistics. Second, it helps to construct quantities which behaves like GPQs and can be used to construct CIs. The quantities based on fiducial empirical distribution is called fiducial empirical quantities (FEQs).

The remainder of the article proceeds as follows. In Sec. 2, we adopt fiducial approach to form FEQs for the performance measures of $M/G/1$. Assuming equilibrium, we obtain CIs for the five performance measures in Sec. 3. Bootstrap confidence intervals are constructed in Sec. 4. Simulated numerical examples are presented in Sec. 5. The performances of the new CIs are studied and compared to those of the existing bootstrap CIs and asymptotically normal CIs. We further use an efficient criterion, namely *relative coverage* (see Ke et al., 2008) to evaluate the performance of the methods. It will be seen that FEQs can yield new CIs dominate the bootstrap CIs in relative coverage for performance measures of an $M/G/1$ queueing system in most of the cases. Section 6 concludes the article.

2. FEQs for ρ, L, L_q, W, and W_q

In this section, a general method for constructing fiducial empirical quantities of performance measures for an $M/G/1$ queueing system is presented. As mentioned in Sec. 1, $\{X_i, i = 1, 2, \ldots, m\}$ are independent exponentially distributed. Hence, the statistics $T = 2m\overline{X} = 2\sum_{i=1}^{m} X_i$ is sufficient for the parameter λ. They can be expressed in the pivotal model, $\lambda T = E$, where $E \sim \chi^2(2m)$. This pivotal model can be converted to

$$\hat{\lambda} = E/t, \tag{2}$$

where t is the observed value of T.

Notice that performance measures are functions of $E(Y) = \int_0^\infty t dH(t)$ and $E(Y^2) = \int_0^\infty t^2 dH(t)$. Let $\mu_1 = E(Y)$, $\mu_2 = E(Y^2)$. Substituting $\widetilde{H}_n(t; \underline{\mathrm{U}}, \underline{\mathrm{y}})$ described in (1) for $H(t)$, μ_1 can be estimated by

$$\hat{\mu}_1 = \int_0^\infty t d\widetilde{H}_n(t; \underline{\mathrm{U}}, \underline{\mathrm{y}}) = \sum_{i=1}^{n+1} V_i z_i, \tag{3}$$

where $z_i = \frac{y_{(i-1)} + y_{(i)}}{2}$, $V_i \stackrel{d}{=} U_{(i)} - U_{(i-1)}$, for $i = 1, \ldots, n+1$, $y_{(0)} = 0$, $y_{(n+1)} = y_{(n)} + 2k$, $U_{(0)} = 0$, $U_{(n+1)} = 1$. Note that $(V_1, \ldots, V_{n+1})$ is jointly distributed as Dirichlet$(n+1, 1, \ldots, 1)$. Similarly, μ_2 can be estimated by

$$\hat{\mu}_2 = \int_0^\infty t^2 d\widetilde{H}_n(t; \underline{\mathrm{U}}, \underline{\mathrm{y}}) = \sum_{i=1}^{n+1} V_i z_i', \tag{4}$$

where $z_i' = \frac{y_{(i-1)}^2 + y_{(i)}^2 + y_{(i)} y_{(i-1)}}{3}$, for $i = 1, \ldots, n$, $z_{n+1}' = \frac{y_{(n+1)}^2 + y_{(n)}^2}{2}$.

Now consider the choices of k. Usually, k is determined to make the distribution function $\widetilde{H}_n(t; \underline{\mathrm{U}}, \underline{\mathrm{y}})$ satisfy some conditions. Here one can find k so that the conditional expectation $E_U(\hat{\mu}_1)$ is closest to the sample mean, when $\underline{Y} = \underline{y}$.

Rewrite $E_U(\hat{\mu}_1) = \frac{n}{n+1}\bar{y} + \frac{y_{(1)}+y_{(n)}}{2(n+1)} + \frac{k}{n+1}$. According to the above principle, we get the reasonable choice of k.

$$k = \begin{cases} \bar{y} - \dfrac{y_{(1)} + y_{(n)}}{2} & \bar{y} - \dfrac{y_{(1)} + y_{(n)}}{2} > 0 \\ 0 & \bar{y} - \dfrac{y_{(1)} + y_{(n)}}{2} \leq 0 \end{cases} \tag{5}$$

From Sec. 1, we know that all the five performance measures can be expressed as a function of (λ, μ_1, μ_2). We have:

$$\theta_1 = g_1(\lambda, \mu_1, \mu_2) = \lambda E(Y), \quad \theta_2 = g_2(\lambda, \mu_1, \mu_2) = \lambda E(Y) + \frac{\lambda^2 E(Y^2)}{2(1 - \lambda E(Y))},$$

$$\theta_3 = g_3(\lambda, \mu_1, \mu_2) = \frac{\lambda^2 E(Y^2)}{2(1 - \lambda E(Y))}, \quad \theta_4 = g_4(\lambda, \mu_1, \mu_2) = E(Y) + \frac{E(Y^2)}{2(1/\lambda - E(Y))},$$

$$\theta_5 = g_5(\lambda, \mu_1, \mu_2) = \frac{E(Y^2)}{2(1/\lambda - E(Y))}.$$

To construct confidence intervals of above five parameters, we need fiducial empirical quantities (FEQs) defined as follows.

Definition 2.1. Suppose the parameter $\theta = f(\lambda, H)$ is a function of λ and H. The random $\hat{\theta} = f(\hat{\lambda}, \widetilde{H}_n(t; \underline{\mathrm{U}}, \underline{\mathrm{y}}))$ is called as a fiducial empirical quantity (FEQ) for θ, where $\hat{\lambda}$ and $\widetilde{H}_n(t; \underline{\mathrm{U}}, \underline{\mathrm{y}})$ are given in (2) and (1), respectively.

Now we define FEQs

$$R_{g_i}(\hat{\lambda}, \hat{\mu}_1, \hat{\mu}_2) = g_i(\hat{\lambda}, \hat{\mu}_1, \hat{\mu}_2), \tag{6}$$

for $i = 2, \ldots, 5$. In a stationary $M/G/1$ queue, $\lambda E(Y) < 1$ is the condition for stationarity. However, due to the independence of E and $(V_1, \ldots, V_{n+1})$, $\hat{\lambda}\hat{\mu}_1 \geq 1$ may occur according to the expressions (2) and (3). Sometimes this occurrence may lead to the confidence intervals containing values out of the ranges of parameters. Hence, the conditional FEQs of θ_i are constructed by

$$\widetilde{R}_{g_i}(\hat{\lambda}, \hat{\mu}_1, \hat{\mu}) = g_i(\hat{\lambda}, \hat{\mu}_1, \hat{\mu}_2), \quad \text{under the condition } \hat{\lambda}\hat{\mu}_1 < 1.$$

Here, $\hat{\lambda}\hat{\mu}_1 < 1$ is equivalent to $E/t \cdot \sum_{i=1}^{n+1} V_i z_i < 1$. The above notion means random variables E and $(V_1, \ldots, V_{n+1})$ in FEQs take values satisfying $E/t \cdot \sum_{i=1}^{n+1} V_i z_i < 1$.

3. New Confidence Intervals

In this section, we want to construct confidence intervals of parameters through FEQs. Our approach is similar to the one through a generalized pivotal quantity. So a short review of generalized pivotal quantities is given below. Let $X = (X_1, X_2, \ldots, X_n)$ be a random sample from a distribution which depends on unknown parameters $\eta = (\theta, \xi)$, where θ is the parameter of interest and ξ is a vector of nuisance parameters. The random quantity $R(X; x, \eta)$ is said to be a GPQ, where x is an observed value of X, for interval estimation defined in Weeranhandi (1993) if it has the following two properties:

1. For fixed x, $R(X; x, \eta)$ has a distribution free of unknown parameters.
2. $R(x; x, \eta)$ does not depend on nuisance parameters, ξ.

Hannig et al. (2006) singled out a subclass of GPQ, called the Fiducial Generalized Pivotal Quantity (FGPQ). The FGPQ has property 1 of the GPQ and a stronger property 2, which says that $R(x; x, \eta) = \theta$.

Now consider FEQs given in expression (6). Recall $\hat{\lambda} = \frac{E}{t}$, $\hat{\mu}_1 = \sum_{i=1}^{n+1} V_i z_i$ and $\hat{\mu}_2 = \sum_{i=1}^{n+1} V_i z_i$. Because $E \sim \chi^2(2m)$, $(V_1, \ldots, V_{n+1}) \sim \text{Dirichlet}(n+1; 1, \ldots, 1)$, the conditional distributions of $R_{g_i}(\hat{\lambda}, \hat{\mu}_1, \hat{\mu}_2)$ is free of unknown parameters, when t and y are given.

Furthermore, notice that $E \overset{d}{=} \lambda T$ and $U_{(i)} \overset{d}{=} H(Y_{(i)}), i = 1, \ldots, n$. We have $V_i = U_{(i)} - U_{(i-1)} \overset{d}{=} H(Y_{(i)}) - H(Y_{(i-1)})$, for $i = 1, \ldots, n+1$, where $H(Y_{(n+1)}) = 1$. It follows that $\hat{\mu}_1 = \sum_{i=1}^{n+1}(H(Y_{(i)}) - H(Y_{(i-1)}))z_i \triangleq \hat{\mu}_1(Y; y, H)$, and similarly $\hat{\mu}_2 \triangleq \hat{\mu}_2(Y; y, H)$. With $\hat{\lambda} = \frac{\lambda T}{t} = \hat{\lambda}(T; t, \lambda)$, FEQ $R_{g_i}(\hat{\lambda}, \hat{\mu}_1, \hat{\mu}_2)$ can be expressed as $R_{g_i}(T, Y; t, y, \lambda, H)$, $i = 1, \ldots, 5$, which have the form of a GPQ. The observed values $R_{g_i}(t, y; t, y, \lambda, H)$, $i = 1, \ldots, 5$, don't have the property 2 for H is nonparametric. But they have a weaker property given below.

Proposition 3.1. *The FEQs given in expression* (6) *have property*

$$\lim_{n\to\infty} R_{g_i}(t, y; t, y, \lambda, H) \xrightarrow{P} \theta_i, \quad i = 1, \ldots, 5.$$

The FEQs $R_{g_i}(T, Y; t, y, \lambda, H)$, $i = 1, \ldots, 5$ do not depend on nuisance parameters and therefore can be taken as the approximations of the GPQs for θ_i. The proof of Proposition 3.1 is given in the Appendix.

A $1-\alpha$ generalized CI is usually defined as the interval formed by the lower and upper $\alpha/2$ quantiles of a generalized pivotal quantities. Our new CIs are constructed based on the similar idea. Note that there are value ranges for θ_i, $i = 1 \ldots 5$. While the lower or upper $\alpha/2$ quantiles of the FEQs are out of the value ranges, we calculate those of the conditional FEQs. For given two independent samples $(x_1, \ldots, x_m; y_1, \ldots, y_n)$, the CIs for θ_i can be estimated using the following steps.

Algorithm 1.

1. Calculate the statistics $\bar{x}$, $(z_1, \ldots, z_{n+1})$, $(z'_1, \ldots, z'_{n+1})$.
2. Generate $e \sim \chi^2_{2m}$, and $(v_1, \ldots, v_{n+1}) \sim \text{Dirichlet}(n+1, 1, \ldots, 1)$.
3. Compute $r_i = g_i(\frac{e}{t}, \hat{\mu}_1, \hat{\mu}_2)$, where $\hat{\mu}_1 = \sum_{j=1}^{n+1} v_j z_j$ and $\hat{\mu}_2 = \sum_{j=1}^{n+1} v_j z'_j$.
4. Repeat Steps 2–3 for N times, then we get the estimated quantiles of θ_i's FEQ, and rearrange them in an ascending order to get $(r_{i(1)}, \ldots, r_{i(N)})$.
5. The $100(1-\alpha)\%$ new CIs for θ_i is (r_{iL}, r_{iU}), where $r_{iL} = r_{i[N\cdot\alpha/2]}$, $r_{iU} = r_{i[N\cdot(1-\alpha/2)]}$, for $i = 1, \ldots, 5$.
6. If $r_{iL} < 0$, for $i = 2, \ldots, 5$ or $r_{1U} > 1$, continue to compute the conditional quantiles; else, end this algorithm.
7. If $e/t \sum_{j=1}^{n+1} v_j z_j < 1$, compute $r_i = g_i(\frac{e}{t}, \hat{\mu}_1, \hat{\mu}_2)$, where $\hat{\mu}_1 = \sum_{j=1}^{n+1} v_j z_j$ and $\hat{\mu}_2 = \sum_{j=1}^{n+1} v_j z'_j$; else regenerate e and $(v_1, \ldots, v_{n+1})$ as in Step 2. Repeat this step

for N times, then we get the estimated conditional quantiles of θ_i's conditional distribution, and rearrange them in an ascending order to get $(\widetilde{r}_{i(1)}, \ldots, \widetilde{r}_{i(N)})$.

8. The $100(1-\alpha)\%$ new CIs for θ_i is $(\widetilde{r}_{iL}, \widetilde{r}_{iU})$, where $\widetilde{r}_{iL} = \widetilde{r}_{i[N\cdot\alpha/2]}$, $\widetilde{r}_{iU} = \widetilde{r}_{i[N\cdot(1-\alpha/2)]}$, for $i = 1, \ldots, 5$.

4. Bootstrap Confidence Intervals

Chu and Ke (2006) constructed bootstrap confidence intervals for mean sojourn time. This method can be easily applied to the other four performance measures.

We briefly give an introduction of the application of the percentile bootstrap method to the other four performance measures. Given the original data $(X_1, \ldots, X_m)$, $(Y_1, \ldots, Y_n)$. Denote the empirical distributions of $(X_1, \ldots, X_m)$ and $(Y_1, \ldots, Y_n)$ by F_m and G_n, respectively. Suppose that $X_1^*, \ldots, X_m^*$ i.i.d. $\sim F_m$, $Y_1^*, \ldots, Y_n^*$ i.i.d.$\sim G_n$, and that $\overline{X}^*$, $\overline{Y}^*$ are the bootstrap versions of $\overline{X}, \overline{Y}$, respectively.

By the strong law of large numbers, $\overline{X}$, $\overline{Y}$ and $A_2 = \sum_{i=1}^n Y_i^2/n$ are strong consistent estimators of $1/\lambda$, $E(Y)$ and $E(Y^2)$, respectively. Hence, θ_i can be estimated by

$$\hat{\theta}_i = g_i\big(1/\overline{X}, \overline{Y}, A_2\big).$$

The bootstrap version of $\hat{\theta}_i^*$ can be calculated from bootstrap sample as

$$\hat{\theta}_i^* = g_i\big(1/\overline{X}^*, \overline{Y}^*, A_2^*\big),$$

where $A_2^* = \sum_{i=1}^n Y_i^{*2}/n$. Here, $\hat{\theta}_i^*$ is called a bootstrap estimate of $\hat{\theta}_i$. Suppose that B bootstrap samples are available, then B bootstrap estimates, $\hat{\theta}_{i1}^*, \ldots, \hat{\theta}_{iB}^*$, can be computed from the bootstrap samples.

The B bootstrap estimates can be ordered from the smallest to the largest, denoted by $\hat{\theta}_{i(1)}^*, \ldots, \hat{\theta}_{i(B)}^*$, which constitutes an empirical bootstrap distribution of θ_i. The level $1-\alpha$ confidence interval of θ_i can be constructed as

$$\big[\hat{\theta}^*_{i[(\alpha/2)B]}, \hat{\theta}^*_{i[(1-\alpha/2)B]}\big],$$

where $[x]$ denotes the greatest integer less than or equal to x.

5. Examples and Simulations

We conducted three simulation studies to evaluate the performance of the new CIs. For comparison, we also included several bootstrap methods and asymptotically normal method. In our simulation study, the confidence level is $1-\alpha = 0.90$. We consider samples of service data from various service time distributions as follows:

Case 1. $M/M/1$ queueing model, $h(y) = \mu e^{-\mu y}$, $y \geq 0$, $E(Y) = 1/\mu$.

Case 2. $M/E_4/1$ queueing model, $h(y) = \mu^4 y^3 e^{-\mu y}/3!$, $y \geq 0$, $E(Y) = 4/\mu$.

Case 3. $M/H_4/1$ queueing model, $h(y) = \frac{3}{8}\mu_1 e^{-\mu_1 y} + \frac{1}{8}\mu_2 e^{-\mu_2 y} + \frac{1}{4}\mu_3 e^{-\mu_3 y} + \frac{1}{4}\mu_4 e^{-\mu_4 y}$, $y \geq 0$, $E(Y) = \frac{3/8}{\mu_1} + \frac{1/8}{\mu_2} + \frac{1/4}{\mu_3} + \frac{1/4}{\mu_4}$.

Case 4. $M/LN/1$ queueing model, $h(y) = \frac{1}{y\sigma\sqrt{2\pi}}\exp\big\{-\frac{(\ln y-\mu)^2}{2\sigma^2}\big\}$, $y \geq 0$, $E(Y) = e^{\mu+\sigma^2/2}$.

Here, we denote $h(y)$ as the service time density. The first three models are also discussed Chu and Ke (2006). We compared the new CIs with the other methods with $E(Y) = 1$ and the mean arrival rate $\lambda = 0.1$ to 0.9 by increments of 0.2. The parameters are assigned that $\mu = 1$ in Case 1, $\mu = 4$ in Case 2, $\mu_1 = 5/6$, $\mu_2 = 5/2$, $\mu_3 = 3/2$, $\mu_4 = 3/4$ in Case 3, and $\mu = -1/32$, $\sigma = 1/4$ in Case 4. All simulations were conducted with 1,000 replications and implemented using MATLAB R2006a. We calculate coverage probability, average length, and relative coverage for the five performance measures. The coverage probability of the confidence interval is obtained by the percentage of times that the true value of θ_i is covered by the confidence interval. In addition, the average length of the length of the confidence interval is calculated based on the 1,000 replications.

Relative coverage which was first introduced by Ke et al. (2008) is used to evaluate performance of methods. It is defined as the ratio of coverage probability (CP) to average length (AL) of confidence interval, and can be considered as the amount of coverage probability contained by per unit-length confidence interval. The greater the relative coverage is, the better the method. The relative coverage is

Table 1
Coverage probabilities, average lengths, and relative coverages for the 90% confidence interval of $M/M/1$ queueing models

Arrival rate λ	Performance measures	Coverage probability		Average length		Relative coverage	
		FE	PB	FE	PB	FE	PB
0.1	L	0.898[a]	0.858	0.129[a]	0.131	6.972[a]	6.560
	L_q	0.846[a]	0.824	0.029[a]	0.030	29.58[a]	27.84
	W	0.842[a]	0.814	0.870	0.837[a]	0.968	0.973[a]
	W_q	0.812[a]	0.776	0.200	0.193[a]	4.054[a]	4.015
0.3	L	0.874[a]	0.872	0.947[a]	1.026	0.923[a]	0.850
	L_q	0.856	0.862[a]	0.649[a]	0.741	1.318[a]	1.163
	W	0.824	0.832[a]	2.081[a]	2.160	0.396[a]	0.385
	W_q	0.814	0.826[a]	1.468[a]	1.573	0.555[a]	0.525
0.5	L	0.878	0.936[a]	6.297[a]	8.731	0.139[a]	0.099
	L_q	0.874	0.930[a]	5.851[a]	8.437	0.149[a]	0.103
	W	0.858	0.916[a]	9.409[a]	12.61	0.091[a]	0.070
	W_q	0.840	0.912[a]	8.872[a]	12.19	0.095[a]	0.073
0.7	L	0.916[a]	0.904	21.54[a]	23.97	0.041[a]	0.037
	L_q	0.914[a]	0.900	21.07[a]	23.89	0.042[a]	0.038
	W	0.912[a]	0.894	26.07[a]	28.46	0.034[a]	0.031
	W_q	0.916[a]	0.892	25.65[a]	28.32	0.035[a]	0.032
0.9	L	0.872[a]	0.796	47.99	33.58[a]	0.018	0.024[a]
	L_q	0.872[a]	0.796	47.56	33.60[a]	0.018	0.024[a]
	W	0.868[a]	0.756	51.58	35.52[a]	0.017	0.021[a]
	W_q	0.868[a]	0.760	51.26	35.51[a]	0.017	0.021[a]

[a]Indicate the largest coverage probability, the shortest average length, and the largest relative coverage among the three confidence intervals.

computed as follows:

$$\text{Relative Coverage (RC)} = \begin{cases} \dfrac{\text{CP}}{\text{AL}}, & \text{CP} \leq 90\%, \\ \dfrac{90\% - (\text{CP} - 90\%)}{\text{AL}}, & \text{CP} \geq 90\%. \end{cases} \tag{7}$$

We give two examples to illustrate the Fiducial Empirical method and to demonstrate its advantages over the other method. Three major criteria in evaluating these interval estimation methods are:

1. Whether the coverage probabilities are in good agreement with the nominal levels.
2. Whether the average lengths are shorter.
3. If shorter confidence intervals lead to smaller coverage probabilities, or larger coverage probabilities due to longer confidence intervals, we evaluate the performance of confidence intervals by relative coverages.

Example 5.1. The performance of the new CIs is assessed in the above queueing models. The performances of Percentile Bootstrap (PB) and Fiducial Empirical

Table 2
Coverage probabilities, average lengths, and relative coverages for the 90% confidence interval of $M/E_4/1$ queueing models

Arrival rate λ	Performance measures	Coverage probability		Average length		Relative coverage	
		FE	PB	FE	PB	FE	PB
0.1	L	0.895[a]	0.868	0.093	0.094[a]	9.655[a]	9.283
	L_q	0.890[a]	0.868	0.014[a]	0.015	63.57[a]	59.45
	W	0.875[a]	0.869	0.405[a]	0.415	2.160[a]	2.094
	W_q	0.881[a]	0.850	0.082[a]	0.082[a]	10.76[a]	10.37
0.3	L	0.900[a]	0.871	0.476[a]	0.490	1.891[a]	1.779
	L_q	0.890[a]	0.872	0.241[a]	0.254	3.696[a]	3.440
	W	0.871[a]	0.857	0.765[a]	0.777	1.139[a]	1.103
	W_q	0.874[a]	0.866	0.476[a]	0.490	1.836[a]	1.769
0.5	L	0.890	0.957[a]	2.815[a]	3.821	0.316[a]	0.220
	L_q	0.888	0.954[a]	2.446[a]	3.513	0.363[a]	0.241
	W	0.911	0.937[a]	3.328[a]	4.516	0.267[a]	0.191
	W_q	0.882	0.942[a]	3.084[a]	4.294	0.286[a]	0.200
0.7	L	0.933	0.937[a]	13.50[a]	16.18	0.064[a]	0.053
	L_q	0.933	0.936[a]	13.08[a]	16.06	0.066[a]	0.054
	W	0.920	0.932[a]	14.77[a]	17.45	0.060[a]	0.050
	W_q	0.925	0.932[a]	14.58[a]	17.35	0.060[a]	0.050
0.9	L	0.910[a]	0.862	33.50	28.61[a]	0.027	0.030[a]
	L_q	0.910[a]	0.862	33.11	28.60[a]	0.027	0.030[a]
	W	0.910[a]	0.862	34.10	29.23[a]	0.026	0.030[a]
	W_q	0.911[a]	0.861	33.96	29.21[a]	0.026	0.030[a]

[a]Indicates the largest coverage probability, shortest average length, and largest relative coverage among the three confidence intervals.

Table 3
Coverage probabilities, average lengths, and relative coverages for the 90% confidence interval of $M/H_4/1$ queueing models

Arrival rate λ	Performance measures	Coverage probability FE	Coverage probability PB	Average length FE	Average length PB	Relative coverage FE	Relative coverage PB
0.1	L	0.872	0.875[a]	0.148[a]	0.152	5.892[a]	5.753
	L_q	0.814	0.842[a]	0.038[a]	0.040	21.65[a]	21.26
	W	0.814	0.821[a]	0.970	0.967[a]	0.839	0.849[a]
	W_q	0.760	0.790[a]	0.249[a]	0.257	3.047	3.073[a]
0.3	L	0.858	0.876[a]	1.362[a]	1.519	0.630[a]	0.577
	L_q	0.830	0.856[a]	1.0346	1.208[a]	0.803[a]	0.709
	W	0.787	0.822[a]	2.906[a]	3.053	0.271[a]	0.269
	W_q	0.784	0.810[a]	2.269[a]	2.417	0.346[a]	0.335
0.5	L	0.862	0.905[a]	8.406[a]	10.42	0.103[a]	0.086
	L_q	0.855	0.897[a]	8.017[a]	10.18	0.107[a]	0.088
	W	0.835	0.870[a]	12.75[a]	14.99	0.066[a]	0.058
	W_q	0.841	0.871[a]	12.18[a]	14.63	0.069[a]	0.060
0.7	L	0.912[a]	0.907	26.10	25.07[a]	0.034	0.036[a]
	L_q	0.912[a]	0.905	25.67	25.02[a]	0.035	0.036[a]
	W	0.896[a]	0.892	32.40	30.26[a]	0.028	0.030[a]
	W_q	0.907[a]	0.892	31.91	30.15[a]	0.028	0.030[a]
0.9	L	0.864[a]	0.767	47.11	33.42[a]	0.018	0.023[a]
	L_q	0.858[a]	0.767	46.60	33.44[a]	0.018	0.023[a]
	W	0.855[a]	0.740	50.90	34.75[a]	0.017	0.021[a]
	W_q	0.854[a]	0.741	50.32	34.75[a]	0.017	0.021[a]

[a]Indicates the largest coverage probability, shortest average length, and largest relative coverage among the three confidence intervals.

(FE) methods are compared. Samples of interarrival and service times of size $m = 25$, $n = 20$ are generated. For each data set, the CIs for the five parameters are constructed using Algorithm 1 with $N = 10{,}000$ and $B = 10{,}000$ bootstrap samples $(x_1^*, \ldots, x_m^*; y_1^*, \ldots, y_n^*)$ are generated to compute the new CIs and the bootstrap intervals.

Tables 1–4 show the CPs, ALs, and RCs of the two confidence intervals. Examining these simulation results, we get the following conclusions.

(1) We observe that the ALs of both the PB confidence intervals and the new CIs are increasing with λ.
(2) The FEQ-based confidence intervals have shorter ALs in most of the cases of the four queueing models. When $\lambda E(Y)$ is close to 1, for example, $\lambda = 0.9$, $E(Y) = 1$, the coverage probabilities of PB confidence intervals is not very good. The new CIs have better coverage probabilities but the ALs are wider.
(3) When the data is generated from $M/E_4/1$ and $M/LN/1$ queueing models and $\lambda E(Y)$ is not close to 1, the new CIs have CPs closer to the nominal level, with narrow ALs.

Table 4
Coverage probabilities, average lengths, and relative coverages for the 90% confidence interval of $M/LN/1$ queueing models

Arrival rate λ	Performance measures	Coverage probability		Average length		Relative coverage	
		FE	PB	FE	PB	FE	PB
0.1	L	0.902[a]	0.864	0.080[a]	0.083	11.20[a]	10.43
	L_q	0.896[a]	0.838	0.010[a]	0.011	88.71[a]	75.50
	W	0.912[a]	0.840	0.201[a]	0.212	4.418[a]	3.970
	W_q	0.888[a]	0.824	0.053[a]	0.054	16.88[a]	15.20
0.3	L	0.900[a]	0.838	0.385[a]	0.414	2.340[a]	2.023
	L_q	0.882[a]	0.818	0.171[a]	0.197	5.164[a]	4.144
	W	0.868[a]	0.788	0.439[a]	0.449	1.977[a]	1.753
	W_q	0.870[a]	0.820	0.306[a]	0.335	2.839[a]	2.449
0.5	L	0.930	0.938[a]	1.962[a]	2.847	0.443[a]	0.303
	L_q	0.934	0.936[a]	1.625[a]	2.554	0.533[a]	0.338
	W	0.920	0.922[a]	2.006[a]	2.906	0.439[a]	0.302
	W_q	0.940	0.946[a]	1.900[a]	2.822	0.453[a]	0.303
0.7	L	0.948	0.980[a]	10.74[a]	14.52	0.079[a]	0.057
	L_q	0.944	0.980[a]	10.59[a]	14.41	0.081[a]	0.057
	W	0.944	0.986[a]	10.93[a]	14.78	0.078[a]	0.055
	W_q	0.952	0.984[a]	10.75[a]	14.75	0.079[a]	0.055
0.9	L	0.952[a]	0.902	30.78	27.00[a]	0.028	0.033[a]
	L_q	0.952[a]	0.902	30.12	26.98[a]	0.028	0.033[a]
	W	0.956[a]	0.902	30.39	27.27[a]	0.028	0.033[a]
	W_q	0.952[a]	0.902	30.62	27.27[a]	0.028	0.033[a]

[a]Indicates the largest coverage probability, shortest average length, and largest relative coverage among the three confidence intervals.

(4) In the $M/H_4/1$ queueing model, the new CIs perform better than the bootstrap CIs based on ALs. But the CPs of the new CIs for the parameters W, W_q, L, L_q are worse than the PB confidence intervals.
(5) The relative coverages are computed for the cases when it's difficult for us to compare the performance of the two methods. We find that the relative coverages of the new CIs are larger than the bootstrap CIs in $M/M/1$ queueing model in 75% of the cases.

The simulations concerning other bootstrap method (bootstrap-t, bias corrected bootstrap, and standard bootstrap) were also constructed but not been shown here. The ALs and the CPs or the confidence intervals based on them are not so good as the PB confidence intervals.

Example 5.2. Ke and Chu (2006) constructed a confidence interval of traffic intensity (ρ) for a distribution free queueing system. This example compares the performance of the three methods: FE method, PB method, and Ke and Chu's method.

Ke and Chu's method is based on the asymptotically normal estimator of ρ. Let z_α be the upper αth quantile of the standard normal distribution, $\hat{\rho} = \overline{Y}/\overline{X}$,

Table 5
Coverage probabilities, average lengths, and relative coverages of 90% CIs for traffic intensity (ρ) of $M/M/1$ queueing models

Sample size	λ	Coverage probability FE	Coverage probability PB	Coverage probability AN	Average length FE	Average length PB	Average length AN	Relative coverage FE	Relative coverage PB	Relative coverage AN
20	0.1	0.887[a]	0.872	0.866	0.108	0.110	0.106[a]	8.228[a]	7.949	8.209
	0.3	0.872[a]	0.863	0.866	0.325	0.323	0.307[a]	2.686	2.617	2.817[a]
	0.5	0.898[a]	0.876	0.872	0.471[a]	0.549	0.510	1.907[a]	1.597	1.710
	0.7	0.928[a]	0.863	0.878	0.497[a]	0.763	0.734	1.754[a]	1.131	1.197
	0.9	0.941[a]	0.885	0.876	0.461[a]	1.014	0.929	1.863[a]	0.873	0.943
50	0.1	0.899[a]	0.897	0.869	0.068	0.067	0.065[a]	13.18	13.29[a]	13.29
	0.3	0.889[a]	0.888	0.881	0.205	0.203	0.196[a]	4.341	4.385	4.488[a]
	0.5	0.910[a]	0.910[a]	0.875	0.333	0.334	0.324[a]	2.670	2.665	2.701[a]
	0.7	0.924[a]	0.883	0.883	0.393[a]	0.469	0.455	2.230[a]	1.884	1.939
	0.9	0.943[a]	0.882	0.886	0.348[a]	0.602	0.594	2.462[a]	1.466	1.492
80	0.1	0.899	0.894	0.908[a]	0.054	0.053	0.052[a]	16.74	16.90	17.22[a]
	0.3	0.892[a]	0.883	0.886	0.160	0.158	0.155[a]	5.593	5.589	5.720[a]
	0.5	0.912[a]	0.905	0.912[a]	0.265	0.261	0.260[a]	3.348	3.428[a]	3.410
	0.7	0.915[a]	0.895	0.902	0.339[a]	0.371	0.361	2.609[a]	2.410	2.486
	0.9	0.945[a]	0.892	0.895	0.295[a]	0.475	0.462	2.902[a]	1.876	1.939

[a]Indicates the largest coverage probability, shortest average length, and largest relative coverage among the three confidence intervals.

Table 6
Coverage probabilities, average lengths, and relative coverages of 90% CIs for traffic intensity (ρ) of $M/E_4/1$ queueing models

Sample size	λ	Coverage probability FE	Coverage probability PB	Coverage probability AN	Average length FE	Average length PB	Average length AN	Relative coverage FE	Relative coverage PB	Relative coverage AN
20	0.1	0.903[a]	0.866	0.862	0.086	0.088	0.080[a]	10.42	9.80	10.73[a]
	0.3	0.887[a]	0.850	0.873	0.259	0.268	0.247[a]	3.427	3.178	3.536[a]
	0.5	0.896[a]	0.845	0.872	0.405[a]	0.436	0.409	2.210[a]	1.936	2.133
	0.7	0.932[a]	0.878	0.855	0.457[a]	0.614	0.574	1.899[a]	1.431	1.490
	0.9	0.943[a]	0.842	0.880	0.416[a]	0.805	0.731	2.062[a]	1.047	1.204
50	0.1	0.903[a]	0.884	0.878	0.054	0.054	0.053[a]	16.67	16.43	16.76[a]
	0.3	0.898[a]	0.888	0.894	0.159	0.160	0.156[a]	5.644	5.536	5.723[a]
	0.5	0.909[a]	0.897	0.891	0.255[a]	0.266	0.260	3.494[a]	3.379	3.431
	0.7	0.909[a]	0.873	0.882	0.339[a]	0.373	0.366	2.629[a]	2.342	2.413
	0.9	0.938[a]	0.885	0.898	0.307[a]	0.474	0.465	2.810[a]	1.867	1.931
80	0.1	0.900[a]	0.895	0.892	0.042	0.042	0.041[a]	21.63[a]	21.51	21.60
	0.3	0.886[a]	0.871	0.886[a]	0.125	0.125	0.123[a]	7.077	6.974	7.192[a]
	0.5	0.907[a]	0.897	0.888	0.209	0.208	0.206[a]	4.269	4.319[a]	4.313
	0.7	0.900	0.883	0.906[a]	0.282[a]	0.294	0.287	3.190[a]	3.005	3.111
	0.9	0.944[a]	0.898	0.902	0.258[a]	0.376	0.368	3.317[a]	2.388	2.440

[a]Indicates the largest coverage probability, shortest average length, and largest relative coverage among the three confidence intervals.

Table 7

Coverage probabilities, average lengths, and relative coverages of 90% CIs for traffic intensity (ρ) of $M/H_4/1$ queueing models

Sample size	λ	Coverage probability FE	Coverage probability PB	Coverage probability AN	Average length FE	Average length PB	Average length AN	Relative coverage FE	Relative coverage PB	Relative coverage AN
20	0.1	0.877	0.880[a]	0.851	0.111	0.113	0.110[a]	7.908[a]	7.788	7.758
	0.3	0.860	0.865[a]	0.864	0.327[a]	0.339	0.331	2.627[a]	2.553	2.613
	0.5	0.908[a]	0.879	0.868	0.481[a]	0.571	0.541	1.856[a]	1.539	1.603
	0.7	0.928[a]	0.871	0.871	0.503[a]	0.820	0.765	1.734[a]	1.062	1.139
	0.9	0.922[a]	0.880	0.874	0.470[a]	1.042	0.959	1.870[a]	0.845	0.911
50	0.1	0.872	0.868	0.878[a]	0.068	0.067[a]	0.069	12.88	12.90[a]	12.72
	0.3	0.888	0.886	0.893[a]	0.205	0.204[a]	0.208	4.323	4.347[a]	4.297
	0.5	0.894[a]	0.877	0.878	0.335[a]	0.337	0.340	2.666	2.605	2.584[a]
	0.7	0.908[a]	0.889	0.887	0.393[a]	0.470	0.482	2.270[a]	1.892	1.841
	0.9	0.928[a]	0.904	0.879	0.347[a]	0.601	0.626	2.512[a]	1.492	1.401
80	0.1	0.904[a]	0.901	0.891	0.057	0.055[a]	0.055[a]	15.77	16.23	16.35[a]
	0.3	0.885	0.874	0.906[a]	0.170	0.165	0.164[a]	5.221	5.294	5.458[a]
	0.5	0.903[a]	0.887	0.888	0.279	0.275	0.273[a]	3.221	3.223	3.249[a]
	0.7	0.914[a]	0.888	0.885	0.341[a]	0.388	0.386	2.601[a]	2.288	2.293
	0.9	0.934[a]	0.896	0.902	0.298[a]	0.497	0.493	2.902[a]	1.803	1.821

[a]Indicates the largest coverage probability, shortest average length, and largest relative coverage among the three confidence intervals.

$S_X^2 = \frac{1}{m}\sum_{i=1}^{m}(X_i - \overline{X})^2$ and $S_Y^2 = \frac{1}{n}\sum_{i=1}^{n}(Y_i - \overline{Y})^2$. An approximate $1-\alpha$ confidence interval of ρ is

$$\left(\hat{\rho} - z_\alpha\hat{\sigma}/\sqrt{n},\, \hat{\rho} + z_\alpha\hat{\sigma}/\sqrt{n}\right),$$

where $\hat{\sigma}^2 = (\overline{X}^2 S_Y^2 + \overline{Y}^2 S_X^2)/\overline{X}^4$. This method is denoted as asymptotically normal (AN) method.

We generated samples of size $m = n = 20, 50, 80$ from the given interarrival and service time distributions. The Monte Carlo size used to calculate the new CIs is $N = 10{,}000$, and the bootstrap Monte Carlo size is $B = 10{,}000$.

The CPs, ALs and relative coverages are presented in Table 5–7. Simulation results show that the ALs are decreasing with sample size (m, n), but both coverage probabilities and relative coverages are increasing with sample size. The CPs of FE method are the largest of the three methods in most of the cases, but the CPs are farer from the nominal level than the other methods when $\lambda E(Y)$ is close to 1, for example, $\lambda = 0.7$ or 0.9, $E(Y) = 1$. Also in these cases, the FE method has shorter ALs than the other two methods. The AN method performs better than the other two methods when the sample size (m, n) is large enough and $\lambda E(Y)$ is not close to 1. Relative coverages are calculated when needed. The results show that the best performance assessed by relative coverage is played by the FEQ-based method when the sample sizes (m, n) are no more than 50.

6. Conclusions

This article is devoted to construct confidence interval estimation of performance measures of an $M/G/1$ queueing system. Based on modeling, the general service time distribution using fiducial empirical distribution, we construct fiducial empirical quantities for the performance measures. The relationship between generalized pivotal quantity and fiducial empirical quantity was illustrated. Comparative studies are performed to demonstrate the merit of our approach.

From Tables 2 and 4, we speculate that for performance measures L, L_q, W, and W_q, the FEQ-based method proposed in this article performs better than the bootstrap methods when the service data is generated from a unimodal distribution and $\lambda E(Y)$ is not close to 1. From Tables 5–7, we can get the conclusion that for the performance measure ρ, our method outperforms the other methods based on relative coverages in most of the cases.

Our approach can also be extended to other queueing systems. It is possible to model $G/G/1$ systems using fiducial empirical distributions for both the interarrival and service time distributions. In this case, the calculation of FEQs for the performance measures are more complicated. This problem is currently being studied.

Appendix A

In this section, we give the proof of Proposition 3.1 given in Sec. 3. Let $\hat{\mu}_1(y; y, H)$ and $\hat{\mu}_2(y; y, H)$ be the observed values of $\hat{\mu}_1(Y; y, H)$ and $\hat{\mu}_2(Y; y, H)$, respectively. Because $(\widehat{T}; t, \lambda) = \frac{T\lambda}{t}$ and $\hat{\lambda}(t; t, \lambda) = \lambda$, we only need to prove the following.

1. $\lim_{n\to\infty} \hat{\mu}_1(y; y, H) \xrightarrow{P} \mu_1$; 2. $\lim_{n\to\infty} \hat{\mu}_2(y; , y, H) \xrightarrow{P} \mu_2$.

Proof. 1. Note that

$$\hat{\mu}_1(y; y, H) - \mu_1 = \sum_{i=1}^{n}[H(y_{(i)}) - H(y_{(i-1)})] \cdot \frac{y_{(i)} + y_{(i-1)}}{2} - \int_0^{y_{(n)}} tdH(t) + [1 - H(y_{(n)})] \cdot \frac{y_{(n+1)} + y_{(n)}}{2} - \int_{y_{(n)}}^{\infty} tdH(t) \tag{8}$$

Let $I_1 = \sum_{i=1}^{n}[H(y_{(i)}) - H(y_{(i-1)})] \cdot \frac{y_{(i)}+y_{(i-1)}}{2} - \int_0^{y_{(n)}} tdH(t)$, $I_2 = [1 - H(y_{(n)})] \cdot \frac{y_{(n+1)}+y_{(n)}}{2}$, $I_3 = \int_{y_{(n)}}^{\infty} tdH(t)$. We prove the following.

(1) $|I_1| \xrightarrow{\mathrm{P}} 0$, as $n \to \infty$;
(2) $I_2 \xrightarrow{\mathrm{P}} 0$, as $n \to \infty$;
(3) $I_3 \xrightarrow{\mathrm{P}} 0$, as $n \to \infty$.

For the proof of (1), notice that

$$\begin{aligned}|I_1| &\le \sum_{i=1}^{n} \int_{y_{(i-1)}}^{y_{(i)}} \left|\frac{y_{(i)} + y_{(i-1)}}{2} - t\right| dH(t) \\ &\le \sum_{i=1}^{n} \frac{y_{(i)} - y_{(i-1)}}{2} \int_{y_{(i-1)}}^{y_{(i)}} dH(t)\end{aligned}$$

$$\leq \frac{\max(y_{(i)} - y_{(i-1)})}{2} \cdot H(y_{(n)})$$
$$\leq \frac{\max(y_{(i)} - y_{(i-1)})}{2}. \tag{9}$$

Therefore, we only need to establish $\max(Y_{(i)} - Y_{(i-1)}) \xrightarrow{P} 0$ to prove $|I_1| \xrightarrow{P} 0$. For every $\varepsilon > 0$,

$$P\Big(\max(Y_{(i)} - Y_{(i-1)}) < \varepsilon\Big) \leq P\left(\sum_{i=1}^{n}(Y_{(i)} - Y_{(i-1)}) < n\varepsilon\right) \leq P(Y_{(n)} < n\varepsilon). \tag{10}$$

Notice that $P(Y_{(n)} < n\varepsilon) = H^n(n\varepsilon)$. Clearly,

$$H(n\varepsilon) = 1 - (1 - H(n\varepsilon)) = 1 - \int_{n\varepsilon}^{\infty} dH(t) \geq 1 - \int_{n\varepsilon}^{\infty} \frac{t}{n\varepsilon} dH(t) = 1 - \frac{1}{n\varepsilon}\int_{n\varepsilon}^{\infty} t dH(t)$$

It is known that $\int_0^\infty t dH(t) = \mu_1$. For every $\delta > 0$, there exists M, such that $\int_M^\infty t dH(t) < \delta$. It can be seen that $\int_{n\varepsilon}^\infty t dH(t) \leq \int_M^\infty t dH(t) < \delta$, when n is appropriate large. Then

$$\lim_{n\to\infty} H^n(n\varepsilon) \geq \lim_{n\to\infty}\left(1 - \frac{\delta}{n\varepsilon}\right)^n = e^{-\frac{\delta}{\varepsilon}}.$$

Let $\delta \to 0$, we get $\lim_{n\to\infty} H^n(n\varepsilon) = 1$. Hence,

$$\frac{Y_{(n)}}{n} \xrightarrow{P} 0, \quad \text{as } n \to \infty. \tag{11}$$

With regard to (9)–(11), we arrive at $|I_1| \xrightarrow{P} 0$.

We continue with the proof of (2).

When $k = 0$, $I_2 = [1 - H(y_{(n)})] \cdot y_{(n)}$. Therefore,

$$P(I_2 > \varepsilon) = P(I_2 > \varepsilon, k = 0) + P(I_2 > \varepsilon, k > 0)$$
$$\leq P([1 - H(y_{(n)})] \cdot y_{(n)} > \varepsilon) + P(k > 0). \tag{12}$$

Rewrite $[1 - H(y_{(n)})] \cdot y_{(n)} = n[1 - H(y_{(n)})] \cdot \frac{y_{(n)}}{n}$. Recall that $\frac{Y_{(n)}}{n} \xrightarrow{P} 0$, as $n \to \infty$. It remains to establish that $n[1 - H(Y_{(n)})] = O_p(1)$ and $P(k > 0) \to 0$. We shall prove $n[1 - H(Y_{(n)})] = O_p(1)$ first. Notice that

$$P([1 - H(Y_{(n)})]n < z) = P\left((1 - U_{(n)}) < \frac{z}{n}\right)$$
$$= 1 - P\left(U_{(n)} \leq 1 - \frac{z}{n}\right) = 1 - \left(1 - \frac{z}{n}\right)^n. \tag{13}$$

As $n \to \infty$, $1 - \left(1 - \frac{z}{n}\right)^n \to 1 - e^{-z}$. Therefore, $[1 - H(Y_{(n)})]n \xrightarrow{d} Z$, as $n \to \infty$. Here, Z is exponential distributed with mean 1. So, it suffices to show that $n[1 - H(Y_{(n)})] = O_p(1)$.

Next, $k > 0$ is equivalent to $\bar{y} - \frac{y_{(1)}+y_{(n)}}{2} > 0$. To make the proof clear, we only consider the support of H is infinite. For every $c > 0$,

$$P(Y_{(n)} > c) = 1 - [H(c)]^n$$

Because $H(c) < 1$, we have $\lim_{n\to\infty} P(Y_{(n)} > c) = 1$. Thus,

$$Y_{(n)} \xrightarrow{P} +\infty, \quad \text{as } n \to +\infty. \tag{14}$$

By the law of large numbers, $\overline{Y} \xrightarrow{P} \mu_1$, as $n \to \infty$. Hence,

$$\overline{Y} - \frac{Y_{(1)} + Y_{(n)}}{2} \xrightarrow{P} -\infty. \tag{15}$$

Consequently, $P(k > 0) \to 0$. Combining (11)–(13) and (15), the proof of (2) is completed.

The proof of (3) is a straightforward result of Eq. (14) and $\int_0^\infty t dH(t) = \mu_1$. Then we establish the desired conclusion 1.

2. The proof of $\hat{\mu}_2 \xrightarrow{P} \mu_2$ is similar as 1. Note that

$$\begin{aligned}\hat{\mu}_2(y; y, H) - \mu_2 &= \sum_{i=1}^{n}[H(y_{(i)}) - H(y_{(i-1)})] \cdot \frac{y_{(i)}^2 + y_{(i-1)}^2 + y_{(i)}y_{(i-1)}}{3} \\ &\quad - \int_0^{y_{(n)}} t^2 dH(t) + [1 - H(y_{(n)})] \cdot \frac{y_{(n+1)}^2 + y_{(n)}^2}{2} \\ &\quad - \int_{y_{(n)}}^{\infty} t^2 dH(t)\end{aligned} \tag{16}$$

Let $J_1 = \sum_{i=1}^n [H(y_{(i)}) - H(y_{(i-1)})] \cdot \frac{y_{(i)}^2+y_{(i-1)}^2+y_{(i)}y_{(i-1)}}{3} - \int_0^{y_{(n)}} t^2 dH(t)$, $J_2 = [1 - H(y_{(n)})] \cdot \frac{y_{(n+1)}^2+y_{(n)}^2}{2}$, $J_3 = \int_{y_{(n)}}^\infty t^2 dH(t)$. In the following, we prove:

(1) $|J_1| \xrightarrow{\mathrm{P}} 0$, as $n \to \infty$;
(2) $J_2 \xrightarrow{\mathrm{P}} 0$, as $n \to \infty$;
(3) $J_3 \xrightarrow{\mathrm{P}} 0$,as $n \to \infty$.

By an argument similar to that of 1.(1), we have:

$$\begin{aligned}|J_1| &\le \sum_{i=1}^{n} \int_{y_{(i-1)}}^{y_{(i)}} \left| \frac{y_{(i)}^2 + y_{(i-1)}^2 + y_{(i)}y_{(i-1)}}{3} - t^2 \right| dH(t) \\ &\le \sum_{i=1}^{n} \frac{(2y_{(i)}^2 - y_{(i-1)}y_{(i)} - y_{(i-1)}^2)}{3} \int_{y_{(i-1)}}^{y_{(i)}} dH(t) \\ &\le \sum_{i=1}^{n} \frac{(2y_{(i)} + y_{(i-1)}) \cdot (y_{(i)} - y_{(i-1)})}{3} \int_{y_{(i-1)}}^{y_{(i)}} dH(t) \\ &\le y_{(n)} \cdot \max(y_{(i)} - y_{(i-1)}) \cdot H(y_{(n)}) \\ &\le y_{(n)} \cdot \max(y_{(i)} - y_{(i-1)}).\end{aligned} \tag{17}$$

Let us also note that, similarly to Eq. (10),

$$P(Y_{(n)} \max(Y_{(i)} - Y_{(i-1)}) < \varepsilon) \leq P\left(Y_{(n)} \sum_{i=1}^{n}(Y_{(i)} - Y_{(i-1)}) < n\varepsilon\right) \leq P(Y_{(n)}^2 < n\varepsilon). \quad (18)$$

Notice that $P(Y_{(n)} < \sqrt{n\varepsilon}) = H^n(\sqrt{n\varepsilon})$. From $\mu_2 < \infty$, we have:

$$\frac{Y_{(n)}^2}{n} \xrightarrow{P} 0, \quad \text{as } n \to \infty. \quad (19)$$

With regard to (17)–(19), we arrive at $|J_1| \xrightarrow{P} 0$.

The proofs of (2) and (3) are similar as those in 1 and are omitted.

Then we establish the desired conclusion 2. □

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