Math 542-Modern Algebra II

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Problem:
(Mon Feb 3) Suppose G_1, G_2, H_1, H_2 are finite abelian groups, $G_1 \times G_2 \cong H_1 \times H_2$ and $G_1 \cong H_1$. Prove that $G_2 \cong H_2$.

Give a counterexample if the word finite is dropped, i.e., $G_1 \times G_2 \cong H_1 \times H_2$ and $G_1 \cong H_1$ but G_2 is not isomorphic to H_2.

Solution:
If we view our common group $G_1 \times G_2 \cong H_1 \times H_2$ as the product of cyclic groups of prime power, then by taking the quotient of this group by $G_1 \cong H_1$, we are essentially striping away the factors of our common group which make up the cyclic prime decomposistion of $G_1 \cong H_1$. It is clear that in both cases, H and G, our remaining group will have the same structure as in the other case. In other words, if $G_1 \times G_2 \cong H_1 \times H_2$ were both isomorphic to:

$$(\mathbb{Z}/p_1^{e_1}\mathbb{Z}) \times \cdots \times (\mathbb{Z}/p_i^{e_i}\mathbb{Z}) \times (\mathbb{Z}/q_1^{f_1}\mathbb{Z}) \times \cdots \times (\mathbb{Z}/q_j^{f_j}\mathbb{Z})$$

Where the factors of prime power p^k would be isomorphic to G_1, H_1, then the quotient of this group by either G_1 or H_1 would leave:

$$(\mathbb{Z}/q_1^{f_1}\mathbb{Z}) \times \cdots \times (\mathbb{Z}/q_j^{f_j}\mathbb{Z}).$$

Which is clearly ismorphic to both G_2 and H_2.

As a counterexample, consider $G = G_1 \times G_2 = \mathbb{Z} \times e$ and $H = H_1 \times H_2 = \mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. We can see that these two groups are isomorphic, if we map $H_1 \times H_2$ by $h \mapsto 2h_1 + h_2$. In other words, double the value of h_1 in order to map it to the corresponding even integer, and if h_2 is equal to one, add 1 to this even integer in order to get the coresponding odd integer. It is clear that $G_2 = e \ncong (\mathbb{Z}/2\mathbb{Z}) = H_2$, hence we have our counterexample.