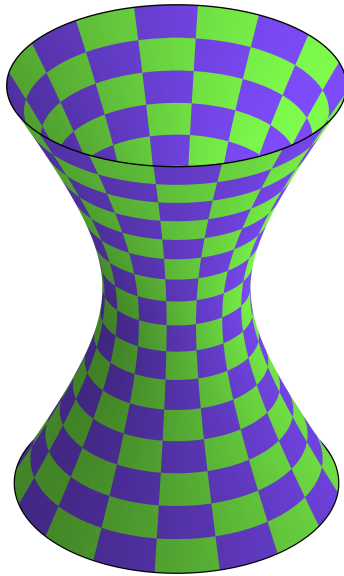


# Holographic Methods for Condensed Matter Physics

Petter Sätterskog

May 17, 2013



## Abstract

The physics of certain condensed matter systems is not well understood due to strong coupling preventing perturbative descriptions and in certain cases also numerical simulations. The AdS/CFT correspondence might allow a non-perturbative description of these systems in terms of a dual weakly coupled system. In this thesis the AdS/CFT correspondence is used to model a high- $T_c$  superconductor by a field theory outside a black hole in AdS space. The frequency dependent conductivity is calculated using this model and a superconducting phase is shown to appear below a critical temperature. These computations are described in detail in the first part of the thesis. In the second part of the thesis is, in the spirit of effective field theory, a higher curvature correction included on the gravity side of the correspondence. The correction is shown to give a Drude peak and its properties are examined. Another way to introduce a Drude peak is by introducing a periodic lattice[1], as was recently done by Horowitz et al. Our way of obtaining the Drude peak is computationally much simpler than the periodic lattice and might be a useful effective description.

# Acknowledgements

Thanks Mum.

# Contents

<b>1</b>	<b>Introduction</b>	<b>4</b>
1.1	The Correspondence . . . . .	5
1.2	Applications . . . . .	6
<b>2</b>	<b>Application to Two-Dimensional Condensed Matter Systems</b>	<b>8</b>
2.1	Symmetry Assumptions . . . . .	9
2.2	A Lagrangian . . . . .	9
2.3	Equations of Motion . . . . .	10
2.4	Parameters . . . . .	11
2.4.1	$\kappa$ . . . . .	11
2.4.2	$\Lambda$ . . . . .	12
2.4.3	$q$ . . . . .	12
2.4.4	$m^2$ . . . . .	12
<b>3</b>	<b>Solution of the Classical Bulk Theory</b>	<b>13</b>
3.1	Definitions . . . . .	13
3.2	Metric . . . . .	14
3.3	Equations of Motion . . . . .	16
3.4	AdS Boundary behaviour of Fields . . . . .	17
3.4.1	Boundary conditions at Horizon . . . . .	18
3.5	Expectation Values of Field Theory Operators . . . . .	19
3.5.1	Choice of scalar mass $m$ . . . . .	20
3.6	Horizon Behaviour of Bulk Fields . . . . .	21
3.7	Numerical Solution . . . . .	22
3.8	Thermodynamic Variables . . . . .	22
3.8.1	Gravitational Free Energy and Entropy . . . . .	22
3.8.2	Free Energy of Scalar and Electromagnetic Fields . . . . .	23
3.9	Field Theory Electrical Potential . . . . .	25

3.10	Field Theory Scalar Operator . . . . .	26
3.11	Electrical Conductivity . . . . .	26
3.11.1	Definition of Electrical Conductivity and its Properties	26
3.11.2	Calculating the Holographic Conductivity . . . . .	28
3.11.3	Comparison with Experimental Results . . . . .	28
3.12	Consistency Check using a Conductivity Sum Rule . . . . .	30
<b>4</b>	<b>Extended Lagrangian</b>	<b>31</b>
4.1	Drude Model . . . . .	31
4.2	Higher Order Maxwell Term . . . . .	32
4.3	Asymptotic Behavior for $\alpha_2 \gg T_c^?$ , TODO . . . . .	34
4.4	Disussion . . . . .	34
4.4.1	Comparison with Results using a Periodic Lattice . . .	34
<b>5</b>	<b>Summary</b>	<b>37</b>
5.1	Outlook . . . . .	37
	<b>Appendices</b>	<b>38</b>
.1	Conventions in this Report . . . . .	39
.2	Boundary Behavior of Bulk Fields . . . . .	39
.3	Computer-Aided Analytical Calculations . . . . .	43

# Chapter 1

## Introduction

The AdS/CFT correspondence was conjectured in 1997 by Juan Maldacena. It relates the physics of a string theory in Anti de Sitter space (AdS) to a conformal field theory (CFT) on the boundary of the AdS space.

AdS is the maximally symmetric space of constant negative curvature. AdS spaces have many interesting properties which will not be described here in detail. A visualization of a two-dimensional AdS space can be made by embedding it as a hyperbola in a three-dimensional space-time with two time directions. The inherited metric is then that of two-dimensional AdS-space with one time direction. See figure on the front of this report. The two time directions are there in the horizontal plane. This embedding gives a periodic time which is not necessary. AdS space has, as opposed to Minkowski space, a boundary at spatial infinity. This boundary often called the *conformal boundary* can be reached by a massless particle in finite time and boundary conditions must be specified here for a field theory in AdS space. The action of the isometry group of AdS space on the conformal boundary is the conformal group, thereof the name.

The string theory in the AdS space is gravitational and thus perturbs the AdS space. The correspondence only requires an asymptotically AdS space. The boundary of the AdS space where the CFT lives is a Minkowski space of one spatial dimension lower. This difference in dimension is why this approach is called *holographic*. The AdS isometry group requires conformal symmetry of the boundary theory which thus is a conformal field theory. A conformal field theory is a quantum field theory with invariance under conformal transformations. The conformal group is the Poincaré group with

dilations and special conformal transformations added. Dilations are scaling transformations. The word *conformal* comes from the angle-preserving property of these transformations.

This means that the CFT can be scaled by a factor  $s$ , TODO write.

There is no proof of the correspondence but it has been extensively tested. The field theory of the original conjecture[2] was a supersymmetric Yang-Mills theory. Extensions of the conjecture has later been made and we will here use a field theory without supersymmetry. That the extended correspondence also holds is motivated in for example [3].

The strength of the duality comes from that the bulk theory is weakly coupled when the boundary theory is strongly coupled and vice versa. This lets us solve otherwise computationally unavailable problems on the strongly coupled side by solving them on the weakly coupled side.

## 1.1 The Correspondence

The correspondence can be formulated through

$$Z_{bulk}(\phi_0) = \left\langle \exp(i \int d^d x \sqrt{g_0} \phi_0 \mathcal{O}) \right\rangle_{FT} \quad (1.1)$$

[4]. Here  $Z_{bulk}(\phi_0)$  is the partition function for the bulk theory with boundary condition at the conformal boundary  $\phi_0$ . The partition function is defined as TODO.

The expectation value on the right-hand side is of a field theory at a temperature given by the periodicity of the path integral for the partition function. The field  $\phi_0$  is source to the operator  $\mathcal{O}$ ,

$$\mathcal{O} = \frac{\delta S_{CFT}}{\delta \phi_0} \quad (1.2)$$

where  $S_{CFT}$  is the CFT action.

The boundary theory becomes classical for a gauge boundary theory with a large number of colors, a large  $N$ . We do not have a large- $N$  theory but a similar effect is expected for strongly coupled boundary theories so a classical bulk theory is assumed. See for example [3] for a thorough treatment of

when the bulk theory can be considered classical. The partition function is then given in a semi-classical limit by

$$Z_{bulk}(\phi_0) = C \exp(iS_c) \quad (1.3)$$

where  $S_c$  is the bulk theory action for the classical periodic path in Euclidean time[4] and  $C$  is a constant. This path has boundary condition at the conformal boundary described by  $\phi_0$ .

Expectation values of the CFT operator  $\mathcal{O}(x)$  can be calculated by

$$\begin{aligned} \frac{\delta S_c(\phi_0)}{\delta \phi_0(x)}|_{\phi_0=0} &= -i \frac{\delta \log Z_{bulk}(\phi_0)}{\delta \phi_0(x)}|_{\phi_0=0} \\ &= -i \frac{\delta \log \langle \exp(i \int d^d x \sqrt{g_0} \phi_0 \mathcal{O}) \rangle_{FT}}{\delta \phi_0(x)}|_{\phi_0=0} \\ &= \frac{\langle \sqrt{g_0} \phi_0 \exp(i \int d^d x \sqrt{g_0} \phi_0 \mathcal{O}) \rangle_{FT}}{\langle \exp(i \int d^d x \sqrt{g_0} \phi_0 \mathcal{O}) \rangle_{FT}}|_{\phi_0=0} \\ &= \langle \sqrt{g_0} \mathcal{O}(x) \rangle_{FT} \end{aligned} \quad (1.4)$$

where the first equality comes from the semi-classical approximation and the second equality from (1.1).

## 1.2 Applications

The correspondence can be used both ways but we will consider a strongly coupled boundary theory. Conformal field theories are characterized by not having any specific length scale. Physics at critical points often have this property. A critical point can be a thermodynamic phase transition or a quantum phase transition. The characteristic length goes to infinity as the critical point is approached and the length scale disappears. The physics near a critical point can be expected to be similar to the critical system and finding the critical behaviour is then of interest.

Examples of strongly coupled systems exhibiting critical behaviour are, quark-gluon plasmas[5], high  $T_c$  superconductors[4], and possibly graphene [4].



We will hereafter focus on high  $T_c$  superconductors. These superconductors are layered and the electrons effectively moves in two dimensions. They are believed to be both strongly coupled and scale-invariant[4].

## Chapter 2

# Application to Two-Dimensional Condensed Matter Systems

We wish to model a high  $T_c$  superconductor. Conventional superconductors are well described by the BCS theory where the electrons, photons and phonons are the degrees of freedom of interest. The importance of the phonon interactions was understood from the isotope effect, the mass of the atoms in the lattice changed the superconductivity behavior. The isotope effect is though much weaker, TODO cite, in high temperature superconductors and the phonons are thus not believed to be important for high temperature superconductivity. The important degrees of freedom are the electrons and the photons. The electrons are, just as in BCS theory, expected to form Cooper-pairs, TODO cite. These are pairs of electrons of opposite spin but otherwise in the same state effectively becoming spin 0 particles. Our high temperature superconductor model will thus contain two fields, a spin 1 field  $A_a$  for the photons and a spin 0 field  $\psi$  for the Cooper-pairs.

The superconductor lives in 2+1 dimensional flat space. We will use coordinates  $x, y$  for the spatial directions and  $t$  for time. The extra dimension in the AdS dual will be parametrised by the coordinate  $z$ . See Appendix .1 for details on how indices are labeled and ordered in this report.

## 2.1 Symmetry Assumptions

The bulk theory should have the same symmetries as the boundary theory. We therefore impose a  $U(1)$  gauge symmetry of the complex  $\psi$  field. Lorentz invariance will also be used for both theories even though relativistic phenomena hardly are important for superconductivity.

## 2.2 A Lagrangian

There are many different ways to construct a bulk Lagrangian for the fields  $A_a$  and  $\psi$  and the metric  $g_{ab}$ . A Lagrangian previously used successfully to model two-dimensional electron condensates[6],[1] will initially be used here.

$$\mathcal{L} = \frac{1}{2\kappa} (R - 2\Lambda) - \frac{1}{4} F_{ab} F^{ab} - m^2 |\psi|^2 - |D_a \psi|^2 \quad (2.1)$$

This is obtained using Wilsonian naturalness meaning that the lowest order terms obeying all symmetries are used. A higher order term will be investigated in Chapter 4.

The action  $S$  is calculated from this Lagrangian from

$$S = \int d^{d+1}x \sqrt{g} \mathcal{L} + S_{\text{boundary}}. \quad (2.2)$$

where  $g$  is the absolute value of the determinant of the metric tensor,  $g = |\det g_{ab}|$ .  $S_{\text{boundary}}$  is a boundary term that is needed to cancel divergences when integrating the action towards the boundary. It does not affect the equations of motion but is needed to get normalizeable modes.

The first term of the Lagrangian is an Einstein-Hilbert term with a cosmological constant  $\Lambda$ . A negative cosmological constant gives a asymptotically anti-de-Sitter space as required.  $R$  is the Ricci scalar curvature obtained from the metric  $g_{ab}$ . The constant  $\kappa$  determines the coupling between the metric and the other fields.

The second term is an ordinary Maxwell term where the electromagnetic tensor  $F_{ab}$  is the exterior derivative of the electromagnetic field tensor,  $F_{ab} = \partial_a A_b - \partial_b A_a$ .

The third and fourth terms are the kinetic and mass terms for the scalar field respectively.  $D_a$  is the gauge covariant derivative  $D_a = \nabla_a - iqA_a$ .  $\nabla_a$  is the covariant derivative, see Appendix .1. This minimal gauge coupling makes the Lagrangian invariant under a U(1) gauge transformation

$$\psi \rightarrow e^{i\theta(x)}\psi \quad (2.3)$$

$$A_a \rightarrow A_a + \frac{1}{q}\nabla_a\theta(x). \quad (2.4)$$

This Lagrangian also possess conformal invariance. This means that the Lagrangian is unchanged by the transformation  $g_{ab} \rightarrow f(x)g_{ab}$ . TODO check. The Lagrangian is also manifestly Lorentz invariant imposing Lorentz invariance of the boundary theory.

## 2.3 Equations of Motion

The bulk equations of motion are obtained by varying the bulk Lagrangian with respect to all the fields. This can be done with the Euler-Lagrange equation since the action does not contain any higher derivatives. The Euler-Lagrange equation for a scalar field  $\chi$  states

$$\nabla_a \left( \frac{\partial \mathcal{L}}{\partial (\nabla_a \chi)} \right) - \frac{\partial \mathcal{L}}{\partial \chi} = 0. \quad (2.5)$$

First vary  $\psi$ . This gives

$$(m^2 - \nabla^2 + q^2 A^2 + iq(\nabla_a A^a)) \psi = 0. \quad (2.6)$$

Varying  $A_a$  gives these equations of motion

$$-\nabla_a F^{ab} + 2q^2 |\psi|^2 A^b + iq(\bar{\psi} \nabla^b \psi - \psi \nabla^b \bar{\psi}) = 0 \quad (2.7)$$

A real  $\psi$  simplifies calculations and that can be obtained since the gauge invariance lets us relate any configuration with a real one through a gauge transformation. The Lorentz gauge,

$$\nabla_a A^a = 0, \quad (2.8)$$

removes the last term in the parenthesis of the equation of motion for  $\psi$ . The equation of motion for  $\psi$  does not mix the real and imaginary parts

after this choice and  $\psi$  can be taken to be real since a global shift of phase does not affect  $A_a$ , see (2.4). The gauge is still not completely fixed, a gauge transformation  $\theta(x)$  such that  $\nabla_a \nabla^a \theta(x) = 0$  can still be done without violating the gauge condition, (2.8). The equations of motion are

$$\begin{aligned} (m^2 - \nabla^2 + q^2 A^2) \psi &= 0 \\ -\nabla_a F^{ab} + 2q^2 \psi^2 A^b &= 0. \end{aligned} \tag{2.9}$$

after choosing the Lorentz gauge and a real  $\psi$ .

## 2.4 Parameters

There are multiple unknown parameters in the bulk Lagrangian. These must be investigated to find values that give us the boundary theory we are interested in. The Lagrangian contains the parameters  $\kappa$ ,  $\Lambda$ ,  $m^2$ ,  $q$ . Some of these parameters might be redundant since we can make different symmetry transformations of fields and coordinates. The physics of the bulk are treated in the classical limit and the Lagrangian can thus be changed as long as the equations of motion for  $\psi$  and  $A_a$  are left unchanged.

### 2.4.1 $\kappa$

The Einstein-Hilbert term of the Lagrangian makes the theory gravitational.  $\kappa$  is proportional to Newton's gravitational constant. A small  $\kappa$  gives the probe limit where the metric equations of motion can be solved independently of the other fields. This can be understood by varying the Lagrangian with respect to the metric, the Einstein-Hilbert part gives a term inversely proportional to  $\kappa$  and the rest of the Lagrangian gives the stress-energy tensor independent of  $\kappa$ .

This greatly simplifies calculations and will therefore be used throughout this work. It is though not guaranteed that the interesting boundary theories are dual to bulk theories in the probe limit. Earlier studies have though found that interesting boundary systems can be obtained by treating a bulk in the probe limit. A superconducting condensate develops for low temperatures in the work by S. Hartnoll, C. Herzog and G. Horowitz [6] where the bulk is treated in the probe limit.  $\kappa \rightarrow 0$  is a fixed-point of the theory so the physics is independent of the exact value of  $\kappa$  as long as we are in the probe limit.

### 2.4.2 $\Lambda$

Scale-invariance of the system lets us choose an arbitrary  $\lambda$ . Two systems with different  $\lambda$  can be shown to be equal by a rescaling.  $\lambda$  sets a length scale  $L$

$$L = \sqrt{-\frac{3}{\Lambda}} \quad (2.10)$$

to which other parameters, e.g  $m^2$ , can be related. Scale-invariance can thus not be used to choose those parameters freely. The factor 3 is used so that we later will see that  $L$  becomes the AdS radius.

$\lambda$  will be set to a convenient number in numerical calculations but kept in calculations for clarity.

### 2.4.3 $q$

$q$  sets the strength of the gauge coupling and is thus the charge of the scalar field. Considering  $\tilde{\psi} = q\psi$  and  $\tilde{A}_a = qA_a$  as the fields gives a Lagrangian of the same form but divided by  $q^2$  except for the term originally containing  $q^2$  which is divided by  $q^4$ . Multiplying the Lagrangian by a constant doesn't affect the equations of motion so the system can be solved for any value of  $q$ . Other solutions can then be obtained by rescaling the fields.

### 2.4.4 $m^2$

$m$  is the mass of the scalar field in the bulk. What values of  $m$  that are suitable will be investigated later when solving the equations of motion in the bulk.

## Chapter 3

# Solution of the Classical Bulk Theory

We wish to compute expectation values of the CFT. This will be done through the correspondence (??). The bulk theory must then be solved so that the partition function there can be obtained. The bulk theory was assumed to be classical in Section 1.1 so the system is solved by finding the equations of motion for the fields and solving them. This lets us calculate the partition function and use it to find CFT expectation values through ??.

### 3.1 Definitions

The Lagrangian describes a general system so there are many solutions to the equations of motion. We wish to investigate two properties of a superconductor, the development of a condensate at low temperatures and the conductivity at different frequencies. We are interested in a superconductor subject to spatially uniform conditions, the applied electric field is uniform and the chemical potential is uniform. The atomic lattice and its imperfections are thus not accounted for but interesting superconductivity behaviour has been obtained anyways [6]. It is thus enough to look at a system with translational symmetry in the  $x$  and  $y$  directions. A rotationally invariant superconductor will further be studied. The system is subject to conditions constant in time, e.g. no time-dependent chemical potential. This lets us assume time-independence while solving the non-linear field equations.

The conductivity is the linear electrical current response to an applied transverse electrical field. We can apply this in the  $x$  direction due to the rotational symmetry. We let the applied field have a harmonic time dependence  $\exp(it\omega)$  so we can get the response function in the frequency domain. The linear response is sought so the applied field should be infinitesimal. The applied field breaks the rotational and time symmetries but since it is infinitesimal and we are not interested in the effect it has on the other fields it can be neglected while calculating them. The applied field is later added with the other fields as background solutions.

The electrical field in the  $x$ -direction is  $E_x = F_{xt} = \nabla_x A^t - \nabla_t A^x$ . Translational symmetry gives  $E_x = \nabla_t A^x$ .

These limitations lets us do the following definitions

$$\begin{aligned} ds^2 &= g_{zz}(z)dz^2 + g_{tt}(z)dt^2 + g_{xx}(z)(dx^2 + dy^2) \\ \psi &= \psi(z) \\ A_a &= (A_z(z), \phi(z), A_x(z) \exp(it\omega), 0) \end{aligned} \tag{3.1}$$

where  $\phi(z)$  is infinitesimal. The gauge condition requires

$$\nabla_a A^a = \partial_a A^a + \Gamma^a_{ba} A^b = 0 \tag{3.2}$$

this gives a homogeneous first-order linear ordinary differential equation for  $A_z(z)$ . The remaining gauge symmetry can be used to set  $A_z(z) = 0$  for a  $z$  and  $A_z(z)$  can thus be taken to be identically 0 for all  $z$ .

The explicit  $z$  and  $t$ -dependence of these functions will hereafter be omitted.

## 3.2 Metric

We wish to find the metric saddle-point of the periodic imaginary time path integral. The bulk equation of motion for the metric  $g_{ab}$  is the Einstein equation with a cosmological constant

$$R_{ab} - \frac{1}{2}g_{ab}R + g_{ab}\lambda = \kappa T_{ab} \tag{3.3}$$

where  $R_{ab}$  is the Ricci curvature tensor and  $T_{ab}$  is the stress-energy tensor. We assumed the probe limit in Section ?? and therefore neglect the right



hand side of this equation. We want a translationally invariant solution in the  $t$ ,  $x$ , and  $y$  directions that is asymptotically AdS. The solution is known to be a black hole[3], the Schwarzschild metric in AdS space. The metric has the following form in a particular choice of coordinates where the radial coordinate  $z$  is 0 at the boundary and  $z_h$  at the horizon

$$g_{ab}dx^a dx^b = \frac{L^2}{z^2} \left( \frac{dz^2}{f(z)} - f(z)dt^2 + dx^2 + dy^2 \right). \quad (3.4)$$

Here  $f(z) = 1 - z^3 z_h^{-3}$ .  $f(z)$  approaches 1 at the boundary and the space is asymptotically AdS. There is a horizon at  $z = z_h$  where  $f(z_h) = 0$ . The space behind the horizon can not affect the physics of the boundary and can thus be neglected in our calculations. This solution is periodic in imaginary time. Consider the near-horizon metric where

$$f(z) = f(z_h) - (z_h - z)f'(z_h) + \mathcal{O}((z_h - z)^2) \approx 3(1 - zz_h^{-1}) \quad (3.5)$$

Do the change of variables  $\rho^2 = \frac{4L^2}{3}(1 - zz_h^{-1})$ . This gives  $f(z) \approx \rho^2 \frac{9}{4L^2}$  and  $\rho^2 d\rho^2 = dz^2 z_h^{-2} \frac{4L^4}{9}$ . The near-horizon metric is then

$$\begin{aligned} g_{ab}dx^a dx^b &= \frac{L^2}{z_h^2} \left( \frac{\rho^2 d\rho^2}{z_h^{-2} \frac{4L^4}{9} \rho^2 \frac{9}{4L^2}} - \rho^2 \frac{9}{4L^2} dt^2 + dx^2 + dy^2 \right) \\ &= d\rho^2 - \rho^2 \frac{9}{4z_h^2} dt^2 + \frac{L^2}{z_h^2} (dx^2 + dy^2). \end{aligned} \quad (3.6)$$

Now extend this to imaginary time  $\tau = it$

$$g_{ab}dx^a dx^b = d\rho^2 + \rho^2 \left( \frac{3}{2z_h} d\tau \right)^2 + \frac{L^2}{z_h^2} (dx^2 + dy^2). \quad (3.7)$$

The near horizon metric is then that of a Euclidean plane in polar coordinates. There is thus a deficit angle unless  $\frac{3}{2z_h}\tau$  has a periodicity of  $2\pi$ . The imaginary time has periodicity  $\beta$  so we thus have

$$\frac{3}{2z_h} = \frac{2\pi}{\beta} \quad (3.8)$$

This gives the relationship between  $z_h$  and the temperature

$$T = \frac{3}{4\pi z_h}. \quad (3.9)$$

This expression for the temperature agrees with the Beckenstein-Hawking temperature of a black hole.

The backreaction  $\Delta g_{ab}$  from the non-zero fields will be of order  $\kappa T_{ab}$  according to (3.3). The Einstein equation is obtained by varying the Lagrangian with respect to  $g_{ab}$  so  $\Delta S \propto \kappa^{-1} \Delta g_{ab}^2$  for the solution. We thus have that  $\Delta S \propto \kappa T_{ab}^2$  and the backreaction can safely be neglected also when calculating the action from different field configurations.

This background metric can now be used instead of solving the equations of motion for the metric together with the fields. The gravitational part of the Lagrangian must be kept when calculating the value of the total action which is dominated by the gravitational part.

The horizon  $z_h$  and the curvature length  $L$  set length scales in the metric. Length units in the numerical solution can be chosen such that  $z_h = 1$ . This means that we for different temperatures have different units since  $z_h$  is related to the temperature. We will have to convert between these units when comparing results from different temperatures.

### 3.3 Equations of Motion

The equations of motion for  $\psi(z)$ ,  $\phi(z)$  and  $A_x(z)$  can now be obtained. Inserting (3.1) into the equations of motion (2.9) and using the metric (3.4) gives

$$\left\{ \begin{array}{l} \left( q^2 z^2 \phi^2 - L^2 m^2 f + z f (z f' - 2f) \partial_z + z^2 f^2 \partial_z \partial_z \right) \psi = 0 \end{array} \right. \quad (3.10)$$

$$\left\{ \begin{array}{l} \left( -2q^2 \psi^2 L^2 + z^2 f \partial_z \partial_z \right) \phi = 0 \end{array} \right. \quad (3.11)$$

$$\left\{ \begin{array}{l} \left( -2q^2 \psi^2 L^2 f + z^2 \omega^2 + z^2 f f' \partial_z + z^2 f^2 \partial_z \partial_z \right) A_x = 0 \end{array} \right. \quad (3.12)$$

The formulas in Appendix .3 have here been used.

### 3.4 AdS Boundary behaviour of Fields

A Frobenius expansion of these equations can be done at the boundary,  $z = 0$ . The leading behaviours of the functions are

$$\begin{cases} \psi = \psi_{(0)} \left( \frac{z}{z_h} \right)^{\Delta_\psi} & (3.13) \\ \phi = \phi_{(0)} \left( \frac{z}{z_h} \right)^{\Delta_\phi} & (3.14) \\ A_x = A_{x(0)} \left( \frac{z}{z_h} \right)^{\Delta_{A_x}} & (3.15) \end{cases}$$

where  $\Delta_\psi$ ,  $\Delta_\phi$  and  $\Delta_{A_x}$  are constants that are to be determined. This is a slight assumption since not all functions have this type of leading behaviour. Entering this in the equations of motion yields

$$\begin{cases} q^2 z^2 \phi_{(0)}^2 s^{2\Delta_\phi} - L^2 m^2 f + f(zf' - 2f)\Delta_\psi + f^2 \Delta_\psi (\Delta_\psi - 1) = 0 & (3.16) \\ -2q^2 \psi_{(0)}^2 s^{2\Delta_\psi} L^2 + f \Delta_\phi (\Delta_\phi - 1) = 0 & (3.17) \\ -2q^2 \psi_{(0)}^2 s^{2\Delta_\psi} L^2 f + z^2 \omega^2 + z f f' \Delta_{A_x} + f^2 \Delta_{A_x} (\Delta_{A_x} - 1) = 0. & (3.18) \end{cases}$$

where  $s = z z_h^{-1}$ . This immediately gives  $\Delta_\psi \geq 0$  and  $1 + \Delta_\phi \geq 0$ . Assume the strict inequalities. The leading order behaviour is then

$$\begin{cases} -L^2 m^2 - 2\Delta_\psi + \Delta_\psi (\Delta_\psi - 1) = 0 & (3.19) \\ \Delta_\phi (\Delta_\phi - 1) = 0 & (3.20) \\ \Delta_{A_x} (\Delta_{A_x} - 1) = 0. & (3.21) \end{cases}$$

with solutions

$$\begin{cases} \Delta_\psi = \frac{3}{2} \pm \sqrt{\frac{9}{4} + L^2 m^2} & (3.22) \\ \Delta_\phi = 0, 1 & (3.23) \\ \Delta_{A_x} = 0, 1. & (3.24) \end{cases}$$

Now assume  $\Delta_\psi = 0$ . (3.17) gives

$$-2q^2 \psi_{(0)}^2 L^2 + \Delta_\phi (\Delta_\phi - 1) = 0 \quad (3.25)$$

while (3.16) gives  $\Delta_\phi = -1$ . We then have

$$\begin{cases} q^2 \phi_{(0)}^2 z_h^2 = L^2 m^2 \end{cases} \quad (3.26)$$

$$\begin{cases} q^2 \psi_{(0)}^2 L^2 = 1 \end{cases} \quad (3.27)$$

$$\begin{cases} \Delta_{A_x}(\Delta_{A_x} - 1) = 2. \end{cases} \quad (3.28)$$

with solutions  $\Delta_{A_x} = -1, 2$ . First assuming  $\Delta_\phi = -1$  yields the same result. There are though no solutions to (3.26) for the negative  $m^2$  we later will consider and infinities are encountered when calculating the action for these solutions so they will not be considered. All useful solutions are thus given by equation 3.22 to 3.24.

### 3.4.1 Boundary conditions at Horizon

The same kind of expansion can be made at the horizon but there are some simplifying conditions. The time component of the metric vanishes at the horizon,  $f(z_h) = 0$ . This means that  $A_t(z_h)$  must be zero because a finite  $A_t(z_h)$  would give a finite Wilson loop around the periodic imaginary time circle whose length in time is 0. A Wilson loop is contrary to  $A_t$  a physical quantity ( $A_t$  is gauge-dependent). This gives a singular gauge connection which is unwanted[4]. We thus have  $\phi(z_h) = 0$ . Expand the fields as

$$\begin{cases} \psi = \psi_{(1)} s^{\Upsilon_\psi} \end{cases} \quad (3.29)$$

$$\begin{cases} \phi = \phi_{(1)} s^{\Upsilon_\phi} \end{cases} \quad (3.30)$$

$$\begin{cases} A_x = A_{x(1)} s^{\Upsilon_{A_x}} \end{cases} \quad (3.31)$$

where  $s$  now is  $(1 - z/z_h)$  and  $\Upsilon_\phi > 0$ . The function  $f$  can be expanded as  $f = 3s - 3s^2 + s^3$ . Insert these leading terms in the equations of motion. We get

$$\begin{cases} q^2 z^2 \phi_{(1)}^2 s^{2\Upsilon_\phi} + 9z_h^2 \Upsilon_\psi + z_h^2 9\Upsilon_\psi(\Upsilon_\psi - 1) = 0 \end{cases} \quad (3.32)$$

$$\begin{cases} -2q^2 \psi_{(1)}^2 s^{2\Upsilon_\psi} L^2 + z_h^2 3s^{-1} \Upsilon_\phi(\Upsilon_\phi - 1) = 0 \end{cases} \quad (3.33)$$

$$\begin{cases} -6q^2 \psi_{(1)}^2 s^{2\Upsilon_\psi+1} L^2 + z_h^2 \omega^2 + z_h^2 9\Upsilon_{A_x} + z_h^2 9\Upsilon_{A_x}(\Upsilon_{A_x} - 1) = 0. \end{cases} \quad (3.34)$$

Solving for the leading terms of these equations gives

$$\begin{cases} \Upsilon_\psi = 0 & (3.35) \\ \Upsilon_\phi = 1 & (3.36) \\ \Upsilon_{A_x} = \pm \frac{i\omega}{3}. & (3.37) \end{cases}$$

The two possible  $\Upsilon_{A_x}$  represent solutions going into or coming out of the horizon.  $A_x(z, t)$  is close to the horizon

$$s^{\pm \frac{i\omega}{3}} \exp(i\omega t) = \exp\left(i\omega\left(t \pm \frac{\log s}{3}\right)\right). \quad (3.38)$$

The phase is constant for  $s = \exp(\mp 3t)$  so the plus sign in (3.37) gives the ingoing solution.

### 3.5 Expectation Values of Field Theory Operators

These expectation values are calculated using (??) and (??).

$$\begin{aligned} \langle \mathcal{O}(\psi(x)) \rangle &= -i \frac{\delta}{\delta J(x)} \log(Z_{\text{CFT}}[J])|_{J=0} \\ &\stackrel{\text{GKPW}}{=} -i \frac{\delta}{\delta J(x)} \log(Z_{\text{Bulk}}[J])|_{J=0} \\ &\stackrel{\text{classical}}{=} \frac{\delta}{\delta J(x)} S_c|_{J=0} \end{aligned} \quad (3.39)$$

where  $J$  is the field coupling to the operator  $\mathcal{O}$  in the field theory.  $S_c$  here is the on-shell action. The functional derivative is thus the change in the classical action when the boundary value of the fields are changed. The total change in the partition function is needed and the whole field solutions change when changing the boundary conditions so this must be taken into account. The derivative becomes

$$\frac{\delta}{\delta J(x)} S_c|_{J=0} = \int d^{d+1}y \sqrt{g} \left( \frac{\partial \mathcal{L}(y)}{\partial \phi_i(y)} \frac{\partial \phi_i(y)}{\partial J(x)} + \frac{\partial \mathcal{L}(y)}{\partial (\nabla_a \phi_i(y))} \frac{\partial (\nabla_a \phi_i(y))}{\partial J(x)} \right) \quad (3.40)$$

where  $i$  goes over all fields. Here the Lagrangian is assumed to only depend on the fields and their first derivatives. Now integrate by parts

$$\begin{aligned} \frac{\delta}{\delta J(x)} S_c|_{J=0} &= \int d^{d+1}y \sqrt{g} \left( \frac{\partial \mathcal{L}(y)}{\partial \phi_i(y)} - \nabla_a \frac{\partial \mathcal{L}(y)}{\partial (\nabla_a \phi_i(y))} \right) \frac{\partial(\phi_i(y))}{\partial J(x)} \\ &+ \int_{\partial \text{AdS}} d^d y \sqrt{g_0} n_a \frac{\partial \mathcal{L}(y)}{\partial (\nabla_a \phi_i(y))} \frac{\partial(\phi_i(y))}{\partial J(x)} \end{aligned} \quad (3.41)$$

where  $n_a$  is an outward normal to the boundary of AdS. The first integral vanishes since the fields obey the Euler-Lagrange equation.

### 3.5.1 Choice of scalar mass $m$

The mass squared of a scalar field in flat space must be non-negative for stability. This is though not the case in a space with negative curvature. The Breitenlohner-Freedman bound (BF) is a lower bound on  $m^2$  of a massive scalar field in AdS space with . It requires

$$L^2 m^2 \geq -\frac{d^2}{4} \quad (3.42)$$

for stability[7]. The scalar field  $\psi$  should obey this bound far away from the black-hole for normalizable modes, TODO explain. We would though like a spontaneous symmetry breaking of  $\psi$  near the black hole (low energies) corresponding to the electron condensate [8]. This can happen because the coupling of  $\psi$  to  $A_a$  gives  $\psi$  an effective mass that might break the BF bound near the black hole.

The effective mass is given by:

$$m_{\text{eff}}^2 = m^2 + A_a A^a = m^2 - \frac{z^2}{L^2(1 - z^d z_h^{-d})} \phi^2 \quad (3.43)$$

This can for large enough values of  $\phi$  break the BF-bound, see Fig. 3.1. Consider the trivial, uncondensed, solution (??). When does this give an effective mass breaking the BF-bound and possibly enabling an additional condensed solution? The location of the effective mass minimum,  $z_0$ , can be found by differentiating (3.43) by  $z$  and using (??),

$$\frac{z_0}{z_h} = \frac{1}{3} \left( \sqrt[3]{37 + 9\sqrt{17}} - \frac{2}{\sqrt[3]{37 + 9\sqrt{17}}} - 2 \right). \quad (3.44)$$

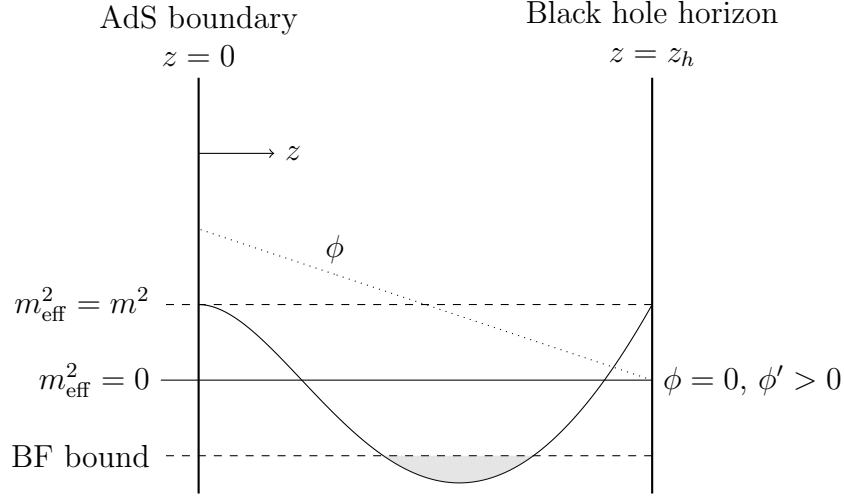


Figure 3.1: Schematic plot of how the the effective mass breaks the BF bound outside the horizon. A value of  $\phi$  has been assumed.

The effective mass breaks the BF-bound at  $z_0$  when

$$\frac{\mu}{T} > \frac{2\pi}{\sqrt{3}} \sqrt{\frac{4L^2 m^2 + 9}{8 - \sqrt[3]{142 - 34\sqrt{17}} - \sqrt[3]{142 + 34\sqrt{17}}}} \quad (3.45)$$

where (3.9) has been used.

Consider another possible term in the Lagrangian, an electromagnetic field  $A_\mu$ . The Lagrangian is  $-\frac{1}{4}F_{ab}F^{ab}$  where  $F_{ab} = \partial_a A_b - \partial_b A_a$ . Consider fields with  $t$ ,  $x_1$ , and  $x_2$  symmetry.

TODO, check. It is easily shown that the behaviour of these fields is the same when there is an interaction present as will be considered later.

### 3.6 Horizon Beaviour of Bulk Fields

TODO: Wilson loop= $\phi = 0$ . Ingoing boundary conditions. Looking at the equations of motion for  $z \rightarrow z_h$  gives when  $\phi(z_h) = 0$

$$\psi(z_h) = \frac{-3z_h \psi'(z_h)}{L^2 m^2} \quad (3.46)$$

## 3.7 Numerical Solution

Short description of numerics, link to source. Refer to sum rule for accuracy test. TODO

## 3.8 Thermodynamic Variables

TOD, compare with experiments. The free energy,  $A = -T \log Z$ , is through the GKPW equation the same for the bulk and the boundary theory. This can be calculated in the classical limit in the bulk.

$$A = -T \log Z \stackrel{\text{classical}}{=} -iTS_c \quad (3.47)$$

Here  $S_c$  is the classical periodic time action. The classical field solutions only depend on the  $z$  coordinate and are thus proportional to  $V = -i\beta V_2$  where  $V_2$  is the area considered in coordinates  $x_1, x_2$ . This gives the free energy per surface area

$$\frac{A}{V_2} = - \int_0^{z_h} dz \sqrt{-g} \mathcal{L} + V^{-1} S_{\text{bdy}} \quad (3.48)$$

Consider the case where  $m^2 = -2$ . Then  $\Delta_0 = 1$  and the boundary term is given by (12).

### 3.8.1 Gravitational Free Energy and Entropy

The action is dominated by the gravitational part since we are in the probe limit. The expression for the temperature (3.9) can be used to calculate the free energy from the action at finite temperature

$$\begin{aligned} \frac{A_{\text{grav}}}{V_2} &= -T \frac{-1}{\kappa} \sqrt{\frac{-2\Lambda}{d(d-1)}} \left( \frac{d}{4\pi T L} \right)^{-d} \\ &= \frac{(4\pi)^d T^d L^{d-1}}{d^d \kappa} \end{aligned} \quad (3.49)$$

The entropy  $H$  can now be calculated from the free energy

$$H = -\frac{\partial A}{\partial T} = -\frac{(4\pi)^d T^{d-1} L^{d-1}}{d^{d-1} \kappa}. \quad (3.50)$$

TODO, Beckenstein-Hawking



### 3.8.2 Free Energy of Scalar and Electromagnetic Fields

It is important to calculate the free energy not only of the gravitational part of the Lagrangian since we have multiple solutions of the field equations and boundary values at the same temperature. Which solution is physical can be found by finding the one of lowest free energy. Here we neglect the contribution from any back-reaction on the metric. The back-reaction is small since  $\kappa$  is small but a small  $\kappa$  also makes the effect of the back-reaction on the free energy large. TODO, motivate or say that we leave this open.

The free energy of the trivial solution (??) can be found analytically. For this solution we have

$$\begin{aligned}\psi &= 0 \\ \psi' &= 0 \\ \phi &= \mu - \mu \left( \frac{z}{z_h} \right)^{d-2} \\ \phi' &= -(d-2) \frac{\mu}{z_h} \left( \frac{z}{z_h} \right)^{d-3}\end{aligned}\tag{3.51}$$

TODO, explain d-3

$$\begin{aligned}\frac{A}{V_{\text{SC}}} &= - \int_0^{z_h} dz \sqrt{-g} \mathcal{L} + V_{\text{SC}}^{-1} S_{\text{bdy}} \\ &= \int_0^{z_h} dz \left( \frac{z}{L} \right)^{-d-1} \frac{1}{4} F_{ab} F^{ab} \\ &= \int_0^{z_h} dz \left( \frac{z}{L} \right)^{-d-1} \frac{1}{2} F_{zt}^2 g^{zz} g^{tt} \\ &= - \int_0^{z_h} dz \left( \frac{z}{L} \right)^{-d-1+4} \left( \frac{z}{z_h} \right)^{2(d-3)} (d-2)^2 \frac{\mu^2}{2z_h^2} \\ &= - L^{d-3} z_h^{-2d+4} (d-2)^2 \frac{\mu^2}{2} \int_0^{z_h} dz z^{d-3} \\ &= - L^{d-3} z_h^{-d+2} (d-2) \frac{\mu^2}{2} \\ &= - L^{d-3} \left( \frac{4\pi T}{d} \right)^{d-2} (d-2) \frac{\mu^2}{2}\end{aligned}\tag{3.52}$$

Interpretation of Multiple Solutions The multiple solutions obtained

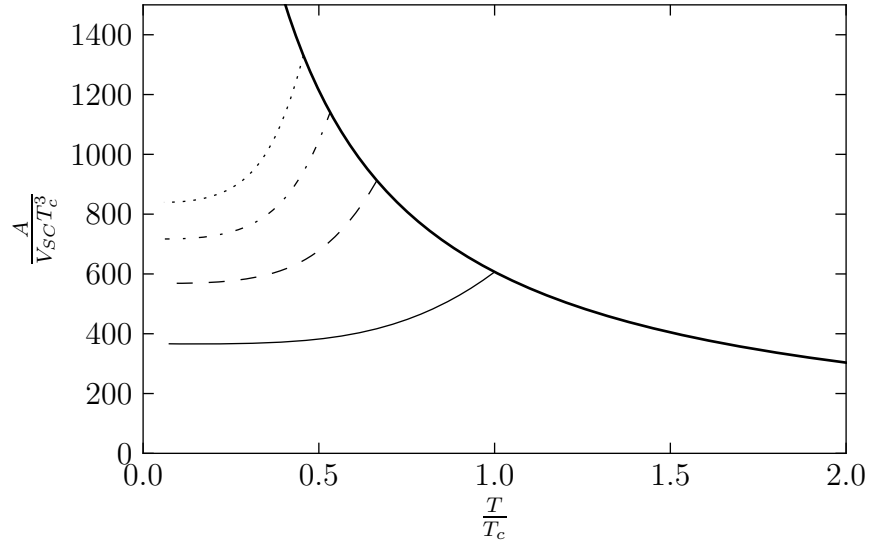


Figure 3.2: Curves corresponding to different solutions at the same temperature. The trivial solution is shown as the thick solid line. The other curves correspond to numerical solutions. The lowest one corresponds to the first root and the other roots follow in order.

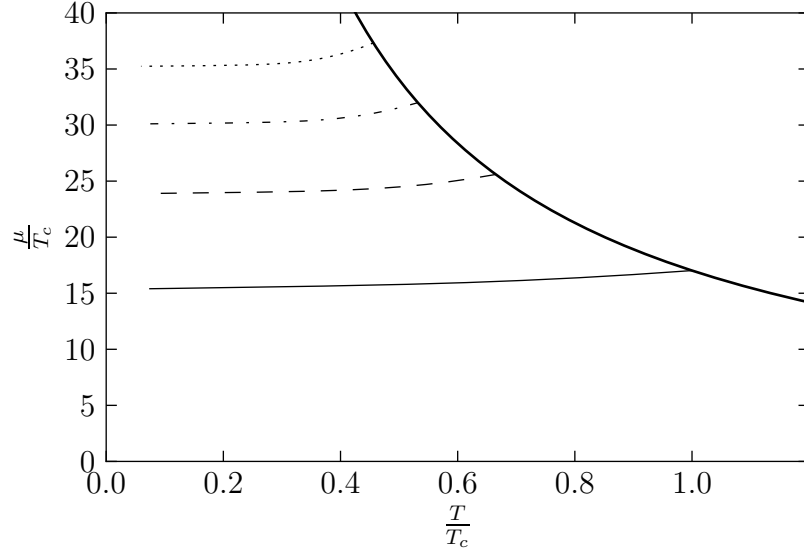


Figure 3.3: Curves corresponding to different solutions at the same temperature. The trivial solution is shown as the thick solid line. The other curves correspond to numerical solutions. The lowest one corresponds to the first root and the other roots follow in order.

### 3.9 Field Theory Electrical Potential

TODO, introduce  $\rho$ ,  $\mu$ ... The charge density  $\rho$  of the trivial solution (??) is through (??) given by

$$\rho = \frac{\mu}{z_h}. \quad (3.53)$$

The chemical potential of the trivial solution can thus be calculated given the temperature and charge density

$$\mu = \frac{d\rho}{4\pi T}. \quad (3.54)$$

The chemical potential of the numerical solutions can be obtained from the boundary values. The result is seen in Fig. 3.3.

Comparison with Experiments Results TODO

## 3.10 Field Theory Scalar Operator

TODO, write in more detail. TODO, write in some clear way about the dimensionless quantities. The  $\psi$  condensate should be obtained without any source. Boundary conditions without source,  $\psi_i = 0$ , are chosen at the boundary.  $\psi_1$ ,  $\mu$ , and  $\rho$  are then unknown. The boundary condition that  $\phi = 0$  at the horizon is actually a double boundary condition due to the singular form of the equations of motion there. There are thus only two free parameters and it is easiest to start integrating from there. A numerical integrator is used to integrate from the horizon to the boundary. The source term is read off and the horizon conditions are adapted until the source term is zero. This leaves a one dimensional space of solutions since we have two parameters to vary at the horizon but just one constraint on the boundary.  $\mu$ ,  $\rho$  and  $\epsilon_{ij}\psi_j$  can be read off at the boundary.

The fields all have the unit of inverse length and  $\rho$  thus has unit inverse length squared. Solutions can be obtained at any temperature by choosing different values of  $z_h$ . The temperature can be chosen such that the charge density  $\rho$  is constant for all solutions. The numerical solution gives  $\rho z_h^2$ . Choosing a constant  $\rho = 1$  gives  $z_h$  and thus the temperature. We call the highest temperature obtained  $T_c$ .

Comparison with Experiments Results TODO

## 3.11 Electrical Conductivity

The conductivity of a superconductor can easily be measured experimentally for a wide range of frequencies and it is therefore an interesting property to calculate from our model of a superconductor. The agreement in different frequency ranges tells us about similarities and differences between our model and the experimental superconductors.

### 3.11.1 Definition of Electrical Conductivity and its Properties

TODO, def from L The current density,  $\mathbf{J}$  is a vector quantity defined as the flow of charge. The rate of charge passing through a surface at  $x$  of length

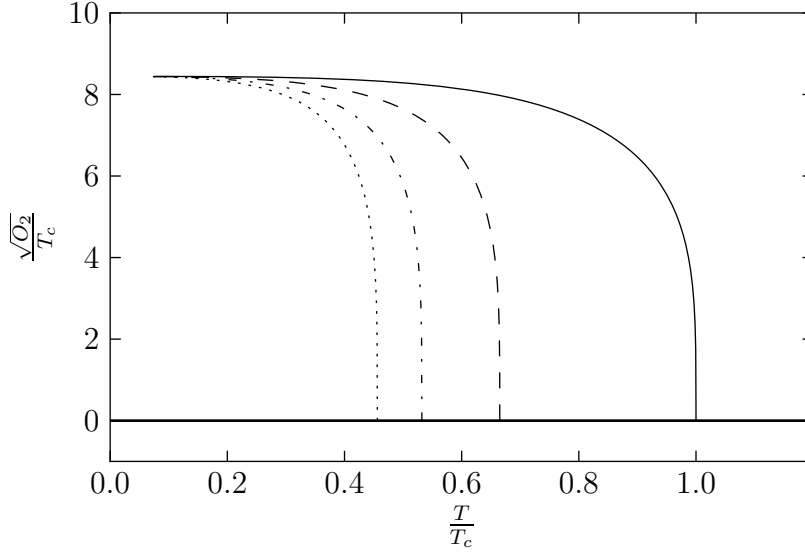


Figure 3.4: Curves corresponding to different solutions at the same temperature.

$d\mathbf{l}$  and with normal  $\mathbf{n}$  is given by  $d\mathbf{l} \cdot \mathbf{J}(x)$ . We define conductivity  $\sigma$  as the linear response function for the current density  $J_x$  with the applied electrical field  $E_x$  as source

$$\sigma(\omega) = \frac{J_x(\omega)}{E_x(\omega)}. \quad (3.55)$$

Here the direction  $x$  has been chosen for concreteness but since we consider two-dimensional systems with rotational symmetry we need only consider one direction. These functions of  $\omega$  are the Fourier transforms of the time-dependent quantities. The considered electrical field  $E_x$  is further assumed to be translationally symmetric giving a still translationally symmetric system. The current in the time domain can be obtained from the conductivity and the applied field through a inverse Fourier transform

$$J_x(t) = \int_{-\infty}^{\infty} E_x(t - \tau) \sigma(\tau) d\tau. \quad (3.56)$$

Causality implies that  $\sigma(\tau) = 0$  for  $\tau < 0$  since the current would otherwise depend on future values of the electrical field. The conductivity can using

this be written

$$\sigma(\omega) = \int_0^\infty \sigma(\tau) \exp(i\tau\omega) d\tau \quad (3.57)$$

and thus has an analytical extension to the upper half of the complex plane. Both the current,  $J_x(t)$ , and applied field,  $E_x(t)$ , are real quantities which makes  $\sigma(\tau)$  also real and thus  $\text{Re}(\sigma(\omega))$  an even function and  $\text{Im}(\sigma(\omega))$  odd. These properties of  $\sigma(\omega)$  give the Kramers–Kronig relations

$$\begin{aligned} \text{Re}(\sigma(\omega)) &= \frac{2}{\pi} \int_0^\infty \frac{\omega' \text{Im}(\sigma(\omega'))}{\omega'^2 - \omega^2} \\ \text{Im}(\sigma(\omega)) &= -\frac{2}{\pi} \int_0^\infty \frac{\omega \text{Re}(\sigma(\omega'))}{\omega'^2 - \omega^2} \end{aligned} \quad (3.58)$$

These relations state that the real part of the conductivity uniquely determines the imaginary part and vice versa.

### 3.11.2 Calculating the Holographic Conductivity

We want to apply a homogeneous but time-varying electrical field on the superconductor and measure the induced current. The two-dimensional superconductor is rotationally symmetric so we can apply the field in any direction. We choose

$$E_x = E_0 \exp(-i\omega t). \quad (3.59)$$

The response function can be obtained by using this time-dependence. This electrical field corresponds to the electromagnetic potential

$$A = (A_z, \phi, A_x, A_y) \quad (3.60)$$

Applying an electric field in The obtained conductivity for temperatures above the  $T_c$  is  $\sigma(\omega) = 1$ . This agrees with TODO. The conductivity changes when the condensate forms. An energy gap  $\Delta$  forms and the conductivity for  $\omega < \Delta$  goes to 0 as the temperature is lowered. The superconductivity is not immediately evident from the obtained conductivity curves. There is a delta-function at  $\omega = 0$  since translational invariance of the boundary theory has been assumed. ODO, add DC temperature dependence? energy gap size?

### 3.11.3 Comparison with Experimental Results

TODO. graphene?

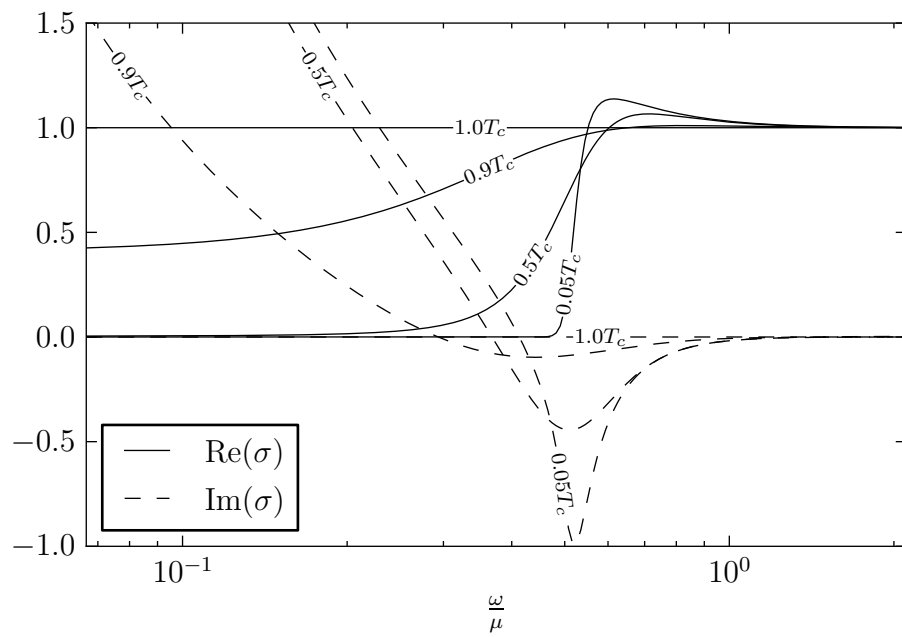


Figure 3.5: Real and imaginary part of the conductivity for different temperatures. Higher temperatures above  $T_c$  give the same conductivity as for the  $T = T_c$  case.

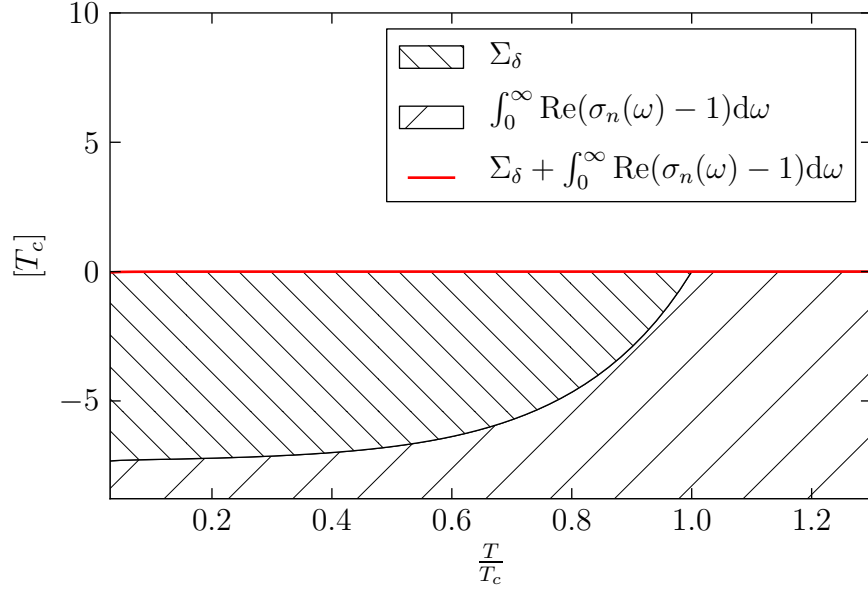


Figure 3.6: Integral

### 3.12 Consistency Check using a Conductivity Sum Rule

The Kramers-Kronig relations and the decoupling of the electrons from the system at high frequencies can be used to prove a sum rule for the conductivity [9]. The rule states

$$\int_0^\infty \text{Re}(\sigma(\omega))d\omega = C \quad (3.61)$$

where  $C$  is a constant depending on what system we are considering



# Chapter 4

## Extended Lagrangian

Many assumptions have been made in the previous study. The most simple Lagrangian (2.1) has been used and translational symmetry has been assumed. This has given us a boundary theory with a scalar field that condenses below a critical temperature as expected for a superconductor. The conductivity shows both similarities and differences with that of high- $T_c$  superconductors. A  $\delta$ -function develops at  $\omega = 0$  for  $T < T_c$  giving infinite DC conductivity. An evident difference is the lack of a so-called Drude peak at low frequencies. A Drude peak is an increase in conductivity for low frequencies in metals that can be well modeled by the Drude model of conductivity[10], thereof the name. The Drude model agrees with experiments on cuprates above  $T_c$ [11].

### 4.1 Drude Model

The Drude model is obtained by treating the charge carriers classically. They are expected to obey the differential equation

$$\frac{dv}{dt} = \frac{q}{m}E - \frac{1}{\tau}v \quad (4.1)$$

Solving this for harmonic  $E = E_0 \exp(-i\omega t)$  gives

$$v = \frac{\tau q E_0}{m(1 - i\omega\tau)} \exp(-i\omega t). \quad (4.2)$$

This gives the conductivity

$$\sigma(\omega) = \frac{J(\omega)}{E(\omega)} = \frac{v(\omega)q\rho}{E(\omega)} = \frac{\tau\rho q^2}{m(1 - i\omega\tau)} = \frac{\sigma_0}{1 - i\tau\omega}. \quad (4.3)$$

where  $\rho$  is the density of charge carriers of charge  $q$ .

## 4.2 Higher Order Maxwell Term

Different generalizations of the standard Lagrangian  $L$  have been studied, TODO cite. Tobias Wenger studied three different extensions propose by Sachdev, TODO cite. An increase in conductivity for low frequencies similar to a Drude peak, courtesy of Hugo Strand, was observed for one of the extensions. This will here be studied in more detail. The extended Lagrangian has a higher order Maxwell term and another parameter  $\alpha_2$  with dimension TODO.

$$\mathcal{L} = \frac{1}{2\kappa} (R - 2\Lambda) - \frac{1}{4} F_{ab} F^{ab} - m^2 \psi \bar{\psi} - D_a \psi \overline{D^a \psi} + \alpha_2 F_b^a F_c^b F_d^c F_a^d \quad (4.4)$$

The equations of motion now become TODO.

TODO: Boundary, Horizon behaviour.

This higher order term introduces a perturbation in the low frequency conductivity above the critical point. The perturbation is very close to what is expected from the Drude model of electrical conductivity. This agrees very well with what is obtained for low frequencies above the critical point, especially for high values of TODO. See Fig. 4.1.

The low frequency limit of the conductivity  $\sigma_0$  seems to depend linearly on  $\alpha_2$  for values of  $\alpha_2$  larger than TODO. See Fig 4.2. The proportionality constant depends on the temperature, see Fig 4.3, and seems to closely follow a power-law, see Fig 4.4.  $\sigma_0$  thus seems to closely follow

$$\sigma_0 = C\alpha_2 \left( \frac{T}{T_c} \right)^{-4/3} \quad (4.5)$$

for values of  $\alpha_2 \gg TODO$ .

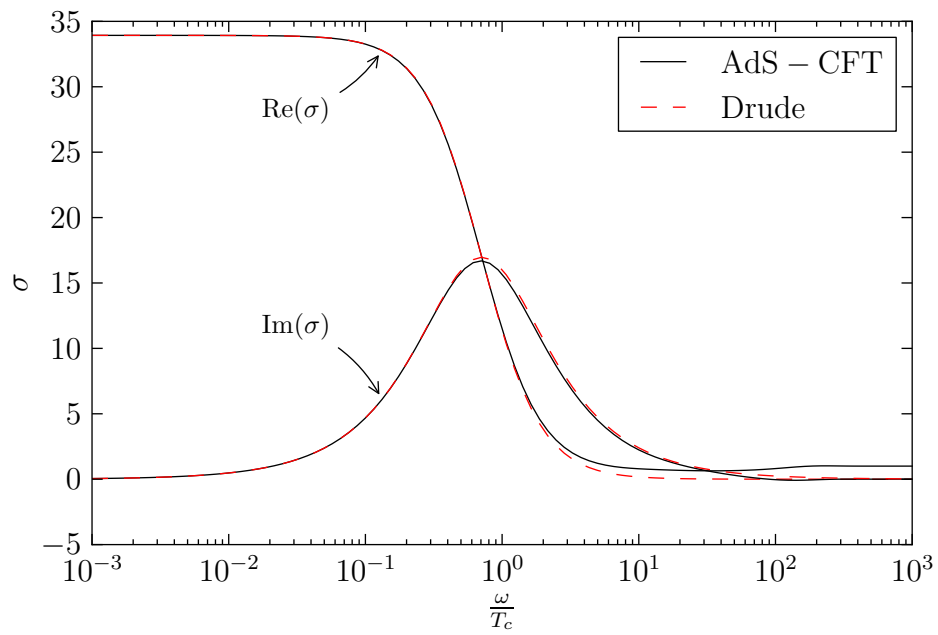


Figure 4.1: Conductivity for  $\alpha_2 = 1$  and  $T = T_c$ . The Drude parameters have been obtained from the lowest frequency data-point.

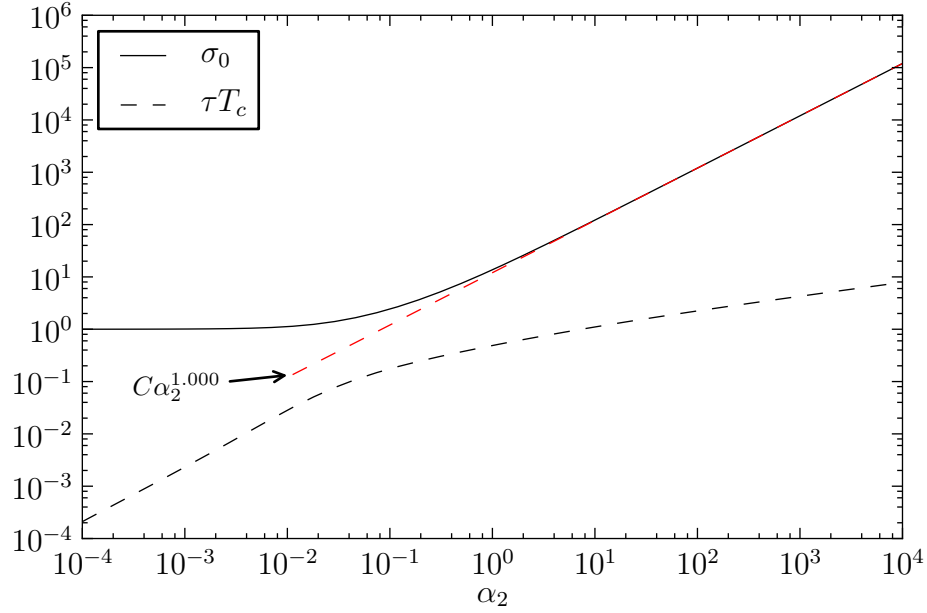


Figure 4.2: Drude parameters as functions of  $\alpha_2$  at  $T = 2T_c$ .

### 4.3 Asymptotic Behavior for $\alpha_2 \gg T_c^?$ , TODO

Consistency Check using a Conductivity Sum Rule A test of the numerics was again performed by verifying the TODO...

### 4.4 Disussion

The peak in the Drude model is due to imperfections in the lattice

#### 4.4.1 Comparison with Results using a Periodic Lat-tice

TODO

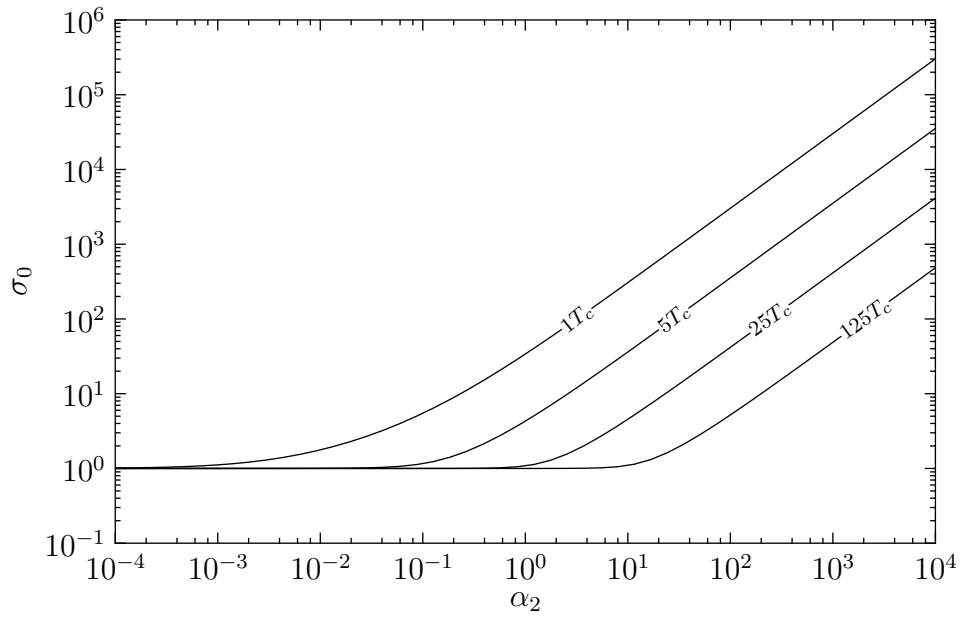


Figure 4.3:  $\omega \rightarrow 0$  limit of the conductivity as a function of  $\alpha_2$  for different temperatures.

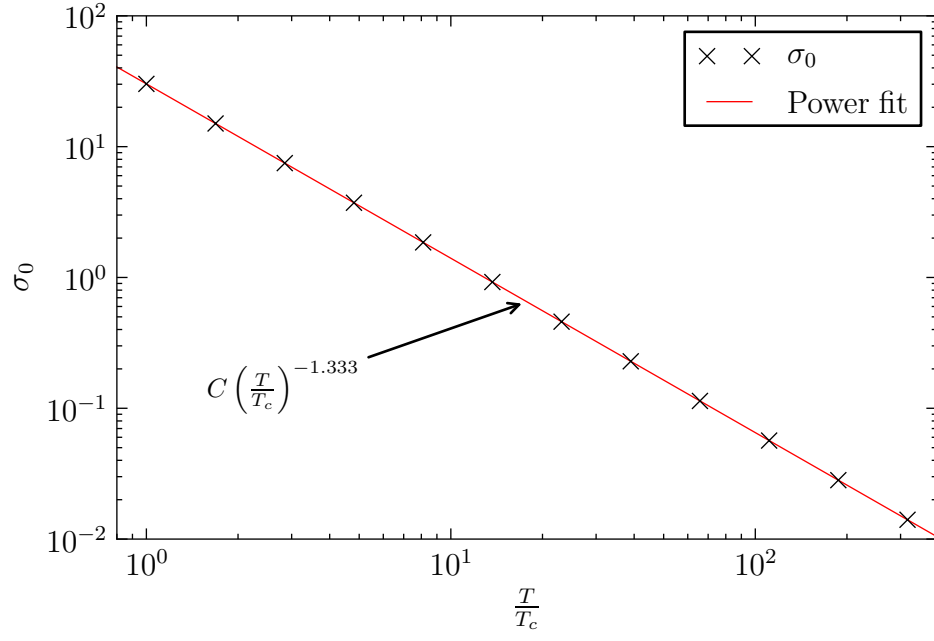


Figure 4.4:  $\sigma_0/\alpha_2$  as a function of temperature for large  $\alpha_2$ .

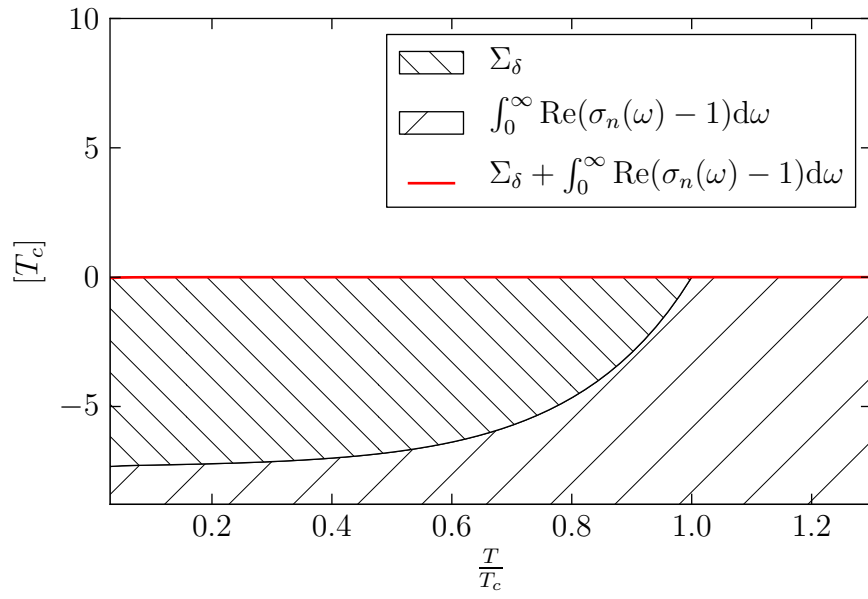


Figure 4.5: Integral, TODO plot new

# Chapter 5

## Summary

TODO

### 5.1 Outlook

TODO

# Appendices



## .1 Conventions in this Report

The AdS space will be referred to as the bulk and the boundary conformal field theory will be referred to as the boundary theory or the superconductor. Vector quantities not involving time components will in the boundary theory be written with boldface,  $\mathbf{E}, \mathbf{J}$ .

Tensor indices in the  $d + 1$ -dimensional bulk theory will be TODO latin letters,  $a, b, c, \dots$ . Tensor indices in the  $d$ -dimensional boundary theory will be TODO greek letters,  $\mu, \nu, \dots$ . Tensors written in component form will have it's components ordered as  $\mu = t, x, y$  on the boundary and  $a = z, t, x, y$  in the bulk. The metric sign convention for positive space-like distances will be chosen.

The action is calculated from the Lagrangian as

$$S = \int d^{d+1}x \sqrt{g} \mathcal{L} + S_{\text{boundary}}. \quad (1)$$

This is independent of coordinates but makes us use covariant derivatives for finding the equations of motion. The square root could instead be absorbed in the Lagrangian and the space time be considered flat. This has been done in computer-aided calculations of the equations of motion.

## .2 Boundary Behavior of Bulk Fields

The bulk Lagrangian considered will contain different fields and depends both on the fields and their first derivatives. Consider a field  $\psi$  with a kinetic term  $-(\partial_a \psi)^2$  and a potential term  $V(\psi)$ . The classical solution is the one that extremizes the action. The action integral contains the metric as an integration measure

$$S = \int d^{d+1}x \sqrt{|\det g_{ab}|} \mathcal{L} \equiv \int d^{d+1}x \sqrt{g} \mathcal{L}. \quad (2)$$

The Euler-Lagrange equation is obtained by varying the action. The integration measure can then be regarded as part of the Lagrangian or covariant derivatives can be used in the derivation of the Euler-Lagrange equation. The measure becomes when using the metric (3.4)  $L^{d+1} z^{-d-1}$ . The Euler-

Lagrange equation gives

$$0 = \partial_a \left( \frac{\partial(z^{-d-1}(V(\psi) - (\partial_b \psi)^2))}{\partial(\partial_a \phi)} \right) - \frac{\partial(z^{-d-1}(V(\psi) - (\partial_b \psi)^2))}{\partial \phi} = \quad (3)$$

$$= -\partial_a (z^{-d-1} 2\partial^a \psi) - z^{-d-1} V'(\psi)$$

We will be interested in boundary systems with translational symmetry so  $\psi$  is assumed to be a function of  $z$ , TODO motivate more. The equation of motion then becomes

$$0 = -\partial_z (z^{-d-1} 2g^{zz} \partial_z \psi) - \frac{V'(\psi)}{z^{d+1}} =$$

$$-\partial_z (z^{-d-1} 2(z^2 L^{-2} f(z)) \partial_z \psi) - \frac{V'(\psi)}{z^{d+1}} =$$

$$-z^{-d-1} 2z^2 L^{-2} f(z) \psi'' - L^{-2} ((-d+1)z^{-d} 2f(z) + z^{-d+1} 2f'(z)) \psi' - \frac{V'(\psi)}{z^{d+1}} \quad (4)$$

This gives a second order differential equation for  $\psi(z)$

$$0 = -z^2 2f(z) \psi'' - ((-d+1)z 2f(z) + z^2 2f'(z)) \psi' - L^2 V'(\psi) \quad (5)$$

Now consider the boundary,  $z = 0$ . Our metric is required to be asymptotically AdS so  $f(0) \rightarrow 1, z f'(0) \rightarrow 0$ .  $\psi$  can be expanded at the boundary as a Laurent series. Call the lowest exponent in this series  $\Delta$ .  $\psi$  will then behave as  $z^\Delta$  near the boundary. This should solve the differential equation in the near boundary limit. Insertion of  $z^\Delta$  into the differential equation and taking the limit of small  $z$  gives

$$0 = -z^2 2\Delta(\Delta-1)z^{\Delta-2} - ((-d+1)2z + z^2 2f'(z)) \Delta z^{\Delta-1} - L^2 V'(z^\Delta) \quad (6)$$

$$= z^\Delta (-2\Delta(\Delta-1) - 2(-d+1)\Delta) - L^2 V'(z^\Delta).$$

Now consider a potential for a massive scalar field,  $V(\psi) = -m^2 \psi^2 + \mathcal{O}(\psi^3)$ . We then get the following equation for  $\Delta$

$$0 = \Delta^2 - d\Delta - L^2 m^2 \quad (7)$$

in the limit  $z \rightarrow 0$ . This has solutions

$$\Delta = \frac{d \pm \sqrt{d^2 + 4L^2 m^2}}{2}. \quad (8)$$

$\psi$  thus goes as  $z^{\Delta_0}$  where  $\Delta_0$  is the smaller solution and  $\Delta_1$  the larger. The leading behaviour of  $\psi$  near  $z = 0$  is

$$\psi = \psi_0 \left( \frac{z}{L} \right)^{\Delta_0} + \psi_1 \left( \frac{z}{L} \right)^{\Delta_1} \quad (9)$$

unless  $\Delta_1 - \Delta_0 \geq 1$  and further terms from the series corresponding to  $\Delta_0$  must be included.

What will the contribution to the action from this solution be? Consider

the action contribution from the region  $z \in [\epsilon, \delta]$  where  $\delta$  is small and  $\epsilon \rightarrow 0$ .

$$\begin{aligned}
S_{[\epsilon, \delta]} &= \int_{z \in [\epsilon, \delta]} d^{d+1}x \sqrt{g} \mathcal{L} = \\
&= V \int_{\epsilon}^{\delta} dz \left( \frac{z}{L} \right)^{-d-1} (-m^2 \psi^2 - (\partial_a \psi)^2) = \\
&= V \int_{\epsilon}^{\delta} dz \left( \frac{z}{L} \right)^{-d-1} \left( -m^2 (\psi_0 \left( \frac{z}{L} \right)^{\Delta_0} + \psi_1 \left( \frac{z}{L} \right)^{\Delta_1})^2 - (\partial_a (\psi_0 \left( \frac{z}{L} \right)^{\Delta_0} + \psi_1 \left( \frac{z}{L} \right)^{\Delta_1}))^2 \right) = \\
&= V \int_{\epsilon}^{\delta} dz \left( \frac{z}{L} \right)^{-d-1} \left[ -m^2 \left( \psi_0^2 \left( \frac{z}{L} \right)^{2\Delta_0} + \psi_1^2 \left( \frac{z}{L} \right)^{2\Delta_1} + 2\psi_0 \psi_1 \left( \frac{z}{L} \right)^{\Delta_0 + \Delta_1} \right) \right. \\
&\quad \left. - g^{zz} L^{-2} (\Delta_0 \psi_0 \left( \frac{z}{L} \right)^{\Delta_0 - 1} + \Delta_1 \psi_1 \left( \frac{z}{L} \right)^{\Delta_1 - 1})^2 \right] = \\
&= V \int_{\epsilon}^{\delta} dz \left( \frac{z}{L} \right)^{-d-1} \left[ -m^2 \left( \psi_0^2 \left( \frac{z}{L} \right)^{2\Delta_0} + \psi_1^2 \left( \frac{z}{L} \right)^{2\Delta_1} + 2\psi_0 \psi_1 \left( \frac{z}{L} \right)^{\Delta_0 + \Delta_1} \right) \right. \\
&\quad \left. - g^{zz} L^{-2} \left( \Delta_0^2 \psi_0^2 \left( \frac{z}{L} \right)^{2(\Delta_0 - 1)} + \Delta_1^2 \psi_1^2 \left( \frac{z}{L} \right)^{2(\Delta_1 - 1)} + 2\Delta_0 \Delta_1 \psi_0 \psi_1 \left( \frac{z}{L} \right)^{\Delta_0 + \Delta_1 - 2} \right) \right] = \\
&= V \int_{\epsilon}^{\delta} dz \left[ -m^2 \left( \psi_0^2 \left( \frac{z}{L} \right)^{2\Delta_0 - d - 1} + \psi_1^2 \left( \frac{z}{L} \right)^{2\Delta_1 - d - 1} + 2\psi_0 \psi_1 \left( \frac{z}{L} \right)^{-1} \right) \right. \\
&\quad \left. - L^{-2} \left( \Delta_0^2 \psi_0^2 \left( \frac{z}{L} \right)^{2\Delta_0 - d - 1} + \Delta_1^2 \psi_1^2 \left( \frac{z}{L} \right)^{2\Delta_1 - d - 1} - 2L^2 m^2 \psi_0 \psi_1 \left( \frac{z}{L} \right)^{-1} \right) \right] = \\
&= V \int_{\epsilon}^{\delta} dz \left( (-m^2 - \Delta_0^2 L^{-2}) \psi_0^2 \left( \frac{z}{L} \right)^{2\Delta_0 - d - 1} + (-m^2 - \Delta_1^2 L^{-2}) \psi_1^2 \left( \frac{z}{L} \right)^{2\Delta_1 - d - 1} \right) = \\
&= V L^{-2} (\Delta_1 - \Delta_0) \int_{\epsilon}^{\delta} dz \left( \Delta_0 \psi_0^2 \left( \frac{z}{L} \right)^{2\Delta_0 - d - 1} - \Delta_1 \psi_1^2 \left( \frac{z}{L} \right)^{2\Delta_1 - d - 1} \right) = \\
&= V L^{-1} (\Delta_1 - \Delta_0) \left( -\frac{\Delta_0 \psi_0^2}{2\Delta_0 - d} \left( \frac{\epsilon}{L} \right)^{2\Delta_0 - d} + \frac{\Delta_1 \psi_1^2}{2\Delta_1 - d} \left( \frac{\epsilon}{L} \right)^{d - 2\Delta_0} \right) + \text{finite} \\
&= V L^{-1} \left( \Delta_0 \psi_0^2 \left( \frac{\epsilon}{L} \right)^{2\Delta_0 - d} - \Delta_1 \psi_1^2 \left( \frac{\epsilon}{L} \right)^{d - 2\Delta_0} \right) + \text{finite}
\end{aligned} \tag{10}$$

Here  $\Delta_0 + \Delta_1 = d$  and  $\Delta_0 \Delta_1 = -L^2 m^2$  have been used. One of these two terms diverges as  $\epsilon \rightarrow 0$ . The term with  $\psi_0$  diverges since  $2\Delta_0 - d = -\sqrt{d^2 + 4L^2 m^2}$ . The action from the near boundary thus diverges. This can be remedied by having a boundary term in the action that exactly cancels this divergence.

The boundary term must thus evaluate to

$$-\Delta_0 V L^{-1} \psi_0^2 \left( \frac{\epsilon}{L} \right)^{2\Delta_0 - d} \quad (11)$$

near the boundary. A boundary term like this can be constructed using  $\psi = \psi_0 L^{-\Delta_0} \epsilon^{\Delta_0}$  near the boundary and  $\sqrt{\gamma} = L^d z^{-d}$  where  $\gamma$  is the determinant of the metric induced on the boundary by  $g_{ab}$ . The boundary term then becomes

$$S_{\text{bdy}} = - \int_{z=\epsilon} d^d x \Delta_0 L^{-1} \psi^2 \sqrt{\gamma} \quad (12)$$

This addition to the Lagrangian is Lorentz invariant and it also has conformal invariance.

### .3 Computer-Aided Analytical Calculations

The Christoffel symbols for the AdS black hole were calculated using SymPy. All the non-zero components are shown below.

$$\begin{aligned} \Gamma_{zzz} &= \frac{L^2 (-zf'(z) - 2f(z))}{2z^3 f^2(z)} = \frac{L^2 z_h^3 (5z^3 - 2z_h^3)}{2z^3 (z^6 - 2z^3 z_h^3 + z_h^6)} \\ \Gamma_{ztt} &= \frac{L^2 (zf'(z) - 2f(z))}{2z^3} = \frac{L^2 (-z^3 - 2z_h^3)}{2z^3 z_h^3} \\ \Gamma_{zxx} &= \frac{L^2}{z^3} \\ \Gamma_{zyy} &= \frac{L^2}{z^3} \\ \Gamma_{ttz} = \Gamma_{tzt} &= \frac{L^2 (-zf'(z) + 2f(z))}{2z^3} = \frac{L^2 (z^3 + 2z_h^3)}{2z^3 z_h^3} \\ \Gamma_{xxz} = \Gamma_{xzx} &= -\frac{L^2}{z^3} \\ \Gamma_{yyz} = \Gamma_{yzy} &= -\frac{L^2}{z^3} \end{aligned} \quad (13)$$

which gives

$$R = \frac{-z^2 f''(z) + 6zf'(z) - 12f(z)}{L^2} = -\frac{12}{L^2} \quad (14)$$

Some Christoffel symbol contractions are useful

$$\begin{aligned}
\Gamma^a_{az} &= -\frac{4}{z} \\
\Gamma^a_{at} &= 0 \\
\Gamma^a_{ax} &= 0 \\
\Gamma^a_{ay} &= 0 \\
g^{ab}\Gamma^z_{ab} &= \frac{z(-zf'(z) + 2f(z))}{L^2} = \frac{z(z^3 + 2z_h^3)}{L^2 z_h^3} \\
g^{ab}\Gamma^t_{ab} &= 0 \\
g^{ab}\Gamma^x_{ab} &= 0 \\
g^{ab}\Gamma^y_{ab} &= 0.
\end{aligned} \tag{15}$$

Using these one obtains

$$\nabla_a \nabla^a \chi = \left( \partial_a \partial^a + \frac{z(zf'(z) - 2f(z))}{L^2} \partial_z \right) \chi. \tag{16}$$

for a scalar field  $\chi$ . The non-zero components of the electromagnetic tensor after making the definitions in Section 3.1 are

$$\begin{aligned}
F_{zt}(z) &= -F_{tz}(z) = \phi'(z) \\
F_{zx}(z, t) &= -F_{xz}(z, t) = A'_x(z) \exp(it\omega) \\
F_{tx}(z, t) &= -F_{xt}(z, t) = i\omega A_x(z) \exp(it\omega)
\end{aligned} \tag{17}$$

Another useful quantity is  $\nabla_a F^{ab}$

$$\nabla_a F^{ab} = \partial_a F^{ab} + \Gamma^a_{ca} F^{cb} + \Gamma^b_{ca} F^{ac} = \partial_a F^{ab} + \Gamma^a_{ca} F^{cb} \tag{18}$$

# Bibliography

- [1] Gary T. Horowitz and Jorge E. Santos. General Relativity and the Cuprates. 2013.
- [2] Juan Martin Maldacena. The Large N limit of superconformal field theories and supergravity. *Adv.Theor.Math.Phys.*, 2:231–252, 1998.
- [3] John McGreevy. Holographic duality with a view toward many-body physics. *Adv.High Energy Phys.*, 2010:723105, 2010.
- [4] Sean A. Hartnoll. Lectures on holographic methods for condensed matter physics. *Class.Quant.Grav.*, 26:224002, 2009.
- [5] Romuald A. Janik and Robi Peschanski. Asymptotic perfect fluid dynamics as a consequence of ads/cft correspondence. *Phys. Rev. D*, 73:045013, Feb 2006.
- [6] Sean A. Hartnoll, Christopher P. Herzog, and Gary T. Horowitz. Building a Holographic Superconductor. *Phys.Rev.Lett.*, 101:031601, 2008.
- [7] M. Kleban, M. Porrati, and R. Rabadan. Stability in asymptotically AdS spaces. *JHEP*, 0508:016, 2005.
- [8] Steven S. Gubser. Breaking an Abelian gauge symmetry near a black hole horizon. *Phys.Rev.*, D78:065034, 2008.
- [9] Richard A. Ferrell and Rolfe E. Glover. Conductivity of superconducting films: A sum rule. *Phys. Rev.*, 109:1398–1399, Feb 1958.
- [10] P. Drude. Zur Elektronentheorie der Metalle. *Annalen der Physik*, 306:566–613, 1900.

- [11] H. L. Liu, M. Quijada, D. B. Romero, D. B. Tanner, A. Zibold, G. L. Carr, H. Berger, L. Forró, L. Mihaly, G. Cao, B.-H. O, J. T. Markert, J. P. Rice, M. J. Burns, and K. A. Delin. Drude behavior in the far-infrared conductivity of cuprate superconductors. *Annalen der Physik*, 518:606–618, July 2006.