Complex Numbers $\mathbb{C}$

$$\mathbb{C} = \{x+iy : x,y \in \mathbb{R}\}$$
$$i^2 = -1$$

Addition & Multiplication:

If $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$

$(x_i, y_i \in \mathbb{R})$

$$z_1 + z_2 = (x_1+x_2) + i(y_1+y_2)$$

$$z_1 z_2 = (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1)$$

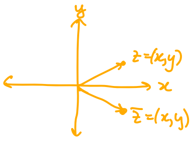
If $z_2 \neq 0$

$$\frac{z_1}{z_2} = \frac{x_1+iy_1}{x_2+iy_2} = \frac{x_1x_2+y_1y_2}{x_2^2+y_2^2} + i\,\frac{x_2y_1 - x_1y_2}{x_2^2+y_2^2}$$

$$|z| = \sqrt{x^2+y^2}$$

↳ E.g. circle is $\{z : |z| = 1\}$

Complex conjugate of z: $\bar{z} = x - iy$



Properties of conjugate:

$\bar{\bar{z}} = z$

$|\bar{z}| = |z|$

$\overline{z_1+z_2} = \bar{z}_1 + \bar{z}_2$

$\overline{z_1z_2} = \bar{z}_1\bar{z}_2$

$z\bar{z} = |z|^2$

If $x \in \mathbb{R}$, $\sqrt{x^2} = |x|$

Thm: Let $z = x+iy \in \mathbb{C}$, $x,y \in \mathbb{R}$

$$|z| = \sqrt{x^2+y^2} \geq \sqrt{x^2} = |x|, \quad \sqrt{y^2} = |y|$$

$$\downarrow$$

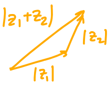
$$\begin{cases} |z| \geq |x| \geq x, -x \\ |z| \geq |y| \geq y, -y \end{cases}$$

$\mathrm{Re}(z) \leq |z|$

The triangle inequalities: Let $z_1, z_2 \in \mathbb{C}$, then

$$|z_1+z_2| \leq |z_1| + |z_2|$$

and $|z_1 - z_2| \leq |z_1| + |z_2|$



Proof: $|z_1+z_2|^2 = (z_1+z_2)(\bar{z}_1+\bar{z}_2) = |z_1|^2 + |z_2|^2 + z_1\bar{z}_2 + z_2\bar{z}_1 = |z_1|^2 + |z_2|^2 + z_1\bar{z}_2 + \overline{z_1\bar{z}_2} = |z_1|^2+|z_2|^2 + 2\,\mathrm{Re}\{z_1\bar{z}_2\} \leq |z_1|^2+|z_2|^2 + 2|z_1\bar{z}_2| = |z_1|^2 + |z_2|^2 + 2|z_1||z_2| = (|z_1|+|z_2|)^2$

Q.E.D.

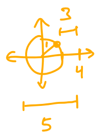
↳ Consequence: If $z_1, z_2 \in \mathbb{C}$, then $\big||z_1| - |z_2|\big| \leq |z_1 + z_2|$

Proof: $|z_1| = |z_1 - z_2 + z_2| \leq |z_1 - z_2| + |z_2|$

$$\left.\begin{array}{l} |z_1| - |z_2| \leq |z_1 - z_2| \\ \text{Thus } |z_2| - |z_1| \leq |z_2 - z_1| = |z_1 - z_2| \end{array}\right\} \therefore \big||z_1| - |z_2|\big| \leq |z_1 - z_2|$$

Q.E.D.

Ex) Suppose $|z| = 1$

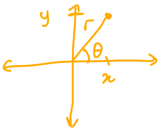


by triangle inequality $|z-4| \leq 5$ → Furthest point on unit circle → can be used to get farthest dist.

by consequence of triangle inequality $\big||z| - |4|\big| \geq 3$ → Closest point on unit circle → can be used to get closest dist.

Exponential (Polar) form of $z \in \mathbb{C}$:

Let $z = x + iy \neq 0$, $x, y \in \mathbb{R}$



(r, θ) are polar coordinates of z

arguments (angles) of $z = \{\theta + 2\pi n : n \in \mathbb{Z}\}$

By Argument of z we mean $-\pi < Arg(z) \leq \pi$ → also called Principal Argument

↳ Ex: $Arg(1-i) = -\frac{\pi}{4}$

$arg(1-i) = \{-\frac{\pi}{4} + 2\pi n : n \in \mathbb{Z}\}$

$x = r\cos\theta$
$y = r\sin\theta$ → $z = r(\cos\theta + i\sin\theta)$

Euler's notation: $e^{i\theta} = \cos\theta + i\sin\theta$

Products & Quotients of Exponentials:

If $z \neq 0$, $r = |z|$, θ is an argument of z, then $z = re^{i\theta}$

$$e^{i\theta_1}e^{i\theta_2} = (\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) = \cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2) = e^{i(\theta_1+\theta_2)}$$

$$\frac{e^{i\theta_1}}{e^{i\theta_2}} = \frac{e^{i\theta_1}}{e^{i\theta_2}} \cdot \frac{e^{-i\theta_2}}{e^{-i\theta_2}} = \frac{e^{i(\theta_1-\theta_2)}}{1} = e^{i(\theta_1-\theta_2)}$$

De Moivre's formula

$(e^{i\theta})^n = e^{in\theta}$, $n \in \mathbb{Z}$

↳ $z = re^{i\theta} \rightarrow z^n = r^n e^{in\theta}$

Ex: Solve $z^2 = -4$

Let $z = re^{i\theta}$ → $r^2 e^{i2\theta} = -4 = 4e^{i\pi}$ → $\cancel{4}e^{i2\theta} = \cancel{4}e^{i\pi}$

$|r^2 e^{i2\theta}| = |4e^{i\pi}|$ $\quad e^{i(2\theta - \pi)} = 1$

$r^2 = 4 \rightarrow r = 2$ $\quad \cos(2\theta - \pi) + i\sin(2\theta - \pi) = 1$

$\left.\begin{matrix}\cos(2\theta-\pi) = 1 \\ \sin(2\theta - \pi) = 0\end{matrix}\right\}$ $2\theta - \pi \in \{2n\pi : n \in \mathbb{Z}\}$

$\theta \in \{(n + \frac{1}{2})\pi : n \in \mathbb{Z}\}$

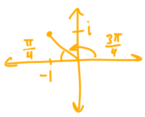
$z \in \{2e^{i(n+\frac{1}{2})\pi}, n \in \mathbb{Z}\} = \{2e^{i\pi/2}, 2e^{i3\pi/2}\}$

(the only 2 distinct elements)

Ex: Find $(-1+i)^{1/5}$ → we need to find z st $z^5 = -1 + i$

Look for z in the form $z = re^{i\theta}$

$z^5 = r^5 e^{i5\theta} = -1 + i$



$r^5 = \sqrt{2} \rightarrow r = 2^{\frac{1}{10}}$

$5\theta \in \{\frac{3\pi}{4} + 2\pi n : n \in \mathbb{Z}\}$

$\theta \in \{\frac{3\pi}{20}, \frac{3\pi}{20} + \frac{2\pi}{5}, \frac{3\pi}{20} + \frac{4\pi}{5}, \frac{3\pi}{20} + \frac{6\pi}{5}, \frac{3\pi}{20} + \frac{8\pi}{5}, \frac{3\pi}{20} + 2\pi, \dots\}$

$z \in \{2^{\frac{1}{10}}e^{i3\pi/20}, 2^{\frac{1}{10}}e^{i11\pi/20}, 2^{\frac{1}{10}}e^{i19\pi/20}, 2^{\frac{1}{10}}e^{i27\pi/20}, 2^{\frac{1}{10}}e^{i35/20}\}$ → The five roots

nth roots of unity:

Solve $z^n = 1$

$z = re^{i\theta}$ $\quad r^n e^{in\theta} = 1 = 1 \cdot e^{i \cdot 0}$

$r = 1$, $n\theta \in \{2k\pi : k \in \mathbb{Z}\}$

$\theta \in \{\frac{2k}{n}\pi : k \in \mathbb{Z}\}$

↓

$\theta \in \{0, \frac{2\pi}{n}, \frac{4\pi}{n}, \dots, \frac{2(n-1)}{n}\pi\}$

Sets in complex plane :

Topology of the plane:

An ε neighborhood of a point $z_0 \in \mathbb{C}$ is the set $\{z : |z-z_0| < \varepsilon\}$

A deleted neighborhood of $z_0 \in \mathbb{C}$ ← (w/o z_0 itself)

is a set of the form $\{z : 0 < |z-z_0| < \varepsilon\}$

Let $S \subseteq \mathbb{C}$ be a set

neighborhood (abbreviation)

A point $z_0 \in \mathbb{C}$ is called an interior point of S if S contains a nbhd of z_0

A point $z_0 \in \mathbb{C}$ is called an exterior point of S if there is a nbhd of S which contains no point of S

A point $z_0 \in \mathbb{C}$ is called a boundary point of S if every nbhd of z_0 intersects both S and S^c

Ex: Let $S = \{z : |z| < 1\}$ $\quad int(S) = S \quad ext(S) = \{z : |z| > 1\} \quad boundary(S) = \{z : |z| = 1\}$

Ex: Let $S = \{1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}, \dots\}$ Find boundary of S

↳ Boundary of S is $S \cup \{0\}$

We say a set $S \in \mathbb{C}$ is an open set if S contains none of its boundary points → Usual def : a set is open if every point in the set is an interior point

We say a set $S \in \mathbb{C}$ is a closed set if S contains all of its boundary points → a closed set is a set which is the complement of an open set

Ex: $\{z : |z| < 1\}$ is open

Ex: $\{z : 1 \le |z| \le 2\}$ is closed

Ex: $\{z : 1 < |z| \le 2\}$ is neither open nor closed

Ex: $\mathbb{Q} = \{\frac{m}{n} : m, n \in \mathbb{Z}, n \neq 0\}$ $\quad$ boundary of $\mathbb{Q} = \mathbb{R}$

A point z_0 is called an accumulation point of S if every deleted neighborhood of z_0 contains a point of S

Ex: $A = \{z : |z| < 1\}$

Accumulation points of $A = \{z : |z| \le 1\}$

Ex: $S = \{1, 2, 3, \dots\}$ → S has no accumulation points

Ex: Let $S = \{z_1, \dots, z_n\}$ be a finite set

Suppose z_0 is an acc. pnt. of S → every nbhd. has pnt. of S → can take nbhds smaller than any given point infinitely → must have ∞ points in S

$\boxed{\text{If a set } S \text{ has an accumulation point, then } S \text{ has an } \infty \text{ \# of points}}$

Let z_0 be an acc. pnt. of S.

If s is ext. pnt. → $\exists$ nbhd of s that does not intersect S → not acc. pnt.

$\boxed{\text{So if } z_0 \text{ is acc. pnt.} \Rightarrow \text{must be either interior or boundary point}}$

An open set D is connected if for any 2 points $A, B \in D$, there is a polygonal path in D from A to B

↳ path consisting of line segments

A set that is open and connected is called a domain

Suppose $S \subseteq \mathbb{C}$ is a set. The closure of S is $\bar{S} = S \cup boundary(S)$

Complex-Valued Functions: $f(z) = f(x+iy) = u(x,y) + iv(x,y)$

$$u(x,y) = \mathrm{Re}\ f(x+iy)$$
$$v(x,y) = \mathrm{Im}\ f(x+iy)$$

Limits of functions:

We say $\lim_{z\to z_0} f(z) = w_0$ if for any $\varepsilon>0$, there is $\delta>0$ st.

when $0<|z-z_0|<\delta$, then $|f(z)-w_0|<\varepsilon$

Ex: Show that the $\lim_{z\to z_0} 2z+i = 2z_0+i$

Sol: Let $\varepsilon>0$

We want $|f(z)-w_0| = |2z+i-2z_0-i| = 2|z-z_0| < \varepsilon$

Chose $\delta = \frac{\varepsilon}{2}$

If $|z-z_0|<\delta = \frac{\varepsilon}{2} \to 2|z-z_0|<\varepsilon$

Ex: Consider $\lim_{z\to 0} \frac{\bar{z}}{z}$

$$\left.\begin{aligned} &\lim_{\substack{z\to 0\\ z=x}} \frac{\bar{z}}{z} = \lim_{x\to 0} \frac{\bar{x}}{x} = 1 \\ &\lim_{\substack{z\to 0\\ z=y}} \frac{\bar{z}}{z} = \lim_{y\to 0} \frac{-iy}{iy} = -1 \end{aligned}\right\} \lim_{z\to 0} \frac{\bar{z}}{z} \text{ does not exist}$$

Thm 1: Suppose $f(z) = u(x,y) + iv(x,y)$

Let $z_0 = x_0 + iy_0$, $w_0 = u_0 + iv_0$

Limit of real part   Limit of imaginary part

$\lim_{z\to z_0} f(z) = w_0$ iff $\lim_{(x,y)\to(x_0,y_0)} u(x,y) = u_0$ and $\lim_{(x,y)\to(x_0,y_0)} v(x,y) = v_0$

Proof: Recall that if $a,b\in\mathbb{R}$, then $|a|,|b| \le |a+ib| \le |a|+|b|$ $\quad \begin{cases} |a+ib|^2 = a^2+b^2 \\ (|a|+|b|)^2 = a^2+b^2+2|a||b| \end{cases}$

Part 1)

Suppose $\lim_{z\to z_0} f(z) = w_0$. Let $\varepsilon>0$. There is $\delta>0$ st if $0<|z-z_0|<\delta$, $|f(z)-w_0|<\varepsilon$

$$|f(z)-w_0| = |u(x,y)-u_0+i(v(x,y)-v_0)| \ge |u(x,y)-u_0|, |v(x,y)-v_0|$$

When $|z-z_0|<\delta$ (i.e. $|(x,y)-(x_0,y_0)|<\delta$), $|u(x,y)-u_0|, |v(x,y)-v_0| \le |f(z)-w_0| \le \varepsilon \to \lim_{(x,y)\to(x_0,y_0)} u(x,y) = u_0$ and $\lim_{(x,y)\to(x_0,y_0)} v(x,y) = v_0$

Part 2)

Suppose $\lim_{(x,y)\to(x_0,y_0)} u(x,y) = u_0$ and $\lim_{(x,y)\to(x_0,y_0)} v(x,y) = v_0$

Let $\varepsilon>0$. $\exists\, \delta_1, \delta_2$ st. when $0<|z-z_0|<\delta_1$, $|u(x,y)-u_0|<\varepsilon$

and when $0<|z-z_0|<\delta_2$, $|v(x,y)-v_0|<\varepsilon$

Let $\delta = \min\{\delta_1, \delta_2\}$

Then when $0<|z-z_0|<\delta$, $|u(x,y)-u_0|<\varepsilon$ and $|v(x,y)-v_0|<\varepsilon$

and therefore $|f(z)-w_0| = |u(x,y)-u_0+i(v(x,y)-v_0)| \le |u(x,y)-u_0| + |v(x,y)-v_0| < \varepsilon+\varepsilon = 2\varepsilon$

Thm 2: Suppose $\lim_{z\to z_0} f(z) = w_1$, $\lim_{z\to z_0} g(z) = w_2$

Then $\lim_{z\to z_0} [f(z)+g(z)] = w_1+w_2$

$\lim_{z\to z_0} [f(z)g(z)] = w_1 w_2$

$\lim_{z\to z_0} \left[\frac{f(z)}{g(z)}\right] = \frac{w_1}{w_2}$ if $w_2\ne 0$

We say complex-valued function f is <u>continuous</u> at z_0 if $\lim_{z\to z_0} f(z) = f(z_0)$

Thm: If f & g are continuous at z_0, then $f+g$, fg, $\frac{f}{g}$ (if $g(z_0)\ne 0$) are continuous at z_0

Thm: Suppose f is continuous at z_0 and $f(z_0)\ne 0$, there exists a neighborhood of z_0 where $f(z)\ne 0$

Proof: Let $\varepsilon = \frac{|f(z_0)|}{2}$

Since $\lim_{z\to z_0} f(z) = f(z_0)$, there is $\delta>0$ st. when $|z-z_0|<\delta$, $|f(z)-f(z_0)| < \frac{|f(z_0)|}{2}$

Let $|z-z_0|<\delta$, $|f(z)| = |f(z_0)+(f(z)-f(z_0))| \ge |f(z_0)| - |f(z)-f(z_0)| \ge |f(z_0)| - \frac{|f(z_0)|}{2} = \frac{|f(z_0)|}{2} > 0$

Thm: Let f be cont. at z_0. Then $\exists M>0$ st. $|f(z)|\le M$ for z on a neighborhood of z_0 (function is bounded at continuous points)

Proof: $\lim_{z\to z_0} f(z) = f(z_0)$ (def. of continuity)

$\downarrow$

Let $\varepsilon = 1$. There is $\delta > 0$ st. if $|z-z_0|<\delta$, then $|f(z)-f(z_0)|<1$

If $|z_0 - z|<\delta$, $|f(z)| = |(f(z)-f(z_0)) + f(z_0)| \le |f(z)-f(z_0)| + |f(z_0)| < 1 + |f(z_0)| = M$

Differentiable Functions

Let f be a function defined in a neighborhood of z_0

We say f is differentiable at z_0 if $\displaystyle\lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0} = f'(z_0)$ exists.

↳ If it exists, we say f has a derivative at z_0

↳ Notation: Let $\Delta z = z - z_0$ $\to z = z_0 + \Delta z$

$$\lim_{z\to z_0}\frac{f(z)-f(z_0)}{z-z_0} = \lim_{\Delta z\to 0}\frac{f(z_0+\Delta z)-f(z_0)}{\Delta z}$$

Ex: $f(z) = c \in \mathbb{C}$

Let $z \in \mathbb{C}$

$$\lim_{h\to 0}\frac{f(z+h)-f(z)}{h} = \lim_{h\to 0}\frac{c-c}{h} = 0$$ $\therefore f$ is differentiable at any z and $f'(z)=0$

Ex: $g(z) = az+b$, $a,b\in\mathbb{C}$

$$\lim_{h\to 0}\frac{g(z+h)-g(z)}{h} = \lim_{h\to 0}\frac{ah}{h} = a$$ $\therefore g$ is differentiable on $\mathbb{C}$ and $g'(z)=a$

Ex: $f(z)=\bar{z}$. Let $z\in\mathbb{C}$

$$\lim_{h\to 0}\frac{f(z+h)-f(z)}{h} = \lim_{h\to 0}\frac{\bar{z}+\bar{h}-\bar{z}}{h} = \lim_{h\to 0}\frac{\bar{h}}{h} = \lim_{h\to 0}\frac{x-iy}{x+iy}$$ $\displaystyle\lim_{h\to 0}\frac{\bar{h}}{h}$ DNE $\therefore f$ is not differentiable at any point in $\mathbb{C}$

Ex: $g(z)=z^2$. Let $z\in\mathbb{C}$

$$\lim_{h\to 0}\frac{g(z+h)-g(z)}{h} = \lim_{h\to 0}\frac{(z+h)^2-z^2}{h} = \lim_{h\to 0}\frac{2zh+h^2}{h} = \lim_{h\to 0} 2z+h = 2z$$

Generally: $g(z)=z^n$. Let $z\in\mathbb{C}$

$$\lim_{h\to 0}\frac{(z+h)^n-z^n}{h} = \lim_{h\to 0}\frac{nz^{n-1}h+\binom{n}{2}z^{n-2}h^2+\dots+h^n}{h} = \lim_{h\to 0} nz^{n-1}+\binom{n}{2}z^{n-2}h+\dots+h^{n-1} = nz^{n-1}$$ $\therefore f$ is differentiable on $\mathbb{C}$ and $f'(z)=nz^{n-1}$

Thm: If f is differentiable at z_0, then it is continuous at z_0

Proof: $f(z)-f(z_0) = \left(\frac{f(z)-f(z_0)}{z-z_0}\right)(z-z_0)$

↳ $\displaystyle\lim_{z\to z_0}(f(z)-f(z_0)) = \lim_{z\to z_0}\left(\frac{f(z)-f(z_0)}{z-z_0}\right)\lim_{z\to z_0}(z-z_0)$

$\therefore \displaystyle\lim_{z\to z_0} f(z) = \lim_{z\to z_0}\left[(f(z)-f(z_0))+f(z_0)\right] = \lim_{z\to z_0}[f(z)-f(z_0)] + \lim_{z\to z_0} f(z_0) = 0 + f(z_0)$ $\therefore f$ is continuous at z_0

Thm: Suppose f & g are differentiable at z_0

Then so is $f+g$, cf $(c\in\mathbb{C})$, $f\cdot g$, $\frac{f}{g}$ (if $g(z_0)\neq 0$)

Moreover $(f\pm g)'(z_0) = f'(z_0) \pm g'(z_0)$

$(f\cdot g)'(z_0) = f'(z_0)g(z_0) + f(z_0)g'(z_0)$

$$\left(\frac{f}{g}\right)'(z_0) = \frac{f'(z_0)g(z_0) - f(z_0)g'(z_0)}{g(z_0)^2}$$

Proof: $$\frac{f(z)g(z)-f(z_0)g(z_0)}{z-z_0} = \frac{f(z)g(z)-f(z_0)g(z)+f(z_0)g(z)-f(z_0)g(z_0)}{z-z_0} = g(z)\left(\frac{f(z)-f(z_0)}{z-z_0}\right) + f(z_0)\left(\frac{g(z)-g(z_0)}{z-z_0}\right)$$

As $z\to z_0$, this approaches $g(z_0)f'(z_0)+f(z_0)g'(z_0)$

$\uparrow$ Since g is continuous

(it is the product of differentiable functions)

Ex: $z^2 = z\cdot z$ is differentiable on $\mathbb{C}$ $\to$ z^n is differentiable on $\mathbb{C}$

If $n\in\{-1,-2,\dots\}$ $z^n = \frac{1}{z^{-n}}$ $\to$ z^n is differentiable on $\mathbb{C}$

Thm: If g is differentiable at z_0 and f is differentiable at $g(z_0)$, then $f\circ g(z) = f(g(z))$ is differentiable at z_0 and $(f\circ g)'(z_0) = f'(g(z_0))\, g'(z_0)$

Proof: $\dfrac{f(g(z)) - f(g(z_0))}{z - z_0} = \dfrac{f(g(z)) - f(g(z_0))}{g(z) - g(z_0)} \dfrac{g(z) - g(z_0)}{z - z_0}$

As $z \to z_0$, $g(z) \to g(z_0)$ (since it is continuous), so the expression becomes $f'(g(z_0))\, g'(z_0)$

Cauchy-Riemann Equations:

Suppose $f(z) = u(x,y) + iv(x,y)$, $z = x+iy$

is differentiable at $z_0 = x_0 + iy_0 = (x_0, y_0)$

Then $\lim\limits_{\substack{h\to 0}} \dfrac{f(z_0+h) - f(z_0)}{h} = f'(z_0)$

$h \in \mathbb{C}$

Then $\lim\limits_{s\to 0} \dfrac{f(z_0+s) - f(z_0)}{h} = f'(z_0)$, that is

$s \in \mathbb{R}$

$(z_0 + s = (x_0, y_0) + (s, 0) = (x_0+s, y_0))$

$$\lim_{s\to 0} \frac{u(x_0+s, y_0) + iv(x_0+s, y_0) - u(x_0, y_0) - iv(x_0, y_0)}{s} = f'(z_0)$$

$$\lim_{s\to 0} \left[\frac{u(x_0+s, y_0) - u(x_0, y_0)}{s} + i \frac{v(x_0+s, y_0) - v(x_0, y_0)}{s} \right] = f'(z_0) = u_0 + iv_0$$

$\downarrow$

$u_x(x_0, y_0) = u_0 \qquad v_x(x_0, y_0) = v_0$

and so $f'(z_0) = u_0 + iv_0 = u_x(x_0, y_0) + iv_x(x_0, y_0)$

Let $h = it$, $t \in \mathbb{R}$, $t \to 0$

$$\lim_{t\to 0} \frac{f(z+it) - f(z_0)}{it} = f'(z_0) = u_0 + iv_0$$

$(z + it = (x_0, y_0) + (0, t) = (x_0, y_0 + t))$

$$\lim_{t\to 0} \left[\frac{u(x_0, y_0+t) - u(x_0, y_0)}{it} + i \frac{v(x_0, y_0+t) - v(x_0, y_0)}{it} \right] = u_0 + iv_0$$

$\frac{1}{i} u_y(x_0, y_0) + v_y(x_0, y_0) = u_0 + iv_0 = f'(z_0)$

$u_0 = v_y(x_0, y_0) \qquad v_0 = -u_y(x_0, y_0)$

and so $f'(z_0) = u_0 + iv_0 = v_y(x_0, y_0) - iu_y(x_0, y_0)$

So if f is differentiable at z_0

$v_y(x_0, y_0) - iu_y(x_0, y_0) = u_x(x_0, y_0) + iv_y(x_0, y_0)$

$$\boxed{\begin{aligned} u_x(x_0, y_0) &= v_y(x_0, y_0) \\ u_y(x_0, y_0) &= -v_x(x_0, y_0) \end{aligned}}$$ Cauchy-Riemann Equations

We proved: If $f = u + iv$ is differentiable at $z_0 = x_0 + iy_0$, then u_x, u_y, v_x, v_y exist at (x_0, y_0) and they satisfy the C-R equations at z_0, that is:

$$\begin{cases} u_x(x_0, y_0) = v_y(x_0, y_0) \\ u_y(x_0, y_0) = -v_x(x_0, y_0) \end{cases}$$

You can use this thm. to check if a function is NOT differentiable at z_0 (not necessarily the converse)

Ex: Let $f(z) = |z|^2 = x^2 + y^2$

$u_x = 2x \qquad u_y = 2y$

$v_x = 0 \qquad v_y = 0$

$$\begin{cases} 2x = 0 \\ 2y = 0 \end{cases}$$

This function is not differentiable at all points outside the origin.

Thm: Determining when function is differentiable

Suppose $f(z)=u(x,y)+iv(x,y)$. If:

u_x, u_y exist in a nbhd of (x_0,y_0) $(z_0=x_0+iy_0)$ and are continuous at (x_0,y_0) and

$u_x(x_0,y_0)=v_y(x_0,y_0)$

$u_y(x_0,y_0)=-v_x(x_0,y_0)$

the f is differentiable at z_0 and

$f'(z_0)=u_x(x_0,y_0)+iv_x(x_0,y_0)$

We say f is <u>analytic</u> at z_0 if f is differentiable in a nbhd of z_0

We say f is <u>entire</u> if it is analytic in $\mathbb{C}$

Ex: $f(z)=e^x e^{iy}=e^x(\cos y+i\sin y)=u(x,y)+iv(x,y)$

$u(x,y)=e^x\cos y$, $v(x,y)=e^x\sin y$

$u_x(x,y)=e^x\cos y$ $\quad v_x(x,y)=e^x\sin y$

$u_y(x,y)=-e^x\sin y$ $\quad v_y(x,y)=e^x\cos y$

u_x,u_y,v_x,v_y are cont. on $\mathbb{C}$

C-R equations hold on $\mathbb{C}$

$\therefore f$ is differentiable on $\mathbb{C}$ $\rightarrow$ f is entire

Ex: Let $g(z)=|z|^2=x^2+y^2$

$u(x,y)=x^2+y^2$, $v(x,y)=0$

$u_x(x,y)=2x$ $\quad v_x=0$

$u_y(x,y)=2y$ $\quad v_y=0$

u_x,u_y,v_x,v_y are cont. on $\mathbb{C}$

$$\begin{cases} u_x(x,y)=v_y(x,y) \\ u_y(x,y)=v_y(x,y)\end{cases} \rightarrow \begin{cases} 2x=0 \\ 2y=0\end{cases} \rightarrow x=0,\ y=0$$

g is not differentiable at any $z\neq 0$

g is differentiable at 0

g is not analytic at any point

Ex: $f(z)=\frac{z^2+1}{z^2-1}$

f is differentiable on $\mathbb{C}\setminus\{-1,1\}$

f is analytic on $\mathbb{C}\setminus\{-1,1\}$

Thm: If f and g are analytic at z_0, then so are $f\pm g$, $f\cdot g$, $\frac{f}{g}$ $(g(z_0)\neq 0)$

Thm: If $h: D\in\mathbb{C}\rightarrow\mathbb{R}$ is a real-valued function, D is a <u>domain</u> (open & connected) and $h_x(x,y)=0=h_y(x,y)$ on D, then h is constant

Proof:

Prove $h(A)=h(B)$

Choose polygonal path $A\rightarrow B$ consisting of only vertical & horizontal components

By mean value thm, h is constant on each horizontal & vertical segment

Thm: Suppose f is analytic on a domain D and $f'(z)=0$ for all $z\in D$

then f is constant.

Proof: Let $z=x+iy\in D$

$0=f'(z)=u_x(x,y)+iv_x(x,y)$ $\rightarrow$ $u_x(x,y)=0$, $v_x(x,y)=0$

B/c f is differentiable at (x,y), by C-R eqs, $u_y(x,y)=0$, $v_y(x,y)=0$

$\therefore$ by previous thm. u & v are constant $\rightarrow f$ is constant

Thm: Suppose $f=u+iv$ and $\bar{f}=u-iv$ are analytic on a domain D. Then f is constant $\rightarrow$ Unless f is a constant, $\bar{f}$ is never analytic

Proof: f analytic on $D\rightarrow u_x=v_y$, $u_y=-v_x$ on D

$\bar{f}$ analytic on $D\rightarrow u_x=-v_y$, $u_y=v_x$ on D

$v_y=-v_y\rightarrow v_y=0$

$v_x=-v_x\rightarrow v_x=0$ $\quad$ on D

$u_x=u_y=0$

u,v constant $\rightarrow f$ constant $\quad$ (or $f'=u_x+iv_x=0\rightarrow f$ constant)