Convergence of the Simultaneous Algebraic Reconstruction Technique (SART)

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***Abstract*—Computed tomography (CT) has been extensively studied for years and widely used in the modern society. Although the filtered back-projection algorithm is the method of choice by manufacturers, efforts are being made to revisit iterative methods due to their unique advantages, such as superior performance with incomplete noisy data. In 1984, the simultaneous algebraic reconstruction technique (SART) was developed as a major refinement of the algebraic reconstruction technique (ART). However, the convergence of the SART has never been established since then. In this paper, the convergence is proved under the condition that coefficients of the linear imaging system are nonnegative. It is shown that from any initial guess the sequence generated by the SART converges to a weighted least square solution.**

***Index Terms*—Algebraic reconstruction technique (ART), computed tomography (CT), emission tomography, expectation maximization (EM), inverse problem, simultaneous algebraic reconstruction technique (SART).**

I. Introduction

Computed tomography (CT) has been extensively studied for years and widely used. Although the filtered back-projection method has been the method of choice by CT manufacturers, efforts are being made to revisit iterative methods [1]–[7]. Relative to closed-form solutions such as the filtered back-projection algorithm, the iterative approach has a major potential to achieve a superior performance in handling incomplete, noisy, and dynamic data. Although the iterative approach is generally slow, the computing technology is coming to the point that commercial implementation of iterative methods becomes practical for important radiological applications, including image noise suppression, metal artifact reduction, CT fluoroscopic imaging, and so on. More importantly, the theoretical findings accumulated over the past three decades have greatly improved our knowledge of the iterative methods, and formed a solid foundation for further advancement.

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Historically, the algebraic reconstruction technique (ART) was the first algorithm applied in CT [8]. In 1984, the simultaneous algebraic reconstruction technique (SART) was proposed as a major refinement of the ART [9]. Since then, the SART remains a powerful tool for iterative reconstruction. Recently, the SART has been used in a number of studies with impressive results [10]–[14]. However, we noticed that the convergence of the SART has never been established.

In the following, we will prove that the SART does converge, assuming that coefficients of the linear imaging system are in the nonnegative space, as in the cases of transmission and emission CT. In Section II, we will review the SART, describe our assumptions, and define some notations. In Section III, we will establish several properties of the SART. In Section IV, we will prove the convergence of the SART. In Section V, some relevant issues are discussed.

II. Preliminaries

We model a linear imaging system as follows:

$$Ax = b \tag{1}$$

where $A = (A_{i,j})$ denotes an $M \times N$ matrix, $b = (b_1 \cdots b_M)^{tr} \in \mathbf{R}^M$ observed data, $x = (x_1 \cdots x_N)^{tr} \in \mathbf{R}^N$ an underlying image, and ^{tr} the transpose of a vector/matrix. We define

$$A_{i,+} = \sum_{j=1}^{N} A_{i,j} \text{ for } i = 1, \ldots, M \tag{2}$$

$$A_{+,j} = \sum_{i=1}^{M} A_{i,j} \text{ for } j = 1, \ldots, N \tag{3}$$

$$\bar{b}(x) = Ax. \tag{4}$$

The SART, proposed in [9] and generalized in [15], is

$$x_j^{(k+1)} = x_j^{(k)} + \frac{\omega}{A_{+,j}} \sum_{i=1}^{M} \frac{A_{i,j}}{A_{i,+}} \left(b_i - \bar{b}_i(x^{(k)}) \right) \tag{5}$$

for $k = 0, 1, \ldots$, where ω denotes a relaxation parameter in (0,2). The original SART in [9]was with $\omega = 1$.

Our assumptions are as follows.

- $A_{i,j} \geq 0$, for $i = 1, \ldots, M$ and $j = 1, \ldots, N$.
- $A_{+,j} \neq 0$ and $A_{i,+} \neq 0$ for $j = 1, \ldots, N$ and $i = 1, \ldots, M$, respectively.

Some comments on the assumptions are in order. The first assumption requests that coefficients of the imaging system be nonnegative. The nonnegativity is valid in many imaging problems, such as transmission and emission CT. The second assumption appears restrictive, but it is really not. Actually, $A_{+,j} \neq 0$ means that x is measured at each x_j. On the other hand, the third assumption $A_{i,+} \neq 0$ means that every b_i carries a certain amount of information about x.

The following notations are defined for later use. In either $\mathbf{R}^N$ or $\mathbf{R}^M$, the canonical Euclidean norm is represented by $\|\cdot\|$, the canonical Euclidean inner product by $\langle\cdot,\cdot\rangle$, and the zero element by θ. Let V be the diagonal matrix with diagonal element $A_{+,j}$, and W be the diagonal matrix with diagonal element $1/A_{i,+}$. The matrices V and W induce the following inner products on $\mathbf{R}^N$ and $\mathbf{R}^M$:

$$\langle x,x\rangle_V = \langle Vx,x\rangle, \text{ for } x \in \mathbf{R}^N \tag{6}$$

$$\langle y,y\rangle_W = \langle Wy,y\rangle, \text{ for } y \in \mathbf{R}^M. \tag{7}$$

The corresponding V- and W-norms are denoted as $\|\cdot\|_V$ and $\|\cdot\|_W$, respectively. In terms of V and W, the SART formula becomes

$$x^{(k+1)} = x^{(k)} + \omega V^{-1}A^{tr}W(b - Ax^{(k)}). \tag{8}$$

We use $\mathbf{R}^N$ and $\mathbf{R}^M$ to denote the canonical Euclidean spaces with the canonical Euclidean inner products $\langle\cdot,\cdot\rangle$. Also, we have the spaces $\mathcal{X}$ and $\mathcal{Y}$ that are $\mathbf{R}^N$ with the inner product $\langle\cdot,\cdot\rangle_V$ and $\mathbf{R}^M$ with the inner product $\langle\cdot,\cdot\rangle_W$, respectively.

III. Properties of the SART

Let $N(A)$ be the null space of A. By the orthogonal decomposition theorem

$$\mathbf{R}^N = N(A) \oplus N(A)^{\perp}. \tag{9}$$

We use $B^{\perp}$ to denote the orthogonal complement of a subspace B in either $\mathbf{R}^N$ or $\mathbf{R}^M$. Let $R(A^{tr})$ be the range of A^{tr}. It is well-known that $N(A)^{\perp} = R(A^{tr})$.

Let us consider the following weighted least square functional:

$$L(x) = \sum_{i=1}^{M} \frac{1}{A_{i,+}} \left(b_i - \bar{b}_i(x)\right)^2 = \|b - Ax\|_W^2 \tag{10}$$

for $x \in \mathbf{R}^N$. It turns out later that the sequence generated by the SART converges to a minimizer of this functional. The gradient of L is

$$\nabla L(x) = -2A^{tr}W(b - Ax). \tag{11}$$

All the minimizers of L satisfy the following normal equation:

$$A^{tr}WAx = A^{tr}Wb. \tag{12}$$

Since $\langle A^{tr}WAx, x\rangle = \|Ax\|_W^2$, it is easy to show that $N(A^{tr}WA) = N(A)$. Since $A^{tr}Wb \in R(A^{tr}) = N(A)^{\perp}$, there must exist a solution to (12).

Let S be the set of all solutions to (12). It can be easily verified that $L(x)$ is convex in x. Hence, a solution to (12) is also a minimizer of L on $\mathbf{R}^N$, and vice versa. It is straightforward that S is also the set of all minimizers of L on $\mathbf{R}^N$ and a closed convex set of $\mathbf{R}^N$.

The SART can be written as the following gradient-based scheme:

$$x^{(k+1)} = x^{(k)} - \frac{\omega}{2}V^{-1}\nabla L(x^{(k)}). \tag{13}$$

Let

$$F(x) = \omega V^{-1}A^{tr}W(b - Ax) \tag{14}$$

the SART becomes

$$x^{(k+1)} = x^{(k)} + F\left(x^{(k)}\right). \tag{15}$$

The descending of L with the SART iteration can be estimated by the following proposition.

Proposition 1:

$$L\left(x^{(k+1)}\right) - L(x^{(k)}) \leq -\alpha\|X^{(k+1)} - X^{(k)}\|_v^2 \tag{16}$$

for $k = 0, 1, \ldots$, where $\alpha = 2/\omega - 1 > 0$.

Proof:

$$\begin{aligned} L\left(x^{(k+1)}\right) - L(x^{(k)}) &= -\sum_{i=1}^{M} \frac{1}{A_{i,+}} \\ &\quad \cdot \left(2b_i - \bar{b}_i\left(x^{(k)}\right) - \bar{b}_i\left(x^{(k+1)}\right)\right) \\ &\quad \cdot \left(\bar{b}_i\left(x^{(k+1)}\right) - \bar{b}_i\left(x^{(k)}\right)\right) \\ &= -\sum_{i=1}^{M} \frac{1}{A_{i,+}} \\ &\quad \cdot \left(2b_i - 2\bar{b}_i\left(x^{(k)}\right) \right.\\ &\qquad \left. -\bar{b}_i\left(F(x^{(k)})\right)\right) \\ &\quad \cdot \bar{b}_i\left(F(x^{(k)})\right) \\ &= \sum_{i=1}^{M} \frac{1}{A_{i,+}} \left(\bar{b}_i\left(F(x^{(k)})\right)\right)^2 \\ &\quad - 2\sum_{i=1}^{M} \frac{1}{A_{i,+}} \\ &\quad \cdot \left(b_i - \bar{b}_i\left(x^{(k)}\right)\right) \cdot \bar{b}_i\left(F(x^{(k)})\right). \end{aligned}$$

For the first term, by the convexity of the function $t \mapsto t^2$, it is

$$\sum_{i=1}^{M} A_{i,+} \left(\sum_{j=1}^{N} \frac{A_{i,j}}{A_{i,+}} F_j\left(x^{(k)}\right)\right)^2 \tag{17}$$

$$\leq \sum_{i=1}^{M} A_{i,+} \sum_{j=1}^{N} \frac{A_{i,j}}{A_{i,+}} \left[F_j\left(x^{(k)}\right)\right]^2 \tag{18}$$

$$= \sum_{j=1}^{N} A_{+,j} F_j\left(x^{(k)}\right)^2 = \left\|x^{(k+1)} - x^{(k)}\right\|_V^2. \tag{19}$$

The second term is

$$
\begin{aligned}
&2\sum_{i=1}^{M}\frac{1}{A_{i,+}}\left(b_i-\bar{b}_i\left(x^{(k)}\right)\right)\cdot\sum_{j=1}^{N}A_{i,j}F_j\left(x^{(k)}\right)\\
&\quad=2\sum_{j=1}^{N}F_j\left(x^{(k)}\right)\sum_{i=1}^{M}\frac{A_{i,j}}{A_{i,+}}\left(b_i-\bar{b}_i\left(x^{(k)}\right)\right)\\
&\quad=\frac{2}{\omega}\sum_{j=1}^{N}A_{+,j}\left[F_j\left(x^{(k)}\right)\right]^2=\frac{2}{\omega}\left\|x^{(k+1)}-x^{(k)}\right\|_V^2.
\end{aligned}
$$

Therefore, the conclusion follows immediately. ■

Corollary 2: We have the following corollaries:

A) $\{L(x^{(k)})\}_{k=0}^{\infty}$ is bounded;

B) $L(x^{(k+1)})\leq L(x^{(k)})$;

C)

$$
L\left(x^{(k+1)}\right)+\alpha\sum_{j=0}^{k}\left\|x^{(j+1)}-x^{(j)}\right\|_V^2\leq L(x_0); \tag{20}
$$

D) $\sum_{j=0}^{\infty}\|x^{(j+1)}-x^{(j)}\|_V^2$ converges;

E)

$$
\left\|x^{(k+1)}-x^{(k)}\right\|_V\longrightarrow 0 \text{ as } k\longrightarrow\infty. \tag{21}
$$

Proof: A and B are obvious. As for C, D and E, note

$$
\begin{aligned}
L\left(x^{(1)}\right)+\alpha\left\|x^{(1)}-x^{(0)}\right\|_V^2&\leq L\left(x^{(0)}\right),\\
&\vdots\\
L\left(x^{(k+1)}\right)+\alpha\left\|x^{(k+1)}-x^{(k)}\right\|_V^2&\leq L\left(x^{(k)}\right).
\end{aligned}
$$

Summing up all these inequalities, we have

$$
L\left(x^{(k+1)}\right)+\alpha\sum_{j=0}^{k}\left\|x^{(j+1)}-x^{(j)}\right\|_V^2\leq L\left(x^{(0)}\right). \tag{22}
$$

Now, D is obvious. Then, E follows immediately from D. ■

IV. Convergence of the SART

Now consider the Hilbert space $\mathcal{X}$. By orthogonal decomposition, we have

$$
\mathcal{X}=N(A)\oplus N(A)^{\perp_V} \tag{23}
$$

where $B^{\perp_V}$ denotes the orthogonal complement of a subspace B in $\mathcal{X}$. Since S is also a closed convex set in $\mathcal{X}$, it must contain a unique element x^* with the minimal V-norm. Clearly, $x^*\in N(A)^{\perp_V}$.

By the normal equation (12), the SART can be written as

$$
x^{(k+1)}-x^*=x^{(k)}-x^*+\omega V^{-1}A^{tr}WA\left(x^*-x^{(k)}\right). \tag{24}
$$

Let $z^{(k)}=x^{(k)}-x^*$, we have

$$
z^{(k+1)}=z^{(k)}-\omega V^{-1}A^{tr}WAz^{(k)}. \tag{25}
$$

The inner product of (25) with $z^{(k)}$ is

$$
\begin{aligned}
\left\langle Vz^{(k+1)},z^{(k)}\right\rangle&=\left\langle Vz^{(k)},z^{(k)}\right\rangle\\
&\quad-\omega\left\langle A^{tr}WAz^{(k)},z^{(k)}\right\rangle\\
&=\left\|z^{(k)}\right\|_V^2-\omega\left\|Az^{(k)}\right\|_W^2.
\end{aligned}
$$

The left hand is

$$
\begin{aligned}
\left\langle Vz^{(k+1)},z^{(k)}\right\rangle&=\left\langle Vz^{(k+1)},z^{(k+1)}\right\rangle\\
&\quad+\left\langle Vz^{(k+1)},z^{(k)}-z^{(k+1)}\right\rangle\\
&=\left\|z^{(k+1)}\right\|_V^2-\left\|z^{(k)}-z^{(k+1)}\right\|_V^2\\
&\quad+\left\langle Vz^{(k)},z^{(k)}-z^{(k+1)}\right\rangle\\
&=\left\|z^{(k+1)}\right\|_V^2-\left\|z^{(k)}-z^{(k+1)}\right\|_V^2\\
&\quad+\omega\left\langle Vz^{(k)},V^{-1}A^{tr}WAz^{(k)}\right\rangle\\
&=\left\|z^{(k+1)}\right\|_V^2-\left\|z^{(k)}-z^{(k+1)}\right\|_V^2\\
&\quad+\omega\left\langle z^{(k)},A^{tr}WAz^{(k)}\right\rangle\\
&=\left\|z^{(k+1)}\right\|_V^2-\left\|z^{(k)}-z^{(k+1)}\right\|_V^2\\
&\quad+\omega\left\|Az^{(k)}\right\|_W^2.
\end{aligned}
$$

Therefore

$$
\left\|z^{(k+1)}\right\|_V^2=\|z^k\|_V^2+\left\|z^{(k)}-z^{(k+1)}\right\|_V^2-2\omega\left\|Az^{(k)}\right\|_W^2 \tag{26}
$$

and

$$
\left\|z^{(k+1)}\right\|_V^2\leq\|z^k\|_V^2+\left\|z^{(k)}-z^{(k+1)}\right\|_V^2. \tag{27}
$$

Proposition 3: The sequence $\{z^k\}$ is bounded.

Proof: By (27) and using the same method as in the proof of Corollary 2, we have

$$
\begin{aligned}
\left\|z^{(1)}\right\|_V^2&\leq\left\|z^{(0)}\right\|_V^2+\left\|z^{(1)}-z^{(0)}\right\|_V^2,\\
&\vdots\\
\left\|z^{(k+1)}\right\|_V^2&\leq\left\|z^{(k)}\right\|_V^2+\left\|z^{(k+1)}-z^{(k)}\right\|_V^2.
\end{aligned}
$$

Summing up all these inequalities, we have

$$
\begin{aligned}
\left\|z^{(k+1)}\right\|_V^2&\leq\left\|z^{(0)}\right\|_V^2+\sum_{j=0}^{k}\left\|z^{(j+1)}-z^{(j)}\right\|_V^2\\
&\leq\left\|z^{(0)}\right\|_V^2+\sum_{j=0}^{\infty}\left\|x^{(j+1)}-x^{(j)}\right\|_V^2.
\end{aligned}
$$

The conclusion follows by Corollary 2. ■

Let $r_k=\|z^{(k)}\|_V^2$. The next proposition establishes the convergence of $\{r_k\}$.

Proposition 4: The sequence $\{r_k\}$ is convergent, i.e., the limit $\lim_{k\to\infty} r_k$ exists.

Proof: Since $\{r_k\}$ is bounded, both the limit superior $\hat{r} = \limsup_{k\to\infty} r_k$ and limit inferior $\check{r} = \liminf_{k\to\infty} r_k$ exist. We can show that $\hat{r} = \check{r}$ as follows. For any $p > q$, we have

$$r_p \le r_q + \sum_{j=q}^{\infty} \left\| x^{(j+1)} - x^{(j)} \right\|_V^2 .$$

Fixing q and taking the limit superior with respect to p, we obtain

$$\hat{r} \le r_q + \sum_{j=q}^{\infty} \left\| x^{(j+1)} - x^{(j)} \right\|_V^2 .$$

Then, taking the limit inferior with respect to q

$$\hat{r} \le \check{r} + \lim_{q\to\infty} \sum_{j=q}^{\infty} \left\| x^{(j+1)} - x^{(j)} \right\|_V^2 .$$

Because $\sum_{j=0}^{\infty} \|x^{(j+1)} - x^{(j)}\|_V^2$ exists

$$\lim_{q\to\infty} \sum_{j=q}^{\infty} \left\| x^{(j+1)} - x^{(j)} \right\|_V^2 = 0.$$

Therefore, $\hat{r} = \check{r}$. ■

Proposition 5: The following series is convergent, i.e., the sum is finite

$$\sum_{k=0}^{\infty} \left\| Az^{(k)} \right\|_W^2 . \tag{28}$$

Proof: By (26)

$$\left\| Az^{(k)} \right\|_W^2 = \frac{1}{2\omega} \left(r_k - r_{k+1} + \left\| z^{(k)} - z^{(k+1)} \right\|_V^2 \right).$$

The conclusion follows from Corollary 2 D and Proposition 4. ■

Before we prove the main theorem on the convergence of the SART, we need the following lemma.

Lemma 6: There exists a positive constant λ such that

$$\|Av\|_W \ge \lambda \|v\|_V \text{ for } v \in N(A)^{\perp_V}. \tag{29}$$

Proof: We will prove the inequality by contradiction. If the inequality is false, there must exist $v_n \in N(A)^{\perp_V}$ such that

$$\|Av_n\|_W \le \frac{1}{n} \|v_n\|_V, \text{ for } n = 1, \ldots .$$

Dividing both sides by $\|v_n\|_V$ and setting $u_n = (v_n / \|v_n\|_V)$, we have

$$\|Au_n\|_W \le \frac{1}{n}. \tag{30}$$

Note that $u_n \in N(A)^{\perp_V}$ and $\|u_n\|_V = 1$. Because $\{u_n\}$ is bounded, there exists a convergent subsequence. Let it be $\{u_{n_j}\}$ and assume that $u_{n_j} \longrightarrow u_0$. By continuity of the norm function, $\|u_0\|_V = 1$. By the closeness of $N(A)^{\perp_V}$, $u_0 \in N(A)^{\perp_V}$. By (30), we have $\|Au_0\|_W = 0$. Therefore, $Au_0 = \theta$, which implies that $u_0 \in N(A)$. Hence, $u_0 \in N(A) \cap N(A)^{\perp_V} = \{\theta\}$. Clearly, $u_0 = \theta$ is in contradiction with $\|u_0\|_V = 1$. ■

Let P and Q be the orthogonal projection from $\mathcal{X}$ to $N(A)$ and $N(A)^{\perp_V}$, respectively, we are now ready to prove the following main theorem.

Theorem 7: The sequence $x^{(k)}$ generated by the SART (5) converges to $P[x_0] + x^* \in S$, for any $\omega \in (0, 2)$, where x^* is the solution of the normal equation (12) with the minimum V-norm. The limit is a solution to the normal (12), hence a global minimizer of L.

Proof: For any $u \in N(A)$ and any $x \in \mathbf{R}^N$

$$\begin{aligned} \langle F(x), u \rangle_V =& \langle VF(x), u \rangle \\ =& \langle \omega A^{tr} W(b - Ax), u \rangle \\ =& \omega \langle W(b - Ax), Au \rangle = 0. \end{aligned}$$

Therefore, $F(x) \in N(A)^{\perp_V}$. By (15), we have $P[x^{(k+1)}] = P[x^{(k)}] + P[F(x^{(k)})] = P[x^{(k)}]$, for $k = 0, \ldots$. Hence, $P[x^{(k+1)}] = P[x^{(0)}]$. By construction, $x^* \in N(A)^{\perp_V}$. Therefore

$$\begin{aligned} Az^{(k)} =& A \left(P\left[x^{(k+1)}\right] + Q\left[x^{(k+1)}\right] - x^* \right) \\ =& AQ\left[x^{(k+1)} - x^*\right] \end{aligned}$$

noting $AP[x^{(k+1)}] = \theta$ and $Q[x^*] = x^*$. Since $Q[x^{(k+1)} - x^*] \in N(A)^{\perp_V}$, by Lemma 6

$$\left\| Az^{(k)} \right\|_W \ge \lambda \left\| Q\left[x^{(k+1)} - x^*\right] \right\|_V .$$

By Proposition 5, $Q[x^{(k+1)}] \longrightarrow Q[x^*] = x^*$. Since $x^{(k+1)} = P[x^{(k+1)}] + Q[x^{(k+1)}]$, the conclusion follows immediately. ■

V. Discussion and Conclusion

The SART is a counterpart of the EM for emission tomography, although the latter has been remarkably successful and been studied more thoroughly than the former. The following statistical viewpoint is helpful to understand the significance of the SART: the SART and the EM formula produce the maximum likelihood estimates in the Gaussian and Poisson environments respectively, which are both practically important. Furthermore, the SART has the capability to solve the linear system in the real space, which accommodates imaging models for optical microscopy and so on. This is a major freedom relative to the EM which is handicapped when some data are negative.

By the end of the review process of this paper, we were kindly informed by Censor and Elfving that they obtained a proof of the convergence of the SART [16], independent of our work. It is interesting that the methods of both proofs are quite different. More general iterative schemes are studied in [16], while our method can also be used to study the convergence of general iterative schemes in the form of (8). Furthermore, Theorem 7 characterizes the structure of the final limit and its dependence on the initial value, which is instructive in practical applications. E.g., one corollary from Theorem 7, not trivial at all, indicates that the final limit is continuous with respect to the initial guess.

When the coefficients of the imaging system may take negative values, $A_{i,+}$ may be negative. In this case, the weighted least squares functional $L(x)$ is nonconvex. Hence, the original SART must be refined. In [16], the convergence of SART was established with

$$A_{i,+} = \sum_{j=1}^{N} |A_{i,j}| \text{ for } i = 1, \ldots, M \tag{31}$$

$$A_{+,j} = \sum_{i=1}^{M} |A_{i,j}| \text{ for } j = 1, \ldots, N. \tag{32}$$

After tracing the proof carefully, we found that the only place involving the above definitions is from (17) to (18), where we need to change $A_{i,j}$ to $|A_{i,j}|$ in (18). With the above modifications only, our proof remains valid. Hence, the same convergent result can be obtained.

As a side note, we point out that the convergence of the SART can also be proved using a general theorem of Bialy [17], which involves more advanced mathematics on spectral decomposition, as well as the corresponding definitions. We have decided to present the above relatively elementary proof for easier understanding and wider readership. As an extension to the above results and a direction for further work, there are a number of major issues to be addressed for ordered-subset (or block-iterative) versions of the SART. These issues include convergence of the ordered-subset SART variants, optimization of the control mechanism, and selection of the relaxation coefficients [18], [19].

In conclusion, we have established the convergence of the SART. It is shown that the sequence generated by the SART converges to a minimizer of a weighted least square functional from any initial guess.

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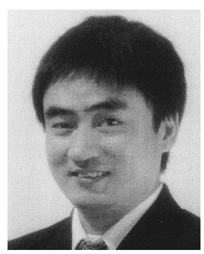

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