Light-Matter Interactions for People in a Hurry

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November 2023

0 Preface

The aim of this text is to explore the nature of light-matter interactions. This will ideally be done at a level that can be understood by an undergraduate student familiar with elements of multivariable calculus. The text will seek to paint a general picture of how light interacts with matter, but will not go into the full depth and formalisms of optics as it is conventionally taught. The goal of this text is merely to serve as an intuitive and relatively concise introduction.

I plan to structure the text as follows:

1. Electromagnetic waves in a vacuum.
2. Electromagnetic waves in a material.
3. Elements of nonlinear optics.
4. Elements of quantum optics.
5. (Addendum) Other formulations of electromagnetism and waves (e.g. geometric algebra).

All of these sections will be brief and will attempt to work as an elementary but rigorous introduction.

Currently the first 3 sections are (mostly) complete.

Contents

1 Electromagnetic Waves in a Vacuum

1.1 Fields

We begin our discussion of electromagnetism by reviewing the concept of fields. In physics, a field is simply a function with a value at every point in space. Mathematically, it is a function of the variables x, y, z, and perhaps time. The output of a field can be either a scalar or a vector. To distinguish between the two, in this text vector quantities will be printed in boldface. It turns out that in many situations it is easier to look at the way physical things interact by considering first how they generate a particular field, and subsequently how they are affected by the field.

The formalism of fields is how the subject of electromagnetism is best approached. We say that there exist two fields: the electric field, **E**, and the magnetic field, **B**. These two fields are closely related. In fact, the magnetic field is simply the relativistic manifestation of the electric field. The relations between the two fields that we will soon uncover will hint at this fact.

1.2 Maxwell's Equations

The behavior of the electric and magnetic fields is described by Maxwell's equations:

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{1.1}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \tag{1.2}$$

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \tag{1.3}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{1.4}$$

Here **J** is the electric current density (total current per unit area) and ρ is the electric charge density (total charge per unit volume). It is in principle possible to add a magnetic "current" term into equation 1 and a magnetic "charge" term into equation 4. However, as far as we know, magnetic "charges," called magnetic monopoles, do not exist. We therefore set these theoretical terms to zero.

As mentioned before, to fully describe electromagnetism we also need to know how physical things react to these fields. This reaction is given by the Lorentz force:

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \tag{1.5}$$

Here q is the electric charge of the object in question and **v** is its velocity. For now we will focus on the electric and magnetic fields, but it is good to keep this expression in mind.

1.3 Maxwell's Equations in a Vacuum

Let us consider Maxwell's equations in the absence of charge and current. This means that $\mathbf{J} = 0$ and $\rho = 0$. The equations then reduce to:

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{1.6}$$

$$\nabla \times \mathbf{B} = \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \tag{1.7}$$

$$\nabla \cdot \mathbf{E} = 0 \tag{1.8}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{1.9}$$

Except for a negative sign and a factor of $\mu_0\epsilon_0$, these equations are symmetrical! In fact, it is only arbitrary unit conventions that introduce the $\mu_0\epsilon_0$ factor. We can resolve this stylistic issue by defining $c^2 = \frac{1}{\mu_0\epsilon_0}$ and $\mathbf{H} = c\mathbf{B}$. Using these conventions, i.e. Gaussian units, the equations can then be rewritten as:

$$c\nabla \times \mathbf{E} = -\frac{\partial \mathbf{H}}{\partial t} \tag{1.10}$$

$$c\nabla \times \mathbf{H} = \frac{\partial \mathbf{E}}{\partial t} \tag{1.11}$$

$$\nabla \cdot \mathbf{E} = 0 \tag{1.12}$$

$$\nabla \cdot \mathbf{H} = 0 \tag{1.13}$$

Note that other conventions sometimes use **B** and **H** to refer to different quantities.

We begin by qualitatively examining these equations. Assume that somewhere in space there is a positive electric field. This means that there is a positive curl of the electric field. Equation 10 tells us that since there is a positive curl of the electric field, the magnetic field must be decreasing. As the magnetic field decreases, it eventually becomes negative, as does its curl. Equation 11 tells us that because the curl of the magnetic field is now negative, the electric field must start decreasing. When the electric field in turn decreases past zero and becomes negative, equation 10 tells us that the magnetic field must now start increasing. We can see that this pattern continues cyclically. What we have just described is an electromagnetic wave!

1.4 Solving Maxwell's Equations in a Vacuum

Armed with this new intuition, we can now look for a solution that fits equations 10-13 to describe an electromagnetic wave in a vacuum. To achieve this goal, we must find a way to manipulate equations 10 and 11 into two equations each of which relates **E** or **B** to itself. We can do this with the use of the following convenient identity:

$$\nabla \times (\nabla \times \ ) = \nabla(\nabla \cdot \ ) - \nabla^2 \tag{1.14}$$

∇^2 is the vector Laplacian.

We can take the curl of both sides of equations 10 and 11, substitute the right hand side of equations 11 and 10 for the curl of **H** and **E** respectively, and substitute identity 14 for the left hand side of the equations. Note that the $\nabla(\nabla\cdot\ )$ term becomes zero because there are no charges (equations 12-13). We are left with the following two wave equations:

$$\nabla^2\mathbf{E} = \frac{1}{c^2}\frac{\partial^2\mathbf{E}}{\partial t^2} \tag{1.15}$$

$$\nabla^2\mathbf{H} = \frac{1}{c^2}\frac{\partial^2\mathbf{H}}{\partial t^2} \tag{1.16}$$

Since these equations are the same, we will only consider the electric field equation (equation 15) for now. We can guess a solution in the form:

$$\mathbf{E} = \mathbf{E_0}e^{i(k_x x+k_y y+k_z z-\omega t)} = \mathbf{E_0}e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} \tag{1.17}$$

where $r = (x, y, z)$ is the position vector and $k = (k_x, k_y, k_z)$ is the so called wavevector.

We express the solution in the form of a complex exponential in order to forgo doing unnecessary trigonometry. We also allow $\mathbf{E_0}$ to be a complex vector. Since the electric field must be real, we can simply take the real part of this expression once we have gone through with our calculations. Alternatively, we can note that a more general solution is a linear combination of particular solutions:

$$\mathbf{E} = \mathbf{E_0}e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} + \mathbf{E_1}e^{i(\mathbf{k}\cdot\mathbf{r}+\omega t)} + \mathbf{E_2}e^{i(-\mathbf{k}\cdot\mathbf{r}-\omega t)} + \mathbf{E_3}e^{i(-\mathbf{k}\cdot\mathbf{r}+\omega t)} \tag{1.18}$$

We can then impose the constraint that **E** must be real, which means that $\mathbf{E_3}$ must be a complex conjugate of $\mathbf{E_0}$ and $\mathbf{E_2}$ must be a complex conjugate of $\mathbf{E_1}$. Doing this accomplishes the same thing as simply taking the real part of the complex exponential.

1.5 Plane Harmonic Waves

How do we interpret our solution $\mathbf{E} = \mathbf{E_0}e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}$?

For simplicity, let us first look at a snapshot of this equation at time t=0. Consider under what conditions **E** is constant. **E** will have the same value when the complex exponential is constant–that is when $\mathbf{k}\cdot\mathbf{r} = constant$. This is the equation for a plane in 3D space perpendicular to **k**. These planes are called planes of equal phase. Moreover, the values of the complex exponential will repeat for constants separated by 2π. Hence, we have a plane wave. The wave can be pictured as regularly spaced infinite planes of constant electric fields moving forwards.

Now that we have described the shape of the wave, we will seek to describe how it travels. How can we define the velocity of this wave?

Consider a position **r**. We can ask how far we must displace in space, $\mathbf{\Delta r}$, to reach a point where the electric field after some time Δt is the same as our original electric field at position $\mathbf{r}$. We can interpret this as a plane of equal phase moving with velocity $\mathbf{v} = \frac{\mathbf{\Delta r}}{\Delta t}$ along the vector $\mathbf{k}$.

In a unit of time, a plane of equal phase will travel $\mathbf{v}$ units. We must therefore have $\mathbf{k} \cdot \mathbf{r} - \omega t = \mathbf{k} \cdot (\mathbf{r} + \mathbf{v}) - \omega(t+1)$.

From this expression we can see that $\mathbf{k} \cdot \mathbf{v} - \omega = 0$.

Since $\mathbf{k}$ and $\mathbf{v}$ are parallel, we can express the magnitude of the velocity in terms of the magnitude of the wavevector $\mathbf{k}$ as $v = \frac{\omega}{k}$.

We now know how ω and k are related to wave speed, but in order to find what the wave speed is, we must again turn to Maxwell's equations. Let us plug our solution for $\mathbf{E}$ into the wave equation (15). We get $-\mathbf{k} \cdot \mathbf{k}\mathbf{E} = -\omega^2 \frac{1}{c^2}\mathbf{E}$. This reduces to $\frac{w^2}{k^2} = c^2$, from which we can see that $v = c$. Now it hopefully makes sense why we chose to use the letter c in the definition $c^2 = \frac{1}{\mu_0 \epsilon_0}$. c is the speed of light in a vacuum!

1.6 The Relationship Between the Electric and Magnetic Fields

Up until now we have only considered the electric field $\mathbf{E}$. In this section we will look at the magnetic field as well.

The wave equation for the magnetic field (equation 16) is the same as the wave equation for the electric field, so our previous discussion applies in the same way to $\mathbf{H}$. We will write the solution to the electric and magnetic field equations as:

$$\mathbf{E} = \mathbf{E_0} e^{i(\hat{\mathbf{k_E}} \cdot \mathbf{r} - ct)} \tag{1.19}$$

$$\mathbf{H} = \mathbf{H_0} e^{i(\hat{\mathbf{k_H}} \cdot \mathbf{r} - ct)} \tag{1.20}$$

where $\hat{\mathbf{k_E}}$ and $\hat{\mathbf{k_H}}$ are the unit vectors in the direction of the electric and magnetic fields respectively. By rewriting our solutions in this form, we have incorporated our finding that the wave travels at speed c.

In this form, there is no apparent dependence of $\mathbf{H}$ on $\mathbf{E}$ or vice versa. To determine how they are related we must, as before, turn to Maxwell's equations.

Let's substitute our solutions for $\mathbf{E}$ and $\mathbf{H}$ into Maxwell's equations in a vacuum (equations 10-13). The expressions we get are

$$\hat{\mathbf{k_E}} \times \mathbf{E} = \mathbf{H} \tag{1.21}$$

$$\hat{\mathbf{k_H}} \times \mathbf{H} = -\mathbf{E} \tag{1.22}$$

$$\hat{\mathbf{k_E}} \cdot \mathbf{E} = 0 \tag{1.23}$$

$$\hat{\mathbf{k_H}} \cdot \mathbf{H} = 0 \tag{1.24}$$

Let us parse these equations.

- Equation 23 tells us that the electric field is orthogonal to $\hat{\mathbf{k_E}}$.
- Equation 24 analogously tells us that the magnetic field is orthogonal to $\hat{\mathbf{k_H}}$.

- Equation 21 tells us that the magnetic field is orthogonal to both the electric field and $\hat{\mathbf{k_E}}$.
- Equation 22 tells us that the electric field is orthogonal to the magnetic field and $\hat{\mathbf{k_H}}$.

Putting all of this together, we see that $\hat{\mathbf{k_E}}$ and $\hat{\mathbf{k_H}}$ must be parallel. Since these are unit vectors, they must in fact be identical. In other words, the magnetic and electric fields can both be expressed with the same wavevector, $\mathbf{k}$.

The wavevector (and so the direction that the wave is traveling in), the magnetic field, and the electric field are all orthogonal to each other. In other words, the direction the wave is traveling in (the wavevector) and $\mathbf{E}$ fully determine the direction of the magnetic field. In order to signify the direction that the wave oscillates in, a quantity we call the polarization, we thus only need to consider either the direction of the electric or the magnetic field. We arbitrarily define the wave's polarization to be the direction of $\mathbf{E}$.

How are the magnitudes of the electric and magnetic fields related? To answer this we can use our realization that $\mathbf{k}$, $\mathbf{E}$, and $\mathbf{H}$ are orthogonal. With this in mind, equation 21 reduces to:

$$E = H \tag{1.25}$$

This means that the electric and magnetic fields oscillate in phase, and that their magnitudes are completely determined by each other. Note that earlier we had defined $\mathbf{H} = c\mathbf{B}$, so the magnetic field of a plane electromagnetic wave in SI units is given by $\mathbf{B} = \frac{1}{c}\mathbf{E}$. Since the magnetic field is fully determined by the electric field in this manner, we can restrict our further analyses to the electric field, knowing that the magnetic field will behave in the same way.

1.7 Introducing a Dielectric Medium

Let us now consider what happens when light propagates in a dielectric (insulator). Whenever a free charge is introduced in such a material, surrounding polar molecules will surround and "shield" the charge. The net "bound charge," given by the sum of the free charge and its polar molecule "shield," will therefore be less than the free charge. We can define the ratio between this net charge density and the free charge density as $\frac{\rho_{net}}{\rho_{free}} = \frac{\epsilon_0}{\epsilon}$, where ϵ is called the permittivity of the dielectric.

Consider the divergence Maxwell's equation $\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$. It is easier to deal with ρ_{free} then it is to deal with ρ_{net}, so we can rewrite the equation as follows:

$$\nabla \cdot \mathbf{E} = \frac{\rho_{net}}{\epsilon_0} = \frac{\rho_{free}}{\epsilon} \tag{1.26}$$

Note that this implies that the electric field in a material is less than the electric field would be in a vacuum by a factor of $\frac{\epsilon_0}{\epsilon}$. We can thus substitute ϵ for ϵ_0 wherever it appears in Maxwell's equations, and sweep the bound charges under the rug.

A similar modification can be made to account for the extra magnetic field produced by the tiny "bound currents" caused by electrons spinning around within their atoms. We can substitute μ, the permeability of a material, for μ_0 to sweep the bound currents under the rug as well.

These substitution carry the important implication that light travels at a speed $v = \frac{1}{\sqrt{\mu\epsilon}}$ in a material. We can represent this "slowing down" of light by defining the index of refraction as $n = \frac{c}{v} = \sqrt{\frac{\mu\epsilon}{\mu_0\epsilon_0}}$.

It is important to note here that light itself does not slow down–it still travels at the speed c between the individual atoms of a material. The vibrations of the bound charges induced by the light, however, interfere with it and kick the phase of the EM wave backwards or forwards slightly, changing its apparent speed. This is discussed in more detail in chapter 2.

1.8 Reflection and Refraction

Consider an electromagnetic wave incident on a boundary between two mediums. Lets say that light travels at a speed v_1 in the first medium and at a speed v_2 in the second medium. In general, part of the incident wave will be reflected back into the first medium while part will be transmitted into the second medium. The 3 corresponding wave equations are

$$\mathbf{E_i} = \mathbf{E_{0i}} e^{i(\frac{\hat{\mathbf{k_i}}}{v_1}\cdot\mathbf{r}-t)} \tag{1.27}$$

$$\mathbf{E_r} = \mathbf{E_{0r}} e^{i(\frac{\hat{\mathbf{k_r}}}{v_1}\cdot\mathbf{r}-t)} \tag{1.28}$$

$$\mathbf{E_t} = \mathbf{E_{0t}} e^{i(\frac{\hat{\mathbf{k_t}}}{v_2}\cdot\mathbf{r}-t)} \tag{1.29}$$

At the boundary, the condition $\mathbf{E_i} + \mathbf{E_r} = \mathbf{E_t}$ must be satisfied at any given time. For this to condition to hold, the exponentials must be equal:

$$e^{i(\frac{\hat{\mathbf{k_i}}}{v_1}\cdot\mathbf{r}-t)} = e^{i(\frac{\hat{\mathbf{k_r}}}{v_1}\cdot\mathbf{r}-t)} = e^{i(\frac{\hat{\mathbf{k_t}}}{v_2}\cdot\mathbf{r}-t)} \tag{1.30}$$

We can simplify this to:

$$\frac{\hat{\mathbf{k_i}}}{v_1}\cdot\mathbf{r} = \frac{\hat{\mathbf{k_r}}}{v_1}\cdot\mathbf{r} = \frac{\hat{\mathbf{k_t}}}{v_2}\cdot\mathbf{r} \tag{1.31}$$

From this expression, it is clear that the wavevectors all lie on the same plane. We can use this fact to chose a fitting coordinate system: let us set the plane that the wavevectors lie on–called the plane of incindence–to be the xy plane and the boundary to be the xz plane. We can consider a 2D slice of this coordinate system, where the boundary plane becomes the x axis.

We will denote the angle that the incident wavevector makes with the vertical θ_i, the angle between the reflected wavevector and the vertical θ_r, and the angle between the transmitted wavevector and the vertical ϕ.

Equation 30 tells us that the projections of the wavevectors onto the boundary axis must be equal. Since the incident and reflected waves both travel at the same velocity v_1, this must mean that $\theta_i = \theta_r$. We will therefore drop the subscript and simply write θ.

From equation 30, we also know that that

$$\frac{\sin(\theta)}{v_1} = \frac{\sin(\phi)}{v_2} \tag{1.32}$$

which we can rewrite as

$$n_1 \sin(\theta) = n_2 \sin(\phi) \tag{1.33}$$

where n_1 and n_2 are the indices of refraction of the two materials.

1.9 Magnitudes in Reflection and Refraction

We have just worked out how the direction and speed of an electromagnetic wave changes when it enters a medium, however we have yet to describe how much of the wave is transmitted and how much is reflected. To do this we must consider two cases:

In the transverse electric (TE) mode, the electric field is orthogonal to the plane of incidence (parallel to the boundary plane).

In the transverse magnetic (TM) mode, the magnetic field is orthogonal to the plane of incidence (parallel to the boundary plane).

A general electromagnetic wave can be broken down into the sum of a TE and TM component (the two modes form an orthogonal basis).

1.9.1 Transverse Electric Mode

The electric fields all point in the same direction orthogonal to the plane of incidence, so their amplitudes must satisfy the condition:

$$E_i + E_r = E_t \tag{1.34}$$

The direction of the magnetic fields is given by the right hand rule. If we do this and set the components of the magnetic fields parallel to the boundary plane equal to each other, we find that their amplitudes must satisfy the condition:

$$-B_i \cos(\theta) + B_r \cos(\theta) = -B_t \cos(\phi) \tag{1.35}$$

Note that since light travels at some speed $v = \frac{1}{\sqrt{\mu\epsilon}}$ in a medium, the definition of $\mathbf{H}$ becomes $\mathbf{H} = v\mathbf{B}$. We can therefore rewrite formula 35 as:

$$-\frac{H_i}{v_1}\cos(\theta) + \frac{H_r}{v_1}\cos(\theta) = -\frac{H_t}{v_2}\cos(\phi) \tag{1.36}$$

Since n is defined as $\frac{c}{v}$, this equation can be rewritten as

$$-n_1 H_i \cos(\theta) + n_2 H_r \cos(\theta) = -H_t \cos(\phi) \tag{1.37}$$

Recall from equation 25 that $E = H$. We therefore have the condition:

$$-n_1 E_i \cos(\theta) + n_2 E_r \cos(\theta) = -E_t \cos(\phi) \tag{1.38}$$

Equations 34 and 38 are together sufficient to determine the magnitude of the reflected and transmitted electromagnetic field. To get a more intuitive view, however, we can define the reflection and transmission coefficients as $r_s = \frac{E_r}{E_i}$ and $t_s = \frac{E_t}{E_i}$ respectively.

From equations 34 and 38 we get

$$r_e = \frac{n_1 \cos(\theta) - n_2 \cos(\phi)}{n_1 \cos(\theta) + n_2 \cos(\phi)} \tag{1.39}$$

$$t_e = \frac{n_1 \cos(\theta) + n_2 \cos(\phi)}{2n_1 \cos(\theta)} \tag{1.40}$$

1.9.2 Transverse Magnetic Mode

We can determine the boundary conditions in the transverse magnetic mode using the right hand rule as before.

$$B_i - B_r = B_t \tag{1.41}$$

$$E_i \cos(\theta) + E_r \cos(\theta) = E_t \cos(\phi) \tag{1.42}$$

Following the same reasoning as before, equation 41 becomes

$$n_1 E_i - n_1 E_r = n_2 E_t \tag{1.43}$$

From equations 42 and 43, we can now derive the coefficients of reflection and transmission

$$r_m = \frac{n_1 \cos(\phi) - n_2 \cos(\theta)}{n_1 \cos(\phi) + n_2 \cos(\theta)} \tag{1.44}$$

$$t_m = \frac{n_1 \cos(\phi) + n_2 \cos(\theta)}{n_1 \cos(\theta)} \tag{1.45}$$

2 Electromagnetic Waves In Matter

2.1 Preface

In this section we will consider how EM waves propagate through matter. We will looks at three types of matter: dielectrics (insulators), conductors, and crystals.

2.2 Introduction

To tackle this topic, we will need to bring back Maxwell's equations. Recall that these are:

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{2.1}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \tag{2.2}$$

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \tag{2.3}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{2.4}$$

Previously we dealt with Maxwell's equations in a vacuum, where there were no charges to provide currents. Now, however, there will be currents from the motion of the material's electrons. We will first need to consider how an oscillating electromagnetic wave makes the charges in a material move, and then figure out what effect the resulting current has on the electromagnetic wave.

2.3 Dielectrics

In a dielectric, electrons are bound to atoms. The binding force can be represented as an elastic force $-\kappa \mathbf{x}$, where $\mathbf{x}$ is the displacement of an electron from its equilibrium positions and κ is some proportionality constant. Additionally, when in motion the electrons will radiate energy–we can model this by introducing a damping force $-m\gamma\mathbf{v}$, where $\mathbf{v}$ is the velocity of an electron, m is its mass, and γ is some damping factor.

Let us now suppose that there is an electromagnetic wave propagating through the medium. The electromagnetic field will apply the Lorentz force $\mathbf{F} = e(\mathbf{E} + \mathbf{v} \times \mathbf{B})$ where e is the charge of an electron. Recall, however, that the magnetic field $\mathbf{B}$ is given by the electric field divided by speed of the electromagnetic wave. This speed is usually very large, so the magnetic field is negligible compared to the electric field. Thus, the force on an electron reduces to $\mathbf{F} = e\mathbf{E}$.

With these realizations, we can write an electron's equation of motion:

$$m\frac{d^2\mathbf{x}}{dt^2} = -\kappa \mathbf{x} - m\gamma \frac{d\mathbf{x}}{dt} + e\mathbf{E} \tag{2.5}$$

where m is the mass of an electron. This simplifies to

$$\frac{d^2\mathbf{x}}{dt^2} + \gamma \frac{d\mathbf{x}}{dt} + \omega_0^2 \mathbf{x} = \frac{e}{m}\mathbf{E} \tag{2.6}$$

where we have defined $\omega_0^2 = \frac{\kappa}{m}$. ω_0 can be interpreted as the angular frequency of the oscillation without the damping and driving forces.

We can now assume that the electric field at the electron's position is $\mathbf{E} = \mathbf{E_0} e^{-i\omega t}$ and that the displacement takes the same form $\mathbf{x} = \mathbf{x_0} e^{-i\omega t}$. Substituting these into equation 6, we get

$$-\omega^2 \mathbf{x_0} e^{-i\omega t} - i\omega\gamma \mathbf{x_0} e^{-i\omega t} + \omega_0^2 \mathbf{x_0} e^{-i\omega t} = \frac{e}{m} \mathbf{E_0} e^{-i\omega t} \tag{2.7}$$

which simplifies to the expression

$$-\omega^2 \mathbf{x_0} - i\omega\gamma \mathbf{x_0} + \omega_0^2 \mathbf{x_0} = \frac{e}{m} \mathbf{E_0} \tag{2.8}$$

Thus we find that

$$\mathbf{x} = \frac{e/m}{\omega_0^2 - \omega^2 - i\omega\gamma} \mathbf{E} \tag{2.9}$$

2.3.1 Current Density

Consider now a snapshot in time where the electrons are traveling at some velocity $\mathbf{v}$. How can we determine the current density?

Picture a cylinder of electrons with cross-sectional area A. This cylinder is moving at the velocity v along its long axis. Take the are currently at the front face of the cylinder. After some amount of time Δt, the thickness of the section of cylinder that will move through this area is $v\Delta t$. The charge contained in this section of the cylinder is $Q = \rho_e \cdot A \cdot v\Delta t$, where ρ_e is the electron charge density (charge per unit volume). The current is therefore $\mathbf{I} = \frac{Q}{\Delta t} = \rho_e A \mathbf{v}$. The current density (current per cross-sectional area is then)

$$\mathbf{J} = \frac{\mathbf{I}}{A} = \rho_e \mathbf{v} \tag{2.10}$$

Keep this expression in mind–we will invoke it in a number of places.

2.3.2 Solving Maxwell's Equations in a Dielectric

Returning to our problem, we can now find the current density.

$$\mathbf{J} = \rho_e \mathbf{v} = \rho_e \frac{d\mathbf{x}}{dt} \tag{2.11}$$

Substituting x given by equation 9, we get

$$\mathbf{J} = \frac{\rho_e e/m}{\omega_0^2 - \omega^2 - i\omega\gamma} \frac{d\mathbf{E}}{dt} \tag{2.12}$$

We can rewrite this in a slightly more convenient way

$$\mathbf{J} = \frac{Ne^2/m}{\omega_0^2 - \omega^2 - i\omega\gamma} \frac{d\mathbf{E}}{dt} \tag{2.13}$$

where N is the number of electrons per unit volume (so that $\rho_e = Ne$). We are now have everything we need to solve Maxwell's equations.

Let us begin by taking the curl of equation 1:

$$\nabla \times (\nabla \times \mathbf{E}) = -\frac{\partial \nabla \times \mathbf{B}}{\partial t} \tag{2.14}$$

We can now substitute equation 2 for $\nabla \times \mathbf{B}$.

$$\nabla \times (\nabla \times \mathbf{E}) = -\mu_0 \frac{\partial \mathbf{J}}{\partial t} - \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2} \tag{2.15}$$

Recall the identity

$$\nabla \times (\nabla \times\ ) = \nabla(\nabla \cdot\ ) - \nabla^2 \tag{2.16}$$

Since we are dealing with an electrically neutral medium, the overall charge density will be $\rho = 0$, so $\nabla \cdot \mathbf{E} = 0$. Using these two facts to simplify the left hand side of equation 15 and plugging in our expression for current (equation 13) into the right hand side, we get

$$\nabla^2 \mathbf{E} = \left(\frac{\mu_0 N e^2/m}{\omega_0^2 - \omega^2 - i\omega\gamma} + \mu_0 \epsilon_0\right) \frac{\partial^2 \mathbf{E}}{\partial t^2} \tag{2.17}$$

Which we can rewrite more neatly as

$$\nabla^2 \mathbf{E} = \frac{1}{c^2}\left(\frac{Ne^2}{m\epsilon_0} \cdot \frac{1}{\omega_0^2 - \omega^2 - i\omega\gamma} + 1\right) \frac{\partial^2 \mathbf{E}}{\partial t^2} \tag{2.18}$$

For simplicity, we can assume a homogeneous plane harmonic wave traveling in one direction, such that $\mathbf{E} = \mathbf{E_0} e^{i(Kz - \omega t)}$, we get

$$K^2 = \frac{1}{c^2}\left(\frac{Ne^2}{m\epsilon_0} \cdot \frac{1}{\omega_0^2 - \omega^2 - i\omega\gamma} + 1\right)\omega^2 \tag{2.19}$$

Notice that the relation between k and ω is no longer trivial. The speed of the wave now depends on its frequency, leading to dispersion.

Notice also that the wavenumber K is now in general complex. We can thus write it as $K = k + i\alpha$, where k is the real part of K and α is the imaginary part.

Plugging this into our expression for $\mathbf{E}$, we get

$$\mathbf{E} = \mathbf{E_0} e^{-\alpha z} e^{i(kz - \omega t)} \tag{2.20}$$

The $e^{-\alpha z}$ term tells us that the amplitude of the wave decreases exponentially with distance into the material. In other words, as an electromagnetic wave travels through the medium, its energy is absorbed by the medium.

With our complex wavenumber, we can define a complex index of refraction:

$$\frac{\omega}{K} = \frac{c}{\mathcal{N}} \tag{2.21}$$

which simplifies to

$$\mathcal{N} = \frac{c}{\omega} K \tag{2.22}$$

We can rewrite the complex index of refraction in terms of its real and imaginary parts:

$$\mathcal{N} = n + i\kappa \tag{2.23}$$

Using expression 22, these can be written in terms of the components of the complex wavenumber:

$$n = \frac{c}{\omega} k \tag{2.24}$$

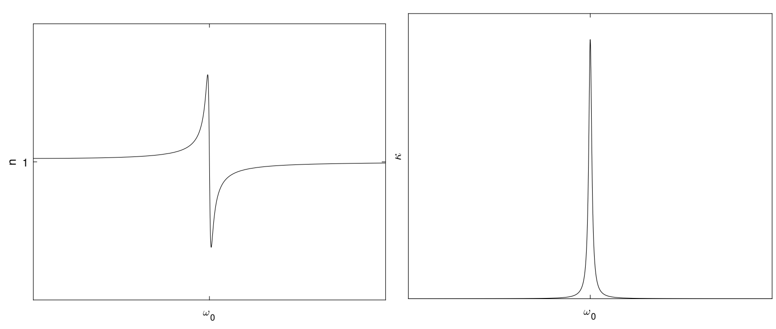
$$\kappa = \frac{c}{\omega} \alpha \tag{2.25}$$

Thus n is our real refractive index and acts as a measure of the wave's speed, while κ–called the extinction coefficient–is a measure of how quickly the wave amplitude decays.

Let us examine the complex index of refraction in more detail. We will begin by using equation 19 and 22 to write the expression for the complex index of refraction:

$$\mathcal{N}^2 = \frac{Ne^2}{m\epsilon_0} \cdot \frac{1}{\omega_0^2 - \omega^2 - i\omega\gamma} + 1 \tag{2.26}$$

Now let us plot the refractive index (the real part of $\mathcal{N}$) and the extinction coefficient (the imaginary part of $\mathcal{N}$).



((a)) n=Real{$\mathcal{N}$} ((b)) κ=Imag{$\mathcal{N}$}

Figure 2.1: 2 Figures side by side

We see that the refractive index n is above 1 and increases as the frequency of light approaches the resonant frequency ω_0. This causes dispersion–i.e. light with different frequencies is diffracted differently. We will frequently see this gradual increase of the real index of refraction with frequency. However when we get very close to the resonant frequency, the dispersion relation becomes "anomalous" and the index of refraction quickly drops. This anomalous dispresion can be demonstrated experimentally in some materials.

We also see that the absorption coefficient κ is high near the resonant frequency and drops to 0 elsewhere. This means that light whose frequency is near the resonant frequency of the electrons is absorbed while light with sufficiently different frequencies can pass through the material. This should make intuitive sense–if the frequency of light is different from the resonant frequency, the electron oscillations cannot acquire a large amplitude (think of pushing a swing out of sync), meaning that little energy needs to be lost to driving their oscillations. If light is at the resonant frequency, a lot of energy goes into building up large electron oscillations (think of pushing a swing in sync).

2.3.3 Different Binding Strengths

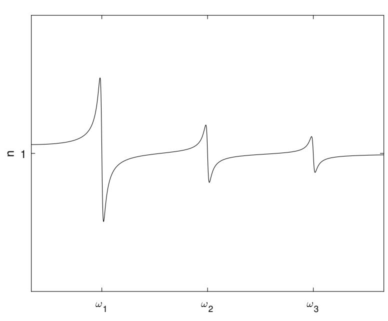
So far we have assumed that all electrons have the same binding strength, however this is not necessarily true. In general, we can assume that a fraction f_1 of the electrons are bound such that they have a resonant frequency ω_1, a fraction f_2 has resonant frequency ω_2, and so on. The resulting dispersion relation is then

$$K^2 = \frac{1}{c^2}(1 + \frac{Ne^2}{m\epsilon_0} \cdot \sum_j \frac{f_j}{\omega_j^2 - \omega^2 - i\omega\gamma_j})\omega^2 \tag{2.27}$$

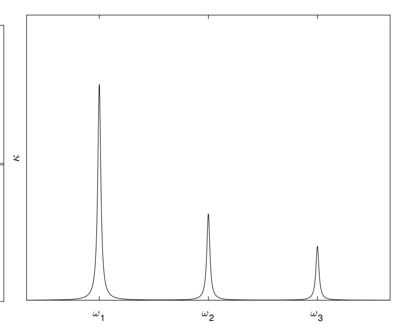
or equivalently

$$\mathcal{N}^2 = 1 + \frac{Ne^2}{m\epsilon_0} \cdot \sum_j \frac{f_j}{\omega_j^2 - \omega^2 - i\omega\gamma_j} \tag{2.28}$$

Below is a plot of the index of refraction and coefficient of extinction for a dielectric material with three resonant frequencies ω_1, ω_2, and ω_3.



((a)) n=Real{$\mathcal{N}$}



((b)) κ=Imag{$\mathcal{N}$}

Figure 2.2: 2 Figures side by side

We can see from figure b that the material has absorption bands around the resonant frequencies. We an also see that the resonant frequencies break up the index of refraction into separate regimes.

2.4 Conductors

2.4.1 Current Density

In a conductor, electrons are not bound to atoms. The elastic binding force therefore disappears, and we are left with only the damping force and the driving electromagnetic field. For reasons that will soon become clear, instead of writing the damping force as $-m\gamma\mathbf{v}$, we will write it as $-m\tau^{-1}\mathbf{v}$. The equation of motion for an electron is therefore.

$$m\frac{d\mathbf{v}}{dt} + m\tau^{-1}\mathbf{v} = e\mathbf{E} \tag{2.29}$$

Assuming $\mathbf{E} = \mathbf{E_0}e^{-i\omega t}$ and $\mathbf{v} = \mathbf{v_0}e^{-i\omega t}$, equation 27 becomes

$$-im\omega\mathbf{v} + m\tau^{-1}\mathbf{v} = e\mathbf{E} \tag{2.30}$$

from which we get

$$\mathbf{v} = \frac{e/m}{\tau^{-1} - i\omega}\mathbf{E} \tag{2.31}$$

The current density is therefore given by

$$\mathbf{J} = \rho_e\mathbf{v} = \frac{\rho_e e/m}{\tau^{-1} - i\omega}\mathbf{E} = \frac{Ne^2/m}{\tau^{-1} - i\omega}\mathbf{E} \tag{2.32}$$

where N is the number of electrons per unit volume. This gives us the steady state solution, but there will also in general be transient terms. These terms come from the linearity of the equation of motion (29). Namely, if we take any $\mathbf{v}$ that is a solution to the associated homogeneous differential equation

$$m\frac{d\mathbf{v}}{dt} + m\tau^{-1}\mathbf{v} = 0 \tag{2.33}$$

then adding that $\mathbf{v}$ to the exponential solution we found is equivalent to adding 0 to the right hand side of equation 29. Thus, we still have a valid solution.

It can be easily verified that the solution to equation 33 is $\mathbf{v} = \mathbf{v_0}e^{-\frac{t}{\tau}}$. It is now clear why we chose to use the letter τ–it is a measure of the time that it takes for the transient term in the current generated by the electrons to decay! This is often called the relaxation time.

Since the transient term decays exponentially, and since the relaxation time is usually very short, we only need to consider the steady-state solution we found for $\mathbf{J}$.

2.4.2 EM Wave in a Conductor

We now have everything we need to solve Maxwell's equations in a conductor. We use the same method that we used in section 2.3.2 to get

$$\nabla^2\mathbf{E} = \frac{1}{c^2}\left(\frac{Ne^2}{m\epsilon_0} \cdot \frac{1}{\tau^{-1} - i\omega}\frac{\partial\mathbf{E}}{\partial t} + \frac{\partial^2\mathbf{E}}{\partial t^2}\right) \tag{2.34}$$

Once again assuming a plane harmonic wave with an electric field of the form $\mathbf{E} = \mathbf{E_0} e^{i(Kz - \omega t)}$, we get an expression for the complex wavenumber and the complex index of refraction

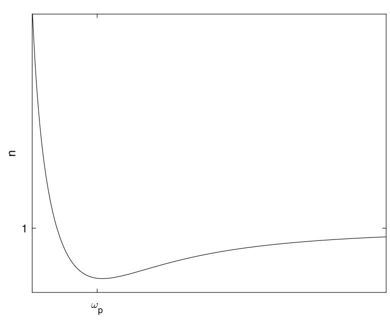
$$K^2 = \frac{1}{c^2}\left(\frac{Ne^2}{m\epsilon_0} \cdot \frac{-i\omega}{\tau^{-1} - i\omega} + \omega^2\right) \tag{2.35}$$

$$\begin{aligned} N^2 &= \frac{Ne^2}{m\epsilon_0} \cdot \frac{-i}{\omega\tau^{-1} - i\omega^2} + 1 \\ &= 1 - \frac{Ne^2}{m\epsilon_0} \cdot \frac{1}{\omega^2 + i\omega\tau^{-1}} \end{aligned} \tag{2.36}$$

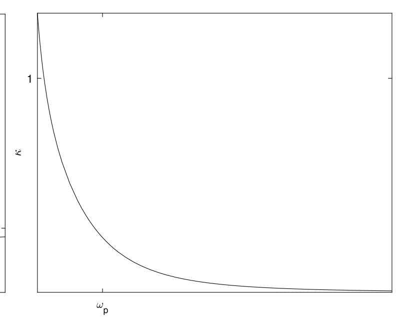
We often simplify this formula by defining a quantity called plasma frequency as $\omega_p^2 = \frac{Ne^2}{m\epsilon_0}$. Our expression can then be rewritten as

$$\mathcal{N} = 1 - \frac{\omega_p}{\omega^2 + i\omega\tau^{-1}} \tag{2.37}$$

As before, we can plot the index of refraction and extinction coefficient to get a picture of how light interacts with conductors.



((a)) n=Real{$\mathcal{N}$}



((b)) κ=Imag{$\mathcal{N}$}

Figure 2.3: 2 Figures side by side

As the frequency of light drops below the plasma frequency of a conductor, the conductor's index of refraction and extinction coefficient rapidly increase. This makes conductors opaque and highly reflective. Above the plasma frequency, however, the coefficient of extinction drops to zero and the material is transparent.

Plasma frequencies for typical metals are of the order of 10^{15} Hz, which corresponds to visible and near ultraviolet light. This is why most metals will tend to be appear opaque and reflective.

2.5 Poor Conductors and Semiconductors

In a poorly conducting or semiconducting material, both the bound and free electrons can contribute to the electromagnetic field. Combining our analysis from the preceding sections, the complex index of refraction of such a material is

$$\mathcal{N} = 1 - \frac{\omega_p^2}{\omega^2 + i\omega\tau^{-1}} + \frac{Ne^2}{m\epsilon_0} \cdot \sum_j \frac{f_j}{\omega_j^2 - \omega^2 - i\omega\gamma_j} \tag{2.38}$$

2.6 Reflection and Refraction

Let us now consider reflection and refraction with a complex index of refraction. For simplicity, let's assume that incident light is coming from a non-absorbing material (i.e. a material with a real index of refraction) and shining onto the boundary of a material with complex index of refraction $\mathcal{N} = n + i\kappa$ and a complex wavevector $\mathbf{K_t} = \mathbf{k_t} + i\alpha$. The incident, reflected, and transmitted waves can be written correspondingly as

$$\mathbf{E_i} = \mathbf{E_{i0}} e^{i(\mathbf{k_1}\cdot\mathbf{r} - \omega t)} \tag{2.39}$$

$$\mathbf{E_r} = \mathbf{E_{r0}} e^{i(\mathbf{k_1}\cdot\mathbf{r} - \omega t)} \tag{2.40}$$

$$\mathbf{E_t} = \mathbf{E_{t0}} e^{i(\mathbf{K_t}\cdot\mathbf{r} - \omega t)} = \mathbf{E_{t0}} e^{-\alpha\cdot\mathbf{r}} e^{i(\mathbf{k_t}\cdot\mathbf{r} - \omega t)} \tag{2.41}$$

As before, the exponentials must be equal at the boundary, leading to the requirement

$$\mathbf{k_1} \cdot \mathbf{r} = \mathbf{K_t} \cdot \mathbf{r} = (\mathbf{k_t} + i\alpha) \cdot \mathbf{r} \tag{2.42}$$

Setting the real and imaginary portions of this equation equal, we get

$$\mathbf{k_1} \cdot \mathbf{r} = \mathbf{k_t} \cdot \mathbf{r} \tag{2.43}$$

$$0 = \alpha \cdot \mathbf{r} \tag{2.44}$$

The expression $\alpha \cdot \mathbf{r} = 0$ tells us that α is orthogonal to the boundary plane. The planes of constant amplitudes–that is the planes where $\alpha \cdot \mathbf{r} = \textit{constant}$–must therefore be parallel to the boundary plane.

We can also see that $\mathbf{k_t}$ can point in any direction. Thus, in general $\mathbf{k_t}$ does not have to point in the same direction as α. A wave where this is the case is called inhomogeneous.

If we denote the angle of incidence as θ and the angle of refraction as ϕ, equation 44 becomes

$$k_1 \sin(\theta) = k_t \sin(\phi) \tag{2.45}$$

To find the angle of refraction from this expression, we still need to know k_t. To find k_t, we must go back to the wave equation

$$\nabla^2 \mathbf{E} = \frac{\mathcal{N}^2}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} \tag{2.46}$$

Plugging in our solution for $\mathbf{E_t}$, we get

$$\mathbf{K} \cdot \mathbf{K} = \frac{\mathcal{N}^2}{c^2}\omega^2 = \mathcal{N}^2 k_1^2 \tag{2.47}$$

where we have used $k_1 = \omega/c$. Note that in this case we are denoting $c = 1/\sqrt{\mu\epsilon}$, that is–the speed of light in the nonabsorbing medium.

Writing out the expression in terms of its real and imaginary parts we get

$$(\mathbf{k_t} + i\alpha) \cdot (\mathbf{k_t} + i\alpha) = (n + i\kappa)^2 k_1^2 \tag{2.48}$$

We can then equate the real and imaginary parts to get

$$\mathbf{k_t} \cdot \mathbf{k_t} - \alpha \cdot \alpha = k_t^2 - \alpha^2 = (n^2 - \kappa^2) k_1^2 \tag{2.49}$$

$$\mathbf{k_t} \cdot \alpha = k_t \alpha \cos(\phi) = n\kappa k_1^2 \tag{2.50}$$

These two expressions can be reduced to

$$k_t \cos(\phi) + i\alpha = k_1 \sqrt{\mathcal{N}^2 - \sin^2(\theta)} \tag{2.51}$$

Thus we can find k_t and the angle of refraction from equations 46 and 52.

We can greatly simplify further analysis by defining a complex angle of refraction

$$\mathcal{N} = \frac{\sin(\theta)}{\sin(\phi)} \tag{2.52}$$

The sin of a complex angle is defined as $\sin(\theta) = \frac{i}{2}(e^{-i\theta} - e^{i\theta})$. Equivalently you can treat it as plugging in a complex number into the Taylor expansion of sin.

Equation 53 can be equivalently rewritten as

$$\cos(\phi) = \sqrt{1 - \frac{\sin^2(\theta)}{\mathcal{N}^2}} \tag{2.53}$$

This in turn can be used to write another expression for the complex index of refraction:

$$\mathcal{N} = \frac{k_t \cos(\phi) + i\alpha}{k_1 \cos(\phi)} \tag{2.54}$$

This formula will make dealing with reflection and refraction much nicer.

2.6.1 Amplitudes in Reflection and Refraction

To tackle the problem of amplitudes of the reflected and refracted waves, we will first determine how the $\mathbf{E}$ and $\mathbf{B}$ are related. Recall from the first section that $\mathbf{H} = \hat{k} \times \mathbf{E}$. Recall also that $\mathbf{H} = v\mathbf{B}$, $v = \frac{\omega}{k}$. We thus have

$$\mathbf{B_i} = \frac{1}{\omega}\mathbf{k_i} \times \mathbf{E_i} \tag{2.55}$$

$$\mathbf{B_r} = \frac{1}{\omega}\mathbf{k_r} \times \mathbf{E_r} \tag{2.56}$$

$$\mathbf{B_t} = \frac{1}{\omega}\mathbf{K_t} \times \mathbf{E_t} = \frac{1}{\omega}(\mathbf{k_t} + i\alpha_t) \times \mathbf{E_t} \tag{2.57}$$

As before, we will consider the transverse electric (TE) and transverse magnetic (TM) polarization.

In the TE case we have the conditions

$$E_i + E_r = E_t \tag{2.58}$$

$$-B_i \cos(\theta) + B_r \cos(\theta) = B_{t\ tangential} \tag{2.59}$$

Equation 60 becomes

$$-k_1 E_i \cos(\theta) + k_1 E_r \cos(\theta) = -k_t E_t \cos(\phi) - i\alpha_t E_t \tag{2.60}$$

We can now use expression 55 to rewrite this as

$$-E_i \cos(\theta) + E_r \cos(\theta) = -\mathcal{N} \cos(\phi)\mathbf{E_t} \tag{2.61}$$

From equations 59 and 62, we now find the coefficient of reflection

$$r_s = \frac{E_r}{E_i} = \frac{\cos(\theta) - \mathcal{N}\cos(\phi)}{\cos(\theta) + \mathcal{N}\cos(\phi)} \tag{2.62}$$

The analysis for the transverse magnetic case is very similar, and yields a coefficient of reflection of

$$r_p = \frac{-\mathcal{N}\cos(\theta) + \cos(\phi)}{\mathcal{N}\cos(\theta) + \cos(\phi)} \tag{2.63}$$

The amplitudes of the refracted wave can be similarly found from the amplitude of the reflected wave and the boundary conditions.

2.7 Qualitative Analysis

We will now look at examples of these properties in a hypothetical dielectric and conductor.

The MATLAB script used to generate these plots can be found at `https://github.com/DennisChunikhin/Light-Matter-Interactions/blob/main/optical_properties.m`.

2.7.1 Dielectric

Let's first look at a typical dielectric. Our hypothetical material will have an electron number density of 10^{27}. Half of the electrons will have a resonant angular frequency of $\omega_1 = 3.5 \times 10^{15}$ 1/s (corresponding to about 540 nm light) and half will have a resonant frequency of $\omega_2 = 7 \times 10^{15}$ 1/s (corresponding to about 270 nm light).

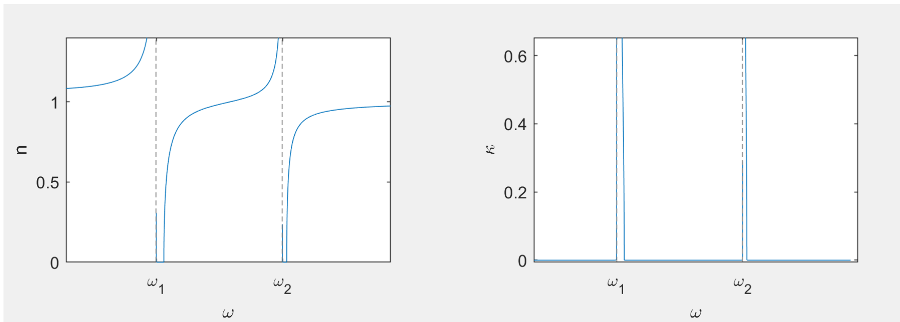


Figure 2.4: The real index of refraction and coefficient of extinction of the hypothetical dielectric.

As we saw earlier, the dielectric material exhibits sharp peaks in the coefficient of extinction near the resonant frequencies where a lot of energy is lost to driving the large electron vibrations. The real index of refraction falls into 3 regions, in each of which it gradually increases with light frequency (explaining the phenomenon of dispersion) until it reaches an anomalous region near the resonant frequencies, where it sharply drops.

Let's now graph the reflectance for light at one of the resonant frequencies.

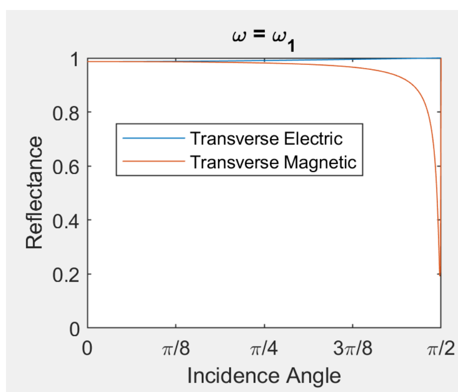


Figure 2.5: The reflectance of the hypothetical dielectric for light at the resonant frequency ω_1.

Nearly all of the light is reflected. This will generally be the case whenever the coefficient of extinction is high.

Interestingly, at a grazing angle of incidence significantly less of the transverse magnetic polarization is reflected. Light reflected at grazing incidence will therefore be partially polarized.

What about light below the resonant frequency?

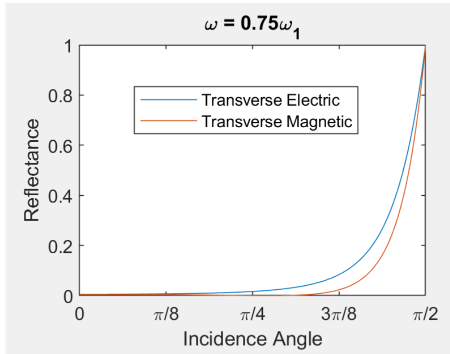


Figure 2.6: The reflectance of the hypothetical dielectric for light below the resonant frequency ω_1.

Except for a high reflectance at grazing incidence, most of the light is transmitted through the dielectric. This makes intuitive sense, as when the frequency is below resonance, electrons do not build up a great oscillation and therefore don't emit much of a wave, leaving the incident light fairly undisturbed.

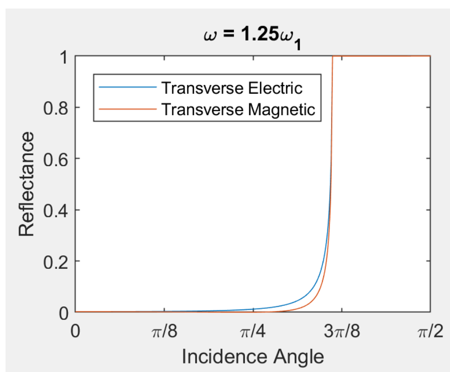
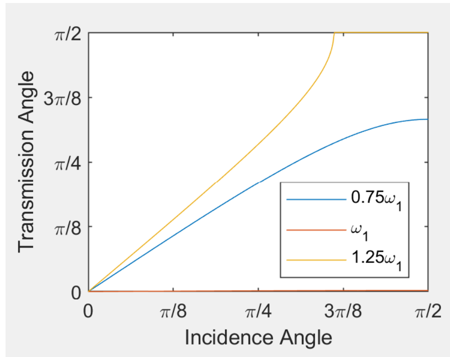


Figure 2.7: The reflectance of the hypothetical dielectric for light between the resonant frequencies ω_1 and ω_2.

We observe similar behavior for light at higher non-resonant frequencies, with the difference that in certain situations, there is a critical angle above which we get total reflection.

We can complete this discussion by plotting the angle of refraction for the three frequencies of light we discussed.



This provides us with another qualitative picture that supports our conclusions.

2.7.2 Conductor

Our hypothetical conductor will have a typical plasma frequency of 10^{15} Hz and a decay time of 6×10^{-13} seconds.

We start as before, by plotting the real index of refraction and coefficient of extinction.

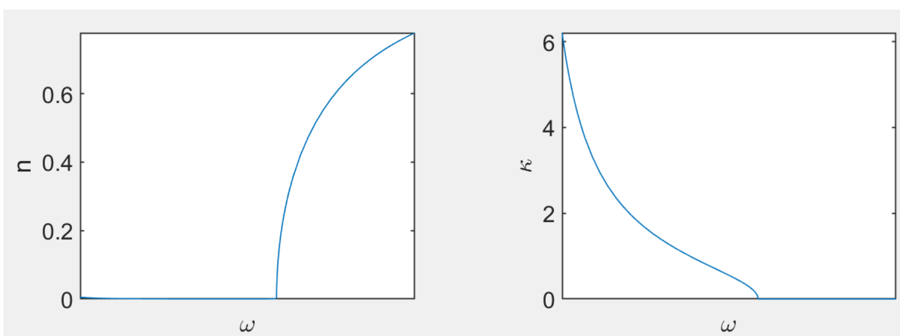


Figure 2.8: Real index of refraction and coefficient of extinction for the hypothetical conductor.

The graphs are broken into two regions meeting at the plasma frequency. Below the plasma frequency, the metal has a nearly zero index of refraction and high coefficient of extinction, indicating that it should be highly reflective. Above the plasma frequency, the coefficient of extinction drops down to near zero, indicating that the material should be transparent.

The reflectance at the plasma frequency is plotted below.

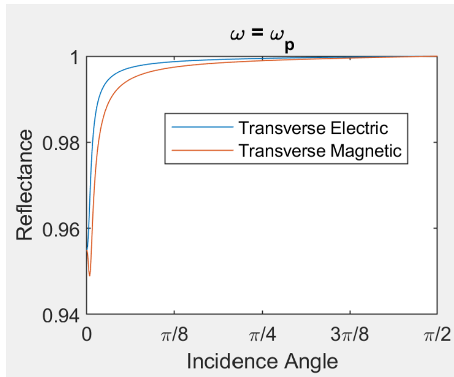


Figure 2.9: Reflectance of the hypothetical conductor for light at its plasma frequency.

It is indeed very close to 1. If we decrease the frequency of light further, the reflectance will continue to approach 1. Since 10^{15} Hz is a typical plasma frequency for many metals and represents light in the UV range, we can now understand why most metals appear shiny.

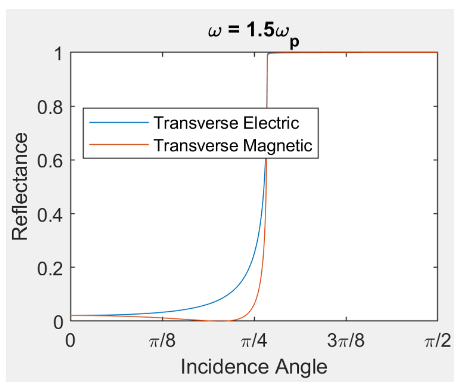


Figure 2.10: Reflectance of the hypothetical conductor for light above its plasma frequency.

Above the plasma frequency, metals will be almost entirely transparent up to some critical angle.

We will conclude by plotting the angle of transmission.

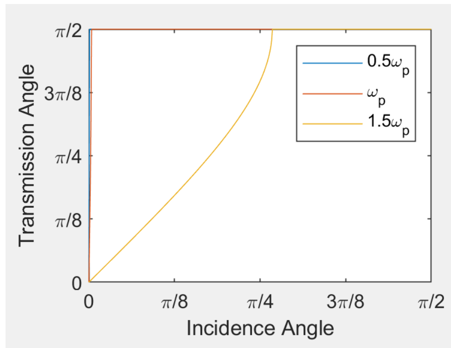


Figure 2.11: Angle of transmission for the hypothetical conductor.

This graph gives us the same information.

3 Nonlinear Optics

3.1 Electric Polarization

In section 2, we implicitly averaged over the motions of individual valence electrons and neglected inner electrons since they are bound much more tightly to atoms. However higher electric field energies can affect inner electrons and create complications in our formalism, so it becomes more useful to instead treat atoms as dipoles and talk about them in agregate.

The electric dipole moment is a measure of the separation of electric charges. Two point charges +q and -q separated by distance d constitute the dipole $\mathbf{p} = q\mathbf{d}$, where $\mathbf{d}$ is the directed displacement vector pointing from -q to +q.

When dealing with many atoms in aggregate, we define a quantity called electric polarization as the volume density of electric dipoles: $\mathbf{P} = \frac{\mathbf{p}}{V}$.

A passing electric field induces electric dipoles in a dielectric material. When we have an EM wave with a changing electric field, the electric dipoles also change in time (since, as we have discussed in the previous section, the oscillating electric field causes bound electrons to oscillate). This in turn creates an electric current. Recall that the current density is $\mathbf{J} = \rho_b\mathbf{v}$, where ρ_b is the charge density of the bound charges and $\mathbf{v}$ is their velocity. In terms of electric dipoles, we can thus see that

$$\mathbf{J} = \rho_b\mathbf{v} = \frac{q}{V}\frac{d\mathbf{d}}{dt} \tag{3.1}$$

Recalling that $\mathbf{P} = \frac{q\mathbf{d}}{V}$, we see that the bound electrons generate current density

$$\mathbf{J_b} = \frac{d\mathbf{P}}{dt} \tag{3.2}$$

What happens if the electric polarization is not uniform? If this is the case, we can take an arbitrary volume, and may find some net charge within it as a result of more charges entering from one side than exiting from another.

Let's first let's imagine how much charge moves through a surface. From the formula $\mathbf{P} = \frac{q\mathbf{d}}{V}$ we see that the charge should be proportional to the component of polarization tangent to the normal vector of the area: $dQ = -\mathbf{P} \cdot \mathbf{dA}$. Over a closed surface, this becomes

$$\Delta Q = -\int_S \mathbf{P} \cdot \mathbf{dA} \tag{3.3}$$

By Gauss' theorem, we have

$$\int_S \mathbf{P} \cdot \mathbf{dA} = \int_V \nabla \cdot \mathbf{P} dV \tag{3.4}$$

We also have

$$\Delta Q = \int_V \rho_b dV \tag{3.5}$$

Combining equations 4 and 5, we get

$$\int_V \rho_b dV = -\int_V \nabla \cdot \mathbf{P} dV \tag{3.6}$$

$$\rho_b = -\nabla \cdot \mathbf{P} \tag{3.7}$$

Thus we have an expression for the bound charge density.

3.1.1 Magnetization

We can analogously define the magnetization of a magnetic material to be the volume density of magnetic dipoles $\mathbf{M} = \frac{\mathbf{m}}{V}$.

We must now make a more formal distinction between **B** and **H**. **B** in the way we used and will continue to use it represents the magnetic flux density. In a magnetic material, however, it is easier to deal with a quantity called the magnetic field strength. It is conventional to denote this quantity using the letter **H**, which we will do (note that this is different to how we have used **H** in previous sections). In a vacuum, the magnetic flux density and the magnetic field strength are related by $\mathbf{H} = \frac{\mathbf{B}}{\mu_0}$. More generally, in a magnetic material we define **H** so that

$$\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M}) \tag{3.8}$$

3.1.2 Maxwell's Equations

In Maxwell's equations, we can split up $\rho = \rho_{free} + \rho_b$ and $J = J_{free} + J_b$. Substituting expressions 2 and 3 into Maxwell's equations, we then get

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{3.9}$$

$$\nabla \times (\mathbf{H} + \mathbf{M}) = (\mathbf{J_{external}} + \frac{\partial \mathbf{P}}{\partial t}) + \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \tag{3.10}$$

$$\nabla \cdot \mathbf{E} = \frac{\rho_{free}}{\epsilon_0} - \frac{\nabla \cdot \mathbf{P}}{\epsilon_0} \tag{3.11}$$

$$\nabla \cdot \mathbf{H} = -\nabla \cdot \mathbf{M} \tag{3.12}$$

$\nabla \times \mathbf{M}$, called the magnetization current density, can be thought of as the magnetization's contribution to the external current density. We can simply combine this and $\mathbf{J_{external}}$ into a single current density **J**, leaving us with

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{3.13}$$

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{P}}{\partial t} + \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \tag{3.14}$$

$$\nabla \cdot \mathbf{E} = \frac{\rho_{free}}{\epsilon_0} - \frac{\nabla \cdot \mathbf{P}}{\epsilon_0} \tag{3.15}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{3.16}$$

Often times a quantity called electric displacement is defined as $\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P}$ to compress equation 14 and 15 into

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \tag{3.17}$$

$$\nabla \cdot \mathbf{D} = \rho_{free} \tag{3.18}$$

For our purposes, however, it will be easiest to deal with the electric polarization.

3.2 Higher Harmonic Generation

In section 2, we built a simplified model of a dielectric wherein the outer electrons were bound to the nuclei by a spring force. As the strength of the electric field in an EM wave increases, however, this model starts to break down. Particularly, as the electric field amplitude reaches the binding strength of the atoms, the inner electrons start oscillating as well, some atoms get ionized, and in general the dynamics get intractably complicated.

To skirt this issue, we can simply say that the electric polarization $\mathbf{P}$ becomes some complicated function of $\mathbf{E}$.

As we know, any function can be represented by its Taylor expansion. Our complicated electric polarization therefore becomes

$$\mathbf{P} = \epsilon_0 \left(\chi_1 \mathbf{E} + \chi_2 \mathbf{E}^2 + \chi_3 \mathbf{E}^3 + \chi_4 \mathbf{E}^4 + \cdots \right) \tag{3.19}$$

The Taylor coefficients χ are called electric susceptibilities. For most materials, they will drop off quite quickly with increasing order. To fully understand how the susceptibilities relate to the properties of a material requires the study of light propagation in crystals, which I have not yet included in this document.

What happens when we have an electromagnetic wave with electric field $\mathbf{E} = \mathbf{E_0} e^{i(kx - \omega t)}$? The induced polarization is

$$\mathbf{P} = \epsilon_0 \left(\chi_1 \mathbf{E_0} e^{i(kx-\omega t)} + \chi_2 \mathbf{E_0}^2 e^{i(2kx - 2\omega t)} + \chi_3 \mathbf{E_0}^3 e^{i(3kx - 3\omega t)} + \cdots \right) \tag{3.20}$$

We can see that there are components of the electric polarization that oscillate at frequencies 2ω, 3ω, and so on. From our modified Maxwell's equations incorporating the electric polarization, we see that each of these must induce an electromagnetic wave oscillating at the same frequency. Thus, our electromagnetic wave obtains higher harmonics oscillating at integer multiples of the original frequency.

As was mentioned, in reality the amplitudes of the polarization terms, and therefore these higher harmonics, falls of extremely quickly with increasing order. Nevertheless, certain materials have, for example, a non-negligible χ_2, and can therefore be used to double the frequency of light by inducing the second harmonic.

3.3 Plasma Wakefield Acceleration

If the electromagnetic field of an EM wave is strong enough (as might be the case for a powerful laser), it can rip electrons free of their atoms, creating a plasma. The propagation of light in a plasma can then create some extremely interesting and useful effects. One of these, called wakefield acceleration, can be used to accelerate charged particles extremely rapidly.

Wakefield acceleration involves firing a very powerful and very short (usually in the 10s of femtoseconds) laser pulse into a plasma. The light pulse creates a wave (wake) in the plasma, which travels ever so slightly slower than the light pulse itself. Electrons, or any charged particles injected into the wake, can then accelerate by "surfing" it.

To discuss this process, we will first try to understand what happens when a laser pulse is fired into a plasma.

3.3.1 The Ponderomotive Force

Most laser light comes in the form of a Gaussian wave. A Gaussian wave is a good approximation of a plane wave, however the planes of equal phase have a slight curvature, and, most importantly, the intensity of the light decreases as a Gaussian function with distance from the center (called the optical axis). We can qualitatively show that this means that charged particles will be pushed away from the optical axis.

Picture a particle sitting somewhere along the optical axis wave. Let's say that the electric field begins to accelerate the particle away from the axis. By the time the electric field has reversed its direction, the particle will have moved some distance away and is now in a location where the electric field is weaker. Thus, the electron is pulled back less than it was initially pushed away. This process repeats, pushing the electron away from the optical axis throughout the course of many oscillations. This "average push" can be quantified as the ponderomotive force.

To quantify this force, we will start by examining the motion of an electron in the x direction. Consider a nonuniform oscillating electric field

$$E(x) = g(x)\cos(\omega t) \tag{3.21}$$

As in our discussion from section 2, the electric field will apply a Lorentz force on the electron, causing it to oscillate along with the field. We will assume that these oscillations are fast enough that throughout each oscillation, a particle stays in approximately the same region (that is, the length scale of $g(x)$ is much greater than that of the fast oscillations).

This assumption allows us to break up the electron's motion into a fast oscillation, and slow drift:

$$x = x_f + x_s \tag{3.22}$$

The equation of motion of the electron is therefore

$$m(\ddot{x}_f + \ddot{x}_s) = qE(x) \tag{3.23}$$

We can approximate E(x) using the first two terms of its Taylor expansion around x_s

$$E(x) = [g(x_s) + x_f g'(x_s)] \cos(\omega t) \tag{3.24}$$

Since x_f represents the quick oscillations while x_s represents a more gradual drift, we can assume that $x_f << x_s$. This implies that the x_f term in equation 24 is negligible, leaving us with

$$E(x) \approx g(x_s) \cos(\omega t) \tag{3.25}$$

We can use the fact that $\ddot{x}_s << \ddot{x}_f$ (since we've defined x_s to be the gradual drift and x_f to be the rapid oscillation) to approximate the electron's equation of motion (equation 23) as

$$m\ddot{x}_f \approx qg(x_s) \cos(\omega t) \tag{3.26}$$

We can then say that x_s is essentially constant on the timescale of x_f oscillations and, treating it as a constant variable, take two integrals with respect to time to get

$$x_f = -\frac{q}{m} \cdot \frac{g(x_s)}{\omega^2} \cos(\omega t) \tag{3.27}$$

Plugging $\ddot{x}_f$ and x_f into the force equation (23) and using the Taylor approximation of E (equation 24), we get

$$\ddot{x}_s + \frac{q}{m} g(x_s) \cos(\omega t) = \frac{q}{m} [g(x_s) + x_f g'(x_s)] \cos(\omega t) \tag{3.28}$$

This reduces to

$$\ddot{x}_s = \frac{q}{m} x_f g'(x_s) \cos(\omega t) \tag{3.29}$$

When we plug in the formula for x_f, we are left with

$$\ddot{x}_s = -\frac{q^2}{m^2\omega^2} g(x_s) g'(x_s) \cos^2(\omega t) \tag{3.30}$$

If we average over one oscillation, $\cos^2(\omega t)$ becomes $1/2$ and so our expression becomes

$$\ddot{x}_s = -\frac{q^2}{2m^2\omega^2} g(x_s) g'(x_s) \tag{3.31}$$

Notice that this is the same as

$$\ddot{x}_s = -\frac{q^2}{4m^2\omega^2} \frac{d}{dx} g(x_s)^2 \tag{3.32}$$

This expression represents the average drift of the particle. We formulate this as a ponderomotive force, which we define as

$$F_p(x_s) = m\ddot{x}_s = -\frac{q^2}{4m\omega^2} \frac{d}{dx} g(x_s)^2 \tag{3.33}$$

We can apply this same logic to the y and z directions and get analogous expressions. Noting that the function g is the amplitude of the electric field, we can package all of these expressions into one:

$$\mathbf{F_p}(x, y, z) = -\frac{q^2}{4m\omega^2}\nabla E(x, y, z)^2 \tag{3.34}$$

where E is the amplitude of the electric field.

What does this expression tell us? The gradient of E^2 points in the direction of the sharpest increase in electric field intensity. The converse is true for the negative gradient, meaning that charged particles are pushed towards wherever the electric field is weakest. Quite interestingly, the expression also tells us that this is true for both negatively and positively charged particles.

3.3.2 Plasma Wake Formation

In the case of our light pulse, the electric field is strongest at its center on the optical axis. The electrons in front of the light pulse will therefore be pushed forwards and away from the optical axis by the ponderomotive force.

As the electrons are swept from the optical axis, they bunch up in negatively charged "walls". Repelled by each other and attracted to the electron deficiency left near the axis, the electrons subsequently rush back inwards, where they again tightly bunch up. They now repel each other once again, and this cycle of moving towards and away from the optical axis continues periodically. The resulting effect is a wave of electron density following the light pulse.

Any electrons that end up stuck just behind a region of electron deficiency and just in front of a dense bunch of electrons in this wave are accelerated forwards, in the light pulse's direction of propagation.

3.3.3 Particle in Cell Simulations

Let us now see whether our qualitative description holds in a simulation.

Simulating every individual electron in a plasma is of course intractable. Thus, a family of simulation techniques called particle in cell simulations is often employed. A particle in cell simulation lumps electrons into 'macroparticles' representing groups of electrons, and breaks up space into a discrete grid. In each timestep, the simulation places field strengths on the grid using particle positions, solves the Maxwell's equations, interpolates the electromagnetic field at the particle positions, and pushes the particles using the fields. This is a cyclic process which quite nicely mirrors our intuition for dealing with fields in the first place.

We will use the Fourier-Bessel Particle-In-Cell (FBPIC) simulation. The method and its implementation in python is documented at `https://fbpic.github.io/index.html`.

3.3.4 Sample Simulation

We will simulate firing a 16 femtosecond long laser pulse into a plasma with an electron density of 4×10^{24} electrons per m^3. The wavelength of our laser will be 800 nm (red to infrared light), and the peak amplitude of its electric field will be about $4 \times 10^{12} \frac{N}{C}$. For a Gaussian laser pulse, this corresponds to a total power in the neighborhood of around 1 Terawatt.

The simulation script and results are included in the github repository `https://github.com/DennisChunikhin/Light-Matter-Interactions`.

We'll start by familiarizing ourselves with the laser pulse. The pulse travels in the positive z direction and is polarized in the x direction. The FBPIC simulation uses cylindrical coordinates, so we will plot z (distance along the optical axis) vs. r (distance from the optical axis). We will visualize the electric field amplitude using a color bar. The simulation window will move along with the laser (this feature is another optimization of FBPIC).

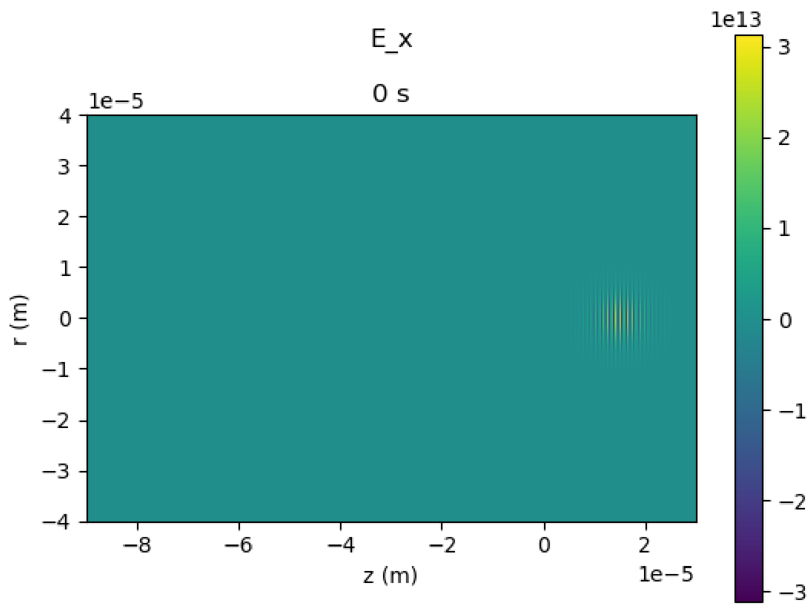


Figure 3.1: x component of the electric field at time 0s.

We can see the laser pulse at about the 2×10^{-5} meter mark. As expected, we have a sinusoidally alternating approximately planar wave whose intensity fades off from its center.

We are more interested in what happens to the plasma, so we will now skip a few frames forward and plot the electron charge density.

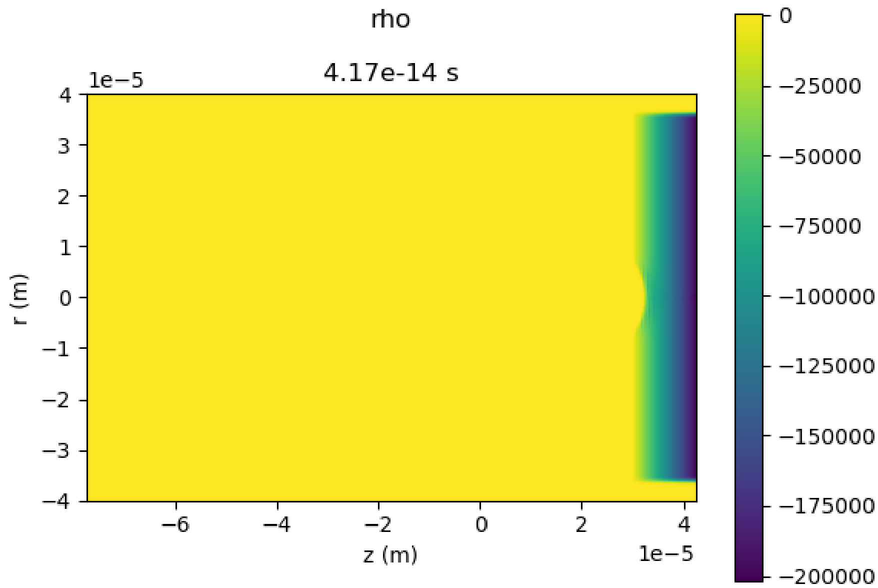


Figure 3.2: Electron charge density at time 41.7 femtoseconds.

The blue blob of negative charge density represents the start of the plasma. Near the optical axis, we see a small divot–this is the laser pulse beginning to interact with the plasma.

Skipping forward a few more time steps, we can see the effect of the ponderomotive force clearing electrons from the optical axis.

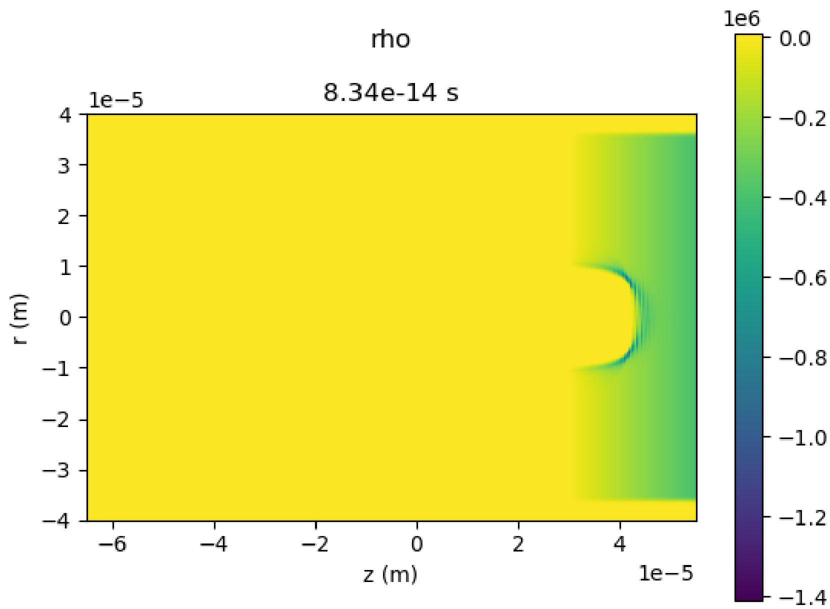


Figure 3.3: Electron charge density at time 83.4 femtoseconds.

Nearly all electrons have been cleared from the optical axis in the laser's wake, and these electrons have started bunching up. The resulting electrostatic force should now pull these electrons back towards the axis. Going forward a few more frames, we can see this starting to happen.

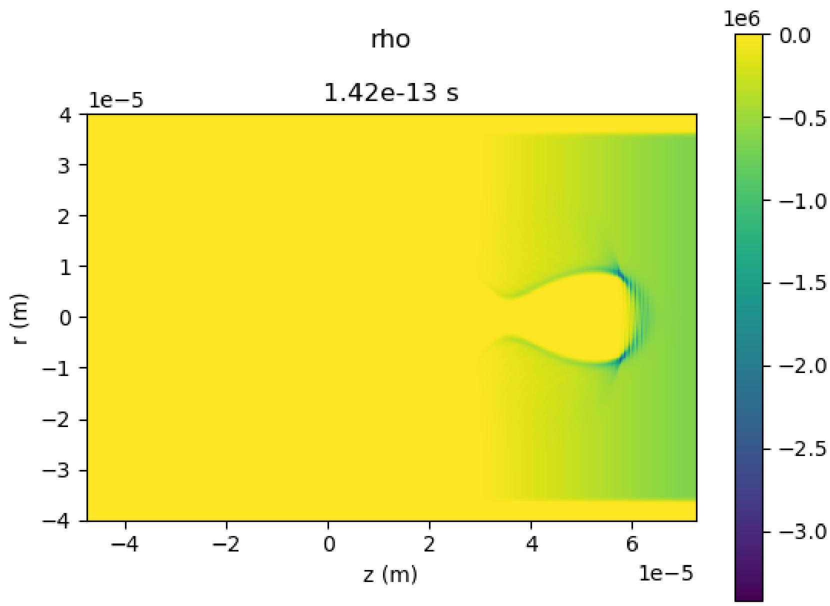


Figure 3.4: Electron charge density at time 142 femtoseconds.

Going forward a few timesteps again, we see that these electrons indeed bunch up on the optical axis.

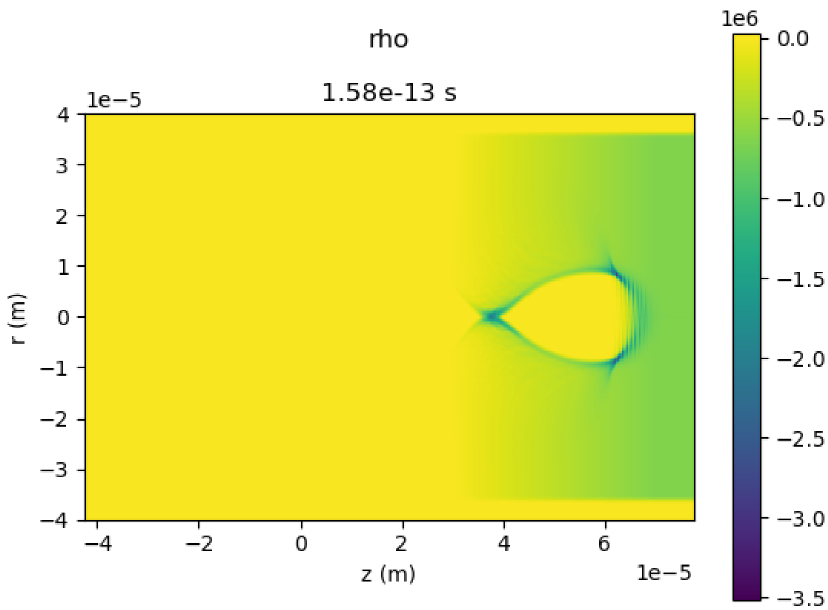


Figure 3.5: Electron charge density at time 158 femtoseconds.

This behavior repeats periodically, creating a string of ”bubbles” (often called ”buckets”) following the laser pulse, each nearly cleared of all electrons in the center and surrounded by negatively charged ”walls.”

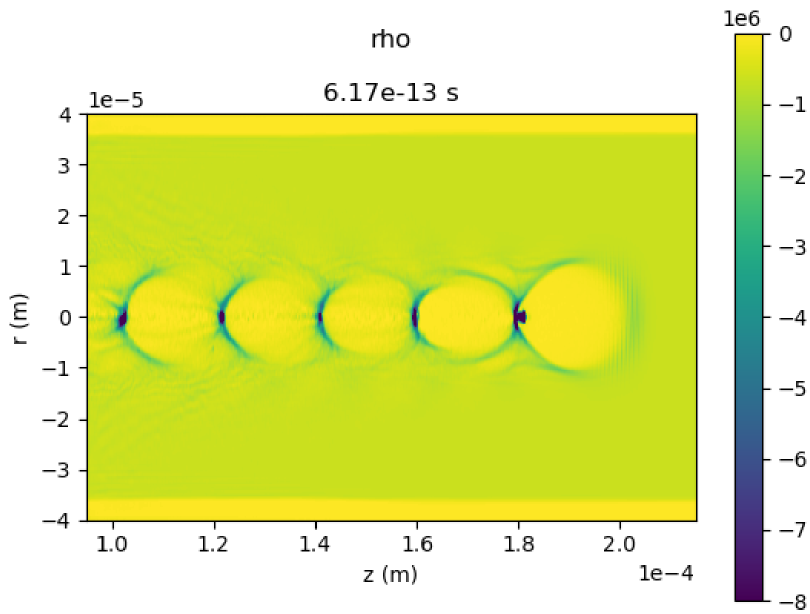


Figure 3.6: Electron charge density at time 61.7 picoseconds.

A nice animated figure showing this entire process is available at `https://github.com/DennisChunikhin/Light-Matter-Interactions/blob/main/Particle%20in%20Cell/Figures/rho.gif`.

Notice the negatively charged blue blob near the back of the rightmost bucket (at about the 1.8×10^{-4} meter mark). This represents a group of electrons trapped in the plasma wake. These electrons are repelled by the negatively charged region behind them and attracted to the electron deficiency in front of them. This creates a very intense acceleration gradient.

To better illustrate this, we can plot the electron momenta in the z direction normalized in units of $m_e c^2$.

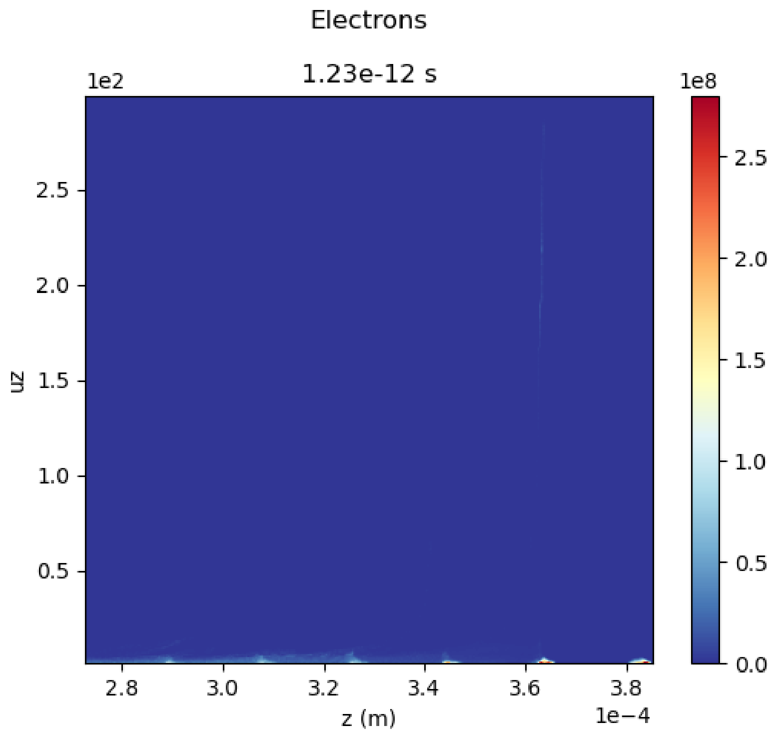


Figure 3.7: A 2D histogram of electron momenta in the z direction in units of $m_e c^2$. The color bar measures how many electrons are in each bin (i.e. the combined weights of the macroparticles in that bin).

We indeed see that bunches of electrons near the back of each bucket are accelerated. The effect is by far the most pronounced for the bucket just behind the laser pulse, where we have a large clump of trapped electrons.

An animated version of this plot is available at `https://github.com/DennisChunikhin/Light-Matter-Interactions/blob/main/Particle%20in%20Cell/Figures/electrons.gif`.

3.3.5 Describing the Plasma Wake

In cases where the laser intensity is below a certain threshold, the plasma wake behaves linearly and can be described using Poisson's equation for a cold fluid.

This is however beyond the scope of this paper (for now).