Mappings

Linear Mapping (Affine Mappings):

$T(z) = az+b$, $a \neq 0$

$a, b \in \mathbb{C}$

$a = |a|e^{i\theta}$

Let $T_1(z) = e^{i\theta} z$ → Rotation by θ

$T_2(z) = |a| z$ → Dialates z by a factor of $|a|$

$T_3(z) = z+b$ → Translation

$T(z) = az+b = T_3 \circ T_2 \circ T_1(z)$

Takes a line to a line, a circle to a circle

Proof:

Let L be a line. $L: \{(x,y): Ax+By+C=0,\ A,B,C \in \mathbb{R}\}$

Let $z = x+iy$. Then $z \in L$ iff $A\left(\frac{z+\bar{z}}{2}\right) + B\left(\frac{z-\bar{z}}{2i}\right) + C = 0$

$$A(z+\bar{z}) - Bi(z-\bar{z}) + 2C = 0$$

$$(A-Bi)z + (A+Bi)\bar{z} + 2C = 0$$

$$(A-Bi)z + \overline{(A-Bi)z} = -2C$$

$$2\,\mathrm{Re}[(A-Bi)z] = -2C$$

$$\mathrm{Re}[(A-Bi)z] = -C$$

Let $\alpha = A-Bi \in \mathbb{C}$, $-C \in \mathbb{R}$ $\Big\}$ $\mathrm{Re}(\alpha z) = C$ ← we've renamed the constant (its really $-C$)

∴ L has equ.: $\mathrm{Re}(\alpha z) = C$

$T_1(z) = w = e^{i\theta} z \to z = we^{-i\theta}$

$T_1(L) = \mathrm{Re}(\alpha e^{-i\theta} w) = \mathrm{Re}(\alpha' w) = C \in \mathbb{R}$

∴ $T_1(L)$ is a line

$T_2(z) = rz = w$, $r > 0$

$z = \frac{w}{r}$ $T_2(L) = \mathrm{Re}\left(\frac{\alpha}{z} w\right) = C$ ∴ $T_2(L)$ is a line

$T_3(z) = z+b = w \Rightarrow z = w-b$

$T_3(L) = \mathrm{Re}(\alpha(w-b)) = C$

$\Rightarrow \mathrm{Re}(\alpha w) = C + \mathrm{Re}(\alpha b) \in \mathbb{R}$ ∴ $T_3(L)$ is a line

Hence $T(L) = T_3 \circ T_2 \circ T_1(L)$ is a line

Proof for a circle:

Circle C: $(x-x_0)^2 + (y-y_0)^2 = R^2$

$z_0 = x_0 + iy_0$

C: $|z-z_0|^2 = R^2$

$$(z-z_0)(\bar{z}-\bar{z}_0) = R^2$$

$$|z|^2 - \bar{z}_0 z - z_0\bar{z} = R^2 - |z_0|^2$$

$T_1(z) = e^{i\theta} z = w$

$z = e^{-i\theta} w$

$|z-z_0|^2 = R^2 \Leftrightarrow |e^{-i\theta} w - z_0|^2 = R^2 \Leftrightarrow |w - z_0 e^{i\theta}|^2 R^2$ ∴ $T_1(C)$ is a circle

$T_2(z) = rz = w$ $z = \frac{w}{r}$

$\left|\frac{w}{r} - z_0\right|^2 = R^2 \Leftrightarrow |w - rz_0|^2 = (rR)^2$ ∴ $T_2(C)$ is a circle

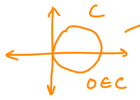
$T_3(z) = z+b = w$ $z = w-b$

$|w-(z_0+b)|^2 = R^2$ ∴ $T_3(C)$ is a circle

So $T(C)$ is a circle

Let $T(z) = \frac{1}{z} = w$

$\frac{1}{z} = \frac{\bar{z}}{|z|^2}$



As $z \to 0$, $T(C) \to \infty$
We'll find $T(C)$ is line if $0 \in C$

we'll find $T(C_2)$ is circle if $0 \notin C_2$

Let $T(z) = \frac{1}{z} = w$

Let L be a line. $L: Re(\alpha z) = c \in \mathbb{R}$

$z = \frac{1}{w}$

$Re\left(\frac{\alpha}{w}\right) = c \iff \frac{\alpha}{w} + \frac{\bar{\alpha}}{\bar{w}} = 2c$

$\alpha \bar{w} + \bar{\alpha} w = 2c|w|^2$

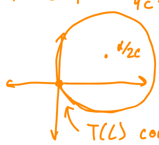
Case i) $c = 0$ $(0 \in L)$:

$\alpha \bar{w} + \bar{\alpha} w = 0 \iff Re(\bar{\alpha} w) = 0 \in \mathbb{R}$ which is a line L' w/ $0 \in L'$

Case ii) $c \neq 0$ $(0 \notin L)$

$\alpha \bar{w} + \bar{\alpha} w = 2c|w|^2 \iff |w|^2 - \frac{\bar{\alpha}}{2c} w - \frac{\alpha}{2c} \bar{w} = 0 \iff \left|w - \frac{\alpha}{2c}\right|^2 = \frac{|\alpha|^2}{4c^2}$

$\therefore$ $T(L)$ is the circle $\left|w - \frac{\alpha}{2c}\right|^2 = \left(\frac{|\alpha|}{2c}\right)^2$



$T(L)$ contains 0

Suppose C is a circle. Then $T(C)$ is a line or a circle
If $0 \in C$, then $T(C)$ is a line
If $0 \notin C$, then $T(C)$ is a circle

Can show this as an exercise

Let $G = \left\{ T(z) = \frac{az+b}{cz+d} : ad - bc \neq 0 \right\}$

Called Fractional linear transformations (FLTs), bilinear maps, Möbius transformations

If $c = 0$, then $d \neq 0$ $T(z) = \frac{a}{d} z + \frac{b}{d}$ which is linear

If $a = 0$, then $T(z) = \frac{b}{cz+d} = S_1 \circ S_2(z)$

$S_2(z) = cz + d$, $S_1(z) = \frac{b}{z}$

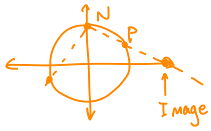
If $c \neq 0$, and $a \neq 0$, then $T(z) = \frac{a}{c}\left(\frac{z + b/a}{z + d/c}\right) = \frac{a}{c}\left(\frac{z + d/c + b/a - d/c}{z + \frac{d}{c}}\right) = \frac{a}{c} + a\left(\frac{\frac{b}{a} - \frac{d}{c}}{cz+d}\right) = T_3 \circ T_2 \circ T_1(z)$

Any $T \in G$ maps a line or a circle to a line or a circle

If $T, S \in G$, then $T \circ S \in G$ (so G is a group)

Extended Plane = $\mathbb{C} \cup \{\infty\} = \mathbb{C}_\infty$

Let $\mathbb{R}_\infty = \mathbb{R} \cup \{\infty\}$



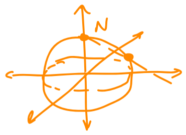
We've mapped every point except N

As we approach N, the image $\to \infty$

So the North pole is mapped to ∞

$\therefore \mathbb{R} \cup \{\infty\}$ is a circle!

By similar logic, the extended complex plane is a sphere!



Since ∞ is just the North pole, and this is arbitrary, ∞ is just a point like any other #

Thm: Let $T \in G$. Then $T: \mathbb{C}_\infty = \mathbb{C} \cup \{\infty\} \to \mathbb{C}_\infty$ is one-to-one and onto.

Moreover, $T^{-1} \in G$

Proof: Ontoness: Let $w \in \mathbb{C}_\infty$

Want $z \in \mathbb{C}_\infty$ st. $\frac{az+b}{cz+d} = w \Rightarrow az+b = czw + dw \Rightarrow z(a-cw) = dw-d \Rightarrow z = \frac{dw-d}{-cw+a}$

If $w \in \mathbb{C}$, $w \neq a/c$, then $z \in \frac{dw-b}{-cw+a} \in \mathbb{C}$ and $T(z) = w$

Set $T(\infty) = \frac{a}{c}$

$$\lim_{z\to\infty} \left(\frac{az+b}{cz+d}\right) = \lim_{z\to\infty} \left(\frac{a+\frac{b}{z}}{c+\frac{d}{z}}\right) = \frac{a}{c}$$

Thus $T: \mathbb{C}_\infty \to \mathbb{C}_\infty$ is onto

$T^{-1}(w) = \frac{dw-b}{-cw+a}$, $T^{-1}(\infty) = -\frac{d}{c}$

$\therefore T$ is 1-1. $T(z_1) = T(z_2) \Rightarrow T^{-1}(T(z_1)) = T^{-1}(T(z_2)) \Rightarrow z_1 = z_2$

Note that $T^{-1} \in G$

Let C be a circle or a line in $\mathbb{C}_\infty$

3 distinct points in $\mathbb{C}_\infty$ determine C

Def: Let $T \in G$. We say $z \in \mathbb{C}_\infty$ is a fixed point of T if $T(z) = z$

Ex) Assume $T(z) = az+b \neq Id$

If $z \in \mathbb{C}$ and $az+b = z$, then $(a-1)z = -b$

$\Rightarrow z = -\frac{b}{a-1}$ is a fixed point if $a \neq 1$

$|T(z)| = |az+b| \geq |z| - |b| \to \infty$ as $|z| \to \infty$

$\Rightarrow T(\infty) = \infty$

$\therefore T(z) = az+b$ has 2 fixed points, $-\frac{b}{a-1}$, ∞ if $a \neq 1$

1 fixed point, ∞, if $a = 1$

Thm: Let $T(z) = \frac{az+b}{cz+d}$ be a FLT

Then if $T \neq$ the identity, T has at most 2 fixed points in $\mathbb{C}_\infty$

Proof: Case 1: Assume $c = 0$. The $T(z) = \frac{a}{d}z + \frac{b}{d}$, and $a, d \neq 0$ since $ad - bc = ad \neq 0$

We saw that if $T \neq Id$, T has ≤ 2 fixed points

Case 2: Assume $c \neq 0$. Suppose $T(z) = z$. Then $\frac{az+b}{cz+d} = z \Rightarrow cz^2 + (d-a)z - b = 0$ $(c \neq 0)$. This has ≤ 2 solns. in $\mathbb{C}$

Since $\lim_{|z| \to \infty} \frac{az+b}{cz+d} = \frac{a}{c} \neq \lim_{|z| \to \infty} z$, $T(\infty) \neq \infty$ $\therefore$ T has ≤ 2 fixed points

Corollary: Suppose $T, S \in G$. Assume z_1, z_2, z_3 are 3 points in $\mathbb{C}_\infty$ and $T(z_i) = S(z_i)$, $1 \leq i \leq 3$

Then $T \equiv S$

Proof: $S^{-1} \circ T(z_i) = z_i$, $1 \leq i \leq 3$

$\Rightarrow S^{-1} \circ T(z) = z \quad \forall z$

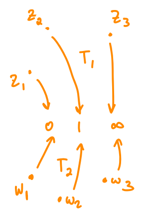
$\Rightarrow T(z) = S(z) \quad \forall z$

Lemma: Let $z_1, z_2, z_3 \in \mathbb{C}_\infty$ (distinct)

$w_1, w_2, w_3 \in \mathbb{C}_\infty$ (distinct)

Then $\exists\, T$ FLT s.t. $T(z_i) = w_i$, $1 \leq i \leq 3$. It has to be <u>unique</u>.

Proof:



$T_1, T_2 \in G$

$T_2^{-1} \circ T_1 : z_i \to w_i$

$\therefore$ just find $T_1 \in G$ s.t. $T_1(z_1) = 0$, $T_1(z_2) = 1$, $T_1(z_3) = \infty$

$$T_1(z) = \frac{z_2 - z_3}{z_2 - z_1}\left(\frac{z - z_1}{z - z_3}\right) \quad \text{(and similarly construct } T_2 \text{ for } w\text{)}$$

And thus you can get the transformations by composition!

Let z_1, z_2, z_3 determine C (C is a line or circle)

Let w_1, w_2, w_3 determine C' (C' is a line or circle)

Let S be FLT mapping z_i to w_i

Since $S(w_i) \in S(C)$, $S(C) = C'$

S is one-to-one

Formula for a FLT T that maps 3 distinct $z_1, z_2, z_3 \in \mathbb{C}_\infty$ to 3 distinct $w_1, w_2, w_3 \in \mathbb{C}_\infty$

$w = T(z)$ is given by $$\left(\frac{w_2 - w_3}{w_2 - w_1}\right)\left(\frac{w - w_1}{w - w_3}\right) = \left(\frac{z_2 - z_3}{z_2 - z_1}\right)\left(\frac{z - z_1}{z - z_3}\right)$$

If $z_1 = \infty$

Solve: $$\left(\frac{w_2 - w_3}{w_2 - w_1}\right)\left(\frac{w - w_1}{w - w_3}\right) = \frac{z_2 - z_3}{z - z_3}$$

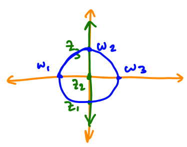
If $z_2 = \infty$, $w_1 = \infty$

Solve: $$\frac{w_2 - w_3}{w - w_3} = \frac{z - z_1}{z - z_3}$$

Ex) Find a FLT $T \ni T(-i) = -1$, $T(0) = i$, $T(i) = 1$

$z_1 = -i$, $z_2 = 0$, $z_3 = i$

$w_1 = -1$, $w_2 = i$, $w_3 = 1$



Maps a line to a circle

$$\left(\frac{i-1}{i+1}\right)\left(\frac{w+1}{w-1}\right) = \frac{-i}{i}\left(\frac{z+i}{z-i}\right)$$

$(i-1)(w+1)(z-i) = -(z+i)(w-1)(i+1)$

$(i-1)(wz - iw + z - i) = -(i+1)(wz + wi - z - i)$

$w\left[(i-1)z - i(i-1) + (i+1)z + i(i+1)\right] = z\left[-(i-1) - (i+1)\right] + i(i-1) + i(i+1)$

$w\left[2iz + 2i\right] = -2iz - 2 \quad \Rightarrow w = \frac{-iz - 1}{iz + i}$ so $T(z) = \frac{-iz-1}{iz+i}$

$T(\{x + iy : x > 0\}) = ?$

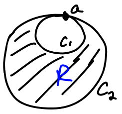
$\left|T(1)\right| = \left|\frac{-i-1}{2i}\right| = \left|-\frac{1}{2} - \frac{1}{2i}\right| = \sqrt{\frac{1}{4} + \frac{1}{4}} < 1$

Ex) Find a FLT that maps $\mathbb{R}_\infty = \mathbb{R} \cup \{\infty\}$ onto $\{z : |z| = 1\}$

Let $z_1 = 0$, $z_2 = 1$, $z_3 = \infty$

$w_1 = -1$, $w_2 = i$, $w_3 = 1$

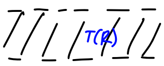
$$\left(\frac{w_2 - w_3}{w_2 - w_1}\right)\left(\frac{w - w_1}{w - w_3}\right) = \frac{z - z_1}{z_2 - z_1} \implies w = \frac{z - i}{z + i} = T(z)$$



Take FLT that maps a to ∞ (e.g. $\frac{1}{z-a}$)

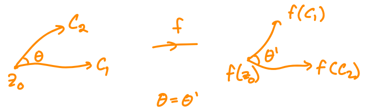
Since $a \in C_1, C_2$ and $T(a) = \infty$, $T(C_1)$ and $T(C_2)$ are lines

$T(C_1)$ and $T(C_2)$ don't intersect since $C_1 \neq C_2 \Rightarrow T(C_1)$ & $T(C_2)$ are parallel



$\rightarrow$ Much nicer region!

We say a function $f : D \to \mathbb{C}$ is <u>conformal</u> at $z_0 \in D$ if f preserves angles at z_0



FLT: $T = \frac{az+b}{cz+d}$ $\qquad T' = \frac{ad-bc}{(cz+d)^2}$

Thm: Suppose f is analytic at z_0. Then f is conformal (preserves angles) at z_0.

Proof: Suppose C_1, C_2 are curves thru z_0, say $C_1(t_0) = z_0 = C_2(t_0)$

We have to show: Angle between $C_1'(t_0)$ and $C_2'(t_0)$ = Angle between $\frac{d}{dt} f(C_1(t))\big|_{t=t_0}$ and $\frac{d}{dt} f(C_2(t))\big|_{t=t_0}$

TODO: Look over, and check this

$$\frac{d}{dt} f(C_1(t))\Big|_{t=t_0} = f'(z_0)\, C_1'(t_0) \qquad \frac{d}{dt} f(C_2(t))\Big|_{t=t_0} = f'(z_0)\, C_2'(t_0)$$

Let $C_1'(t_0) = |C_1'(t_0)| e^{i\theta_1}$, $C_2'(t_0) = |C_2'(t_0)| e^{i\theta_2}$

Let $f'(z_0) = re^{i\theta}$, $r \neq 0$

$$f'(z_0)\, C_1'(t_0) = r\,|C_1'(t_0)|\, e^{i(\theta_1 + \theta)}$$

$$f'(z_0)\, C_2'(t_0) = r\,|C_2'(t_0)|\, e^{i(\theta_2 + \theta)}$$

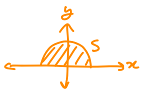
Thus, f preserves angles at z_0

Ex) Let $f(z) = \frac{az+b}{cz+d}$, $ad - bc \neq 0$

$$f'(z) = \frac{ad-bc}{(cz+d)^2} \neq 0$$

$\therefore$ a FLT preserves angles

Ex) Let S be the semidisc $\{z : |z| < 1, \operatorname{Im} z \geq 0\}$



Find a conformal map from S onto $H = \{z = x+iy : y \geq 0\}$

Soln: map $-1 \to \infty$ by a conformal map

$f_1(z) = \frac{1}{z+1}$ (is a FLT $\therefore$ conformal)

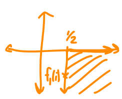
$f_1(0) = 1 \in \mathbb{R}_\infty$, $f(1) = \frac{1}{2} \in \mathbb{R}_\infty$ $\quad \therefore f_1(x\text{-axis}) = \mathbb{R}_\infty$

$f_1(C_1)$ is part of a line $\quad f_1(x\text{-axis}) \cap f_1(C_1) \ni f_1(1) = \frac{1}{2}, \infty$

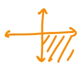
Since f_1 preserves angle at 1, $f_1(C_1) \subseteq$ part of $\{x = \frac{1}{2}\}$

C_1 is connected $\therefore f_1(C_1)$ is connected

$\infty, \frac{1}{2} \in f_1(C_1) \qquad f_1(i) = \frac{1}{i+1} = \frac{1-i}{2} = \frac{1}{2} - \frac{i}{2}$



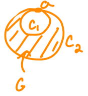
Let $f_2(z) = z - \frac{1}{2}$



$f_2 \circ f_1(S)$

Let $f_3(z) = iz$

$$f(z) = f_4 \circ f_3 \circ f_2 \circ f_1(z) = (f_3 \circ f_2 \circ f_1(z))^2 = (i f_2 \circ f_1(z))^2 = \left[i(f_1(z) - \tfrac{1}{2})\right]^2 = \boxed{-\left(\frac{1}{z+1} - \frac{1}{2}\right)^2}$$

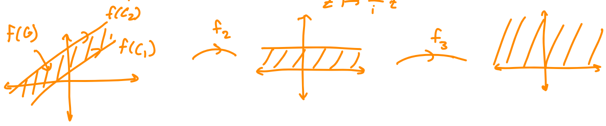
Ex) 

Find a conformal map $f: G \to$ 

$f_1(z) = \frac{1}{z-a}$

$f(C_1), f(C_2)$ are parallel

$z \mapsto \frac{\pi}{i} z$



Ex) $f = u + iv \qquad \Delta u = 0 = \Delta v$

$u(x,y) = e^x \sin y - xy + x + 2$

Find $v(x,y)$ so that $u+iv$ is analytic

$\Delta u = 0$

$v_y(x,y) = u_x(x,y) = e^x \sin y - y + 1$

$v(x,y) = -e^x \cos y - \frac{y^2}{2} + y + h(x)$

$v_x = -e^x \cos y + h'(x) = -e^x \cos y + x \Rightarrow h'(x) = x \qquad h(x) = \frac{x^2}{2} + C$

$v(x,y) = e^{-x} \cos y - \frac{y^2}{2} + y + \frac{x^2}{2} + C$