ST227 Table

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Disclaimer: The descriptions provided below are *NOT* rigorous definitions and should *NOT* be used for revising theory. They serve a purpose of a small cheat sheet with intuitive explanations behind the notions in the module, that could be used during assignments, lectures, tutorials or practicing applied questions for exam.

Notation

Symbol	Name	Description
S	State Space	Possible values taken by the Markov chain
P	Transition matrix	A matrix containing the probabilities of going between state spaces, rows being "old" and columns being "new" spaces, all 1-step
P_{ij}	Transition probability	A probability of Markov Chain going from state i to state j in 1-step
P_{ij}^n	Transition probability	A probability of Markov Chain going from state i to state j in n-steps
λ	Initial distribution	A vector containing distribution of the first state of the chain
λ_i	Initial probability of state i	A probability of state i being the first state of the Markov's chain, i'th entry of λ
d_i	State Period	Greatest Common divisor of all loops leading from state i to i
H_j	State first hitting time	Number of steps until state j is hit. Is 0 if the start point is state j
H_A	Set first hitting time	Number of steps until set A is hit. Is 0 if the start point is a state in set A
h_{ij}	State hitting probability	Probability of reaching state j from state i.
h_{iA}	Set hitting probability	Probability of entering a set A from state i.
η_{ij}	State expected hitting time	Expected number of steps until state j is reached from state i
η_{iA}	Set expected hitting time	Expected number of steps until set A is reached from state i
T_i	First Return time	Number of steps until state i loops on itself
f_i	Return probability	probability of chain returning back to state i from i, basically h_{ii}
m_i	Expected return time	Expected number of steps until state i loops on itself, basically η_{ii}
V_j	Number of visits	Number of visits to state j
V_A	Number of visits	Number of visits to set of states A
$\mathcal{T}$	Set of Transient States	Subset of S, containing all transient states
μ	Invariant Measure	A vector of positive entries, describing the measure, or "weight" of each state at every step of the chain. Has a property of $\mu = \mu P^n$

Symbol	Name	Description
π	Invariant Distribution	A special case of invariant measure, distribution describing the distribution of the chain at every step of the chain, has a property of $\pi = \pi P^n$ and $\sum_{i \in S} \pi_i = 1$. Not every chain has π. Time-reversed chains have the same stationary distributions.
γ_i^k	Expected number of visits	Expected number of visits to state i before chain returns to state k
Π	Long Term Transition Matrix	$\Pi = \lim_{n \to \infty} P^n$, with all rows equal to π. Describes the behaviour of the chain after arbitrarily many steps. Not applicable to every Markov Chain.
$G = (V, E)$	Graph	Graph G, where V is a set of vertices and $E \subset S \times S$, set of all edges. All graphs in this module are connected and weighted
V	Set of all vertices	Set of all vertices in a graph. In this module, $V = S$
E	Set of all edges	Set containing all connections in a graph, is a subset of $S \times S$. As all the graphs are connected, for any 2 points $x, y \in V$ there exists a subset of E connecting them
$w(x, y)$	Weight between vertices x and y	A non-negative real number, describing the weight of an edge. $w(x, y) = 0$ if $x, y \notin E$, and $w(x, y) > 0$ otherwise.
$W(x)$	Sum of weights of edges connected to vertex x	Positive(due to connected graph assumption) number, describing the sum of all edges connecting x with the rest of the graph: $W(x) = \sum_{y \in V} w(x, y) < \infty \ \forall x \in V$
Y	Offspring Distribution	Distribution describing the number of children in the branching process. X_n, number of offspring at time n, is described as $X_n = \sum_{k=1}^{X_{n-1}} Y_k$, where Y_i, $\forall i \in \mathbb{N}$ are independent copies of Y.
$G(s)$	Probability generating function(of Y)	Notion similar to moment generating function. Calculated by $G(s) = \mathbb{E}[s^Y], \quad \lvert s \rvert < 1.$
$F_n(s)$	Probability generating function(of X_n)	Notion similar to moment generating function. Calculated by $F_n(s) = \mathbb{E}[s^{X_n} \lvert X_0 = 1], \quad \lvert s \rvert < 1, \quad n \in \mathbb{N} \cup \{0\}.$
α	Extinction Probability	Probability that the branching process ends up with no offspring. Defined as $\alpha = \mathbb{P}(H_0 < \infty \lvert X_0 = 1)$. Can be interpreted as $h_{1,0}$. Smallest solution for equation $G(s) = s$ on the interval $[0, 1]$.
α_n	Extinction Probability at time n	Probability that the branching process is extinct at time n. Follows a property $\alpha_n \geq \alpha_{n+1}$. The sequence $(\alpha_n)_{n \geq 0}$ converges to α.

Probability generation function properties

Assuming the notation from the table above:

1. $G(s) = \sum_{k=0}^{\infty} s^k \mathbb{P}(Y = k)$ from definition of expectation.
2. $G(0) = \mathbb{P}(Y = k)$ from 1 by $s = 0$.
3. $G(1) = \mathbb{P}(Y < \infty) = 1$ from convergence of expectation.
4. $G^{(1)}(1) = \mathbb{E}[Y]$ from differentiation of the Power Series.
5. $F_n(s) = \sum_{k=0}^{\infty} s^k \mathbb{P}(X_n = k \lvert X_0 = 1)$ from definition of expectation.
6. $F_0(s) = s, \quad s \in [0, 1]$ following from 5 by subbing in $n = 0$.
7. $F_1(s) = \mathbb{E}[s^{X_1} \lvert X_0 = 1] = \mathbb{E}[s^Y \lvert X_0 = 1] = G(s), \quad s \in [0, 1]$ from definiton of Y.
8. $F_n(0) = \mathbb{P}(X_n = 0 \lvert X_0 = 1), \quad n \in \mathbb{N} \cup \{0\}$ from 5 by subbing in $s = 0$.

9. $F_n^{(1)}(1) = \mathbb{E}[X_n | X_0 = 1], \quad n \in \mathbb{N} \cup \{0\}$ from differentiation of the Power Series.
10. Proposition 6.1.1, will be added later.
11.
12.