Geodesic Checks in the Weyl Frame for Log-Time Reparameterization

Addendum to LTQG

October 15, 2025

Setup

Consider a spatially flat FLRW metric in proper time $\tau > 0$:

$$ds^2 = -d\tau^2 + a(\tau)^2 dx^2, \quad a(\tau) = \left(\frac{\tau}{\tau_*}\right)^p, \; p > 0. \tag{1}$$

Introduce the log-time $\sigma = \log(\tau/\tau_0)$ so that $\tau = \tau_0 e^\sigma$. Define the Weyl-rescaled *log-time frame* metric $\tilde{g}$ with conformal factor $\Omega(\tau) = \tau^{-1}$:

$$\tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu} = \tau^{-2} g_{\mu\nu}\,, \qquad d\tilde{s}^2 = -\tau^{-2} d\tau^2 + \tau^{-2} a(\tau)^2 dx^2. \tag{2}$$

In σ-coordinates this is

$$d\tilde{s}^2 = -d\sigma^2 + \left(\frac{\tau_0}{\tau_*}\right)^{2p} e^{2(p-1)\sigma}\, dx^2. \tag{3}$$

Null geodesics

Null geodesics are conformally invariant as curves (their images are preserved) though the affine parameter rescales. For radial null curves ($dx = dr$),

$$0 = -d\sigma^2 + C^2 e^{2(p-1)\sigma} dr^2, \qquad C := \left(\frac{\tau_0}{\tau_*}\right)^p. \tag{4}$$

Hence $dr/d\sigma = \pm C^{-1} e^{-(p-1)\sigma}$ and the coordinate distance from $\sigma = -\infty$ to a finite σ_1 is

$$\Delta r = \int_{-\infty}^{\sigma_1} C^{-1} e^{-(p-1)\sigma}\, d\sigma = \begin{cases} \infty, & p \le 1, \\ \dfrac{1}{C(p-1)} e^{-(p-1)\sigma_1}, & p > 1. \end{cases} \tag{5}$$

Thus for radiation ($p = \frac{1}{2}$) and matter ($p = \frac{2}{3}$), the comoving distance to the past boundary $\sigma \to -\infty$ diverges in the Weyl frame.

Affine parameter near $\sigma \to -\infty$

Let λ be an affine parameter for the null geodesic in the Weyl frame. For a metric of the form $d\tilde{s}^2 = -d\sigma^2 + B(\sigma)^2 dr^2$, radial nulls satisfy $dr/d\sigma = \pm 1/B(\sigma)$ and one finds

$$\frac{d^2\sigma}{d\lambda^2} + \frac{B'(\sigma)}{B(\sigma)} \left(\frac{d\sigma}{d\lambda}\right)^2 = 0 \quad \Rightarrow \quad \frac{d\sigma}{d\lambda} \propto B(\sigma)^{-1}. \tag{6}$$

With $B(\sigma) = C\, e^{(p-1)\sigma}$ we obtain $d\lambda \propto e^{-(p-1)\sigma} d\sigma$; hence

$$\int_{-\infty}^{\sigma_1} d\lambda \propto \int_{-\infty}^{\sigma_1} e^{-(p-1)\sigma} d\sigma = \begin{cases} \infty, & p \le 1, \\ \dfrac{1}{p-1} e^{-(p-1)\sigma_1}, & p > 1. \end{cases} \tag{7}$$

Conclusion: for $p \le 1$ (including radiation and matter), past-directed null geodesics in the Weyl frame have *infinite* affine length to $\sigma \to -\infty$; i.e., the past boundary is null-geodesically complete in this frame.

Timelike geodesics (comoving observers)

Consider comoving worldlines ($r = \text{const}$). In the Weyl frame their proper time is simply $\tilde{\tau} = \sigma$ up to an additive constant, because $d\tilde{s}^2 = -d\sigma^2$. Therefore the proper time to the past boundary is

$$\Delta\tilde{\tau} = \int_{-\infty}^{\sigma_1} d\sigma = \infty. \tag{8}$$

Thus comoving timelike geodesics are also complete to the past in the Weyl frame.

Proposition 1 (Geodesic completeness criteria in the Weyl frame)**.** *For spatially flat FLRW with $a(\tau) \propto \tau^p$ and conformal rescaling $\tilde{g} = \tau^{-2} g$:*

- *Past null geodesics are complete for $p \le 1$.*
- *Comoving timelike geodesics are complete for all $p > 0$.*

Remark 1 (Interpretation for LTQG)**.** *The σ-time boundary ($\sigma \to -\infty$) corresponds to $\tau \to 0^+$. In the Weyl frame, standard early-time cosmologies ($p \le 1$) become null- and timelike-complete towards this boundary. This matches the "asymptotic silence" narrative: dynamics* slow *in σ and geodesic paths accumulate infinite parameter length before reaching $\sigma = -\infty$.*