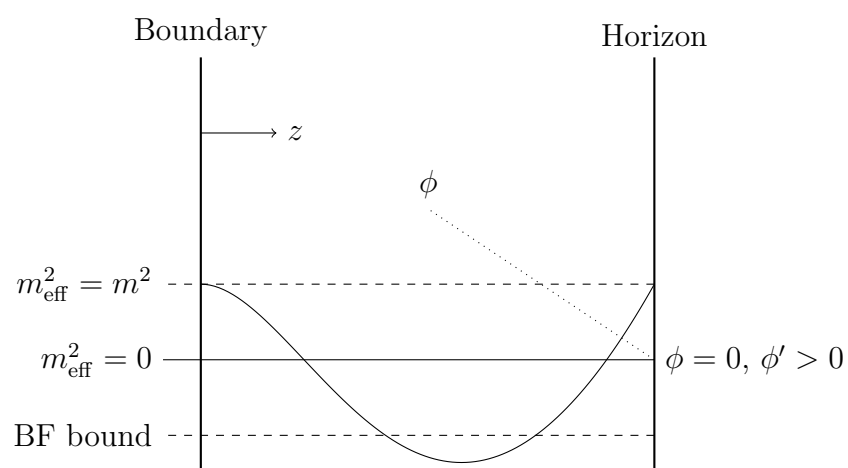


Holographic Superconductivity

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Chapter 1

Introduction

The AdS-CFT correspondence was conjectured in 1997 [1]. It relates the physics of a string theory in Anti de Sitter space (AdS) to a conformal field theory (CFT) on the boundary of the AdS space.

Strong coupling

High TC Superconductors

1.1 The Correspondence

The correspondence can be formulated by the GKPW equation [2]

$$Z_{\text{CFT}} = Z_{\text{strings}} \tag{1.1}$$

where Z_{CFT} is the partition function of the boundary theory and Z_{strings} is the partition function of the bulk theory. The relation between the Lagrangians of the theories is unknown but the boundary values of fields in the bulk theory correspond to fields in the CFT. TODO Lorentz invariance, relativity, causality, locality?

Chapter 2

Application to Two-Dimensional Electron Condensates

2.1 A Lagrangian

TODO, Strong coupling, classical limit. TODO, motivate, reference. The Lagrangian used is

$$\mathcal{L} = \frac{1}{2\kappa} (R - 2\Lambda) - \frac{1}{4} F_{ab} F^{ab} - m^2 \psi \bar{\psi} - D_a \psi \overline{D^a \psi} + \gamma C_{abcd} F^{ab} F^{cd} + \alpha_1 (F_{ab} F^{ab})^2 + \alpha_2 F_b^a F_c^b F_d^c F_a^d \quad (2.1)$$

TODO, describe term by term. The first term is an Einstein-Hilbert term with a cosmological constant Λ . where D_a is the gauge covariant derivative $D_a = \nabla_a - iqA_a$ and $F_{ab} = \partial_a A_b - \partial_b A_a$. This Lagrangian is invariant under a U(1) gauge transformation

$$\psi \rightarrow e^{i\theta(x)} \psi \quad (2.2)$$

$$A_a \rightarrow A_a + \frac{1}{q} \nabla_a \theta(x). \quad (2.3)$$

This lets us make a choice of gauge, $\nabla_a A^a = 0$, the Lorentz gauge. The gauge is still not completely fixed, a gauge transformation $\theta(x)$ such that $\nabla_a \nabla^a \theta(x) = 0$ can still be done without violating the gauge condition. The Lagrangian is also evidently Lorentz invariant imposing Lorentz invariance

of the boundary theory. This Lagrangian also possess conformal invariance. This means that the Lagrangian is unchanged by the transformation $g_{ab} \rightarrow f(x)g_{ab}$.

2.2 Symmetry Assumptions

2.3 Equations of motion

Describe why ψ can be considered real.

First consider a system with $A_r = A_x = A_y = 0$. The field equations obtained by varying the fields ψ, ϕ are respectively

$$\begin{aligned}
0 = \psi & \left(2L^4 m_2 z^3 z h^3 - 2L^4 m_2 z h^6 + 2L^2 \phi^2 z^2 z h^6 + 2L^2 z^5 z h^3 \text{Ax}^2(z, t) - 2L^2 z^2 z h^6 \text{Ax}^2(z, t) \right) \\
& + (2L^2 z^8 - 4L^2 z^5 z h^3 + 2L^2 z^2 z h^6) \frac{\partial^2}{\partial z^2} \psi \\
& + (2L^2 z^7 + 2L^2 z^4 z h^3 - 4L^2 z z h^6) \frac{\partial}{\partial z} \psi
\end{aligned} \tag{2.4}$$

$$\begin{aligned}
0 = \phi & \left(2L^2 \psi^2 z^3 z h^3 - 2L^2 \psi^2 z h^6 \right) \\
& + (z^8 - 2z^5 z h^3 + z^2 z h^6) \frac{\partial^2}{\partial z^2} \phi.
\end{aligned} \tag{2.5}$$

TODO, boundary conditions.

These equations have the trivial solution

$$\begin{aligned}
\psi &= 0 \\
\phi &= \mu - \mu \frac{z}{z_h}
\end{aligned} \tag{2.6}$$

2.4 Partition Function

The partition function is a concept from statistical physics. It is for a quantum-mechanical system defined as

$$Z(\beta) = \text{tr}(e^{-\beta \hat{H}}) \tag{2.7}$$

where \hat{H} is the time independent Hamiltonian and $\beta = (k_B T)^{-1}$ where k_B is Boltzmann's constant and T is the temperature. Hereafter we let $k_B = 1$ meaning that we measure temperature in units of what energy it corresponds

to. This is similar to the trace of the time-evolution operator $\hat{U}(t_2, t_1)$ evolving a state from time t_1 to t_2

$$\text{tr}(\hat{U}(t_2, t_1)) = \text{tr}(e^{-i\frac{(t_2-t_1)\hat{H}}{\hbar}}) = f(t_2 - t_1). \quad (2.8)$$

We will hereafter let $\hbar = 1$ by measuring energy in units of inverse time. The partition function can then be obtained as the analytical extension of f ,

$$Z(\beta) = f(-i\beta). \quad (2.9)$$

The trace of the time evolution operator can be calculated as an integral over configuration space which in our case will be fields configurations ψ ,

$$f(t) = \int \mathcal{D}[\psi] \langle \psi | \hat{U}(t, 0) | \psi \rangle. \quad (2.10)$$

The time-evolution operator can be calculated using Feynman's path integrals [3],

$$\langle \psi_2 | \hat{U}(t, 0) | \psi_1 \rangle = \int_{\psi_1}^{\psi_2} \mathcal{D}[\psi(t)] e^{iS[\psi(t)]} \quad (2.11)$$

where $S[\psi(t)]$ is the action of the path through field configurations $\psi(t)$. Combining these results tells us that the trace $f(t)$ can be calculated as a periodic time path integral with period t

$$f(t) = \int_0^t \mathcal{D}[\psi(t)] e^{iS[\psi(t)]}. \quad (2.12)$$

This means that $Z(\beta)$ can be obtained by calculating a path integral where the time is imaginary and periodic. The action must be analytically extended to imaginary time $\tau = it$.

$$S[x(\tau)] = \int_0^\beta d\tau \mathcal{L} \quad (2.13)$$

The CFT we are interested in lives on the boundary of an AdS theory. The metric of the CFT is the metric induced from the bulk theory. The boundary is time-like and the time periodicity of the two theories are thus the same. This means that they are at the same temperatures.

Only paths of extremal action will contribute to this in the classical limit because of the oscillatory behavior of the exponential. Let $\psi_c(t)$ be the path

that extremizes the action and expand the exponent as a Taylor series around this

$$S[\psi_c(t) + \psi(t)] = S[\psi_c(t)] + \frac{\psi(t)^2}{2!} \frac{\delta^2}{\delta\psi(t)^2} S[\psi_c(t)] + \frac{\partial_t \psi(t)^2}{2!} \frac{\delta^2}{\delta(\partial_t \psi(t))^2} S[\psi_c(t)] + O(\psi(t)^3) \quad (2.14)$$

The functional derivative of the action is just the ordinary derivative of the Lagrangian function. TODO The partition function is in the classical limit

$$Z(\beta) = f(-i\beta) \stackrel{\text{classical}}{=} e^{iS_c} \quad (2.15)$$

where S_c is the action of classical periodic path with period $-i\beta$.

2.5 Expectation Values

The boundary values of the bulk fields correspond to fields in the CFT. Expectation values of observables in the CFT can be calculated using a generating functional $Z[J]$. This is a partition function for a system with a perturbed Lagrangian $\mathcal{L}_J(x) = \mathcal{L}(x) + J(x)O(x)$. Here $O(x)$ is a local operator on the fields. The generating functional can be regarded an expectation value of a system with the original Lagrangian \mathcal{L} .

$$\begin{aligned} Z[J] &= \int_0^{-i\beta} \mathcal{D}[\psi(t)] e^{i \int \mathcal{L}(\psi(x)) + J(x)O(\psi(x))} \\ &= Z[0] \int_0^{-i\beta} \mathcal{D}[\psi(t)] \frac{e^{i \int \mathcal{L}(\psi(x))}}{Z[0]} e^{i \int J(x)O(\psi(x))} \\ &= Z[0] \langle e^{i \int J(x)O(\psi(x))} \rangle \end{aligned} \quad (2.16)$$

Taking a functional derivative of this gives:

$$\begin{aligned} \frac{\delta}{\delta J} \log(Z[J])|_{J=0} &= \frac{Z[0] \langle iO(\psi(x)) e^{i \int J(x)O(\psi(x))} \rangle}{Z[0] \langle e^{i \int J(x)O(\psi(x))} \rangle} \Big|_{J=0} \\ &= i \langle O(\psi(x)) \rangle \end{aligned} \quad (2.17)$$

The partition functions of the bulk and boundary theories are the same even for a perturbed Lagrangian. Both Lagrangians are then perturbed. Expectation values of operators of the boundary theory can thus be calculated using the partition function for the bulk theory. It is though not trivial to figure out what fields in the bulk theory corresponds to what operators in the boundary theory

Chapter 3

Solution of the Classical Bulk Theory

We wish to compute expectation values of the CFT. The field theory is a strongly coupled quantum field theory and the expectation values are thus difficult to compute directly. The expectation values can though be calculated using (2.17) and the partition function can be obtained from the GKPW equation, (1.1).

The partition function of the bulk theory must then be calculated. This is in our case easier, TODO reference, since this theory can be treated in a classical limit. But the equations of motion must still be calculated and the solutions must be translated to expectation values for the CFT. This section is devoted to the theory of how this is done.

3.1 Metric

The precise form of the bulk Lagrangian that corresponds to the boundary theory we are interested in is unknown. The string theory is gravitational and the space is asymptotically AdS. The action contains an Einstein-Hilbert term with a cosmological constant

$$\Lambda = -\frac{d(d-1)}{2L^2} \tag{3.1}$$

where L is a length scale and d is the dimension of the boundary. A Gibbons-Hawking-York boundary term is also needed [4], [5] TODO, read, comment.

$$S_{\text{grav}} = \frac{1}{2\kappa} \int d^{d+1}x \sqrt{-g} (R - 2\Lambda) - \frac{1}{\kappa} \int d^d x \sqrt{\gamma} \left(-\gamma^{ab} \nabla_a n_b + \frac{d-1}{L} \right) \quad (3.2)$$

Here R is the Ricci scalar. The gravitational constant κ (related to Newton's constant by $\kappa = 8\pi G$) is assumed to be small so that we are in a so called "probe limit". We will later see when this assumption can't be trusted, TODO. n_a is a normal vector at the boundary. The probe limit lets us solve the equations of motion for the metric independently of the other fields. We are looking for an equilibrium solution at a finite temperature T and the solution is then known to be a black hole (TODO ref), the Schwarzschild metric in AdS space. The metric has the following form in a particular choice of coordinates where the radial coordinate z is 0 at the boundary and z_h at the horizon

$$g_{ab} dx^a dx^b = \frac{L^2}{z^2} \left(\frac{dz^2}{f(z)} - f(z) dt^2 + d\mathbf{x}^2 \right). \quad (3.3)$$

Here $f(z) = 1 - z^d z_h^{-d}$. The vector \mathbf{x} thus has $d-1$ components. $f(z)$ approaches 1 at the boundary and the space is asymptotically AdS.

All coordinates are evidently of the same unit. The gravitational part of the Lagrangian can now be removed and this background metric can be used instead of solving the equations of motion for the metric together with the fields. This gravitational part of the Lagrangian must though be kept when calculating the value of the total action. The Ricci scalar of this metric is

$$R = -\frac{d(d+1)}{L^2} = 2\Lambda \frac{d+1}{d-1} \quad (3.4)$$

TODO: does the region behind the horizon contribute? The action can be calculated using this. Consider the space $(z, t, \mathbf{x}) \in [\epsilon, z_h] \times V$ corresponding to a d -volume of the boundary theory V . The action boundary density is

then

$$\begin{aligned}
\frac{S_{\text{grav}}}{V} &= \frac{\Lambda}{\kappa} \left(\frac{d+1}{d-1} - 1 \right) \int_{\epsilon}^{z_h} dz \sqrt{-g} + \frac{1}{\kappa} \sqrt{\gamma} \left(-\gamma^{ab} \partial_a dz_b + \frac{d-1}{L} \right) \\
&= \frac{\Lambda}{\kappa} \frac{2}{d-1} \int_{\epsilon}^{z_h} dz \left(\frac{L}{z} \right)^{d+1} + \frac{1}{\kappa} \left(\frac{\epsilon}{L} \right)^{-d} \frac{d-1}{L} \\
&= \frac{L\Lambda}{\kappa} \frac{2}{d(d-1)} \left(\left(\frac{\epsilon}{L} \right)^{-d} - \left(\frac{z}{L} \right)^{-d} \right) \\
&= -\frac{1}{L\kappa} \left(\left(\frac{\epsilon}{L} \right)^{-d} - \left(\frac{z_h}{L} \right)^{-d} \right)
\end{aligned} \tag{3.5}$$

The total action density is obtained as $\epsilon \rightarrow 0$. . TODO, imaginary time, deficit angle gives T .

$$T = \frac{d}{4\pi z_h} \tag{3.6}$$

3.2 Boundary Behavior

The bulk Lagrangian considered will contain different fields and depends both on the fields and their first derivatives. Consider a field ψ with a kinetic term $-(\partial_a \psi)^2$ and a potential term $V(\psi)$. The classical solution is the one that extremizes the action. The action integral contains the metric as an integration measure

$$S = \int d^{d+1}x \sqrt{|\det g_{ab}|} \mathcal{L} \equiv \int d^{d+1}x \sqrt{g} \mathcal{L}. \tag{3.7}$$

The Euler-Lagrange equation is obtained by varying the action. The integration measure can then be regarded as part of the Lagrangian or covariant derivatives can be used in the derivation of the Euler-Lagrange equation. The measure becomes when using the metric (3.3) $L^{d+1} z^{-d-1}$. The Euler-Lagrange equation gives

$$\begin{aligned}
0 &= \partial_a \left(\frac{\partial(z^{-d-1}(V(\psi) - (\partial_b \psi)^2))}{\partial(\partial_a \phi)} \right) - \frac{\partial(z^{-d-1}(V(\psi) - (\partial_b \psi)^2))}{\partial \phi} = \\
&= -\partial_a (z^{-d-1} 2\partial^a \psi) - z^{-d-1} V'(\psi)
\end{aligned} \tag{3.8}$$

We will be interested in boundary systems with translational symmetry so ψ is assumed to be a function of z , TODO motivate more. The equation of motion then becomes

$$\begin{aligned}
0 &= -\partial_z \left(z^{-d-1} 2g^{zz} \partial_z \psi \right) - \frac{V'(\psi)}{z^{d+1}} = \\
&= -\partial_z \left(z^{-d-1} 2(z^2 L^{-2} f(z)) \partial_z \psi \right) - \frac{V'(\psi)}{z^{d+1}} = \\
&= -z^{-d-1} 2z^2 L^{-2} f(z) \psi'' - L^{-2} \left((-d+1) z^{-d} 2f(z) + z^{-d+1} 2f'(z) \right) \psi' - \frac{V'(\psi)}{z^{d+1}}
\end{aligned} \tag{3.9}$$

This gives a second order differential equation for $\psi(z)$

$$0 = -z^2 2f(z) \psi'' - \left((-d+1) z 2f(z) + z^2 2f'(z) \right) \psi' - L^2 V'(\psi) \tag{3.10}$$

Now consider the boundary, $z = 0$. Our metric is required to be asymptotically AdS so $f(0) \rightarrow 1, z f'(0) \rightarrow 0$. ψ can be expanded at the boundary as a Laurent series. Call the lowest exponent in this series Δ . ψ will then behave as z^Δ near the boundary. This should solve the differential equation in the near boundary limit. Insertion of z^Δ into the differential equation and taking the limit of small z gives

$$\begin{aligned}
0 &= -z^2 2\Delta(\Delta-1) z^{\Delta-2} - \left((-d+1) 2z + z^2 2f'(z) \right) \Delta z^{\Delta-1} - L^2 V'(z^\Delta) \\
&= z^\Delta \left(-2\Delta(\Delta-1) - 2(-d+1)\Delta - L^2 V'(z^\Delta) \right).
\end{aligned} \tag{3.11}$$

Now consider a potential for a massive scalar field, $V(\psi) = -m^2 \psi^2 + \mathcal{O}(\psi^3)$. We then get the following equation for Δ

$$0 = \Delta^2 - d\Delta - L^2 m^2 \tag{3.12}$$

in the limit $z \rightarrow 0$. This has solutions

$$\Delta = \frac{d \pm \sqrt{d^2 + 4L^2 m^2}}{2}. \tag{3.13}$$

ψ thus goes as z^{Δ_0} where Δ_0 is the smaller solution and Δ_1 the larger. The leading behaviour of ψ near $z = 0$ is

$$\psi = \psi_0 \left(\frac{z}{L} \right)^{\Delta_0} + \psi_1 \left(\frac{z}{L} \right)^{\Delta_1} \tag{3.14}$$

unless $\Delta_1 - \Delta_0 \geq 1$ and further terms from the series corresponding to Δ_0 must be included.

What will the contribution to the action from this solution be? Consider the action contribution from the region $z \in [\epsilon, \delta]$ where δ is small and $\epsilon \rightarrow 0$.

$$\begin{aligned}
S_{[\epsilon, \delta]} &= \int_{z \in [\epsilon, \delta]} d^{d+1}x \sqrt{g} \mathcal{L} = \\
&= V \int_{\epsilon}^{\delta} dz \left(\frac{z}{L} \right)^{-d-1} (-m^2 \psi^2 - (\partial_a \psi)^2) = \\
&= V \int_{\epsilon}^{\delta} dz \left(\frac{z}{L} \right)^{-d-1} \left(-m^2 (\psi_0 \left(\frac{z}{L} \right)^{\Delta_0} + \psi_1 \left(\frac{z}{L} \right)^{\Delta_1})^2 - (\partial_a (\psi_0 \left(\frac{z}{L} \right)^{\Delta_0} + \psi_1 \left(\frac{z}{L} \right)^{\Delta_1}))^2 \right) = \\
&= V \int_{\epsilon}^{\delta} dz \left(\frac{z}{L} \right)^{-d-1} \left[-m^2 \left(\psi_0^2 \left(\frac{z}{L} \right)^{2\Delta_0} + \psi_1^2 \left(\frac{z}{L} \right)^{2\Delta_1} + 2\psi_0 \psi_1 \left(\frac{z}{L} \right)^{\Delta_0 + \Delta_1} \right) \right. \\
&\quad \left. - g^{zz} L^{-2} (\Delta_0 \psi_0 \left(\frac{z}{L} \right)^{\Delta_0 - 1} + \Delta_1 \psi_1 \left(\frac{z}{L} \right)^{\Delta_1 - 1})^2 \right] = \\
&= V \int_{\epsilon}^{\delta} dz \left(\frac{z}{L} \right)^{-d-1} \left[-m^2 \left(\psi_0^2 \left(\frac{z}{L} \right)^{2\Delta_0} + \psi_1^2 \left(\frac{z}{L} \right)^{2\Delta_1} + 2\psi_0 \psi_1 \left(\frac{z}{L} \right)^{\Delta_0 + \Delta_1} \right) \right. \\
&\quad \left. - g^{zz} L^{-2} \left(\Delta_0^2 \psi_0^2 \left(\frac{z}{L} \right)^{2(\Delta_0 - 1)} + \Delta_1^2 \psi_1^2 \left(\frac{z}{L} \right)^{2(\Delta_1 - 1)} + 2\Delta_0 \Delta_1 \psi_0 \psi_1 \left(\frac{z}{L} \right)^{\Delta_0 + \Delta_1 - 2} \right) \right] = \\
&= V \int_{\epsilon}^{\delta} dz \left[-m^2 \left(\psi_0^2 \left(\frac{z}{L} \right)^{2\Delta_0 - d - 1} + \psi_1^2 \left(\frac{z}{L} \right)^{2\Delta_1 - d - 1} + 2\psi_0 \psi_1 \left(\frac{z}{L} \right)^{-1} \right) \right. \\
&\quad \left. - L^{-2} \left(\Delta_0^2 \psi_0^2 \left(\frac{z}{L} \right)^{2\Delta_0 - d - 1} + \Delta_1^2 \psi_1^2 \left(\frac{z}{L} \right)^{2\Delta_1 - d - 1} - 2L^2 m^2 \psi_0 \psi_1 \left(\frac{z}{L} \right)^{-1} \right) \right] = \\
&= V \int_{\epsilon}^{\delta} dz \left((-m^2 - \Delta_0^2 L^{-2}) \psi_0^2 \left(\frac{z}{L} \right)^{2\Delta_0 - d - 1} + (-m^2 - \Delta_1^2 L^{-2}) \psi_1^2 \left(\frac{z}{L} \right)^{2\Delta_1 - d - 1} \right) = \\
&= V L^{-2} (\Delta_1 - \Delta_0) \int_{\epsilon}^{\delta} dz \left(\Delta_0 \psi_0^2 \left(\frac{z}{L} \right)^{2\Delta_0 - d - 1} - \Delta_1 \psi_1^2 \left(\frac{z}{L} \right)^{2\Delta_1 - d - 1} \right) = \\
&= V L^{-1} (\Delta_1 - \Delta_0) \left(-\frac{\Delta_0 \psi_0^2}{2\Delta_0 - d} \left(\frac{\epsilon}{L} \right)^{2\Delta_0 - d} + \frac{\Delta_1 \psi_1^2}{2\Delta_1 - d} \left(\frac{\epsilon}{L} \right)^{d - 2\Delta_0} \right) + \text{finite} \\
&= V L^{-1} \left(\Delta_0 \psi_0^2 \left(\frac{\epsilon}{L} \right)^{2\Delta_0 - d} - \Delta_1 \psi_1^2 \left(\frac{\epsilon}{L} \right)^{d - 2\Delta_0} \right) + \text{finite}
\end{aligned} \tag{3.15}$$

Here $\Delta_0 + \Delta_1 = d$ and $\Delta_0 \Delta_1 = -L^2 m^2$ have been used. One of these

two terms diverges as $\epsilon \rightarrow 0$. The term with ψ_0 diverges since $2\Delta_0 - d = -\sqrt{d^2 + 4L^2m^2}$. The action from the near boundary thus diverges. This can be remedied by having a boundary term in the action that exactly cancels this divergence.

The boundary term must thus evaluate to

$$-\Delta_0 V L^{-1} \psi_0^2 \left(\frac{\epsilon}{L} \right)^{2\Delta_0 - d} \quad (3.16)$$

near the boundary. A boundary term like this can be constructed using $\psi = \psi_0 L^{-\Delta_0} \epsilon^{\Delta_0}$ near the boundary and $\sqrt{\gamma} = L^d z^{-d}$ where γ is the determinant of the metric induced on the boundary by g_{ab} . The boundary term then becomes

$$S_{\text{bdy}} = - \int_{z=\epsilon} d^d x \Delta_0 L^{-1} \psi^2 \sqrt{\gamma} \quad (3.17)$$

This in addition to the Lagrangian is Lorentz invariant and it also has conformal invariance.

Consider another possible term in the Lagrangian, an electromagnetic field A_μ . The Lagrangian is $-\frac{1}{4} F_{ab} F^{ab}$ where $F_{ab} = \partial_a A_b - \partial_b A_a$. Consider fields with t , x_1 , and x_2 symmetry.

TODO, check. It is easily shown that the behaviour of these fields is the same when there is an interaction present as will be considered later.

3.3 Horizon Behaviour

TODO: Wilson loop= $\oint \phi = 0$. Ingoing boundary conditions. Looking at the equations of motion for $z \rightarrow z_h$ gives when $\phi(z_h) = 0$

$$\psi(z_h) = \frac{-3z_h \psi'(z_h)}{L^2 m^2} \quad (3.18)$$

3.4 Expectation Values of Boundary Operators

These expectation values are calculated using (2.17) and (2.15).

$$\langle O(\psi(x)) \rangle = -i \frac{\delta}{\delta J(x)} \log(Z[J])|_{J=0} \stackrel{\text{GKPW}}{=} -i \frac{\delta}{\delta J(x)} \log(Z[J])|_{J=0} \stackrel{\text{classical}}{=} \frac{\delta}{\delta J(x)} S_c|_{J=0} \quad (3.19)$$

The functional derivative is thus the change in the classical action when the boundary value of the fields are changed. The total change in the partition function is needed and the whole field solutions change when changing the boundary conditions so this must be taken into account. The derivative becomes

$$\frac{\delta}{\delta J(x)} S_c|_{J=0} = \int d^{d+1}y \sqrt{g} \left(\frac{\partial \mathcal{L}(y)}{\partial \phi_i(y)} \frac{\partial \phi_i(y)}{\partial J(x)} + \frac{\partial \mathcal{L}(y)}{\partial (\partial_a \phi_i(y))} \frac{\partial (\partial_a \phi_i(y))}{\partial J(x)} \right) \quad (3.20)$$

where i goes over all fields. Here the Lagrangian is assumed to only depend on the fields and their first derivatives. Now let $\mathcal{L}' = \sqrt{g}\mathcal{L}$ and integrate by parts

$$\begin{aligned} \frac{\delta}{\delta J(x)} S_c|_{J=0} = & \int d^{d+1}y \left(\frac{\partial \mathcal{L}'(y)}{\partial \phi_i(y)} - \partial_a \frac{\partial \mathcal{L}'(y)}{\partial (\partial_a \phi_i(y))} \right) \frac{\partial (\phi_i(y))}{\partial J(x)} \\ & + \int_{\partial \text{AdS}} d^d y n_a \frac{\partial \mathcal{L}'(y)}{\partial (\partial_a \phi_i(y))} \frac{\partial (\phi_i(y))}{\partial J(x)} \end{aligned} \quad (3.21)$$

where n_a is an outward normal to the boundary of AdS. The first integral vanishes since the fields obey the Euler-Lagrange equation.

3.5 Parameters

There are multiple unknown parameters. These must be investigated to find values that give us the boundary theory we are interested in. The Lagrangian contains the parameters κ , L , m^2 , q , α_1 , α_2 , and γ . The solution of the equations of motion for the metric we used in the probe limit also contains a temperature dependent constant z_h . Some of these parameters might be redundant since we can make different transformations of fields and coordinates. The physics of the bulk are treated in the classical limit and the Lagrangian can thus be changed as long as the equations of motion for ψ and A_a are left unchanged. The physics are independent of the exact value of κ as long as we are in the probe limit.

The horizon z_h and the curvature length L set length scales in the metric. Length units in the numerical solution can be chosen such that $z_h = 1$. This means that we for different temperatures have different units since z_h

is related to the temperature. We will have to convert between these units when comparing results from different temperatures.

The factor L^2 in the metric does not affect the theory since the Lagrangian has conformal invariance. TODO, true? Let $L = 1$ hereafter.

Considering $q\psi$ and qA_a as the fields gives a Lagrangian of the same form but with different constants α_1 , α_2 , and γ and the whole Lagrangian is divided by q^2 except for the term originally containing q^2 which is divided by q^4 . Multiplying the Lagrangian by a constant doesn't affect the equations of motion so $q = 1$ can be assumed without getting a less general Lagrangian. It should be noted that a different value of q would give $q\psi$ as solution instead where ψ is the solution of the equations of motion when $q = 1$. The same is true for A_a .

The Breitenlohner-Freedman bound (BF) is a lower bound on m^2 of a massive scalar field in AdS space. It requires

$$L^2 m^2 \geq -\frac{d^2}{4} \quad (3.22)$$

for stability[6], TODO explain. The scalar field ψ should obey this bound far away from the black-hole for normalizeable modes, TODO explain. We would though like a symmetry breaking of ψ near the black hole corresponding to the electron condensate [7]. This can happen because the coupling of ψ to A_a gives ψ an effective mass.

The effective mass is given by:

$$m_{eff}^2 = m^2 + A_a A^a = m^2 - \frac{z^2}{L^2(1 - z^d z_h^{-d})} \phi^2 \quad (3.23)$$

The bulk field equations are obtained by varying the bulk Lagrangian with respect to all the fields. This can be done with the Euler-Lagrange equation since the action does not contain any higher derivatives. The Euler-Lagrange equation for a scalar field ϕ states

$$\partial_a \left(\frac{\partial \mathcal{L}}{\partial(\partial_a \phi)} \right) - \frac{\partial \mathcal{L}}{\partial \phi} = 0 \quad (3.24)$$

ϕ is 0 at the horizon but increases as ψ is non-zero. This effective mass breaks the BF bound since g^{tt} diverges at the horizon. TODO, talk to Ulf

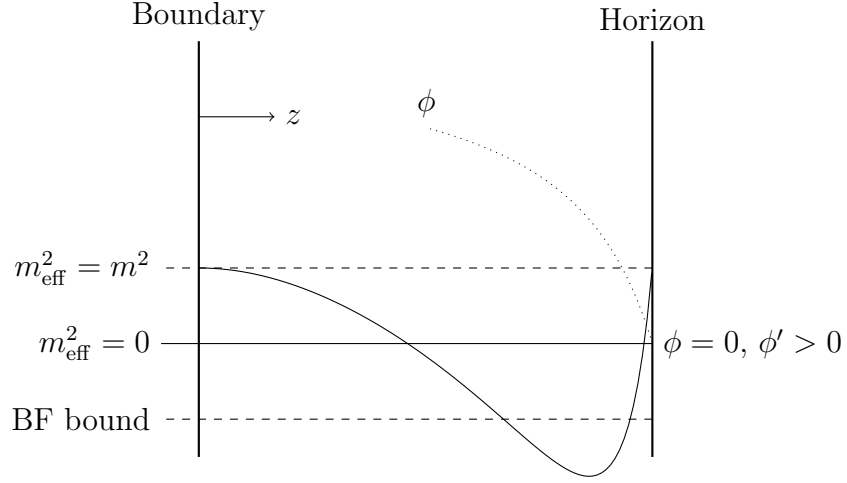


Figure 3.1: Schematic plot of how the the effective mass breaks the BF bound outside the horizon. A value of ϕ has been assumed.

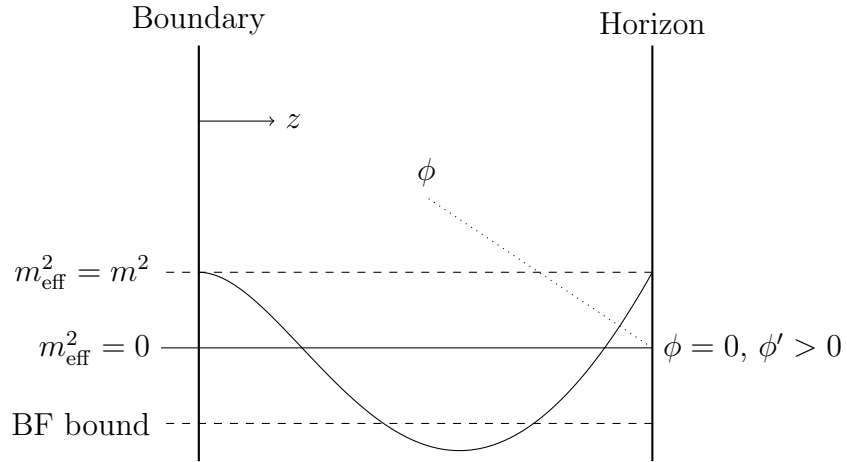


Figure 3.2: Schematic plot of how the the effective mass breaks the BF bound outside the horizon. A value of ϕ has been assumed.

and do calculations. **Skip the following?:**

Our space, (3.3), is asymptotically AdS with $d = 3$ at the boundary. But what happens near the horizon? Let $s = z_h - z$. Expand $f(z_h - s)$ around $s = 0$.

$$f(z_h - s) = d \frac{s}{z_h} + O(s^2 z_h^{-2}) \quad (3.25)$$

The metric now becomes

$$\begin{aligned} g_{ab} dx^a dx^b &= \frac{L^2}{(z_h - s)^2} \left(\frac{ds^2}{d \frac{s}{z_h}} - d \frac{s}{z_h} dt^2 + d\mathbf{x}^2 \right) \\ &= \frac{L^2}{z_h^2} \left(\frac{z_h ds^2}{ds} - \frac{ds dt^2}{z_h} + d\mathbf{x}^2 \right) \end{aligned} \quad (3.26)$$

Summary, fields, parameters, units

3.6 Thermodynamic Variables

The free energy, $A = -T \log Z$, is through the GKPW equation the same for the bulk and the boundary theory. This can be calculated in the classical limit in the bulk.

$$A = -T \log Z \stackrel{\text{classical}}{=} -i T S_c \quad (3.27)$$

Here S_c is the classical periodic time action. The classical field solutions only depend on the z coordinate and are thus proportional to $V = -i\beta V_2$ where V_2 is the area considered in coordinates x_1, x_2 . This gives the free energy per surface area

$$\frac{A}{V_2} = - \int_0^{z_h} dz \sqrt{-g} \mathcal{L} + V^{-1} S_{\text{bdy}} \quad (3.28)$$

Consider the case where $m^2 = -2$. Then $\Delta_0 = 1$ and the boundary term is given by (3.17).

3.6.1 Gravitational Free Energy

The action is dominated by the gravitational part since we are in the probe limit. The expression for the temperature (3.6) can be used to calculate the

free energy from the action at finite temperature

$$\begin{aligned}\frac{A_{\text{grav}}}{V_2} &= -T \frac{-1}{\kappa} \sqrt{\frac{-2\Lambda}{d(d-1)}} \left(\frac{d}{4\pi T L} \right)^{-d} \\ &= \frac{(4\pi)^d T^d L^{d-1}}{d^d \kappa}\end{aligned}\tag{3.29}$$

The entropy H can now be calculated from the free energy

$$H = -\frac{\partial A}{\partial T} = -\frac{(4\pi)^d T^{d-1} L^{d-1}}{d^{d-1} \kappa}.\tag{3.30}$$

TODO, Beckenstein-Hawking

3.6.2 Free Energy of Scalar and Electromagnetic Fields

It is important to calculate the free energy not only of the gravitational part of the Lagrangian since we have multiple solutions of the field equations and boundary values at the same temperature. Which solution is physical can be found by finding the one of lowest free energy. Here we neglect the contribution from any back-reaction on the metric. The back-reaction is small since κ is small but a small κ also makes the effect of the back-reaction on the free energy large. TODO, motivate or say that we leave this open.

The free energy of the trivial solution (2.6) can be found analytically. For this solution we have

$$\begin{aligned}\psi &= 0 \\ \psi' &= 0 \\ \phi &= \mu - \mu \left(\frac{z}{z_h} \right)^{d-2} \\ \phi' &= -(d-2) \frac{\mu}{z_h} \left(\frac{z}{z_h} \right)^{d-3}\end{aligned}\tag{3.31}$$

TODO, explain d-3

$$\begin{aligned}
\frac{A}{V_{\text{SC}}} &= - \int_0^{z_h} dz \sqrt{-g} \mathcal{L} + V_{\text{SC}}^{-1} S_{\text{bdy}} \\
&= \int_0^{z_h} dz \left(\frac{z}{L} \right)^{-d-1} \frac{1}{4} F_{ab} F^{ab} \\
&= \int_0^{z_h} dz \left(\frac{z}{L} \right)^{-d-1} \frac{1}{2} F_{zt}^2 g^{zz} g^{tt} \\
&= - \int_0^{z_h} dz \left(\frac{z}{L} \right)^{-d-1+4} \left(\frac{z}{z_h} \right)^{2(d-3)} (d-2)^2 \frac{\mu^2}{2z_h^2} \quad (3.32) \\
&= - L^{d-3} z_h^{-2d+4} (d-2)^2 \frac{\mu^2}{2} \int_0^{z_h} dz z^{d-3} \\
&= - L^{d-3} z_h^{-d+2} (d-2) \frac{\mu^2}{2} \\
&= - L^{d-3} \left(\frac{4\pi T}{d} \right)^{d-2} (d-2) \frac{\mu^2}{2}
\end{aligned}$$

3.7 Operator Expectation Values

TODO, introduce $\rho, \mu...$ The charge density ρ of the trivial solution (2.6) is through (??) given by

$$\rho = \frac{\mu}{z_h}. \quad (3.33)$$

The chemical potential of the trivial solution can thus be calculated given the temperature and charge density

$$\mu = \frac{d\rho}{4\pi T}. \quad (3.34)$$

The chemical potential of the numerical solutions can be obtained from the boundary values. The result is seen in Fig. 3.4.

3.8 Conductivity

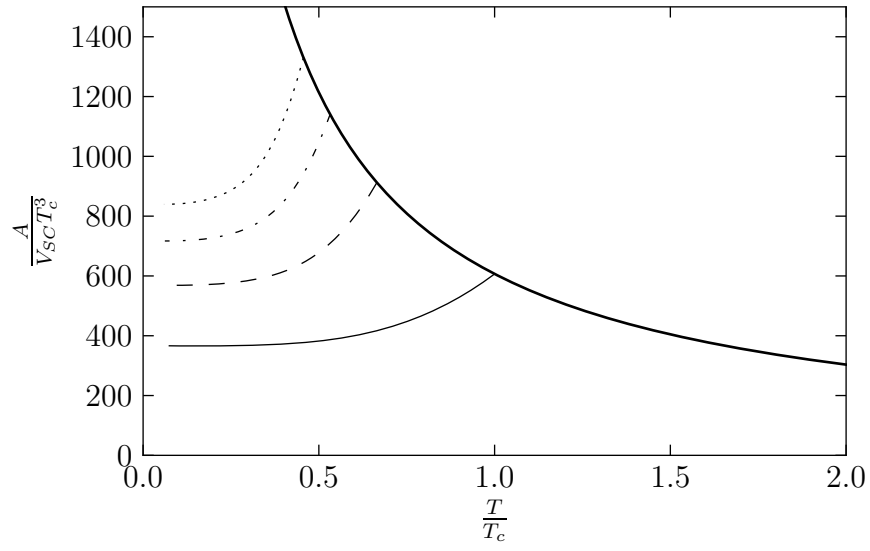


Figure 3.3: Curves corresponding to different solutions at the same temperature. The trivial solution is shown as the thick solid line. The other curves correspond to numerical solutions. The lowest one corresponds to the first root and the other roots follow in order.

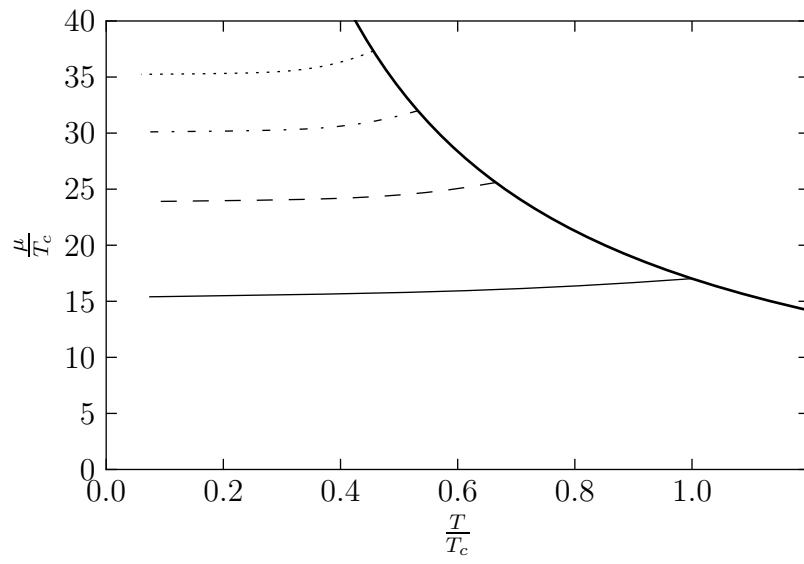


Figure 3.4: Curves corresponding to different solutions at the same temperature. The trivial solution is shown as the thick solid line. The other curves correspond to numerical solutions. The lowest one corresponds to the first root and the other roots follow in order.

Chapter 4

Numerical Solution of Bulk Equations of Motion

4.1 Condensate

TODO, write in more detail. TODO, write in some clear way about the dimensionless quantities. The ψ condensate should be obtained without any source. Boundary conditions without source, $\psi_i = 0$, are chosen at the boundary. ψ_1 , μ , and ρ are then unknown. The boundary condition that $\phi = 0$ at the horizon is actually a double boundary condition due to the singular form of the equations of motion there. There are thus only two free parameters and it is easiest to start integrating from there. A numerical integrator is used to integrate from the horizon to the boundary. The source term is read off and the horizon conditions are adapted until the source term is zero. This leaves a one dimensional space of solutions since we have two parameters to vary at the horizon but just one constraint on the boundary. μ , ρ and $\epsilon_{ij}\psi_j$ can be read off at the boundary.

The fields all have the unit of inverse length and ρ thus has unit inverse length squared. Solutions can be obtained at any temperature by choosing different values of z_h . The temperature can be chosen such that the charge density ρ is constant for all solutions. The numerical solution gives ρz_h^2 . Choosing a constant $\rho = 1$ gives z_h and thus the temperature. We call the highest temperature obtained T_c .

Chapter 5

Results

The numerical results are here presented.

5.1 Condensate

This section shows results from Electrical Conductivity The conductivity of a superconductor can easily be measured experimentally for a wide range of frequencies and it is therefore an interesting property to calculate from our model of a superconductor. The agreement in different frequency ranges tells us about similarities and differences between our model and the experimental superconductors.

5.1.1 Definition of Electrical Conductivity and its Properties

The current density, \mathbf{J} is a vector quantity defined as the flow of charge. The rate of charge passing through a surface at x of length dl and with normal \mathbf{n} is given by $dl\mathbf{n} \cdot \mathbf{J}(x)$. We define conductivity σ as the linear response function for the current density J_x with the applied electrical field E_x as source

$$\sigma(\omega) = \frac{J_x(\omega)}{E_x(\omega)}. \quad (5.1)$$

Here the direction x has been chosen for concreteness but since we consider two-dimensional systems with rotational symmetry we need only consider

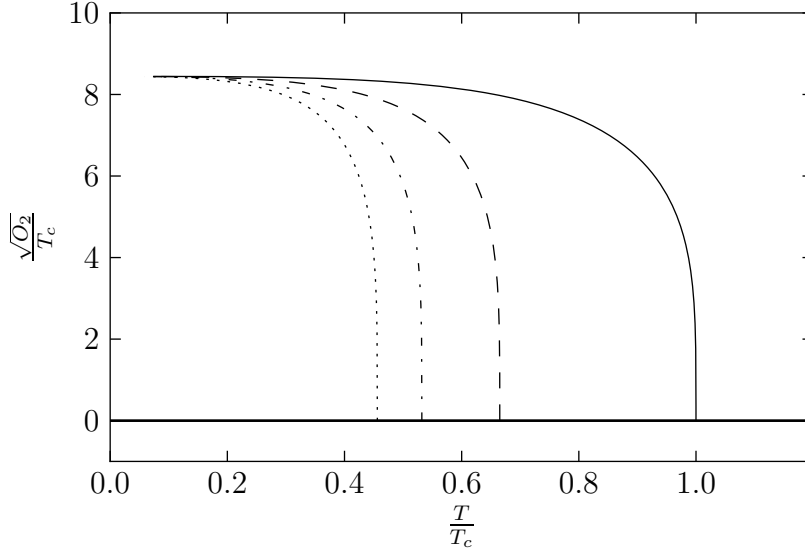


Figure 5.1: Curves corresponding to different solutions at the same temperature.

one direction. These functions of ω are the Fourier transforms of the time-dependent quantities. The considered electrical field E_x is further assumed to be translationally symmetric giving a still translationally symmetric system. The current in the time domain can be obtained from the conductivity and the applied field through a inverse Fourier transform

$$J_x(t) = \int_{-\infty}^{\infty} E_x(t - \tau) \sigma(\tau) d\tau. \quad (5.2)$$

Causality implies that $\sigma(\tau) = 0$ for $\tau < 0$ since the current would otherwise depend on future values of the electrical field. The conductivity can using this be written

$$\sigma(\omega) = \int_0^{\infty} \sigma(\tau) \exp(i\tau\omega) d\tau \quad (5.3)$$

and thus has an analytical extension to the upper half of the complex plane. Both the current, $J_x(t)$, and applied field, $E_x(t)$, are real quantities which makes $\sigma(\tau)$ also real and thus $\text{Re}(\sigma(w))$ an even function and $\text{Im}(\sigma(\omega))$ odd.

These properties of $\sigma(\omega)$ gives the Kramers–Kronig relations

$$\begin{aligned}\text{Re}(\sigma(\omega)) &= \frac{2}{\pi} \int_0^\infty \frac{\omega' \text{Im}(\sigma(\omega'))}{\omega'^2 - \omega^2} \\ \text{Im}(\sigma(\omega)) &= -\frac{2}{\pi} \int_0^\infty \frac{\omega \text{Re}(\sigma(\omega'))}{\omega'^2 - \omega^2}\end{aligned}\tag{5.4}$$

These relations state that the real part of the conductivity uniquely determines the imaginary part and vice versa.

5.1.2 Calculating the Holographic Conductivity

The obtained conductivity for temperatures above the T_c is $\sigma(\omega) = 1$. This agrees with TODO. The conductivity changes when the condensate forms. An energy gap Δ forms and the conductivity for $\omega < \Delta$ goes to 0 as the temperature is lowered. The superconductivity is not immediately evident from the obtained conductivity curves. There is a delta-function at $\omega = 0$ since translational invariance of the boundary theory has been assumed. Consistency Check using a Sum Rule The Kramers-Kronig relations and the decoupling of the electrons from the system at high frequencies can be used to prove a sum rule for the conductivity [8]. The rule states

$$\int_0^\infty \text{Re}(\sigma(\omega)) d\omega = C\tag{5.5}$$

where C is a constant depending on what system we are considering

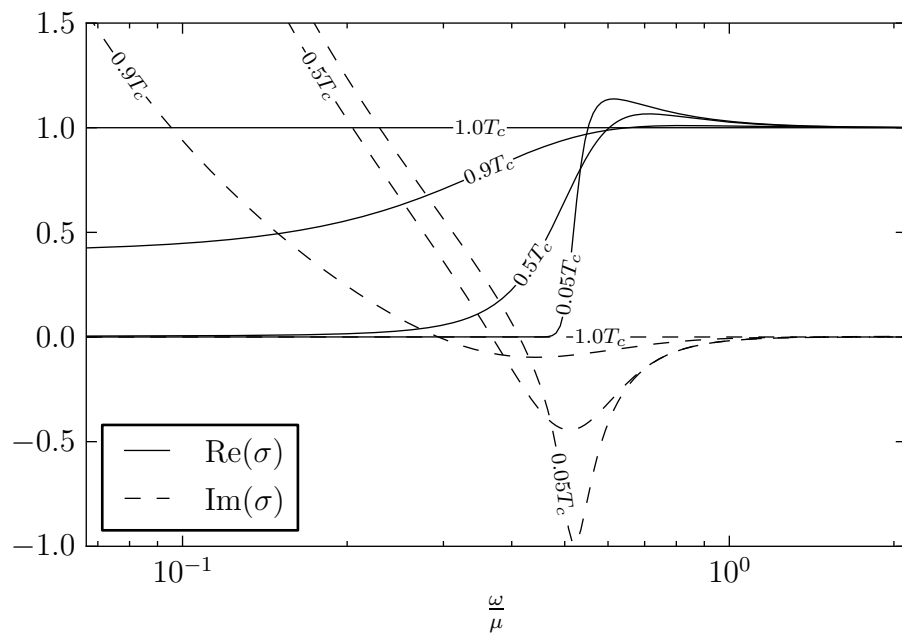


Figure 5.2: Real and imaginary part of the conductivity for different temperatures. Higher temperatures above T_c give the same conductivity as for the $T = T_c$ case.

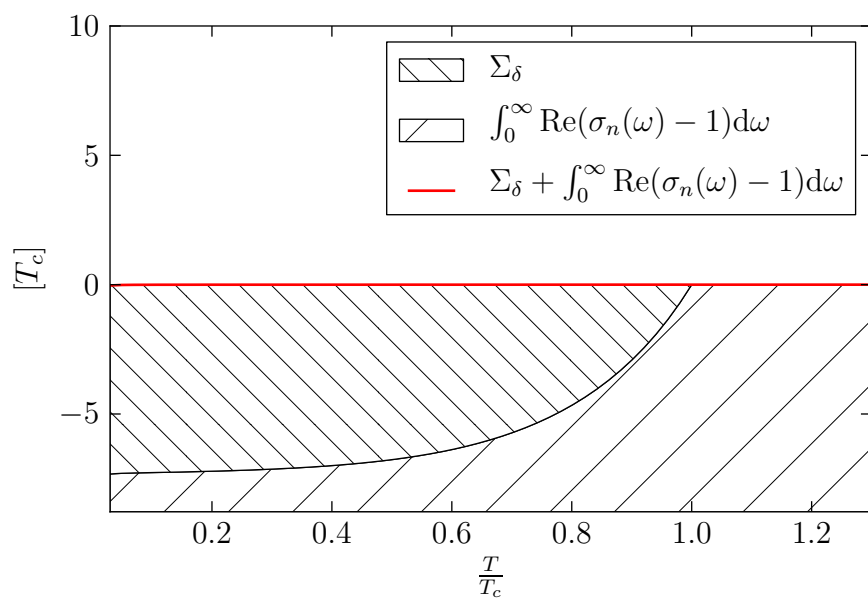


Figure 5.3: Integral

Chapter 6

Higher Order Term

Many assumptions have been made in the previous study. The most simple Lagrangian (??) has been used and translational symmetry has been assumed. This has given us a boundary theory with a scalar field that condenses below a critical temperature as expected for a superconductor. The conductivity shows both similarities and differences with that of high- T_c superconductors. A δ -function develops under T_c giving infinite DC conductivity. A difference that is evident is the lack of a so-called Drude peak at low frequencies. A Drude peak is an increase in conductivity for low frequencies in metals that can be well modeled by the Drude model of conductivity[9], thereof the name. The Drude model agrees with experiments on cuprates above T_c [10].

6.1 Drude Model

The Drude model is obtained by treating the charge carriers classically. They are expected to obey the differential equation

$$\frac{dv}{dt} = \frac{q}{m}E - \frac{1}{\tau}v \quad (6.1)$$

Solving this for harmonic $E = E_0 \exp(-i\omega t)$ gives

$$v = \frac{\tau q E_0}{m(1 - i\omega\tau)} \exp(-i\omega t). \quad (6.2)$$

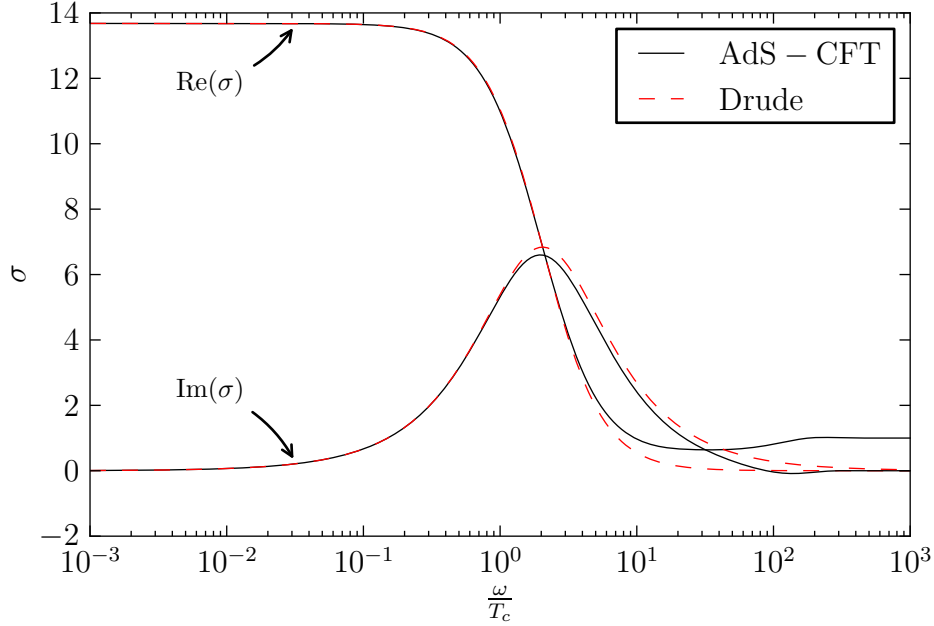


Figure 6.1: Conductivity for $\alpha_2 = 1$ and $T = 2T_c$. The Drude parameters have been obtained from the lowest frequency data-point.

This gives the conductivity

$$\sigma(\omega) = \frac{J(\omega)}{E(\omega)} = \frac{v(\omega)q\rho}{E(\omega)} = \frac{\tau\rho q^2}{m(1 - i\omega\tau)} = \frac{\sigma_0}{1 - i\tau\omega}. \quad (6.3)$$

where ρ is the density of charge carriers of charge q . This higher order term introduces a perturbation in the low frequency conductivity above the critical point. The perturbation is very close to what is expected from the Drude model of electrical conductivity. This agrees very well with what is obtained for low frequencies above the critical point, especially for high values of α_2 . See Fig. 6.1.

The low frequency limit of the conductivity σ_0 seems to depend linearly on α_2 for values of α_2 larger than α_{2c} . See Fig 6.2. The proportionality constant depends on the temperature, see Fig 6.3, and seems to closely follow a power-law, see Fig 6.4. σ_0 thus seems to closely follow

$$\sigma_0 = C\alpha_2 \left(\frac{T}{T_c}\right)^{-4/3} \quad (6.4)$$

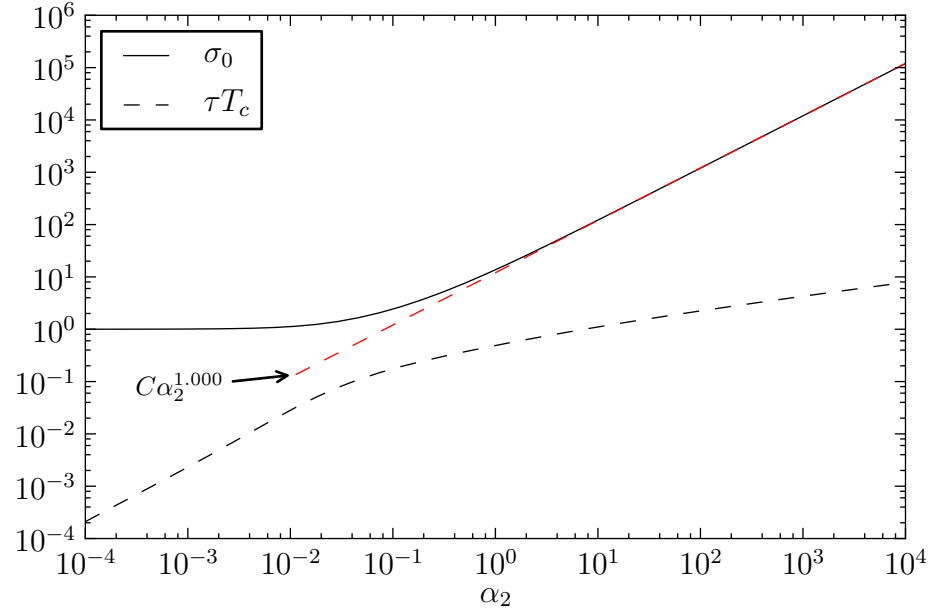


Figure 6.2: Drude parameters as functions of α_2 at $T = 2T_c$.

for values of $\alpha_2 \gg TODO$.

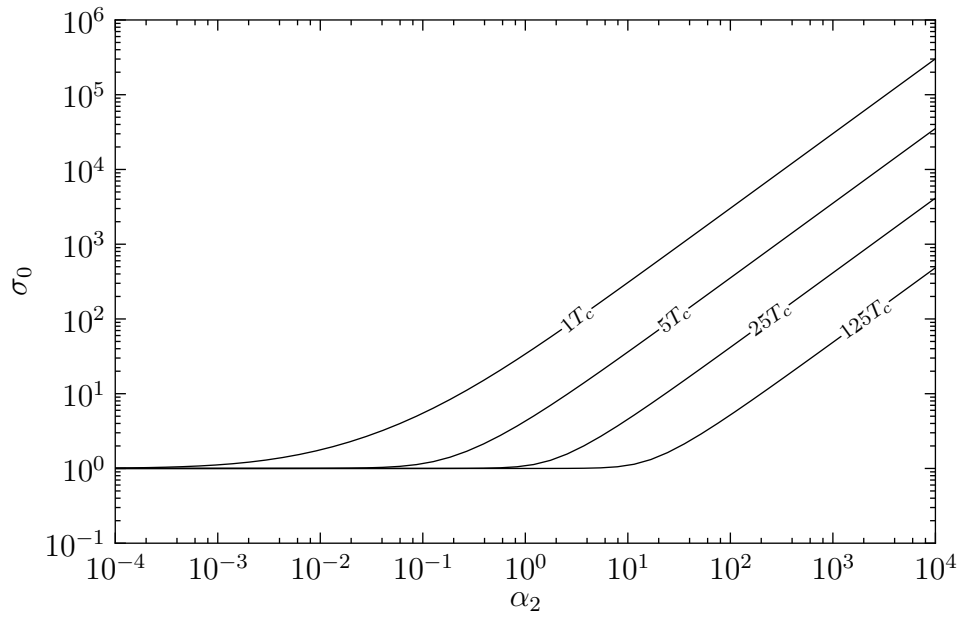


Figure 6.3: $\omega \rightarrow 0$ limit of the conductivity as a function of α_2 for different temperatures.

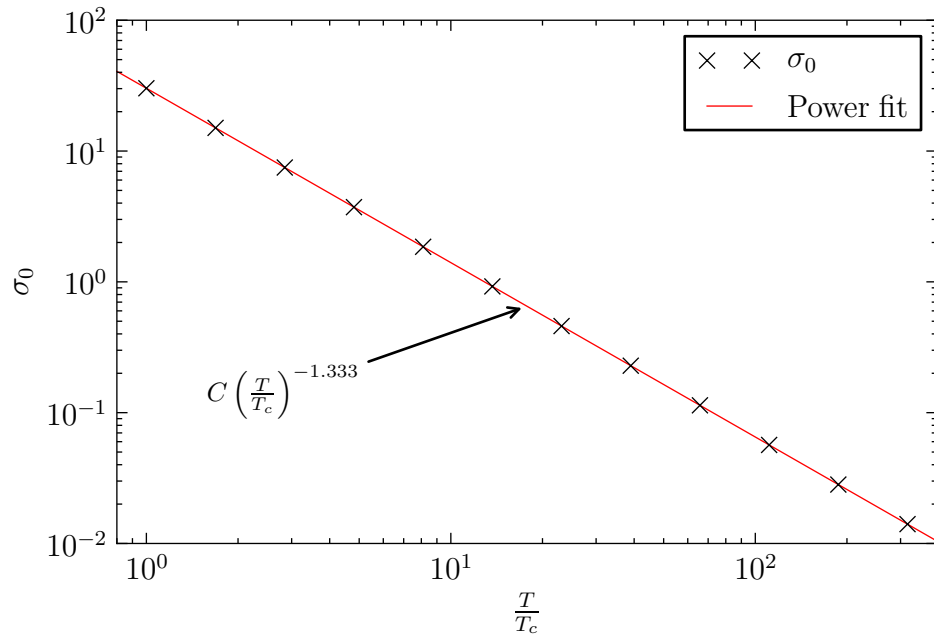


Figure 6.4: σ_0/α_2 as a function of temperature for large α_2 .

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