For $z = x+iy \in \mathbb{C}$, we want to define $e^{z} = e^{x+iy}$

↳ Want $e^{z} = e^{x+iy} = e^{x}e^{iy}$

Def: for $z = x+iy \in \mathbb{C}$, define $e^{z} = e^{x}e^{iy} = e^{x}(\cos y + i\sin y)$

For $z_1 = x_1+iy_1$, $z_2 = x_2+iy_2 \in \mathbb{C}$

$$e^{z_1}e^{z_2} \underset{\text{by def.}}{=} e^{x_1}e^{iy_1}e^{x_2}e^{iy_2} = e^{(x_1+x_2)}e^{i(y_1+y_2)} = e^{(x_1+x_2)+i(y_1+y_2)} = e^{z_1+z_2} \rightarrow \boxed{e^{z_1+z_2} = e^{z_1}e^{z_2}}$$

$|e^{z}| = |e^{x}|\,|e^{iy}| = |e^{x}| > 0 \quad \therefore e^{z} \neq 0$ for any z

$$\frac{e^{z_1}}{e^{z_2}} = \frac{e^{x_1}e^{iy_1}}{e^{x_2}e^{iy_2}} = e^{x_1-x_2}e^{i(y_1-y_2)} = e^{x_1+iy_1-x_2-iy_2} = e^{z_1-z_2} \rightarrow \boxed{e^{z_1-z_2} = \frac{e^{z_1}}{e^{z_2}}}$$

↳ In particular $\boxed{\frac{1}{e^{z}} = e^{-z}}$

Analyticity: We've showed that e^{iy} is entire
↓
$e^{z} = e^{x}(\cos y + i\sin y)$ is entire

$$\frac{d}{dz}e^{z} = u_x(x,y) + i v_x(x,y) = e^{z} \rightarrow \boxed{(e^{z})' = e^{z}}$$

For $n \in \mathbb{Z}$, $e^{z+2n\pi i} = e^{z}e^{2n\pi i} = e^{z}\cdot 1 = e^{z} \rightarrow \boxed{e^{z} \text{ is not one-to-one}}$

↳ We say e^{z} is periodic w/ period $2\pi i$

$e^{(2n+1)\pi i} = e^{\pi i} = -1 \quad \forall n \in \mathbb{Z}$

Ex: Solve $e^{z} = 1+\sqrt{3}i = 2e^{i\frac{\pi}{3}}$

Let $z = x+iy$ be a soln.

$e^{x}e^{iy} = 2e^{i\frac{\pi}{3}}$

↳ $|e^{x}e^{iy}| = |2e^{i\frac{\pi}{3}}| \rightarrow e^{x} = 2 \rightarrow x = \ln 2$

$e^{iy} = e^{i\frac{\pi}{3}} \rightarrow e^{i(y-\frac{\pi}{3})} = 1 \rightarrow y - \frac{\pi}{3} \in \{2n\pi : n \in \mathbb{Z}\}$

$\therefore$ the solns. are $\{\ln 2 + i(\frac{\pi}{3} + 2\pi n) : n \in \mathbb{Z}\}$

Given any $w \neq 0$, there is a soln. of $e^{z} = w$

$z \in \{\ln(|w|) + i(\mathrm{Arg}(w) + 2n\pi) : n \in \mathbb{Z}\}$

Complex Analogue of the Logarithm:

Recall that for $y > 0$, if $e^{x} = y$ & $x \in \mathbb{R}$, then $x = \ln y$

Let $w \in \mathbb{C}$, $w \neq 0$. If $e^{z} = w$, then $z \in \{\ln(|w|) + i(\mathrm{Arg}(w) + 2n\pi) : n \in \mathbb{Z}\}$, $-\pi < \mathrm{Arg}(w) < \pi$

We define the multi-valued logarithmic function by $\log(z) = \ln|z| + i\arg(z)$, $z \neq 0$ $\quad (\arg(z) \in \{\mathrm{Arg}(z) + 2n\pi : n \in \mathbb{Z}\})$

We call the function $\mathrm{Log}(z) = \ln|z| + i\,\mathrm{Arg}(z)$ the principal branch of the logarithmic function

Ex: Find $\log(-1)$ and $\mathrm{Log}(-1)$

$\mathrm{Log}(-1) = \ln(|-1|) + i\,\mathrm{Arg}(-1) = i\pi$

$\log(-1) = \{\pi i + 2n\pi i : n \in \mathbb{Z}\}$

Ex: $\mathrm{Log}(-1-\sqrt{3}i) = \ln|-1-\sqrt{3}i| + i\,\mathrm{Arg}(-1-\sqrt{3}i) = \ln 2 - \frac{2\pi}{3}i$

$\log(-1-\sqrt{3}i) = \{\ln 2 + i(-\frac{2\pi}{3} + 2n\pi) : n \in \mathbb{Z}\}$

Consider $\underbrace{Log(z) = ln(|z|) + i\,Arg(z)}$, $z \neq 0$

Continuous everywhere except negative x axis ($Arg(z)$ has jump discontinuity on $-x$ axis)

If $z \in (-\infty, 0)$, then

$$\lim_{\substack{u+iv=w\to z \\ v>0}} Arg(w) = \pi \quad \text{, while}$$

$$\lim_{\substack{u+iv=w\to z \\ v<0}} Arg(w) = -\pi$$

$Log\, z$ is cont. only on $\mathbb{C}\setminus(-\infty, 0]$

Recall that $\displaystyle\lim_{w\to w_0} \frac{e^{w}-e^{w_0}}{w-w_0} = e^{w_0}$ ← Continuity

Let $a \in \mathbb{C}\setminus(-\infty,0]$

$$\frac{e^{Log z} - e^{Log a}}{Log z - Log a} = \frac{z-a}{Log z - Log a} = \frac{1}{\frac{Log z - Log a}{z-a}} \xrightarrow[\substack{\text{as } z\to a \\ \text{Since Log is} \\ \text{cont. at } a}]{} e^{Log a} = a$$

$$\Downarrow$$

$$\lim_{z\to a} \frac{Log z - Log a}{z-a} = \frac{1}{a}$$

$\therefore$ $Log\, z$ is differentiable & $\therefore$ analytic on $\mathbb{C}\setminus(-\infty,0]$

Moreover, $\displaystyle\frac{d}{dz} Log(z) = \frac{1}{z} \quad \forall z \in \mathbb{C}\setminus(-\infty,0]$

Any other log is also analytic on $\mathbb{C}\setminus(-\infty,0]$ and $\displaystyle\frac{d}{dz} log(z) = \frac{1}{z}$ } Since you are only adding a constant to $Log\, z$

The power Functions:

Want to define z^c, $z, c \in \mathbb{C}$

Motivation: $z^4 = e^{ln(z^4)} = e^{4 ln z}$

↳ This suggests $z^c = e^{c\, log z}$

Def: For $c \in \mathbb{C}$, $z \neq 0$, define the multi-valued function $z^c = e^{c\,log z} = e^{c(ln|z| + i\,arg(z))} = e^{c(ln|z| + i(Arg(z) + 2n\pi))}$, $n \in \mathbb{Z}$

Note that z^c is cont. on $\mathbb{C}\setminus(-\infty,0]$

$z^c = e^{c\,log z}$ is analytic on $\mathbb{C}\setminus(-\infty,0]$

The principal value/principal branch of z^c, $P.V.(z^c) = e^{c\,Log z}$

If $z \neq 0$, $a, b \in \mathbb{C}$

$$\frac{z^a}{z^b} = \frac{e^{a\,log z}}{e^{b\,log z}} = e^{(a-b)\,log z} = z^{a-b}$$

$$z^a z^b = e^{a\,log z} e^{b\,log z} = e^{(a+b)\,log z} = z^{a+b}$$

For $z \in \mathbb{C}\setminus(-\infty,0]$, $\displaystyle\frac{d}{dz} z^c = \frac{d}{dz} e^{c\,log z} = \frac{c}{z} e^{c\,log z} = \frac{c z^c}{z} = c z^{c-1}$

Ex: $i^i = e^{i\,log i} = e^{i(ln|i| + i\,arg(z))} = \overbrace{e^{-(\frac{\pi}{2} + 2n\pi)}}^{\text{Infinitely many solns.}}$, $n \in \mathbb{Z}$

Ex: $(1-i)^{\frac{3}{2}} = e^{\frac{3}{2}log(1-i)} = e^{\frac{3}{2}(ln|1-i| + i\,arg(1-i))} = e^{\frac{3}{2}ln\sqrt{2} + \frac{3}{2}i(-\frac{\pi}{4} + 2n\pi)}$, $n \in \mathbb{Z}$

$$= \underbrace{e^{\frac{3}{2}ln\sqrt{2}}\, e^{i(-\frac{3\pi}{8} + 3n\pi)}}_{2 \text{ solns.}}$$

Ex: $4^{\frac{1}{2}} = e^{\frac{1}{2}log 4} = e^{\frac{1}{2}(ln|4| + i\,arg(4))} = e^{\frac{1}{2}(ln 4 + i2n\pi)}$, $n \in \mathbb{Z}$

$= e^{\frac{1}{2}ln 4} e^{in\pi}$, $n \in \mathbb{Z}$

$= \{2, -2\}$

Ex: $4^2 = e^{2 log 4} = e^{2(ln 4 + i\,arg(4))} = e^{2 ln 4 + i4n\pi}$, $n \in \mathbb{Z}$

$= e^{2 ln 4}$

Trigonometric Functions:

Recall For $x \in \mathbb{R}$, $e^{ix} = \cos x + i \sin x$ $\therefore$ $e^{-ix} = \cos x - i \sin x$

$\rightarrow \sin x = \dfrac{e^{ix} - e^{-ix}}{2i}$, $x \in \mathbb{R}$

$\rightarrow \cos x = \dfrac{e^{ix} + e^{-ix}}{2}$, $x \in \mathbb{R}$

Want to define $\sin z$ & $\cos z$ for $z \in \mathbb{C}$

Def: For $z \in \mathbb{C}$, $\sin z = \dfrac{e^{iz} - e^{-iz}}{2i}$

$\cos z = \dfrac{e^{iz} + e^{-iz}}{2}$ $\Big\}$ $\sin z$ & $\cos z$ are entire

$$\frac{d}{dz}\sin z = \frac{d}{dz}\left(\frac{e^{iz} - e^{-iz}}{2i}\right) = \frac{ie^{iz} + e^{-iz}}{2i} = \frac{e^{iz} + e^{-iz}}{2} = \cos z$$

$$\frac{d}{dz}\cos z = -\sin z$$

Periodicity $\sin(z + 2n\pi) = \dfrac{e^{i(z+2n\pi)} - e^{-i(z+2n\pi)}}{2i} = \dfrac{e^{iz} - e^{-iz}}{2i} = \sin(z)$

for $n \in \mathbb{Z}$, $z \in \mathbb{C}$

likewise, $\cos(z + 2n\pi) = \cos(z)$

For $z \in \mathbb{C}$

$(\sin z)^2 + (\cos z)^2 = 1$

$|\sin z|$ can be > 1

For $y \in \mathbb{R}$

$$|\sin(iy)| = \left|\frac{e^{-y} - e^{y}}{2i}\right| = \frac{1}{2}\left|\underbrace{e^{-y} e^{y}}_{\to\infty \text{ as } y\to\infty}\right|$$

$$\sin(z_1 \pm z_2) = \sin z_1 \cos z_2 \pm \sin z_2 \cos z_1$$

$$\cos(z_1 \pm z_2) = \cos z_1 \cos z_2 \mp \sin z_1 \sin z_2$$

Harmonic Functions:

We say a function

$$h: D \in \mathbb{C} \to \mathbb{R}$$

is harmonic on D if

$$\Delta h(x,y) = h_{xx}(x,y) + h_{yy}(x,y) = 0 \quad \text{for all } (x,y) \in D$$

the Laplace operator

Physical meaning of Δ:

$h(x,y)$ = the temperature at (x,y) at steady state

Then $\Delta h(x,y) = 0$. That is, h is harmonic

Thm:

Let $f(z) = u(x,y) + i v(x,y)$ be analytic on an open set D

Then $u(x,y) = Re[f]$ and

$v(x,y) = Im[f]$ are harmonic on D.

Proof: $u_x(x,y) = v_y(x,y)$ and

$u_y(x,y) = -v_x(x,y)$, $(x,y) \in D$

$u_{xx}(x,y) = v_{yx}(x,y)$ and

$u_{yy}(x,y) = -v_{xy}(x,y)$, $(x,y) \in D$

$\downarrow$

$\Delta u = u_{xx}(x,y) + u_{yy}(x,y) = 0 \quad \forall (x,y) \in D \quad \therefore u$ is harmonic on D.

Likewise, $\Delta v(x,y) = 0$

Ex of harmonic functions:

constant, $x = Re[x+iy]$, $x^2 - y^2 = Re[z^2]$, $xy = \frac{1}{2} Im(z^2)$, $e^x \cos y = Im(e^z)$, $e^x \sin y = Im(e^z)$, $\ln|z| = Re(\operatorname{Log} z)$

Ex: Let $g(x,y) = x^2 + y$

Is there f analytic on $\mathbb{C}$ s.t.

$$Re(f(x,y)) = g(x,y)$$

$\Delta g(x,y) = 2 \neq 0 \quad \therefore g$ cannot be the real part of an analytic fn.

If a statement is true for the real part of all analytic functions, it must also be true for the imaginary part of all analytic fns.

$f = u + iv$ analytic

$-if = v - iu$ analytic (product of analytic fns)

$Im(f) = Re(-if)$

Integration:

Let $w(t) = u(t) + iv(t)$, $a \le t \le b$

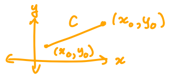
Def: $\int_a^b w(t)\,dt = \int_a^b (u(t)+iv(t))\,dt = \int_a^b u(t)\,dt + i\int_a^b v(t)\,dt$

Single variable

Parametrizing curves:

$z(t) = x(t) + iy(t)$

Ex:



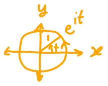
Parametrize C: $x_0 + iy_0 + t(x_1 - x_0 + i(y_1 - y_0))$, $0 \le t \le 1$

$= x(t) + iy(t)$

$\hookrightarrow x(t) = x_0 + (x_1 - x_0)t$

$y(t) = y_0 + (y_1 - y_0)t$, $0 \le t \le 1$

Ex:

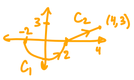


The circle $x^2+y^2=1$: $z(t) = e^{it} = \cos t + i\sin t$, $0 \le t \le 2\pi$

Ex: $|z - z_0| = R$

$z(t) = z_0 + Re^{it}$, $0 \le t \le 2\pi$

Ex:



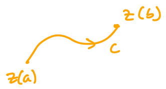
$C = C_1 \cup C_2$

$z(t) = \begin{cases} 2e^{i\pi}, & \pi \le t \le 2\pi \\ 2+2t+i(3t), & 0 \le t \le 1 \end{cases}$

Contour Integral:

Let $f(z)$ be continuous on a curve C parametrized by $C: z(t) = x(t) + iy(t)$, $a \le t \le b$

We define $\int_C f(z)\,dz = \int_a^b f(z(t))\,z'(t)\,dt$



Ex) Let $f(z) = \frac{1}{z}$ and C be the semicircle from -1 to 1

$C: z(t) = e^{it}$, $0 \le t \le \pi$

$z'(t) = ie^{it}$

$\int_C \frac{1}{z}\,dz = \int_0^\pi \frac{1}{z(t)} z'(t)\,dt = \int_0^\pi \frac{ie^{it}}{e^{it}}\,dt = i\int_0^\pi dt = i\pi$

Ex) Given $C: z(t)$, $a \le t \le b$

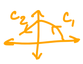
$-C$ will denote the same curve but reversed

$-C$ is parametrized by $w(t) = z(a+b-t)$

$w(a) = z(b)$, $w(b) = z(a)$

$$\boxed{\int_{-C} f(z)\,dz = -\int_C f(z)\,dz}$$

Ex) $C = C_1 \cup C_2$



$$\boxed{\int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz}$$

Upper Bound on modulus of an integral

Lemma: Let $w(t) = u(t) + iv(t)$ be cont. on $[a,b]$. Then

$$\left|\int_a^b w(t)\,dt\right| \le \int_a^b |w(t)|\,dt$$

Proof 1: $\int_a^b w(t)\,dt = re^{i\theta}$, $-\pi < \theta < \pi$

$\left|\int_a^b w(t)\,dt\right| = r = \int_a^b e^{-i\theta} w(t)\,dt = \int_a^b \mathrm{Re}\{e^{-i\theta} w(t)\}\,dt \le \int_a^b |\mathrm{Re}\{e^{-i\theta} w(t)\}|\,dt \le \int_a^b |e^{-i\theta} w(t)|\,dt = \int_a^b |w(t)|\,dt$

Real #, so = to its real part

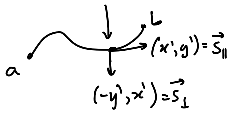
$z = x+iy$, $|\mathrm{Re}\{z\}| = |x| \le |z|$

Q.E.D.

Proof 2: $\left|\int_a^b w(t)\,dt\right| \simeq \left|\sum_{i=1}^{n} w(t_i)\Delta t_i\right| \underset{\text{Triangle inequality}}{\le} \sum_{i=1}^{n} \left|w(t_i)\Delta t_i\right| \simeq \int_a^b |w(t)|\,dt$

$\bar{F}=(u,-v)$

$f=(u,v)$



$f=u+iv$ $\quad$ $C: z(t)=x(t)+iy(t)$

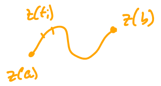
$$\int_C f(z)\,dz=\int_a^b (u+iv)(x'+iy')\,dt=$$

$$=\int_a^b [ux'-vy']\,dt+i\int_a^b [vx'+uy']\,dt$$

$$=\int_C \bar{F}\cdot d\vec{s}_{\parallel}+i\int_C \bar{F}\cdot d\vec{s}_{\perp}$$

Length of a Curve:

Let C be a curve parametrized by $z(t)$, $a\le t\le b$



$$L\approx\sum_{i=1}^{n}|z(t_i)-z(t_{i-1})| \simeq \int_a^b |z'(t)|\,dt$$

The length of C,

$$L=\int_a^b |z'(t)|\,dt$$

M-L estimate:

Let f be a continuous function on curve C of length L

Suppose $|f(z)|\le M$ for all $z\in C$. Then

$$\left|\int_C f(z)\,dz\right|\le ML$$

Proof: Let C be parametrized by $z(t)$, $a\le t\le b$

$$\left|\int_C f(z)\,dz\right|=\left|\int_a^b f(z)z'(t)\,dt\right| \underset{\text{by the lemma}}{\le} \int_a^b |f(z)||z'(t)|\,dt \le M\int_a^b |z'(t)|\,dt=ML$$

Ex) Let $C: z(t)=4e^{it}$, $0\le t\le \frac{\pi}{2}$

Estimate $\int_C \frac{z-1}{z^2+2}\,dz$

$L(C)=2\pi$

$$\left.\begin{array}{l} |z-1|\le|z|+|-1|=4+1=5 \\ |z^2+2|\ge|z^2|-|2|=16-2=14 \rightarrow \frac{1}{|z^2+2|}\ge 14 \end{array}\right\} \left|\frac{z-1}{z^2+2}\right|\le\frac{5}{14}$$

$$\boxed{\text{By ML thm } \int_C \frac{z-1}{z^2+2}\,dz \le \frac{10\pi}{14}}$$