Math 542-Modern Algebra II

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Problem:
(Mon Mar 3) Prove for any prime p and positive integer n that p divides $\binom{p^n}{k}$ for any k with $0 < k < p^n$.

Solution:
First note that $p^n!$ can be divided by p^c where

$$c = \frac{n(n+1)}{2}$$

.

Also note that $p^n!$ is the smallest number with this property. Suppose p does not divide p^n choose k, then this implies that $p^n | k!(p^n - k)!$. However, since $\binom{p^n}{k}$ must be an integer, this implies that $k!(p^n - k)!$ must equal $p^n!$. This is not true, however, since $k!$ is the product of the first k integers, and

$$\frac{p^n!}{(p^n - k)!}$$

is the product of the integers from $p^n - k + 1$ to p^n, which makes it clear that

$$k! \neq \frac{p^n!}{(p^n - k)!}$$

And hence we have a contradiction, and p must divide $\binom{p^n}{k}$.