Holographic Methods for Condensed Matter Physics

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Introduction

The AdS/CFT correspondence was conjectured in 1998 by Juan Maldacena.
It will here be used to model a high T_c superconductor by a gravitational theory.
The application will be described and the results will be presented.
The work initially closely follows that of Sean Hartnoll [1].

Outline

- What is AdS?
- The AdS/CFT correspondence
- High T_c superconductors
- Bulk model of superconductor
- Application of the correspondence
- Results: condensate developement, conductivity
- Extended model
- Results: conductivity, fit of Drude model
- Conclusion

Anti de Sitter space (AdS)

Einsteins equation with negative cosmological constant Λ

$$R_{ab} - \frac{1}{2} g_{ab} R + g_{ab} \Lambda = \kappa T_{ab} \tag{1}$$

has AdS as solution, just like $\Lambda = 0$ gives flat space.

$$g_{ab} \mathrm{d}x^a \mathrm{d}x^b = \frac{L^2}{z^2} \left(\frac{\mathrm{d}z^2}{f(z)} - f(z) \mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 \right). \tag{2}$$

where $f(z) = 1 - z^3 z_h^{-3}$.

The AdS/CFT correspondence

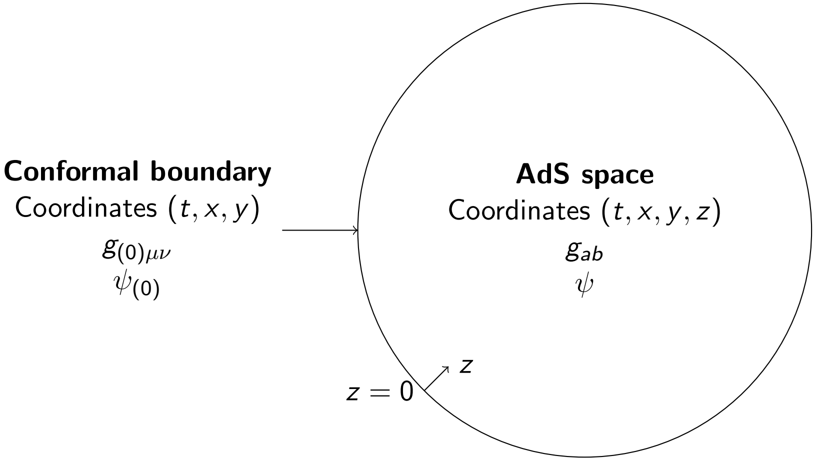
Gravitational theory in AdS space / Conformal field theory

$$Z_{\mathrm{bulk}}(\delta\psi_{(0)}) = \left\langle \exp(i \int \mathrm{d}^d x \sqrt{g_0}\, \delta\psi_{(0)} \mathcal{O}) \right\rangle_{\mathrm{CFT}} \tag{3}$$

The equations of motion in the bulk give the following boundary behaviour

$$\psi(z) = \left(\frac{z}{L}\right)^{\Delta_{(0)}} \psi_{(0)} + \left(\frac{z}{L}\right)^{\Delta_{(1)}} \psi_{(1)} + \dots \tag{4}$$

Overview


Conformal boundary
Coordinates (t, x, y)
$g_{(0)\mu\nu}$
$\psi_{(0)}$
AdS space
Coordinates (t, x, y, z)
g_{ab}
ψ
z
$z = 0$

CFT operators

Background fields in the CFT are sources to operators

$$\mathcal{O} = \frac{\delta S_{\mathrm{CFT}}}{\delta \psi_{(0)}} \tag{5}$$

Expectation values of these can be calculated as

$$\begin{aligned}
& - i \frac{\delta \log Z_{\mathrm{bulk}}(\psi_{(0)})}{\delta \psi_{(0)}(x)}|_{\psi_{(0)}=0} \\
&= -i \frac{\delta \log \left\langle \exp(i \int \mathrm{d}^d x \sqrt{g_{(0)}} \psi_{(0)} \mathcal{O}) \right\rangle_{\mathrm{CFT}}}{\delta \psi_{(0)}(x)}|_{\psi_{(0)}=0} \\
&= \frac{\left\langle \mathcal{O}(x) \exp(i \int \mathrm{d}^d x \sqrt{g_{(0)}} \psi_{(0)} \mathcal{O}) \right\rangle_{\mathrm{CFT}}}{\left\langle \exp(i \int \mathrm{d}^d x \sqrt{g_{(0)}} \psi_{(0)} \mathcal{O}) \right\rangle_{\mathrm{CFT}}}|_{\psi_{(0)}=0} \\
&= \langle \mathcal{O}(x) \rangle_{\mathrm{CFT}}
\end{aligned} \tag{6}$$

Classical approximation

Boundary gauge theory with many colors, large N.

$\Longrightarrow$

Classical bulk theory.

We do not even have an N. A similar effect is expected for certain strongly coupled gauge theories. See e.g. *Holographic duality with a view toward many-body physics* by John McGreevy for a motivation of why we can assume a classical bulk.

This is a major headache for AdS/CFT, the bulk theory should be a stringy quantum gravity theory without the large N and not very useful.

The strength of the AdS/CFT approach to condensed matter physics comes from that we can do otherwise intractable calculations on strongly coupled systems.

Classical Approximation for Obtaining Expectation Values

The way to calculate expectation values shown earlier needed the bulk partition function.
This can be obtained in a classical limit as

$$Z_{\mathrm{bulk}}(\psi_{(0)}) = C \exp(iS_c) \tag{7}$$

S_c is the saddle point action of the Lagrangian. We then have

$$\langle \mathcal{O}(x) \rangle_{\mathrm{CFT}} = \frac{\delta S_c(\psi_{(0)})}{\delta \psi_{(0)}(x)}\Big|_{\psi_{(0)}=0} \tag{8}$$

This lets us calculate CFT expectation values by solving the bulk equations of motion.

Expectation Values from Boundary Behaviour of Bulk Fields

Using the equations of motion and boundary behaviour of fields one arrives at

$$\langle \mathcal{O}(x) \rangle_{\mathrm{CFT}} = \frac{\delta S_c(\psi_{(0)})}{\delta \psi_{(0)}(x)} |_{\psi_{(0)}=0} = \frac{2\psi_{(1)}}{L} \tag{9}$$

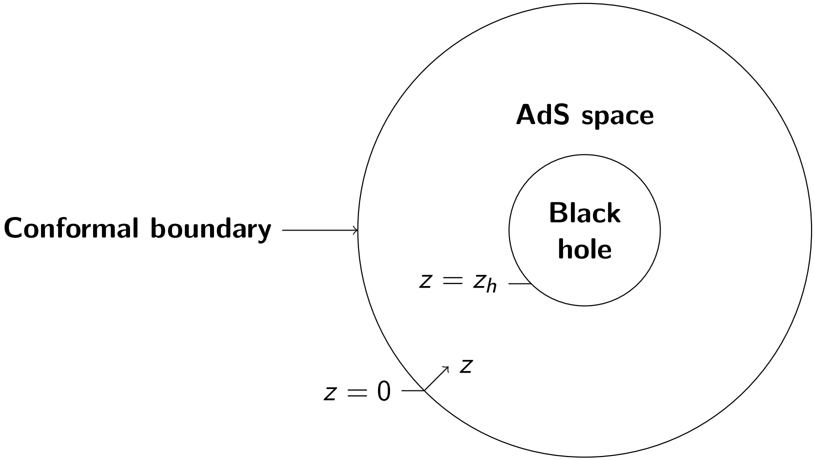
Suitable boundary terms to the Lagrangian must be considered for arriving at this expression. Read my report for more details :). Similar expressions can be obtained for tensor fields, e.g.

$$\langle J_a \rangle_{\mathrm{CFT}} = \frac{A_{a(1)}}{L} \tag{10}$$

Temperature

Field theory expectation values at a finite temperature are calculated using a Euclidean path integral with periodic time.
This corresponds to a metric periodic in Euclidean time on the AdS side of the correspondence.
The solution to the Einstein equation satisfying this boundary condition is a black hole in AdS space.

Black Hole



High T_c Superconductors

Two-dimensional layered materials.
Much higher T_c (>130K) than what is allowed by BCS theory ($\approx$ 30K).
No satisfying explanation of the phenomenon yet, possibly due to strong coupling preventing a perturbative description and numerical simulations.
Strongly coupled systems can be modeled by AdS/CFT.

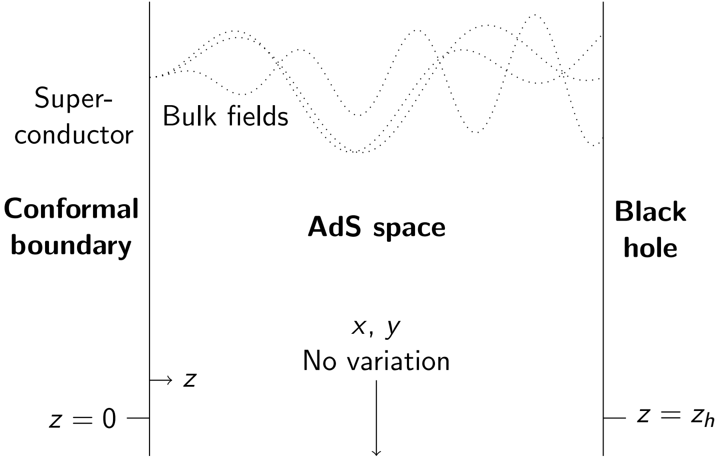
Effect not expected to depend on lattice vibrations, pure electromagnetic phenomenon?
Superconductivity possibly occurs near quantum critical point and thus exhibits scale invariance.
Breaking of U(1) gauge symmetry as in BCS case.

Formulating the AdS Theory Corresponding to a High T_c Superconductor

ψ_0 - Scalar field appearing below T_c.
A_{0a} - Electromagnetic potential.
No phonons, no lattice, no spatial variation, no magnetic field.

CFT	AdS
1+2 dimensions	1+3 dimensions
temperature T	horizon at $z_h = \frac{3}{4\pi T}$
$\mathcal{O}$, ψ_0	$\psi(z)$
J_a, $A_{a,(0)}$	$A(t,z)$
Strongly coupled quantum theory	**Classical theory**

Overview



Super-
conductor
Bulk fields
Conformal
boundary
AdS space
Black
hole
x, y
No variation
z
z = 0
z = z_h

Bulk Theory

A bulk theory Lagrangian of the following form has been used.

$$\mathcal{L} = \quad \frac{1}{2\kappa}\left(R - 2\Lambda\right) - \frac{1}{4}F_{ab}F^{ab} - m^2|\psi|^2 - |D_a\psi|^2 \tag{11}$$

Includes the lowest order terms obeying symmetry requirements. A higher order term will later be included.

Equations of Motion

The equations of motion are obtained by varying the Lagrangian with respect to the different fields,

$$\begin{aligned} \left(m^2 - \nabla^2 + q^2 A^2 + iq(\nabla_a A^a)\right)\psi = 0 \\ -\nabla_a F^{ab} + 2q^2|\psi|^2 A^b + iq\left(\overline{\psi}\nabla^b\psi - \psi\nabla^b\overline{\psi}\right) = 0. \end{aligned} \tag{12}$$

After using the metric, translational invariance, harmonic time-dependence, infinitesimal A_x, and making a choice of gauge we have

$$\begin{cases} \Big(q^2 z^2 \phi^2 - L^2 m^2 f + zf(zf' - 2f)\partial_z + z^2 f^2 \partial_z \partial_z\Big)\psi = 0 & (13) \\ \Big(-2q^2\psi^2 L^2 + z^2 f \partial_z \partial_z\Big)\phi = 0 & (14) \\ \Big(-2q^2\psi^2 L^2 f + z^2\omega^2 + z^2 ff'\partial_z + z^2 f^2 \partial_z\partial_z\Big)A_x = 0. & (15) \end{cases}$$

This is a system of non-linear ordinary differential equations.

Solving Ordinary Differential Equations

There is a trivial solution to the equations

$$\begin{cases} \psi(z) = 0 \\ \phi(z) = \mu(1 - z/z_h) \\ A_x(z) = \left[\exp\left(-\sqrt{3} \tan^{-1} \frac{z_h + 2z}{z_h\sqrt{3}} \right) \frac{z_h - z}{\sqrt{z^2 + zz_h + z_h^2}} \right]^{\frac{i\omega z_h}{3}} \end{cases}$$

Finding an analytical solution with $\psi \neq 0$ is hard so a numerical integrator has been used for this.

Results of Scalar Operator

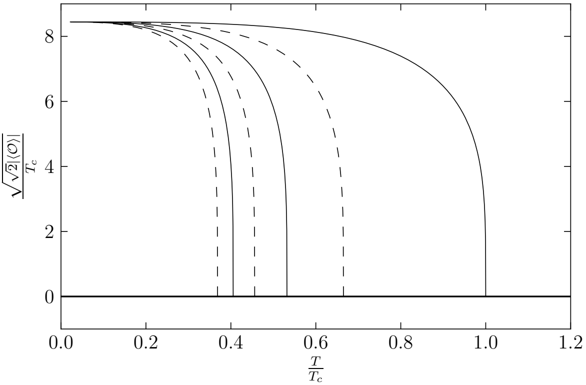


Figure: Expectation value of CFT operator $\mathcal{O}$ at different T and constant μ. The multiple curves correspond to multiple solutions at the same temperature. The dashed lines have different signs of the expectation value of $\mathcal{O}$ and the horizon boundary condition $\psi(z_h)$. Further solutions (here ommited) are obtained for lower temperature following the trend shown here.

Results of Conductivity

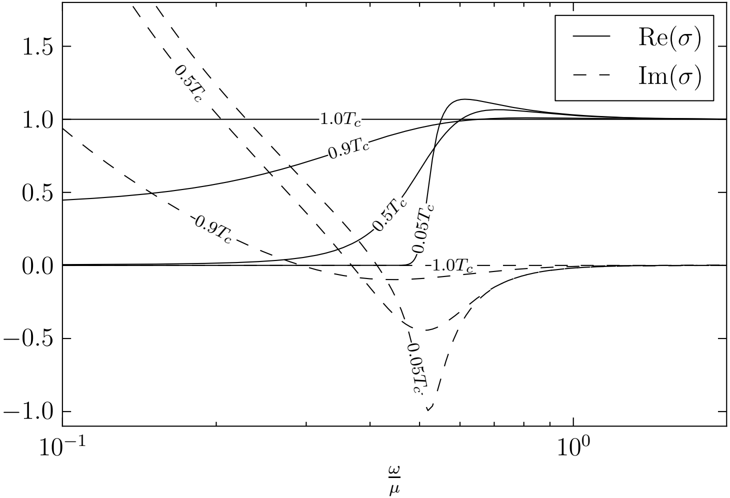


Figure: Real and imaginary part of the conductivity for different temperatures. ρ is here constant.

δ-function at $\omega = 0$

The Kramers-Kronig relation relates the real and imaginary part of response functions of causal systems. This can be used to show that there is a δ-function in the conductivity at $\omega = 0$.

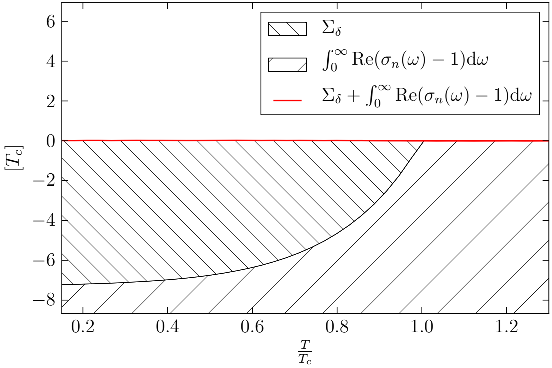
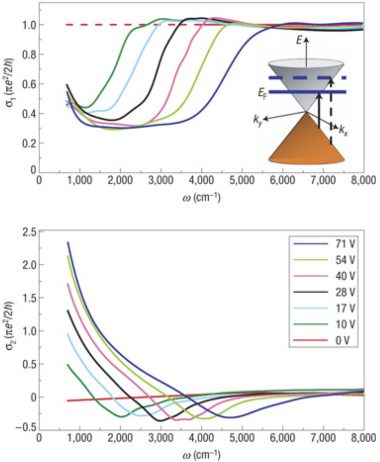
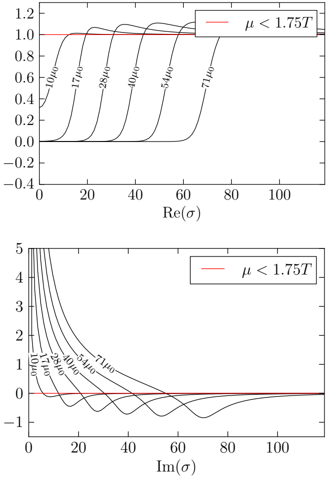


Figure: The two contributions to the integral in the modified Ferrell-Glover-Tinkham sum rule for different temperatures. The red line is expected to be precisely at 0 for perfect numerics

Comparison with Experiments on Graphene





$$\mu_0 = 1.9T$$

1

[1] Image stolen from *Dirac charge dynamics in graphene by infrared spectroscopy*, Li et al. ,Nature Physics

Motivating Extension to Lagrangian

Horowitz et al. [2] have studied the same Lagrangian but with a varying background electrical potential to model a periodic lattice.

They obtained a Drude peak for low frequencies.

Drude model

The charge carriers are assumed to obey the equation of motion

$$\frac{\mathrm{d}v}{\mathrm{d}t} = \frac{q}{m}E - \frac{1}{\tau}v. \tag{16}$$

This gives the conductivity

$$\sigma(\omega) = \frac{J(\omega)}{E(\omega)} = \frac{\sigma_0}{1 - i\tau\omega}. \tag{17}$$

Extended Lagrangian

Myers et al. [3] have studied what higher order corrections of F could be added to the Lagrangian.
These were previously studied by Tobias Wenger, Chalmers University of Technology. One induced what looked like a Drude peak at low frequencies.

$$\mathcal{L}_{\text{ext}} = \mathcal{L} + \alpha_2 F^a{}_b F^b{}_c F^c{}_d F^d{}_a \tag{18}$$

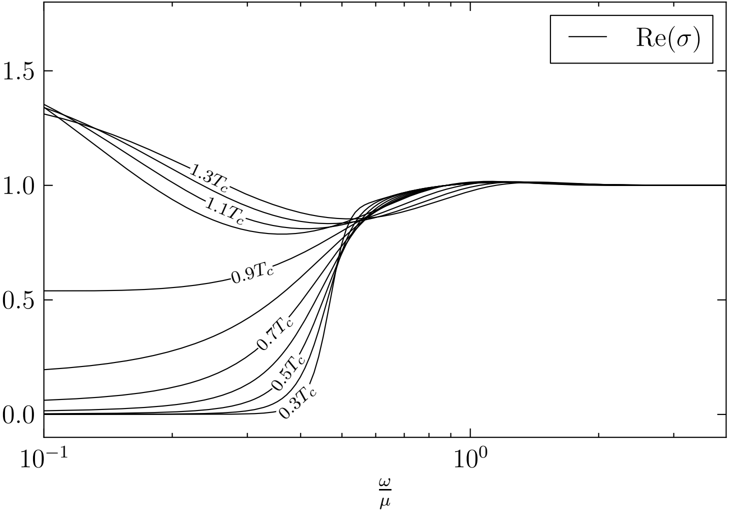


Figure: Real part of the conductivity for different temperatures using the extended Lagrangian with $\alpha_2 = 0.01L^4$. ρ is here constant.

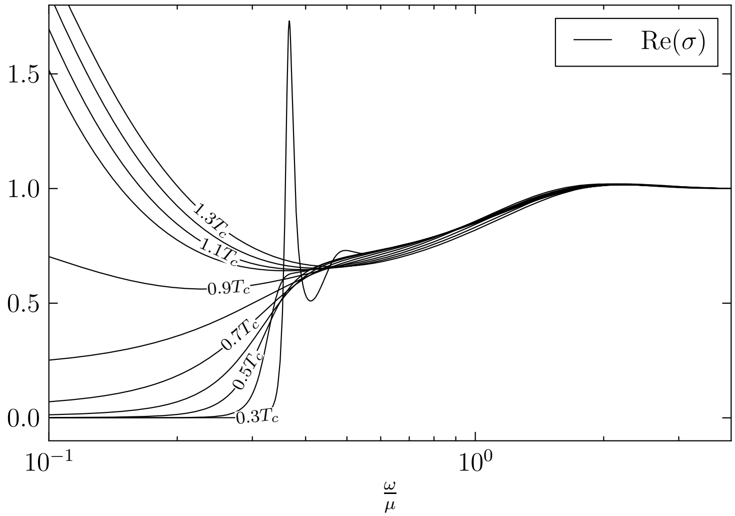


Figure: Real part of the conductivity for different temperatures using the extended Lagrangian with $\alpha_2 = 0.1L^4$. ρ is here constant.

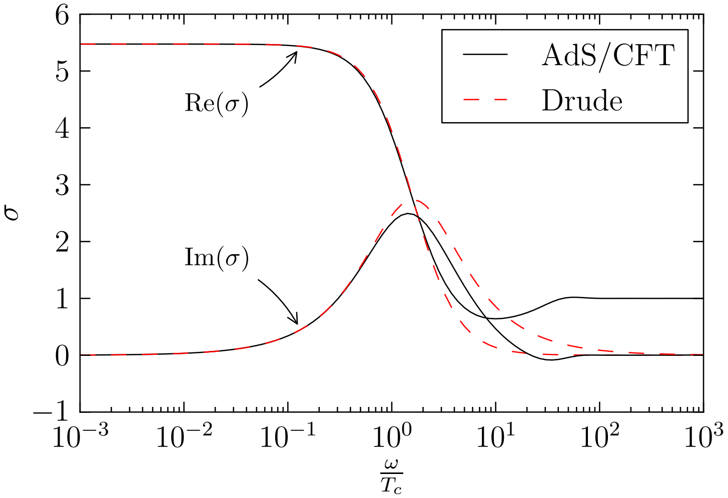


Figure: Conductivity for $\alpha_2 = 0.1L^4$ and $T = T_c$ together with Drude model fit.

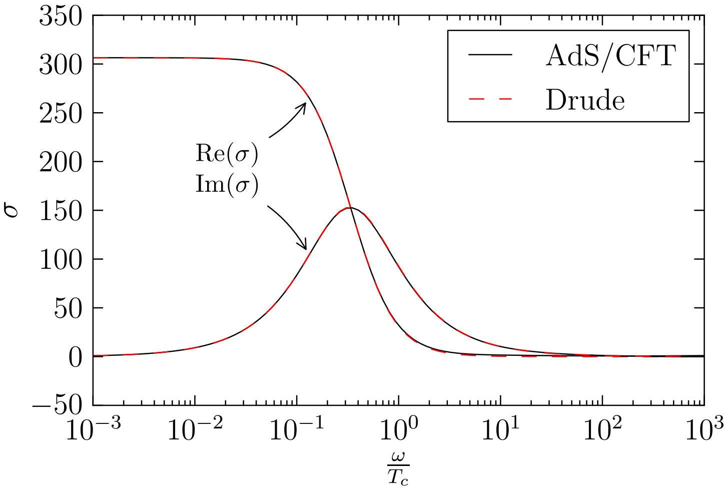


Figure: Conductivity for $\alpha_2 = 10L^4$ and $T = T_c$ together with Drude model fit.

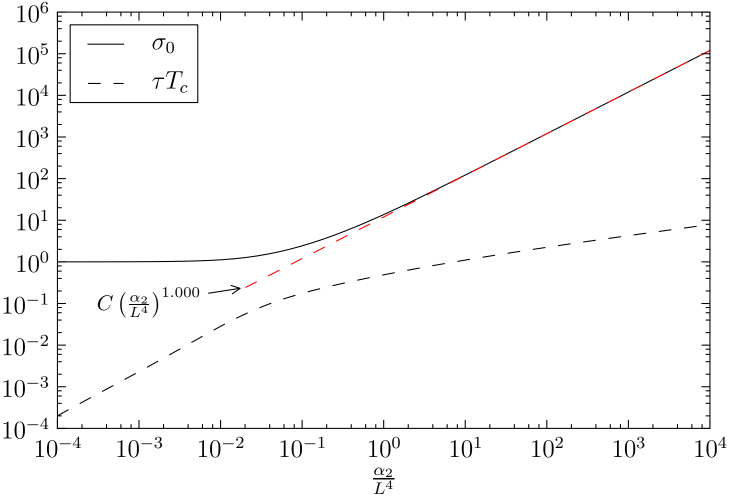


Figure: Drude parameters as functions of α_2 at $T = 2T_c$.

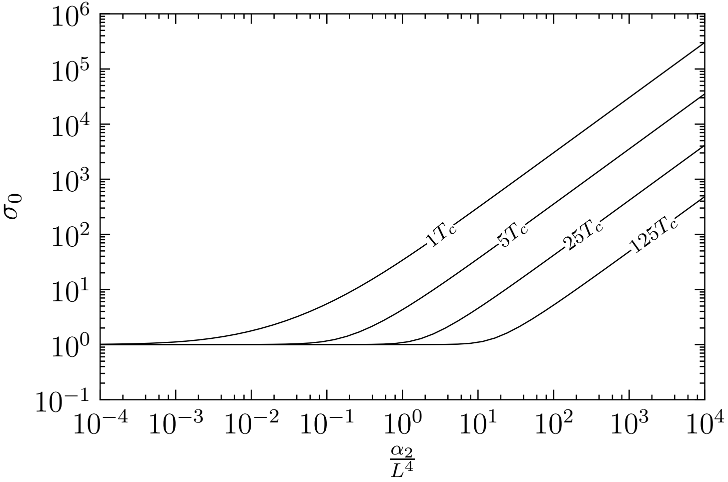


Figure: $\omega \to 0$ limit of the conductivity as a function of α_2 for different temperatures.

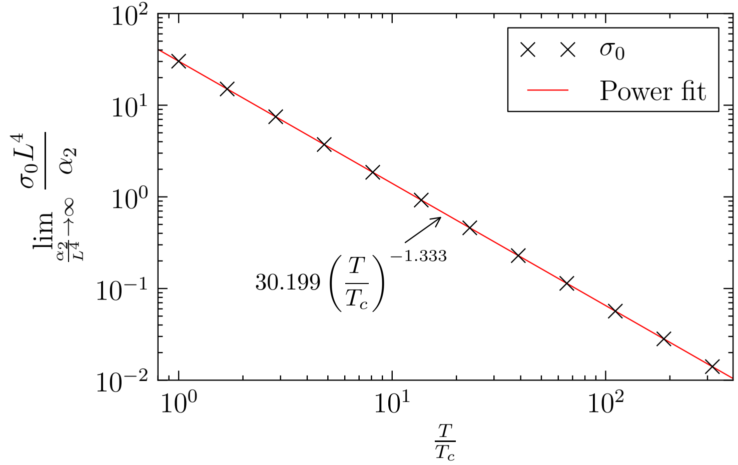


Figure: $\sigma_0 L^4/\alpha_2$ as a function of temperature for large α_2.

This analysis gives the asymptotic behaviour of the DC-conductivity

$$\sigma_0 = C \frac{\alpha_2}{L^4} \left(\frac{T}{T_c} \right)^{-4/3} . \tag{19}$$

This gives a $T^{4/3}$ dependence on the scattering rate $1/\tau$, whereas a T^1 dependence is observed experimentally for the cuprates [4]. Our way of obtaining the Drude peak is computationally much simpler than the periodic lattice and might be a useful effective description.

Summary

- We have modeled a high T_c superconductor using the AdS/CFT correspondence.
- An scalar field coupled to an electromagnetic field has been used.
- A superconducting phase and conductance gap is shown to appear below T_c
- The conductivity sum rule is satisfied indicating correct and accurate calculations
- An addition is made to the Lagrangian giving a peak in the conductivity
- The extended Lagrangian gives a conductivity closer to experiments above T_c since we have a conductivity peak around $\omega = 0$
- The Drude model gives a surprisingly accurate fit to this peak

Questions? (Please not too hard[2])

[2] large N...

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