Math 542-Modern Algebra II

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Problem:
(Mon Jan 27) Prove that for any n there is only one Abelian group (up to isomorphism) of size n iff n is square-free. Square-free mean that no p^2 divides n for p a prime.

We will make use of the following theorem, which was proved in class.

Theorem:
$\mathbb{Z}_n \times \mathbb{Z}_m \cong \mathbb{Z}_{nm}$ iff n, m are relatively prime.

Solution:
Suppose that G is an Abelian of order n where n is square-free. By the fundemental theorem of finite Abelian groups, we can express G as the direct sum of finite cyclic subgroups of prime power order. In this special case:

$$G \cong \mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \times \cdots \times \mathbb{Z}_{p_k}$$

Each p_n is unique. Naturally, $n = p_1 \times \cdots \times p_k$ Hence, we can see that with a simple induction,

$$\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \times \cdots \times \mathbb{Z}_{p_n} \cong \mathbb{Z}_{p_1 p_2} \times \cdots \times \mathbb{Z}_{p_k} \cong \mathbb{Z}_n$$

Where, again, n is the product of $p_1 \dots p_k$.

Conversely, suppose the order n of an Abelian G is not squarefree. Then we can factor G as:

$$G \cong G_{p_1} \times \cdots \times G_{p_k}$$

Where each G_k is the p_k-subgroup of G. Let p_j be our prime whose square (at least) divides n. Then within G_{p_j}, we can a subgroup that is isomorphic to either:

$$\left(G_{p_j} \geq \mathbb{Z}_{p_j} \times \mathbb{Z}_{p_j}\right) \text{ or } \left(G_{p_j} \geq \mathbb{Z}_{p_j^2}\right).$$

And by the theorem above, these two subgroups are distinct, and hence we can have more than one Abelian group of order n.