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An Asymptotically Stable Joint PD Controller for Robot Arms with Flexible Links Under Gravity

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1. Motivation

The regulation problem for articulated mechanical arms is often solved by designing simple control laws which strongly exploit the physical properties of the system. It is well known that a rigid robot can be globally asymptotically stabilized around a given joint configuration via a PD controller on the joint errors, provided that gravity is exactly cancelled by feedback [1]. Under a mild condition on the proportional gain, this scheme can be simplified by performing only a constant gravity compensation at the desired configuration [2]. This result was extended in [3] to the case of robots with elastic joints, under the further assumption that joint stiffness overcomes the gradient of the gravitational term. Asymptotic stability of a joint PD controller for robot arms with flexible links has been recently shown in the absence of gravity [4]. Inspired by the approach of [3], in this work we prove global asymptotic stability of a *joint PD controller*, i.e. avoiding feedback from the elastic coordinates, *with constant gravity compensation* for the full nonlinear model of multilink flexible robots. A structural assumption about link elasticity is required and a mild condition on the proportional gain is derived. The proof goes through a classical Lyapunov argument.

2. Dynamic model of flexible arms

The Lagrangian technique can be used to derive the dynamic model of a robot arm composed of a serial chain of links, some of which are flexible [5]; slender links can be modeled as Euler-Bernoulli beams satisfying proper boundary conditions. While a linear model is in general sufficient to capture the dynamics of each flexible link, the interplay of rigid body motion and flexible deflections in the multilink case gives rise to fully nonlinear dynamic equations.

In order to obtain a finite-dimensional model, let θ denote the n-vector of joint coordinates, and δ the m-vector of link coordinates of an assumed modes description of link deflections; then, the $(n+m)$-vector $q = (\theta^{\mathrm{T}} \;\; \delta^{\mathrm{T}})^{\mathrm{T}}$ characterizes the arm configuration. We suppose to include only bending deformations limited for each link to the plane of rigid motion. The closed-form dynamic equations of the arm can be written as $n+m$ second-order nonlinear differential equations in the general form [6]

$$B(q)\ddot{q} + h(q,\dot{q}) + g(q) + \begin{pmatrix} 0 \\ K\delta + D\dot{\delta} \end{pmatrix} = \begin{pmatrix} u \\ 0 \end{pmatrix}. \quad (1)$$

In (1), the positive definite symmetric inertia matrix B depends in general on both joint (rigid) and link (flexible) coordinates. The vector h contains Coriolis and centrifugal forces that can be factorized as

$$h(q,\dot{q}) = S(q,\dot{q})\dot{q} \quad (2)$$

so that the matrix $\dot{B} - 2S$ is skew-symmetric, similarly to the rigid case [1]. The positive definite (diagonal) matrix D in (1) describes modal damping of the links. Notice that we are describing deformation in a frame which is clamped at the joint actuator side; this implies that the control does not enter directly in the equations of motion for the flexible part.

The terms in (1) deriving from the potential energy U are composed of the gravity contribution U_g and of the elastic contribution U_δ. In view of the small deformation hypothesis, we have that

$$U_\delta = \frac{1}{2}\delta^{\mathrm{T}} K \delta \leq U_{\delta,\max} < \infty, \quad (3)$$

where K is the positive definite symmetric (diagonal) stiffness matrix associated with link elasticity. From (3) it follows that

$$\|\delta\| \leq \sqrt{\frac{2U_{\delta,\max}}{K_M}}, \quad (4)$$

where $\|v\|$ denotes the usual Euclidean norm of a vector v; also, we denote by A_M (A_m) the largest (smallest) eigenvalue of a symmetrix matrix A.

Concerning the gravity contribution, the vector of gravity forces $g = (\partial U_g/\partial q)^{\mathrm{T}}$ can be partitioned as

$$g(q) = \begin{pmatrix} g_\theta(\theta,\delta) \\ g_\delta(\theta) \end{pmatrix}, \quad (5)$$

where the dependence of the lower term is justified by the assumption of small deformation. Further, the vector g satisfies the inequality

$$\left\| \frac{\partial g}{\partial q} \right\| \leq \alpha_0 + \alpha_1\|\delta\| \leq \alpha_0 + \alpha_1\sqrt{\frac{2U_{\delta,\max}}{K_M}} =: \alpha, \quad (6)$$

where $\alpha_0, \alpha_1, \alpha > 0$. This can be easily proven by observing that the gravity term contains only trigonometric functions of θ and linear/trigonometric functions of δ. Also, inequality (4) has been used in (6). As a direct consequence of (6), we have:

$$\|g(q_1) - g(q_2)\| \leq \alpha\|q_1 - q_2\|, \quad \forall q_1, q_2 \in \mathbb{R}^{n+m} \quad (7)$$

We remark that the above arguments and what follows can be easily modified to include also an explicit dependence of g_δ in (5) from δ.

3. Asymptotically stable joint PD control

Consider the control law

$$u = K_P(\theta_{\mathrm{des}} - \theta) - K_D\dot{\theta} + g_\theta(\theta_{\mathrm{des}}, \delta_{\mathrm{des}}), \quad (8)$$

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with $K_P > O$ (at least), $K_D > O$, and being δ_{des} defined by

$$\delta_{\text{des}} = -K^{-1} g_\delta(\theta_{\text{des}}). \tag{9}$$

The equilibrium states of the closed-loop system (1,8) satisfy the equations

$$g_\theta(\theta,\delta) = K_P(\theta_{\text{des}} - \theta) + g_\theta(\theta_{\text{des}}, \delta_{\text{des}}) \tag{10a}$$
$$g_\delta(\theta) = -K\delta. \tag{10b}$$

It is easy to recognize that (10b) has a unique solution δ for any value of $\theta \in \mathbb{R}^n$. Adding $K\delta_{\text{des}} + g_\delta(\theta_{\text{des}}) = 0$ to the right-hand side of (10b) yields

$$\begin{aligned} K_q(q_{\text{des}} - q) &:= \begin{pmatrix} K_P & O \\ O & K \end{pmatrix} \begin{pmatrix} \theta_{\text{des}} - \theta \\ \delta_{\text{des}} - \delta \end{pmatrix} \\ &= \begin{pmatrix} g_\theta(\theta,\delta) - g_\theta(\theta_{\text{des}}, \delta_{\text{des}}) \\ g_\delta(\theta) - g_\delta(\theta_{\text{des}}) \end{pmatrix} = g(q) - g(q_{\text{des}}). \end{aligned} \tag{11}$$

Under the assumption that

$$K_{qm} > \alpha, \tag{12}$$

we have, for $q \neq q_{\text{des}}$,

$$\|K_q(q_{\text{des}} - q)\| > \alpha \|q_{\text{des}} - q\| \geq \|g(q) - g(q_{\text{des}})\|, \tag{13}$$

where the last inequality follows from (7). This implies that $q = q_{\text{des}}$, $\dot{q} = 0$ is the *unique* equilibrium state of the closed-loop system (1,8).

Condition (12) will automatically be satisfied, provided that the assumption on the structural link flexibility $K_m > \alpha$ holds, and that the proportional control gain is chosen so that $K_{Pm} > \alpha$. The main result of the work follows.

Theorem. *The equilibrium state $q = q_{\text{des}}$, $\dot{q} = 0$ of system (1) under control (8) is asymptotically stable provided that (12) holds.* ∎

Proof. Consider the energy-based Lyapunov function candidate

$$\begin{aligned} V = &\frac{1}{2}\dot{q}^{\text{T}} B \dot{q} + \frac{1}{2}(q_{\text{des}} - q)^{\text{T}} K_q (q_{\text{des}} - q) \\ &+ U_g(q) - U_g(q_{\text{des}}) + (q_{\text{des}} - q)^{\text{T}} g(q_{\text{des}}) \geq 0, \end{aligned} \tag{14}$$

which vanishes only at the desired equilibrium state, due to (10–13). The time derivative of (14) along the trajectories of the closed-loop system (1,8) is

$$\begin{aligned} \dot{V} &= \dot{q}^{\text{T}} \left(B\ddot{q} + \frac{1}{2}\dot{B}\dot{q} - K_q(q_{\text{des}} - q) + (g(q) - g(q_{\text{des}})) \right) \\ &= \dot{q}^{\text{T}} \left(\begin{pmatrix} K_P(\theta_{\text{des}} - \theta) - K_D\dot{\theta} + g_\theta(q_{\text{des}}) \\ -(K\delta + D\dot{\delta}) \end{pmatrix} - g(q) \right) \\ &\quad - \dot{q}^{\text{T}} \begin{pmatrix} K_P(\theta_{\text{des}} - \theta) \\ K(\delta_{\text{des}} - \delta) \end{pmatrix} + \dot{q}^{\text{T}} \left(g(q) - \begin{pmatrix} g_\theta(q_{\text{des}}) \\ g_\delta(\theta_{\text{des}}) \end{pmatrix} \right), \end{aligned} \tag{15}$$

where identity (2) and the skew-symmetry of the matrix $\dot{B} - 2S$ have been used. Simplifying terms yields

$$\dot{V} = -\dot{\theta}^{\text{T}} K_D \dot{\theta} - \dot{\delta}^{\text{T}} D \dot{\delta} \leq 0, \tag{16}$$

where (9) has been utilized. When $\dot{V} = 0$, it is $\dot{q} = 0$ and the closed-loop system (1,8) becomes

$$B\ddot{q} = \begin{pmatrix} K_P(\theta_{\text{des}} - \theta) + g_\theta(q_{\text{des}}) - g_\theta(q) \\ -(K\delta + g_\delta(\theta)) \end{pmatrix}. \tag{17}$$

In view of the previous equilibrium analysis and of (12), it is $\ddot{q} = 0$ if and only if $q = q_{\text{des}}$, or $\theta = \theta_{\text{des}}$ and $\delta = \delta_{\text{des}}$. Invoking LaSalle invariance set theorem, asymptotic stability of the desired state follows. ∎

4. Discussion

We have presented a simple joint PD control scheme for robots with flexible links which guarantees global asymptotic stability of a desired constant arm configuration in the presence of gravity. The following comments are in order.

- The control law does not require any feedback from the deflection variables, and is composed by a *linear* term plus a nominal feedforward term.
- Satisfaction of the structural assumption $K_m > \alpha$ is not restrictive in general, and depends on the relative importance of stiffness vs. gravity. When compared to the joint elastic case [3], link stiffness is usually much smaller than transmission stiffness but the lightweight nature of the links greatly reduces also the magnitude of the gravity term.
- The knowledge of the link stiffness K and of the complete gravity term g is needed mainly for defining the steady-state deformation δ_{des}. Indeed, uncertainty in the associated model parameters produces a different asymptotically stable equilibrium state. This can be rendered arbitrarily close to the desired one by increasing K_P, provided that the arm is stiff enough.

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