Math 542-Modern Algebra II

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Febuary 5, 2014

Problem:
(Wed Jan 29) Let G be a finite abelian group. Prove that the following are equivalent

1. For every subgroup H of G there is a subgroup K of G with $HK = G$ and $H \cap K = \{e\}$.
2. Every element of G has square-free order.

Solution:
First we will prove the three easier problems, and then we will use those results to prove the problem itself.

First Notice that a cyclic group of the form C_{p^2} where p is prime has an cyclic subgroup isomorphic to C_p, which is generated by the element p. It is clear that C_{p^2} cannot satisfy the Completion Property, since it's only proper subgroup is C_p, which is true since every element of C_{p^2} must have an order of $1, p$ or p^2. Thus, there is no complementary subgroup of C_{p^2} which is needed for the CP.

Next a group of the form $C_p \times C_p$ will undoubtably satisfy CP, since it's only subgroups are its two cyclic factors, and clearly each one is complemented by the other in the context of the CP.

Thirdly, if two groups have coprime orders, then the prime factorization of their direct product will the sum of the ranks of the prime decomposistion of each groups, and hence they will both have CP. Suppose an element of $g_0 \in G$ does not have sqaure free order, and let p be a prime whose square appears $|g_0|$. Then if H is the subgroup of G generated by g_0, then the factorization of H into cyclic subgroups of prime power will contain $\mathbb{Z}_{p^2}$. Hence, G fails to have the Completion Property.

Conversely, suppose that every element of G has square free order. Then in the prime factorization of G into cyclic abelian groups of prime power, no cyclic subgroup will have an order of prime power that isn't prime, because if one did exist, this would imply that an element of G has a non-squarefree order, which would be a contradiction. Hence, G has the Completion Property.