Holographic Methods for Condensed Matter Physics

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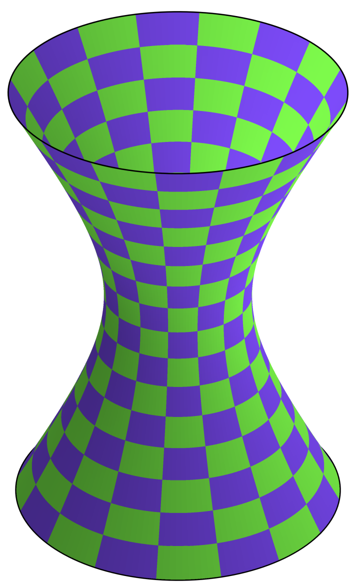

Acknowledgements

Thanks Mum.

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Chapter 1

Introduction

The AdS-CFT correspondence was conjectured in 1997 by Juan Maldacena. It relates the physics of a string theory in Anti de Sitter space (AdS) to a conformal field theory (CFT) on the boundary of the AdS space. AdS is the maximally symmetric space of constant negative curvature. AdS spaces have many interesting properies which will not be described here in detail. A useful visualization of a two-dimensional AdS space can be made by embedding it as a hyperbola in a three-dimensional space-time with two time directions. The inherited metric is then that of two-dimensional AdS-space with one time direction. See figure on the front of this report. The two time directions are there in the horizontal plane. This embedding gives a periodic time which is not necessary. The string theory in the AdS space is gravitational and thus perturbs the AdS space. The correspondence only requires an asymptotically AdS space. The boundary where the conformal field theory lives is then a flat space of one dimension lower. This difference in dimension is why this approach is called *holographic*. The action of the isometry group of AdS space on the AdS boundary is the conformal group, thus requiring conformal symmetry of the boundary theory.
A conformal field theory is a quantum field theory with invariance under conformal transformations. The conformal group is the Poincaré group with dilations and special conformal transformations added. Dilations are scaling transformations. The word *conformal* comes from the angle-preserving property of these transformations.

There is no proof of the correspondence but it has been extensively tested. The field theory of the original conjecture[1] was a supersymmetric Yang-

Mills theory. Extensions of the conjecture has later been made and we will here use a field theory without supersymmetry. That the extended correspondence also holds is motivated in for example [2].

The strength of the duality comes from that the bulk theory is weakly coupled when the boundary theory is strongly coupled and vice versa. This lets us solve otherwise computationally unavailable problems on the strongly coupled side by solving them on the weakly coupled side.

1.1 The Correspondence

The correspondence can be formulated by the GKPW equation [3]

$$Z_{\mathrm{CFT}} = Z_{\mathrm{strings}} \tag{1.1}$$

where Z_{CFT} is the partition function of the boundary theory and Z_{strings} is the partition function of the bulk theory. The exact relation between the Lagrangians of the theories is unknown. Fields in the boundary theory correspond to fields in the bulk theory of same spin.
The boundary theory being scale invariant has different scalings for different operators. Consider an operator $\mathcal{O}$. Scaling the boundary ...TODO TODO Lorentz invariance, relativity, causality, locality?

1.2 Applications

The correspondence can be used both ways but we will consider a strongly coupled boundary theory. Conformal field theories are characterized by not having any specific length scale. Physics at critical points often have this property. A critical point can be a thermodynamic phase transition or a quantum phase transition. The characteristic length goes to infinity as the critical point is approached and the length scale disappears. The physics near a critical point can be expected to be similar to the critical system and finding the critical behaviour is then of interest.

Examples of strongly coupled systems exhibiting critical behaviour are, quark-gluon plasmas[4], high T_c superconductors[5], and possibly graphene

[5].

We will hereafter focus on high T_c superconductors. These superconductors are layered and the electrons effectively moves in two dimensions. They are believed to be both strongly coupled and scale-invariant[5].

Chapter 2

Application to Two-Dimensional Condensed Matter Systems

We wish to model a high T_c superconductor. Conventional superconductors are well described by the BCS theory where the electrons, photons and phonons are the degrees of freedom of interest. The importance of the phonon interactions was understood from the isotope effect, the mass of the atoms in the lattice changed the superconductivity behavior. The isotope effect is though much weaker, TODO cite, in high temperature superconductors and the phonons are thus not believed to be important for high temperature superconductivity. The important degrees of freedom are the electrons and the photons. The electrons are, just as in BCS theory, expected to form Cooper-pairs, TODO cite. These are pairs of electrons of opposite spin but otherwise in the same state effectively becoming spin 0 particles. Our high temperature superconductor model will thus contain two fields, a spin 1 field A_a for the photons and a spin 0 field ψ for the Cooper-pairs.

The superconductor lives in 2+1 dimensional flat space. We will use coordinates x, y for the spatial directions and t for time. The extra dimension in the AdS dual will be parametrised by the coordinate z. See Appendix .1 for details on how indices are labeled and ordered in this report.

2.1 Symmetry Assumptions

The bulk theory should have the same symmetries as the boundary theory. We therefore impose a U(1) gauge symmetry of the complex ψ field. Lorentz invariance will also be used for both theories eventhough relativistic phenomena hardly are important for superconductivity.

2.2 A Lagrangian

There are many different ways to construct a bulk Lagrangian for the fields A_a and ψ and the metric g_{ab}. A Lagrangian previously used succesfully to model two-dimensional electron condensates[6],[7] will initially be used here.

$$\mathcal{L} = \frac{1}{2\kappa}(R - 2\Lambda) - \frac{1}{4}F_{ab}F^{ab} - m^2|\psi|^2 - |D_a\psi|^2 \tag{2.1}$$

This is obtained using Wilsonian naturalness meaning that the lowest order terms obeying all symmetries are used. A higher order term will be investigated in 4.

The action S is calculated from this Lagrangian from

$$S = \int \mathrm{d}^{d+1}x\sqrt{g}\mathcal{L} + S_{\text{boundary}}. \tag{2.2}$$

where g is the absolute value of the determinant of the metric tensor, $g = |\det g_{ab}|$. S_{boundary} is a boundary term that later will be determined. It doesn't affect the equations of motion but is needed for getting normalizeable modes.

The first term of the Lagrangian is an Einstein-Hilbert term with a cosmological constant Λ. A negative cosmological constant gives a asymptotically anti-de-Sitter space as required. R is the Ricci scalar curvature obtained from the nmetric g_{ab}. The constant κ determines the coupling between the metric and the other fields.

The second term is an ordinary Maxwell term where the electromagnetic tensor F_{ab} is the exterior derivative of the electromegnatic field tensor, $F_{ab} = \partial_a A_b - \partial_b A_a$.

The third and fourth terms are the kinetic and mass terms for the scalar field respectively. D_a is the gauge covariant derivative $D_a = \nabla_a - iqA_a$. ∇_a is the covariant derivative, see Appendix .1. This minimal gauge coupling makes the Lagrangian invariant under a U(1) gauge transformation

$$\psi \rightarrow \mathrm{e}^{i\theta(x)}\psi \tag{2.3}$$

$$A_a \rightarrow A_a + \frac{1}{q}\nabla_a\theta(x). \tag{2.4}$$

This Lagrangian also possess conformal invariance. This means that the Lagrangian is unchanged by the transformation $g_{ab} \rightarrow f(x)g_{ab}$. TODO check. The Lagrangian is also manifestly Lorentz invariant imposing Lorentz invariance of the boundary theory.

2.3 Equations of Motion

The bulk equations of motion are obtained by varying the bulk Lagrangian with respect to all the fields. This can be done with the Euler-Lagrange equation since the action does not contain any higher derivatives. The Euler-Lagrange equation for a scalar field χ states

$$\nabla_a\left(\frac{\partial\mathcal{L}}{\partial(\nabla_a\chi)}\right) - \frac{\partial\mathcal{L}}{\partial\chi} = 0. \tag{2.5}$$

First vary ψ. This gives

$$\left(m^2 - \nabla^2 + q^2A^2 + iq(\nabla_aA^a)\right)\psi = 0. \tag{2.6}$$

Varying A_a gives these equations of motion

$$-\nabla_aF^{ab} + 2q^2|\psi|^2A^b + iq\left(\overline{\psi}\nabla^b\psi - \psi\nabla^b\overline{\psi}\right) = 0 \tag{2.7}$$

A real ψ simplifies calculations and that can be obtained since the gauge invariance lets us relate any configuration with a real one through a gauge transformation. The Lorentz gauge,

$$\nabla_aA^a = 0, \tag{2.8}$$

removes the last term in the parenthesis of the equation of motion for ψ. The equation of motion for ψ does not mix the real and imaginary parts after this choice and ψ can be taken to be real since a global shift of phase does not affect A_a, see (2.4). The gauge is still not completely fixed, a gauge transformation $\theta(x)$ such that $\nabla_a \nabla^a \theta(x) = 0$ can still be done without violating the gauge condition, (2.8). The equations of motion are

$$\begin{aligned} \left(m^2 - \nabla^2 + q^2 A^2\right) \psi &= 0 \\ -\nabla_a F^{ab} + 2q^2 \psi^2 A^b &= 0. \end{aligned} \tag{2.9}$$

after choosing the Lorentz gauge and a real ψ.

2.4 Parameters

There are multiple unknown parameters in the bulk Lagrangian. These must be investigated to find values that give us the boundary theory we are interested in. The Lagrangian contains the parameters κ, Λ, m^2, q. Some of these parameters might be redundant since we can make different symmetry transformations of fields and coordinates. The physics of the bulk are treated in the classical limit and the Lagrangian can thus be changed as long as the equations of motion for ψ and A_a are left unchanged.

2.4.1 κ

The Einstein-Hilbert term of the Lagrangian makes the theory gravitational. κ is proportional to Newton's gravitational constant. A small κ gives the probe limit where the metric equations of motion can be solved independently of the other fields. This can be understood by varying the Lagrangian with respect to the metric, the Einstein-Hilbert part gives a term inversely proportional to κ and the rest of the Lagrangian gives the stress-energy tensor independent of κ.

This greatly simplifies calculations and will therefore be used throughout this work. It is though not guaranteed that the interesting boundary theories are dual to bulk theories in the probe limit. Earlier studies have though found that interesting boundary systems can be obtained by treating a bulk in the probe limit. A superconducting condensate developes for low temperatures in the work by S. Hartnoll, C. Herzog and G. Horowitz [6] where the bulk is treated in the probe limit. $\kappa \to 0$ is a fixed-point of the theory so the physics is independent of the exact value of κ as long as we are in the probe limit.

2.4.2 Λ

Scale-invariance of the system lets us choose an arbitrary λ. Two systems with different λ can be shown to be equal by a rescaling. λ sets a length scale L

$$L = \sqrt{-\frac{3}{\Lambda}} \tag{2.10}$$

to which other parameters, e.g m^2, can be related. Scale-invariance can thus not be used to choose those parameters freely. The factor 3 is used so that we later will see that L becomes the AdS radius.
λ will be set to a convenient number in numerical calculations but kept in calculations for clarity.

2.4.3 q

q sets the strength of the gauge coupling and is thus the charge of the scalar field. Considering $\tilde{\psi} = q\psi$ and $\tilde{A}_a = qA_a$ as the fields gives a Lagrangian of the same form but divided by q^2 except for the term originally containing q^2 which is divided by q^4. Multiplying the Lagrangian by a constant doesn't affect the equations of motion so the system can be solved for any value of q. Other solutions can then be obtained by rescaling the fields.

2.4.4 m^2

m is the mass of the scalar field in the bulk. What values of m that are suitable will be investigated later when solving the equations of motion in the bulk.

2.5 Partition Function

The partition function is a concept from statistical physics. It is for a quantum-mechanical system defined as

$$Z(\beta) = \mathrm{tr}(\mathrm{e}^{-\beta\hat{H}}) \tag{2.11}$$

where $\hat{H}$ is the time-independent Hamiltonian and $\beta = (k_B T)^{-1}$ where k_B is Boltzmann's constant and T is the temperature. Hereafter we let $k_B = 1$

meaning that we measure temperature in units of what energy it corresponds to. The partition function is similar to the trace of the time-evolution operator $\hat{U}(t_2, t_1)$ evolving a state from time t_1 to t_2

$$\mathrm{tr}(\hat{U}(t_2, t_1)) = \mathrm{tr}(\mathrm{e}^{-i\frac{(t_2-t_1)\hat{H}}{\hbar}}) = f(t_2 - t_1). \tag{2.12}$$

We will hereafter let $\hbar = 1$ by measuring energy in units of inverse time. The partition function can be obtained as the analytical extension of f,

$$Z(\beta) = f(-i\beta). \tag{2.13}$$

The trace of the time evolution operator can be calculated as an integral over configuration space which in our case will be field configurations Ψ,

$$f(t) = \int \mathcal{D}[\Psi]\langle\Psi|\hat{U}(t, 0)|\Psi\rangle. \tag{2.14}$$

The time-evolution operator matrix elements can be calculated using Feynman's path integrals [8],

$$\langle\Psi_2|\hat{U}(t, 0)|\Psi_1\rangle = \int_{\Psi_1}^{\Psi_2} \mathcal{D}[\Psi(t)]\mathrm{e}^{iS[\Psi(t)]} \tag{2.15}$$

where $S[\Psi(t)]$ is the action of the path $\Psi(t)$ through field configurations. Combining these results tells us that the trace $f(t)$ can be calculated as a periodic time path integral with period t

$$f(t) = \int_0^t \mathcal{D}[\Psi(t)]\mathrm{e}^{iS[\Psi(t)]}. \tag{2.16}$$

This means that $Z(\beta)$ can be obtained by calculating a path integral where the time is imaginary and periodic. The action must be analytically extended to imaginary time $\tau = it$. The metric of the CFT is the metric induced from the bulk theory. The boundary is time-like and the time periodicity of the two theories are thus the same. This means that they are at the same temperatures. The path integral can for a classical theory be calculated using the saddle point approximation. The partition function is then

$$Z(\beta) = f(-i\beta) \overset{\text{classical}}{=} \mathrm{e}^{iS_c} \tag{2.17}$$

where S_c is the action of classical periodic path with period $-i\beta$. For a solution stationary in time the action is $S_c = -i\beta L_c$ where L_c is the imaginary time saddle-point Lagrangian. The classical partition function then becomes

$$Z(\beta) = \mathrm{e}^{\beta L_c} \tag{2.18}$$

2.6 Field Theory Expectation Values

The boundary values of the bulk fields correspond to fields in the CFT. Expectation values of observables in the CFT can be calculated using a generating functional $Z[J]$. This is a partition function for a system with a perturbed Lagrangian $\mathcal{L}_J(x) = \mathcal{L}(x) + J(x)\mathcal{O}(x)$. Here $\mathcal{O}(x)$ is a local operator on the fields. The generating functional can be regarded an expectation value of a system with the original Lagrangian $\mathcal{L}$.

$$\begin{aligned} Z[J] &= \int_0^{-i\beta} \mathcal{D}[\psi(t)] \mathrm{e}^{i\int \mathcal{L}(\psi(x)) + J(x)\mathcal{O}(\psi(x))} \\ &= Z[0] \int_0^{-i\beta} \mathcal{D}[\psi(t)] \frac{\mathrm{e}^{i\int \mathcal{L}(\psi(x))}}{Z[0]} \mathrm{e}^{i\int J(x)\mathcal{O}(\psi(x))} \\ &= Z[0] \langle \mathrm{e}^{i\int J(x)\mathcal{O}(\psi(x))} \rangle \end{aligned} \tag{2.19}$$

Taking a functional derivative of this gives:

$$\begin{aligned} \frac{\delta}{\delta J(x)} \log(Z[J])|_{J=0} &= \frac{Z[0]\langle -ii\mathcal{O}(\psi(x))\mathrm{e}^{i\int J(x)\mathcal{O}(\psi(x))}\rangle}{Z[0]\langle \mathrm{e}^{i\int J(x)\mathcal{O}(\psi(x))}\rangle}\Big|_{J=0} \\ &= \langle \mathcal{O}(\psi(x)) \rangle \end{aligned} \tag{2.20}$$

The partition functions of the bulk and boundary theories are the same even for a perturbed Lagrangian. Both Lagrangians are then perturbed. Expectation values of operators of the boundary theory can thus be calculated using the partition function for the bulk theory. The next chapter will show how to solve the bulk theory and obtain CFT observables.

Chapter 3

Solution of the Classical Bulk Theory

We wish to compute expectation values of the CFT. This will be done through the correspondence (1.1). The bulk theory must then be solved so that the partition function there can be obtained. The bulk theory was assumed to be classical in Section 1.1 so the system is solved by finding the equations of motion for the fields and solving them. This lets ut calculate the partition function and use it to find CFT expectation values through 2.20.

3.1 Definitions

The Lagrangian describes a general system so there are many solutions to the equations of motion. We wish to investigate two properties of a superconductor, the development of a condensate at low temperatures and the conductivity at different frequencies. We are interested in a superconductor subject to spatially uniform conditions, the applied electric field is uniform and the chemical potential is uniform. The atomic lattice and its imperfections are thus not accounted for but interesting superconductivity behaviour has been obtained anyways [6]. It is thus enough to look at a system with translational symmetry in the x and y directions. A rotationally invariant superconductor will further be studied. The system is subject to conditions constant in time, e.g. no time-dependent chemical potential. This lets us assume time-independence while solving the non-linear field equations.

The conductivity is the linear electrical current response to an applied transverse electrical field. We can apply this in the x direction due to the rotational symmetry. We let the applied field have a harmonic time dependence $\exp(it\omega)$ so we can get the response function in the frequency domain. The linear response is sought so the applied field should be infinitesimal. The applied field breaks the rotational and time symmetries but since it is infinitesimal and we are not interested in the effect is has on the other fields it can be neglected while calculating them. The applied field is later added with the other fields as background solutions.
The electrical field in the x-direction is $E_x = F_{xt} = \nabla_x A^t - \nabla_t A^x$. Translational symmetry gives $E_x = \nabla_t A^x$.

These limitations lets us do the following definitions

$$\begin{aligned} \mathrm{d}s^2 &= g_{zz}(z)\mathrm{d}z^2 + g_{tt}(z)\mathrm{d}t^2 + g_{xx}(z)(\mathrm{d}x^2 + \mathrm{d}y^2) \\ \psi &= \psi(z) \\ A_a &= (A_z(z), \phi(z), A_x(z)\exp(it\omega), 0) \end{aligned} \tag{3.1}$$

where $\phi(z)$ is infinitesimal. The gauge condition requires

$$\nabla_a A^a = \partial_a A^a + \Gamma^a{}_{ba} A^b = 0 \tag{3.2}$$

this gives a homogeneous first-order linear ordinary differential equation for $A_z(z)$. The remaining gauge symmetry can be used to set $A_z(z) = 0$ for a z and $A_z(z)$ can thus be taken to be identically 0 for all z.

The explicit z and t-dependence of these functions will hereafter be ommited.

3.2 Metric

We wish to find the metric saddle-point of the periodic imaginary time path integral. The bulk equation of motion for the metric g_{ab} is the Einstein equation with a cosmological constant

$$R_{ab} - \frac{1}{2} g_{ab} R + g_{ab}\lambda = \kappa T_{ab} \tag{3.3}$$

where R_{ab} is the Ricci curvature tensor and T_{ab} is the stress-energy tensor. We assumed the probe limit in Section **??** and therefore neglect the right hand side of this equation. We want a translationally invariant solution in the t, x, and y directions that is asymptotically AdS. The solution is known to be a black hole[2], the Schwarzschild metric in AdS space. The metric has the following form in a particular choice of coordinates where the radial coordinate z is 0 at the boundary and z_h at the horizon

$$g_{ab}\mathrm{d}x^a\mathrm{d}x^b = \frac{L^2}{z^2}\left(\frac{\mathrm{d}z^2}{f(z)} - f(z)\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2\right). \tag{3.4}$$

Here $f(z) = 1 - z^3 z_h^{-3}$. $f(z)$ approaches 1 at the boundary and the space is asymptotically AdS. There is a horizon at $z = z_h$ where $f(z_h) = 0$. The space behind the horizon can not affect the physics of the boundary and can thus be neglected in our calculations. This solution is periodic in imaginary time. Consider the near-horizon metric where

$$f(z) = f(z_h) - (z_h - z)f'(z_h) + \mathcal{O}((z_h - z)^2) \approx 3(1 - zz_h^{-1}) \tag{3.5}$$

Do the change of variables $\rho^2 = \frac{4L^2}{3}(1 - zz_h^{-1})$. This gives $f(z) \approx \rho^2 \frac{9}{4L^2}$ and $\rho^2\mathrm{d}\rho^2 = \mathrm{d}z^2 z_h^{-2}\frac{4L^4}{9}$. The near-horizon metric is then

$$\begin{aligned} g_{ab}\mathrm{d}x^a\mathrm{d}x^b &= \frac{L^2}{z_h^2}\left(\frac{\rho^2\mathrm{d}\rho^2}{z_h^{-2}\frac{4L^4}{9}\rho^2\frac{9}{4L^2}} - \rho^2\frac{9}{4L^2}\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2\right) \\ &= \mathrm{d}\rho^2 - \rho^2\frac{9}{4z_h^2}\mathrm{d}t^2 + \frac{L^2}{z_h^2}\left(\mathrm{d}x^2 + \mathrm{d}y^2\right). \end{aligned} \tag{3.6}$$

Now extend this to imaginary time $\tau = it$

$$g_{ab}\mathrm{d}x^a\mathrm{d}x^b = \mathrm{d}\rho^2 + \rho^2\left(\frac{3}{2z_h}\mathrm{d}\tau\right)^2 + \frac{L^2}{z_h^2}\left(\mathrm{d}x^2 + \mathrm{d}y^2\right). \tag{3.7}$$

The near horizon metric is then that of a Euclidean plane in polar coordinates. There is thus a deficit angle unless $\frac{3}{2z_h}\tau$ has a periodicity of 2π. The imaginary time has periodicity β so we thus have

$$\frac{3}{2z_h} = \frac{2\pi}{\beta} \tag{3.8}$$

This gives the relationship between z_h and the temperature

$$T = \frac{3}{4\pi z_h}. \tag{3.9}$$

This expression for the temperature agrees with the Beckenstein-Hawking temperature of a black hole.
The backreaction Δg_{ab} from the non-zero fields will be of order κT_{ab} according to (3.3). The Einstein equation is obtained by varying the Lagrangian with respect to g_{ab} so $\Delta S \propto \kappa^{-1}\Delta g_{ab}^2$ for the solution. We thus have that $\Delta S \propto \kappa T_{ab}^2$ and the backreaction can safely be neglected also when calculating the action from different field configurations.

This background metric can now be used instead of solving the equations of motion for the metric together with the fields. The gravitational part of the Lagrangian must be kept when calculating the value of the total action which is dominated by the gravitational part.

The horizon z_h and the curvature length L set length scales in the metric. Length units in the numerical solution can be chosen such that $z_h = 1$. This means that we for different temperatures have different units since z_h is related to the temperature. We will have to convert between these units when comparing results from different temperatures.

3.3 Equations of Motion

The equations of motion for $\psi(z)$, $\phi(z)$ and $A_x(z)$ can now be obtained. Inserting (3.1) into the equations of motion (2.9) and using the metric (3.4) gives

$$\begin{cases} \Big(q^2 z^2 \phi^2 - L^2 m^2 f + zf(zf' - 2f)\partial_z + z^2 f^2 \partial_z \partial_z\Big)\psi = 0 & (3.10) \\ \Big(-2q^2\psi^2 L^2 + z^2 f \partial_z \partial_z\Big)\phi = 0 & (3.11) \\ \Big(-2q^2\psi^2 L^2 f + z^2\omega^2 + z^2 f f' \partial_z + z^2 f^2 \partial_z \partial_z\Big)A_x = 0 & (3.12) \end{cases}$$

The formulas in Appendix .2 have here been used.

3.4 AdS Boundary behaviour of Fields

A Frobenious expansion of these equations can be done at the boundary, $z = 0$. The leading behaviours of the functions are

$$\begin{cases} \psi = \psi_{(0)} \left(\dfrac{z}{z_h}\right)^{\Delta_\psi} & (3.13) \\ \phi = \phi_{(0)} \left(\dfrac{z}{z_h}\right)^{\Delta_\phi} & (3.14) \\ A_x = A_{x(0)} \left(\dfrac{z}{z_h}\right)^{\Delta_{A_x}} & (3.15) \end{cases}$$

where Δ_ψ, Δ_ϕ and Δ_{A_x} are constants that are to be determined. This is a slight assumption since not all functions have this type of leading behaviour. Entering this in the equations of motion yields

$$\begin{cases} q^2 z^2 \phi_{(0)}^2 s^{2\Delta_\phi} - L^2 m^2 f + f(zf' - 2f)\Delta_\psi + f^2 \Delta_\psi(\Delta_\psi - 1) = 0 & (3.16) \\ -2q^2 \psi_{(0)}^2 s^{2\Delta_\psi} L^2 + f\Delta_\phi(\Delta_\phi - 1) = 0 & (3.17) \\ -2q^2 \psi_{(0)}^2 s^{2\Delta_\psi} L^2 f + z^2\omega^2 + zff'\Delta_{A_x} + f^2 \Delta_{A_x}(\Delta_{A_x} - 1) = 0. & (3.18) \end{cases}$$

where $s = zz_h^{-1}$. This immidiatly gives $\Delta_\psi \geq 0$ and $1 + \Delta_\phi \geq 0$. Assume the strict inequalities. The leading order behaviour is then

$$\begin{cases} -L^2 m^2 - 2\Delta_\psi + \Delta_\psi(\Delta_\psi - 1) = 0 & (3.19) \\ \Delta_\phi(\Delta_\phi - 1) = 0 & (3.20) \\ \Delta_{A_x}(\Delta_{A_x} - 1) = 0. & (3.21) \end{cases}$$

with solutions

$$\begin{cases} \Delta_\psi = \dfrac{3}{2} \pm \sqrt{\dfrac{9}{4} + L^2 m^2} & (3.22) \\ \Delta_\phi = 0, 1 & (3.23) \\ \Delta_{A_x} = 0, 1. & (3.24) \end{cases}$$

Now assume $\Delta_\psi = 0$. (3.17) gives

$$-2q^2 \psi_{(0)}^2 L^2 + \Delta_\phi(\Delta_\phi - 1) = 0 \tag{3.25}$$

while (3.16) gives $\Delta_\phi = -1$. We then have

$$\begin{cases} q^2\phi_{(0)}^2 z_h^2 = L^2 m^2 & (3.26) \\ q^2\psi_{(0)}^2 L^2 = 1 & (3.27) \\ \Delta_{A_x}(\Delta_{A_x} - 1) = 2. & (3.28) \end{cases}$$

with solutions $\Delta_{A_x} = -1, 2$. First assuming $\Delta_\phi = -1$ yields the same result. There are though no solutions to (3.26) for the negative m^2 we later will consider and infinities are encountered when calculating the action for these solutions so they will not be considered. All useful solutions are thus given by equation 3.22 to 3.24.

3.4.1 Boundary conditions at Horizon

The same kind of expansion can be made at the horizon but there are some simplifying conditions. The time component of the metric vanishes at the horizon, $f(z_h) = 0$. This means that $A_t(z_h) = \phi(z_h)$ must be zero for $A^t(z_h)$ to be finite. Expand the fields as

$$\begin{cases} \psi = \psi_{(1)} s^{\Upsilon_\psi} & (3.29) \\ \phi = \phi_{(1)} s^{\Upsilon_\phi} & (3.30) \\ A_x = A_{x(1)} s^{\Upsilon_{A_x}} & (3.31) \end{cases}$$

where s now is $(1-z/z_h)$. The function f can be expanded as $f = 1-z^3 z_h^{-3} = 1-(1-s)^3 = 3s - 3s^2 + s^3$. Insert these leading terms in the equations of motion. We get

$$\begin{cases} q^2 z^2 \phi_{(1)}^2 s^{2\Upsilon_\phi} + 9z_h^2\Upsilon_\psi + z_h^2 9\Upsilon_\psi(\Upsilon_\psi - 1) = 0 & (3.32) \\ -2q^2\psi_{(1)}^2 s^{2\Upsilon_\psi} L^2 + z_h^2 3s^{-1}\Upsilon_\phi(\Upsilon_\phi - 1) = 0 & (3.33) \\ -6q^2\psi_{(1)}^2 s^{2\Upsilon_\psi+1} L^2 + z_h^2\omega^2 + z_h^2 9\Upsilon_{A_x} + z_h^2 9\Upsilon_{A_x}(\Upsilon_{A_x} - 1) = 0. & (3.34) \end{cases}$$

Assume $2\Upsilon_\psi + 1 > 0$. The leading terms are then

$$\begin{cases} \Upsilon_\psi = 0 & (3.35) \\ \Upsilon_\phi(\Upsilon_\phi - 1) = 0 & (3.36) \\ \omega^2 + 9\Upsilon_{A_x}^2 = 0. & (3.37) \end{cases}$$

This has solutions

$$\begin{cases} \Upsilon_\psi = 0 & (3.38) \\ \Upsilon_\phi = 1 & (3.39) \\ \Upsilon_{A_x} = \pm\frac{i\omega}{3}. & (3.40) \end{cases}$$

The two possible Υ_{A_x} represent ingoing and outgoing solutions. $A_x(z,t)$ is close to the horizon

$$s^{\pm\frac{i\omega}{3}}\exp(i\omega t) = \exp\left(i\omega(t \pm \frac{\log s}{3})\right) \tag{3.41}$$

The phase is constant for

$$s = \exp(\mp 3t) \tag{3.42}$$

These equations have the trivial solution

$$\begin{aligned} \psi =& 0 \\ \phi = \mu - \mu\frac{z}{z_h} \end{aligned} \tag{3.43}$$

3.4.2 Choice of scalar mass m

The mass squared of a scalar field in flat space must be non-negative for stability. This is though not the case in a space with negative curvature. The Breitenlohner-Freedman bound (BF) is a lower bound on m^2 of a massive scalar field in AdS space with . It requires

$$L^2m^2 \geq -\frac{d^2}{4} \tag{3.44}$$

for stability[9]. The scalar field ψ should obey this bound far away from the black-hole for normalizeable modes, TODO explain. We would though like a spontanous symmetry breaking of ψ near the black hole (low energies) corresponding to the electron condensate [10]. This can happen because the coupling of ψ to A_a gives ψ an effective mass that might break the BF bound near the black hole.
The effective mass is given by:

$$m_{\text{eff}}^2 = m^2 + A_aA^a = m^2 - \frac{z^2}{L^2(1 - z^d z_h^{-d})}\phi^2 \tag{3.45}$$

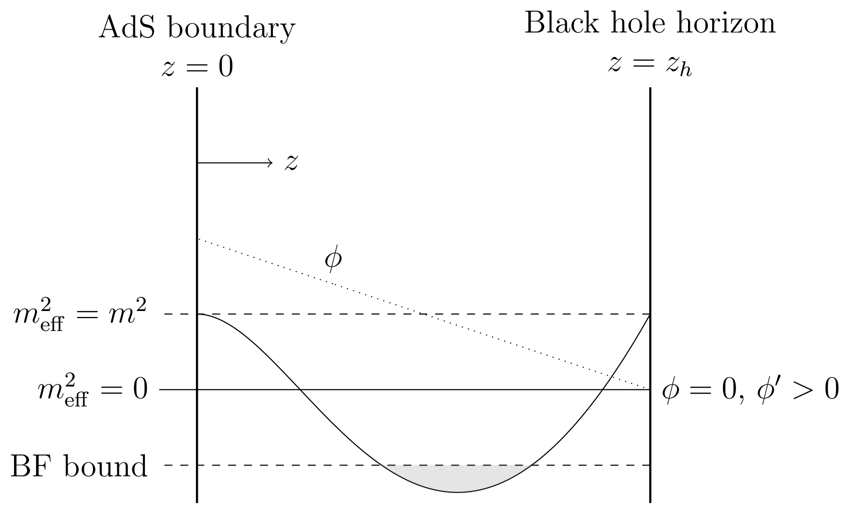

Figure 3.1: Schematic plot of how the the effective mass breaks the BF bound outside the horizon. A value of ϕ has been assumed.

This can for large enough values of ϕ break the BF-bound, see Fig. 3.1. Consider the trivial, uncondensed, solution (3.43). When does this give an effective mass breaking the BF-bound and possibly enabling an additional condensed solution? The location of the effective mass minimum, z_0, can be found by differentiating (3.45) by z and using (3.43),

$$\frac{z_0}{z_h} = \frac{1}{3}\left(\sqrt[3]{37 + 9\sqrt{17}} - \frac{2}{\sqrt[3]{37 + 9\sqrt{17}}} - 2\right). \tag{3.46}$$

The effective mass breaks the BF-bound at z_0 when

$$\frac{\mu}{T} > \frac{2\pi}{\sqrt{3}}\sqrt{\frac{4L^2m^2 + 9}{8 - \sqrt[3]{142 - 34\sqrt{17}} - \sqrt[3]{142 + 34\sqrt{17}}}} \tag{3.47}$$

where (3.9) has been used.

3.5 Boundary Behavior of Bulk Fields

The bulk Lagrangian considered will contain different fields and depends both on the fields and their first derivatives. Consider a field ψ with a kinetic term $-(\partial_a\psi)^2$ and a potential term $V(\psi)$. The classical solution is the one that extremizes the action. The action integral contains the metric as an integration measure

$$S = \int \mathrm{d}^{d+1}x\sqrt{|\det g_{ab}|}\mathcal{L} \equiv \int \mathrm{d}^{d+1}x\sqrt{g}\mathcal{L}. \tag{3.48}$$

The Euler-Lagrange equation is obtained by varying the action. The integration measure can then be regarded as part of the Lagrangian or covariant derivatives can be used in the derivation of the Euler-Lagrange equation. The measure becomes when using the metric (3.4) $L^{d+1}z^{-d-1}$. The Euler-Lagrange equation gives

$$\begin{aligned} 0 = \partial_a\left(\frac{\partial(z^{-d-1}(V(\psi)-(\partial_b\psi)^2))}{\partial(\partial_a\phi)}\right) - \frac{\partial(z^{-d-1}(V(\psi)-(\partial_b\psi)^2))}{\partial\phi} = \\ = -\partial_a\left(z^{-d-1}2\partial^a\psi\right) - z^{-d-1}V'(\psi) \end{aligned} \tag{3.49}$$

We will be interested in boundary systems with translational symmetry so ψ is assumed to be a function of z, TODO motivate more. The equation of motion then becomes

$$\begin{aligned} 0 = -\partial_z\left(z^{-d-1}2g^{zz}\partial_z\psi\right) - \frac{V'(\psi)}{z^{d+1}} = \\ -\partial_z\left(z^{-d-1}2(z^2L^{-2}f(z))\partial_z\psi\right) - \frac{V'(\psi)}{z^{d+1}} = \\ -z^{-d-1}2z^2L^{-2}f(z)\psi'' - L^{-2}\left((-d+1)z^{-d}2f(z) + z^{-d+1}2f'(z)\right)\psi' - \frac{V'(\psi)}{z^{d+1}} \end{aligned} \tag{3.50}$$

This gives a second order differential equation for $\psi(z)$

$$0 = -z^2 2f(z)\psi'' - \left((-d+1)z2f(z) + z^2 2f'(z)\right)\psi' - L^2V'(\psi) \tag{3.51}$$

Now consider the boundary, $z = 0$. Our metric is required to be asymptotically AdS so $f(0) \to 1, zf'(0) \to 0$. ψ can be expanded at the boundary as a Laurent series. Call the lowest exponent in this series Δ. ψ will then behave as z^Δ near the boundary. This should solve the differential equation in the near boundary limit. Insertion of z^Δ into the differential equation and taking the limit of small z gives

$$\begin{aligned} 0 &= -z^2 2\Delta(\Delta-1)z^{\Delta-2} - \left((-d+1)2z + z^2 2f'(z)\right)\Delta z^{\Delta-1} - L^2V'(z^\Delta) \\ &= z^\Delta\left(-2\Delta(\Delta-1) - 2(-d+1)\Delta\right) - L^2V'(z^\Delta). \end{aligned} \tag{3.52}$$

Now consider a potential for a massive scalar field, $V(\psi) = -m^2\psi^2 + \mathcal{O}(\psi^3)$. We then get the following equation for Δ

$$0 = \Delta^2 - d\Delta - L^2m^2 \tag{3.53}$$

in the limit $z \to 0$. This has solutions

$$\Delta = \frac{d \pm \sqrt{d^2 + 4L^2m^2}}{2}. \tag{3.54}$$

ψ thus goes as z^{Δ_0} where Δ_0 is the smaller solution and Δ_1 the larger. The leading behaviour of ψ near $z = 0$ is

$$\psi = \psi_0 \left(\frac{z}{L}\right)^{\Delta_0} + \psi_1 \left(\frac{z}{L}\right)^{\Delta_1} \tag{3.55}$$

unless $\Delta_1 - \Delta_0 >= 1$ and further terms from the series corresponding to Δ_0 must be included.

What will the contribution to the action from this solution be? Consider the action contribution from the region $z \in [\epsilon, \delta]$ where δ is small and $\epsilon \to 0$.

$$
\begin{aligned}
S_{[\epsilon,\delta]} =& \int_{z\in[\epsilon,\delta]} \mathrm{d}^{d+1}x\sqrt{g}\mathcal{L} = \\
=& V\int_\epsilon^\delta \mathrm{d}z \left(\frac{z}{L}\right)^{-d-1} \left(-m^2\psi^2 - (\partial_a\psi)^2\right) = \\
=& V\int_\epsilon^\delta \mathrm{d}z \left(\frac{z}{L}\right)^{-d-1} \left(-m^2(\psi_0\left(\frac{z}{L}\right)^{\Delta_0} + \psi_1\left(\frac{z}{L}\right)^{\Delta_1})^2 - (\partial_a(\psi_0\left(\frac{z}{L}\right)^{\Delta_0} + \psi_1\left(\frac{z}{L}\right)^{\Delta_1}))^2\right) = \\
=& V\int_\epsilon^\delta \mathrm{d}z \left(\frac{z}{L}\right)^{-d-1} \Big[- m^2\left(\psi_0^2\left(\frac{z}{L}\right)^{2\Delta_0} + \psi_1^2\left(\frac{z}{L}\right)^{2\Delta_1} + 2\psi_0\psi_1\left(\frac{z}{L}\right)^{\Delta_0+\Delta_1}\right) \\
&- g^{zz}L^{-2}(\Delta_0\psi_0\left(\frac{z}{L}\right)^{\Delta_0-1} + \Delta_1\psi_1\left(\frac{z}{L}\right)^{\Delta_1-1})^2\Big] = \\
=& V\int_\epsilon^\delta \mathrm{d}z \left(\frac{z}{L}\right)^{-d-1} \Big[- m^2\left(\psi_0^2\left(\frac{z}{L}\right)^{2\Delta_0} + \psi_1^2\left(\frac{z}{L}\right)^{2\Delta_1} + 2\psi_0\psi_1\left(\frac{z}{L}\right)^{\Delta_0+\Delta_1}\right) \\
&- g^{zz}L^{-2}\left(\Delta_0^2\psi_0^2\left(\frac{z}{L}\right)^{2(\Delta_0-1)} + \Delta_1^2\psi_1^2\left(\frac{z}{L}\right)^{2(\Delta_1-1)} + 2\Delta_0\Delta_1\psi_0\psi_1\left(\frac{z}{L}\right)^{\Delta_0+\Delta_1-2}\right)\Big] = \\
=& V\int_\epsilon^\delta \mathrm{d}z \Big[- m^2\left(\psi_0^2\left(\frac{z}{L}\right)^{2\Delta_0-d-1} + \psi_1^2\left(\frac{z}{L}\right)^{2\Delta_1-d-1} + 2\psi_0\psi_1\left(\frac{z}{L}\right)^{-1}\right) \\
&- L^{-2}\left(\Delta_0^2\psi_0^2\left(\frac{z}{L}\right)^{2\Delta_0-d-1} + \Delta_1^2\psi_1^2\left(\frac{z}{L}\right)^{2\Delta_1-d-1} - 2L^2m^2\psi_0\psi_1\left(\frac{z}{L}\right)^{-1}\right)\Big] = \\
=& V\int_\epsilon^\delta \mathrm{d}z \left((-m^2 - \Delta_0^2L^{-2})\psi_0^2\left(\frac{z}{L}\right)^{2\Delta_0-d-1} + (-m^2 - \Delta_1^2L^{-2})\psi_1^2\left(\frac{z}{L}\right)^{2\Delta_1-d-1}\right) = \\
=& VL^{-2}(\Delta_1 - \Delta_0)\int_\epsilon^\delta \mathrm{d}z \left(\Delta_0\psi_0^2\left(\frac{z}{L}\right)^{2\Delta_0-d-1} - \Delta_1\psi_1^2\left(\frac{z}{L}\right)^{2\Delta_1-d-1}\right) = \\
=& VL^{-1}(\Delta_1 - \Delta_0)\left(-\frac{\Delta_0\psi_0^2}{2\Delta_0 - d}\left(\frac{\epsilon}{L}\right)^{2\Delta_0-d} + \frac{\Delta_1\psi_1^2}{2\Delta_1 - d}\left(\frac{\epsilon}{L}\right)^{d-2\Delta_0}\right) + \text{finite} \\
=& VL^{-1}\left(\Delta_0\psi_0^2\left(\frac{\epsilon}{L}\right)^{2\Delta_0-d} - \Delta_1\psi_1^2\left(\frac{\epsilon}{L}\right)^{d-2\Delta_0}\right) + \text{finite}
\end{aligned}
\tag{3.56}
$$

Here $\Delta_0 + \Delta_1 = d$ and $\Delta_0\Delta_1 = -L^2m^2$ have been used. One of these two terms diverges as $\epsilon \to 0$. The term with ψ_0 diverges since $2\Delta_0 - d = -\sqrt{d^2 + 4L^2m^2}$. The action from the near boundary thus diverges. This can be remedied by having a boundary term in the action that exactly cancels this divergence.

The boundary term must thus evaluate to

$$-\Delta_0 V L^{-1}\psi_0^2\left(\frac{\epsilon}{L}\right)^{2\Delta_0-d} \tag{3.57}$$

near the boundary. A boundary term like this can be constructed using $\psi = \psi_0 L^{-\Delta_0}\epsilon^{\Delta_0}$ near the boundary and $\sqrt{\gamma} = L^d z^{-d}$ where γ is the determinant of the metric induced on the boundary by g_{ab}. The boundary term then becomes

$$S_{\mathrm{bdy}} = -\int_{z=\epsilon} \mathrm{d}^d x \Delta_0 L^{-1}\psi^2\sqrt{\gamma} \tag{3.58}$$

This is addition to the Lagrangian is Lorentz invariant and it also has conformal invariance.
Consider another possible term in the Lagrangian, an electromagnetic field A_μ. The Lagrangian is $-\frac{1}{4}F_{ab}F^{ab}$ where $F_{ab} = \partial_a A_b - \partial_b A_a$. Consider fields with t, x_1, and x_2 symmetry.
TODO, check. It is easily shown that the behaviour of these fields is the same when there is an interaction present as will be considered later.

3.6 Horizon Beaviour of Bulk Fields

TODO: Wilson loop=¿$\phi = 0$. Ingoing boundary conditions. Looking at the equations of motion for $z \to z_h$ gives when $\phi(z_h) = 0$

$$\psi(z_h) = \frac{-3z_h\psi'(z_h)}{L^2 m^2} \tag{3.59}$$

3.7 Numerical Solution

Short description of numerics, link to source. Refer to sum rule for accuracy test. TODO

3.8 Thermodynamic Variables

TOD, compare with experiments. The free energy, $A = -T\log Z$, is through the GKPW equation the same for the bulk and the boundary theory. This can be calculated in the classical limit in the bulk.

$$A = -T\log Z \stackrel{\text{classical}}{=} -iTS_c \tag{3.60}$$

Here S_c is the classical periodic time action. The classical field solutions only depend on the z coordinate and are thus proportional to $V = -i\beta V_2$ where V_2 is the area considered in coordinats x_1, x_2. This gives the free energy per surface area

$$\frac{A}{V_2} = -\int_0^{z_h} \mathrm{d}z\sqrt{-g}\mathcal{L} + V^{-1}S_{\mathrm{bdy}} \tag{3.61}$$

Consider the case where $m^2 = -2$. Then $\Delta_0 = 1$ and the boundary term is given by (3.58).

3.8.1 Gravitational Free Energy and Entropy

The action is dominated by the gravitational part since we are in the probe limit. The expression for the temperature (3.9) can be used to calculate the free energy from the action at finite temperature

$$\begin{aligned}\frac{A_{\mathrm{grav}}}{V_2} &= -T\frac{-1}{\kappa}\sqrt{\frac{-2\Lambda}{d(d-1)}}\left(\frac{d}{4\pi TL}\right)^{-d} \\ &= \frac{(4\pi)^d T^d L^{d-1}}{d^d\kappa}\end{aligned} \tag{3.62}$$

The entropy H can now be calculated from the free energy

$$H = -\frac{\partial A}{\partial T} = -\frac{(4\pi)^d T^{d-1} L^{d-1}}{d^{d-1}\kappa}. \tag{3.63}$$

TODO, Beckenstein-Hawking

3.8.2 Free Energy of Scalar and Electromagnetic Fields

It is important to calculate the free energy not only of the gravitational part of the Lagrangian since we have multiple solutions of the field equations and boundary values at the same temperature. Which solution is physical can be found by finding the one of lowest free energy. Here we neglect the contribution from any back-reaction on the metric. The back-reaction is small since κ is small but a small κ also makes the effect of the back-reaction on the free energy large. TODO, motivate or say that we leave this open.

The free energy of the trivial solution (3.43) can be found analytically. For this solution we have

$$\begin{aligned}
\psi &= 0 \\
\psi' &= 0 \\
\phi &= \mu - \mu\left(\frac{z}{z_h}\right)^{d-2} \\
\phi' &= -(d-2)\frac{\mu}{z_h}\left(\frac{z}{z_h}\right)^{d-3}
\end{aligned} \tag{3.64}$$

TODO, explain d-3

$$\begin{aligned}
\frac{A}{V_{\mathrm{SC}}} =& -\int_0^{z_h} \mathrm{d}z\sqrt{-g}\mathcal{L} + V_{\mathrm{SC}}^{-1}S_{\mathrm{bdy}} \\
=& \int_0^{z_h} \mathrm{d}z\left(\frac{z}{L}\right)^{-d-1}\frac{1}{4}F_{ab}F^{ab} \\
=& \int_0^{z_h} \mathrm{d}z\left(\frac{z}{L}\right)^{-d-1}\frac{1}{2}F_{zt}^2 g^{zz}g^{tt} \\
=& -\int_0^{z_h} \mathrm{d}z\left(\frac{z}{L}\right)^{-d-1+4}\left(\frac{z}{z_h}\right)^{2(d-3)}(d-2)^2\frac{\mu^2}{2z_h^2} \\
=& -L^{d-3}z_h^{-2d+4}(d-2)^2\frac{\mu^2}{2}\int_0^{z_h}\mathrm{d}z z^{d-3} \\
=& -L^{d-3}z_h^{-d+2}(d-2)\frac{\mu^2}{2} \\
=& -L^{d-3}\left(\frac{4\pi T}{d}\right)^{d-2}(d-2)\frac{\mu^2}{2}
\end{aligned} \tag{3.65}$$

Interpretation of Multiple Solutions The multiple solutions obtained

3.9 Expectation Values of Field Theory Operators

These expectation values are calculated using (2.20) and (2.17).

$$\langle O(\psi(x))\rangle = -i\frac{\delta}{\delta J(x)}\log(Z[J])|_{J=0} \overset{\mathrm{GKPW}}{=} -i\frac{\delta}{\delta J(x)}\log(Z[J])|_{J=0} \overset{\mathrm{classical}}{=} \frac{\delta}{\delta J(x)}S_c|_{J=0} \tag{3.66}$$

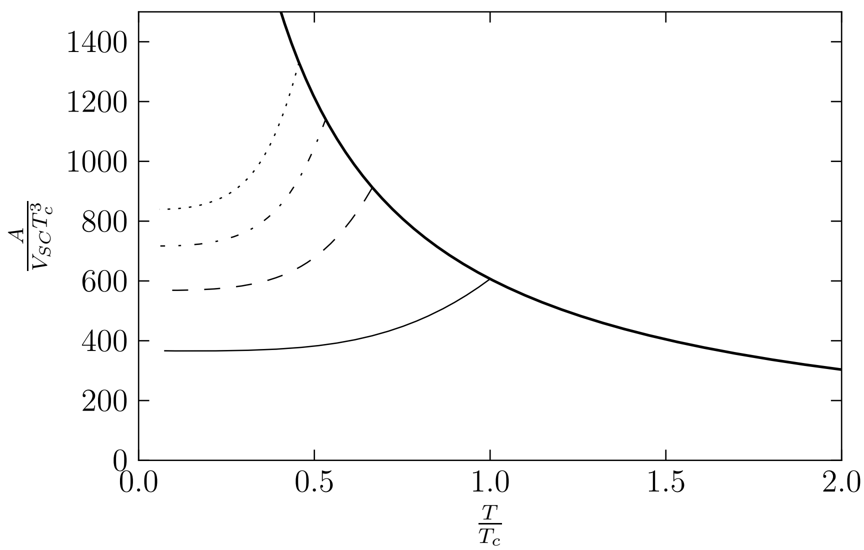

Figure 3.2: Curves corresponding to different solutions at the same temperature. The trivial solution is shown as the thick solid line. The other curves correspond to numerical solutions. The lowest one corresponds to the first root and the other roots follow in order.

The functional derivative is thus the change in the classical action when the boundary value of the fields are changed. The total change in the partition function is needed and the whole field solutions change when changing the boundary conditions so this must be taken into account. The derivative becomes

$$\frac{\delta}{\delta J(x)} S_c|_{J=0} = \int \mathrm{d}^{d+1} y \sqrt{g} \left(\frac{\partial \mathcal{L}(y)}{\partial \phi_i(y)} \frac{\partial \phi_i(y)}{\partial J(x)} + \frac{\partial \mathcal{L}(y)}{\partial(\partial_a \phi_i(y))} \frac{\partial(\partial_a \phi_i(y))}{\partial J(x)} \right) \tag{3.67}$$

where i goes over all fields. Here the Lagrangian is assumed to only depend on the fields and their first derivatives. Now let $\mathcal{L}' = \sqrt{g}\mathcal{L}$ and integrate by parts

$$\begin{aligned} \frac{\delta}{\delta J(x)} S_c| = \int \mathrm{d}^{d+1} y \left(\frac{\partial \mathcal{L}'(y)}{\partial \phi_i(y)} - \partial_a \frac{\partial \mathcal{L}'(y)}{\partial(\partial_a \phi_i(y))} \right) \frac{\partial(\phi_i(y))}{\partial J(x)} \\ + \int_{\partial \mathrm{AdS}} \mathrm{d}^d y n_a \frac{\partial \mathcal{L}'(y)}{\partial(\partial_a \phi_i(y))} \frac{\partial(\phi_i(y))}{\partial J(x)} \end{aligned} \tag{3.68}$$

where n_a is an outward normal to the boundary of AdS. The first integral vanishes since the fields obey the Euler-Lagrange equation.

3.10 Field Theory Electrical Potential

TODO, introduce ρ, μ... The charge density ρ of the trivial solution (3.43) is through (**??**) given by

$$\rho = \frac{\mu}{z_h}. \tag{3.69}$$

The chemical potential of the trivial solution can thus be calculated given the temperature and charge density

$$\mu = \frac{d\rho}{4\pi T}. \tag{3.70}$$

The chemical potential of the numerical solutions can be obtained from the boundary values. The result is seen in Fig. 3.3.

Comparison with Experiments Results TODO

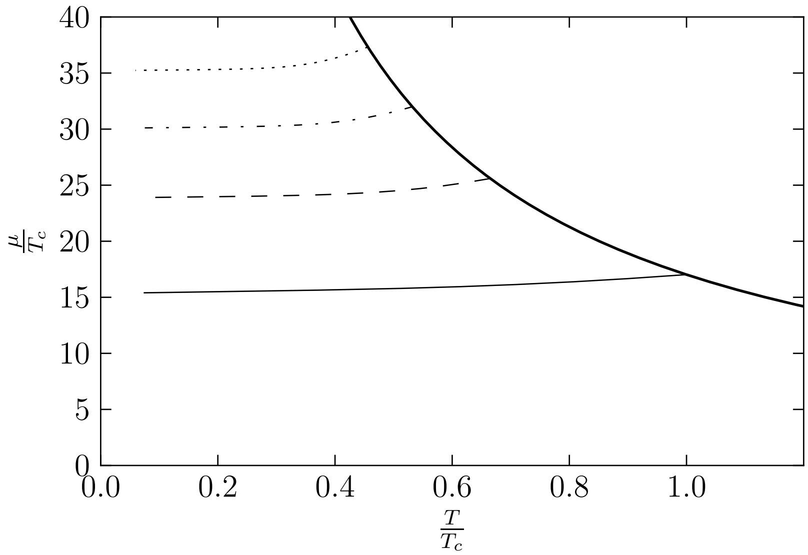

Figure 3.3: Curves corresponding to different solutions at the same temperature. The trivial solution is shown as the thick solid line. The other curves correspond to numerical solutions. The lowest one corresponds to the first root and the other roots follow in order.

3.11 Field Theory Scalar Operator

TODO, write in more detail. TODO, write in some clear way about the dimensionless quantities. The ψ condensate should be obtained without any source. Boundary conditions without source, $\psi_i = 0$, are chosen at the boundary. ψ_1, μ, and ρ are then unknown. The boundary condition that $\phi = 0$ at the horizon is actually a double boundary condition due to the singular form of the equations of motion there. There are thus only two free parameters and it is easiest to start integrating from there. A numerical integrator is used to integrate from the horizon to the boundary. The source term is read off and the horizon conditions are adapted until the source term is zero. This leaves a one dimensional space of solutions since we have two parameters to vary at the horizon but just one constraint on the boundary. μ, ρ and $\epsilon_{ij}\psi_j$ can be read of at the boundary.

The fields all have the unit of inverse length and ρ thus has unit inverse length squared. Solutions can be obtained at any temperature by choosing different values of z_h. The temperature can be chosen such that the charge density ρ is constant for all solutions. The numerical solution gives ρz_h^2. Choosing a constant $\rho = 1$ gives z_h and thus the temperature. We call the highest temperature obtained T_c.

Comparison with Experiments Results TODO

3.12 Electrical Conductivity

The conductivity of a superconductor can easily be measured experimentally for a wide range of frequencies and it is therefore an interesting property to calculate from our model of a superconductor. The agreement in different frequency ranges tells us about similarities and differences between our model and the experimental superconductors.

3.12.1 Definition of Electrical Conductivity and its Properties

TODO, def from L The current density, $\mathbf{J}$ is a vector quantity defined as the flow of charge. The rate of charge passing through a surface at x of length

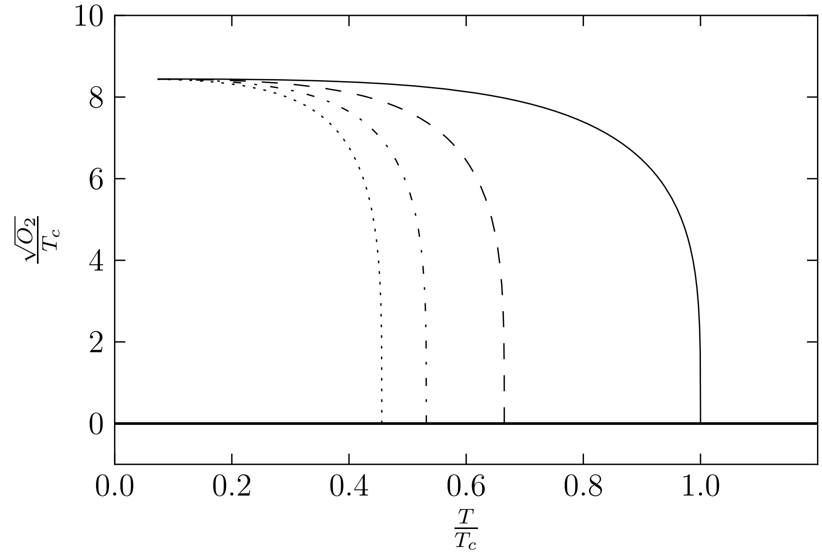

Figure 3.4: Curves corresponding to different solutions at the same temperature.

$\mathrm{d}l$ and with normal $\mathbf{n}$ is given by $\mathrm{d}l\mathbf{n} \cdot \mathbf{J}(x)$. We define conductivity σ as the linear response function for the current density J_x with the applied electrical field E_x as source

$$\sigma(\omega) = \frac{J_x(\omega)}{E_x(\omega)}. \tag{3.71}$$

Here the direction x has been chosen for concreteness but since we consider two-dimensional systems with rotational symmetry we need only consider one direction. These functions of ω are the Fourier transforms of the time-dependent quantities. The considered electrical field E_x is further assumed to be translationally symmetric giving a still translationally symmetric system. The current in the time domain can be obtained from the conductivity and the applied field through a inverse Fourier transform

$$J_x(t) = \int_{-\infty}^{\infty} E_x(t-\tau)\sigma(\tau)\mathrm{d}\tau. \tag{3.72}$$

Causality implies that $\sigma(\tau) = 0$ for $\tau < 0$ since the current would otherwise depend on future values of the electrical field. The conductivity can using this be written

$$\sigma(\omega) = \int_0^\infty \sigma(\tau) \exp(i\tau\omega) \mathrm{d}\tau \tag{3.73}$$

and thus has an analytical extension to the upper half of the complex plane. Both the current, $J_x(t)$, and applied field, $E_x(t)$, are real quantities which makes $\sigma(\tau)$ also real and thus $\mathrm{Re}(\sigma(w))$ an even function and $\mathrm{Im}(\sigma(\omega))$ odd. These properties of $\sigma(\omega)$ give the Kramers–Kronig relations

$$\begin{aligned}\mathrm{Re}(\sigma(\omega)) &= \frac{2}{\pi} \int_0^\infty \frac{\omega' \mathrm{Im}(\sigma(\omega'))}{\omega'^2 - \omega^2} \\ \mathrm{Im}(\sigma(\omega)) &= -\frac{2}{\pi} \int_0^\infty \frac{\omega \mathrm{Re}(\sigma(\omega'))}{\omega'^2 - \omega^2}\end{aligned} \tag{3.74}$$

These relations state that the real part of the conductivity uniquely determines the imaginary part and vice versa.

3.12.2 Calculating the Holographic Conductivity

We want to apply a homogeneous but time-varying electrical field on the superconductor and measure the induced current. The two-dimensional superconductor is rotationally symmetric so we can apply the field in any direction. We choose

$$E_x = E_0 \exp(-i\omega t). \tag{3.75}$$

The response function can be obtained by using this time-dependence. This electrical field corresponds to the electromagnetic potential

$$A = (A_z, \phi, A_x, A_y) \tag{3.76}$$

Applying an electric field in The obtained conductivity for temperatures above the T_c is $\sigma(\omega) = 1$. This agrees with TODO. The conductivity changes when the condensate forms. An energy gap Δ forms and the conductivity for $\omega < \Delta$ goes to 0 as the temperature is lowered. The superconductivity is not immidiately evident from the obtained conductivity curves. There is a delta-function at $\omega = 0$ since translational invariance of the boundary theory has been assumed. ODO, add DC temperature dependence? energy gap size?

3.12.3 Comparison with Experimental Results

TODO. graphene?

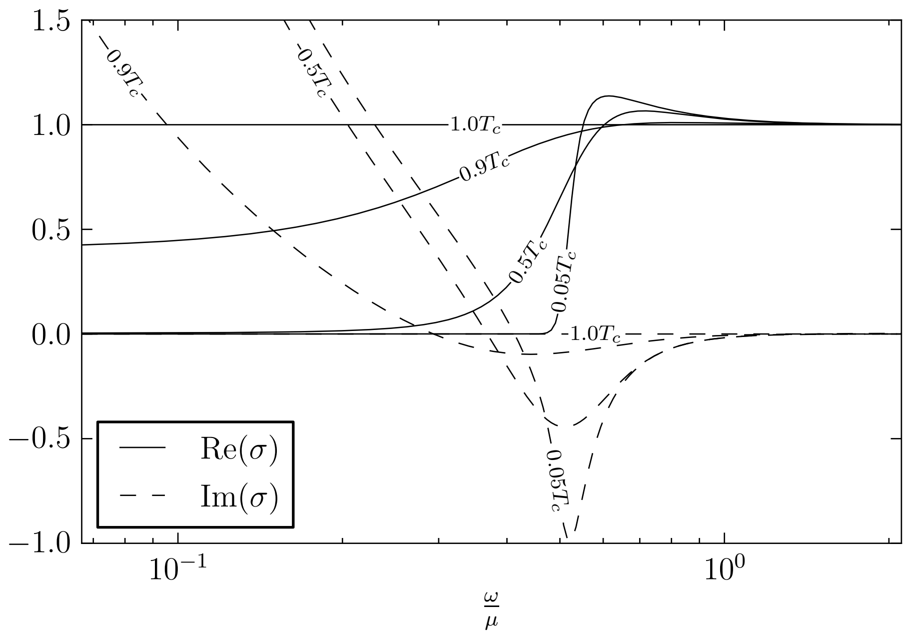

Figure 3.5: Real and imaginary part of the conductivity for different temperatures. Higher temperatures above T_c give the same conductivity as for the $T = T_c$ case.

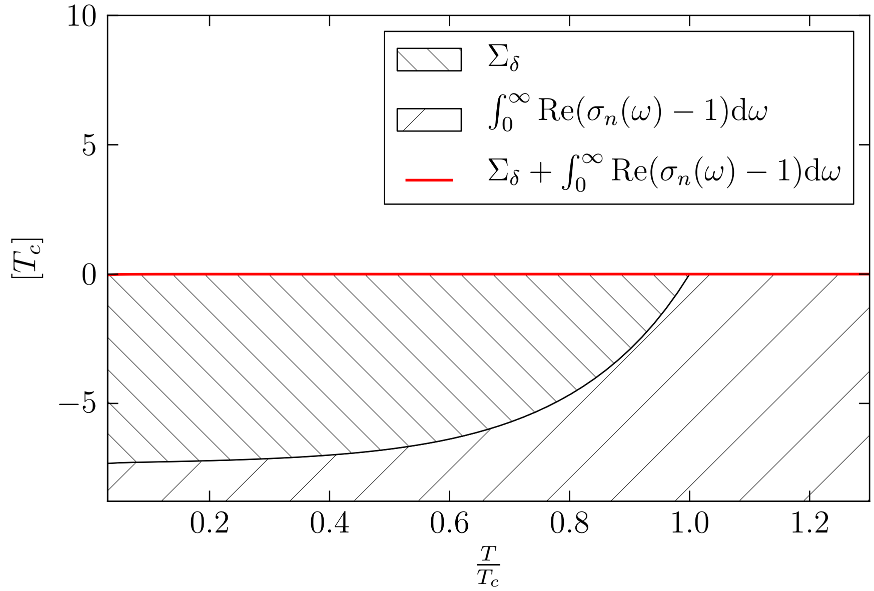

Figure 3.6: Integral

3.13 Consistency Check using a Conductivity Sum Rule

The Kramers-Kroning relations and the decoupeling of the electrons from the system at high frequencies can be used to prove a sum rule for the conductivity [11]. The rule states

$$\int_0^\infty \mathrm{Re}(\sigma(\omega))\mathrm{d}\omega = C \tag{3.77}$$

where C is a constant depending on what system we are considering

Chapter 4

Extended Lagrangian

Many assumptions have been made in the previous study. The most simple Lagrangian (2.1) has been used and translational symmetry has been assumed. This has given us a boundary theory with a scalar field that condeses below a critical temperature as expected for a superconductor. The conductivity shows both similarities and differences with that of high-T_c superconductors. A δ-function developes at $\omega = 0$ for $T < T_c$ giving infinite DC conductivity. An evident difference is the lack of a so-called Drude peak at low frequencies. A Drude peak is an increase in conductivity for low frequencies in metals that can be well modeled by the Drude model of conductivity[12], thereof the name. The Drude model agrees with experiments on cuprates above T_c[13].

4.1 Drude Model

The Drude model is obtained by treating the charge carriers classically. They are expected to obey the differential equation

$$\frac{\mathrm{d}v}{\mathrm{d}t} = \frac{q}{m}E - \frac{1}{\tau}v \tag{4.1}$$

Solving this for harmonic $E = E_0 \exp(-i\omega t)$ gives

$$v = \frac{\tau q E_0}{m(1 - i\omega\tau)} \exp(-i\omega t). \tag{4.2}$$

This gives the conductivity

$$\sigma(\omega) = \frac{J(\omega)}{E(\omega)} = \frac{v(\omega)q\rho}{E(\omega)} = \frac{\tau\rho q^2}{m(1-i\omega\tau)} = \frac{\sigma_0}{1-i\tau\omega}. \tag{4.3}$$

where ρ is the density of charge carriers of charge q.

4.2 Higher Order Maxwell Term

Different generalizations of the standard Lagrangian L have been studied, TODO cite. Tobias Wenger studied three different extensions propose by Sachdev, TODO cite. An increase in conductivity for low frequencies similar to a Drude peak, courtesy of Hugo Strand, was observed for one of the extensions. This will here be studied in more detail.
The extended Lagrangian has a higher order Maxwell term and another parameter α_2 with dimension TODO.

$$\mathcal{L} = \frac{1}{2\kappa}(R-2\Lambda) - \frac{1}{4}F_{ab}F^{ab} - m^2\psi\overline{\psi} - D_a\psi\overline{D^a\psi} + \alpha_2 F^a_b F^b_c F^c_d F^d_a \tag{4.4}$$

The equations of motion now become TODO.
TODO: Boundary, Horizon behaviour.

This higher order term introduces a perturbation in the low frequency conductivity above the critical point. The perturbation is very close to what is expected from the Drude model of electrical conductivity. This agrees very well with what is obtained for low frequencies above the critical point, especially for high values of TODO. See Fig. 4.1.

The low frequency limit of the conductivity σ_0 seems to depend linearly on α_2 for values of α_2 larger than TODO. See Fig 4.2. The proportionality constant depends on the temperature, see Fig 4.3, and seems to closely follow a power-law, see Fig 4.4. σ_0 thus seems to closely follow

$$\sigma_0 = C\alpha_2 \left(\frac{T}{T_c}\right)^{-4/3} \tag{4.5}$$

for values of $\alpha_2 >> TODO$.

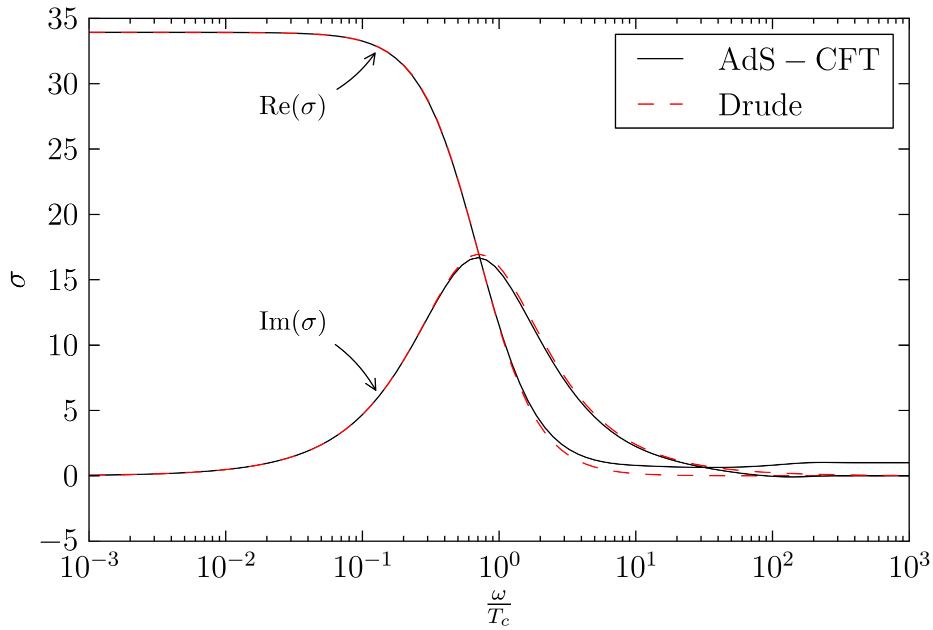

Figure 4.1: Conductivity for $\alpha_2 = 1$ and $T = T_c$. The Drude parameters have been obtained from the lowest frequency data-point.

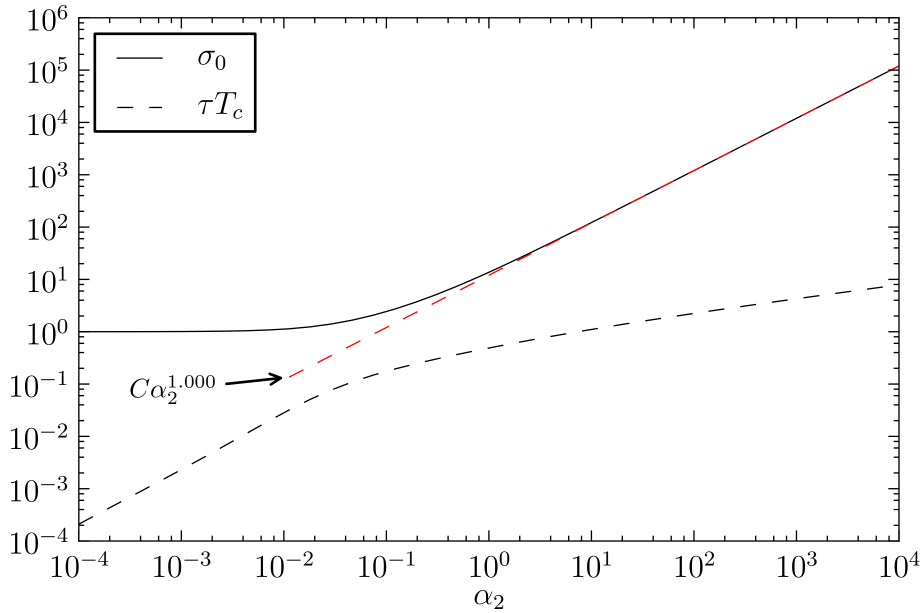

Figure 4.2: Drude parameters as functions of α_2 at $T = 2T_c$.

4.3 Asymptotic Behavior for $\alpha_2 >> T_c^?$,TODO

Consistency Check using a Conductivity Sum Rule A test of the numerics was again performed by verifying the TODO...

4.4 Disussion

The peak in the Drude model is due to imperfections in the lattice

4.4.1 Comparison with Results using a Periodic Lattice

TODO

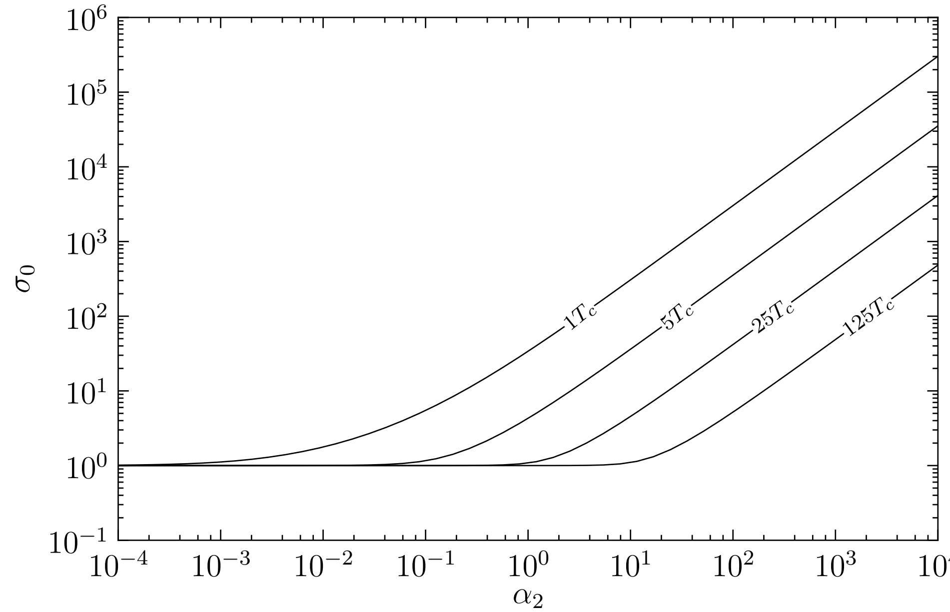

Figure 4.3: $\omega \to 0$ limit of the conductivity as a function of α_2 for different temperatures.

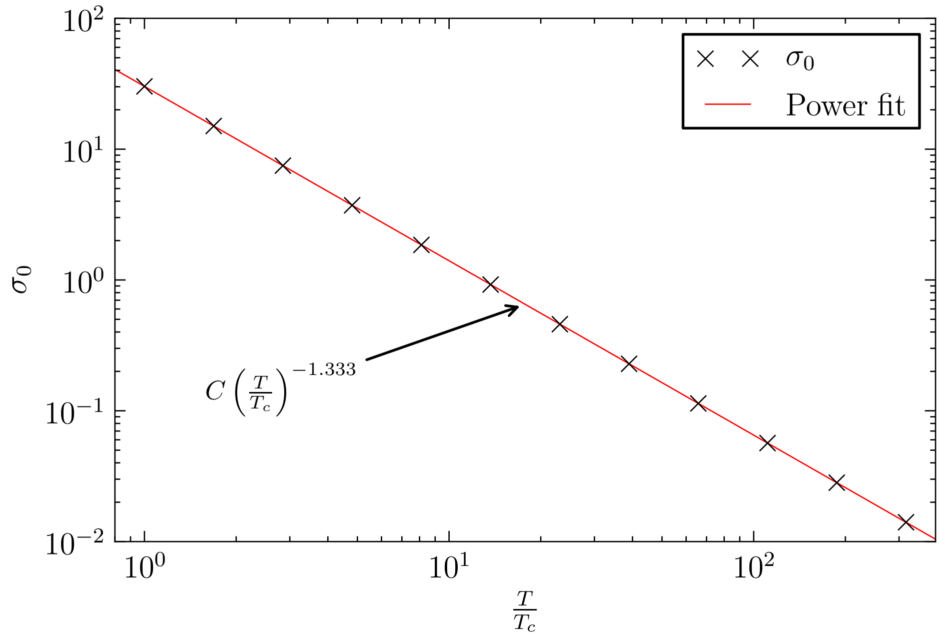

Figure 4.4: σ_0/α_2 as a function of temperature for large α_2.

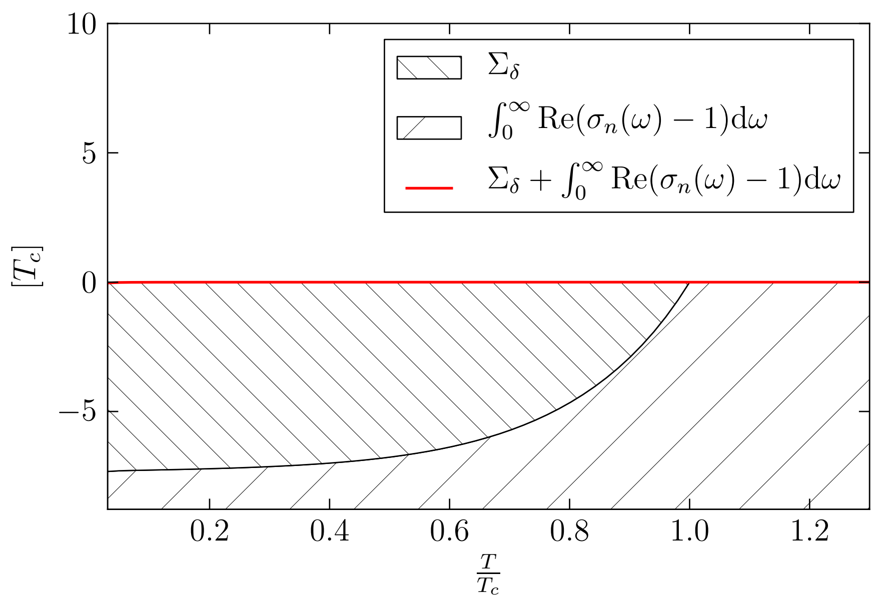

Figure 4.5: Integral, TODO plot new

Chapter 5

Summary

TODO

5.1 Outlook

TODO

Appendices

.1 Conventions in this Report

The AdS space will be refered to as the bulk and the boundary conformal field theory will be referred to as the boundary theory or the superconductor. Vector quantities not involving time components will in the boundary theory be written with boldface, $\mathbf{E}, \mathbf{J}$.
Tensor indices in the $d+1$-dimensional bulk theory will be TODO latin letters, $a, b, c, ...8$. Tensor indices in the d-dimensional boundary theory will be TODO greek letters, $\mu, \nu, ...$. Tensors written in component form will have it's components ordered as $\mu = t, x, y$ on the boundary and $a = z, t, x, y$ in the bulk. The metric sign convention for positive space-like distances will be chosen.
The action is calculated from the Lagrangian as

$$S = \int \mathrm{d}^{d+1}x\sqrt{g}\mathcal{L} + S_{\text{boundary}}. \tag{1}$$

This is independent of coordinates but makes us use covariant derivatives for finding the equations of motion. The square root could instead be absorbed in the Lagrangian and the space time be considered flat. This has been done in computer-aided calculations of the equations of motion.

.2 Computer-Aided Analytical Calculations

The Christoffel symbols for the AdS black hole were calculated using SymPy. All the non-zero components are shown below.

$$
\begin{aligned}
\Gamma_{zzz} &= \frac{L^2\left(-zf'(z) - 2f(z)\right)}{2z^3 f^2(z)} = \frac{L^2 z_h^3 \left(5z^3 - 2z_h^3\right)}{2z^3\left(z^6 - 2z^3 z_h^3 + z_h^6\right)} \\
\Gamma_{ztt} &= \frac{L^2\left(zf'(z) - 2f(z)\right)}{2z^3} = \frac{L^2\left(-z^3 - 2z_h^3\right)}{2z^3 z_h^3} \\
\Gamma_{zxx} &= \frac{L^2}{z^3} \\
\Gamma_{zyy} &= \frac{L^2}{z^3} \\
\Gamma_{ttz} = \Gamma_{tzt} &= \frac{L^2\left(-zf'(z) + 2f(z)\right)}{2z^3} = \frac{L^2\left(z^3 + 2z_h^3\right)}{2z^3 z_h^3} \\
\Gamma_{xxz} = \Gamma_{xzx} &= -\frac{L^2}{z^3} \\
\Gamma_{yyz} = \Gamma_{yzy} &= -\frac{L^2}{z^3}
\end{aligned}
\tag{2}
$$

which gives

$$
R = \frac{-z^2 f''(z) + 6zf'(z) - 12f(z)}{L^2} = -\frac{12}{L^2} \tag{3}
$$

Some Christoffel symbol contractions are useful

$$
\begin{aligned}
{\Gamma^a}_{az} &= -\frac{4}{z} \\
{\Gamma^a}_{at} &= 0 \\
{\Gamma^a}_{ax} &= 0 \\
{\Gamma^a}_{ay} &= 0 \\
g^{ab}{\Gamma^z}_{ab} &= \frac{z\left(-zf'(z) + 2f(z)\right)}{L^2} = \frac{z\left(z^3 + 2z_h^3\right)}{L^2 z_h^3} \\
g^{ab}{\Gamma^t}_{ab} &= 0 \\
g^{ab}{\Gamma^x}_{ab} &= 0 \\
g^{ab}{\Gamma^y}_{ab} &= 0.
\end{aligned}
\tag{4}
$$

Using these one obtains

$$\nabla_a \nabla^a \chi = \left(\partial_a \partial^a + \frac{z \left(z f' \left(z \right) - 2 f \left(z \right) \right)}{L^2} \partial_z \right) \chi. \tag{5}$$

for a scalar field χ. The non-zero components of the electromagnetic tensor after making the definitions in Section 3.1 are

$$\begin{aligned} F_{zt}(z) &= -F_{tz}(z) = \phi'(z) \\ F_{zx}(z,t) &= -F_{xz}(z,t) = A'_x(z) \exp(it\omega) \\ F_{tx}(z,t) &= -F_{xt}(z,t) = i\omega A_x(z) \exp(it\omega) \end{aligned} \tag{6}$$

Another useful quantity is $\nabla_a F^{ab}$

$$\nabla_a F^{ab} = \partial_a F^{ab} + \Gamma^a{}_{ca} F^{cb} + \Gamma^b{}_{ca} F^{ac} = \partial_a F^{ab} + \Gamma^a{}_{ca} F^{cb} \tag{7}$$

Bibliography

[1] Juan Martin Maldacena. The Large N limit of superconformal field theories and supergravity. *Adv.Theor.Math.Phys.*, 2:231–252, 1998.

[2] John McGreevy. Holographic duality with a view toward many-body physics. *Adv.High Energy Phys.*, 2010:723105, 2010.

[3] Edward Witten. Anti-de Sitter space and holography. *Adv.Theor.Math.Phys.*, 2:253–291, 1998.

[4] Romuald A. Janik and Robi Peschanski. Asymptotic perfect fluid dynamics as a consequence of ads/cft correspondence. *Phys. Rev. D*, 73:045013, Feb 2006.

[5] Sean A. Hartnoll. Lectures on holographic methods for condensed matter physics. *Class.Quant.Grav.*, 26:224002, 2009.

[6] Sean A. Hartnoll, Christopher P. Herzog, and Gary T. Horowitz. Building a Holographic Superconductor. *Phys.Rev.Lett.*, 101:031601, 2008.

[7] Gary T. Horowitz and Jorge E. Santos. General Relativity and the Cuprates. 2013.

[8] R.P. Feynman and A.R. Hibbs. *Quantum mechanics and path integrals.* International series in pure and applied physics. McGraw-Hill, 1965.

[9] M. Kleban, M. Porrati, and R. Rabadan. Stability in asymptotically AdS spaces. *JHEP*, 0508:016, 2005.

[10] Steven S. Gubser. Breaking an Abelian gauge symmetry near a black hole horizon. *Phys.Rev.*, D78:065034, 2008.

[11] Richard A. Ferrell and Rolfe E. Glover. Conductivity of superconducting films: A sum rule. *Phys. Rev.*, 109:1398–1399, Feb 1958.

[12] P. Drude. Zur Elektronentheorie der Metalle. *Annalen der Physik*, 306:566–613, 1900.

[13] H. L. Liu, M. Quijada, D. B. Romero, D. B. Tanner, A. Zibold, G. L. Carr, H. Berger, L. Forró, L. Mihaly, G. Cao, B.-H. O, J. T. Markert, J. P. Rice, M. J. Burns, and K. A. Delin. Drude behavior in the far-infrared conductivity of cuprate superconductors. *Annalen der Physik*, 518:606–618, July 2006.