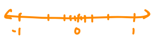
We say f has an isolated singularity at z_0 if there is a deleted disc $0<|z-z_0|<R$ where f is analytic

↳ Ex) $\frac{z}{z(z^2+1)}$ has isolated singularities at $0, i, -i$

Ex) $\operatorname{Log} z$ → 0 is NOT an isolated singularity

Ex) $f(z) = \frac{1}{\sin\left(\frac{\pi}{z}\right)}$ $\sin\left(\frac{\pi}{z}\right)=0$ iff $\frac{\pi}{z} = n\pi$, $n \in \mathbb{Z}$

$\therefore$ f has singularities at $\left\{\frac{1}{n} : n \in \mathbb{Z}\right\} \cup \{0\}$



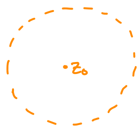
0 is not an isolated singularity since in any disc centered at 0 there is $\frac{1}{n}$

The rest are isolated singularities

Residue:

Suppose f has an isolated singularity at z_0.

Then $\exists R>0$ st. f is analytic on $\{z : 0<|z-z_0|<R\}$



Then f has a Laurent series of the form

$$f(z) = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots$$
$$+ \frac{b_1}{z-z_0} + \frac{b_2}{(z-z_0)^2} + \frac{b_3}{(z-z_0)^3} + \cdots + \frac{b_n}{(z-z_0)^n} + \cdots$$

let C be a simple closed curve (positively oriented) in $0<|z-z_0|<R$ with z_0 inside C

$$\int_C f(z)\,dz = b_1 \int_C \frac{1}{z-z_0}\,dz = 2\pi i b_1$$

↑

The other terms $=0$ b/c they have antiderivative

$$b_1 = \frac{1}{2\pi i}\int_C \frac{1}{z-z_0}\,dz$$ is called the residue of f at z_0 (denoted $\operatorname{Res}(f, z_0)$)

Ex) Let $C : \{z : |z|=1\}$, +ve

Compute $\int_C \frac{e^z-1}{z^3}\,dz$ using residues

$\frac{e^z-1}{z^3}$ has an isolated singularity at 0 $\therefore$ it has a Laurent series on $0<|z|<\infty$

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \cdots$$

$$\frac{e^z-1}{z^3} = \frac{1}{z^2} + \frac{1}{2!\,z} + \frac{1}{3!} + \frac{z}{4!} + \cdots$$

↑ Residue

$$\therefore \int_C \frac{e^z-1}{z^3}\,dz = 2\pi i \cdot \frac{1}{2!} = \pi i$$

Ex) Compute $\int_C \cos\left(\frac{1}{z^2}\right)dz$, $C : |z|=1$, +ve

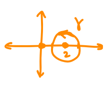
$z=0$ is an isol. sing.

$$\cos(z) = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots$$

$$\cos\left(\frac{1}{z^2}\right) = 1 - \frac{1}{2!\,z^4} + \frac{1}{4!\,z^8} - \cdots$$

$\therefore \operatorname{Res}\left(\cos\frac{1}{z^2}, 0\right) = 0$ $\therefore \int_C \cos\left(\frac{1}{z^2}\right)dz = 0$

Ex) $\int_\gamma \frac{1}{z(z-2)^5}\,dz$, $\gamma : |z-2|=1$, +ve



↳ The only isol. sing. in γ is $z=2$

$$\therefore \int_\gamma \frac{1}{z(z-2)^5}\,dz = 2\pi i \operatorname{Re}\left(\frac{1}{z(z-2)^5}, 2\right)$$

Write Laurents series of $\frac{1}{z(z-2)^5}$ in $0<|z-2|<1$

$\frac{1}{z}$ is analytic on $|z-2|<1$ → $\frac{1}{z} = \frac{1}{z-2+2} = \frac{1}{2}\frac{1}{1+\frac{z-2}{2}} = \frac{1}{2}\sum_{n=1}^{\infty}(-1)^n\left(\frac{z-2}{2}\right)^n = \sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+1}}(z-2)^n$, $|z-2|<2$

$\frac{1}{z(z-2)^5} = \frac{1}{(z-2)^5}\sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+1}}(z-2)^n$ → Residue is $\frac{1}{2^5}$ → $\int_\gamma \frac{1}{z(z-2)^5}\,dz = \frac{2\pi i}{2^5} = \frac{\pi i}{16}$

If f has an isolated sing. at z_0, the $f(z)$ has a Laurent series

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \underbrace{\frac{b_1}{z-z_0} + \frac{b_2}{(z-z_0)^2} + \cdots + \frac{b_n}{(z-z_0)^n} + \cdots}_{\text{Principal Part of } f \text{ at } z_0} \quad \text{in } 0<|z-z_0|<R_2$$

If every $b_n = 0$, z_0 is removable singularity

If an infinite number of b_n are nonzero, z_0 is essential singularity

If there is a finite # of nonzero b_n, there $\exists m$ s.t. $b_m \neq 0$ and $b_{m+1} = b_{m+2} = \cdots = 0$

In this case, z_0 is a pole of order m

↳ A pole of order 1 is a simple pole

Thm: Let z_0 be an isolated sing. of f. Then the following are equivalent:

a) f has a pole of order $m \geq 1$ at z_0

b) $f(z) = \dfrac{g(z)}{(z-z_0)^m}$, g analytic at z_0 and $g(z_0) \neq 0$

If (a) or (b): $\text{Res}(f, z_0) = \dfrac{g^{(m-1)}(z_0)}{(m-1)!}$

Proof:

Suppose $f(z) = \dfrac{g(z)}{(z-z_0)^m}$, g is analytic at z_0 and $g(z_0) \neq 0$

Let g be analytic on $|z-z_0| < R$

Then f has an isolated sing. at z_0

$g(z) = c_0 + c_1(z-z_0) + \cdots + c_k(z-z_0)^k + \ldots \qquad |z-z_0| < R$

$$c_d = \frac{g^{(d)}(z_0)}{d!}$$

On $0 < |z-z_0| < R$

$$f(z) = \frac{c_0}{(z-z_0)^m} + \frac{c_1}{(z-z_0)^{m-1}} + \cdots + \frac{c_{m-1}}{z-z_0} + c_m + c_{m+1}(z-z_0) + \cdots$$

$g(z_0) = c_0 \neq 0 \quad \therefore$ f has a pole of order m at z_0

$$\text{Res}(f, z_0) = c_{m-1} = \frac{g^{(m-1)}(z_0)}{(m-1)!}$$

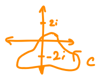
Ex) $f(z) = \dfrac{z}{z^4} = \dfrac{1}{z^3} \rightarrow$ has pole of order 3 at 0.

Ex) $f(z) = \dfrac{z^3+1}{z^2+4} = \dfrac{z^3+1}{(z-2i)(z+2i)} \rightarrow$ isolated singularities at $\pm 2i$

At 2i: $\quad f(z) = \dfrac{\overbrace{\frac{z^3+1}{z+2i}}^{\rightarrow\, g(z)}}{z-2i}$

g is analytic at 2i, $g(2i) \neq 0 \quad \therefore$ f has a pole of order 1 at 2i

$\text{Res}(f, 2i) = g(2i) \rightarrow \displaystyle\int_c f(z)\,dz = 2\pi i\, g(2i)$



Ex) $f(z) = \dfrac{z^4+2z}{(z+i)^3} \quad$ ($z^4+2z \rightarrow g(z)$)

$g(-i) = 1-2i \neq 0 \quad \therefore$ f has a pole of order 3 at $-i$

and $\text{Res}(f, -i) = \dfrac{g^{(2)}(-i)}{2!}$

Ex) $h(z) = \dfrac{1-\cos z}{z^3}$

$$1-\cos z = 1 - \left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \ldots\right) = \frac{z^2}{2!} - \frac{z^4}{4!} + \cdots = z^2\underbrace{\left(\frac{1}{2!} - \frac{z^2}{4!} + \cdots\right)}_{g(z)} = z^2 g(z)$$

$h(z) = \dfrac{z^2 g(z)}{z^3} = \dfrac{g(z)}{z} \rightarrow$ g analytic at 0, $g(0) \neq 0 \qquad \therefore$ h has a pole of order 1 at 0

Ex) $f(z) = \dfrac{1}{z^2 \sin z}$

Consider 0: $\quad f(z) = \dfrac{\frac{1}{\sin z}}{z} \nearrow \dfrac{1}{\sin z}$ not analytic at 0

$$f(z) = \frac{1}{z^2\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \ldots\right)} = \frac{1}{z^3\left(1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \ldots\right)} = \frac{1}{z^3 g(z)} = \frac{\overbrace{\frac{1}{g(z)}}^{\rightarrow \text{ analytic at 0, } \neq 0 \text{ at } 0}}{z^3}$$

$\therefore$ f has a pole of order 3 at 0.

Order of poles:

$\frac{1}{z}, \frac{1}{z^2}, \cdots, \frac{1}{z^m}$

order 1 2 , ... , m

$\lim_{z\to 0} \left|\frac{1}{z}\right| = \lim_{z\to 0} \left|\frac{1}{z^2}\right| = \infty$

The order of a pole measures the rate at which f approaches infinity there: higher order → approaches ∞ faster

If $f'(z_0) = \cdots = f^{(m-1)}(z_0) = 0$ and $f^{(m)}(z_0) \neq 0$, we say f has a zero of order m at z_0

If $m=1$, we say f has a simple zero at z_0

Ex) $f(z) = (z-2)^3$ $\quad f(2) = f'(2) = f^{(2)}(2) = 0$

$f^{(3)}(2) = 3! \neq 0$

f has a zero of order 3 at 2

Thm: Let f be analytic at z_0. The following are equivalent:

a) f has a zero of order m at z_0

b) There is g analytic at z_0 such that

$f(z) = g(z)(z-z_0)^m$ on $|z-z_0| < R$, where $g(z_0) \neq 0$

Proof: a→b:

Since f is analytic at z_0, $\exists R > 0$ st. $f(z) = a_0 + a_1(z-z_0) + \cdots + a_k(z-z_0)^k + \cdots$, $|z-z_0| < R$, $a_i = \frac{f^{(i)}(z_0)}{i!}$

By hypothesis, $f(z) = a_m(z-z_0)^m + a_{m+1}(z-z_0)^{m+1} + \cdots = (z-z_0)^m\left[a_m + a_{m+1}(z-z_0) + \cdots\right] = (z-z_0)^m g(z)$

$a_0 = a_1 = \cdots = a_{m-1} = 0$

g is analytic at z_0 and $g(z_0) = a_m \neq 0$

b→a: $f(z) = g(z)(z-z_0)^m$, $g(z_0) \neq 0$

$g(z) = c_0 + c_1(z-z_0) + c_2(z-z_0)^2 + \cdots$

$f(z) = c_0(z-z_0)^m + c_1(z-z_0)^{m+1} + \cdots = 0 + 0(z-z_0) + \cdots + 0(z-z_0)^{m-1} + c_0(z-z_0)^m + c_1(z-z_0)^{m+1} + \cdots$

f has zero of order m at z_0

Ex) $f(z) = z^4 - 1$ $\quad f(1) = 0$ $\quad f'(1) = 4 \neq 0$ $\quad \therefore$ f has a zero of order 1 at 1

Ex) $f(z) = z(z-1)^4$ → $z \neq 0$ at $z=1$, so by thm f has zero of order 4 at 1

z analytic at 1

order is a measure of how fast the function approaches 0.

Thm: Let f be analytic at z_0

Suppose $f(z_0)=0$ but f is not identically 0 on any nbhd. of z_0

Then $\exists\, r>0$ st. $f(z)\neq 0$ for any $z\in\{z:\ 0<|z-z_0|<r\}$

Proof: $\exists\, R>0$ st. f is analytic on $|z-z_0|<R$

$f(z)=a_0+a_1(z-z_0)+a_2(z-z_0)^2+\cdots$ on $|z-z_0|<R$

$$a_k=\frac{f^{(k)}(z_0)}{k!}$$

If $a_j=0\ \forall j$, then $f\equiv 0$ on $|z-z_0|<R$ which contradicts the hypothesis on f

$\therefore \exists\, j$ st. $a_j\neq 0$

$f(z_0)=a_0=0$

Let m be the smallest st. $a_m=\frac{f^{(m)}(z_0)}{m!}\neq 0$

$\therefore f(z)=a_m(z-z_0)^m+a_{m+1}(z-z_0)^{m+1}+\cdots,\quad a_m\neq 0$

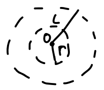
$f(z)=(z-z_0)^m\underbrace{\left(a_m+a_{m+1}(z-z_0)+\cdots\right)}_{g(z)}=(z-z_0)^m g(z)$ where g is analytic on $|z-z_0|<R$ and $g(z_0)=a_m\neq 0$

Since g is continuous at z_0, there is $0<r<R$ st. $g(z)\neq 0$ for any z in $|z-z_0|<r$

Hence $f(z)$ has no zero in $0<|z-z_0|<r$

Corollary: Let f be analytic on $|z|<1$

Suppose $f(z)=0\ \forall z\in L$, L is a segment as shown:



Then $f(z)=0\ \forall z$ in $|z|<1$

Corollary: Suppose f is analytic on $|z|<1$

suppose $\exists\{z_k\}$ distinct, $z_k\to 0$ st. $f(z_k)=0$

Then $f(z)\equiv 0\ \forall z$

That is, unless a function is identically 0, the zeros of an analytic functions are isolated

Zeros and Poles:

Thm: Suppose $p(z)$ and $q(z)$ are analytic at z_0,

$p(z_0)\neq 0$ and $q(z)$ has a zero of order m at z_0

Then $\frac{p(z)}{q(z)}$ has a pole of order m at z_0

Proof: Since $q^{m}(z_0)\neq 0$, z_0 is an isolated zero of $q(z)$ $\therefore \frac{p(z)}{q(z)}$ has an isolated singularity at z_0

q has a zero of order m at $z_0 \to q(z)=(z-z_0)^m h(z)$ where h is analytic at z_0 and $h(z_0)\neq 0$

$\therefore \frac{p(z)}{q(z)}=\frac{p(z)}{(z-z_0)^m h(z)}=\frac{p(z)/h(z)}{(z-z_0)^m}$ Since $\frac{p(z)}{h(z)}$ is analytic at z_0 and $\frac{p(z_0)}{h(z_0)}\neq 0$, $\frac{p(z)}{q(z)}$ has a pole of order m at z_0

Q.E.D.

Ex) $\frac{1}{1-\cos z}=\frac{p(z)}{q(z)}$ $\quad p(0)=1\neq 0$ $\quad q(z)$ has a zero of order 2 at 0 $\therefore \frac{1}{1-\cos z}$ has a pole of order 2 at 0.

Thm: Let p and q be analytic at z_0, $p(z_0)\neq 0$

Suppose $q(z_0)=0$, $q'(z_0)\neq 0$ ($m=1$, z_0 is a simple 0)

Then $\frac{p(z)}{q(z)}$ has a simple pole at z_0 and $Res\left(\frac{p(z)}{q(z)}, z_0\right)=\frac{p(z_0)}{q'(z_0)}$

Proof: $q(z)=q(z_0)+q'(z_0)(z-z_0)+\frac{q^{(2)}(z_0)}{2!}(z-z_0)^2+\cdots=(z-z_0)h(z)$, $\quad h(z_0)=q'(z_0)$

$$\frac{p(z)}{q(z)}=\frac{p(z)}{(z-z_0)h(z)}=\frac{\frac{p(z)}{h(z)}}{z-z_0}=\frac{\frac{p(z_0)}{h(z_0)}+c_1(z-z_0)+\cdots}{z-z_0}=\frac{\frac{p(z_0)}{h(z_0)}}{z-z_0}+c_1+c_2(z-z_0)+\cdots$$

$\therefore Res\left(\frac{p(z)}{q(z)}, z_0\right)=\frac{p(z_0)}{h(z_0)}=\frac{p(z_0)}{q'(z_0)}$

Q.E.D.

Ex) $f(z)=\frac{\cos z}{\sin z}$ $\quad \cos\pi\neq 0$ $\quad \sin\pi=0$ $\quad (\sin z)'(\pi)\neq 0$, $\quad Res(f,\pi)=\frac{\cos\pi}{\cos\pi}=1$

Ex) $g(z) = \frac{z - \sin z}{z^2 \sin z}$ $\pi - \sin\pi \neq 0$ $(z^2\sin z)(\pi) = 0$, $(z^2 \sin z)'(\pi) \neq 0$ $\therefore \ Res(g, \pi) = \frac{\pi - \sin\pi}{-\pi^2} = -\frac{1}{\pi}$

Essential singularities far more complex!

Thm: If f has a removable singularity at z_0 then f is bounded near z_0

Proof: f has isolated sing. at z_0 → the Laurent series of f on $0<|z-z_0|<R$ is $f(z) \equiv a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots$

Let $g(z) = a_0 + a_1(z-z_0) + \cdots$, g is analytic on $|z-z_0|<R$ → g is cont. at z_0 $\therefore \exists r>0, \exists M>0$ st. $|g(z)|<M$ for all z, $|z-z_0| \le r$

→ on $0<|z-z_0| \le r$, $|f(z)| = |g(z)| \le M$

Thm (Riemann's Removable Singularity Theorem): Suppose $\exists M, R>0$ st. $|f(z)| \le M$ on $0<|z-z_0|<R$

Then z_0 is removable for f.

Let f have an isolated singularity at z_0. Suppose $|f(z)| \le M$ on $0<|z-z_0|<R$.
Then z_0 is a removeable singularity.

Proof: $f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \frac{b_1}{z-z_0} + \frac{b_2}{(z-z_0)^2} + \cdots +$ where $b_n = \frac{1}{2\pi i} \int_{C_r} \frac{f(z)}{(z-z_0)^{-n+1}} dz$, $n = 1,2,3 \ldots$

$C_r : \{ z : |z-z_0| = r \}$, +ve , $0<r<R$

$|b_n| \le \frac{1}{2\pi} \cdot \frac{M}{r^{-n+1}} 2\pi r = M r^n$ $\forall \, 0<r<R$

Let $r \to 0$. We get $|b_n| = 0$ $\forall \, n = 1,2,3, \ldots$

Q.E.D.

We can weaken this assumption. E.g.: Assume f has an isolated sing. at z_0 & $|f(z)| \le \frac{1}{|z-z_0|^p}$ $0<p<1$, for $0<|z-z_0|<R$ ← Do this exercise

Show that f has a removeable sing. at z_0

Thm: Suppose f has an essential singularity at z_0. Say f is analytic on $0<|z-z_0|<R$. Then for every $w \in \mathbb{C}$, and any $\delta, \varepsilon > 0$, there is z st. $|z-z_0|<\delta$ and $|f(z) - w| < \varepsilon$

Ex) Let $\varepsilon_n = \frac{1}{n}$, $n = 1,2,3, \ldots$
$\delta_n = \frac{1}{n}$, $n = 1,2,3, \ldots$

Given $w \in \mathbb{C}$, there is $|z_n - z_0| < \frac{1}{n} = \delta_n$ st. $|f(z_n) - w| \le \frac{1}{n} = \varepsilon_n$ $\forall n = 1,2,3, \ldots$

That is, if f has an essential sing. at z_0, then for any $w \in \mathbb{C}$, $\exists$ a sequence $\{z_n\}$ such that $z_n \to z_0$ and $f(z_n) \to w$

Proof by contradiction: Suppose the thm. is not valid. Then $\exists \, \varepsilon, \delta > 0$ st. for any z in $0<|z-z_0|<\delta$, $|f(z) - w| \ge \varepsilon$

Let $g(z) = \frac{1}{f(z) - w}$ g is analytic on $0<|z-z_0|<\delta$

g has an isolated sing. at z_0.

For $0<|z-z_0|<\delta$, $|g(z)| = \frac{1}{|f(z) - w|} \le \frac{1}{\varepsilon}$. Thus g is bounded on $0<|z-z_0|<\delta$

By the Riemann removeable sing. thm., g has a removeable sing. at z_0

So we may assume that g is analytic on $|z-z_0|<\delta$

$g(z) = \frac{1}{f(z) - w}$, for $0<|z-z_0|<\delta$, and so $g(z) \neq 0$ when $0<|z-z_0|<\delta$

$g(z) = g(z_0) + g'(z_0)(z-z_0) + \cdots + \frac{g^{(k)}(z_0)}{k!}(z-z_0)^k + \cdots$ for $|z-z_0|<\delta$, $\exists k \ge 1$ st. $g^{(k)}(z_0) \neq 0$

$g(z) = (z-z_0)^k h(z)$, h analytic on $|z-z_0|<\delta$, $h(z_0) \neq 0$

$f(z) - w = \frac{1}{g(z)} = \frac{\frac{1}{h(z)}}{(z-z_0)^k}$ for $z \neq 0$

$f(z) = w + \frac{\frac{1}{h(z)}}{(z-z_0)^k}$ → f has a pole at z_0 → Contradiction!

Thm: f has a pole at z_0 iff $\lim_{z \to z_0} |f(z)| = \infty$

Proof: Suppose f has a pole of order m $(m \geq 1)$ at z_0

By a thm., $f(z) = \dfrac{g(z)}{(z-z_0)^m}$, $g(z_0) \neq 0$, g analytic on $|z-z_0| < R$ $\therefore$ $\lim_{z \to z_0} |f(z)| = \lim_{z \to z_0} \dfrac{|g(z)|}{|z-z_0|^m} \to \infty$

Suppose $\lim_{z \to z_0} |f(z)| = \infty$

z_0 cannot be a removeable sing. since f is not bounded. If z_0 were an essential sing, there would $\exists \{z_k\} \to z_0$ st. $f(z_k) \to 1$. But $\lim_{k \to \infty} |f(z_k)| = \infty$ $\therefore$ it cannot be essential.

Hence z_0 is a pole.

Q.E.D.

Application: Suppose f is entire and $\lim_{|z| \to \infty} |f(z)| = \infty$.

Then f is a polynomial

Proof: $f(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n + \cdots$, $z \in \mathbb{C}$

Let $g(z) = f\left(\frac{1}{z}\right)$ g is analytic on $0 < |z|$, and so has an isol. sing. at 0.

$\lim_{z \to 0} |g(z)| = \infty$ $\therefore$ by a thm. g has a pole of order $m \geq 1$ at 0

$g(z) = f\left(\frac{1}{z}\right) = a_0 + \frac{a_1}{z} + \frac{a_2}{z^2} + \cdots + \frac{a_n}{z^n} + \cdots$ and g has a pole of order m at 0

$\to a_k = 0 \quad \forall\, k \geq m+1$

$\to g(z) = f\left(\frac{1}{z}\right) = a_0 + \frac{a_1}{z} + \frac{a_2}{z^2} + \cdots + \frac{a_m}{z^m}$, $0 < |z|$

$\to f(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_m z^m$

Let $P(z) = a_0 + a_1 z + \cdots + a_n z^n$, $n \geq 1$ is a polynomial, $a_n \neq 0$

For $|z|$ large, $|P(z)| \sim |z|^n$

More precisely, $\exists\ A, B > 0$ and $R > 0$ s.t. for $|z| \geq R$, $B|z|^n \leq |P(z)| \leq A|z|^n$

Proof: $|P(z)| = |a_0 + a_1 z + \cdots + a_n z^n| \leq |z|^n \left(\frac{|a_0|}{|z|^n} + \frac{|a_1|}{|z|^{n-1}} + \cdots + \frac{|a_{n-1}|}{|z|} + |a_n| \right) = A|z|^n$ $\therefore$ when $|z| \geq 1$,

$\leq |z|^n \left(|a_0| + |a_1| + \cdots + |a_{n-1}| + |a_n| \right) = A|z|^n$

$|P(z)| = |z|^n \left| \frac{a_0}{z^n} + \frac{a_1}{z^{n-1}} + \cdots + \frac{a_{n-1}}{z} + a_n \right| \geq |z|^n \left(|a_n| - \left| \frac{a_0}{z^n} + \frac{a_1}{z^{n-1}} + \cdots + \frac{a_{n-1}}{z} \right| \right)$

Since $\lim_{|z| \to \infty} \left(\frac{a_0}{z^n} + \frac{a_1}{z^{n-1}} + \cdots + \frac{a_{n-1}}{z} \right) = 0$, $\exists\ R_1$ s.t. $\left| \frac{a_0}{z^n} + \cdots + \frac{a_{n-1}}{z} \right| < \frac{|a_n|}{2}$ when $|z| \geq R_1$

$\therefore$ when $|z| \geq R_1$, $|P(z)| \geq |z|^n \left(|a_0| - \frac{|a_0|}{2} \right) = \frac{|a_0|}{2} |z|^n = B|z|^n$

Let $R = \max\{1, R_1\}$. Then when $|z| > R$, $B|z|^n \leq |P(z)| \leq A|z|^n$

Residue Integrals

Type I integral: $\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} dx$, $P(x), Q(x)$ polynomial, no common factors, $\deg Q \geq \deg P + 2$, $Q(x) \neq 0$ for $x \in \mathbb{R}$

Let $f(z) = \frac{P(z)}{Q(z)}$ f is analytic except where $Q(z) = 0$

$Q(z)$ has only a finite # of zeros

For $R > 0$, let $C_R = [-R, R] \cup \Gamma_R$, $\Gamma_R = \{ z = x + iy : |z| = R, y \geq 0 \}$

Choose R big enough so that all the zeros of Q in the upper half plane are inside C_R

Let $\{z_1, \ldots, z_n\}$ be all the zeros of Q inside C_R

$$\int_{C_R} f(z) dz = 2\pi i \sum_{j=1}^{m} \mathrm{Res}(f, z_j)$$

$$\int_{C_R} f(z) dz = \int_{-R}^{R} f(x) dx + \int_{\Gamma_R} f(z) dz$$

$$\left| \int_{\Gamma_R} f(z) dz \right| \leq \pi R \max_{|z|=R} |f(z)|$$

Recall that $\exists\ R_1 > 0$, $A, B > 0$ $\ni$ when $|z| \geq R_1$, $|P(z)| \leq A|z|^k$ and $|Q(z)| \geq B|z|^k$

$\therefore$ when $|z| \geq R_1$, $|f(z)| \leq \frac{A|z|^k}{B|z|^m}$

Let $R \geq R_1$. Then $\left| \int_{\Gamma_R} f(z) dz \right| \leq \pi R \max_{|z|=R} |f(z)| \leq \frac{\pi A}{B} R \cdot \frac{R^k}{R^m} = \frac{\pi A}{B} \cdot \frac{1}{R^{m-k-1}}$ $(m - k - 1 \geq 1)$

$$\therefore \lim_{R \to \infty} \int_{\Gamma_R} f(z) dz = 0$$

$$\lim_{R \to \infty} \int_{-R}^{R} f(x) dx = \int_{-\infty}^{\infty} f(x) dx$$

Thus $\lim_{R \to \infty} \int_{C_R} f(z) dz = \int_{-\infty}^{\infty} f(x) dx$

Hence, $\boxed{\int_{-\infty}^{\infty} f(x) dx = 2\pi i \sum_j \mathrm{Res}(f, z_j)}$

In $\mathrm{Im}\{z\} > 0$

Ex) $\int_{-\infty}^{\infty} \frac{x^2}{x^4 + 16} dx$

$P(x) = x^2$, $Q(x) = x^4 + 16$ — P, Q no common factors

$\deg Q \geq \deg P + 2$ — P, Q polynomial

$Q(x) \neq 0\ \forall x \in \mathbb{R}$

$f(z) = \frac{z^2}{z^4 + 16}$ $\quad z^4 = -16 \to z = 2e^{i\theta}$, $\theta \in \left\{ \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4} \right\}$

$\mathrm{Res}\left(\frac{z^2}{z^4+16}, z_1 \right) = \frac{z_1^2}{4z_1^3} = \frac{1}{4z_1} = \frac{1}{2} e^{-i\frac{\pi}{4}} = \frac{1}{8} \left(\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2} \right)$

$\mathrm{Res}\left(\frac{z^2}{z^4+16}, z_2 \right) = \frac{1}{4z_2} = \frac{1}{8} e^{-i\frac{3\pi}{4}}$

Same for z_3, z_4, but yields $\mathrm{Im} < 0$, so we ignore

Type II : $\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} \sin(ax)\, dx$, $\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} \cos(ax)\, dx$, $a > 0$

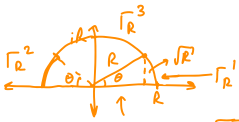
P, Q polynomial, $Q(x) \neq 0 \ \forall x \in \mathbb{R}$. P, Q no common factors. $\deg Q \geq \deg P + 1$

$$\frac{P(z)}{Q(z)} \sin(az) = \frac{P(z)}{Q(z)} \left(\frac{e^{iaz} - e^{-iaz}}{2i} \right) \rightarrow \text{Cannot bound } \frac{P(z)}{Q(z)} \sin z$$

We work with $\int \frac{P(z)}{Q(z)} e^{iaz}\, dz$

$|e^{iaz}| = e^{-y} \leq 1$ for $y \geq 0$

$$\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} e^{iax}\, dx = \int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} \cos(ax)\, dx + \int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} \sin(ax)\, dx$$



$\Gamma_R = \{ z = x + iy : |z| = R, y \geq 0 \}$

$\Gamma_R = \Gamma_R^1 \cup \Gamma_R^2 \cup \Gamma_R^3$

$\sqrt{R^2 - R} = R\sqrt{1 - \frac{1}{R}}$

If $R \geq 2 \rightarrow \frac{1}{R} \leq \frac{1}{2} \rightarrow 1 - \frac{1}{R} \geq \frac{1}{2}$

$\therefore$ when $R \geq 2$, $x \geq \frac{R}{2}$

$\underbrace{\sin\theta - \sin 0}_{\text{Mean Value thm.}} = (\cos t)\,\theta \geq \theta \cos\theta = \theta \frac{x}{R} \geq \frac{\theta}{2}$

$0 \leq t \leq \theta \rightarrow \cos t \geq \cos\theta$

$\therefore \sin\theta \geq \frac{\theta}{2}$ if $R \geq 2$

$\text{length}(\Gamma_R^1) = \text{length}(\Gamma_R^2) = R\theta \leq R \cdot \frac{2}{\sqrt{R}} = 2\sqrt{R}$

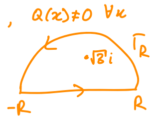
Let $f(x) = \frac{P(x)}{Q(x)}$, P, Q polynomial, $Q(x) \neq 0 \ \forall x \in \mathbb{R}$, $\deg Q > \deg P + 1$

Ex) $\int_0^{\infty} \frac{x \sin(2x)}{x^2+3}\, dx = \frac{1}{2} \int_{-\infty}^{\infty} \frac{x \sin(2x)}{x^2+3}$

Function is even

If the difference in $\deg P, Q$ is 1, we can use this reasoning of dividing Γ_R into 3 parts

$P(x) = x$, $Q(x) = x^2 + 3$, $\deg Q = \deg P + 1$, $Q(x) \neq 0 \ \forall x$



$C_R = [-R, R] \cup \Gamma_R$

$$\int_{C_R} \frac{z}{z^2+3} e^{i2z}\, dz = \int_{-R}^{R} \frac{x}{x^2+3} e^{i2x}\, dx + \int_{\Gamma_R} \frac{z}{z^2+3} e^{i2z}\, dz$$

$$\int_{\Gamma_R} \frac{z}{z^2+3} e^{i2z}\, dz = \int_{\Gamma_R^1} + \int_{\Gamma_R^2} + \int_{\Gamma_R^3}$$

$$\left| \int_{\Gamma_R^3} \frac{z}{z^2+3} e^{i2z}\, dz \right| \leq L(\Gamma_R^3) \cdot \frac{R}{R^2-3} e^{-2\sqrt{R}} \leq \pi \frac{R^2}{R^2-3} e^{-2\sqrt{R}} = \pi \frac{1}{1 - \frac{3}{R^2}} e^{-2\sqrt{R}} \rightarrow 0 \text{ as } R \rightarrow \infty$$

On Γ_R^3, $|e^{i2z}| = e^{-2y} \leq e^{-2\sqrt{R}}$

$|z^2+3| \geq |z|^2 - 3 = R^2 - 3$

$e^{-2y} \leq 1$
$y \geq 0$

$$\left| \int_{\Gamma_R^3} \frac{z}{z^2+3} e^{2iz}\, dz \right| \leq L(\Gamma_R^1) \cdot \frac{R}{R^2-3} \max_{z \in \Gamma_R^1} |e^{i2z}| \leq \frac{2\sqrt{R}\, R}{R^2-3} \cdot 1 \rightarrow 0 \text{ as } R \rightarrow \infty$$

likewise $\int_{\Gamma_R^2} \rightarrow 0$ as $R \rightarrow \infty$

$$\therefore \int_{-\infty}^{\infty} \frac{x}{x^2+3} e^{i2x}\, dx = 2\pi i \operatorname{Res}\left(\frac{z e^{i2z}}{z^2+3}, \sqrt{3}i \right) = \left. \frac{z e^{2iz}}{2z} \right|_{z=\sqrt{3}i} = \frac{1}{2i} e^{-2\sqrt{3}} (2\pi i)$$

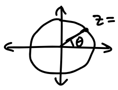
$$= \int_{-\infty}^{\infty} \frac{x}{x^2+3} \cos(2x)\, dx + i \int_{-\infty}^{\infty} \frac{x}{x^2+3} \sin(2x)\, dx$$

Ex) $\displaystyle\int_{-\infty}^{\infty} \frac{\cos(2x)}{(x^2+4)^2}\,dx$ When diff. in deg $P, Q > 1$, we don't need to divide Γ_R into parts

$$\int_{\Gamma_R} \frac{e^{i2z}}{(z^2+4)^2}\,dz \qquad |e^{i2z}| = e^{-2y} \le 1 \quad (y \ge 0)$$

$$\mathrm{Res}\left(\frac{e^{i2z}}{(z^2+4)^2}, 2i\right) = \mathrm{Res}\left(\frac{\frac{e^{i2z}}{(z+2i)^2}}{(z-2i)^2}, 2i\right) = \mathrm{Res}\left(\frac{g(z)}{(z-2i)^2}, 2i\right) = g'(2i)$$

<u>Type III</u>: $\displaystyle\int_0^{2\pi} F(\cos\theta, \sin\theta)\,d\theta$ → we want to write this as $\int f(z)\,dz$



$$\left.\begin{aligned} z &= e^{i\theta} = \cos\theta + i\sin\theta \\ \frac{1}{z} &= \cos\theta - i\sin\theta \end{aligned}\right\} \quad \begin{aligned} \cos\theta &= \frac{1}{2}\left(z + \frac{1}{z}\right) \\ \sin\theta &= \frac{1}{2i}\left(z - \frac{1}{z}\right) \end{aligned}$$

$$z = e^{i\theta} \to dz = ie^{i\theta}d\theta = iz\,d\theta \longrightarrow d\theta = \frac{1}{iz}\,dz$$

Ex) $\displaystyle\int_0^{2\pi} \frac{1}{1+a\sin\theta}\,d\theta$, $-1<a<1$

$$= \frac{1}{i}\int_{|z|=1} \frac{1}{1+\frac{a}{2i}\left(z-\frac{1}{z}\right)} \cdot \frac{1}{z}\,dz = \int_{|z|=1} \frac{1}{iz + \frac{az^2-a}{2}}\,dz = \int_{|z|=1} \frac{2}{2iz+az^2-a}\,dz =$$

$$\downarrow$$

$$az^2+2iz-a: \quad \frac{-2i \pm \sqrt{-4+4a^2}}{2a} = i\left(\frac{-1 \pm \sqrt{1-a^2}}{a}\right)$$

$$\left| i\,\frac{(-1-\sqrt{1-a^2})}{a} \right| > 1$$

$$\left| i\,\frac{(-1+\sqrt{1-a^2})}{a} \right| < 1$$

Zeros:

Take $f(z)=z^2(z-1)^4e^z$

We say 0 is a zero of f of multiplicity 2

f is called meromorphic on D if f is either analytic or has a pole at each $z\in D$

Suppose f has a zero of order m at a

Then $\mathrm{Res}\left(\frac{f'(z)}{f(z)}, a\right)=m$

If f has a pole of order k at a, then $\mathrm{Res}\left(\frac{f'(z)}{f(z)}, a\right)=-k$

$$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}\,dz = Z-P$$

Ex) $f(z)=\dfrac{z^2(z-1)^4}{(z+2)(z+1)^3}$

In $|z|<2$ the # of zeros of f, $Z(f)=2+4=6$, the # of poles of f, $P(f)=3$

Thm: (i) Suppose f has a zero of order m at a. Then $\mathrm{Res}\left(\frac{f'}{f}, a\right)=m$

(ii) pole of order m at a. Then $\mathrm{Res}\left(\frac{f'}{f}, a\right)=-m$

Proof: (i) f has a zero of order m at a,

$\exists g$ analytic in a nbhd of a st. $f(z)=g(z)(z-a)^m$, $g(a)\neq 0$

$$f'(z)=m(z-a)^{m-1}g(z)+(z-a)^m g'(z) \Rightarrow \frac{f'(z)}{f(z)}=\frac{m(z-a)^{m-1}g(z)+(z-a)^m g'(z)}{g(z)(z-a)^m}=\frac{m}{z-a}+\underbrace{\frac{g'(z)}{g(z)}}_{\text{Is analytic at } a} \quad (g(a)\neq 0)$$

$$\therefore \mathrm{Res}\left(\frac{f'}{f}, a\right)=m$$

(ii) $\exists$ g analytic at a, $g(a)\neq 0$ st. $f(z)=\dfrac{g(z)}{(z-a)^m}=g(z)(z-a)^{-m}$

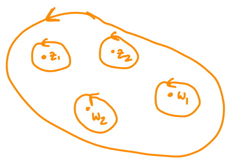
$$f'(z)=g'(z)(z-a)^{-m}-mg(z)(z-a)^{-m-1} \Rightarrow \frac{f'(z)}{f(z)}=\underbrace{\frac{g'(z)}{g(z)}}_{\text{analytic at } a}-\frac{m}{z-a} \qquad \therefore \mathrm{Res}\left(\frac{f'}{f}, a\right)=-m$$

Thm (The Argument Principle): Let f be analytic on and inside a simple closed curve C (+vely oriented) except for poles $w_1, w_2, \dots, w_k$ inside C

Let the distinct zeros of f inside C be $z_1, \dots, z_\ell$, and assume $f(z)\neq 0$ $\forall z\in C$

Then $\displaystyle\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}\,dz = Z-P$ = # of zeros of f inside C − # of poles of f inside C

Proof:



Let $\gamma_1, \dots, \gamma_\ell$ be small pairwise disjoint circles centered at $z_1, \dots, z_\ell$

Let $C_1, \dots, C_k$ be small pairwise disjoint circles centered at $w_1, \dots, w_k$

By the residue thm. $\displaystyle\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}\,dz = \sum_{j=1}^{\ell}\mathrm{Res}\left(\frac{f'}{f}, z_j\right)+\sum_{i=1}^{k}\mathrm{Res}\left(\frac{f'}{f}, w_i\right) = \sum_{j=1}^{\ell}\mathrm{order}(z_j)-\sum_{i=1}^{k}\mathrm{order}(w_i) = Z-P$

Corollary: If f is analytic on and in C and $f(z)\neq 0$ $\forall z\in C$, then $\displaystyle\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}\,dz = Z$ of f inside C

Rouche's Thm.: Let f, g be analytic on and inside a simple closed curve C, and suppose $|g(z)| < |f(z)|$ $\forall z \in C$

Then $\underbrace{Z(f+g)}_{\text{\# zeros inside } C} = Z(f)$

Proof: For each $0 \le t \le 1$, let $h_t(z) = f(z) + t\,g(z)$

For each t, h_t is analytic on and inside C

$$Z(h_t) = \frac{1}{2\pi i} \int \frac{f'(z) + t g'(z)}{f(z) + t g(z)}\, dz$$

For $z \in C$, $f(z) + t g(z) = 0 \Rightarrow |f(z)| = t|g(z)| > |g(z)|$. Impossiple

$\therefore f(z) + t g(z) \neq 0 \;\; \forall z \in C$

Let $S(t) = Z(h_t)$

$S(t): [0,1] \longrightarrow \{0, 1, 2, 3, \dots\}$

TODO: Prove

$S(t)$ is a continuous function of t

By the intermediate value thm., $S(t) \equiv$ const.

$\Rightarrow S(0) = S(1)$

$Z(f) = Z(f+g)$

Ex) Find the # of zeros of $z^7 + 4z^3 + 2z - 1$ in $|z| < 1$

Write $z^7 + 4z^3 + 2z - 1 = f + g$, $|f|_{|z|=1} > |g|_{|z|=1}$

$f(z) = 4z^3$, $g(z) = z^7 + 2z - 1$, $Z(f) = 3$

on $|z| = 1$, $|f(z)| = 4$, $|g(z)| \le 1 + 1 + 1 = 3 < 4$

$\therefore Z(z^7 + 4z^3 + 2z - 1) = Z(f) = 3$

Ex) Find the # of zeros of $z^7 + 4z^3 + 2z - 1$ in $|z| < 2$

$f(z) = z^7$ $\qquad g(z) = 4z^3 + z - 1$

On $|z| = 2$, $|f(z)| = 2^7 > |g(z)|$

$Z(f) = 7$

$\therefore Z(z^7 + 4z^3 + 2z - 1) = 7$

Ex) Find the # of zeros of $z^7 + 4z^3 - z - 1$ in $1 < |z| < 2$

It would be $7 - 3 = 4$

(see last 2 examples)

Ex) 8) $|f(z)| < 1$ on $|z| = 1$

Show $f(z) = z^2$ has 2 solns. in $|z| < 1$

$Z(f(z) - z^2) = ?$

$|z^2| > |f(z)|$ on $|z| = 1$

$Z(z^2) = 2$

$\therefore Z(z^2 - f(z)) = 2$