

Deep Learning & Applied AI

Deep generative models

Emanuele Rodolà
rodola@di.uniroma1.it



SAPIENZA
UNIVERSITÀ DI ROMA

Generative models

Overall idea:

Learn a **distribution** from some given training samples, and generate new samples from the same distribution.

Generative models

Overall idea:

Learn a **distribution** from some given training samples, and generate new samples from the same distribution.



Generative models

What does it mean to learn a **distribution**?

$$\mathbb{R}^{320 \times 240}$$

Generative models

What does it mean to learn a **distribution**?

x
•

$\mathbb{R}^{320 \times 240}$

Generative models

What does it mean to learn a **distribution**?



$$\mathbb{R}^{320 \times 240}$$

Generative models

What does it mean to learn a **distribution**?



$\mathbb{R}^{320 \times 240}$

Generative models

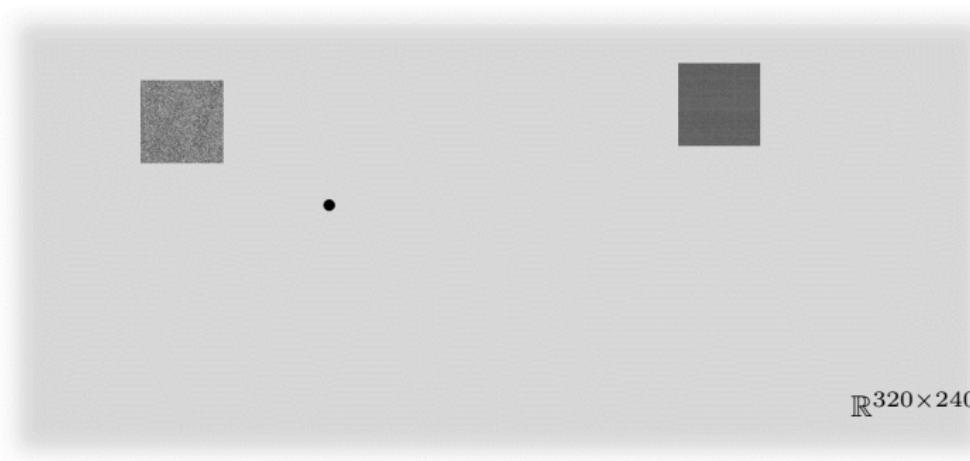
What does it mean to learn a **distribution**?



$\mathbb{R}^{320 \times 240}$

Generative models

What does it mean to learn a **distribution**?



$$\mathbb{R}^{320 \times 240}$$

Generative models

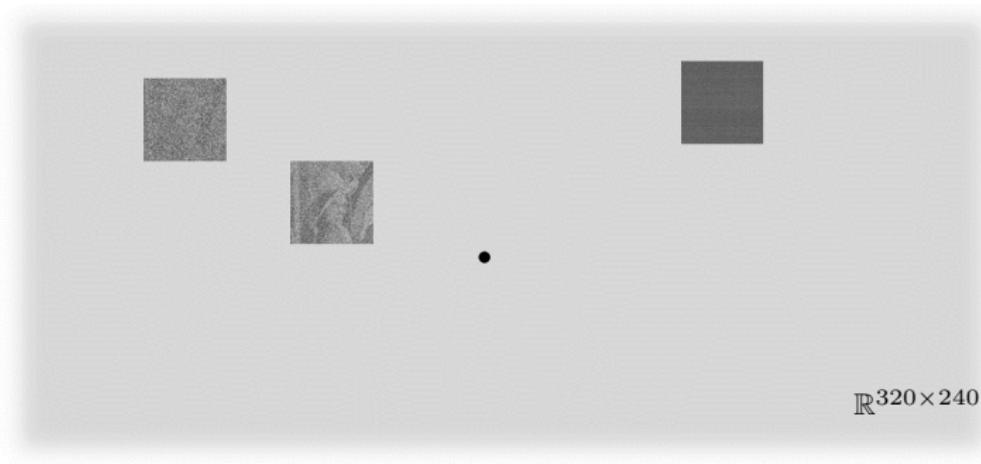
What does it mean to learn a **distribution**?



$\mathbb{R}^{320 \times 240}$

Generative models

What does it mean to learn a **distribution**?



Generative models

What does it mean to learn a **distribution**?



$$\mathbb{R}^{320 \times 240}$$

Generative models

What does it mean to learn a **distribution**?



$\mathbb{R}^{320 \times 240}$

Generative models

What does it mean to learn a **distribution**?



$$\mathbb{R}^{320 \times 240}$$

Generative models

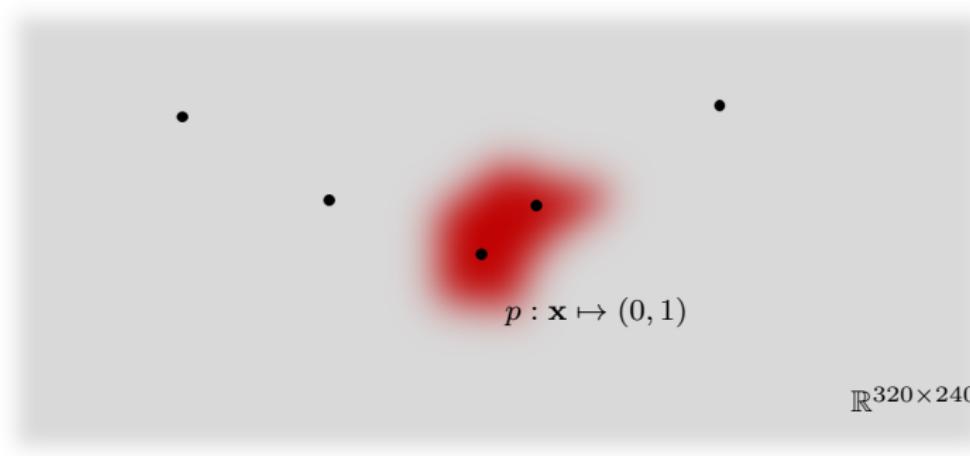
What does it mean to learn a **distribution**?



$\mathbb{R}^{320 \times 240}$

Generative models

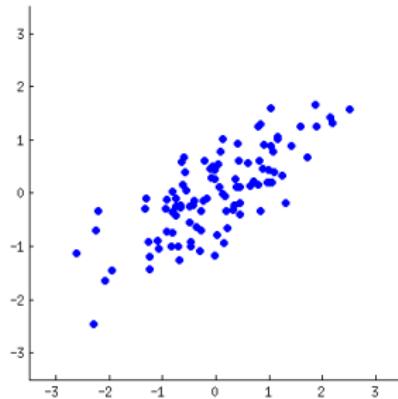
What does it mean to learn a **distribution**?



The probability distribution p is defined on the entire data space.

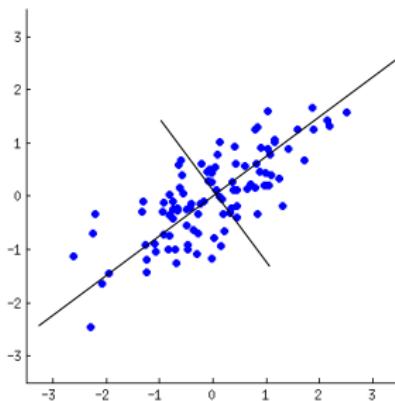
Principal component analysis (PCA)

Regard our data as n points in \mathbb{R}^d :



Principal component analysis (PCA)

Regard our data as n points in \mathbb{R}^d :

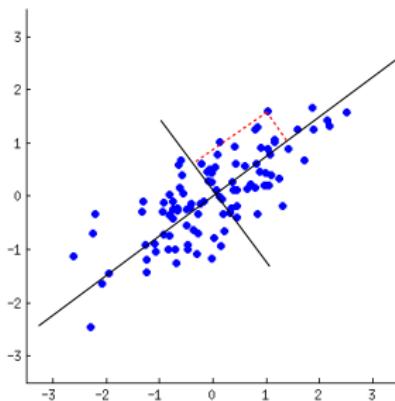


Overall idea:

- Find $k \leq d$ **orthogonal directions** with the most variance.

Principal component analysis (PCA)

Regard our data as n points in \mathbb{R}^d :



Overall idea:

- Find $k \leq d$ **orthogonal directions** with the most variance.
- **Project** all the data points onto these directions.

Principal component analysis (PCA)

In matrix notation:

$$\underbrace{\begin{pmatrix} \text{---} & \mathbf{x}_1^\top & \text{---} \\ & \vdots & \\ \text{---} & \mathbf{x}_n^\top & \text{---} \end{pmatrix}}_{n \times d}$$

Principal component analysis (PCA)

In matrix notation:

$$\underbrace{\begin{pmatrix} \text{---} & \mathbf{x}_1^\top & \text{---} \\ \vdots & & \vdots \\ \text{---} & \mathbf{x}_n^\top & \text{---} \end{pmatrix}}_{n \times d} \underbrace{\begin{pmatrix} | & & | \\ \mathbf{w}_1 & \cdots & \mathbf{w}_k \\ | & & | \end{pmatrix}}_{d \times k}$$

Principal component analysis (PCA)

In matrix notation:

$$\underbrace{\begin{pmatrix} _ & \mathbf{x}_1^\top & _ \\ _ & \vdots & _ \\ _ & \mathbf{x}_n^\top & _ \end{pmatrix}}_{n \times d} \underbrace{\begin{pmatrix} | & & | \\ \mathbf{w}_1 & \cdots & \mathbf{w}_k \\ | & & | \end{pmatrix}}_{d \times k} = \underbrace{\begin{pmatrix} _ & \mathbf{z}_1^\top & _ \\ _ & \vdots & _ \\ _ & \mathbf{z}_n^\top & _ \end{pmatrix}}_{n \times k}$$

Principal component analysis (PCA)

In matrix notation:

$$\underbrace{\begin{pmatrix} _ & \mathbf{x}_1^\top & _ \\ _ & \vdots & _ \\ _ & \mathbf{x}_n^\top & _ \end{pmatrix}}_{n \times d} \underbrace{\begin{pmatrix} | & & | \\ \mathbf{w}_1 & \cdots & \mathbf{w}_k \\ | & & | \end{pmatrix}}_{d \times k} = \underbrace{\begin{pmatrix} _ & \mathbf{z}_1^\top & _ \\ _ & \vdots & _ \\ _ & \mathbf{z}_n^\top & _ \end{pmatrix}}_{n \times k}$$

Assuming $\mathbf{W}^\top \mathbf{W} = \mathbf{I}$, for $k = d$ we get:

$$\mathbf{X}^\top \mathbf{W} = \mathbf{Z}^\top$$

$$\mathbf{X} = \mathbf{WZ}$$

Principal component analysis (PCA)

In matrix notation:

$$\underbrace{\begin{pmatrix} _ & \mathbf{x}_1^\top & _ \\ _ & \vdots & _ \\ _ & \mathbf{x}_n^\top & _ \end{pmatrix}}_{n \times d} \underbrace{\begin{pmatrix} | & & | \\ \mathbf{w}_1 & \cdots & \mathbf{w}_k \\ | & & | \end{pmatrix}}_{d \times k} = \underbrace{\begin{pmatrix} _ & \mathbf{z}_1^\top & _ \\ _ & \vdots & _ \\ _ & \mathbf{z}_n^\top & _ \end{pmatrix}}_{n \times k}$$

Assuming $\mathbf{W}^\top \mathbf{W} = \mathbf{I}$, for $k < d$ we get:

$$\mathbf{X}^\top \mathbf{W} = \mathbf{Z}^\top$$

$$\mathbf{X} \approx \mathbf{WZ}$$

Principal component analysis (PCA)

In matrix notation:

$$\underbrace{\begin{pmatrix} _ & \mathbf{x}_1^\top & _ \\ \vdots & & \vdots \\ _ & \mathbf{x}_n^\top & _ \end{pmatrix}}_{n \times d} \underbrace{\begin{pmatrix} | & & | \\ \mathbf{w}_1 & \cdots & \mathbf{w}_k \\ | & & | \end{pmatrix}}_{d \times k} = \underbrace{\begin{pmatrix} _ & \mathbf{z}_1^\top & _ \\ \vdots & \vdots & \vdots \\ _ & \mathbf{z}_n^\top & _ \end{pmatrix}}_{n \times k}$$

Assuming $\mathbf{W}^\top \mathbf{W} = \mathbf{I}$, for $k < d$ we get:

$$\mathbf{X}^\top \mathbf{W} = \mathbf{Z}^\top \quad \text{projection}$$

$$\mathbf{X} \approx \mathbf{W}\mathbf{Z} \quad \text{reconstruction}$$

Principal component analysis (PCA)

In matrix notation:

$$\underbrace{\begin{pmatrix} _ & \mathbf{x}_1^\top & _ \\ \vdots & & \vdots \\ _ & \mathbf{x}_n^\top & _ \end{pmatrix}}_{n \times d} \underbrace{\begin{pmatrix} | & & | \\ \mathbf{w}_1 & \cdots & \mathbf{w}_k \\ | & & | \end{pmatrix}}_{d \times k} = \underbrace{\begin{pmatrix} _ & \mathbf{z}_1^\top & _ \\ \vdots & \vdots & \vdots \\ _ & \mathbf{z}_n^\top & _ \end{pmatrix}}_{n \times k}$$

Assuming $\mathbf{W}^\top \mathbf{W} = \mathbf{I}$, for $k < d$ we get:

$$\mathbf{X}^\top \mathbf{W} = \mathbf{Z}^\top \quad \text{projection}$$

$$\mathbf{X} \approx \mathbf{W}\mathbf{Z} \quad \text{reconstruction}$$

We call the columns of \mathbf{W} principal components.

They are unknown and must be computed.

Principal component analysis (PCA)

We seek the **direction w** (a column of \mathbf{W}) that:

- Minimizes the **projection/reconstruction error**.
- Maximizes the **variance** of the projected data.

Principal component analysis (PCA)

We seek the **direction w** (a column of \mathbf{W}) that:

- Minimizes the **projection/reconstruction error**.
- Maximizes the **variance** of the projected data.

Principal component analysis (PCA)

Assume the data points \mathbf{X} are **centered** at zero.

For a given \mathbf{w} , the projection of all n points onto \mathbf{w} is $\mathbf{X}^\top \mathbf{w}$.

Principal component analysis (PCA)

Assume the data points \mathbf{X} are **centered** at zero.

For a given \mathbf{w} , the projection of all n points onto \mathbf{w} is $\mathbf{X}^\top \mathbf{w}$.

The **variance** to maximize is $\|\mathbf{X}^\top \mathbf{w}\|_2^2$:

$$(\mathbf{X}^\top \mathbf{w})^\top (\mathbf{X}^\top \mathbf{w})$$

Principal component analysis (PCA)

Assume the data points \mathbf{X} are **centered** at zero.

For a given \mathbf{w} , the projection of all n points onto \mathbf{w} is $\mathbf{X}^\top \mathbf{w}$.

The **variance** to maximize is $\|\mathbf{X}^\top \mathbf{w}\|_2^2$:

$$(\mathbf{X}^\top \mathbf{w})^\top (\mathbf{X}^\top \mathbf{w}) = \mathbf{w}^\top (\mathbf{X} \mathbf{X}^\top) \mathbf{w}$$

Principal component analysis (PCA)

Assume the data points \mathbf{X} are **centered** at zero.

For a given \mathbf{w} , the projection of all n points onto \mathbf{w} is $\mathbf{X}^\top \mathbf{w}$.

The **variance** to maximize is $\|\mathbf{X}^\top \mathbf{w}\|_2^2$:

$$(\mathbf{X}^\top \mathbf{w})^\top (\mathbf{X}^\top \mathbf{w}) = \mathbf{w}^\top \underbrace{(\mathbf{X} \mathbf{X}^\top)}_{\mathbf{C}} \mathbf{w}$$

where $\mathbf{C} \in \mathbb{R}^{d \times d}$ is the symmetric **covariance matrix**.

Principal component analysis (PCA)

Assume the data points \mathbf{X} are **centered** at zero.

For a given \mathbf{w} , the projection of all n points onto \mathbf{w} is $\mathbf{X}^\top \mathbf{w}$.

The **variance** to maximize is $\|\mathbf{X}^\top \mathbf{w}\|_2^2$:

$$(\mathbf{X}^\top \mathbf{w})^\top (\mathbf{X}^\top \mathbf{w}) = \mathbf{w}^\top \underbrace{(\mathbf{X} \mathbf{X}^\top)}_{\mathbf{C}} \mathbf{w}$$

where $\mathbf{C} \in \mathbb{R}^{d \times d}$ is the symmetric **covariance matrix**.

We want to solve the problem:

$$\max_{\mathbf{w}} \mathbf{w}^\top \mathbf{C} \mathbf{w} \quad \text{s.t. } \|\mathbf{w}\|_2 = 1$$

Principal component analysis (PCA)

Assume the data points \mathbf{X} are **centered** at zero.

For a given \mathbf{w} , the projection of all n points onto \mathbf{w} is $\mathbf{X}^\top \mathbf{w}$.

The **variance** to maximize is $\|\mathbf{X}^\top \mathbf{w}\|_2^2$:

$$(\mathbf{X}^\top \mathbf{w})^\top (\mathbf{X}^\top \mathbf{w}) = \mathbf{w}^\top \underbrace{(\mathbf{X} \mathbf{X}^\top)}_{\mathbf{C}} \mathbf{w}$$

where $\mathbf{C} \in \mathbb{R}^{d \times d}$ is the symmetric **covariance matrix**.

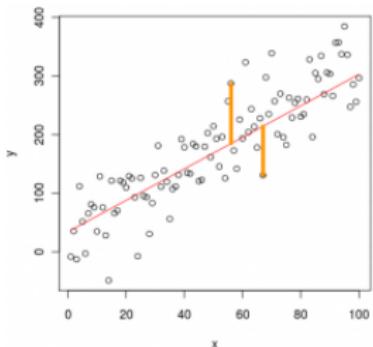
We want to solve the problem:

$$\max_{\mathbf{w}} \mathbf{w}^\top \mathbf{C} \mathbf{w} \quad \text{s.t. } \|\mathbf{w}\|_2 = 1$$

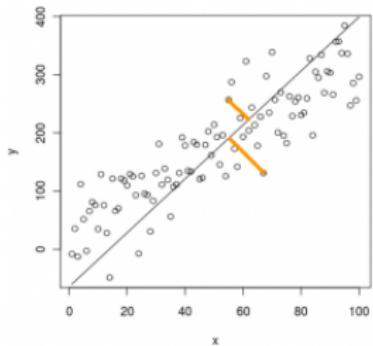
The solution is $\mathbf{w} = \text{principal eigenvector of } \mathbf{C}$ (**Courant minmax principle**), and the value $\mathbf{w}^\top \mathbf{C} \mathbf{w}$ is the corresponding **eigenvalue**.

PCA is not linear regression

With linear regression we measure the error along the y coordinate:



With PCA we measure the error orthogonal to the principal direction:



PCA as a generative model

Given the \mathbf{W} satisfying, for the observations \mathbf{X} :

$$\mathbf{X}^\top \mathbf{W} = \mathbf{Z}^\top \quad \text{projection}$$

$$\mathbf{X} \approx \mathbf{W}\mathbf{Z} \quad \text{reconstruction}$$

We can generate new data just by sampling $\mathbf{z}_{\text{new}} \in \mathbb{R}^k$ and computing:

$$\mathbf{x}_{\text{new}} = \mathbf{W}\mathbf{z}_{\text{new}}$$

PCA as a generative model

Given the \mathbf{W} satisfying, for the observations \mathbf{X} :

$$\mathbf{X}^\top \mathbf{W} = \mathbf{Z}^\top \quad \text{projection}$$

$$\mathbf{X} \approx \mathbf{W}\mathbf{Z} \quad \text{reconstruction}$$

We can generate new data just by sampling $\mathbf{z}_{\text{new}} \in \mathbb{R}^k$ and computing:

$$\mathbf{x}_{\text{new}} = \mathbf{W}\mathbf{z}_{\text{new}}$$

Example:

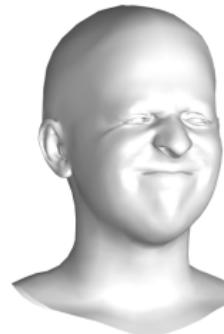


Data point \mathbf{x}_1



Generated

$$\mathbf{x}_{\text{new}} = \frac{1}{2}\mathbf{W}(\mathbf{z}_1 + \mathbf{z}_2)$$



Data point \mathbf{x}_2

Codes

Consider again the relations:

$$\mathbf{W}^\top \mathbf{x} = \mathbf{z} \quad \text{projection}$$

$$\mathbf{x} \approx \mathbf{W}\mathbf{z} \quad \text{reconstruction}$$

From a different perspective, PCA gives us a [parametric model](#).

Codes

Consider again the relations:

$$\begin{aligned}\mathbf{W}^\top \mathbf{x} &= \mathbf{z} && \text{encoding} \\ \mathbf{x} &\approx \mathbf{W}\mathbf{z} && \text{decoding}\end{aligned}$$

From a different perspective, PCA gives us a [parametric model](#).

Each data point \mathbf{x} is transformed into a low-dimensional code $\mathbf{z} \in \mathbb{R}^k$, where the dimension $k < d$ is fixed.

The [encoding](#) and [decoding](#) procedures are linear.

Codes

Consider again the relations:

$$\begin{aligned}\mathbf{W}^\top \mathbf{x} &= \mathbf{z} && \text{encoding} \\ \mathbf{x} &\approx \mathbf{W}\mathbf{z} && \text{decoding}\end{aligned}$$

From a different perspective, PCA gives us a [parametric model](#).

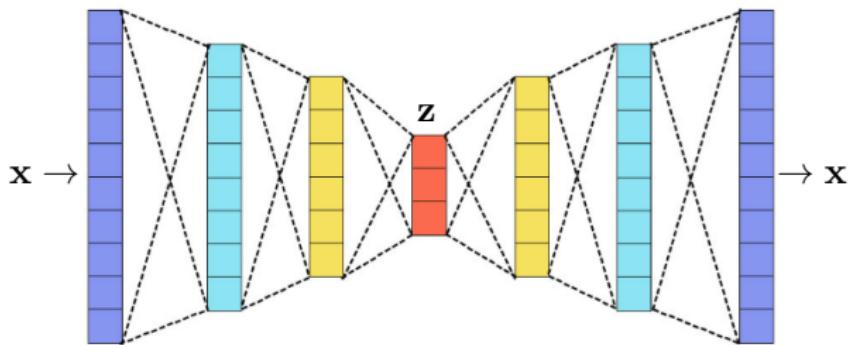
Each data point \mathbf{x} is transformed into a low-dimensional code $\mathbf{z} \in \mathbb{R}^k$, where the dimension $k < d$ is fixed.

The [encoding](#) and [decoding](#) procedures are linear.

How to generalize this idea?

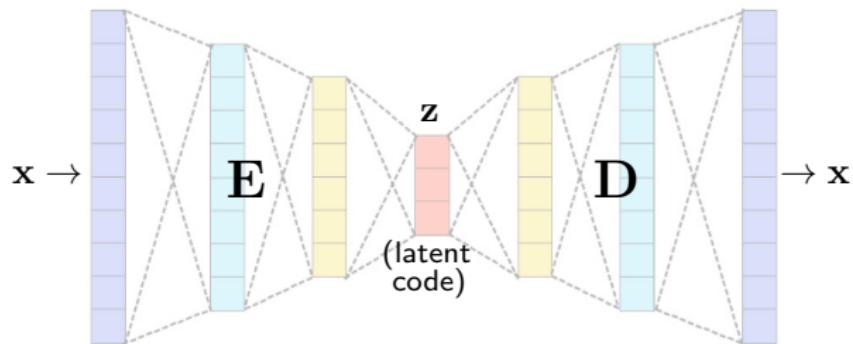
Autoencoders (AE)

We can construct powerful parametric models using deep nets.



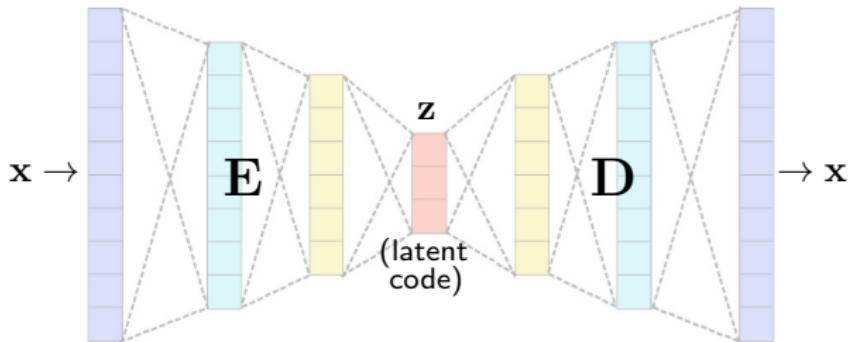
Autoencoders (AE)

We can construct powerful parametric models using deep nets.



Autoencoders (AE)

We can construct powerful parametric models using deep nets.

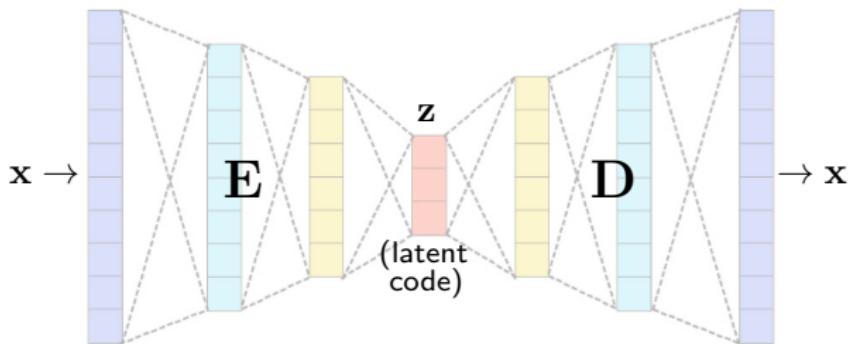


For a given dataset $\{\mathbf{x}_i\}$, we require the encoder E and decoder (or generator) D to minimize the **reconstruction loss**:

$$\ell_\Theta = \sum_i \|\mathbf{x}_i - D_\Theta(E_\Theta(\mathbf{x}_i))\|$$

Autoencoders (AE)

We can construct powerful parametric models using deep nets.



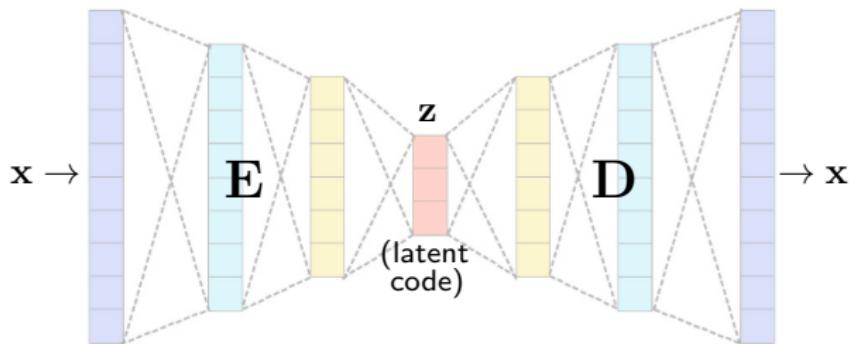
For a given dataset $\{\mathbf{x}_i\}$, we require the encoder E and decoder (or generator) D to minimize the **reconstruction loss**:

$$\ell_\Theta = \sum_i \|\mathbf{x}_i - D_\Theta(E_\Theta(\mathbf{x}_i))\|$$

The choice of the metric depends on the data and on the task.

Autoencoders (AE)

We can construct powerful parametric models using deep nets.



For a given dataset $\{\mathbf{x}_i\}$, we require the encoder E and decoder (or generator) D to minimize the **reconstruction loss**:

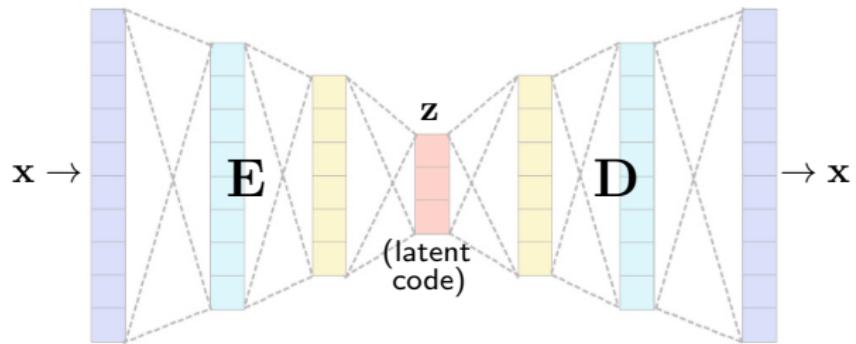
$$\ell_\Theta = \sum_i \|\mathbf{x}_i - D_\Theta(E_\Theta(\mathbf{x}_i))\|$$

The choice of the metric depends on the data and on the task.

If the layers are **linear**, the codes \mathbf{z}_i span exactly the same space as PCA.

Autoencoders (AE)

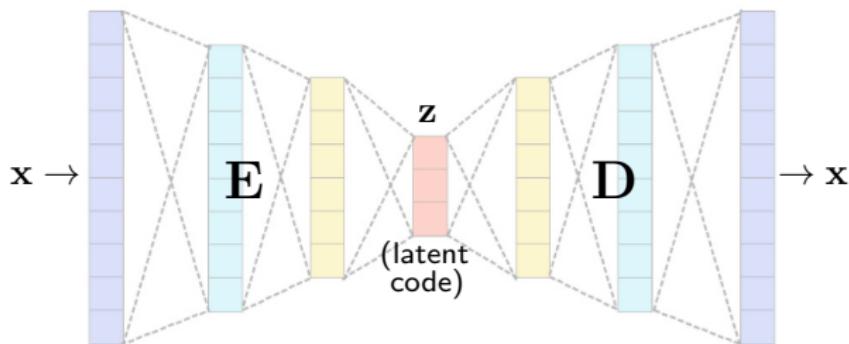
We can construct powerful parametric models using deep nets.



- The **bottleneck** prevents trivial solutions.

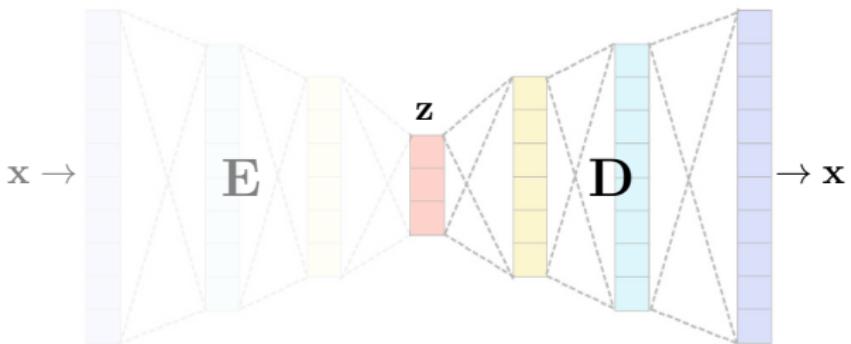
Autoencoders (AE)

We can construct powerful parametric models using deep nets.



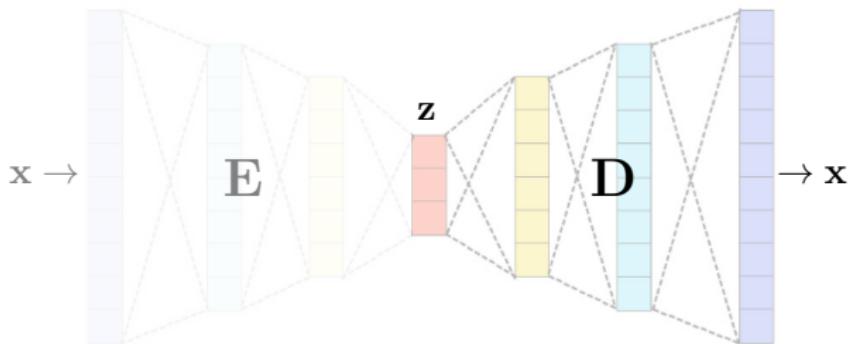
- The **bottleneck** prevents trivial solutions.
- The **task** is not important here: it is always reconstruction.
Once the AE is trained, we are interested in the structure of the **latent space** and in using E, D for new tasks.

Manifold hypothesis



The decoder performs a mapping from a low-dimensional **latent space** to a high-dimensional **embedding** of observed data.

Manifold hypothesis



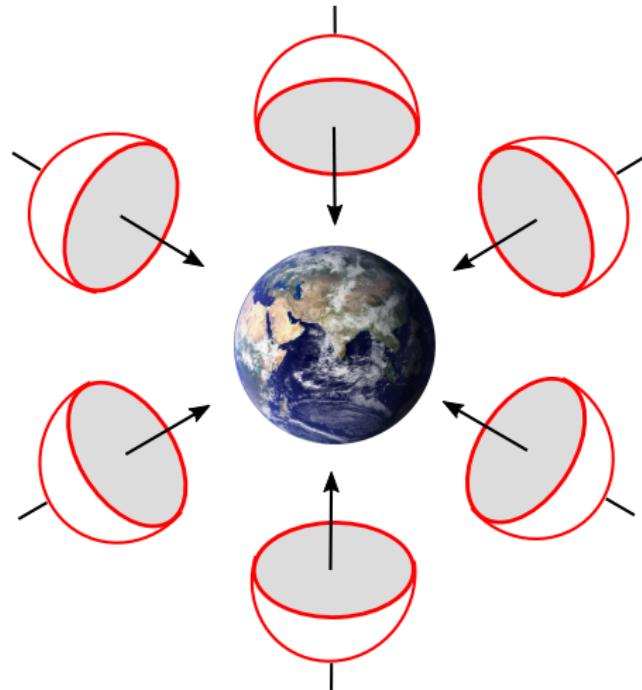
The decoder performs a mapping from a low-dimensional **latent space** to a high-dimensional **embedding** of observed data.

The latent space is Euclidean.

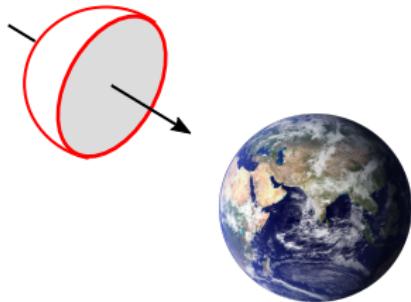
The data embedding space is curved (**manifold hypothesis**).

Manifolds

Manifolds are unions of **charts**:



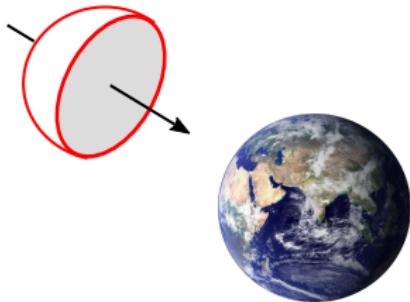
2D manifolds (surfaces)



chart

Each **chart** can be seen as a mapping $\phi : \mathbb{R}^2 \rightarrow \mathcal{S} \subset \mathbb{R}^3$.

2D manifolds (surfaces)

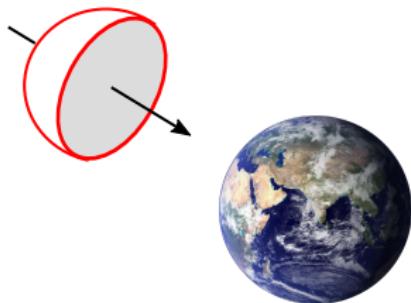


chart

Each **chart** can be seen as a mapping $\phi : \mathbb{R}^2 \rightarrow \mathcal{S} \subset \mathbb{R}^3$.

We require ϕ to be **smooth** and **invertible** (diffeomorphism).

2D manifolds (surfaces)



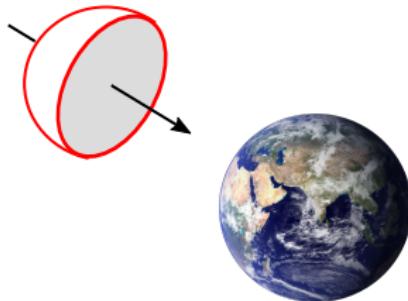
chart

Each **chart** can be seen as a mapping $\phi : \mathbb{R}^2 \rightarrow \mathcal{S} \subset \mathbb{R}^3$.

We require ϕ to be **smooth** and **invertible** (diffeomorphism).

- The domain of ϕ is the **parametric space** and is Euclidean.

2D manifolds (surfaces)



chart

Each **chart** can be seen as a mapping $\phi : \mathbb{R}^2 \rightarrow \mathcal{S} \subset \mathbb{R}^3$.

We require ϕ to be **smooth** and **invertible** (diffeomorphism).

- The domain of ϕ is the **parametric space** and is Euclidean.
- The image of ϕ is the **embedding** and is a surface.

Manifolds

Manifolds can be k -dimensional, meaning that we have charts:

$$\phi : \mathbb{R}^k \rightarrow \mathcal{M} \subset \mathbb{R}^d \quad \text{with } k < d$$

Manifolds

Manifolds can be k -dimensional, meaning that we have charts:

$$\phi : \mathbb{R}^k \rightarrow \mathcal{M} \subset \mathbb{R}^d \quad \text{with } k < d$$

The parametrization is **not unique**:

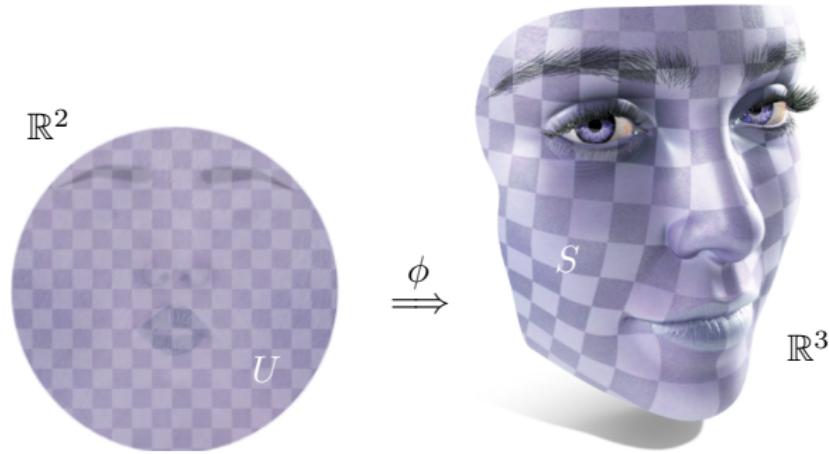


Figure by Keenan Crane

Manifolds

Manifolds can be k -dimensional, meaning that we have charts:

$$\phi : \mathbb{R}^k \rightarrow \mathcal{M} \subset \mathbb{R}^d \quad \text{with } k < d$$

The parametrization is **not unique**:

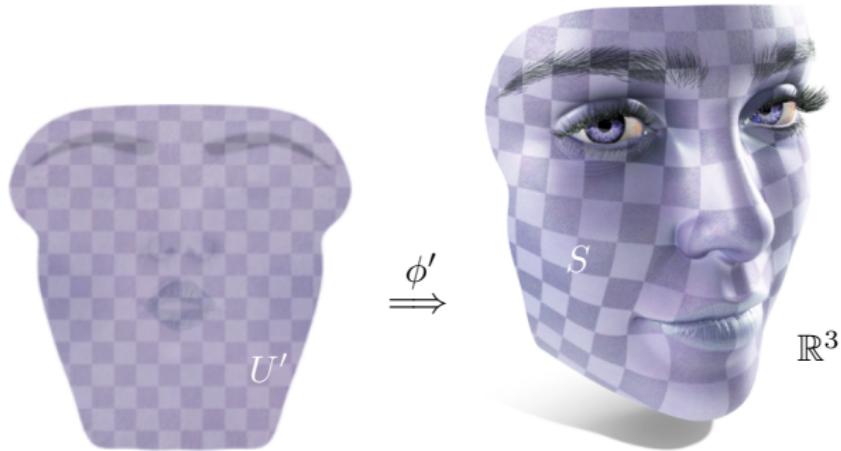


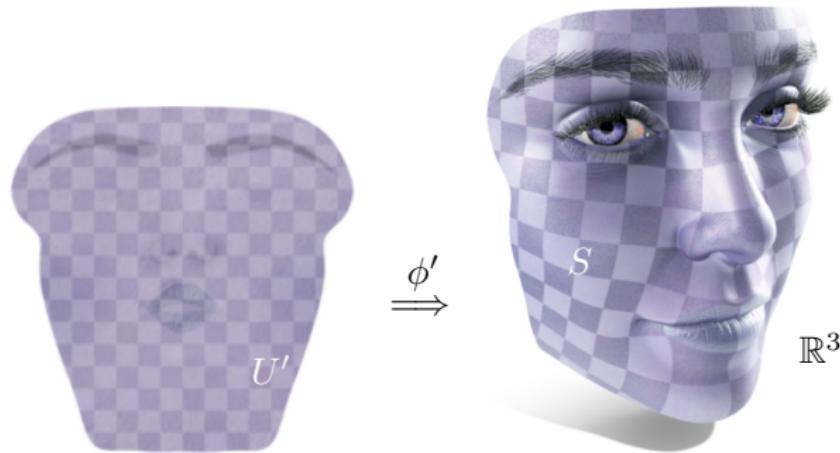
Figure by Keenan Crane

Manifolds

Manifolds can be k -dimensional, meaning that we have charts:

$$\phi : \mathbb{R}^k \rightarrow \mathcal{M} \subset \mathbb{R}^d \quad \text{with } k < d$$

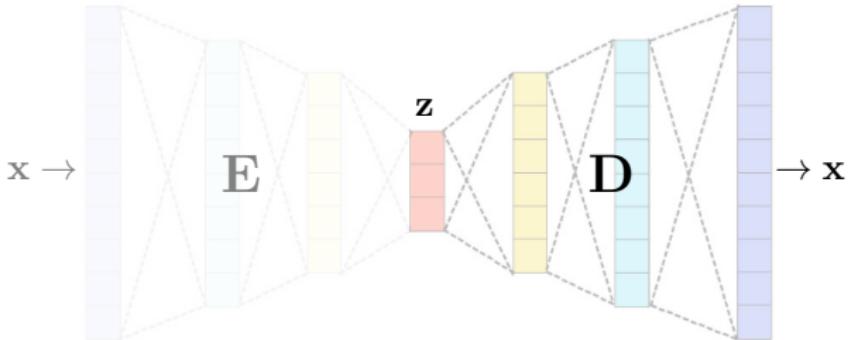
The parametrization is **not unique**:



However, all encode exactly **the same** geometric information.

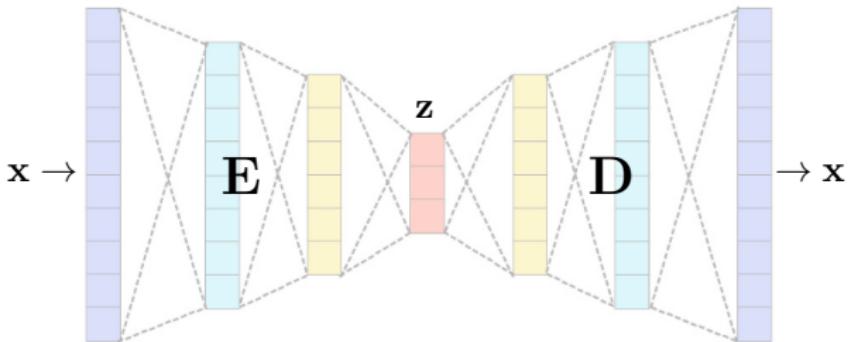
Figure by Keenan Crane

Manifolds and generative models



The decoder $D : \mathbb{R}^k \rightarrow \mathbb{R}^d$ is a chart from the latent space spanned by the codes \mathbf{z} to the data space of the inputs \mathbf{x} .

Manifolds and generative models

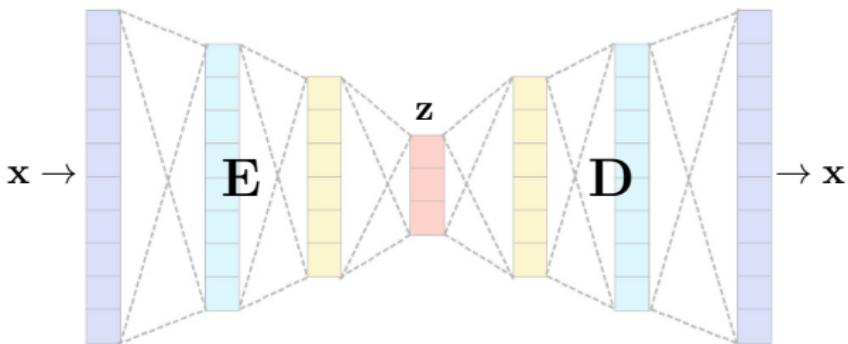


The decoder $D : \mathbb{R}^k \rightarrow \mathbb{R}^d$ is a chart from the latent space spanned by the codes \mathbf{z} to the data space of the inputs \mathbf{x} .

It is differentiable.

It is invertible via the encoder E .

Manifolds and generative models



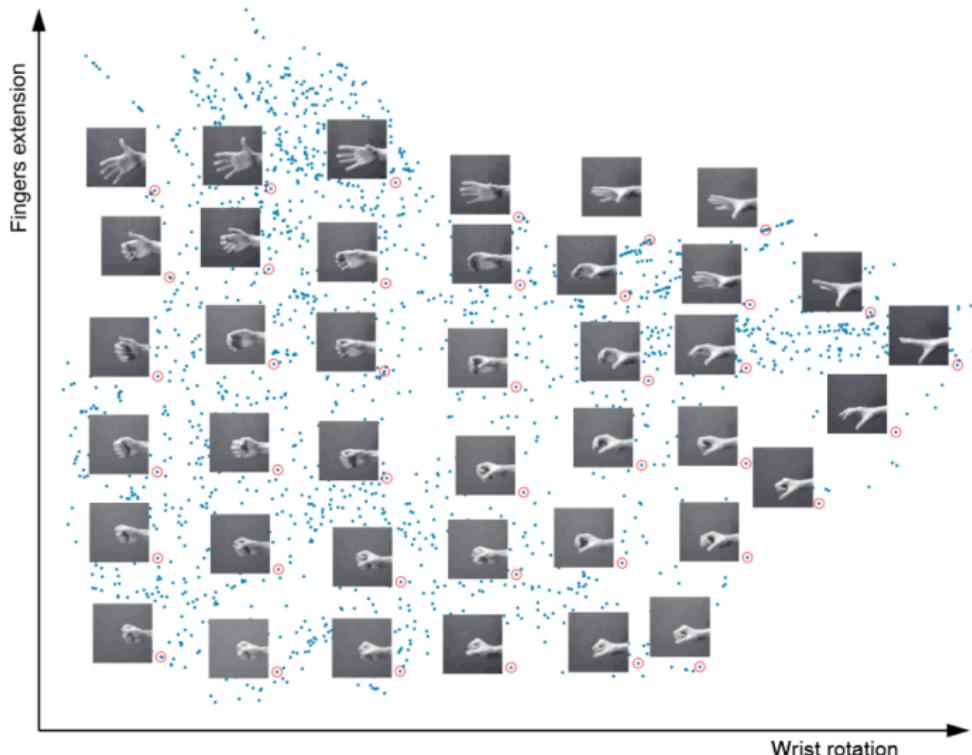
The decoder $D : \mathbb{R}^k \rightarrow \mathbb{R}^d$ is a chart from the latent space spanned by the codes z to the data space of the inputs x .

It is differentiable.

It is invertible via the encoder E .

PCA puts the data on a **linear** (flat) manifold, since D simply performs a linear combination of orthogonal vectors.

Manifolds and generative models



Variational autoencoders (VAE)

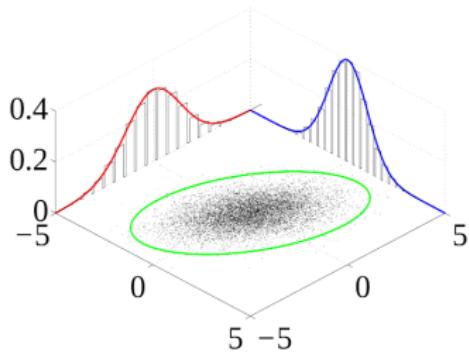
Autoencoders can map similar inputs to far regions of the latent space.

Variational autoencoders (VAE)

Autoencoders can map similar inputs to far regions of the latent space.

A **variational** autoencoder constructs a probability distribution on the latent space.

- The data is seen as a **sampling** of the learned distribution.
- The distribution is fixed and decided **a priori** (e.g. Gaussian).



Entropy and divergence

The information carried by an event \mathbf{x} can be quantified as:

$$I(\mathbf{x}) = -\log p(\mathbf{x})$$

Entropy and divergence

The information carried by an event \mathbf{x} can be quantified as:

$$I(\mathbf{x}) = -\log p(\mathbf{x})$$

The average information encoded in p is its **entropy**:

$$H(p) = - \sum_{\mathbf{x}} p(\mathbf{x}) \log p(\mathbf{x})$$

Entropy and divergence

The information carried by an event \mathbf{x} can be quantified as:

$$I(\mathbf{x}) = -\log p(\mathbf{x})$$

The average information encoded in p is its **entropy**:

$$H(p) = - \sum_{\mathbf{x}} p(\mathbf{x}) \log p(\mathbf{x})$$

Given two distributions p and q , the **Kullback-Leibler divergence**:

$$KL(p\|q) \approx H(q) - H(p)$$

measures their dissimilarity in terms of their entropy.

Entropy and divergence

The information carried by an event \mathbf{x} can be quantified as:

$$I(\mathbf{x}) = -\log p(\mathbf{x})$$

The average information encoded in p is its **entropy**:

$$H(p) = - \sum_{\mathbf{x}} p(\mathbf{x}) \log p(\mathbf{x})$$

Given two distributions p and q , the **Kullback-Leibler divergence**:

$$KL(p\|q) \approx - \sum q(\mathbf{x}) \log q(\mathbf{x}) + \sum p(\mathbf{x}) \log p(\mathbf{x})$$

measures their dissimilarity in terms of their entropy.

Entropy and divergence

The information carried by an event \mathbf{x} can be quantified as:

$$I(\mathbf{x}) = -\log p(\mathbf{x})$$

The average information encoded in p is its **entropy**:

$$H(p) = - \sum_{\mathbf{x}} p(\mathbf{x}) \log p(\mathbf{x})$$

Given two distributions p and q , the **Kullback-Leibler divergence**:

$$KL(p\|q) = - \sum_{\mathbf{x}} p(\mathbf{x}) \log q(\mathbf{x}) + \sum_{\mathbf{x}} p(\mathbf{x}) \log p(\mathbf{x}) \geq 0$$

measures their dissimilarity in terms of their entropy.

Entropy and divergence

The information carried by an event \mathbf{x} can be quantified as:

$$I(\mathbf{x}) = -\log p(\mathbf{x})$$

The average information encoded in p is its **entropy**:

$$H(p) = - \sum_{\mathbf{x}} p(\mathbf{x}) \log p(\mathbf{x})$$

Given two distributions p and q , the **Kullback-Leibler divergence**:

$$KL(p\|q) = \sum_{\mathbf{x}} p(\mathbf{x}) \log \frac{p(\mathbf{x})}{q(\mathbf{x})} \geq 0$$

measures their dissimilarity in terms of their entropy.

Entropy and divergence

The information carried by an event \mathbf{x} can be quantified as:

$$I(\mathbf{x}) = -\log p(\mathbf{x})$$

The average information encoded in p is its **entropy**:

$$H(p) = - \sum_{\mathbf{x}} p(\mathbf{x}) \log p(\mathbf{x})$$

Given two distributions p and q , the **Kullback-Leibler divergence**:

$$KL(p\|q) = - \sum_{\mathbf{x}} p(\mathbf{x}) \log \frac{q(\mathbf{x})}{p(\mathbf{x})} \geq 0$$

measures their dissimilarity in terms of their entropy.

Variational inference

In our scenario: $\textcolor{green}{x}$ is a given data point, and $\textcolor{red}{z}$ is a latent code.

Variational inference

In our scenario: $\textcolor{green}{x}$ is a given data point, and $\textcolor{red}{z}$ is a latent code.

We define a parametric **probabilistic encoder** as the distribution:

$$p_{\theta}(\textcolor{red}{z}|\textcolor{green}{x})$$

Variational inference

In our scenario: x is a given data point, and z is a latent code.

We define a parametric **probabilistic encoder** as the distribution:

$$p_{\theta}(\text{z}|\text{x}) = \frac{p_{\theta}(\text{x}|\text{z})p_{\theta}(\text{z})}{p_{\theta}(\text{x})}$$

Variational inference

In our scenario: $\textcolor{blue}{x}$ is a given data point, and $\textcolor{red}{z}$ is a latent code.

We define a parametric **probabilistic encoder** as the distribution:

$$p_{\theta}(\textcolor{red}{z}|\textcolor{blue}{x}) = \frac{p_{\theta}(\textcolor{blue}{x}|\textcolor{red}{z})p_{\theta}(\textcolor{red}{z})}{p_{\theta}(\textcolor{blue}{x})} = \frac{p_{\theta}(\textcolor{blue}{x}, \textcolor{red}{z})}{p_{\theta}(\textcolor{blue}{x})}$$

Variational inference

In our scenario: \mathbf{x} is a given data point, and \mathbf{z} is a latent code.

We define a parametric **probabilistic encoder** as the distribution:

$$p_{\theta}(\mathbf{z}|\mathbf{x}) = \frac{p_{\theta}(\mathbf{x}|\mathbf{z})p_{\theta}(\mathbf{z})}{p_{\theta}(\mathbf{x})} = \frac{p_{\theta}(\mathbf{x}, \mathbf{z})}{p_{\theta}(\mathbf{x})}$$

However, computing:

$$p_{\theta}(\mathbf{x}) = \int p_{\theta}(\mathbf{x}|\mathbf{z})p_{\theta}(\mathbf{z})d\mathbf{z}$$

is a high-dimensional **intractable** integral over the entire **latent space**.

Variational inference

In our scenario: \mathbf{x} is a given data point, and \mathbf{z} is a latent code.

We define a parametric **probabilistic encoder** as the distribution:

$$p_{\theta}(\mathbf{z}|\mathbf{x}) = \frac{p_{\theta}(\mathbf{x}|\mathbf{z})p_{\theta}(\mathbf{z})}{p_{\theta}(\mathbf{x})} = \frac{p_{\theta}(\mathbf{x}, \mathbf{z})}{p_{\theta}(\mathbf{x})}$$

Instead, compute an approximation:

$$q_{\phi}(\mathbf{z}|\mathbf{x}) \approx p_{\theta}(\mathbf{z}|\mathbf{x})$$

where $q_{\phi}(\mathbf{z}|\mathbf{x})$ has a fixed parametric form w.r.t. ϕ .

Variational inference

In our scenario: \mathbf{x} is a given data point, and \mathbf{z} is a latent code.

We define a parametric **probabilistic encoder** as the distribution:

$$p_{\theta}(\mathbf{z}|\mathbf{x}) = \frac{p_{\theta}(\mathbf{x}|\mathbf{z})p_{\theta}(\mathbf{z})}{p_{\theta}(\mathbf{x})} = \frac{p_{\theta}(\mathbf{x}, \mathbf{z})}{p_{\theta}(\mathbf{x})}$$

Instead, compute an approximation $q_{\phi^*}(\mathbf{z}|\mathbf{x}) \approx p_{\theta}(\mathbf{z}|\mathbf{x})$ with:

$$\phi^* = \arg \min_{\phi, \theta} KL(q_{\phi}(\mathbf{z}|\mathbf{x}) \| p_{\theta}(\mathbf{z}|\mathbf{x}))$$

where $q_{\phi}(\mathbf{z}|\mathbf{x})$ has a fixed parametric form w.r.t. ϕ .

Variational inference

In our scenario: \mathbf{x} is a given data point, and \mathbf{z} is a latent code.

We define a parametric **probabilistic encoder** as the distribution:

$$p_{\theta}(\mathbf{z}|\mathbf{x}) = \frac{p_{\theta}(\mathbf{x}|\mathbf{z})p_{\theta}(\mathbf{z})}{p_{\theta}(\mathbf{x})} = \frac{p_{\theta}(\mathbf{x}, \mathbf{z})}{p_{\theta}(\mathbf{x})}$$

Instead, compute an approximation $q_{\phi^*}(\mathbf{z}|\mathbf{x}) \approx p_{\theta}(\mathbf{z}|\mathbf{x})$ with:

$$\phi^* = \arg \min_{\phi, \theta} - \sum_{\mathbf{z}} q_{\phi}(\mathbf{z}|\mathbf{x}) \log \frac{p_{\theta}(\mathbf{z}|\mathbf{x})}{q_{\phi}(\mathbf{z}|\mathbf{x})}$$

where $q_{\phi}(\mathbf{z}|\mathbf{x})$ has a fixed parametric form w.r.t. ϕ .

Variational inference

In our scenario: \mathbf{x} is a given data point, and \mathbf{z} is a latent code.

We define a parametric **probabilistic encoder** as the distribution:

$$p_{\theta}(\mathbf{z}|\mathbf{x}) = \frac{p_{\theta}(\mathbf{x}|\mathbf{z})p_{\theta}(\mathbf{z})}{p_{\theta}(\mathbf{x})} = \frac{p_{\theta}(\mathbf{x}, \mathbf{z})}{p_{\theta}(\mathbf{x})}$$

Instead, compute an approximation $q_{\phi^*}(\mathbf{z}|\mathbf{x}) \approx p_{\theta}(\mathbf{z}|\mathbf{x})$ with:

$$\phi^* = \arg \min_{\phi, \theta} - \sum_{\mathbf{z}} q_{\phi}(\mathbf{z}|\mathbf{x}) \log \frac{p_{\theta}(\mathbf{x}, \mathbf{z})/p_{\theta}(\mathbf{x})}{q_{\phi}(\mathbf{z}|\mathbf{x})}$$

where $q_{\phi}(\mathbf{z}|\mathbf{x})$ has a fixed parametric form w.r.t. ϕ .

Variational inference

In our scenario: \mathbf{x} is a given data point, and \mathbf{z} is a latent code.

We define a parametric **probabilistic encoder** as the distribution:

$$p_{\theta}(\mathbf{z}|\mathbf{x}) = \frac{p_{\theta}(\mathbf{x}|\mathbf{z})p_{\theta}(\mathbf{z})}{p_{\theta}(\mathbf{x})} = \frac{p_{\theta}(\mathbf{x}, \mathbf{z})}{p_{\theta}(\mathbf{x})}$$

Instead, compute an approximation $q_{\phi^*}(\mathbf{z}|\mathbf{x}) \approx p_{\theta}(\mathbf{z}|\mathbf{x})$ with:

$$\phi^* = \arg \min_{\phi, \theta} - \sum_{\mathbf{z}} q_{\phi}(\mathbf{z}|\mathbf{x}) \log \frac{p_{\theta}(\mathbf{x}, \mathbf{z})}{q_{\phi}(\mathbf{z}|\mathbf{x})} \frac{1}{p_{\theta}(\mathbf{x})}$$

where $q_{\phi}(\mathbf{z}|\mathbf{x})$ has a fixed parametric form w.r.t. ϕ .

Variational inference

In our scenario: \mathbf{x} is a given data point, and \mathbf{z} is a latent code.

We define a parametric **probabilistic encoder** as the distribution:

$$p_{\theta}(\mathbf{z}|\mathbf{x}) = \frac{p_{\theta}(\mathbf{x}|\mathbf{z})p_{\theta}(\mathbf{z})}{p_{\theta}(\mathbf{x})} = \frac{p_{\theta}(\mathbf{x}, \mathbf{z})}{p_{\theta}(\mathbf{x})}$$

Instead, compute an approximation $q_{\phi^*}(\mathbf{z}|\mathbf{x}) \approx p_{\theta}(\mathbf{z}|\mathbf{x})$ with:

$$\phi^* = \arg \min_{\phi, \theta} - \sum_{\mathbf{z}} q_{\phi}(\mathbf{z}|\mathbf{x}) \left(\log \frac{p_{\theta}(\mathbf{x}, \mathbf{z})}{q_{\phi}(\mathbf{z}|\mathbf{x})} + \log \frac{1}{p_{\theta}(\mathbf{x})} \right)$$

where $q_{\phi}(\mathbf{z}|\mathbf{x})$ has a fixed parametric form w.r.t. ϕ .

Variational inference

In our scenario: \mathbf{x} is a given data point, and \mathbf{z} is a latent code.

We define a parametric **probabilistic encoder** as the distribution:

$$p_{\theta}(\mathbf{z}|\mathbf{x}) = \frac{p_{\theta}(\mathbf{x}|\mathbf{z})p_{\theta}(\mathbf{z})}{p_{\theta}(\mathbf{x})} = \frac{p_{\theta}(\mathbf{x}, \mathbf{z})}{p_{\theta}(\mathbf{x})}$$

Instead, compute an approximation $q_{\phi^*}(\mathbf{z}|\mathbf{x}) \approx p_{\theta}(\mathbf{z}|\mathbf{x})$ with:

$$\phi^* = \arg \min_{\phi, \theta} - \sum_{\mathbf{z}} q_{\phi}(\mathbf{z}|\mathbf{x}) \left(\log \frac{p_{\theta}(\mathbf{x}, \mathbf{z})}{q_{\phi}(\mathbf{z}|\mathbf{x})} - \log p_{\theta}(\mathbf{x}) \right)$$

where $q_{\phi}(\mathbf{z}|\mathbf{x})$ has a fixed parametric form w.r.t. ϕ .

Variational inference

In our scenario: \mathbf{x} is a given data point, and \mathbf{z} is a latent code.

We define a parametric **probabilistic encoder** as the distribution:

$$p_{\theta}(\mathbf{z}|\mathbf{x}) = \frac{p_{\theta}(\mathbf{x}|\mathbf{z})p_{\theta}(\mathbf{z})}{p_{\theta}(\mathbf{x})} = \frac{p_{\theta}(\mathbf{x}, \mathbf{z})}{p_{\theta}(\mathbf{x})}$$

Instead, compute an approximation $q_{\phi^*}(\mathbf{z}|\mathbf{x}) \approx p_{\theta}(\mathbf{z}|\mathbf{x})$ with:

$$\phi^* = \arg \min_{\phi, \theta} - \sum_{\mathbf{z}} q_{\phi}(\mathbf{z}|\mathbf{x}) \log \frac{p_{\theta}(\mathbf{x}, \mathbf{z})}{q_{\phi}(\mathbf{z}|\mathbf{x})} + \sum_{\mathbf{z}} q_{\phi}(\mathbf{z}|\mathbf{x}) \log p_{\theta}(\mathbf{x})$$

where $q_{\phi}(\mathbf{z}|\mathbf{x})$ has a fixed parametric form w.r.t. ϕ .

Variational inference

In our scenario: \mathbf{x} is a given data point, and \mathbf{z} is a latent code.

We define a parametric **probabilistic encoder** as the distribution:

$$p_{\theta}(\mathbf{z}|\mathbf{x}) = \frac{p_{\theta}(\mathbf{x}|\mathbf{z})p_{\theta}(\mathbf{z})}{p_{\theta}(\mathbf{x})} = \frac{p_{\theta}(\mathbf{x}, \mathbf{z})}{p_{\theta}(\mathbf{x})}$$

Instead, compute an approximation $q_{\phi^*}(\mathbf{z}|\mathbf{x}) \approx p_{\theta}(\mathbf{z}|\mathbf{x})$ with:

$$\phi^* = \arg \min_{\phi, \theta} - \sum_{\mathbf{z}} q_{\phi}(\mathbf{z}|\mathbf{x}) \log \frac{p_{\theta}(\mathbf{x}, \mathbf{z})}{q_{\phi}(\mathbf{z}|\mathbf{x})} + \log p_{\theta}(\mathbf{x}) \sum_{\mathbf{z}} q_{\phi}(\mathbf{z}|\mathbf{x})$$

where $q_{\phi}(\mathbf{z}|\mathbf{x})$ has a fixed parametric form w.r.t. ϕ .

Variational inference

In our scenario: \mathbf{x} is a given data point, and \mathbf{z} is a latent code.

We define a parametric **probabilistic encoder** as the distribution:

$$p_{\theta}(\mathbf{z}|\mathbf{x}) = \frac{p_{\theta}(\mathbf{x}|\mathbf{z})p_{\theta}(\mathbf{z})}{p_{\theta}(\mathbf{x})} = \frac{p_{\theta}(\mathbf{x}, \mathbf{z})}{p_{\theta}(\mathbf{x})}$$

Instead, compute an approximation $q_{\phi^*}(\mathbf{z}|\mathbf{x}) \approx p_{\theta}(\mathbf{z}|\mathbf{x})$ with:

$$\phi^* = \arg \min_{\phi, \theta} - \sum_{\mathbf{z}} q_{\phi}(\mathbf{z}|\mathbf{x}) \log \frac{p_{\theta}(\mathbf{x}, \mathbf{z})}{q_{\phi}(\mathbf{z}|\mathbf{x})} + \log p_{\theta}(\mathbf{x}) \sum_{\mathbf{z}} q_{\phi}(\mathbf{z}|\mathbf{x})$$

where $q_{\phi}(\mathbf{z}|\mathbf{x})$ has a fixed parametric form w.r.t. ϕ .

Variational inference

In our scenario: \mathbf{x} is a given data point, and \mathbf{z} is a latent code.

We define a parametric **probabilistic encoder** as the distribution:

$$p_{\theta}(\mathbf{z}|\mathbf{x}) = \frac{p_{\theta}(\mathbf{x}|\mathbf{z})p_{\theta}(\mathbf{z})}{p_{\theta}(\mathbf{x})} = \frac{p_{\theta}(\mathbf{x}, \mathbf{z})}{p_{\theta}(\mathbf{x})}$$

Instead, compute an approximation $q_{\phi^*}(\mathbf{z}|\mathbf{x}) \approx p_{\theta}(\mathbf{z}|\mathbf{x})$ with:

$$\phi^* = \arg \min_{\phi, \theta} - \sum_{\mathbf{z}} q_{\phi}(\mathbf{z}|\mathbf{x}) \log \frac{p_{\theta}(\mathbf{x}, \mathbf{z})}{q_{\phi}(\mathbf{z}|\mathbf{x})} + \log p_{\theta}(\mathbf{x})$$

where $q_{\phi}(\mathbf{z}|\mathbf{x})$ has a fixed parametric form w.r.t. ϕ .

Variational inference

In our scenario: \mathbf{x} is a given data point, and \mathbf{z} is a latent code.

We define a parametric **probabilistic encoder** as the distribution:

$$p_{\theta}(\mathbf{z}|\mathbf{x}) = \frac{p_{\theta}(\mathbf{x}|\mathbf{z})p_{\theta}(\mathbf{z})}{p_{\theta}(\mathbf{x})} = \frac{p_{\theta}(\mathbf{x}, \mathbf{z})}{p_{\theta}(\mathbf{x})}$$

Instead, compute an approximation $q_{\phi^*}(\mathbf{z}|\mathbf{x}) \approx p_{\theta}(\mathbf{z}|\mathbf{x})$ with:

$$\phi^* = \arg \max_{\phi, \theta} \sum_{\mathbf{z}} q_{\phi}(\mathbf{z}|\mathbf{x}) \log \frac{p_{\theta}(\mathbf{x}, \mathbf{z})}{q_{\phi}(\mathbf{z}|\mathbf{x})} - \log p_{\theta}(\mathbf{x})$$

where $q_{\phi}(\mathbf{z}|\mathbf{x})$ has a fixed parametric form w.r.t. ϕ .

Variational inference

In our scenario: \mathbf{x} is a given data point, and \mathbf{z} is a latent code.

We define a parametric **probabilistic encoder** as the distribution:

$$p_{\theta}(\mathbf{z}|\mathbf{x}) = \frac{p_{\theta}(\mathbf{x}|\mathbf{z})p_{\theta}(\mathbf{z})}{p_{\theta}(\mathbf{x})} = \frac{p_{\theta}(\mathbf{x}, \mathbf{z})}{p_{\theta}(\mathbf{x})}$$

Instead, compute an approximation $q_{\phi^*}(\mathbf{z}|\mathbf{x}) \approx p_{\theta}(\mathbf{z}|\mathbf{x})$ with:

$$\phi^* = \arg \max_{\phi, \theta} \sum_{\mathbf{z}} q_{\phi}(\mathbf{z}|\mathbf{x}) \log \frac{p_{\theta}(\mathbf{x}, \mathbf{z})}{q_{\phi}(\mathbf{z}|\mathbf{x})} - \cancel{\log p_{\theta}(\mathbf{x})} \xrightarrow{\text{relax}}$$

where $q_{\phi}(\mathbf{z}|\mathbf{x})$ has a fixed parametric form w.r.t. ϕ .

Variational inference

In our scenario: \mathbf{x} is a given data point, and \mathbf{z} is a latent code.

We define a parametric **probabilistic encoder** as the distribution:

$$p_{\theta}(\mathbf{z}|\mathbf{x}) = \frac{p_{\theta}(\mathbf{x}|\mathbf{z})p_{\theta}(\mathbf{z})}{p_{\theta}(\mathbf{x})} = \frac{p_{\theta}(\mathbf{x}, \mathbf{z})}{p_{\theta}(\mathbf{x})}$$

Instead, compute an approximation $q_{\phi^*}(\mathbf{z}|\mathbf{x}) \approx p_{\theta}(\mathbf{z}|\mathbf{x})$ with:

$$\phi^* = \arg \max_{\phi, \theta} ELBO_{\phi, \theta}(\mathbf{x}) - \cancel{\log p_{\theta}(\mathbf{x})}^{\text{relax}}$$

where $q_{\phi}(\mathbf{z}|\mathbf{x})$ has a fixed parametric form w.r.t. ϕ and:

$$ELBO_{\phi, \theta}(\mathbf{x}) = \sum_{\mathbf{z}} q_{\phi}(\mathbf{z}|\mathbf{x}) \log \frac{p_{\theta}(\mathbf{x}, \mathbf{z})}{q_{\phi}(\mathbf{z}|\mathbf{x})} \leq \log p_{\theta}(\mathbf{x})$$

is called the **Evidence variational Lower BOund**.

Variational autoencoder (VAE)

$$\max_{\phi, \theta} ELBO_{\phi, \theta}(\mathbf{x})$$

Variational autoencoder (VAE)

$$\max_{\phi, \theta} \sum_{\mathbf{z}} q_{\phi}(\mathbf{z}|\mathbf{x}) \log \frac{p_{\theta}(\mathbf{x}, \mathbf{z})}{q_{\phi}(\mathbf{z}|\mathbf{x})}$$

Variational autoencoder (VAE)

$$\max_{\phi, \theta} \sum_{\mathbf{z}} q_{\phi}(\mathbf{z}|\mathbf{x}) \log \frac{p_{\theta}(\mathbf{x}|\mathbf{z}) p_{\theta}(\mathbf{z})}{q_{\phi}(\mathbf{z}|\mathbf{x})}$$

Variational autoencoder (VAE)

$$\max_{\phi, \theta} \sum_{\mathbf{z}} q_{\phi}(\mathbf{z}|\mathbf{x}) \left(\log p_{\theta}(\mathbf{x}|\mathbf{z}) + \log \frac{p_{\theta}(\mathbf{z})}{q_{\phi}(\mathbf{z}|\mathbf{x})} \right)$$

Variational autoencoder (VAE)

$$\max_{\phi, \theta} \sum_{\mathbf{z}} q_{\phi}(\mathbf{z}|\mathbf{x}) \log p_{\theta}(\mathbf{x}|\mathbf{z}) + \sum_{\mathbf{z}} q_{\phi}(\mathbf{z}|\mathbf{x}) \log \frac{p_{\theta}(\mathbf{z})}{q_{\phi}(\mathbf{z}|\mathbf{x})}$$

Variational autoencoder (VAE)

$$\max_{\phi, \theta} \mathbb{E}_{q_\phi(\mathbf{z}|\mathbf{x})} \log p_\theta(\mathbf{x}|\mathbf{z}) - KL(q_\phi(\mathbf{z}|\mathbf{x}) \| p_\theta(\mathbf{z}))$$

Variational autoencoder (VAE)

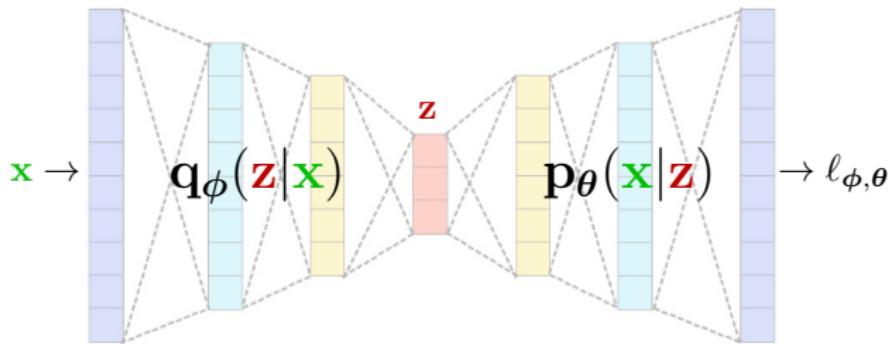
$$\max_{\phi, \theta} \underbrace{\mathbb{E}_{q_\phi(\mathbf{z}|\mathbf{x})} \log p_\theta(\mathbf{x}|\mathbf{z})}_{\text{likelihood of observing } \mathbf{x} \text{ given } \mathbf{z}} - KL(q_\phi(\mathbf{z}|\mathbf{x}) \| p_\theta(\mathbf{z}))$$

Variational autoencoder (VAE)

$$\max_{\phi, \theta} \underbrace{\mathbb{E}_{q_\phi(\mathbf{z}|\mathbf{x})} \log p_\theta(\mathbf{x}|\mathbf{z})}_{\text{likelihood of observing } \mathbf{x} \text{ given } \mathbf{z}} - \underbrace{KL(q_\phi(\mathbf{z}|\mathbf{x})||p_\theta(\mathbf{z}))}_{\text{ensures } q_\phi(\mathbf{z}|\mathbf{x}) \approx p_\theta(\mathbf{z})}$$

Variational autoencoder (VAE)

$$\ell_{\phi, \theta} = \underbrace{-\mathbb{E}_{q_\phi(\mathbf{z}|\mathbf{x})} \log p_\theta(\mathbf{x}|\mathbf{z})}_{\text{likelihood of observing } \mathbf{x} \text{ given } \mathbf{z}} + \underbrace{KL(q_\phi(\mathbf{z}|\mathbf{x}) \| p_\theta(\mathbf{z}))}_{\text{ensures } q_\phi(\mathbf{z}|\mathbf{x}) \approx p_\theta(\mathbf{z})}$$



Variational autoencoder (VAE)

$$\ell_{\phi, \theta} = \underbrace{-\mathbb{E}_{q_\phi(\mathbf{z}|\mathbf{x})} \log p_\theta(\mathbf{x}|\mathbf{z})}_{\text{likelihood of observing } \mathbf{x} \text{ given } \mathbf{z}} + \underbrace{KL(q_\phi(\mathbf{z}|\mathbf{x}) \| p_\theta(\mathbf{z}))}_{\text{ensures } q_\phi(\mathbf{z}|\mathbf{x}) \approx p_\theta(\mathbf{z})}$$

The prior over the latent variables is Gaussian and has no free parameters:

$$p(\mathbf{z}) = \mathcal{N}_{\mathbf{0}, \mathbf{I}}(\mathbf{z})$$

Variational autoencoder (VAE)

$$\ell_{\phi, \theta} = \underbrace{-\mathbb{E}_{q_\phi(\mathbf{z}|\mathbf{x})} \log p_\theta(\mathbf{x}|\mathbf{z})}_{\text{likelihood of observing } \mathbf{x} \text{ given } \mathbf{z}} + \underbrace{KL(q_\phi(\mathbf{z}|\mathbf{x}) \| p_\theta(\mathbf{z}))}_{\text{ensures } q_\phi(\mathbf{z}|\mathbf{x}) \approx p_\theta(\mathbf{z})}$$

The prior over the latent variables is Gaussian and has **no free parameters**:

$$p(\mathbf{z}) = \mathcal{N}_{\mathbf{0}, \mathbf{I}}(\mathbf{z})$$

The **probabilistic encoder** also generates a Gaussian distribution:

$$q_\phi(\mathbf{z}|\mathbf{x}) = \mathcal{N}_{\boldsymbol{\mu}, \boldsymbol{\sigma}}(\mathbf{z})$$

Variational autoencoder (VAE)

$$\ell_{\phi, \theta} = \underbrace{-\mathbb{E}_{q_\phi(\mathbf{z}|\mathbf{x})} \log p_\theta(\mathbf{x}|\mathbf{z})}_{\text{likelihood of observing } \mathbf{x} \text{ given } \mathbf{z}} + \underbrace{KL(q_\phi(\mathbf{z}|\mathbf{x}) \| p_\theta(\mathbf{z}))}_{\text{ensures } q_\phi(\mathbf{z}|\mathbf{x}) \approx p_\theta(\mathbf{z})}$$

The prior over the latent variables is Gaussian and has no free parameters:

$$p(\mathbf{z}) = \mathcal{N}_{\mathbf{0}, \mathbf{I}}(\mathbf{z})$$

The probabilistic encoder also generates a Gaussian distribution:

$$q_\phi(\mathbf{z}|\mathbf{x}) = \mathcal{N}_{\boldsymbol{\mu}, \boldsymbol{\sigma}}(\mathbf{z})$$



Variational autoencoder (VAE)

$$\ell_{\phi, \theta} = \underbrace{-\mathbb{E}_{q_\phi(\mathbf{z}|\mathbf{x})} \log p_\theta(\mathbf{x}|\mathbf{z})}_{\text{likelihood of observing } \mathbf{x} \text{ given } \mathbf{z}} + \underbrace{KL(q_\phi(\mathbf{z}|\mathbf{x}) \| p_\theta(\mathbf{z}))}_{\text{ensures } q_\phi(\mathbf{z}|\mathbf{x}) \approx p_\theta(\mathbf{z})}$$

The prior over the latent variables is Gaussian and has **no free parameters**:

$$p(\mathbf{z}) = \mathcal{N}_{\mathbf{0}, \mathbf{I}}(\mathbf{z})$$

The **probabilistic encoder** also generates a Gaussian distribution:

$$q_\phi(\mathbf{z}|\mathbf{x}) = \mathcal{N}_{\boldsymbol{\mu}, \boldsymbol{\sigma}}(\mathbf{z})$$

where $\boldsymbol{\mu}, \boldsymbol{\sigma}$ are functions of the input \mathbf{x} and the network parameters ϕ .

The probabilistic encoder outputs $\boldsymbol{\mu}, \boldsymbol{\sigma}$, not \mathbf{z} .

Variational autoencoder (VAE)

$$\ell_{\phi, \theta} = \underbrace{-\mathbb{E}_{q_\phi(\mathbf{z}|\mathbf{x})} \log p_\theta(\mathbf{x}|\mathbf{z})}_{\text{likelihood of observing } \mathbf{x} \text{ given } \mathbf{z}} + \underbrace{KL(q_\phi(\mathbf{z}|\mathbf{x}) \| p_\theta(\mathbf{z}))}_{\text{ensures } q_\phi(\mathbf{z}|\mathbf{x}) \approx p_\theta(\mathbf{z})}$$

The prior over the latent variables is Gaussian and has **no free parameters**:

$$p(\mathbf{z}) = \mathcal{N}_{\mathbf{0}, \mathbf{I}}(\mathbf{z})$$

The **probabilistic encoder** also generates a Gaussian distribution:

$$q_\phi(\mathbf{z}|\mathbf{x}) = \mathcal{N}_{\boldsymbol{\mu}, \boldsymbol{\sigma}}(\mathbf{z})$$

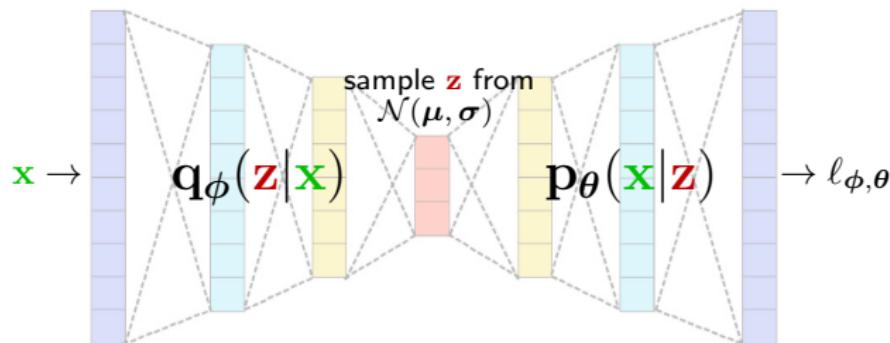
where $\boldsymbol{\mu}, \boldsymbol{\sigma}$ are functions of the input \mathbf{x} and the network parameters ϕ .

The probabilistic encoder outputs $\boldsymbol{\mu}, \boldsymbol{\sigma}$, not \mathbf{z} .

Using Gaussians, the KL term has a closed form.

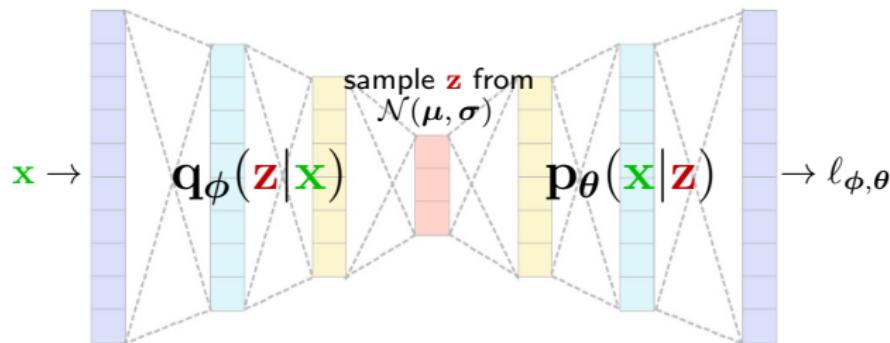
Variational autoencoder (VAE)

$$\ell_{\phi, \theta} = \underbrace{-\mathbb{E}_{q_\phi(\mathbf{z}|\mathbf{x})} \log p_\theta(\mathbf{x}|\mathbf{z})}_{\text{likelihood of observing } \mathbf{x} \text{ given } \mathbf{z}} + \underbrace{KL(q_\phi(\mathbf{z}|\mathbf{x}) \| p_\theta(\mathbf{z}))}_{\text{ensures } q_\phi(\mathbf{z}|\mathbf{x}) \approx p_\theta(\mathbf{z})}$$



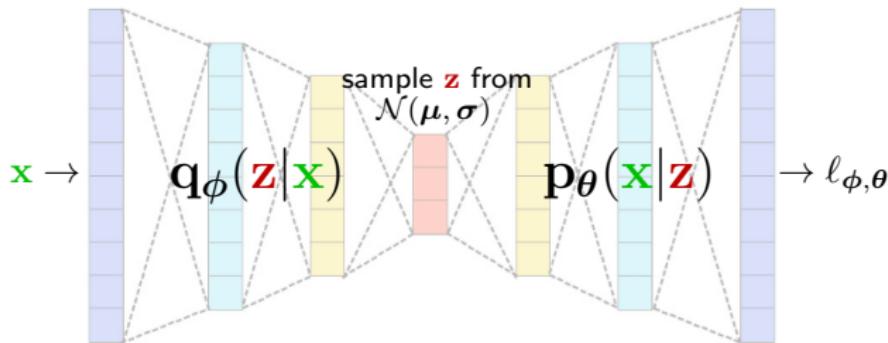
Variational autoencoder (VAE)

$$\ell_{\phi, \theta} = \underbrace{-\mathbb{E}_{q_\phi(\mathbf{z}|\mathbf{x})} \log p_\theta(\mathbf{x}|\mathbf{z})}_{\text{reconstruction loss} \\ (\text{choose your own})} + \underbrace{KL(q_\phi(\mathbf{z}|\mathbf{x}) \| p_\theta(\mathbf{z}))}_{\text{regularizer}}$$



Variational autoencoder (VAE)

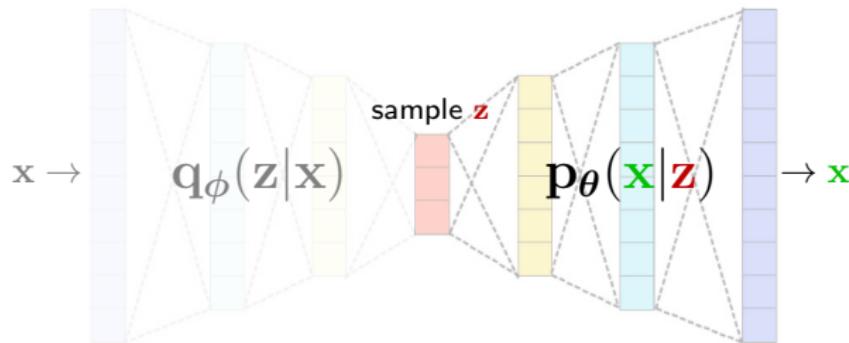
$$\ell_{\phi, \theta} = \underbrace{-\mathbb{E}_{q_\phi(\mathbf{z}|\mathbf{x})} \log p_\theta(\mathbf{x}|\mathbf{z})}_{\text{reconstruction loss} \atop (\text{choose your own})} + \underbrace{KL(q_\phi(\mathbf{z}|\mathbf{x}) \| p_\theta(\mathbf{z}))}_{\text{regularizer}}$$



To distinguish from VAE, classical AE are also called [deterministic AE](#).

Variational autoencoder (VAE)

$$\ell_{\phi, \theta} = \underbrace{-\mathbb{E}_{q_\phi(\mathbf{z}|\mathbf{x})} \log p_\theta(\mathbf{x}|\mathbf{z})}_{\text{reconstruction loss} \atop (\text{choose your own})} + \underbrace{KL(q_\phi(\mathbf{z}|\mathbf{x}) \| p_\theta(\mathbf{z}))}_{\text{regularizer}}$$

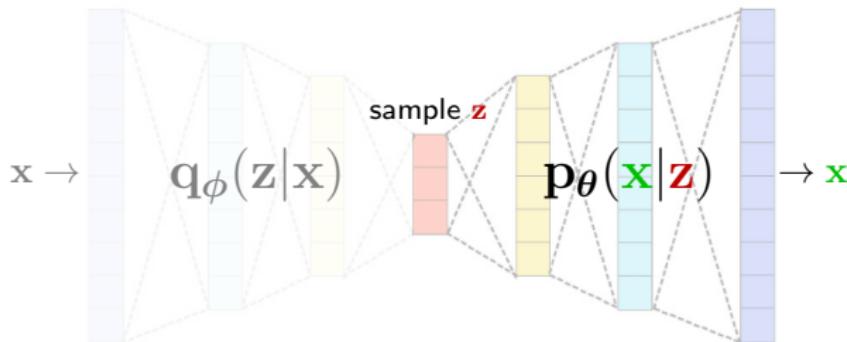


To distinguish from VAE, classical AE are also called **deterministic AE**.

Generation: Sample \mathbf{z} according to the learned distribution.

Variational autoencoder (VAE)

$$\ell_{\phi, \theta} = \underbrace{-\mathbb{E}_{q_\phi(\mathbf{z}|\mathbf{x})} \log p_\theta(\mathbf{x}|\mathbf{z})}_{\text{reconstruction loss} \atop (\text{choose your own})} + \underbrace{KL(q_\phi(\mathbf{z}|\mathbf{x}) \| p_\theta(\mathbf{z}))}_{\text{regularizer}}$$



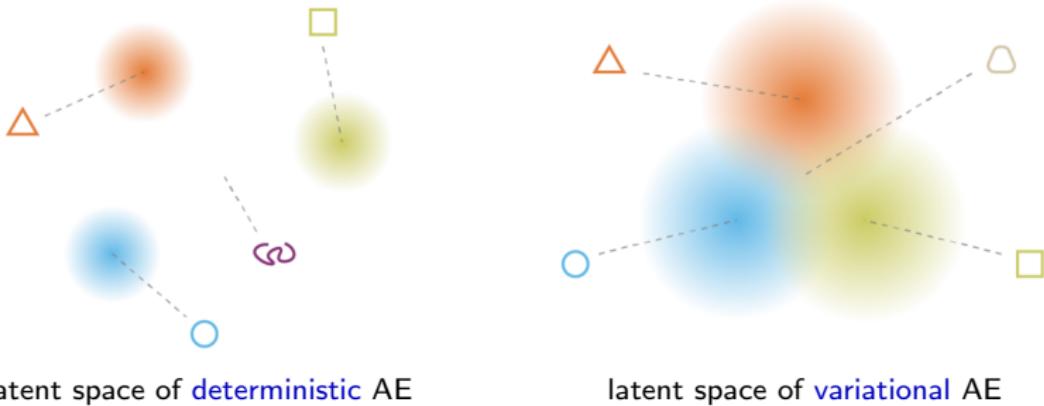
To distinguish from VAE, classical AE are also called **deterministic AE**.

Generation: Sample \mathbf{z} according to the learned distribution.

In principle, **also the decoder outputs a probability distribution**.

If the reconstruction loss does not need a distribution, take $\mathbf{x} := \mu$.

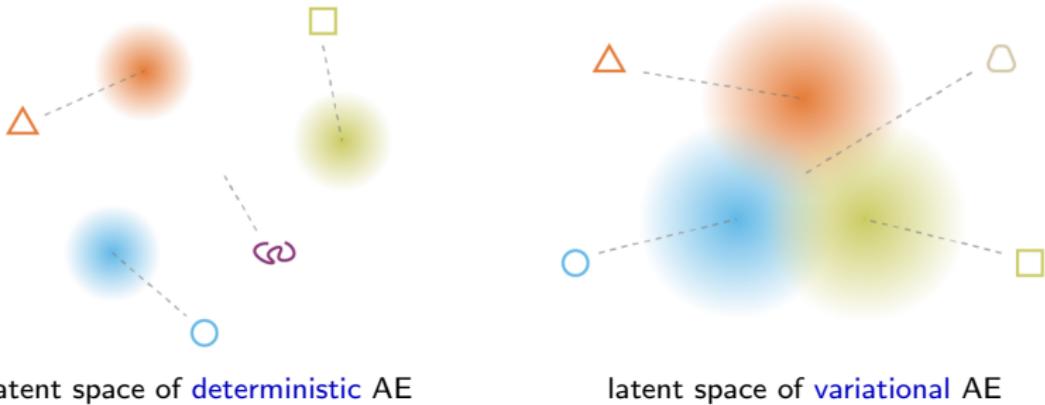
Regularizing effect



The requirement $\mathbf{z} \sim \mathcal{N}(\mathbf{0}, \mathbf{I})$ leads to a compact latent space.

Small distances $\|\mathbf{z}_1 - \mathbf{z}_2\|_2 \Rightarrow$ small changes in the decoded output.

Regularizing effect

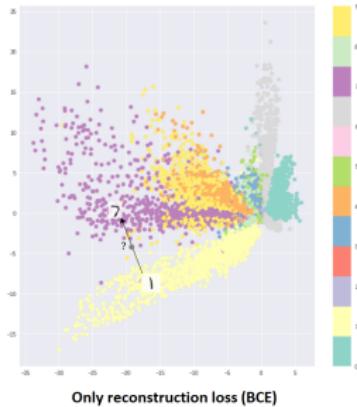


The requirement $\mathbf{z} \sim \mathcal{N}(\mathbf{0}, \mathbf{I})$ leads to a compact latent space.

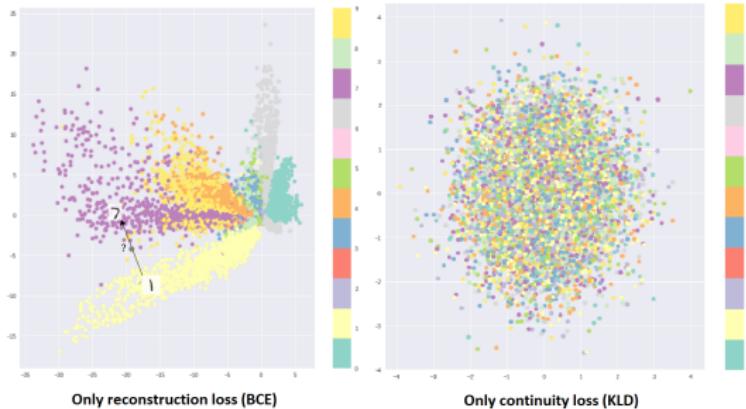
Small distances $\|\mathbf{z}_1 - \mathbf{z}_2\|_2 \Rightarrow$ small changes in the decoded output.

A deterministic AE might generate arbitrary noise.

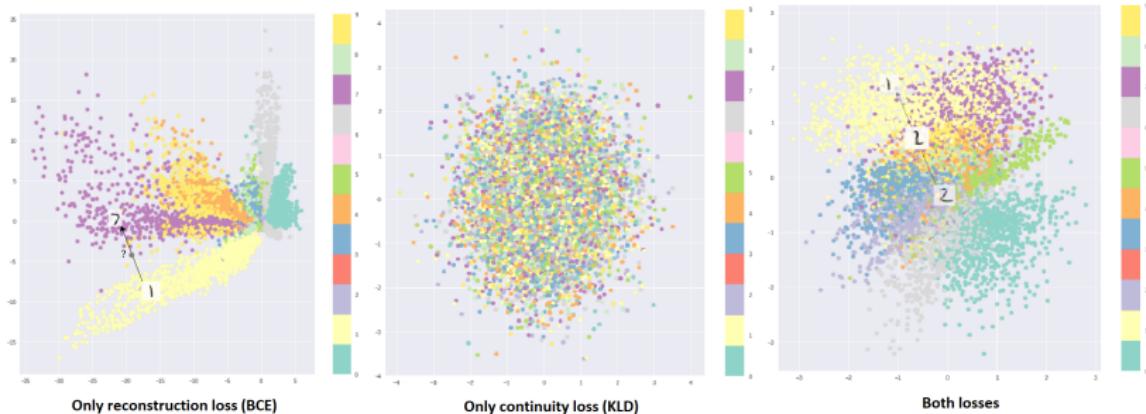
Regularizing effect



Regularizing effect



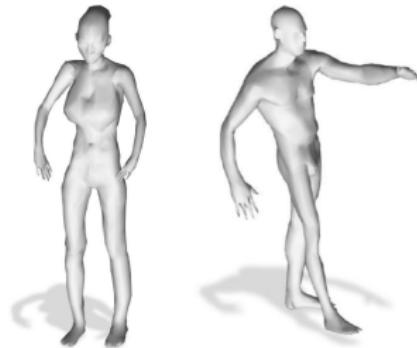
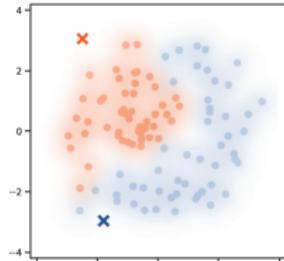
Regularizing effect



The distributional regularizer brings **smoothness** in the learned space.

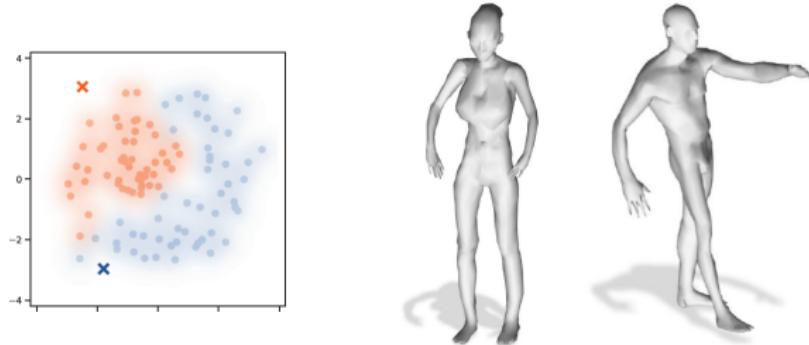
Unlikely samples

Sampling on the **boundaries** of the learned distribution leads to the generation of **unlikely** samples:



Unlikely samples

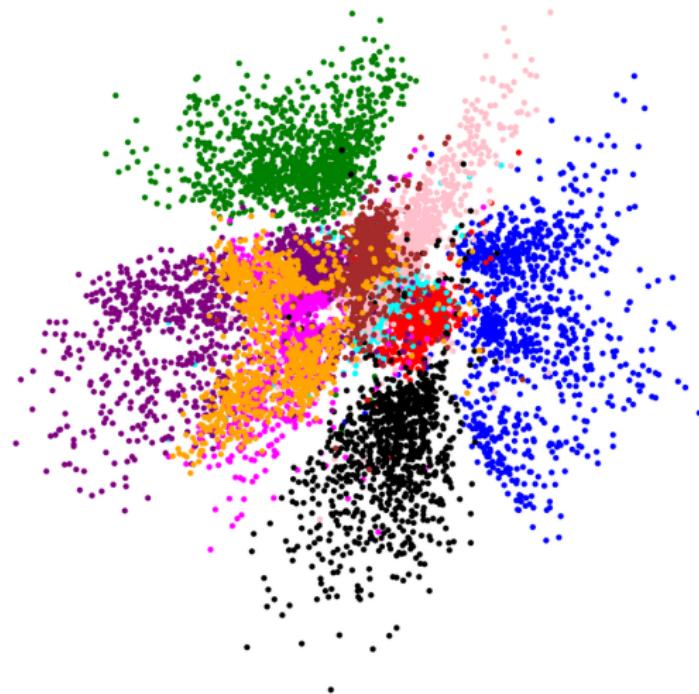
Sampling on the **boundaries** of the learned distribution leads to the generation of **unlikely** samples:



Another relevant phenomenon is **posterior collapse**, where the latents are ignored if the decoder is powerful enough to model the data perfectly.

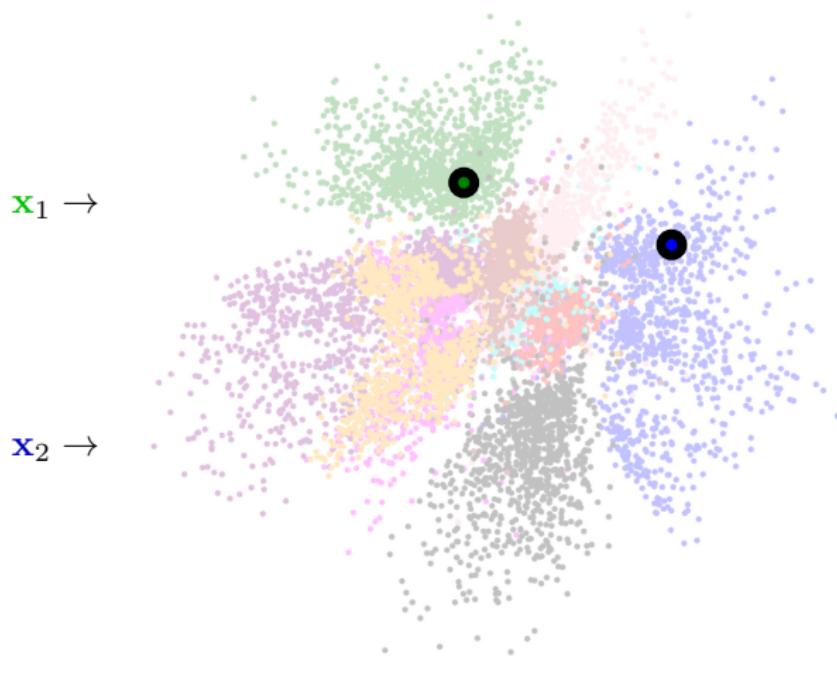
VAE interpolation

Assume we have a trained VAE, colors correspond to different classes:



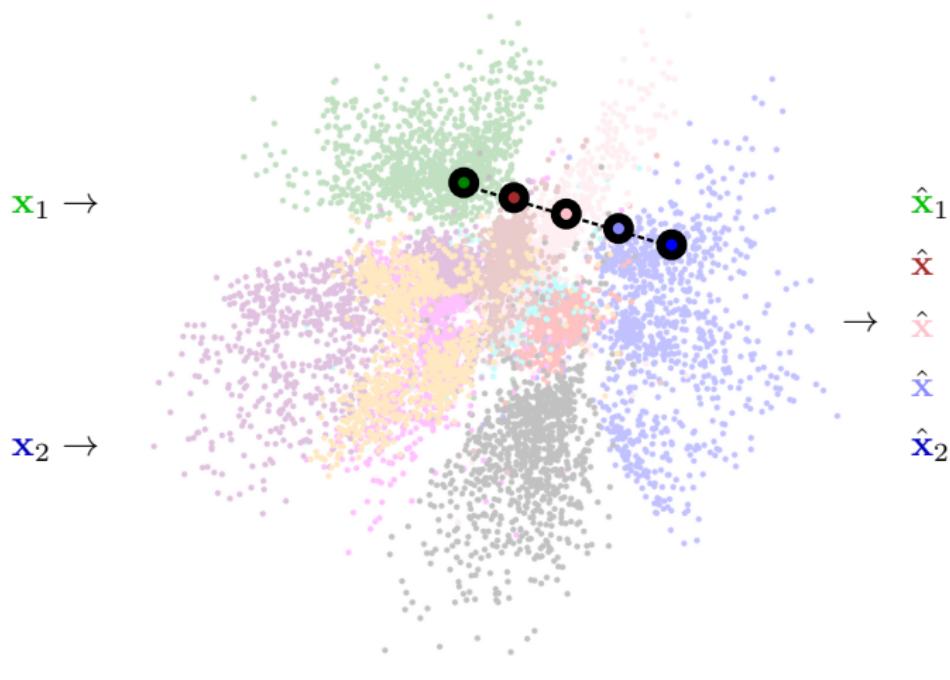
VAE interpolation

Assume we have a trained VAE, colors correspond to different classes:



VAE interpolation

Assume we have a trained VAE, colors correspond to different classes:



The interpolated samples are **not** part of the training set.

VAE interpolation

7

1

VAE interpolation

7 7 7 7 7 7 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1

VAE interpolation

6

7 7 7 7 7 7 1 1 1 1 1 1 1 1 1 1 1 1

VAE interpolation

六四九九九九九九

777777771111111111111111

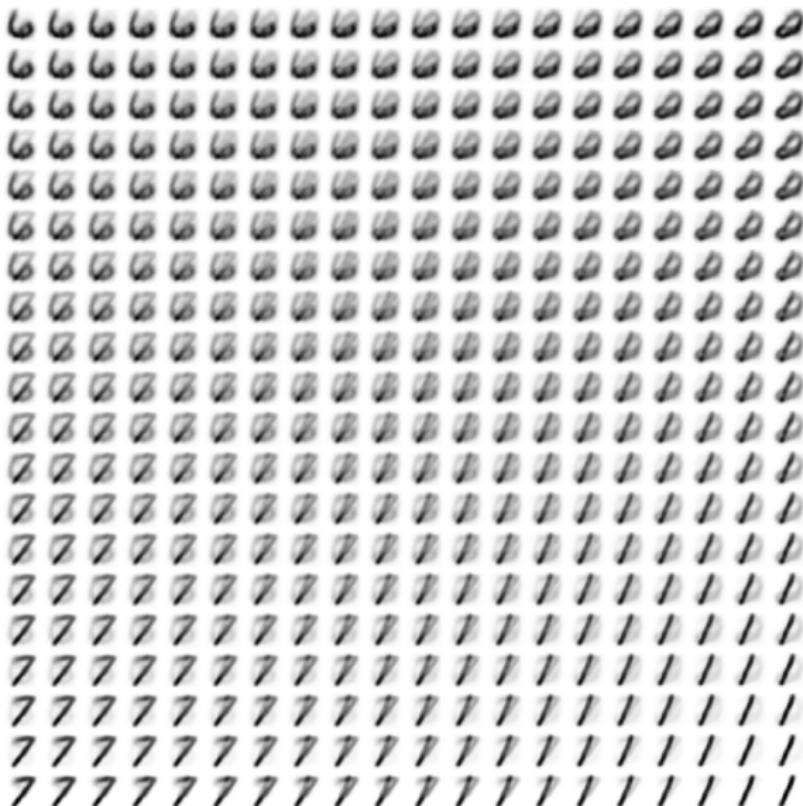
VAE interpolation

A 10x10 grid of handwritten digits, likely generated by a Variational Autoencoder (VAE) during an interpolation process. The digits transition smoothly from a '6' at the top-left to a '7' at the bottom-right, illustrating the latent space manifold.

The digits are arranged in a grid:

- Row 1: 6 6 6 6 6 6 6 6 6 6
- Row 2: 9 4 4 2 2 2 2 2 2 2
- Row 3: 9 2 2 2 2 2 2 2 2 2
- Row 4: 9 9 2 2 2 2 2 2 2 2
- Row 5: 9 9 4 2 2 2 2 2 2 2
- Row 6: 9 9 9 2 2 2 2 2 2 2
- Row 7: 9 9 9 9 2 2 2 2 2 2
- Row 8: 9 9 9 9 9 3 3 3 3 3
- Row 9: 9 9 9 9 9 8 3 3 3 3
- Row 10: 7 9 9 9 9 9 8 3 3 3
- Row 11: 7 9 9 9 9 9 8 8 3 3
- Row 12: 7 9 9 9 9 9 8 8 8 3
- Row 13: 7 9 9 9 9 9 8 8 8 8
- Row 14: 7 9 9 9 9 9 8 8 8 8
- Row 15: 7 9 9 9 9 9 8 8 8 8
- Row 16: 7 9 9 9 9 9 8 8 8 8
- Row 17: 7 9 9 9 9 9 8 8 8 8
- Row 18: 7 9 9 9 9 9 8 8 8 8
- Row 19: 7 9 9 9 9 9 8 8 8 8
- Row 20: 7 9 9 9 9 9 8 8 8 8
- Row 21: 7 9 9 9 9 9 9 9 9 9
- Row 22: 7 9 9 9 9 9 9 9 9 9
- Row 23: 7 9 9 9 9 9 9 9 9 9
- Row 24: 7 9 9 9 9 9 9 9 9 9
- Row 25: 7 9 9 9 9 9 9 9 9 9
- Row 26: 7 9 9 9 9 9 9 9 9 9
- Row 27: 7 9 9 9 9 9 9 9 9 9
- Row 28: 7 9 9 9 9 9 9 9 9 9
- Row 29: 7 9 9 9 9 9 9 9 9 9
- Row 30: 7 7 7 7 7 7 7 7 7 7

Euclidean interpolation

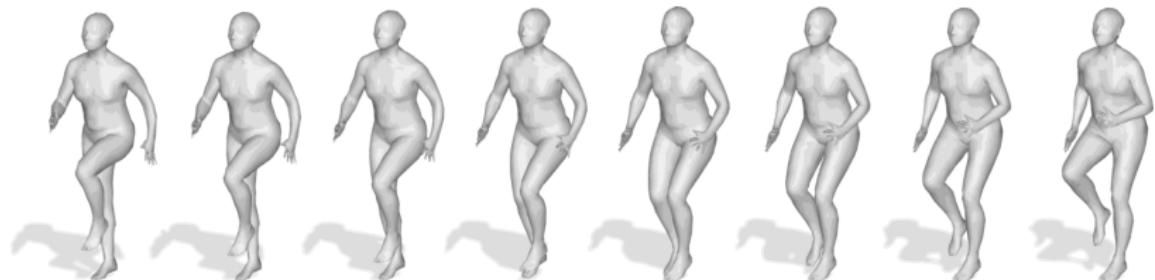


Example: Deformable 3D shapes



Cosmo, Norelli, Halimi, Kimmel, Rodolà, "LIMP: Learning Latent Shape Representations with Metric Preservation Priors", 2020

Example: Deformable 3D shapes



Cosmo, Norelli, Halimi, Kimmel, Rodolà, "LIMP: Learning Latent Shape Representations with Metric Preservation Priors", 2020

Example: Deformable 3D shapes



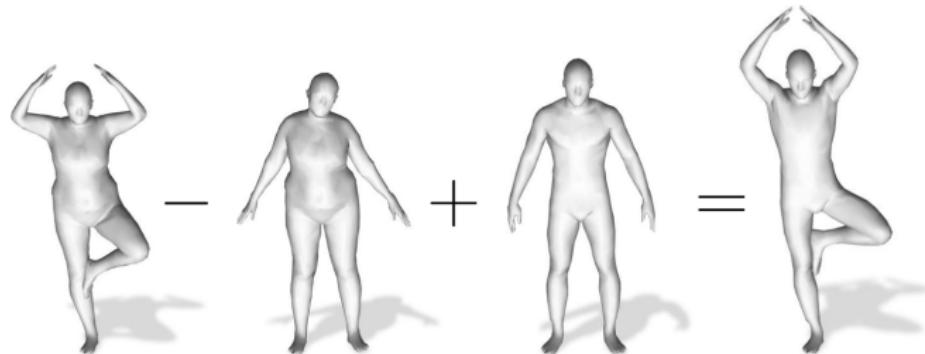
The latent space can be explored along different dimensions
(disentanglement):



Cosmo, Norelli, Halimi, Kimmel, Rodolà, "LIMP: Learning Latent Shape Representations with Metric Preservation Priors", 2020

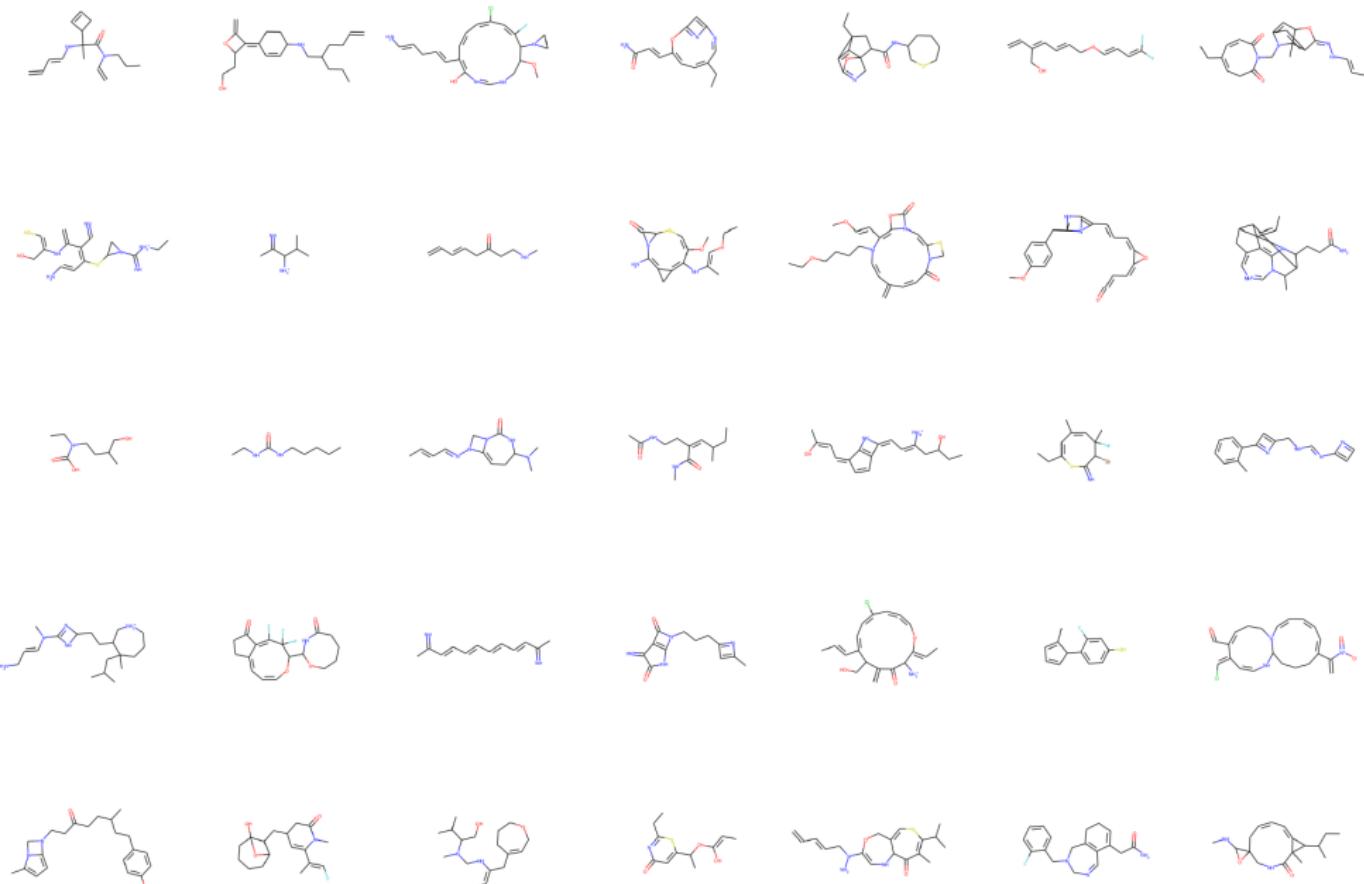
Analogies

Algebraic manipulation of codes leads to the following:



In NLP this is done by manipulating [word embeddings](#):

$$\text{King} - \text{Man} + \text{Woman} = \text{Queen}$$



Bresson, "A Two-Step Graph Convolutional Decoder for Molecule Generation", 2019



Razavi et al, "Generating Diverse High-Fidelity Images with VQ-VAE-2", 2019