Time-Ordering Invariance Under Monotone Clock Changes

Addendum to LTQG: $\sigma = \log(\tau/\tau_0)$

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Context

Let $\tau \in I \subset \mathbb{R}$ with I an interval and let $\sigma = F(\tau)$ be a *monotone* C^1 change of clock with $F' : I \to (0, \infty)$. Consider a (possibly time-dependent) Schrödinger generator $H(\tau)$ on a Hilbert space $\mathcal{H}$. Write $U_\tau(t_2, t_1)$ for the τ-time propagator solving

$$i\,\partial_\tau U_\tau(\tau, \tau_0) = H(\tau)\,U_\tau(\tau, \tau_0), \qquad U_\tau(\tau_0, \tau_0) = \mathbf{1}, \tag{1}$$

and define the σ-time generator by the usual chain rule

$$\tilde{H}(\sigma) := \frac{d\tau}{d\sigma} H(\tau(\sigma)) = \frac{H(\tau(\sigma))}{F'(\tau(\sigma))}. \tag{2}$$

Denote by $U_\sigma(\sigma_2, \sigma_1)$ the corresponding σ-time propagator.

Main Lemma (Time-Ordering Invariance)

Lemma 1 (Chronological invariance under monotone clocks)**.** *If F is strictly increasing, then for any $\tau_1 \le \tau_2$ we have*

$$\mathcal{T}_\tau\left[\exp\Big(-i\int_{\tau_1}^{\tau_2} H(\tau)\,d\tau\Big)\right] \;=\; \mathcal{T}_\sigma\left[\exp\Big(-i\int_{\sigma_1}^{\sigma_2} \tilde{H}(\sigma)\,d\sigma\Big)\right], \tag{3}$$

with $\sigma_j = F(\tau_j)$. In particular, $U_\tau(\tau_2, \tau_1) = U_\sigma(\sigma_2, \sigma_1)$.

Proof. Strict monotonicity implies that the map $\tau \mapsto \sigma = F(\tau)$ preserves order: $\tau_a < \tau_b \iff \sigma_a < \sigma_b$. Hence the chronological partitions used to define Dyson expansions can be transported bijectively. Start from the Dyson series in τ:

$$U_\tau(\tau_2, \tau_1) = \mathbf{1} + \sum_{n\ge1} (-i)^n \int_{\tau_1 \le t_n \le \cdots \le t_1 \le \tau_2} H(t_1)\cdots H(t_n)\,dt_1\cdots dt_n. \tag{4}$$

Change variables $s_k = F(t_k)$; then $dt_k = \frac{dt_k}{ds_k}\,ds_k = \frac{1}{F'(t_k)}\,ds_k$ and the ordered simplex is mapped to $\sigma_1 \le s_n \le \cdots \le s_1 \le \sigma_2$ by monotonicity. Each factor transforms as $H(t_k)\,dt_k = \tilde{H}(s_k)\,ds_k$ by definition of $\tilde{H}$. Therefore every n-simplex integral equals the corresponding σ-time Dyson term, yielding the identity. $\square$

Existence & Uniqueness Hypotheses

Assumption 1 (Bounded case)**.** $H(\tau)$ is strongly measurable and $\sup_{\tau\in I}\|H(\tau)\|<\infty$. Then the Dyson series converges in operator norm and defines a unique unitary propagator.

Assumption 2 (Kato Class (unbounded case))**.** $H(\tau)$ is self-adjoint on a dense domain D (independent of τ), $\tau\mapsto H(\tau)\psi$ is continuous $\forall\psi\in D$, and the family is stable in the sense of Kato (domain invariance and suitable bounds). Under these hypotheses the evolution family U_τ exists and is unique; the reparameterization $\tilde{H}(\sigma)=\frac{d\tau}{d\sigma}H(\tau(\sigma))$ preserves the same class, so U_σ exists and equals U_τ by Lemma 1.

Theorem 1 (Unitary equivalence under monotone reparameterization)**.** *Under either set of hypotheses above, $U_\tau(\tau_2,\tau_1)=U_\sigma(\sigma_2,\sigma_1)$ for $\sigma_j=F(\tau_j)$. Consequently, spectra of Heisenberg-evolved observables and transition probabilities coincide for τ- and σ-descriptions.*

Remark 1 (Application to LTQG)**.** For $F(\tau)=\log(\tau/\tau_0)$ with $\tau>0$ we have $\frac{d\tau}{d\sigma}=\tau$ and hence $\tilde{H}(\sigma)=\tau(\sigma)\,H(\tau(\sigma))$. The lemma shows that the reparameterization leaves all physical predictions invariant while making multiplicative early-time structure additive in σ.