Math 542-Modern Algebra II

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Problem:
(Wed Feb 12)

(a) Prove that there are no simple groups of order either 575 or 272.

(b) For any prime p prove there are no simple groups of order $p(p-1)$ or $p(p+2)$.

Solution:
(a) 272 factors as $2^4 \times 17$. It is clear that the 17-Sylow subgroup is normal, since the number of 17-Sylow subgroups must divide $m = 16$, and also be congruent to one $\pmod{17}$. It is clear, that for $1, 2, 4, 8, 16$, the only number which is congruent to one $\pmod{17}$ is 1, hence there is one 17-Sylow suggroup, and hence it is normal, making it impossible for a group of order 272 to be simple.

575 factors as $5^2 \times 23$. It is clear that the 23-Sylow subgroup is normal, since the number of 23-Sylow subgroups must divide $m = 25$, and also be congruent to one $\pmod{23}$. It is clear, that for $1, 5, 25$, the only number which is congruent to one $\pmod{23}$ is 1. Hence, there is only one 23-Sylow subgroup, and hence it is normal, making it impossible for a group of order 272 to be simple.

(b) For any prime p, a group with order $p(p-1)$ must not be simple, since the number of p-Sylow subgroups must divide $p-1$, and also be congruent to 1 (mod p). Hence, since $1+p$ is the next number above 1 which is 1 (mod p) and yet is larger than $p-1$, the only possible number of p-Sylow subgroups is 1, as all other numbers 1 (mod p) will also be too large to divide $p-1$.

A group of order $p(p+2)$ must not be simple, since the number of p-Sylow subgroups must divide $p+2$, and also be congruent to 1 (mod p). Hence, since $1+p$ is the next number above 1 which is congruent to 1 (mod p) and yet is too large to divide $p+2$, the only possible number of p-Sylow subgroups is 1, as all other numbers 1 (mod p) will also be too large to divide $p+1$. A special case is when $p = 2$, and the order of the group is 8. In this case, the 2-Sylow subgroup is the group itself, and since 8 is a power of 2, this is the only possible p-Sylow subgroup and hence is any groups of order 8 are simple.