Sequences: We say a sequence $z_n = x_n + iy_n$ **converges** to $z = x+iy$ if for any $\varepsilon > 0$,

$\exists N$ such that $|z_N - z| < \varepsilon$ $\forall n \geq N$

Remark: $z_n = x_n + iy_n \to x+iy$
iff $x_n \to x$ and $y_n \to y$

Proof: $0 \le |x_n - x|, |y_n - y| \le |z_n - z| \le |x_n + x| + |y_n + y|$

Ex) Let $|z| < 1$

$\{z^n\} = \{z, z^2, z^3, \dots\}$

Let $z \neq 0$, $|z^n| = |z|^n = e^{\ln(|z|^n)} = e^{n \ln|z|}$

Since $|z| < 1$, $\ln|z| < 0$, so $\lim_{n\to\infty} e^{n\ln|z|} = 0$

Series: We say a series $\sum_{k=1}^{\infty} z_k = \sum_{k=1}^{\infty} (x_k + iy_k)$ **converges** to a sum $S = X + iY$ if the sequence $\{S_N\}$ of partial sums $S_N = \sum_{k=1}^{N} z_k$ converges to S

Remark: The series $\sum_{k=1}^{\infty} z_k$ converges to $S = X+iY$ iff $\sum_{k=1}^{\infty} x_k$ converges to X and $\sum_{k=1}^{\infty} y_k$ converges to Y

Proof: $\sum_{k=1}^{\infty} z_k$ conv. to z $\leftrightarrow$ $S_n \to S = X + iY$ $\overset{\text{Remark 1}}{\longleftrightarrow}$ $\{\sum_{k=1}^{\infty} x_k\} \to X$ and $\{\sum_{k=1}^{\infty} y_k\} \to Y$

$\sum_{k=1}^{\infty} z_k$ conv. to S iff $\{R_N\} = \{S - S_N\}$ conv. to 0

A **power series** centered at z_0 is a series of the form $\sum_{k=0}^{\infty} a_k (z - z_0)^k$

↳ There is some disk in which this series converges

Ex) Lemma: $\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$ for $|z| < 1$ ← Geometric series

Proof: $(1-z)(1 + z + z^2 + \dots + z^N) = 1 - z^{N+1}$
if $z \neq 1$, $S_N = 1 + z + \dots + z^N = \frac{1 - z^{N+1}}{1-z}$

Since $|z| < 1$, $\lim_{N\to\infty} S_N = \frac{1}{1-z}$

If $|z| \geq 1$, Since $\lim_{n\to\infty} z^n \neq 0$, $\sum_{n=1}^{\infty} z^n$ diverges

Thm: Suppose f is analytic in a disc

$\{z : |z - z_0| < R\}$ $(0 < R \le \infty)$

Then $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$ for $|z - z_0| < R$ where $a_n = \frac{f^{(n)}(z_0)}{n!}$ $\forall n = 0, 1, 2, \dots$

Proof: Since f is analytic on $|z - z_0| < R$, we can use the CIF

Assume $z_0 = 0$. Let $|z| < R$. Choose $|z| < r < R$. Let $C = \{w : |w| = r\}$, positively oriented.

By the CIF, $f(z) = \frac{1}{2\pi i} \int_C \frac{f(w)}{w - z} dw$

For $w \in C_R$, $\frac{1}{w-z} = \frac{1}{w}\left(\frac{1}{1 - z/w}\right) = \frac{1}{w} \sum_{k=0}^{n} \frac{z^k}{w^k} = \frac{1}{w} \sum_{k=0}^{n} \left(\frac{z^k}{w^k} + \frac{\left(\frac{z}{w}\right)^{n+1}}{1 - \frac{z}{w}} \right)$

$\left|\frac{z}{w}\right| < \frac{r}{r}$

$\frac{f(w)}{w-z} = \frac{f(w)}{w} \left(\sum_{k=0}^{n} \frac{z^k}{w^k} + \frac{\left(\frac{z}{w}\right)^{n+1}}{1 - \frac{z}{w}} \right) = \sum_{k=0}^{n} \frac{f(w)}{w^{k+1}} z^k + \frac{f(w)\left(\frac{z}{w}\right)^{k+1}}{w - z}$

$\left| \int_{C_R} \frac{f(w)}{w-z} \left(\frac{z}{w}\right)^{n+1} dw \right| \le \frac{2\pi r}{r - |z|} \left(\frac{|z|}{r} \right)^{n+1}$

Since $|z| < r$,
as $n \to \infty$ this $\to 0$

Thus $f(z) = \frac{1}{2\pi i} \sum_{k=0}^{n} \left(\int_{C_R} \frac{f(w)}{w^{k+1}} dw \right) z^k + \frac{1}{2\pi i} \int \frac{f(w)\left(\frac{z}{w}\right)^{k+1}}{w - z} dw$

$|w - z| \ge |w| - |z| = r - |z| > 0$

an $n \to \infty$, This $\to 0$ (by ML estimate)

$f(z) = \frac{1}{2\pi i} \sum_{k=0}^{n} z^k \int_{C_R} \frac{f(w)}{w^{k+1}} dw = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} z^k$

By general CIF

Q.E.D!

If center is not 0, we just look at $f(z + z_0)$

Taylor's Theorem: Suppose f is analytic on $\{z : |z-z_0| < R\}$

Then $f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n \quad \forall z,\ |z-z_0| < R$

and $a_n = \dfrac{f^{(n)}(z_0)}{n!} = \dfrac{1}{2\pi i} \displaystyle\int_C \dfrac{f(z)}{(z-z_0)^{n+1}}\, dz$

In general, given f on $|z-z_0|<R$, let $g(z) = f(z+z_0)$

Then g is analytic on $|z|<R$, $g(z) = \sum_{n=0}^{\infty} a_n z^n$, $|z|<R$, $a_n = \dfrac{g^{(n)}(0)}{n!}$

$f(z) = g(z-z_0) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$ and $f^{(n)}(z_0) = g^{(n)}(0) \quad \forall n$

Ex) $f(z) = e^z$

$\hookrightarrow f^{(n)}(z) = e^z$

$$e^z = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

Ex) $\cos z = f(z)$ is entire

$f'(z) = -\sin z$

$f^{(2)}(z) = -\cos z$

$f^{(3)}(z) = \sin z$

$f^{(4)}(z) = \cos z$

$f(0) = 1 \quad f^{(1)}(0) = 0 \quad f^{(2)}(0) = -1 \quad f^{(3)}(0) = 0 \quad f^{(4)}(0) = 1 \quad \cdots$

$\cos z = 1 - \frac{1}{2!}z^2 + \frac{1}{4!}z^4 - \frac{1}{6!}z^6 + \cdots \quad \forall z \in \mathbb{C}$

$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots$

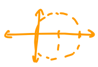
Ex) We saw $\frac{1}{1-z} = 1 + z + z^2 + z^3 + \cdots + z^n + \cdots = \sum_{n=0}^{\infty} z^n$ for $|z|<1$

$\hookrightarrow \frac{1}{1-z}$ is analytic in $|z|<1$

So by Taylor's Thm.: $f(z) = \frac{1}{1-z} = (1-z)^{-1}$ $\quad f^{(1)}(z) = (1-z)^{-2}$ $\quad f^{(2)}(z) = 2(1-z)^{-3}$ $\quad f^{(3)}(z) = 3\cdot 2\cdot (1-z)^{-4}$ $\quad \cdots \quad f^{(n)}(z) = n!(1-z)^{-n-1}$

$f(z) = \sum_{n=0}^{\infty} \frac{n!\, 1^{-n-1}}{n!} z^n = \sum_{n=0}^{\infty} z^n$

Ex) Power series for $f(z) = \frac{1}{z}$
in powers of $z-1$



$\frac{1}{z}$ is analytic on $|z-1|<1$

$\frac{1}{z} = \sum_{n=0}^{\infty} a_n (z-1)^n$ for $|z-1|<1$

Method 1:

use what we found for $\frac{1}{1-z}$

$\frac{1}{z} = \frac{1}{z-1+1} = \frac{1}{1-(1-z)}$

so $\frac{1}{z} = \sum_{n=0}^{\infty} (1-z)^n$ for $|1-z|<1$

$= \sum_{n=0}^{\infty} (-1)^n (z-1)^n$

Method 2:

$f(z) = \frac{1}{z} = z^{-1}$

$f'(z) = -z^{-2}$

$f^{(2)}(z) = 2z^{-3}$

$f^{(3)}(z) = -3\cdot 2\cdot z^{-4}$

$\vdots$

$f^{(n)}(z) = (-1)^n n!\, z^{-n-1}$

$\therefore a_n = \frac{f^{(n)}(1)}{n!} = (-1)^n$

$\frac{1}{z} = \sum_{n=0}^{\infty} (-1)^n (z-1)^n$ for $|z-1|<1$

Ex) Power series for $f(z) = \frac{1}{z}$ centered at 2

$\frac{1}{z} = \sum_{n=0}^{\infty} a_n (z-2)^n$

$\frac{1}{z} = \frac{1}{z-2+2} = \frac{1}{\frac{z-2}{2}+1} = \frac{1}{1-\left(\frac{2-z}{2}\right)}$

$\frac{1}{z} = \frac{1}{2}\sum_{n=0}^{\infty} \frac{(2-z)^n}{2^n}$, for $\left|\frac{2-z}{2}\right| = \frac{|z-2|}{2} < 1 \rightarrow |z-2| < 2$ ✓

Series with Negative Powers:

Ex) $\frac{e^z}{z^2}$ is analytic on $0<|z|<\infty$

$$e^z = \sum_{n=0} \frac{z^n}{n!} = 1+z+\frac{z^2}{2!}+\frac{z^3}{3!}+\cdots$$

$$\frac{e^z}{z^2} = \frac{1}{z^2}+\frac{1}{z}+\frac{1}{2!}+\frac{z}{3!}+\cdots$$

Laurent Series: Suppose f is analytic on the annulus $R_1<|z-z_0|<R_2$ $(R_1\geq 0,\ R_2\leq\infty)$

Then for any z in this annulus,

$$f(z)=\sum_{n=0}^{\infty} a_n(z-z_0)^n + \sum_{n=1}^{\infty}\frac{b_n}{(z-z_0)^n}$$

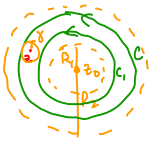
where $a_n = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^{n+1}}dz$ and $b_n = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^{-n+1}}dz$

and C is any simple closed curve in $R_1<|z-z_0|<R_2$ containing z_0

Remark: If f is analytic on $|z-z_0|<R_2$, then $\frac{f(z)}{(z-z_0)^{-n+1}}$ will be analytic on $|z-z_0|<R$ for $n=1,2,3,\ldots$ and so $b_n=0$ $\forall n=1,2,\ldots$

which means $f(z)=\sum_{n=0}^{\infty}a_n(z-z_0)^n$ which is Taylor's Thm.

Proof: Let $R_1<|z-z_0|<R_2$



Let $R_1<r_1<|z-z_0|<r_2<R_2$

$C_1: \{z: |z-z_0|=r_1\}$ $\quad C_2: \{z: |z-z_0|=r_2\}$

Let γ = a circle centered at z inside the region between C_1 & C_2

$\frac{f(w)}{w-z}$ is analytic function of w on the shaded region & on γ, C_1, C_2

By the deformation thm., $\int_{C_2}\frac{f(w)}{w-z}dw - \int_{C_1}\frac{f(w)}{w-z}dw = \int_\gamma \frac{f(w)}{w-z}dw$

We assume $z_0=0$

$$\int_\gamma \frac{f(w)}{w-z}dw = 2\pi i f(z)$$

$\int_{C_2}\frac{f(w)}{w-z}dw$ $\quad (w\in C_2,\ |w|=r_2>|z|)$

$$\frac{1}{w-z}=\frac{1}{w}\left(\frac{1}{1-\frac{z}{w}}\right)=\frac{1}{w}\sum_{k=0}^{\infty}\frac{z^k}{w^k}=\sum_{k=0}^{\infty}\frac{z^k}{w^{k+1}}$$

$$\frac{f(w)}{w-z}=\sum_{n=0}^{\infty}\frac{f(w)}{w^{k+1}}z^k,\quad w\in C_2$$

$$\int_{C_2}\frac{f(w)}{w-z}dz = \sum_{k=0}^{\infty}\int_{C_2}\frac{f(w)}{w^{k+1}}dw\, z^k$$

When $w\in C_1$, $|w|=r_1<|z|$ and so $\frac{|w|}{|z|}<1$

$$\therefore \frac{1}{w-z}=\frac{1}{z}\left(\frac{1}{\frac{w}{z}-1}\right)=-\frac{1}{z}\left(\frac{1}{1-\frac{w}{z}}\right)=-\frac{1}{z}\sum_{k=0}^{\infty}\frac{w^k}{z^k}$$

$$\therefore -\int_{C_1}\frac{f(w)}{w-z}dw = \sum_{k=0}^{\infty}\int_{C_1}f(w)w^k dw\,\frac{1}{z^{k+1}}$$

$$2\pi i f(z) = \sum_{k=0}^{\infty}\int_{C_2}\frac{f(w)}{w^{k+1}}dw\, z^k + \sum_{k=0}^{\infty}\int_{C_1}f(w)w^k dw\, z^{-k-1}$$

Ex) Find Laurent series for $f(z)=\frac{1}{z(1+z^2)}$

↳ Singularities at $\pm i$ and 0

f analytic in $0<|z|<1$

$\frac{1}{1+z^2}$ is analytic in $|z|<1$

$$\frac{1}{1+z^2}=\frac{1}{1-(-z^2)}=\sum_{n=0}^{\infty}(-1)^n z^{2n} \quad \text{since } |z^2|<1$$

$$\frac{1}{z(1+z^2)}=\frac{1}{z}-z+z^3-z^5+\cdots \quad 0<|z|<1$$

f analytic on $0<|z-i|<1$

$$f(z)=\frac{1}{z(z^2+1)}=\frac{1}{z(z+i)(z-i)}=\frac{1}{z-i}\left[\frac{1}{z(z+i)}\right]=\frac{1}{z-i}\left[\frac{A}{z}+\frac{B}{z+i}\right]$$

$$\frac{1}{z}=\frac{1}{(z-i)+i}=\frac{1}{i}\left(\frac{1}{1-\frac{(i-z)}{i}}\right)=\frac{1}{i}\sum_{n=0}^{\infty}\frac{(i-z)^n}{i^n}=\sum_{n=0}^{\infty}\frac{(-1)^n}{i^{n+1}}(z-i)^n$$

$|i-z|<1$

$$\frac{1}{z+i}=\frac{1}{z-i+2i}=\frac{1}{2i}\left[\frac{1}{1-\frac{i-z}{2i}}\right]=\frac{1}{2i}\sum_{n=0}^{\infty}\frac{(-1)^n(z-i)^n}{(2i)^{n+1}}$$

$$\therefore f(z)=\sum_{n=0}^{\infty}a_n(z-i)^n+\sum_{n=1}^{\infty}\frac{b_n}{(z-i)^n}$$

A third possible annulus is $1<|z|<\infty$

Ex) Let $f(z) = \frac{z+1}{z-1}$

Find the Laurent series of f

a) on $0 < |z| < 1$

f analytic on $|z|<1$, $\frac{1}{z-1} = -\frac{1}{1-z} = -\sum_{n=0}^{\infty} z^n$, $|z|<1$ $\quad \therefore \frac{z+1}{z-1} = (z+1)\left(-\sum_{n=0}^{\infty} z^n\right) = -\left(\sum_{n=0}^{\infty} z^{n+1} + \sum_{n=0}^{\infty} z^n\right) = -1 + 2\sum_{n=1}^{\infty} z^n$, $|z|<1$

b) on $1 < |z| < \infty$

$$\frac{1}{z-1} = \frac{1}{z}\left(\frac{1}{1-\frac{1}{z}}\right) = \frac{1}{z}\sum_{n=0}^{\infty}\frac{1}{z^n} \quad \text{since } \frac{1}{|z|}<1$$

$$= \sum_{n=0}^{\infty} \frac{1}{z^{n+1}}$$

$$\frac{z+1}{z-1} = (z+1)\sum_{n=0}^{\infty}\frac{1}{z^{n+1}} = \sum_{n=0}^{\infty}\frac{1}{z^n} + \sum_{n=0}^{\infty}\frac{1}{z^{n+1}} = 1 + 2\sum_{n=1}^{\infty}\frac{1}{z^n} \quad \text{when } |z|>1$$

Ex) Find the Laurent series for $f(z) = \frac{1}{z^2-4}$

a) about $z=0$

f analytic on $|z|<2$

$$f(z) = -\frac{1}{4}\cdot\frac{1}{1-\left(\frac{z}{2}\right)^2} = -\frac{1}{4}\sum_{n=0}^{\infty}\left(\frac{z^2}{4}\right)^n \quad \text{when } \left|\frac{z^2}{4}\right| = \left(\frac{|z|}{2}\right)^2 < 1, \text{ that is, when } |z|<2$$

$$= -\sum_{n=0}^{\infty}\frac{1}{4^{n+1}} z^{2n} \quad \text{when } |z|<2$$

b) about $z=2$

f analytic on $0<|z-2|<4$

$$f = \frac{1}{z^2-4} = \frac{1}{(z-2)(z+2)}$$

$$\frac{1}{z+2} = \frac{1}{(z-2)+4} = \frac{1}{4} = \frac{1}{1+\frac{z-2}{4}} = \frac{1}{4}\frac{1}{1-\frac{2-z}{4}} = \frac{1}{4}\sum_{n=0}^{\infty}\frac{(2-z)^n}{4^n} \quad \text{when } |z-2|<4$$

$$= \sum_{n=0}^{\infty}(-1)^n\frac{(z-2)^n}{4^{n+1}} \quad \text{when } |z-2|<4$$

$$f(z) = \frac{1}{(z-2)(z+2)} = \frac{1}{z-2}\sum_{n=0}^{\infty}(-1)^n\frac{(z-2)^n}{4^{n+1}} = \sum_{n=0}^{\infty}(-1)^n\frac{(z-2)^{n-1}}{4^{n+1}} \quad \text{when } |z-2|<4$$

c) When $|z|>2$

same as previous example

Ex) Find Laurent expansion in powers of z of $f(z) = \frac{1}{z(z-1)(z-2)}$

Soln: use partial fraction decomposition

$$f(z) = \frac{A}{z} + \underbrace{\frac{B}{z-1} + \frac{C}{z-2}}_{\text{use geometric series}}$$

Thm: If a power series $\sum_{n=0}^{\infty} A_n(z-z_0)^n$ converges at z_1 $(z_1 \neq z_0)$, then it converges absolutely at any z where $|z-z_0| < |z_1 - z_0|$

Series of absolute values converges

Proof: Since $\sum_{n=0}^{\infty} a_n(z_1-z_0)^n$ converges, $\lim_{n\to\infty} a_n(z_1-z_0)^n = 0$

and so $\exists M$ st. $|a_n(z_1-z_0)^n| \le M \quad \forall n$

Let $|z-z_0|<|z_1-z_0|$

$0 \le |a_n||z-z_0|^n = |a_n||z_1-z_0|^n \left(\frac{|z-z_0|}{|z_1-z_0|}\right)^n \le M\left(\frac{|z-z_0|}{|z_1-z_0|}\right)^n$ and $\frac{|z-z_0|}{|z_1-z_0|}<1$

Since $\sum_{n=0}^{\infty}\left(\frac{|z-z_0|}{|z_1-z_0|}\right)^n$ converges (geometric w/ $0<r<1$) $\therefore$

By the comparison series test, $\sum_{n=0}^{\infty}|a_n||z-z_0|^n$ converges

We say a sequence of functions $\{f_n(z)\}$ converge uniformly to f on set S if for each $\varepsilon>0$, $\exists N$ st.

$$|f_n(z)-f(z)|<\varepsilon \quad \text{for all } n \ge N, \ \forall z \in S$$

Thm: Suppose $\{f_n\}$ are cont. and f_n converge uniformly to f on an open set D. Then f is cont. on D.

Proof: Let $z_0 \in D$. Let $\varepsilon>0$. Since $f_n \to f$ uniform on D, $\exists N$ st. $\forall n \ge N$, $|f_n(z)-f(z)|<\varepsilon$ $\forall n\ge N$, $\forall z \in D$

Since f_N is cont. at z_0, $\exists \delta>0$ st. $|z-z_0|<\delta \to |f(z)-f(z_0)|<\varepsilon$

For $|z-z_0|<\delta$

$|f(z)-f(z_0)| = |(f(z)-f(z_0)) + (f_N(z)-f_N(z_0)) + (f_N(z_0)-f_N(z))| \le |f(z)-f(z_0)| + |f_N(z)-f_N(z_0)| + |f_N(z_0)-f_N(z)| < 3\varepsilon$

Ex) $f_n(x)=x^n$, $0 \le x \le 1$

If $0 \le x \le 1$, $f_n(x) \to 0$

If $x=1$, $f_n(x)\to 1$

$\therefore f_n(x) \to f(x) = \begin{cases} 0, & 0 \le x < 1 \\ 1, & x=1 \end{cases}$

Converges pointwise

The convergence is not uniform b/c f is not cont.

Thm: Suppose $f_n \to f$ uniformly on a curve C

Then $\int_C f_n(z)\,dz \to \int_C f(z)\,dz$

Proof: $\left|\int_C (f_n(z)-f(z))\,dz\right| \le \underbrace{L(C)\cdot \max_C |f_n - f|}_{ML} \to 0$ by unif. convergence

Given a power Series $\sum_{n=1}^{\infty} a_n(z-z_0)^n$, there is $0 \le R \le \infty$ s.t. the power series converges absolutely on $\{z : |z-z_0|<R\}$ and it diverges on $\{z: |z-z_0|>R\}$

This R is called the radius of convergence of the power series

Thm: Suppose $\sum_{n=1}^{\infty} a_n(z-z_0)^n$ converges on $|z-z_0|<R$. Then given any $0<R_1<R$, the sequence $\{S_N(z) = \sum_{n=0}^{N} a_n(z-z_0)^n\}$ converges uniformly to $S(z)=\sum_{n=0}^{\infty} a_n(z-z_0)^n$ on $|z-z_0| \le R_1$

Proof: Let $R_1 < |z_1 - z_0| < R$

Since $|z_1-z_0|<R$, the series $\sum_{n=1}^{\infty}|a_n||(z_1-z_0)|^n$ converges, that is, $\{S_N(z_1)=\sum_{n=0}^{\infty}|a_n||(z_1-z_0)|^n\}$ converges to $S(z_1)=\sum_{n=0}^{\infty}|a_n||z_1-z_0|^n$

Let $\varepsilon>0$. $\exists k$ st. $\forall N \ge k$, $\sum_{n=N+1}^{\infty}|a_n||z_1-z_0|^n < \varepsilon$

For any z, $|z-z_0| \le R_1$, $|S_N(z)-S(z)| = |\sum_{n=N+1}^{\infty} a_n(z-z_0)^n| \le \sum_{n=N+1}^{\infty}|a_n||z-z_0|^n$ $\quad (|z-z_0|<R_1 \le |z_1-z_0|)$

$\le \sum_{n=N+1}^{\infty}|a_n||z_1-z_0|^n < \varepsilon$

Thm: Suppose $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ converges on $|z-z_0|<R$. Then $f(z)=\sum_{n=0}^{\infty} a_n(z-z_0)^n$ is cont. on $|z-z_0|<R$

Proof: Let $|z_1-z_0|<R$. Let $0<|z_1-z_0|<R_1<R$.

For each $N=1,2,3,\dots$, let $S_N(z)=\sum_{k=0}^{N} a_k(z-z_0)^k$, S_N is clearly cont. on $\mathbb{C}$. We just proved that $\{S_N\}_N$ conv. unif. to $f(z)$ on $|z-z_0| \le R_1$

By a thm., f is cont. on $|z-z_0|<R_1 \Rightarrow f$ is cont at $z_1 \Rightarrow f$ is cont. on $|z-z_0|<R$

Thm: Let $f(z)=\sum_{n=1}^{\infty} a_n(z-z_0)^n$ converge on $|z-z_0|<R$. Then f is analytic on $|z-z_0|<R$.

Proof: Let C be a closed curve in $|z-z_0|<R$

we want to prove $\int_C f(z)dz = 0$

$\{S_N\}_N$ conv. unif. to f on C

$\therefore$ by a theorem $\int_C S_N(z)dz \longrightarrow \int_C f(z)dz$

$\int_C S_N(z)dz = 0 \quad \therefore \int_C f(z)\,dz = 0$

By Morera's thm, f is analytic on $|z-z_0|<R$

Summary:

1) If f is analytic on $|z-z_0|<R$, then $f(z)=\sum_{n=0}^{\infty} a_n(z-z_0)^n$ for $|z-z_0|<R$ and $a_n = \dfrac{f^{(n)}(z_0)}{n!}$ $\forall n$

2) If $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ conv. on $|z-z_0|<R$, then it is analytic on $|z-z_0|<R$

Thm: Let $f(z)=\sum_{n=0}^{\infty} a_n(z-z_0)^n$ on $|z-z_0|<R$. (So f is analytic on $|z-z_0|<R$). Then for any $|z-z_0|<R$, $f'(z)=\sum_{n=1}^{\infty} n a_n(z-z_0)^{n-1}$

Proof: Let $|z-z_0|<R$. Let $C: |w-z_0|=R_1$ where $R_1>|z-z_0|$

Recall that $f'(z)=\frac{1}{2\pi i}\int_C \frac{f(w)}{(w-z)^2}dw =$

Let $S_N(w)=\sum_{n=0}^{\infty} a_n(w-z_0)^n$

$S_N'(w) = \frac{1}{2\pi i}\int_C \frac{S_N(w)}{(w-z)^2}dw$ for $w \in C$

$|w-z| \ge R_1 - R$

$\left|\frac{S_N(w)}{(w-z)^2} - \frac{f(w)}{(w-z)^2}\right| \le \frac{1}{R_1-R}|S_N(w)-f(w)| \to 0$ uniformly on C

$\therefore \int_C \frac{S_N(w)}{(w-z)^2}dw \to \int_C \frac{f(w)}{(w-z)^2}dw \to S_N'(z) \to f'(z)$

$S_N'(z)=\sum_{n=1}^{N} n a_n(z-z_0)^{n-1}$

Thus $\left\{\sum_{n=1}^{N} n a_n(z-z_0)^{n-1}\right\} \to f'(z)$. That is $\sum_{n=1}^{\infty} n a_n(z-z_0)^{n-1} = f'(z)$

Ex) Let $f(z) = \begin{cases} \frac{\sin z}{z} & , z \neq 0 \\ 1 & , z = 0 \end{cases}$

$f : \mathbb{C} \to \mathbb{C}$

f is clearly analytic on $\mathbb{C} \setminus \{0\}$

When $z \neq 0$, $f(z) = \frac{\sin z}{z} = \frac{z - z^3/3! + z^5/5! - \cdots}{z} = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots$

The power series $1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots$ converges on $\mathbb{C}$

It $= f(z)$ for $z \neq 0$

At $z=0$, $1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots = 1 = f(z)$

$\therefore f(z) = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots$ for $z \in \mathbb{C}$

By thm, since the power series converges it is analytic $\to f(z)$ is analytic for $z \in \mathbb{C}$

Ex) Let $g(z) = \begin{cases} \frac{e^z - 1 - z}{z^2} & , z \neq 0 \\ \frac{1}{2} & , z = 0 \end{cases}$

$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$

$e^z - 1 - z = \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$

$g(z) = \frac{e^z - 1 - z}{z^2} = \frac{1}{2!} + \frac{z}{3!} + \cdots$ for $z \neq 0$

$\frac{1}{2!} + \frac{z}{3!} + \cdots$ converges on $\mathbb{C}$ and is $\therefore$ entire

It equals $g(z)$ for $z \neq 0$ and $z = 0$, $\therefore$ $g(z) = \frac{1}{2!} + \frac{z}{3!} + \cdots$ $\forall z \in \mathbb{C}$ and hence is entire