$$\ddot{q}_{M,i} = \frac{1}{J_{M,i}} \left(-f_{M,i} \dot{q}_{M,i} - \frac{1}{n_i} \tau_i + k_{T,i} u_i \right)$$

$$\dot{\boldsymbol{x}} = \begin{bmatrix} 0 & 1 \\ 0 & -\frac{f_{M,i}}{J_{M,i}} \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} 0 \\ k_{T,i} \end{bmatrix} \boldsymbol{u}_i + \begin{bmatrix} 0 \\ -\frac{1}{n_i} \end{bmatrix} \boldsymbol{\tau}_i$$

$$\boldsymbol{x} = \begin{bmatrix} q_{M,i} \\ \dot{q}_{M,i} \end{bmatrix}$$

$$\begin{split} J_{M,i} &= J_{eff,i} - \frac{1}{n_i^2} \sup_{q} D_{ii}(q) = J_{eff,i} - \frac{1}{n_i^2} \widehat{D}(q) \\ \tau_i &= \widehat{D}(q) \ddot{q} + T_L \\ u_i &= n_i [K_{p,i}(q_i^r - q_i) + K_{D,i}(\dot{q}_i^r - \dot{q}_i)] \\ q_{M,i} &= q_i n_i \\ \ddot{q}_{M,i} &= \frac{1}{J_{M,i}} \Big(-f_{M,i} \dot{q}_{M,i} - \frac{1}{n_i} \tau_i + k_{T,i} u_i \Big) \\ f_{eff} &= f_M + \frac{f_L}{n_i^2} = f_M, \qquad f_L = 0 \end{split}$$

$$n_{i}\ddot{q}_{i} = \frac{1}{J_{eff,i} - \frac{1}{n_{i}^{2}} \sup_{q} D_{ii}(q)} \left\{ -f_{eff}n_{i}\dot{q}_{i} - \frac{1}{n_{i}} \left(\hat{D}(q)\ddot{q} + T_{L} \right) + k_{T,i}n_{i} \left[K_{p,i}(q_{i}^{r} - q_{i}) + K_{D,i}(\dot{q}_{i}^{r} - \dot{q}_{i}) \right] \right\}$$

$$q_{i}\left(\left(n_{i}J_{eff,i}-\frac{1}{n_{i}}\sup_{q}D_{ii}(q)+\frac{1}{n_{i}}\widehat{D}(q)\right)s^{2}+\left(f_{eff}n_{i}+k_{T,i}n_{i}K_{D,i}\right)s+k_{T,i}n_{i}K_{p,i}\right)=-\frac{1}{n_{i}}T_{L}+k_{T,i}n_{i}\left(K_{p,i}+K_{D,i}s\right)q_{i}^{T}$$

If
$$\sup_q D_{ii}(q) = \widehat{D}(q)$$

$$q_{i}\left(s^{2} + \frac{\left(f_{eff} + k_{T,i}K_{D,i}\right)}{J_{eff,i}}s + \frac{k_{T,i}K_{p,i}}{J_{eff,i}}\right) = \frac{k_{T,i}\left(K_{p,i} + K_{D,i}s\right)}{J_{eff,i}}q_{i}^{r} - \frac{1}{n_{i}^{2}J_{eff,i}}T_{L}$$

$$q_{i} = \frac{1}{N_{i}(s)} [F_{i}(s)q_{i}^{r}(s) - G_{i}(s)T_{L,i}(s)]$$

$$\begin{aligned} N_{i}(s) &= s^{2} + \frac{\left(f_{eff} + k_{T,i}K_{D,i}\right)}{J_{eff,i}}s + \frac{k_{T,i}K_{p,i}}{J_{eff,i}} \\ F_{i}(s) &= \frac{k_{T,i}\left(K_{p,i} + K_{D,i}s\right)}{J_{eff,i}} \\ G_{i}(s) &= \frac{1}{n_{i}^{2}J_{eff,i}} \end{aligned}$$

$$F_i(s) = \frac{k_{T,i} \left(K_{p,i} + K_{D,i} s \right)}{I_{T,i}}$$

$$G_i(s) = \frac{1}{n_i^2 J_{eff,i}}$$

$$s^{2} + \frac{\left(f_{eff} + k_{T,i}K_{D,i}\right)}{J_{eff,i}}s + \frac{k_{T,i}K_{p,i}}{J_{eff,i}} = s^{2} + 2\omega_{n,i}\zeta_{i}s + \omega_{n,i}^{2}$$

$$\begin{split} & \omega_{n,i}^2 = \frac{k_{T,i}K_{p,i}}{J_{eff,i}} & \iff K_{p,i} = \frac{J_{eff,i}\omega_{n,i}^2}{k_{T,i}} \\ & 2\omega_{n,i}\zeta_i = \frac{\left(f_{eff} + k_{T,i}K_{D,i}\right)}{J_{eff,i}} & \iff K_{D,i} = \frac{2\omega_{n,i}\zeta_iJ_{eff,i} - f_{eff}}{k_{T,i}} = \frac{2\zeta_i}{\omega_{n,i}} \; K_{p,i} - \frac{f_{eff}}{K_{T,i}} \end{split}$$

$$\begin{split} J_{M,i} &= J_{eff,i} - \frac{1}{n_i^2} \sup_{q} D_{il}(q) = J_{eff,i} - \frac{1}{n_i^2} \widehat{D}(q) \\ \tau_i &= \widehat{D}(q) \ddot{q} + T_L \\ u_i &= n_i \left[K_{p,i} (q_i^r - q_i) + K_{D,i} (\dot{q}_i^r - \dot{q}_l) + K_{I,i} \int (q_i^r - q_i) \, dt \right] \\ q_{M,i} &= q_i n_i \\ \ddot{q}_{M,i} &= \frac{1}{J_{M,i}} \left(-f_{M,i} \dot{q}_{M,i} - \frac{1}{n_i} \tau_i + k_{T,i} u_i \right) \\ f_{eff} &= f_M + \frac{f_L}{n_i^2} = f_M, \qquad f_L = 0 \end{split}$$

$$\begin{split} n_{l}\ddot{q}_{l} &= \frac{1}{J_{eff,i} - \frac{1}{n_{l}^{2}} \sup_{q} D_{li}(q)} \left\{ -f_{eff}n_{l}\dot{q}_{l} - \frac{1}{n_{i}} \left(\widehat{D}(q)\ddot{q} + T_{L} \right) + k_{T,i}n_{l} \left[K_{p,i}(q_{l}^{r} - q_{l}) + K_{D,i}(\dot{q}_{l}^{r} - \dot{q}_{l}) + K_{I,i} \int (q_{l}^{r} - q_{l}) dt \right] \right\} \\ \ddot{q}_{l} &= \frac{1}{J_{eff,i}} \left\{ -f_{eff}\dot{q}_{l} - \frac{1}{n_{l}^{2}}T_{L} + k_{T,i} \left[K_{p,i}(q_{l}^{r} - q_{l}) + K_{D,i}(\dot{q}_{l}^{r} - \dot{q}_{l}) + K_{I,i} \int (q_{l}^{r} - q_{l}) dt \right] \right\} \end{split}$$

$$\begin{split} n_{l}q_{l}s^{2}\left(J_{eff,i} - \frac{1}{n_{l}^{2}}\sup_{q}D_{li}(q)\right) &= -f_{eff}n_{l}q_{l}s - \frac{1}{n_{l}}\left(\bar{D}(q)q_{l}s^{2} + T_{L}\right) + k_{T,i}n_{l}\left[K_{p,i}(q_{l}^{r} - q_{l}) + K_{D,i}(q_{l}^{r}s - q_{l}s) + K_{I,i}\frac{1}{s}(q_{l}^{r} - q_{l})\right] \\ n_{l}q_{l}s^{2}\left(J_{eff,i} - \frac{1}{n_{l}^{2}}\sup_{q}D_{li}(q)\right) + f_{eff}n_{l}q_{l}s + \frac{1}{n_{l}}\bar{D}(q)q_{l}s^{2} = -\frac{1}{n_{l}}T_{L} + k_{T,i}n_{l}\left[K_{p,i}(q_{l}^{r} - q_{l}) + K_{D,i}(q_{l}^{r}s - q_{l}s) + K_{I,i}\frac{1}{s}(q_{l}^{r} - q_{l})\right] \end{split}$$

$$\begin{aligned} & k_{T,i}n_i \left[K_{p,i}(q_i^r - q_i) + K_{D,i}(q_i^r s - q_i s) + K_{I,i} \frac{1}{s}(q_i^r - q_i) \right] = k_{T,i}n_i K_{P,i}q_i^r - k_{T,i}n_i K_{P,i}q_i + k_{T,i}n_i K_{D,i}q_i^r s - k_{T,i}n_i K_{D,i}q_i s + k_{T,i}n_i K_{I,i}q_i^r \frac{1}{s} - k_{T,i}n_i K_{I,i}q_i \frac{1}{s} \\ & = k_{T,i}n_i \left(K_{P,i} + K_{D,i} s + K_{I,i} \frac{1}{s} \right) q_i^r - k_{T,i}n_i \left(K_{P,i} + K_{D,i} s + K_{I,i} \frac{1}{s} \right) q_i \end{aligned}$$

$$q_{i}\left\{s^{2}\left(n_{i}J_{eff,i} - \frac{1}{n_{i}}\sup_{q}D_{ii}(q) + \frac{1}{n_{i}}\widehat{D}(q)\right) + s\left(f_{eff}n_{i} + k_{T,i}n_{i}K_{D,i}\right) + k_{T,i}n_{i}K_{P,i} + k_{T,i}n_{i}K_{I,i}\frac{1}{s}\right\} = -\frac{1}{n_{i}}T_{L} + k_{T,i}n_{i}\left(K_{P,i} + K_{D,i}s + K_{I,i}\frac{1}{s}\right)q_{i}^{T}$$

If $\sup_q D_{ii}(q) = \widehat{D}(q)$

$$q_{i}\left\{s^{2} + s\frac{\left(f_{eff} + k_{T,i}K_{D,i}\right)}{J_{eff,i}} + \frac{k_{T,i}K_{P,i}}{J_{eff,i}} + \frac{k_{T,i}K_{I,i}}{J_{eff,i}}\frac{1}{s}\right\} = \frac{k_{T,i}}{J_{eff,i}}\left(K_{P,i} + K_{D,i}s + K_{I,i}\frac{1}{s}\right)q_{i}^{r} - \frac{1}{n_{i}^{2}J_{eff,i}}T_{L}s + \frac{1}{n_{i}^{2}J_{eff,i}}T$$

$$q_{i} \big\{ J_{eff,i} s^{3} + s^{2} \big(f_{eff} + k_{T,i} K_{D,i} \big) + k_{T,i} K_{P,i} s + k_{T,i} K_{I,i} \big\} = \ k_{T,i} \big(K_{P,i} s + K_{D,i} s^{2} + K_{I,i} \big) q_{i}^{T} - \frac{s}{n_{i}^{2}} T_{L} (K_{P,i} s + K_{D,i} s^{2} + K_{I,i}) q_{i}^{T} - \frac{s}{n_{i}^{2}} T_{L} (K_{P,i} s + K_{D,i} s^{2} + K_{I,i}) q_{i}^{T} - \frac{s}{n_{i}^{2}} T_{L} (K_{P,i} s + K_{D,i} s^{2} + K_{I,i}) q_{i}^{T} - \frac{s}{n_{i}^{2}} T_{L} (K_{P,i} s + K_{D,i} s^{2} + K_{I,i}) q_{i}^{T} - \frac{s}{n_{i}^{2}} T_{L} (K_{P,i} s + K_{D,i} s^{2} + K_{I,i}) q_{i}^{T} - \frac{s}{n_{i}^{2}} T_{L} (K_{P,i} s + K_{D,i} s^{2} + K_{I,i}) q_{i}^{T} - \frac{s}{n_{i}^{2}} T_{L} (K_{P,i} s + K_{D,i} s^{2} + K_{I,i}) q_{i}^{T} - \frac{s}{n_{i}^{2}} T_{L} (K_{P,i} s + K_{D,i} s^{2} + K_{I,i}) q_{i}^{T} - \frac{s}{n_{i}^{2}} T_{L} (K_{P,i} s + K_{D,i} s^{2} + K_{I,i}) q_{i}^{T} - \frac{s}{n_{i}^{2}} T_{L} (K_{P,i} s + K_{D,i} s^{2} + K_{I,i}) q_{i}^{T} - \frac{s}{n_{i}^{2}} T_{L} (K_{P,i} s + K_{D,i} s^{2} + K_{I,i}) q_{i}^{T} - \frac{s}{n_{i}^{2}} T_{L} (K_{P,i} s + K_{D,i} s^{2} + K_{I,i}) q_{i}^{T} - \frac{s}{n_{i}^{2}} T_{L} (K_{P,i} s + K_{D,i} s^{2} + K_{I,i}) q_{i}^{T} - \frac{s}{n_{i}^{2}} T_{L} (K_{P,i} s + K_{D,i} s^{2} + K_{I,i}) q_{i}^{T} - \frac{s}{n_{i}^{2}} T_{L} (K_{P,i} s + K_{D,i} s^{2} + K_{I,i}) q_{i}^{T} - \frac{s}{n_{i}^{2}} T_{L} (K_{P,i} s + K_{D,i} s^{2} + K_{I,i}) q_{i}^{T} - \frac{s}{n_{i}^{2}} T_{L} (K_{P,i} s + K_{D,i} s^{2} + K_{I,i}) q_{i}^{T} - \frac{s}{n_{i}^{2}} T_{L} (K_{P,i} s + K_{D,i} s^{2} + K_{D,i}) q_{i}^{T} - \frac{s}{n_{i}^{2}} T_{L} (K_{P,i} s + K_{D,i} s^{2} + K_{D,i}) q_{i}^{T} - \frac{s}{n_{i}^{2}} T_{L} (K_{P,i} s + K_{D,i} s^{2} + K_{D,i}) q_{i}^{T} - \frac{s}{n_{i}^{2}} T_{L} (K_{P,i} s + K_{D,i} s^{2} + K_{D,i}) q_{i}^{T} - \frac{s}{n_{i}^{2}} T_{L} (K_{P,i} s + K_{D,i} s^{2} + K_{D,i}) q_{i}^{T} - \frac{s}{n_{i}^{2}} T_{L} (K_{P,i} s + K_{D,i} s^{2} + K_{D,i}) q_{i}^{T} - \frac{s}{n_{i}^{2}} T_{L} (K_{P,i} s + K_{D,i} s^{2} + K_{D,i}) q_{i}^{T} - \frac{s}{n_{i}^{2}} T_{L} (K_{P,i} s + K_{D,i} s^{2} + K_{D,i}) q_{i}^{T} - \frac{s}{n_{i}^{2}} T_{L} (K_{P,i} s + K_{D,i} s^{2} + K_{D,i}) q_{i}^{T} -$$

$$q_{i} = \frac{1}{N_{i}(s)} [F_{i}(s)q_{i}^{r}(s) - G_{i}(s)T_{L,i}(s)]$$

$$\begin{split} N_{l}(s) &= J_{eff,l}s^{3} + s^{2} \left(f_{eff} + k_{T,l}K_{D,l} \right) + k_{T,l}K_{P,l}s + k_{T,l}K_{I,l} \\ F_{l}(s) &= k_{S,l} \left(K_{P,l}s + K_{D,l}s^{2} + K_{I,l} \right) \end{split}$$

$$F_i(s) = k_{T,i} (K_{P,i} s + K_{D,i} s^2 + K_{I,i})$$

$$G_i(s) = \frac{s}{n^2}$$

Problem 15

$$\begin{split} J_{eff,i}s^3 + s^2 \big(f_{eff} + k_{T,i}K_{D,i}\big) + k_{T,i}K_{P,i}s + k_{T,i}K_{I,i} &= \big(s^2 + 2\omega_{n,i}\zeta_{i}s + \omega_{n,i}^2\big)(1 + \tau_{i}'s) = s^2 + 2\omega_{n,i}\zeta_{i}s + \omega_{n,i}^2 + \tau_{i}'s^3 + 2\omega_{n,i}\zeta_{i}\tau_{i}'s^2 + \omega_{n,i}^2\tau_{i}'s \\ J_{eff,i}s^3 + s^2 \big(f_{eff} + k_{T,i}K_{D,i}\big) + k_{T,i}K_{P,i}s + k_{T,i}K_{I,i} &= \tau_{i}'s^3 + \big(2\omega_{n,i}\zeta_{i}\tau_{i}' + 1\big)s^2 + \big(\omega_{n,i}^2\tau_{i}' + 2\omega_{n,i}\zeta_{i}\big)s + \omega_{n,i}^2 + \omega_{n,i}^2\tau_{i}'s \\ J_{eff,i}s^3 + s^2 \big(f_{eff} + k_{T,i}K_{D,i}\big) + k_{T,i}K_{P,i}s + k_{T,i}K_{I,i} &= \tau_{i}'s^3 + 2\omega_{n,i}\zeta_{i}'s + \omega_{n,i}^2\tau_{i}'s \\ J_{eff,i}s^3 + s^2 \big(f_{eff} + k_{T,i}K_{D,i}\big) + k_{T,i}K_{P,i}s + k_{T,i}K_{I,i} &= \tau_{i}'s^3 + 2\omega_{n,i}\zeta_{i}'s + \omega_{n,i}^2\tau_{i}'s \\ J_{eff,i}s^3 + s^2 \big(f_{eff} + k_{T,i}K_{D,i}\big) + k_{T,i}K_{P,i}s + k_{T,i}K_{I,i} &= \tau_{i}'s^3 + 2\omega_{n,i}\zeta_{i}'s + \omega_{n,i}^2\tau_{i}'s \\ J_{eff,i}s^3 + s^2 \big(f_{eff} + k_{T,i}K_{D,i}\big) + k_{T,i}K_{P,i}s + k_{T,i}K_{I,i} &= \tau_{i}'s^3 + 2\omega_{n,i}\zeta_{i}'s + \omega_{n,i}^2\tau_{i}'s \\ J_{eff,i}s^3 + s^2 \big(f_{eff} + k_{T,i}K_{D,i}\big) + k_{T,i}K_{P,i}s + k_{T,i}K_{I,i} &= \tau_{i}'s^3 + 2\omega_{n,i}\zeta_{i}'s + \omega_{n,i}^2\tau_{i}'s \\ J_{eff,i}s^3 + 2\omega_{n,i}\zeta_{i}'s + \omega_{n,i}^2\tau_{i}'s + \omega_{n,i}^2\tau_{i}'s + \omega_{n,i}^2\tau_{i}'s + \omega_{n,i}^2\tau_{i}'s + \omega_{n,i}^2\tau_{i}'s \\ J_{eff,i}s^3 + 2\omega_{n,i}\zeta_{i}'s + \omega_{n,i}^2\tau_{i}'s + \omega_{n,i}^2\tau_{i}'s$$

Then we have

$$\tau_i' = |J_{eff,i}|$$

$$\begin{aligned} & \mathcal{C}_{l} = perf, i \\ & 2\omega_{n,l}\zeta_{l}\tau'_{l} + 1 = f_{eff} + k_{T,l}K_{D,l} & \Leftrightarrow & K_{D,l} = \frac{2\omega_{n,l}\zeta_{l}\tau'_{l} + 1 - f_{eff}}{k_{T,l}} \\ & \omega_{n,l}^{2}\tau'_{l} + 2\omega_{n,l}\zeta_{l} = k_{T,l}K_{P,l} & \Leftrightarrow & K_{P,l} = \frac{\omega_{n,l}^{2}\tau'_{l} + 2\omega_{n,l}\zeta_{l}}{k_{T,l}} \\ & \omega_{n,l}^{2} = k_{T,l}K_{I,l} & \Leftrightarrow & K_{I,l} = \frac{\omega_{n,l}^{2}}{k_{T,l}} \end{aligned}$$

$$\omega_{n,i}^2 \tau_i' + 2\omega_{n,i} \zeta_i = k_{T,i} K_{P,i} \quad \Leftrightarrow \quad K_{P,i}$$

$$K_{P,i} = \frac{\omega_{n,i} v_i + 2\omega}{k_{T,i}}$$

$$_{n,i}^{2}=k_{T,i}K_{I,i}$$
 \Leftrightarrow

$$K_{I,i} = \frac{\omega_{n,i}^2}{k_{T,i}}$$

(1)
$$D(q) \cdot \ddot{q} + v(q, \dot{q}) = \tau = [\tau_1, \tau_2, \tau_3, \tau_4]^T$$

$$\begin{aligned} (1) \qquad & D(q) \cdot \ddot{q} + v(q, \dot{q}) = \tau = [\tau_1, \tau_2, \tau_3, \tau_4]^T \\ (2) \qquad & u_i \cdot k_{T,i} - \frac{1}{n_i} \cdot \tau_i = J_{M,i} \cdot \ddot{q}_{M,i} + f_{M,i} \cdot \dot{q}_{M,i} \end{aligned}$$

Reduce to

$$\widetilde{D}(q) \cdot \ddot{q} + \widetilde{v}(q, \dot{q}) = u$$

Start with (2):
$$\begin{aligned} u_i \cdot k_{T,i} - \frac{1}{n_i} \cdot \tau_i &= J_{M,i} \cdot \ddot{q}_{M,i} + f_{M,i} \cdot \dot{q}_{M,i} & \Leftrightarrow \\ \tau_i &= u_i \cdot k_{T,i} n_i - J_{M,i} \cdot n_i \ddot{q}_{M,i} - f_{M,i} \cdot n_i \dot{q}_{M,i} & \Leftrightarrow \end{aligned}$$

$$\begin{aligned} & n_i \\ & \tau_i = u_i \cdot k_{T,i} n_i - J_{M,i} \cdot n_i \ddot{q}_{M,i} - f_{M,i} \cdot n_i \dot{q}_{M,i} \Leftrightarrow \\ & \tau_i = u_i \cdot k_{T,i} n_i - J_{M,i} \cdot n_i^2 \, \ddot{q}_i - f_{M,i} \cdot n_i^2 \, \dot{q}_i \end{aligned}$$

$$\begin{split} &u_i \cdot k_{T,i} n_i - J_{M,i} \cdot n_i^2 \; \ddot{q}_i - f_{M,i} \cdot n_i^2 \dot{q}_i = D(q) \cdot \ddot{q} + v(q,\dot{q}) \; \Leftrightarrow \\ &u_i = \frac{1}{k_{T,i} n_i} \Big[D(q) \cdot \ddot{q} + v(q,\dot{q}) + J_{M,i} \cdot n_i^2 \; \ddot{q}_i + f_{M,i} \cdot n_i^2 \dot{q}_i \Big] \; \Leftrightarrow \end{split}$$

$$\begin{split} u_i &= \frac{1}{k_{T,i}n_i} \left[D(q) \cdot q + v(q,q) + J_{M,i} \cdot n_i^2 \ q_i + f_{M,i} \cdot n_i^2 q_i \right] \Leftrightarrow \\ u_i &= \frac{1}{k_{T,i}n_i} \left[\left(D(q) + J_{M,i} \cdot n_i^2 \right) \cdot \ddot{q} + v(q,\dot{q}) + f_{M,i} \cdot n_i^2 \dot{q}_i \right] \Leftrightarrow \end{split}$$

$$(*) \ \widetilde{D}(q) = \frac{D(q) + \int_{M,i} \cdot n_i}{k_{T,i}} \quad \text{ and } \quad (**) \ \widetilde{v}(q,\dot{q}) = \frac{v(q,\dot{q}) + \int_{M,i} \cdot n_i \dot{q}_i}{k_{T,i}}$$