LTQG Quantum Field Theory: Mode Evolution and Particle Creation

Log-Time Quantum Gravity Framework

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Abstract

This document presents the quantum field theory applications of the Log-Time Quantum Gravity (LTQG) framework. We develop the theory of scalar field mode evolution in expanding FLRW backgrounds using log-time coordinates $\sigma = \log(\tau/\tau_0)$. The framework provides natural regularization of particle creation processes, finite Bogoliubov coefficients, and well-behaved mode functions throughout cosmic evolution. We establish the connection between LTQG and standard cosmological QFT, demonstrate Wronskian conservation, and validate the framework through comprehensive numerical calculations. Applications include cosmological particle creation, vacuum decay, and quantum field dynamics in curved spacetime.

Contents

1 Introduction

Quantum field theory in curved spacetime presents fundamental challenges, particularly in cosmological contexts where the background spacetime evolves dynamically. Standard treatments often encounter divergences in particle creation rates, mode function normalization, and vacuum energy calculations. The LTQG framework addresses these issues through the log-time coordinate $\sigma = \log(\tau/\tau_0)$ and associated regularization techniques.

1.1 The Mode Evolution Problem

In expanding FLRW spacetimes, quantum field modes satisfy Klein-Gordon-type equations with time-dependent mass terms arising from the cosmic expansion. For a massless scalar field ϕ in conformal time η:

$$\phi_k'' + \left(k^2 - \frac{a''}{a}\right)\phi_k = 0 \tag{1}$$

where $\phi_k(\eta)$ are the mode functions and $a(\eta)$ is the scale factor. Near the Big Bang, the term a''/a typically diverges, leading to:

- Infinite particle creation rates
- Divergent mode function amplitudes
- Breakdown of Bogoliubov transformation unitarity
- Ill-defined vacuum states

The LTQG approach resolves these issues systematically.

1.2 LTQG Resolution Strategy

The LTQG framework addresses QFT problems through:

1. **Log-Time Coordinates**: Transform to $\sigma = \log(\tau/\tau_0)$ coordinates, regularizing the early universe limit
2. **Effective Mode Equations**: Derive modified Klein-Gordon equations in σ-coordinates with finite effective masses
3. **Asymptotic Silence**: Utilize the property that mode coupling vanishes as $\sigma \to -\infty$
4. **Finite Bogoliubov Coefficients**: Ensure all particle creation processes have finite amplitudes

This approach maintains Lorentz invariance and general covariance while providing mathematical regularity.

2 Scalar Field Theory in Log-Time

2.1 Field Equations in FLRW Spacetime

Consider a massless scalar field ϕ in FLRW spacetime with metric:

$$ds^2 = a(\tau)^2\left[-d\tau^2 + \delta_{ij}dx^i dx^j\right] \tag{2}$$

where τ is conformal time and $a(\tau) = a_0(\tau/\tau_0)^p$ with $p = 2/(3(1+w))$.

The Klein-Gordon equation becomes:

$$\frac{1}{a^2}\partial_\tau\left(a^2\partial_\tau\phi\right) - \nabla^2\phi = 0 \tag{3}$$

2.2 Mode Decomposition

Expanding in plane wave modes:

$$\phi(\tau, \vec{x}) = \int \frac{d^3k}{(2\pi)^3} \left[a_{\vec{k}} u_k(\tau) + a^\dagger_{\vec{k}} u^*_k(\tau) \right] e^{i\vec{k}\cdot\vec{x}} \tag{4}$$

The mode functions $u_k(\tau)$ satisfy:

$$u''_k + \left(k^2 - \frac{a''}{a} \right) u_k = 0 \tag{5}$$

where primes denote derivatives with respect to conformal time τ.

2.3 Transformation to Log-Time

Under the transformation $\sigma = \log(\tau/\tau_0)$, we have $\tau = \tau_0 e^\sigma$ and:

$$\frac{d}{d\tau} = \frac{1}{\tau_0 e^\sigma} \frac{d}{d\sigma} \tag{6}$$

The mode equation transforms to:

Theorem 2.1 (Mode Equation in Log-Time)**.** *In log-time coordinates, the mode equation becomes:*

$$\frac{d^2u_k}{d\sigma^2} + \left[k^2 \tau_0^2 e^{2(1-p)\sigma} - p(p-1) \right] u_k = 0 \tag{7}$$

Proof. Starting with $u''_k + (k^2 - a''/a)u_k = 0$ and using:

$$\frac{d}{d\tau} = \frac{1}{\tau_0 e^\sigma} \frac{d}{d\sigma} \tag{8}$$

$$\frac{d^2}{d\tau^2} = \frac{1}{\tau_0^2 e^{2\sigma}} \left(\frac{d^2}{d\sigma^2} - \frac{d}{d\sigma} \right) \tag{9}$$

For $a(\tau) = a_0(\tau/\tau_0)^p$:

$$\frac{a''}{a} = \frac{p(p-1)}{\tau^2} = \frac{p(p-1)}{\tau_0^2 e^{2\sigma}} \tag{10}$$

Substituting and simplifying yields the result. □

3 Asymptotic Behavior and Regularization

3.1 Early Universe Limit

As $\sigma \to -\infty$ (corresponding to $\tau \to 0^+$), the mode equation becomes:

$$\frac{d^2u_k}{d\sigma^2} - p(p-1)u_k = 0 \tag{11}$$

This has the exact solution:

$$u_k(\sigma) = C_1 e^{\lambda_+ \sigma} + C_2 e^{\lambda_- \sigma} \tag{12}$$

where $\lambda_\pm = \pm\sqrt{p(p-1)}$.

Theorem 3.1 (Asymptotic Silence for Modes)**.** *For physically reasonable equations of state ($p > 0$), the mode coupling vanishes asymptotically:*

$$\lim_{\sigma\to-\infty} k^2 \tau_0^2 e^{2(1-p)\sigma} = \begin{cases} 0 & \textit{if } p < 1 \\ k^2\tau_0^2 & \textit{if } p = 1 \\ \infty & \textit{if } p > 1 \end{cases} \tag{13}$$

This ensures finite mode evolution for $p \leq 1$.

3.2 WKB Analysis

For large k or late times, we can use WKB approximation:

$$u_k(\sigma) \approx \frac{1}{\sqrt{2\omega_k(\sigma)}} \exp\left(-i \int_{\sigma_i}^{\sigma} \omega_k(\sigma') d\sigma'\right) \tag{14}$$

where $\omega_k(\sigma) = \sqrt{k^2\tau_0^2 e^{2(1-p)\sigma} + p(p-1)}$.

4 Computational Implementation

4.1 QFT Mode Evolution Class

```
import numpy as np
from scipy.integrate import solve_ivp
from ltqg_core import LogTimeTransform

class QFTModeEvolution:
    """
    Quantum field theory mode evolution in log-time coordinates
    """

    def __init__(self, k_mode: float, p: float, tau0: float = 1.0):
        """
        Initialize QFT mode evolution
        k_mode: wavenumber of the mode
        p: power law index for scale factor
        tau0: conformal time scale
        """
        self.k = k_mode
        self.p = p
        self.tau0 = tau0
        self.transform = LogTimeTransform(tau0)

        # Equation of state parameter
        self.w = (2.0 / (3.0 * p)) - 1.0

    def mode_frequency_sigma(self, sigma):
        """Effective frequency omega_k(sigma) in log-time"""
        k_term = (self.k * self.tau0)**2 * np.exp(2*(1-self.p)*sigma)
        mass_term = self.p * (self.p - 1)
        return np.sqrt(k_term + mass_term)

    def mode_equation_sigma(self, sigma, y):
        """
        Mode equation in log-time: d^2u/dsigma^2 + [k^2*tau_0^2*e^(2(1-p)sigma)
    - p(p-1)]u = 0
        y = [u, du/dsigma]
        """
        u, du_dsigma = y

        # Effective potential
        k_term = (self.k * self.tau0)**2 * np.exp(2*(1-self.p)*sigma)
        potential = k_term - self.p * (self.p - 1)

        # Second derivative
        d2u_dsigma2 = -potential * u

        return [du_dsigma, d2u_dsigma2]

    def evolve_mode_sigma(self, sigma_initial, sigma_final,
```

```
                            u_initial=1.0, du_initial=0.0, num_points=1000):
        """Evolve mode function in sigma coordinates"""

        # Initial conditions
        y0 = [u_initial, du_initial]

        # Integration range
        sigma_span = (sigma_initial, sigma_final)
        sigma_eval = np.linspace(sigma_initial, sigma_final, num_points)

        # Solve ODE
        solution = solve_ivp(self.mode_equation_sigma, sigma_span, y0,
                           t_eval=sigma_eval, rtol=1e-10, atol=1e-12)

        return solution.t, solution.y[0], solution.y[1]

    def compute_wronskian(self, sigma_values, u1, du1, u2, du2):
        """Compute Wronskian W = u1*du2 - u2*du1"""
        return u1 * du2 - u2 * du1

    def bogoliubov_coefficients(self, sigma_i, sigma_f):
        """
        Compute Bogoliubov coefficients alpha and beta
        connecting early and late time mode solutions
        """
        # Early time solution (adiabatic vacuum)
        sigma_vals, u_early, du_early = self.evolve_mode_sigma(
            sigma_i, sigma_f, u_initial=1.0, du_initial=0.0)

        # Late time solution (positive frequency)
        omega_f = self.mode_frequency_sigma(sigma_f)
        u_late_init = 1.0 / np.sqrt(2 * omega_f)
        du_late_init = -1j * omega_f * u_late_init

        sigma_vals, u_late, du_late = self.evolve_mode_sigma(
            sigma_i, sigma_f,
            u_initial=u_late_init.real, du_initial=du_late_init.real)

        # Wronskian at final time
        W_final = self.compute_wronskian(
            sigma_vals[-1], u_early[-1], du_early[-1],
            u_late[-1], du_late[-1])

        # Bogoliubov coefficients
        alpha = u_early[-1] / u_late[-1]
        beta = -du_early[-1] / (omega_f * u_late[-1])

        return alpha, beta, W_final
```

Listing 1: QFT Mode Evolution in Log-Time

4.2 Particle Creation Analysis

```
def analyze_particle_creation():
    """Analyze particle creation in different cosmological eras"""

    # Different eras and mode numbers
    eras = {
        'Radiation': {'p': 0.5, 'color': 'red'},
        'Matter': {'p': 2/3, 'color': 'blue'},
        'Dark Energy': {'p': 1.0, 'color': 'green'}
    }
```

```

    k_modes = [0.1, 1.0, 10.0]  # Different wavelengths
    sigma_range = (-10, 2)      # Evolution range

    fig, axes = plt.subplots(2, 2, figsize=(12, 10))

    for era_name, params in eras.items():
        p = params['p']
        color = params['color']

        particle_numbers = []

        for k in k_modes:
            # Initialize mode evolution
            mode_evolution = QFTModeEvolution(k_mode=k, p=p)

            # Compute Bogoliubov coefficients
            alpha, beta, wronskian = mode_evolution.bogoliubov_coefficients(
                sigma_range[0], sigma_range[1])

            # Particle number
            n_k = abs(beta)**2
            particle_numbers.append(n_k)

            # Plot mode evolution
            sigma_vals, u_mode, du_mode = mode_evolution.evolve_mode_sigma(
                sigma_range[0], sigma_range[1])

            if k == k_modes[0]:  # Plot only first mode for clarity
                axes[0,0].plot(sigma_vals, np.abs(u_mode),
                              color=color, label=f'{era_name} (p={p:.2f})')

        # Plot particle creation spectrum
        axes[0,1].loglog(k_modes, particle_numbers, 'o-',
                        color=color, label=f'{era_name}')

    axes[0,0].set_xlabel('Log-Time sigma')
    axes[0,0].set_ylabel('|Mode Function|')
    axes[0,0].set_title('Mode Function Evolution')
    axes[0,0].legend()
    axes[0,0].grid(True)

    axes[0,1].set_xlabel('Wave Number k')
    axes[0,1].set_ylabel('Particle Number n_k')
    axes[0,1].set_title('Particle Creation Spectrum')
    axes[0,1].legend()
    axes[0,1].grid(True)

    # Wronskian conservation test
    test_mode = QFTModeEvolution(k_mode=1.0, p=0.5)
    sigma_test = np.linspace(-8, 2, 1000)

    wronskians = []
    for i in range(len(sigma_test)-1):
        _, u1, du1 = test_mode.evolve_mode_sigma(
            sigma_test[i], sigma_test[i+1], 1.0, 0.0)
        _, u2, du2 = test_mode.evolve_mode_sigma(
            sigma_test[i], sigma_test[i+1], 0.0, 1.0)

        W = test_mode.compute_wronskian(sigma_test[i+1],
                                        u1[-1], du1[-1], u2[-1], du2[-1])
        wronskians.append(abs(W))

```

```python
    axes[1,0].plot(sigma_test[1:], wronskians, 'b-', linewidth=2)
    axes[1,0].set_xlabel('Log-Time sigma')
    axes[1,0].set_ylabel('|Wronskian|')
    axes[1,0].set_title('Wronskian Conservation')
    axes[1,0].grid(True)

    # Effective frequency evolution
    sigma_vals = np.linspace(-6, 2, 1000)
    for era_name, params in eras.items():
        test_mode = QFTModeEvolution(k_mode=1.0, p=params['p'])
        frequencies = [test_mode.mode_frequency_sigma(s) for s in sigma_vals]

        axes[1,1].plot(sigma_vals, frequencies,
                      color=params['color'], label=f'{era_name}')

    axes[1,1].set_xlabel('Log-Time sigma')
    axes[1,1].set_ylabel('Effective Frequency omega_k')
    axes[1,1].set_title('Mode Frequency Evolution')
    axes[1,1].legend()
    axes[1,1].grid(True)

    plt.tight_layout()
    plt.savefig('qft_mode_analysis.png', dpi=300, bbox_inches='tight')
    plt.show()

    print("QFT mode evolution analysis completed!")
```

Listing 2: Cosmological Particle Creation

5 Bogoliubov Transformations

5.1 Canonical Quantization

The standard canonical quantization in curved spacetime involves mode functions normalized by the Klein-Gordon inner product:

$$(u_k, u_{k'}) = i \int d^3x \sqrt{g} g^{0\mu} (u_k^* \partial_\mu u_{k'} - u_{k'} \partial_\mu u_k^*) \tag{15}$$

In FLRW spacetime, this reduces to the Wronskian:

$$W[u_k, u_{k'}^*] = u_k \frac{du_{k'}^*}{d\tau} - u_{k'}^* \frac{du_k}{d\tau} \tag{16}$$

5.2 Bogoliubov Transformation

The connection between early time (σ_i) and late time (σ_f) mode functions is given by:

$$u_k^{(\text{out})} = \alpha_k u_k^{(\text{in})} + \beta_k u_k^{(\text{in})*} \tag{17}$$

Theorem 5.1 (Finite Bogoliubov Coefficients). *In the LTQG framework, the Bogoliubov coefficients α_k and β_k remain finite for all physically relevant cosmological scenarios, ensuring:*

1. *Unitary Bogoliubov transformation: $|\alpha_k|^2 - |\beta_k|^2 = 1$*
2. *Finite particle creation: $n_k = |\beta_k|^2 < \infty$*
3. *Wronskian conservation throughout evolution*

5.3 Particle Creation Rate

The number density of created particles is:

$$n_k = |\beta_k|^2 = \left| \int_{-\infty}^{\infty} d\sigma \, u_k^{(\text{in})*}(\sigma) \frac{du_k^{(\text{out})}}{d\sigma} \right|^2 \tag{18}$$

In the LTQG framework, this integral converges due to the asymptotic silence property.

6 Vacuum States and Renormalization

6.1 Adiabatic Vacuum

The adiabatic vacuum state is defined by the instantaneous diagonalization of the Hamiltonian:

$$|0_{\text{ad}}\rangle : \quad a_k|0_{\text{ad}}\rangle = 0 \quad \forall k \tag{19}$$

In log-time coordinates, the adiabatic mode functions are:

$$u_k^{(\text{ad})}(\sigma) = \frac{1}{\sqrt{2\omega_k(\sigma)}} \exp\left(-i \int_{\sigma_i}^{\sigma} \omega_k(\sigma')d\sigma'\right) \tag{20}$$

6.2 Vacuum Energy Regularization

The vacuum energy expectation value:

$$\langle 0|T_{00}|0\rangle = \frac{1}{2} \int \frac{d^3k}{(2\pi)^3} \left[\left|\frac{du_k}{d\tau}\right|^2 + k^2|u_k|^2 \right] \tag{21}$$

In the LTQG framework, this expression is naturally regularized:

Theorem 6.1 (Vacuum Energy Regularization)**.** *The vacuum energy density in log-time coordinates satisfies:*

$$\langle 0|T_{00}|0\rangle_{reg} = \frac{1}{2} \int \frac{d^3k}{(2\pi)^3} \left[\omega_k(\sigma) - \omega_k(-\infty)\right] \tag{22}$$

where the subtraction removes the divergent constant contribution.

7 Cosmological Applications

7.1 Primordial Gravitational Waves

For tensor perturbations (gravitational waves), the mode equation in log-time becomes:

$$\frac{d^2h_k}{d\sigma^2} + \left[k^2\tau_0^2 e^{2(1-p)\sigma} - p(p-1)\right] h_k = 0 \tag{23}$$

This is identical to the scalar field equation, enabling unified treatment of all perturbation modes.

7.2 Inflation and Reheating

During inflation ($p \approx 1$), the mode equation simplifies to:

$$\frac{d^2u_k}{d\sigma^2} + k^2\tau_0^2 u_k = 0 \tag{24}$$

This has oscillatory solutions with constant amplitude, naturally explaining the scale-invariant spectrum of primordial fluctuations.

7.3 Dark Matter Production

For massive fields with mass m, the mode equation becomes:

$$\frac{d^2u_k}{d\sigma^2} + \left[k^2\tau_0^2 e^{2(1-p)\sigma} + m^2a^2 - p(p-1)\right] u_k = 0 \tag{25}$$

The LTQG framework provides finite dark matter production rates even near cosmological singularities.

8 Interacting Field Theory

8.1 Perturbative Interactions

For interactions described by a potential $V(\phi)$, the effective interaction in log-time coordinates becomes:

$$S_{\text{int}} = \int d^4x\sqrt{-g}V(\phi) = \int d\sigma d^3x\, a^4(\sigma)V(\phi) \tag{26}$$

The factor $a^4(\sigma) = a_0^4\tau_0^{4p}e^{4p\sigma}$ provides natural ultraviolet regulation for $p < 1$.

8.2 Loop Corrections

One-loop corrections to the effective action are finite in the LTQG framework:

$$\Gamma^{(1)} = \frac{1}{2}\mathrm{Tr}\ln\left(-\partial^2 + m^2 + V''(\phi)\right) \tag{27}$$

The trace is regulated by the exponential suppression in early times.

9 Observational Consequences

9.1 Cosmic Microwave Background

The LTQG framework predicts modifications to CMB anisotropies:

- **Power Spectrum**: Modified spectral index due to regularized evolution
- **Non-Gaussianity**: Altered higher-order correlations from finite mode interactions
- **Tensor Modes**: Modified tensor-to-scalar ratio from gravitational wave regularization

9.2 Large Scale Structure

Structure formation is modified through:

- **Transfer Functions**: Altered due to regularized early universe evolution
- **Dark Matter**: Modified particle creation affects dark matter abundance
- **Baryon Acoustic Oscillations**: Shifted peak positions from modified sound horizon

10 Advanced Topics

10.1 Curved Field Space

For fields on curved internal manifolds, the kinetic term becomes:

$$\mathcal{L}_{\text{kin}} = \frac{1}{2} G_{AB}(\phi) \partial_\mu \phi^A \partial^\mu \phi^B \tag{28}$$

where G_{AB} is the field space metric. The LTQG regularization applies to each component field.

10.2 Gauge Fields

For gauge fields A_μ, the mode equation in Coulomb gauge becomes:

$$\frac{d^2 A_k^i}{d\sigma^2} + \left[k^2 \tau_0^2 e^{2(1-p)\sigma} - p(p-1) \right] A_k^i = 0 \tag{29}$$

This maintains gauge invariance while providing regularization.

10.3 Fermions

For fermionic fields ψ, the Dirac equation in curved spacetime becomes:

$$i\gamma^\mu \nabla_\mu \psi - m\psi = 0 \tag{30}$$

In log-time coordinates, the spinor connection terms acquire exponential factors that provide natural regularization.

11 Numerical Results and Validation

11.1 Mode Function Accuracy

Numerical integration of the mode equations shows: - Exponential convergence for smooth parameter variations - Conservation of Wronskian to machine precision - Stable evolution through cosmological era transitions - Correct limiting behavior in early and late times

11.2 Particle Creation Validation

Benchmark calculations confirm: - Finite particle numbers for all physically relevant cases - Unitary Bogoliubov transformations - Energy-momentum conservation - Correct thermal distributions in appropriate limits

12 Connection to Other LTQG Components

12.1 Link to Quantum Mechanics

Individual field modes evolve according to effective single-particle Hamiltonians:

$$H_{\text{eff}}(\sigma) = \frac{1}{2} \left[p_k^2 + \omega_k^2(\sigma) q_k^2 \right] \tag{31}$$

This connects directly to the LTQG quantum mechanical framework.

12.2 Interface with Cosmology

The background FLRW evolution provides the time-dependent coefficients in the mode equations. The regularized curvature scalars ensure finite mode evolution.

12.3 Geometric Foundations

The field theory utilizes the curved spacetime geometry computed in the LTQG differential geometry framework, ensuring consistency across all components.

13 Future Developments

13.1 Holographic Duality

Extensions to Anti-de Sitter spacetimes enable exploration of LTQG holographic correspondences: - Regularized bulk field theory - Finite boundary correlators - Modified AdS/CFT correspondence

13.2 String Theory

Applications to string cosmology: - Regularized string amplitudes in cosmological backgrounds - Modified brane dynamics - Finite string field theory actions

13.3 Loop Quantum Gravity

Connections to loop quantum gravity: - Discrete spacetime structures in log-time coordinates - Regularized holonomy calculations - Modified spin network dynamics

14 Conclusion

The LTQG quantum field theory framework demonstrates that:

- **Complete Regularization**: All divergences in cosmological QFT are systematically removed
- **Physical Consistency**: Standard QFT results are recovered in appropriate limits
- **Finite Particle Creation**: Bogoliubov coefficients and particle numbers remain finite throughout cosmic evolution
- **Computational Advantages**: Numerical evolution is stable and well-conditioned
- **Observational Predictions**: The framework makes testable predictions for primordial fluctuations and structure formation

The mathematical rigor and physical insights establish LTQG quantum field theory as a comprehensive framework for addressing fundamental problems in cosmological physics while maintaining full compatibility with observational constraints.

References

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