Safe trajectory tracking using closed-form controllers based on control barrier functions

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***Abstract*— This paper considers the problem of guaranteeing avoidance of critical state space regions during tracking of reference trajectories for systems with dynamics equivalent to r-th order decoupled integrators. The necessity to avoid those critical regions during trajectory tracking may arise during the transient phase or because the reference trajectory was planned without taking into account the presence of those critical regions. A typical problem in mobile robotics, taken as reference in this paper, is the avoidance of obstacles in the robot workspace during tracking of reference state space trajectories. The proposed controller ensures a safety clearance from the forbidden regions by filtering out, when appropriate, the component of the tracking command that would eventually lead the system to enter the critical region. The method relies on the construction of first-order control barrier functions and closed-form controllers, with formal proof of safety and stability, and its effective application to wheeled mobile robots and quadrotors is demonstrated through simulation.**

I. Introduction

This paper considers the problem of avoiding regions of the state space of a dynamic system that are considered critical with respect to safety or performance criteria, while tracking reference state trajectories.

This is a problem arising, for example, in mobile robotics, when the vehicle must follow a reference trajectory in an environments cluttered by obstacles. In this case, the reference trajectory might not avoid collisions with the obstacles because all or some of the obstacles were not known a priori. It is therefore necessary to combine a tracking controller with an obstacle avoidance method. Even if the reference trajectory has been planned to be collision-free, collisions with obstacles may still occur during transient. While shaping the transient is possible in some cases, a general method is not available and safety during transient is complicated to guarantee, particularly for high relative degree constraints.

There exist few examples of methods combining tracking and obstacle avoidance with formal guarantees of safety and tracking error boundedness. Among these, the work [1] proposes a method to track the output trajectory, namely the Cartesian position of a reference point on the vehicle, for differential drive mobile robots. With respect to [1], the proposed approach is more general because it can consider the avoidance of generic regions of the state space and, due to the adopted methodology, is amenable to extension to systems with higher relative degree. A more general method, strictly related to the one proposed here, is [2] which also relies on the construction of a safety filter to guarantee obstacle avoidance during trajectory tracking for a quadrotor. The controller, however, relies on online optimization with the avoidance constraint in order to modify the tracking commands when necessary for safety keeping.

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Optimization-based methods are, in fact, usually adopted to avoid collisions with obstacles during tracking [3], to avoid collisions among vehicles, each accomplishing its own output tracking tasks, in a multi-agent scenario [4], [5], to enforce other safety constraints [6], or to rise risk awareness [7]. The approach is to consider the safety requirement, enforced and certified by properly defined barrier functions, as a constraint of the optimization problem. This computational approach, however, does not allow in general to easily predict or shape the produced trajectories, thus resulting in a somewhat difficult physical interpretation of the control action.

Considering systems governed by r-th integrator dynamics, possibly under feedback, in this paper we propose to adopt control barrier functions to formally guarantee the avoidance of regions of the state space, during tracking of reference state trajectories. In particular, the avoidance of obstacles, in the case of a second order model of mobile robots, is obtained by properly defining a control barrier function (CBF) in the state space that takes into account not only the robot position but also its velocity. The proposed controller modifies, when necessary, the trajectory tracking law by filtering out only those components of the tracking control command that would eventually lead the system to a collision. Formal safety guarantees are provided by ensuring forward invariance of the safe region together with conditions for boundedness of the tracking error and stability of the controlled system. Although illustrated for the case of trajectory tracking with obstacle avoidance for mobile robots, the proposed method is general and can be extended to systems with arbitrary order dynamics. Preliminary results are presented for a quadrotor UAV. The contribution of the paper consists in: i) proposing an intuitive approach to the design of first-order CBFs for second order systems to avoid obstacles, and the preliminary extension for application to systems with higher order dynamics; ii) providing explicitly a safe tracking controller in closed-form, which can be proven to coincide with the solution obtained via QP methods; iii) formally proving the stability of the control system.

The presentation is organized as follows. Section II formalizes the problem and illustrates the proposed solution approach, leading to a control law that combines trajectory tracking and obstacle avoidance. The stability of the obtained control system, its generalization to include multiple obstacles and higher order dynamics, are then analysed in Sect. III. The validation results are reported in Sect. IV. Concluding remarks and future developments are proposed in Sect. V.

II. Problem formulation and proposed approach

This section formalizes the problem of safe trajectory planning and illustrates the proposed controller design by making reference to the second order linear dynamic system

$$\ddot{p}(t) = u(t), \quad p \in \mathbb{R}^n, \quad u \in \mathbb{R}^n. \tag{1}$$

It is worth to note that n-dimensional systems of decoupled r-th order integrators can represent the dynamics of nonlinear systems that are exactly linearizable through static or dynamic feedback and these include a wide range of mobile robots like, e.g., unicycle-like, differential drive, car-like robots, standard trailer-truck systems, quadrotors.

Interpreting eq. (1) as the motion dynamics of a single rigid body, p represents the vector of Cartesian coordinates of the body center of mass P, while u is the control input. The task is to track the, not necessarily feasible and not necessarily collision-free, output reference trajectory $p_d(t)$ while guaranteeing safety with respect to collisions with obstacles discovered at the tracking time.

To track the pre-planned output trajectory $p_d(t)$ as close as possible while guaranteeing to stay clear from obstacles even when moving away from $p_d(t)$, the paper proposes to design the control u based on the definition of an appropriate barrier function (see [3] for a brief history and fundamental concepts). Consider the function

$$h(p(t),\dot{p}(t)) = (p(t)-\bar{p})^T(p(t)-\bar{p}) + \mu(p(t)-\bar{p})^T(\dot{p}(t)), \tag{2}$$

where $\bar{p}$ denotes the position of a point obstacle and μ is a positive constant. Equation (2) defines the norm of distance vector from the body center of mass position p to the obstacle position $\bar{p}$ plus a term with sign depending on the relative direction of the distance vector and the velocity $\dot{p}$ of the system. In particular, if the velocity points away from the obstacle, this term adds to the norm of the distance between P and the obstacle, otherwise it is subtracted. The objective is to define the control u so as to guarantee that this function is always bigger than a positive constant δ which defines the obstacle clearance. The system states that satisfy the equation

$$\bar{h}(p(t),\dot{p}(t)) := h(p(t),\dot{p}(t)) - \delta \geq 0 \tag{3}$$

define the *safety region* $\mathcal{S}_{\text{free}} = \{(p,\dot{p}) \in \mathbb{R}^{2n} : \bar{h}(p,\dot{p}) \geq 0\}$. To guarantee safety, we need to prove that $\bar{h}$, defined through the function h in (2), is a CBF, i.e., we need to find u such that [3]:

$$\dot{h}(t) + \alpha(h(t)-\delta) \geq 0.$$

where $\alpha > 0$ is a fixed positive constant. If $\bar{h}$ is a CBF, then a trajectory of the system starting in $\mathcal{S}_{\text{free}}$ is guaranteed to remain in that set.

Dropping the time dependence for brevity, we look for a right-differentiable input u such that

$$2(p-\bar{p})^T(\dot{p}) + \mu\dot{p}^T\dot{p} + \mu(p-\bar{p})^T u + \alpha(h-\delta) \geq 0. \tag{4}$$

Assuming that the initial state is within the *safety region* (3), and considering that the differential condition for $\bar{h}$ reads as

$$\dot{\bar{h}} \geq -\alpha\bar{h}, \tag{5}$$

application of the comparison lemma [8, Lemma 3.4] implies that $\bar{h}(t) \geq e^{-\alpha t}\bar{h}(0) > 0$ for any $t \geq 0$, that is the safety set is forward invariant with $(p(t),\dot{p}(t)) \in \mathcal{S}_{\text{free}}$ for any $t \geq 0$.

Introducing a slight conservatism, a sufficient condition for the forward invariance of the safety set is obtained by determining u such that $\dot{\bar{h}} \geq 0$. If this condition is satisfied, $\bar{h}$ can never become negative. The input u is therefore obtained by satisfying the following inequality

$$2(p-\bar{p})^T\dot{p} + \mu(p-\bar{p})^T u \geq 0. \tag{6}$$

Some preliminary definitions are in order to formulate the control u such that the desired output trajectory p_d is asymptotically tracked while keeping the system safe.

- Define the projection operators

$$\Pi_{p-\bar{p}} := (p-\bar{p})[(p-\bar{p})^T(p-\bar{p})]^{-1}(p-\bar{p})^T \tag{7}$$
$$\Pi^{\perp}_{p-\bar{p}} := I - \Pi_{p-\bar{p}}. \tag{8}$$

  It is easy to verify that $(p-\bar{p})^T\Pi^{\perp}_{p-\bar{p}}w = 0$ for any $w \in \mathbb{R}^n$, i.e., the operator defined in (8) returns the component of w which is orthogonal to $(p-\bar{p})$.
- Let

$$u^\star = \ddot{p}_d + k_d(\dot{p}_d - \dot{p}) + k_p(p_d - p)$$

  be the trajectory tracking control law. Setting $u = u^\star$ in (1), the gains k_p, k_d are chosen such that the closed-loop state transition matrix

$$A_{cl} = \begin{bmatrix} 0 & I \\ -k_pI & -k_dI \end{bmatrix},$$

  where the identity blocks have dimension n, is Hurwitz.
- Decompose the state space $\mathbb{R}^{2n}$ as follows

$$\mathbb{R}^{2n} = \mathcal{D}_{\text{track}}(t) \cup \mathcal{D}_{\perp}(t), \quad \mathcal{D}_{\text{track}}(t) = \mathcal{D}_{u^\star}(t) \cup \mathcal{D}_{\delta_1}.$$

  $\mathcal{D}_{\text{track}}$ is the state subspace where the priority of the control action is the asymptotic tracking of $p_d(t)$. In particular, in region

$$\mathcal{D}_{u^\star}(t) := \{(p,\dot{p}) : (p-\bar{p})^T(\mu u^\star + 2\dot{p}) > 0\}$$

  forward invariance of $\mathcal{S}_{\text{free}}$ during tracking is preserved because the control vector and the current velocity are such that p is moving away from the closest obstacle. When the state belongs to the *conservative* safety region

$$\mathcal{D}_{\delta_1} := \{(p,\dot{p}) : h(p,\dot{p}) > \delta_1 > \delta\}$$

  forward invariance is guaranteed for any direction of motion of the state. The control priority in the region

$$\mathcal{D}_{\perp}(t) = \mathbb{R}^{2n} \setminus \mathcal{D}_{\text{track}}(t)$$

  is instead to avoid obstacles.

Safety during tracking is then enforced by

$$u = \begin{cases} u^\star & \text{if } (p,\dot{p}) \in \mathcal{D}_{\text{track}} \\ -\frac{2}{\mu}\Pi_{p-\bar{p}}\,\dot{p} + \Pi^{\perp}_{p-\bar{p}}u^\star & \text{if } (p,\dot{p}) \in \mathcal{D}_{\perp}. \end{cases} \tag{9}$$

In fact, whenever $(p,\dot{p}) \in \mathcal{D}_{\perp}$, the evaluation of $\dot{\bar{h}}$ yields

$$\begin{aligned}\dot{\bar{h}} &= 2(p-\bar{p})^T\dot{p} + \mu\dot{p}^T\dot{p} \\ &\quad + \mu(p-\bar{p})^T(-\tfrac{2}{\mu}\Pi_{p-\bar{p}}\,\dot{p} + \Pi^{\perp}_{p-\bar{p}}u^\star) = \mu\dot{p}^T\dot{p} \geq 0,\end{aligned}$$

thus showing that the control action prevents the state of the system from approaching further the obstacle. The control policy (9) essentially overrules, when needed, the PID tracking controller $u^\star$ by filtering out the component of the acceleration pointing toward the obstacle. As well known, triggering control switchings based on state conditions is likely to generate chattering. However, this can be avoided by enforcing an hysteresis scheme, for instance by introducing a smooth junction between the two control laws.

Remark 1. *It is worth stressing that the proposed explicit controller* (9)*, which results from the natural intuition of pruning the nominal input components pointing towards* *the obstacle, is actually equivalent to the solution provided by QP methods. For instance, using Karush-Kuhn-Tucker conditions, the control law* (9) *can be shown to be the explicit solution of the QP problem [3, CBF-QP page 3423] subject to the CBF constraint* (6). *Nevertheless, the latter optimization problem alone is not capable to formally guarantee a stable closed-loop behaviour. Conversely, the system driven by our switching control strategy is proven to be asymptotically stable, in a hybrid sense, in Sect. III.* ∘

III. Properties and generalization

A. Stability analysis

To prove that the closed-loop system, driven by the control law (9), is stable it is convenient to use the state space $z = (z_1, z_2) =: (p, \dot{p})$ and recast the system as

$$\begin{array}{ll} \dot{z} = (A - BK)z + B\varphi(t) & \text{if } (z_1, z_2) \in \mathcal{D}_{\text{track}} \\ \dot{z} = Az + Bg(z) + B\Pi^{\perp}_{z_1 - \bar{p}}\varphi(t) & \text{if } (z_1, z_2) \in \mathcal{D}_{\perp} \end{array} \tag{10}$$

with matrices given by

$$A = \begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ I \end{bmatrix}, \quad K = [k_p I \quad k_d I],$$
$$g(z) = -\tfrac{2}{\mu}\Pi_{z_1-\bar{p}} z_2 - \Pi^{\perp}_{z_1-\bar{p}} K z,$$

and where $\varphi(t)$ is the feed-forward term

$$\varphi(t) = \ddot{p}_d(t) + k_d \dot{p}_d(t) + k_p p_d(t)$$

To address stability, consider the unforced system (10), i.e., neglect the external input $\varphi(t)$. Next results show that stability of the closed loop is guaranteed by positive PD gains and μ small enough. In particular, a common Lyapunov function exists for both switching modes in (10).

Theorem 1. *Let us consider the autonomous system*

$$\begin{array}{ll} \dot{z} = (A - BK)z & \text{if } (z_1, z_2) \in \mathcal{D}_{\text{track}} \\ \dot{z} = Az + Bg(z) & \text{if } (z_1, z_2) \in \mathcal{D}_{\perp} \end{array}$$

For any choice of positive gains $k_p, k_d > 0$ and for $\mu < \min\left\{\frac{2}{k_d}, \frac{2k_d}{k_p + k_d^2}\right\}$, the system is stable and, in particular, the set $\mathcal{A} = \{z \in \mathbb{R}^{2n} : z_1 = \gamma\bar{p}, \gamma \in \mathbb{R}, z_2 = 0\}$ is an asymptotically stable attractor.

Proof. Let us consider a quadratic Lyapunov function candidate $V(z) = z^T P z$ with $P = P^T$ of the form

$$P = \begin{bmatrix} P_1 & P_2 \\ P_2 & P_3 \end{bmatrix}$$

where the blocks are given by $P_i = \lambda_i I$, for $i = 1, 2, 3$. The positive definiteness of P is guaranteed by the inequalities

$$\lambda_1 > 0, \quad \lambda_1\lambda_3 - \lambda_2^2 > 0. \tag{11}$$

The derivative of the candidate Lyapunov function along the system trajectories must be negative for asymptotic stability:

$$z^T((A - BK)^T P + P(A - BK))z < 0, \tag{12}$$

which is equivalent to

$$\begin{bmatrix} -2k_p\lambda_2 & \lambda_1 - k_d\lambda_2 - k_p\lambda_3 \\ \lambda_1 - k_d\lambda_2 - k_p\lambda_3 & 2(\lambda_2 - k_d\lambda_3) \end{bmatrix} < 0. \tag{13}$$

The previous inequality is satisfied by selecting $0 < \lambda_2 < k_d\lambda_3$ and $\lambda_1 = k_d\lambda_2 + k_p\lambda_3$. Note that this choice is also compatible with the conditions (11). In particular, the first inequality in (11) is trivially satisfied, whereas the second one reads as $k_p\lambda_3^2 + k_d\lambda_2\lambda_3 - \lambda_2^2 > 0$, which is verified whenever

$$\lambda_2 \in \left(\frac{k_d - \sqrt{k_d^2 + 4k_p}}{2}\lambda_3, \frac{k_d + \sqrt{k_d^2 + 4k_p}}{2}\lambda_3 \right).$$

The lower bound is negative by construction and the upper bound is always larger than $k_d\lambda_3$. It is then sufficient to pick $0 < \lambda_2 < k_d\lambda_3$ to stay within these bounds so as to satisfy simultaneously (11) and (13). The above computations prove that, as long as $z \in \mathcal{D}_{\text{track}}$, $V(z)$ is a strict Lyapunov function for the closed-loop system.

Let us now evaluate the derivative of $V(z)$ along the system trajectories belonging to the set $\mathcal{D}_{\perp}$, which is

$$\dot{V}(z) = z^T P(Az + Bg(z)) + (Az + Bg(z))^T P z. \tag{14}$$

Direct computation of $Az + Bg(z)$ yields

$$Az + Bg(z) = \begin{bmatrix} 0 & I \\ -k_p\Pi^{\perp}_{z_1-\bar{p}} & -k_d\Pi^{\perp}_{z_1-\bar{p}} - \frac{2}{\mu}\Pi_{z_1-\bar{p}} \end{bmatrix} z,$$

which, together with the special structure of P and thanks to the symmetry of the matrices $\Pi^{\perp}_{z_1-\bar{p}}$ and $\Pi_{z_1-\bar{p}}$, implies that (14) reads as

$$\dot{V}(z) = z^T \begin{bmatrix} -2k_p\lambda_2\Pi^{\perp}_{z_1-\bar{p}} & (k_d\lambda_2 + k_p\lambda_3)I - q_2(z) \\ * & 2(\lambda_2 I - q_3(z)) \end{bmatrix} z$$

where $*$ indicates a symmetric quantity and with

$$q_2(z) := ((k_d\lambda_2 + k_p\lambda_3)\Pi^{\perp}_{z_1-\bar{p}} + \tfrac{2}{\mu}\lambda_2\Pi_{z_1-\bar{p}})$$
$$q_3(z) := k_d\lambda_3\Pi^{\perp}_{z_1-\bar{p}} + \tfrac{2}{\mu}\lambda_3\Pi_{z_1-\bar{p}}.$$

We aim at cancelling the off-diagonal terms and, observing that by construction the projection operators satisfy $\Pi^{\perp}_{z_1-\bar{p}} + \Pi_{z_1-\bar{p}} = I$, this cancellation can be achieved if

$$k_d\lambda_2 + k_p\lambda_3 = \frac{2}{\mu}\lambda_2.$$

The above condition, which is feasible and compatible with (13) if μ satisfies the assumptions $\mu < \frac{2}{k_d}$ and $\mu < \frac{2k_d}{k_p + k_d^2}$, yields the following expression for λ_2

$$\lambda_2 = \frac{\mu k_p}{2 - \mu k_d}\lambda_3.$$

Thanks to this choice, $\dot{V}(z)$ becomes

$$\dot{V}(z) = z^T \underbrace{\begin{bmatrix} -2k_p\lambda_2\Pi^{\perp}_{z_1-\bar{p}} & 0 \\ 0 & -\bar{q}_3\lambda_3 I - (\frac{4}{\mu} - 2k_d)\lambda_3\Pi_{z_1-\bar{p}} \end{bmatrix}}_{=:Q(z)} z \tag{15}$$

where $\bar{q}_3 = 2k_d - \frac{2\mu k_p}{2 - \mu k_d} > 0$ and $Q(z) \leq 0$ is a negative semi-definite matrix for any $z \in \mathbb{R}^{2n}$ and for any fixed coefficient $\lambda_3 > 0$. As a consequence, the system trajectories remain bounded and, invoking LaSalle's principle, converge to the largest invariant set contained in $\mathcal{Z} := \{z \in \mathbb{R}^{2n} : \dot{V}(z) = 0\}$. The lower-right block of $Q(z)$ is negative definite for any z, whereas the upper-left block is negative semi-definite, with null-space given by the set $\mathcal{N}_0 = \{z_1 \in \mathbb{R}^n : \Pi^{\perp}_{z_1-\bar{p}} z_1 = 0\}$. On the other hand, $\Pi^{\perp}_{z_1-\bar{p}} z_1 = 0$

if and only if z_1 is a multiple of $\bar{p}$, that is $z_1 = \gamma\bar{p}$ for some $\gamma \in \mathbb{R}$. In conclusion, combining (12) and (14), we have shown that the set $\mathcal{A} = \{z \in \mathbb{R}^{2n} : z_1 = \gamma\bar{p}, z_2 = 0\}$ is a global attractor, with rate of convergence $\beta = \min\{2k_p\lambda_2,\ 2(k_d\lambda_3 - \lambda_2),\ \bar{q}_3\}$. □

Remark 2. *It is worth noticing that, since for $z \in \mathcal{D}_{\delta_1}$ the derivative $\dot{V}(z)$ is strictly negative and the set $\mathcal{S}_{\rm free}$ is invariant, convergence onto the attractor $\mathcal{A}$ may only occur towards the compact set of points of the form $z_\gamma = (\gamma\bar{p}, 0)$ with $\delta \le h(\gamma\bar{p}, 0) \le \delta_1$. In particular, the scale factor γ is admissible only if $\delta \le (\gamma-1)^2\|\bar{p}\|^2 \le \delta_1$.* ◦

The results proved in Theorem 1 imply that the autonomous system trajectory may eventually approach and get stuck on points lying in a limited region aligned with the obstacle $\bar{p}$. This phenomenon can be ruled out by a persistency-of-excitation condition on the feedforward input $\varphi(t)$.

Corollary 1. *Consider the forced system (10). Suppose that the following persistency-of-excitation (PE) condition holds:*

$$\limsup_{t\to+\infty} \|\Pi_{\bar{p}}^{\perp}(\ddot{p}_d(t) + k_d\dot{p}_d(t) + k_p p_d(t))\| \neq 0$$

Then the forced system can not remain at rest in any of the points of the set $\mathcal{A}$, in particular the state z_2 can not be identically zero.

Proof. The statement simply follows observing that assuming $z_2 \equiv 0$ would also imply $\dot{z}_2 \equiv 0$ and that, for initial condition $(z_1, z_2) = (\gamma\bar{p}, 0)$, the latter is equivalent to the identity $\Pi_{\bar{p}}^{\perp}\varphi(t) = 0$ for any t. This contradicts the persistency of excitation assumption and therefore, under such condition, the trajectory necessarily leaves the set $\mathcal{A}$ due to the driving force towards the reference path. □

Remark 3. *Although the fulfillment of the PE condition by the reference trajectory prevents the state z from remaining in the set $\mathcal{A}$, Theorem 1 and Corollary 1 are not sufficient in general to guarantee that the tracking error $e = (e_1, e_2) = (z_1 - p_d, z_2 - \dot{p}_d)$ vanishes asymptotically. In particular the convergence depends on the interplay between the reference trajectory and the obstacle (through the barrier function). On the other hand, whenever the state $z \in \mathcal{D}_{\rm track}$, the error system satisfies the Lyapunov condition $e^T((A - BK)^T P + P(A - BK))e < 0$, which is inherited from (12). Accordingly, if the reference trajectory ultimately lies away from the critical region, i.e., if $h(p_d(t), \dot{p}_d(t)) > \delta_1$ for any $t > \bar{t}$, asymptotic tracking is expected to be attained based on a continuity argument.* ◦

B. Connection with exponential control barrier functions

Although the primary objective of the proposed design is to keep the position of a mobile agent (1) away from obstacles, the considered barrier function depends explicitly also on its velocity. This provides the barrier function with relative degree equal to one. Nonetheless, by ensuring that the proposed function remains positive, collisions are also guaranteed to be avoided. The velocity term acts as a correction of the clearance that is reduced when the vehicle moves away from the obstacle and increased in the opposite case. The inversion of the velocity sign can occur at most on the boundary of the safe ball with radius δ centered at the obstacle position. Alternatively, one could rely on the construction of exponential barrier functions [3], as this would relax the need for a safety test involving velocity (or higher derivatives as in the case of higher order systems, see Section III-D). However, the design of exponential barrier functions is not straightforward and requires the fulfillment of nested invariance conditions (see [3]) which would translate, in our scheme, into nested switching conditions.

C. Multiple obstacles

The developments in Sect. II can be readily extended to encompass scenarios with multiple obstacles. To this end, consider the set of obstacles $\mathcal{O} = \{\bar{p}_i \in \mathbb{R}^n, i = 1, ..., m\}$ and assume that

$$\|\bar{p}_i - \bar{p}_j\|^2 > 2\delta_1 \text{ for any } i, j = 1, ..., m, \tag{16}$$

these conditions being imposed to avoid overlapping of the basins of influence of two or more obstacles. Accordingly, define the family of barrier functions

$$\bar{h}_i(p, \dot{p}) = (p - \bar{p}_i)^T(p - \bar{p}_i) + \mu(p - \bar{p}_i)^T\dot{p} - \delta,\ i = 1, ..., m. \tag{17}$$

A possible way to tackle the problem is to consider the minimum over this family, namely

$$\bar{h}(p(t), \dot{p}(t)) = \min_{i=1,\dots,m} \bar{h}_i(p(t), \dot{p}(t)).$$

However this would result in a non-smooth function, thus making the evaluation of the derivative of $\bar{h}$ along the system trajectories challenging. A better and simple alternative for implementing (17), is to compute the label of the closest obstacle to the current position, i.e.,

$$i^* = \arg\min_{i=1,\dots,m} \|p - \bar{p}_i\|,$$

and to consider *active* the barrier function $\bar{h}_{i^*}$ only. Note that, thanks to the non-overlapping assumption (16), the state $(p, \dot{p})$ must necessarily have a transition from the critical set $\mathcal{D}_{\perp,i}$ to $\mathcal{D}_{\rm track}$ and cannot switch directly from $\mathcal{D}_{\perp,i}$ to $\mathcal{D}_{\perp,j}$ with $j \neq i$. Finally, it is worth stressing that the Lyapunov function $V(\cdot)$ used in the proof of Theorem 1 and the associated convergence rate do not depend on the obstacle location $\bar{p}$, so that such $V(\cdot)$ is a common Lyapunov function for all switched modes of the system.

D. Higher order integrators

The proposed design may also be extended to systems with higher relative degree. Let us briefly sketch the case of a r-th order integrator, whose dynamics is given by

$$p^{(r)} = u$$

It is convenient to adopt a state space representation with $(p, \dot{p}, \ddot{p}, \dots, p^{(r-1)}) = (z_1, z_2, z_3, ..., z_r) = z$ and rewrite the system as

$$\begin{aligned} \dot{z}_j &= z_{j+1} \quad j = 1, ..., r-1 \\ \dot{z}_r &= u \end{aligned} \tag{18}$$

The CBF candidate $\bar{h}(z) = h(z) - \delta$ is considered, with

$$h(z) = (z_1 - \bar{p})^T(z_1 - \bar{p}) + \sum_{j=2}^{r} \mu_j (z_1 - \bar{p})^T z_j \tag{19}$$

where $\mu_j \geq 0$ for $j = 2, ..., r-1$ and $\mu_r > 0$. Differentiating $h(z)$ along the system trajectories yields

$$\begin{aligned}\dot{h}(z,u) &= 2(z_1-\bar{p})^T z_2 + \sum_{j=2}^{r} \mu_j z_2^T z_j \\ &+ \sum_{j=2}^{r-1} \mu_j (z_1-\bar{p})^T z_{j+1} + \mu_r (z_1-\bar{p})^T u.\end{aligned}$$

Denoting by $\mu_1 := 2$, the following input is designed[1] to guarantee that the barrier function (19) is always positive:

$$u^{\dagger} = -\frac{1}{\mu_r}\sum_{j=1}^{r-1} \mu_j z_{j+1} - \frac{\sum_{j=3}^{r} \mu_j z_2^T z_j}{\mu_r h(z)} \left((z_1-\bar{p}) + \sum_{j=2}^{r} \mu_j z_j \right).$$

Let us stress that, as long as the system is in the region $\bar{h}(z) > 0$, that is $h(z) > \delta$, the second term in the right-hand side does not become singular. By direct evaluation, it is easy to check that the input $u^{\dagger}$ guarantees the invariance of the set $\{h(z) - \delta \geq 0\}$, i.e.

$$\dot{h}(z, u^{\dagger}) = \mu_2 z_2^T z_2 \geq 0 \quad \forall z.$$

Similarly to the double integrator case, the control $u^{\dagger}$ can be combined with the projection of the tracking controller $\Pi^{\perp}_{z_1-\bar{p}} u^*$, and input switchings can be triggered by the condition $\{\dot{h}(z, u^*) \leq 0\} \vee \{h(z) \leq \delta_1\}$. In case of obstacle avoidance by a mobile agent, it is important to point out that ensuring invariance of the set $\{h(z) - \delta \geq 0\}$, with $h(z)$ given by (19), does not necessarily imply invariance of the set $\{(z_1-\bar{p})^T(z_1-\bar{p}) - \delta \geq 0\}$. This means that the vehicle can enter any sphere $\mathcal{B}_\delta$ with radius δ and center in any of the obstacles position, before the velocity changes sign. To guarantee that the higher order derivatives appearing in (19) timely anticipate the inversion of motion, a proper choice of the coefficients μ_j is necessary.

In principle exponential CBFs, or high order CBFs at large [9], might be preferable for tackling the setup (18). On the other hand, as already mentioned in Section III-B, using exponential CBFs would introduce more complex switching conditions which could result in a heavier implementation. In either case the control policy guaranteeing avoidance of $\mathcal{B}_\delta$ would be semi-global, as the coefficients μ_j characterizing the CBF (19) or the vector of gains describing the invariance property in [3, Definition 7] are typically dependent on initial conditions.

In this paper, we have taken a practical approach for a preliminary illustration of the effectiveness of the proposed method in solving the trajectory tracking problem for a quadrotor, based on the 4-th order linear system obtained through dynamic feedback transformation, and with coefficient μ_j arbitrarily chosen, assigning a larger weight to the velocity term. In the following Remark 4 we provide a lower bound on the possible penetration of the ball $\mathcal{B}_\delta$, while the simulation results are shown in the Example 4.

Remark 4. *Consider a simplified version of the fourth order barrier function*

$$h(z) = (z_1-\bar{p})^T(z_1-\bar{p}) + \mu(z_1-\bar{p})^T(q z_2 + z_4), \quad (20)$$

which corresponds to (19) with coefficients $\mu_4 = \mu > 0$, $\mu_3 = 0$, *and* $\mu_2 = \mu\, q > 0$. *The condition* $h(z) > \delta$ *does not always ensure in this case that* $|z_1-\bar{p}|^2 > \delta$. *In particular "obstacle penetration" might occur when the following conditions are simultaneously satisfied:*

- $(z_1-\bar{p})^T(q z_2 + z_4) > 0$
- $(z_1-\bar{p})^T z_2 < 0$

By some algebraic manipulations and considering the safety margin $\delta_1 > \delta$, *we can give a lower bound on such penetration, that is*

$$|z_1-\bar{p}|^2 \geq \left(\frac{4\delta_1}{2\mu\zeta + 2\sqrt{4\delta_1 + \mu^2\zeta^2}} \right)^2 =: \omega(\zeta)$$

where $\zeta = |q z_2 + z_4|$. *This inequality allows for the possible tuning of parameters* δ_1, q *and* μ *depending on the initial conditions* $z_2(0), z_4(0)$ *in order to guarantee* $\omega(\zeta) \geq \delta$. ∘

IV. Simulation results

This section shows the simulation results obtained in CoppeliaSim by applying the proposed method to systems of increasing complexity. In particular, the first example considers the second order dynamics (1) as the unconstrained equation of motion of a mass on the plane; the second example considers that same dynamics as the result of the dynamic feedback linearization of a unycicle model [10], notably subject to nonholonomic constraints; the third and forth examples consider the application of the proposed approach to a quadrotor, first relying on a hierarchical control, then resorting to the dynamic feedback linearization of the system equations. Detailed comments follow, and a videoclip showing the simulations results is available at:

http://diag.uniroma1.it/labrob/research/TrajTrackCBF.html

Example 1. In this example, a unconstrained mass has to track an elliptic reference trajectory on the plane in a scenario with 5 static obstacles. The mass initial state is off the trajectory and the obstacles have not been considered in the planning phase. Hence, there is a risk of collision both while moving toward the trajectory and during tracking the reference itself. The controller (9) with the barrier function (2) is applied using the parameters $\delta = 0.5$, $\delta_1 = 3$, $\mu = 0.05$, $\{k_p, k_d\} = \{100, 30\}$. The behaviour is shown in Fig. (1a) and (1c). A persistent deviation from the reference path can be appreciated when the trajectory lies too close to the obstacles.

Example 2. Considering the same scenario of the previous example, the dynamics considered here is that of a differential drive wheeled mobile robot. It is well known that, under input transformation and dynamic feedback [10], the motion of the vehicle can be recast into pair of double-integrators together with a dynamic compensator. The linearizing outputs are the Cartesian coordinates of the vehicle that are kept at a due distance to the obstacles by the proposed method. Fig. (1b) and (1d) show the good performance of the proposed strategy in such more advanced scenario.

Example 3. Let us now consider the more challenging case of tracking control for quadrotor, with an elicoidal reference path. Using a hierarchical control strategy [11], based on a inner loop for attitude control and a outer loop for position and height control, we can still use the CBF (2), applied to

[1] By imposing the condition $\dot{h} \geq 0$.

the system of double-integrators in $\mathbb{R}^3$ corresponding to the outer loop subsystem. The parameters used in the simulation are: $\delta = 0.9$, $\delta_1 = 1.8$, $\mu = 0.05$, $\{k_p, k_d\} = \{100, 30\}$. We considered a scenario with two static obstacles, and the results are shown in Fig. (2a) and (2c). We can see that the obstacles are successfully avoided, and the reference path is safely reached without collisions.

Example 4. In the last example, the same scenario with a different control strategy is considered for the quadrotor, based on dynamic feedback linearization [12]. In particular, the dynamics can be reformulated as a system of 4-th order integrators, for which we used a higher-order barrier function of the form (20) with $\mu = 0.05$ and $q = 8$. The tracking control gains are such that the characteristic polynomial of the closed loop system is Hurwitz with real roots. Results are reported in Fig. (2b) and (2d), showing that the tracking and safety goals are attained, with a reduced transient compared to the previous case.

V. Conclusions

The problem of avoidance of critical state space regions during tracking of reference trajectories has been tackled in this paper. For systems with dynamics equivalent to r-th order decoupled integrators, the proposed controllers are given in closed-form and are based on CBFs ensuring that the state keeps a safety clearance from the forbidden regions during tracking. The main idea of our approach is to filter out the component of the tracking controller that might drive the system towards unsafe conditions, while retaining the tracking action of the harmless components. The efficacy of the method has been validated through numerical simulations in CoppeliaSim, addressing collision avoidance for a unicycle robot in a 2D scenario and for a quadrotor in a 3D scenario. Future developments include the study of constraints with generic relative degree, the generalization of the method to the avoidance of moving obstacles, the combination of the proposed approach with fast motion planning methods.

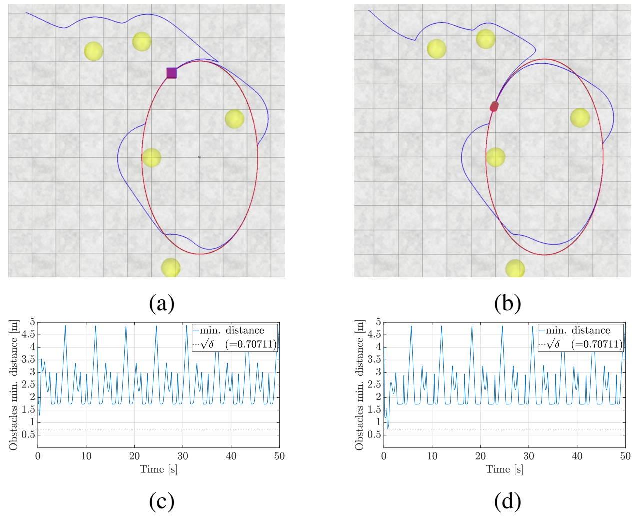

Fig. 1: Planar scenarios. Top: actual trajectory (blue) vs reference trajectory (red) in case of unconstrained mass (a) and unicycle (b). Bottom: minimum Euclidean distance of the unconstrained mass (c) and unicycle (d) from the obstacles placed in the environment.

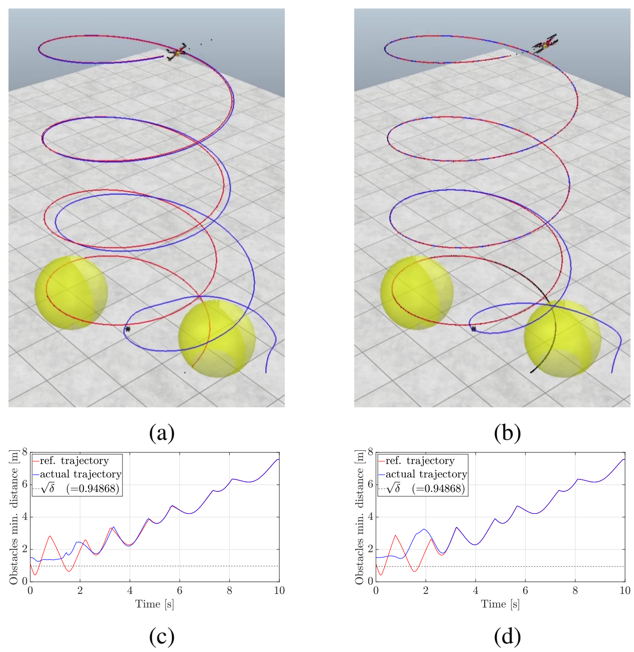

Fig. 2: Quadrotor flight simulation. Top: actual trajectory (blue) vs reference trajectory (red) in case of hierachical control (a) and dynamic feedback linearization (b). Bottom: minimum Euclidean distance to the obstacles: (c) is relative to case (a)and is (d) relative to (b).

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