The Brachistochrone Curve: The Path of Least Time

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This experiment aimed to demonstrate that the brachistochrone, a cycloidal curve, is the path of least time between two points A and B under a constant gravitational field. The points were positioned at different vertical heights and horizontal offsets to ensure a realistic descent scenario. The theoretical prediction was that the cycloid yields the fastest descent and was verified through a comparative time-based demonstration involving three paths: a cycloidal track, a straight line, and an extreme curve with a steep initial slope.

Each path was tested under identical conditions using a roller and timer. The measured descent times were then compared to theoretical expectations. The cycloidal path produced the shortest travel time on average, validating the theoretical solution. Measured times were within experimental error of the predicted values. The results confirm that although the cycloid is not the shortest in distance, it optimizes travel time due to the physics of accelerated motion.

Key words: Brachistochrone, cycloid, least-time path, demo, action, Bernoulli.

I. INTRODUCTION AND THEORY

DEFINITON OF THE PROBLEM

An object of mass m is sliding down a curve from point A to point B. We assume that the points have both a vertical and horizontal offshift, as shown in Figure 1. Now a fairly logical question arises: is there a preferential path to take to minimize time? The answer is yes, and the resulting solution is intriguing.

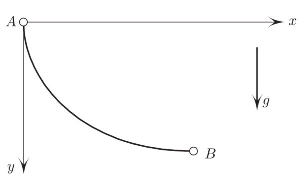

Figure 1: Problem Definition

From Mark Levi's "Solution to the Brachistochrone Problem"

A BRIEF HISTORY OF THE PROBLEM

Johann Bernoulli set this problem as a challenge to the best mathematicians in the world in June, 1696. He gave everybody 6 months to complete the challenge, but no one submitted a solution.

Afterwards, Gottfried Leibniz, a friend of Bernoulli's and one of the fathers of Calculus, pushed Johann to extend the deadline so that foreign mathematicians could submit their solutions as well. The intended target was Isacc Newton, who was inactive at the time and the Warden of the Royal Mint in England.

He got the challenge in the mail and spent the whole day and night solving it, while Bernoulli spent 2 weeks solving the problem. Newton refused to sign his solution, but upon receiving the solution Bernoulli allegedaly said: "I recognize the lion by his claw."

Bernoulli's solution was brilliant and used a phenomenom from optics called "Snell's Law", which governs the behaviour of light when moving from one medium to another.

This solution is easy to grasp, but hard to rigorously prove. For this reason, we have decided to use the Calculus approach, which was the path that Newton took as well.

Before we dive deeper into this question some comments have to be made. We will assume a more ideal system than usual: we will assume that our rollers are point masses with no rolling energy associatted with them. Furthermore, we will assume that friction is negligible, even though it isn't.

The solutions are already quite complex, so we wouldn't want to resort to numeric solutions using programming tools for our report. We found that even with these assumptions the results were accurate.

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THEORY

Let's begin by considering the velocity v of the object:

$$v = \frac{ds}{dt} \tag{1}$$

$ds \equiv$ infinitesimal distance along the curve
$dt \equiv$ infinitesimal time

Now, let's rearrange eq. 1 and solve for dt:

$$dt = \frac{ds}{v}$$

The time travelled is then:

$$\Delta t = \int_A^B dt = \int_A^B \frac{ds}{v} \tag{2}$$

Where we will take A to be at the origin and B to have a position of (x_2, y_2).

Assuming that we start at rest, we can use energy conservation to solve for the velocity in terms of the vertical position y:

$$v(y) = \sqrt{2gy} \tag{3}$$

The infinitesimal displacement element is just the vector sum of the horizontal and vertical displacements:

$$ds = \sqrt{dx^2 + dy^2}$$

Assuming the path is described by $x = x(y)$, we get:

$$ds = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy = \sqrt{1 + x'(y)^2}\, dy \tag{4}$$

$$x'(y) \equiv \frac{dx}{dy}$$

Substituting the results of eq. 3 and Eq. 4 into Eq. 2 yields:

$$\Delta t = \frac{1}{\sqrt{2g}} \int_0^{y_2} \frac{\sqrt{1 + x'(y)^2}}{\sqrt{y}}\, dy$$

Define the integrand as:

$$f(x, x', y) = \frac{\sqrt{1 + x'^2}}{\sqrt{y}} \tag{5}$$

We can use the Euler-Lagrange, which is of the form below (we will not derive the solution in this report):

$$\frac{\partial f}{\partial x} = \frac{d}{dy}\left(\frac{\partial f}{\partial x'}\right)$$

Since of function $f(x, x', y)$ is independent of x, as $x = x(y)$, then we can take the partial derivative and get:

$$\frac{\partial f}{\partial x} = 0 \;\Rightarrow\; \frac{\partial f}{\partial x'} \equiv \text{constant}$$

Solving the PDE above yields:

$$\frac{x'^2}{y(1 + x'^2)} = \frac{1}{2a}$$

We choose the constant in this way, as to incorporate for the radius of the cycloid a. This is similar to the choice of constants for Damped Harmonic Oscillators.

Solving for x', we get:

$$x' = \frac{dx}{dy} = \sqrt{\frac{y}{2a - y}}$$

Solving for x yields:

$$x = \int \sqrt{\frac{y}{2a - y}} dy$$

This is not a trivial integral and requires a trigonometric substitution.

THE CYCLOID: SOLUTION TO THE PROBLEM OF LEAST TIME

To evaluate this integral, use the substitution:

$$y = a(1 - \cos\theta)$$

$$dy = a\, sin\theta\, d\theta$$

We can re-write the integral as:

$$x(\theta) = a \int (1 - \cos\theta)\, d\theta = a(\theta - \sin\theta) + C$$

With the initial condition that we start at the origin $x(0) = 0$, we find:

$$\begin{aligned} x(0) &= a(0 - sin(0) + C = 0 \\ x(0) &= C = 0 \\ C &= 0 \end{aligned}$$

The parametric equations now become:

$$x(\theta) = a(\theta - \sin\theta) \quad (6)$$
$$y(\theta) = a(1 - \cos\theta) \quad (7)$$

These are well-known parametric equations and relate to the equation of a cycloid.

TIME OF DESCENT

Now, let's compute the time of descent along the cycloid.

Let's take the derivative with respect to θ for each variable in the displacement calculation:

$$\frac{dx}{d\theta} = a(1 - \cos\theta) \quad (8)$$
$$\frac{dy}{d\theta} = a\sin\theta \quad (9)$$

The resulting equation will be:

$$\left(\frac{ds}{d\theta}\right)^2 = a^2(1-\cos\theta)^2 + a^2\sin^2\theta$$

$$\left(\frac{ds}{d\theta}\right)^2 = a^2\left[1 - 2\cos\theta + \cos^2\theta + \sin^2\theta\right]$$

$$\left(\frac{ds}{d\theta}\right)^2 = 2a^2(1-\cos\theta) = 4a^2\sin^2\left(\frac{\theta}{2}\right)$$

Thus:

$$\frac{ds}{d\theta} = 2a\sin\left(\frac{\theta}{2}\right)$$

The velocity is:

$$v = \frac{ds}{dt} = \sqrt{2gy} = \sqrt{2ga(1-\cos\theta)}$$

Then:

$$\frac{dt}{d\theta} = \frac{dt}{ds}\cdot\frac{ds}{d\theta} = \frac{2a\sin(\theta/2)}{\sqrt{2ga(1-\cos\theta)}}$$

$$\frac{dt}{d\theta} = \sqrt{\frac{a}{g}}$$

Therefore, the total time is:

$$T = \int_0^\pi \frac{dt}{d\theta}\,d\theta = \sqrt{\frac{a}{g}}\int_0^\pi d\theta = \pi\sqrt{\frac{a}{g}}$$

Where the limits of the integral come from $\theta = 0$, at the top of the curve, and $\theta = \pi$ at the bottom of the curve, as the circle is rolling along the surface.

Although the cycloid isn't the shortest path, it provides the best balance of acceleration and curve length to give the shortest time of descent.

This result shows that the total time is independent of the starting height along the cycloid, depending only on the radius a. This means that no matter where you start on the curve, you will finish the descent in the same amount of time.

II. APPARATUS CONSTRUCTION

The construction of the apparatus was the most extensive and integral component of the project. Our goal was to physically build a Brachistochrone track to experimentally test the principle of least time using actual cycloidal curves.

A. Inspiration

This project was heavily inspired by online science communicators who have brought the beauty of physics to a broader audience. Derek Muller from Veritasium introduced us to the principle of least action and gave us a history on the brachistochrone problem.

Michael Stevens from Vsauce along with Adam Savage built a physical demo to showcase the brachistochrone problem. Their design was the inspiration of our apparatus and we heavily studied how they built each component and made it work together.

Their demo is shown below:

Figure 2: Inspiration for Demo

From Vsauce's "Brachistochrone"

B. Starting Phase

We began the design process by brainstorming and attempting to deconstruct the apparatus conceptually, aiming to recreate it from scratch. Since no comprehensive, step-by-step guides were available online for replicating the demonstration precisely, we conducted an in-depth analysis of the original demonstration video. This involved repeatedly reviewing the footage to identify material types, dimensions, and structural details that could inform our reconstruction strategy.

At the outset, our team had limited experience with fabrication tools or demonstration model construction, which presented several unforeseen challenges. While the setup may appear straightforward at first glance, it involved a number of intricate design considerations and technical obstacles. We developed a couple of sketches to give us an idea of what our prototype would look like:

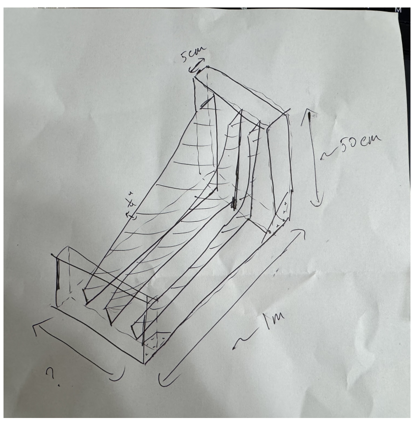

Figure 3: Initial Sketch for Prototype

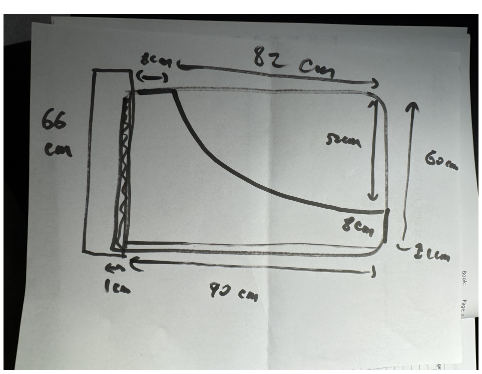

Figure 4: Initial Sketch for Cycloid Curve

C. Building the Frame

We found a lot of support in a trades instructor named Maciej at the Chilliwack Campus of UFV.

He eagerly took us on and helped out a lot during the project.

After meeting with him for the first time for the project, we quickly discarded the idea of building the frame from wood. It was much more time cutting the exact measurements required for the frame than with metal beams.

Figure 5: Extrusion table

We disassembled an old aluminum extrusion table to get the parts for our frame. So, after doing that, we finally had some metal extrusion beams, that would easily slide in together and be connected by metal corners from the table, like how we see in our final design. Additionally, a scrap part of one of the metal beams was used as a stopper pole for the rollers at the end.

After day one, we had a frame:

Figure 6: Completed Frame with Burnt Plexiglass Sheets

D. Acrylic Sheets

For the cycloidal track, we required precisely shaped curves. Initially, Carmen provided a few acrylic sheets for prototyping. To fabricate more curves, we purchased additional sheets online. However, we mistakenly ordered plexiglass instead of acrylic. They are the same material, but one is good for a laser cutter and the other is not.

Using Python scripts, we parameterized the cycloidal curve and converted it into a format readable by Fusion 360 via AutoCAD (DXF file generation). We did this by computing 10,000 points manually using parametric equations.

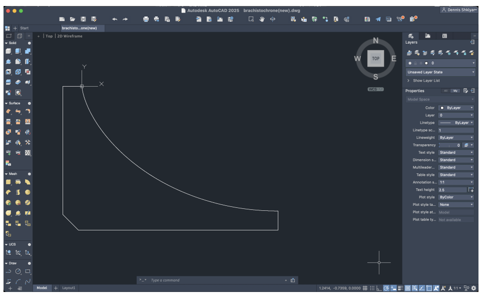

Figure 7: Cycloid draft in AutoCAD

```python
import numpy as np
import ezdxf

# Given radius
R = 0.261

# Define parametric equations
def brachistochrone(t):
    x = R * (t - np.sin(t))
    y = -R * (1 - np.cos(t))
    return x, y

# Generate points with t in the range [0, π] with 10,000 points
t_values = np.linspace(0, np.pi, 10000)  # 10,000 points for very smooth curve
points = [brachistochrone(t) for t in t_values]

# Create DXF file
doc = ezdxf.new()
msp = doc.modelspace()

# Add polyline (smooth curve)
msp.add_lwpolyline(points)

# Save DXF file
doc.saveas("brachistochrone.dxf")
print("DXF file saved as brachistochrone.dxf")
```

```
DXF file saved as brachistochrone.dxf
```

Figure 8: Python Code for generating cycloid in AutoCAD

We used a laser cutter to achieve the accuracy we needed. The acrylic sheets turned out very nicely, but the plexiglass melted on the curve because of the high temperature. This required us to perform manual sanding.

Below is a photo of process of us using a laser cutter:

Figure 9: Laser cutter

We also had another problem:

We made a big mistake by placing the cutting surface right on the edges. This made a couple of the sheets shorter than the others as the plate wasn't placed at the very edges of the laser cutter. This required more refinements using a bandsaw to make them the same height and length.

So in the future, we will draft our documents as shown below:

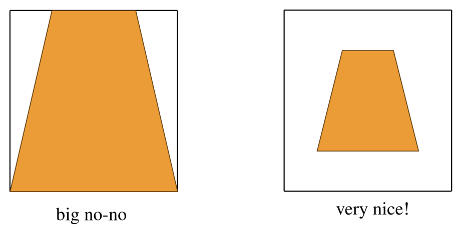

Figure 10: How to draft shapes

E. Painting and Aesthetic Finishing

To address discoloration from the failed plexiglass cuts and to improve visual clarity, the sheets were spray-painted in UFV colors. This improved the overall finish of the apparatus and made the curves easier to identify during presentations and measurements.

F. Rollers and Holders

Initially, we planned to construct the rollers from Delrin, which is a lightweight, durable plastic.

However, issues arose with the CNC machine's compatibility with Delrin. As a result, we pivoted to using solid brass rollers. While heavier, the brass rollers offered excellent consistency and low friction.

The holders posed an additional challenge. The original design, which was acrylic tubing glued to a wooden base was too time-consuming. Ultimately, Maciej fabricated a metal holder out of scrap metal.

The result for the holder is shown below:

Figure 11: Holder

And this is the roller:

Figure 12: Roller

G. Small Nuances

Several refinements were made during assembly. A rubber stopper was added at the base of the track to prevent metal-on-metal contact between the roller and end-stop, reducing both mechanical shock and noise.

The holder design was modified to ensure all rollers could be released from a common, well-defined start position to ensure consistency across trials. They didn't start at the same position because of the geometry of the holder, so we had to put some padding in to ensure that they start at the same height.

H. Final Design

After a month of work and multiple trips to Chilliwack, we put everything together and these were the results:

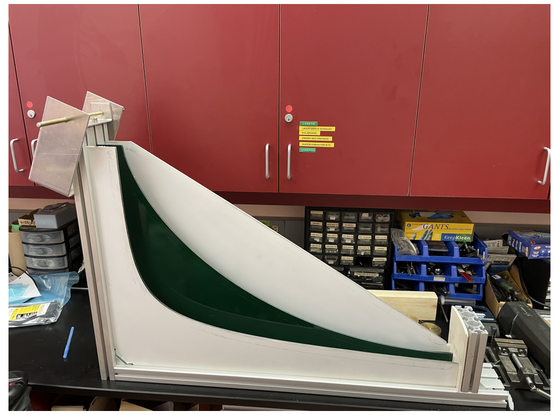

Figure 13: Completed 3-curve demo

Figure 14: Completed cycloid demo

We had two working demos: one to showcase that the cycloid was the path of least time and one to showcase that no matter which point you start on the cycloid, the time taken to descend will always be the same.

III. EXPERIMENTAL PROCEDURE

The demo's were planned ahead of time to demonstrate two phenomenons: to show that the cycloid is the path of least time, i.e. it's the brachistochrone, and to showcase that the cycloid is also an isochrony, i.e. it doesn't matter where you start the descent on the curve, you will finish in the same amount of time.

The first procedure focused on measuring times of descents for the three curves. The descent times were measured using my phone (Iphone 15 Pro Max), as the descent times were too fast to measure accurately with a stopwatch.

We opted to measure each descent separately and took 10 trials for each curve. After recording a descent, we would look at the exact frame Dario would release the roller to the frame it would end up hitting the rubber stopper.

The trials were recorded at 60 fps, but an Instrumentational error wasn't provided for an Iphone camera. Since 1 frame is the lowest we can go, we opted to have 1 frame as our IE.

We made sure that all of the curves were smooth, so we sanded it flush one more time. We sat straight away from the apparatus and recorded our videos to not introduce paralax error.

We also had to make sure that the roller started at rest and that Dario didn't give a push at the beginning, which ruled out some trials.

For our second procedure, we changed the plexiglass sheets to our acrylic sheets and measured 5 different times at 5 different points along the cruve. We had to make sure to only measure in the top half of the curve, as any lower friction would be sufficient.

We proved that the cycloid is an isochrony in this process and also created a demo with 2 identical curves to show that they do arrive at the same time at the end regardless of the starting position.

Our goal was to compare these results to the theoretical time of descent, which we calculated to be:

$$T = \pi\sqrt{\frac{a}{g}}$$

Where we measured the radius of the cycloid a to be:

$$a = 0.260 \pm 0.001\ m$$

The error comes from us measuring the radius with a ruler after it went through the laser cutter.

This radius was chosen to fit the dimensions of our cycloid curve. The circle would roll through an angle of $\theta = \pi$.

So the width would just be the arc length the circle has travelled, which is:

$$W = \pi R$$

We wanted our width to be $W = 0.82\ m$, so our radius would be:

$$R = \frac{W}{\pi} \approx 0.26\, m$$

IV. RESULTS

A. Procedure A

We calculated that the theoretical time of descent was:

$$T_{theor} = 0.511 \pm 0.001\ s$$

The error was calculated using the calculus method with the errors for the radius a and the gravitational acceleration g.

We saw that the average time of descent for each curve was:

Straight line: $T_s = 0.76 \pm 0.04\ s$

Extreme curve: $T_e = 0.58 \pm 0.04\ s$

Cycloid: $T_c = 0.54 \pm 0.04\ s$

The errors were calculated using error propagation for the instrumentational and observational errors of the time.

We can see that on average the cycloid is the fastest curve. An argument can be made that the extreme curve is just as fast within error. However, we saw that due to the geometry of the extreme curve, the roller is in freefall in the beginning of the descent, which gave it an advantage for the descent.

B. Procedure B

We put the roller on 5 random points (on the upper half of the cycloid curve) to prove that it's an *isochrony.*

These are the following times we measured:

All of the values are within error, which proves that the cycloid is an isochrony.

Trial #	Time, t, ±0.04 s
1	0.55
2	0.55
3	0.57
4	0.60
5	0.55

The error is the same as Procedure A and the times were recorded in the same way.

Straight line: $T_s = 0.76 \pm 0.04\ s$

Extreme curve: $T_e = 0.58 \pm 0.04\ s$

Cycloid: $T_c = 0.54 \pm 0.04\ s$

V. CONCLUSION

This experiment confirmed the two theoretical relationships we were after: that the cycloid is the path of least time and that the cycloid is an isochrony.

Furthermore, we built our own apparatus to demonstrate these effects.

The theoretical descent time for the cycloid was shown to be:

$$T_{theor} = 0.511 \pm 0.001\ s$$

The measured values for the 3 curves were shown to be:

Straight line: $T_s = 0.76 \pm 0.04\ s$

Extreme curve: $T_e = 0.58 \pm 0.04\ s$

Cycloid: $T_c = 0.54 \pm 0.04\ s$

The experimental and theoretical values aren't close, as friction and rolling energy were not accounting, but they are within error.

The isochronus property was proven by releasing the roller in 5 different positions along the cycloid curve and proved that all of them were within experimental error.

Future improvements on the apparatus may be made to have all of the curves made of laserable acrylic to ensure even more accurate results.

VI. ACKNOWLEDGMENTS

I would like to express my sincere gratitude to the UFV Physics Department for providing financial support for this project. I am especially thankful to Carmen Herman for her dedicated time, effort, and continuous support throughout the course of this work.

Special thanks are extended to Maciej Kaczor for his invaluable assistance in constructing the experimental apparatus. His expertise and guidance were essential to the success of this project.

I would also like to acknowledge my partners, Madhav Garg and Dario Lopez, for their collaborative efforts in designing the apparatus and conducting the experiment.

VII. APPENDIX

A. Raw Data:

Trial 1	Straight line time, t_s ±0.04s	Cycloid curve, time t_c ±0.04s	Extreme curve, time t_e ±0.04s
1	0.75	0.55	0.55
2	0.74	0.54	0.59
3	0.76	0.55	0.59
4	0.74	0.57	0.57
5	0.75	0.55	0.59
6	0.77	0.53	0.56
7	0.76	0.53	0.58
8	0.73	0.53	0.57
9	0.75	0.53	0.59
10	0.80	0.56	0.56
Average:	0.76	0.54	0.58

Trial #	Time, t, ±0.04 s
1	0.55
2	0.55
3	0.57
4	0.60
5	0.55

B. Error Analysis

Error type	t_a, s	t_b,s	t_c	t, s
Observational	0.03	0.03	0.03	0.03
Instrumental	0.02	0.02	0.02	0.02
Standard deviation	0.02	0.01	0.02	0.02
total	0.04	0.04	0.04	0.04

C. Error Propagation

$$\delta T = \sqrt{\left(\frac{\partial T}{\partial a}\right)^2 (\delta a)^2 + \left(\frac{\partial T}{\partial g}\right)^2 (\delta g)^2} \tag{A1}$$

$$\delta T = \sqrt{\left(\frac{\pi}{2\sqrt{ag}}\right)^2 (\delta a)^2 + \left(-\frac{\pi}{2} \cdot \frac{\sqrt{a}}{g^{3/2}}\right)^2 (\delta g)^2} \tag{A2}$$

$$\delta T = 0.001\,\mathrm{s}$$

[1] J. R. Taylor, Classical Mechanics (University Science Books, Sausalito, CA, 2005).

[2] "The Brachistochrone", Vsauce, https://www.youtube.com/watch?v=skvnj67YGmwt=1426s

[3] "The Closest We've Come to a Theory of Everything", Veritasium, https://www.youtube.com/watch?v=Q10$_s rZ - pbst = 486s$