Problemset3

Dennis Sackmann
Niklas Held
Moritz Gaßmann

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1 Introduction

In this project, we aim to derive the first four moments of the option-implied density in terms of the underlying option prices.
That for, we obtain access to daily implied volatility surface data, for February 2023, for both the SPX and all SP500 constituents. We then try to decompose the implied skewness into its systematic and idiosyncratic parts.
At the end, we try to understand and interpret the difference of skewness of an Index and it's constituents.

2 Computing first four moments

2.1 Introduction to first four moments

The first four moments - mean, variance, skewness and kurtosis - are used because they intuitively describe the characteristics of a distribution without having to specify the entire distribution.

The mean μ is used to describe the expected return of an asset. In the context of option pricing - which relies on the risk-neutral measure - the mean captures the expected change in the logarithm of the asset price.

The Variance σ^2 captures the variability of the underlying asset's return around its mean. A higher variance indicates greater uncertainty and larger potential price swings, which increase the value of options since the chances of profitable movements become large.

Third movement is Skewness $SKEW$, which measures the asymmetry of the return distribution. In option markets, skewness reflects the market's perception of asymmetric risk. Negative skewness indicates a heavier left tail, meaning there's a higher chance of extreme losses.

The last moment, kurtosis $KURT$ measures the propensity of the return distribution to produce extreme values compared to the normal distribution. In the context of options, kurtosis captures the market's expectation of extreme price moves that can have significant impact on option value.

2.2 Formulas

In this analysis, we compute the first four moments over a time horizon of $\tau = 30$ days. This moments are derived from option prices as proposed in BKM (2003).

2.2.1 Risk-Neutral Mean Return $\mu(t, \tau)$

$$\mu(t,\tau) = e^{r\tau} - 1 - \frac{e^{r\tau}}{2}V(t,\tau) - \frac{e^{r\tau}}{6}W(t,\tau) - \frac{e^{r\tau}}{24}X(t,\tau).$$

The formula expresses the expected log-return under the risk-neutral measure, adjusting the risk-free growth rate by accounting for the effects of variance, skewness, and kurtosis to better capture asymmetric return distributions.

2.2.2 Price of the Volatility Contract

$$V(t,\tau) = \int_{S(t)}^{\infty} \frac{2(1 - ln(\frac{K}{S(t)}))}{K^2} C(t,\tau;K)dK + \int_{0}^{S(t)} \frac{2(1 - ln(\frac{S(t)}{K}))}{K^2} P(t,\tau;K)dK$$

The formula represents the return variance, computed as a weighted integral over out-of-the-money call and put option prices, capturing the expected return variance under the risk-neutral measure.

2.2.3 Price of the Cubic Contract

$$W(t,\tau) = \int_{S(t)}^{\infty} \frac{6ln(\frac{K}{S(t)}) - 3(ln(\frac{K}{S(t)}))^2}{K^2} C(t,\tau;K)dK - \int_{0}^{S(t)} \frac{6ln(\frac{S(t)}{K}) + 3(ln(\frac{S(t)}{K}))^2}{K^2} P(t,\tau;K)dk$$

The formula represents the third moment, calculated from a weighted integral over out-of-the-money option prices to capture the asymmetry of the risk-neutral return distribution.

2.2.4 Price of the Quartic Contract

$$X(t,\tau) = \int_{S(t)}^{\infty} \frac{12(ln(\frac{K}{S(t)}))^2 - 4(ln(\frac{K}{S(t)}))^3}{K^2} C(t,\tau;K)dK - \int_{0}^{S(t)} \frac{12(ln(\frac{S(t)}{K}))^2 + 4(ln(\frac{S(t)}{K}))^2}{K^2} P(t,\tau;K)dk$$

The formula represents the fourth moment, calculated from a weighted integral over out-of-money call and put option prices, capturing the fat-tailedness of the risk-neutral return distribution.

2.2.5 Skewness

$$SKEW(t,\tau) = \frac{e^{r\tau}W(t,\tau) - 3\mu(t,\tau)e^{r\tau}V(t,\tau) + 2\mu(t,\tau)^3}{(e^{r\tau}V(t,\tau - \mu(t,\tau)^2)^{3/2}}$$

The formula defines risk-neutral skewness as the **normalized** third moment of the return distribution under the risk-neutral measure, capturing the asymmetry of expected returns over the time horizon.

2.2.6 Kurtosis

$$KURT(t,\tau) = \frac{e^{r\tau}X(t,\tau) - 4\mu(t,\tau)e^{r\tau}W + 6e^{r\tau}\mu(t,\tau)^2V(t,\tau) - 3\mu(t,\tau)^4}{(e^{r\tau}V(t,\tau - \mu(t,\tau)^2)^2}$$

The formula defines risk-neutral kurtosis as the **normalized** fourth moment of the return distribution under the risk-neutral measure, capturing the fat-tailedness of the risk-neutral return distribution.

2.3 Results

The following section shows the results from calculating the four moments of SPX and A, a constituent of the SP500.

Figure 1: SPX moments

Figure 2: A moments

The left figure shows the four moments for the SPX. Interesting to see, are the skewness and kurtosis. The negative skewness indicates the return distribution has a long left tail, meaning large negative returns happen more frequently or are more extreme than positive returns. The very high kurtosis indicates that extreme results (positive or negative) occur more other than in normal distributed markets.

The right figure shows the moments of A. An average, it has a higher skewness, indicating that high negative returns are, compared to SP500, not that frequently or extreme. The kurtosis, which is for most parts at a low level, indicates that extreme return results in general are rare, with some exceptions.

3 Decomposing implied skewness

3.1 Introduction into systematic and idiosyncratic risk

When analyzing the risk or return characteristics of financial assets, especially in the context of a large index like the SP500, it is important to separate the total observed behavior into two key components: **systematic** and **idiosyncratic** parts.

Systematic risk refers to the portion of an asset's risk or skewness that can be explained by movements in the overall market. This risk affects all stocks and cannot be diversified.

Idiosyncratic risk, on the other hand, is asset-specific and independent of the overall market. It can be caused by company-specific events and is not correlated across stocks.

In the context of implied skewness, this decomposition helps us understand how much of the asymmetry in an individual stock's option-implied return distribution is due to general market-wide factors (*systematic skewness*), and how much is due to firm-specific factors (*idiosyncratic skewness*).

3.2 Formulas

Following the idea of linear factor models, we assume that the skewness of an asset's returns is partly driven by the skewness of the market. More precisely, the **systematic skewness** of stock i is defined as:

$$SystematicSkewness_i = \beta_i^3 \cdot MarketSkewness$$

where: β_i is the implied beta of stock i.

The **idiosyncratic skewness** is the component of the stock's total implied skewness that cannot be explained by the systematic part:

$$IdiosyncraticSkewness_i = Skewness_i - SystematicSkewness_i$$

This decomposition enables a better understanding of whether the observed skewness in a stock is driven by market-wide phenomena or by firm-specific factors.

3.3 Results

To illustrate the decomposition of implied skewness into systematic and idiosyncratic components, we select two representative stocks from the S&P 500 index: **A** and **ZTS**. We examine their implied skewness on three trading days within February 2023: the 1st, 17th, and 28th.

Symbol	Date	Skewness	Systematic Skewness	Idiosyncratic Skewness
A	2023-02-01	-0.816370	-0.025804	-0.790565
A	2023-02-17	-1.856511	-0.025804	-1.830707
A	2023-02-28	-0.943818	-0.025804	-0.918014
ZTS	2023-02-01	-0.607399	-0.277048	-0.330351
ZTS	2023-02-17	-3.908887	-0.277048	-3.631840
ZTS	2023-02-28	-1.343074	-0.277048	-1.066026

Table 1: Decomposition of implied skewness into systematic and idiosyncratic components for selected stocks.

The table above shows stock *A* and *ZTS*. We observe that the idiosyncratic skewness dominates, suggesting that most of the asymmetry in their return distribution is driven by firm-specific factors.

The following figure shows the daily evolution of the average implied skewness across all analyzed stocks in February 2023. The daily averages of total, systematic, and idiosyncratic skewness were calculated for this purpose.

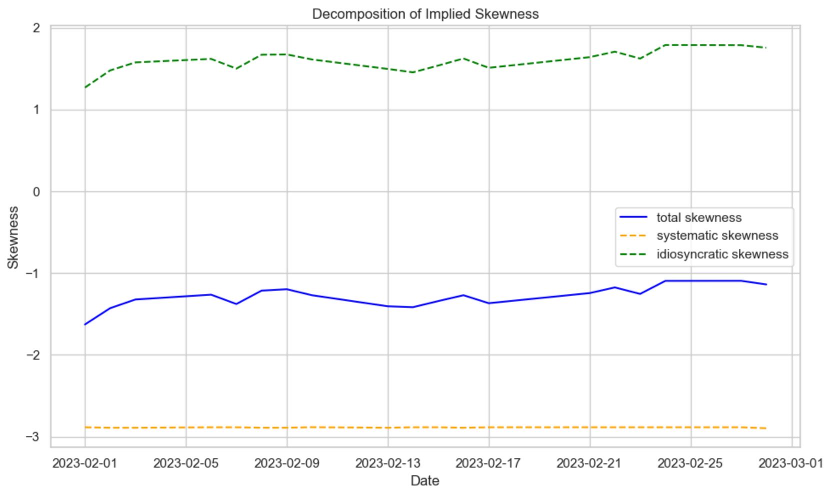

Figure 3: Daily evolution of average implied skewness and its components in February 2023.

The figure clearly shows that the average **systematic skewness** remains relatively constant and highly negative throughout the observed period. This reflects the asymmetric perception of market-wide risk, for example, due to increased demand for protection against extreme negative events.

In contrast, the **idiosyncratic skewness** is consistently positive and significantly higher. This indicates that many individual stocks, despite negative market expectations, exhibit firm-specific factors that positively distort the return distribution. Positive idiosyncratic skewness may reflect optimistic expectations regarding certain companies.

The **total skewness** lies between the two components and results from the combined effects of market and firm-specific influences.

4 Difference Between Index and Single Stock Skewness

4.1 Hypothesis

We hypothesize that the implied skewness of the market index is smaller than that of individual stocks:

$$Skewness_{Index} < Skewness_{SingleStock}$$

This assumption is grounded in the notion that market-wide shocks, such as sudden downturns, create substantial left-tail risk in the index return distribution. In contrast, idiosyncratic shocks affecting individual stocks are more likely to be symmetrically distributed or even positively skewed due to firm-specific upside potential.

4.2 Observation

To verify this, we compare the implied skewness of the SPX with that of 500 individual constituent stocks for February 2023. The results show:

- The average implied skewness of the SPX is **-2.319**.
- The mean skewness across individual stocks is **-1.225**.
- Only **3.8%** of individual stocks exhibit a skewness that is more negative than that of the index.

4.3 Economic Reasoning and Interpretation

There are several theoretical explanations for this result:

1. **Aggregation of Risks:** The market index aggregates both systematic and idiosyncratic risks. While idiosyncratic shocks may average out across firms, systematic shocks—such as recessions or financial crises—generate strong downside risk, leading to persistent negative skewness in the index.
2. **Asymmetric Demand for Protection:** Investors typically demand more downside protection for the overall market than for individual firms, reflected in higher put option prices for indices. This drives more negative implied skewness in index options.
3. **Positive Idiosyncratic Skew:** Many individual stocks exhibit positive or mildly negative skewness due to growth expectations, takeovers, or firm-specific success scenarios. The idiosyncratic component of returns may thus offset the systematic skew in individual stocks.
4. **Portfolio Effects:** While individual stocks may experience random fluctuations in skewness, the index—being a weighted portfolio—is more likely to reflect persistent asymmetries caused by collective behavior, like panic selling or herding in downturns.
5. **Leverage Effects:** Although leverage can create negative skewness at the individual stock level, empirical data (as shown here and in BKM) suggest that the aggregate impact of leverage across firms does not produce individual skews more negative than that of the index.

These findings underline the importance of distinguishing between systematic and idiosyncratic risk when interpreting skewness, and they highlight the limited usefulness of index-based measures when assessing the tail risk of individual equities.