We say F is an antiderivative of f on an open set D if $F'(z) = f(z)$, $z \in D$

↳ $(F(z)+C)' = f(z)$

↗ Many times $\int_{z_1 \to z_2} f(z)\,dz$ is path-independent → depends only on endpoints z_1, z_2

If $\int_C f(z)\,dz$ is path independent in D, then integral on closed curve C_{closed} $\int_{C_{closed}} f(z)\,dz = 0$ and vice versa

Theorem: Let $f(z)$ be a continuous function on domain D. Then the following are equivalent:

(1) f has an antiderivative F on D

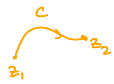
(2) $\int_{C,\, z_1 \to z_2} f(z)\,dz$ is path independent $\forall z_1, z_2 \in D$ and $C \subseteq D$

(3) $\oint_C f(z)\,dz = 0$ for every closed curve $C \subseteq D$

Proof: To prove, we must show $1 \to 2 \to 3 \to 1$

Proof $(1 \to 2)$: Suppose $F'(z) = f(z)$ $\forall z \in D$

Let $z_1, z_2 \in D$ and let C be a curve from z_1 to z_2, $C \subseteq D$



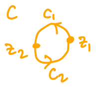
$C: z(t),\ 0 \le t \le 1$

$z(0) = z_1 \quad z(1) = z_2$

$$\int_C f(z)\,dz = \int_0^1 f(z(t))\, z'(t)\,dt = \int_0^1 \frac{d}{dt} F(z(t))\,dt = F(z(1)) - F(z(0)) = F(z_2) - F(z_1)$$

↓ Path independent!

Proof $(2 \to 3)$: Suppose (2). Let C be a closed curve in D.



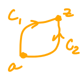
Take $z_1, z_2 \in C$

$C = C_1 \cup C_2$

$$\int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz = \underbrace{\int_{C_1} f(z)\,dz - \int_{-C_2} f(z)\,dz}_{\text{Same start \& end points}} = 0$$

Proof $(3 \to 1)$: Let $a \in D$

Define $F(z) = \int_{a \to z} f(w)\,dw$ for $z \in D$



Let C_1 and C_2 be two curves that go from a to z

Then $\int_{C_1} f(w)\,dw + \int_{-C_2} f(w)\,dw = 0$ by hypothesis

→ $\int_{C_1} f(w)\,dw = \int_{C_2} f(w)\,dw$ $\therefore F(z)$ is well-defined

We will now show that $F'(z)$ exists $\forall z \in D$ and $F'(z) = f(z)$ $\forall z \in D$

$$\frac{F(z)-F(a)}{z-a} - f(a) = \frac{1}{z-a}\int_{a \to z} f(w)\,dw - \frac{1}{z-a}\int_{a \to z} \overset{\swarrow \text{constant}}{f(a)}\,dw = \frac{1}{z-a}\int_{a \to z} (f(w)-f(a))\,dw$$

By ML $\left|\frac{1}{z-a}\int (f(w)-f(a))\,dw\right| \le \frac{L(C)\ \max|f(w)-f(a)|}{|z-a|} \to 0$

we can do this b/c D is open set—eventually as $w \to a$ we will be in nbhd entirely in D.

Since we're interested in $w \to a$, we take C straight line segment: $L(C) = |z-a|$, $|f(w)-f(a)| \to 0$

$\therefore F'$ exists at a and $F'(a) = f(a)$

Ex) $\int_{C_1} (z^2+1)\,dz$

$\frac{d}{dz}\left(\frac{z^3}{3}+z\right) = z^2+1$ on C

Thus, by thm., $\int_{C_1} (z^2+1)\,dz = \left.\frac{z^3}{3}+z\right|_{z_1}^{z_2}$

Ex) $\oint_C \frac{1}{z^2}\,dz = 0$ $\qquad \frac{d}{dz}\left(-\frac{1}{z}\right) = \frac{1}{z^2}$ $\qquad 0 \notin C$

Ex) $\oint \frac{1}{z}\,dz = 2\pi i \neq 0$

$\operatorname{Log} z$ not continuous on $-x$ axis $\rightarrow$ $\frac{1}{z}$ doesn't have antiderivative on C (doesn't exist on $-x$ axis) $\rightarrow$ thm. does not apply!

Ex) $\oint_C e^z\,dz$

$(e^z)' = e^z \quad \forall z \in \mathbb{C}$ $\quad \therefore$ by the thm. $\oint e^z\,dz = 0$

Ex) $\oint_C z\sin(z^2)\,dz$

$\frac{d}{dz}\left(-\frac{\cos(z^2)}{2}\right) = z\sin(z^2)$ on $\mathbb{C}$ $\quad \therefore \int_C z\sin(z^2)\,dz = 0$

Consider $\frac{1}{z}$

Let $C_R = \{z : |z| = R\}$

$z(\theta) = Re^{i\theta}$, $0 \le \theta \le 2\pi$

$\int_{C_R} \frac{1}{z}\,dz = \int_0^{2\pi} \frac{1}{Re^{i\theta}}\, iRe^{i\theta}\,d\theta = 2\pi i$

★

Let $D = \{z : 1 < |z| < 2\}$

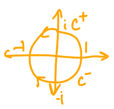
Is there F on D s.t. $F'(z) = \frac{1}{z} \quad \forall z \in D$

If there were such an F, by thm. the intergal over every closed curve would be 0 contrary to what we showed!

$\boxed{\frac{1}{z} \text{ does not have antiderivative (even though it is analytic) in } D!}$

↳ this is b/c $\log z$ is not continuous on D

Ex)



C^+, C^- are curves from 1 to -1

$\int_{C^+} \frac{1}{z}\,dz = \pi i \neq \int_{C^-} \frac{1}{z}\,dz = -\pi i$

Thm: Suppose f is analytic on D

If D has holes, $\int f(z)\,dz$ is not path independent.

If D has no holes, $\int f(z)\,dz$ is path independent

Ex) $\oint_C \frac{1}{z^2}\,dz = 0$ $\quad$ since $\frac{1}{z^2}$ has antiderivative $\frac{d}{dz}\left(-\frac{1}{z}\right) = \frac{1}{z^2}$ on C

Green's Theorem:

Let D be a domain w/ the curve C as its boundary, and D has no holes in it.

Let $P(x,y)$, $Q(x,y)$ have continuous partials on $D \cup C$. Then

$$\int_C P(x,y)\,dx + Q(x,y)\,dy = \iint_D \left(Q_x(x,y) - P_y(x,y)\right)dx\,dy$$

Cauchy-Riemann Theorem:

Suppose $f(z) = u + iv$ is analytic at all points inside and on a simply closed curve C. Then $\int_C f(z)\,dz = 0$

Not self-intersecting

Proof: $z = x + iy$, $dz = dx + i\,dy$

$$\int_C f(z)\,dz = \int_C (u(x,y) + iv(x,y))(dx + i\,dy) = \int_C (u(x,y)\,dx - v(x,y)\,dy) + i\int_C (u(x,y)\,dy + v(x,y)\,dx)$$

If there is no hole in D ★

By Green's thm.: $= \iint_D \underbrace{(-v_x(x,y) - u_y(x,y))}_{=0}\,dx\,dy + i\iint_D \underbrace{(u_x(x,y) - v_y(x,y))}_{=0}\,dx\,dy = 0$ (u & v have partial derivatives b/c f analytic)

By Cauchy-Riemann equations

Remark: Let D be a domain with no hole inside.
Let f be analytic in D
Let $C \subseteq D$ be a closed curve

Then by this thm. $\int_C f(z)\,dz = 0$

$\therefore$ by previous thm.

(1) f has antiderivative everywhere on D

(2) $\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz$, of $C_1, C_2 \subseteq D$ with the same beginning & end point

Ex) Let $C = \{ z(t) = e^{it},\ 0 \le t \le 2\pi \}$

$f(z) = \frac{1}{z}$

$\int_C f(z)\,dz = 2\pi i \neq 0$ thm. does not apply b/c there is a hole at $x = 0$

Ex) $\int_C \frac{1}{z^2}\,dz = 0$ b/c antiderivative exists on C

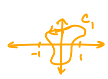
Ex) Let C be any closed curve $\int_C a_0 + a_1 z + \cdots + a_n z^n\,dz = 0$

Ex) Let f be entire. Let C be a closed curve. $\int_C f(z)\,dz = 0$

Ex) $\int_C \frac{1}{z^2 - 1}\,dz$

Not analytic at $x = \pm 1 \rightarrow$ singular at ± 1

For any curve C_1 that does not enclose $z = \pm 1$, $\int_{C_1} \frac{1}{z^2-1}\,dz = 0$



but this does not hold for curves that enclose ± 1

A domain is called simply connected if it has no holes in it

Otherwise, it is called multiply connected

Thm: Let C_1, C_2 be 2 simple closed curves s.t. C_2 is inside C_1 and both are positively oriented

← Allows you to integrate over regions in which the other thms. don't work

Let $f(z)$ be analytic on its domain between C_1 and C_2 and also on C_1 and C_2

← The principle of path deformation

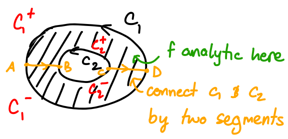
Then $\displaystyle\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz$

Proof:

$$\int_{C_1^+ \cup \overrightarrow{AB} - C_2^+ \cup \overrightarrow{CD}} f(z)\,dz + \int_{C_1^- \cup \overrightarrow{DC} \cup -C_2^- \cup \overrightarrow{BA}} f(z)\,dz$$

$$= 0 + 0$$

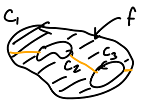
$$\therefore \int_{C_1^+ \cup (-C_2^+) \cup C_1^- \cup (-C_2^-)} f(z)\,dz = \int_{C_1} f(z)\,dz - \int_{C_2} f(z)\,dz = 0$$



Ex) $\displaystyle\int_C \frac{1}{z}\,dz$

by the thm. $\displaystyle\int_C \frac{1}{z}\,dz = \int_{C'} \frac{1}{z}\,dz = 2\pi i$

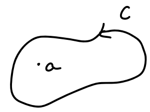
Thm. applies similarly to multiple holes:



Then $\displaystyle\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz + \int_{C_3} f(z)\,dz$

Thm: The Cauchy Integral Formula (CIF)

Let f be analytic inside a simple closed curve C (positively oriented) and on C. Then for any a inside C:



$$\boxed{f(a) = \frac{1}{2\pi i}\int \frac{f(z)}{z-a}\,dz}$$

This is fine since z is on C and a is inside C

↳ So an analytic function is completely determined by its value on the boundary!

f inside C is determined by the values of f just on C

Proof:

Let a be inside C. Let $r>0$ s.t.

$$\int_C \frac{f(z)}{z-a}\,dz = \int_{C_R} \frac{f(z)}{z-a}\,dz, \qquad C_R = \{z : |z-a| = R\} = \{a + re^{it}, 0 \le t \le 2\pi\}$$

(by deformation of path)

$$\int_{C_R} \frac{f(z)-f(a)}{z-a}\,dz + f(a)\int_{C_R}\frac{1}{z-a}\,dz = \int_{C_R}\frac{f(z)-f(a)}{z-a}\,dz + 2\pi i f(a)$$

$$\left|\int_{C_R}\frac{f(z)-f(a)}{z-a}\,dz\right| \le \frac{\max_{C_R}|f(z)-f(a)|}{r}\cdot 2\pi r = 2\pi \max_{z\in C_R}|f(z)-f(a)|$$

↓ As $R\to 0$, this $\to 0$ since f is cont. at a

$$\therefore \int_C \frac{f(z)}{z-a}\,dz = 2\pi i f(a)$$

Ex) $\displaystyle\int_{|z|=4} \frac{e^z}{z-2}\,dz = f(2) = 2\pi i e^2$ ← by the CIF

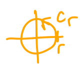
$C: 4e^{it}$ $\quad f(z) = e^z$

Ex) $\displaystyle\int_{|z|=4} \frac{e^z}{z-6} = 0$ ← 0 b/c f is analytic everywhere on & inside C

← CFI does not apply b/c 6 is outside C

Ex) $\displaystyle\int_{|z|=4} \frac{e^z}{z+2}\,dz = 2\pi i e^{-2}$

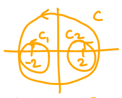
Ex) Let $r>0$. $C_R: |z| = r$



$\displaystyle\int_{C_r}\frac{1}{z}\,dz = 2\pi i$

$f(z)=1$
$a=0$

Ex) Evaluate $\displaystyle\int_{|z|=3}\frac{1}{z^2-4}\,dz$



$$= \int_{C_1}\frac{\frac{1}{z-2}}{z+2}\,dz + \int_{C_2}\frac{\frac{1}{z+2}}{z-2}\,dz = 2\pi i\left[\frac{1}{-4} + \frac{1}{4}\right] = 0$$

$f(z) = \frac{1}{z-2}$, $a=-2$ $\qquad f(z) = \frac{1}{z+2}$, $a=2$

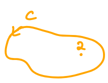
Method 1

$$\int_C \frac{1}{z^2-4}\,dz = \int_C \frac{A}{z-2} - \frac{A}{z+2}\,dz = A\left[\int_C \frac{1}{z-2}\,dz - \int_C \frac{1}{z+2}\,dz\right] = A\left[2\pi i - 2\pi i\right] = 0$$

Partial fraction decomposition $\qquad$ By deformation of path

Method 2

Ex) $\displaystyle\int_C \frac{1}{(z-2)^6}\,dz = 0$



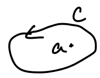
since it has antiderivative on C

If  ? $\displaystyle\int_{C_2}\frac{1}{(z-2)^6}\,dz = 0$

Since $\frac{1}{(z-2)^6}$ analytic inside C

Thm: An extension of the CIF — technically only continuity necessary on curve — find out more (come to office)

Let f be analytic inside and on a simple closed curve (positively oriented) C



Then f is infinitely differentiable inside C and for any a inside C, n=1,2,3,...

$$\boxed{f^{(n)}(a) = \frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-a)^{n+1}}\,dz}$$

Proof: We know $f'(a) = \lim_{w\to a} \frac{f(w)-f(a)}{w-a}$ exists.

By the CIF, $\frac{f(w)-f(a)}{w-a} = \frac{1}{w-a}\int_C f(z)\left(\frac{1}{z-w} - \frac{1}{z-a}\right)dz = \frac{1}{w-a}\int_C \frac{f(z)(w-a)}{(z-w)(z-a)}\,dz = \int_C \frac{f(z)}{(z-w)(z-a)}\,dz$ $\therefore f'(a) = \lim_{w\to a}\int_C \frac{f(z)}{(z-w)(z-a)}\,dz = \int_C \frac{f(z)}{(z-a)^2}\,dz$

We will show next that f' is differential inside C and that $f^{(2)}(a) = \frac{2!}{2\pi i}\int_C \frac{F(z)}{(z-a)^3}\,dz$ for a inside C

Let a be inside C. For w near a,

$$\frac{f'(w)-f'(a)}{w-a} = \frac{1}{2\pi i}\int_C f(z)\left(\frac{1}{(z-w)^2} - \frac{1}{(z-a)^2}\right)dz = \frac{1}{2\pi i}\int_C f(z)\frac{(z-a)^2-(z-w)^2}{(z-w)^2(z-a)^2} = \frac{1}{2\pi i(w-a)}\int_C f(z)\frac{(z-a-(z-w))(z-a+z-w)}{(z-w)^2(z-a)^2}\,dz =$$

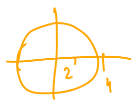
$$= \frac{1}{2\pi i}\int_C f(z)\frac{2z-a-w}{(z-w)^2(z-a)^2}\,dz = \frac{1}{2\pi i}\int_C f(z)\frac{2(z-a)}{(z-a)^4}\,dz = \frac{2}{2\pi i}$$

$\therefore f$ if differentiable at a and whe have formula

Thm: Let f be analytic on a domain D. Then for any n=1,2,3,...
$f^{n}(z)$ exists & is analytic on D.

We saw that f analytic → f' is analytic
↳ And we apply this for arbitrary order of R

Ex) Evaluate $\int_{|z|=4} \frac{\cos z + z^3}{(z-2)^3}\,dz$



2 is inside C

$f(z) = \cos z + z^3$ is analytic inside and on D.

$\therefore$ by extended CIF, $\int_{|z|=4} \frac{\cos z}{(z-2)^3}\,dz = \frac{2\pi i}{2!}f^{(2)}(z)$

Cauchy Estimate: Let f be analytic on and inside the circle $C = \{z : |z-a| < R\}$.
Suppose $|f(z)| \le M_R$ for $|z-a| = R$

Then $\forall$ n=1,2,3,...

$$\boxed{|f^{(n)}(z)| \le \frac{M_R \cdot n!}{R^n}}$$

Proof:

$$|f^{(n)}(a)| = \left|\frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-a)^{n+1}}\,dz\right| \le \frac{n!}{2\pi}\frac{M_R}{R^{n+1}}2\pi R = \frac{M_R \cdot n!}{R^n}$$

Liouville's Theorem: Suppose f is entire and bounded on $\mathbb{C}$ (that is $\exists M>0$ s.t. $|f(z)| \le M\ \forall z\in\mathbb{C}$), then f is constant.

Proof: Let $a\in\mathbb{C}$. Let $R>0$

By Cauchy's estimate: $|f'(a)| \le \frac{M}{R}$ for any R. As $R\to\infty$ $|f'(a)| \le 0 \to f'(a) = 0 \to f$ is constant

Q.E.D.

Lemma: Suppose $|f(z)| \leq |f(z_0)|$ at each point z in some nbhd. $|z-z_0|<\varepsilon$ in which f is analytic.

Then $f(z)$ has the constant value $f(z_0)$ throughout that nbhd.

Proof: Let $z_1 \neq z_0$ be some point in the nbhd.

Let $R = |z_1 - z_0|$

Let $C_R = \{|z-z_0|=R\} \rightarrow z = z_0 + Re^{i\theta} \quad (0 \leq \theta \leq 2\pi)$

By the CIF: $f(z_0) = \frac{1}{2\pi i}\int_{C_R} \frac{f(z)}{z-z_0}\,dz = \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(z_0+Re^{i\theta})}{Re^{i\theta}}\,iRe^{i\theta}d\theta = \frac{1}{2\pi}\int_0^{2\pi} f(z_0+Re^{i\theta})\,d\theta$

$$|f(z_0)| \leq \frac{1}{2\pi}\int_0^{2\pi} |f(z_0+Re^{i\theta})|\,d\theta$$

$$|f(z_0+Re^{i\theta})| \leq |f(z_0)|$$

$$\therefore |f(z_0)| \leq \frac{1}{2\pi}\int_0^{2\pi} |f(z_0+Re^{i\theta})|\,d\theta \leq \frac{1}{2\pi}\int_0^{2\pi}|f(z_0)|\,d\theta = |f(z_0)|$$

$$\therefore \frac{1}{2\pi}\int_0^{2\pi}|f(z_0+Re^{i\theta})|\,d\theta = |f(z_0)|$$

$$\hookrightarrow \int_0^{2\pi} |f(z_0)| - |f(z_0+Re^{i\theta})|\,d\theta = 0$$

Since $|f(z_0)| - |f(z_0+Re^{i\theta})| \geq 0$ throughout the nbhd., $|f(z)| - |f(z_0+Re^{i\theta})| = 0 \rightarrow |f(z_0)| = |f(z_0+Re^{i\theta})|$

$\therefore |f(z_1)| = |f(z)|$ for any circle centered at z_0 & containing z_1 in the nbhd.

$\therefore$ for all z in the nbhd. $|f(z)| = |f(z_0)|$

$\hookrightarrow$ If modulus of analytic function constant $\rightarrow$ that function is constant $\rightarrow f(z) = f(z_0)$ in the nbhd.

Q.E.D.

Maximum Modulus Principle: If a function f is analytic and not constant in a given domain D, then $|f(z)|$ has no maximum value in D. That is, there is no point z_0 in the domain st. $|f(z)| \leq |f(z_0)|$ for all points z in the domain

Proof: Assume analytic function f has maximum at some $z_0 \in D$

Maximum Modulus Principle:

Suppose f is analytic on a domain D.

If $z_0 \in D$ and $|f(z)| \le |f(z_0)|$ $\forall z \in D$, then f is constant in D, $f(z) \equiv f(z_0)$ $\forall z \in D$

Proof: Let $z \in D$

Since D is connected, there exist disks $D_1, \dots, D_n$ st. $D_1 \cap D_2 \neq \emptyset$, $D_2 \cap D_3 \neq \emptyset, \dots, D_{n-1} \cap D_n \neq \emptyset$ w/ z_0 the center of D_1 and z_1 the center of D_n

By the lemma (the disc result), Since $|f(z)|$ on D_1 has max at z_0, $f(z) \equiv f(z_0)$ on D_1

Thus $f(a_1) = f(z_0)$. This means on D_2 $f(z)$ is constant, $f(z) = f(a_1) = f(z_0)$

We continue applying the lemma and arrive at $f(z_1) = f(z_0)$

Ex) If $f(z) = \frac{1}{z}$ on $D = \{z : 0 < |z| < 1\}$,

$|f(z)| = \frac{1}{|z|}$ has no max on D.

Ex) If f is analytic on a bounded set D and cont. on $\bar{D} = D \cup \partial D$,

then $|f(z)|$ always has a max & min value on $\bar{D}$.

By the max modulus principle, if z_0 is a point in $\bar{D}$ occurs, then $z_0 \in \partial D$

Ex) $D = \{z : |z| < 1\} \cup \{z : |z-3| < 1\}$

open, not connected

$D =$  ← max everywhere here

$f(z) = \begin{cases} 1 \\ 2 \end{cases}$

f is analytic on D

$f(z)$ has max at every point in $|z-3| < 1$

but f is not const. on D.

Problem: Suppose f is analytic on domain on D and $\exists z_0 \in D$ st. $|f(z_0)| \le |f(z)|$ $\forall z \in D$

Is z constant on D? No

The minimum modulus principle:

Suppose f is analytic on a domain D and assume f has no zero in D.

If $z_0 \in D$ and $|f(z_0)| \le |f(z)|$ $\forall z \in D$, then f is constant

Proof: Let $g(z) = \frac{1}{f(z)}$

$g(z)$ is analytic on D because f is analytic on D and f has no zeros in D.

$|g(z_0)| \ge |g(z)|$ $\forall z \in D$

By the maximum modulus principle, $g(z)$ is const. on D and so $f(z) = \frac{1}{g(z)}$ is constant on D.

Ex) Suppose f is analytic on a bounded domain D and cont. on $\bar{D}$

Then $|f(z)|$ has a min. on $\bar{D}$. Assume f has no zeros in D.

Then the min of $|f(z)|$ occurs on ∂D.

Harmonic functions have nicer minimum principle:

If $f = u + iv$ is analytic on D a domain.

$-if = -iu + v$

$v = Re\{-if\}$

Extended Liouville's Theorem:

(i) Suppose f is entire and $|f(z)| \le A|z| + B$ $\forall z$ st. $|z| \ge R_0$, $A, B \in \mathbb{C}$

$\exists c_0, c_1 \in \mathbb{C}$ st. $f(z) = c_0 + c_1 z$

} If f is bounded by polynomial growth, f has to be polynomial

Let $z_0 \in \mathbb{C}$. Let $C_R = \{z : |z - z_0| = R\}$

Then by the gen. CIF, $f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_{C_R} \frac{f(z)}{(z-z_0)^{n+1}} dz$

$$|f^{(n)}(z_0)| \le \frac{n!}{2\pi} \cdot \frac{2\pi R}{R^{n+1}} \max_{|z-z_0|=R} |f(z)|$$

When $|z| \ge R_0$, $|f(z)| \le A|z| + B \le A(|z - z_0| + |z_0|) + B = AR + \underbrace{A|z_0| + B}_{\text{const.}}$

$$|f^{(n)}(z_0)| \le \frac{n!}{2\pi} \cdot \frac{2\pi R(AR + A|z_0| + B)}{R^{n+1}} = n!\left(\frac{AR}{R^n} + \frac{AR|z_0| + BR}{R^{n+1}}\right)$$

Let $n = 2$ and $R \to \infty$

$$|f^{(2)}(z_0)| \le 2\left(\frac{AR}{R^2} + \frac{AR|z_0| + BR}{R^3}\right) \to 0$$

$f^{(2)}(z) = 0$ $\forall z \in \mathbb{C}$ $\rightarrow$ 1^{st} derivative is constant $f'(z) = c_1$ $\forall z \in \mathbb{C}$ $\rightarrow f(z) = c_1 z + c_0$

(ii) Suppose f is entire and $|f(z)| \le A|z|^2 + B|z| + C$

Then $f(z) = c_0 + c_1 z + c_2 z^2$

Let $z_0 \in \mathbb{C}$:

$f^{(3)} = \frac{3!}{2\pi i} \int_{C_R} \frac{f(z)}{(z - z_0)^4} dz$, $C_R = \{z : |z - z_0| = R\}$

$|z| \le |z - z_0| + |z_0| = R + |z_0| \le 2R$

Choose R st. $R > |z_0|$. Then $|z|^2 \le (2R)^2 = 4R^2$

$$|f^{(3)}(z_0)| \le \frac{3!}{2\pi} \cdot \frac{2\pi R}{R^4} \max_{z \in C_R} |f(z)| \le \frac{3!}{R^3} \max_{z \in C_R} (A|z|^2 + B|z| + C) \le \frac{3!}{R^3}(A4R^2 + BR + C) \to 0 \text{ as } R \to \infty$$

$f^{(3)}(z) = 0$ $\forall z \in \mathbb{C}$ $\rightarrow f^{(2)} = c_2 \rightarrow f' = c_2 z + c_1 \rightarrow f = c_2 z^2 + c_1 z + c_0$