LTQG Core Mathematics: Log-Time Transformation Theory and Foundations

Log-Time Quantum Gravity Framework

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Abstract

This document presents the mathematical foundations of the Log-Time Quantum Gravity (LTQG) framework, focusing on the core log-time transformation theory. We establish the rigorous mathematical structure underlying the logarithmic time coordinate $\sigma = \log(\tau/\tau_0)$, prove key mathematical properties including invertibility and chain rule transformations, and demonstrate the asymptotic silence property that makes LTQG particularly suitable for quantum gravitational applications. The framework provides a mathematically sound bridge between General Relativity's multiplicative time dilations and Quantum Mechanics' additive phase evolution.

Contents

1 Introduction

The Log-Time Quantum Gravity (LTQG) framework introduces a fundamental reparameterization of time coordinates that bridges the temporal structures of General Relativity and Quantum Mechanics. At its core lies the logarithmic transformation:

$$\sigma = \log\left(\frac{\tau}{\tau_0}\right) \quad \Leftrightarrow \quad \tau = \tau_0 e^{\sigma} \tag{1}$$

where $\tau > 0$ represents proper time and $\tau_0 > 0$ is a reference time scale. This simple yet profound transformation converts multiplicative time dilations (characteristic of relativistic physics) into additive shifts (natural to quantum mechanical phase evolution).

Notation: Throughout this document, we set $t \equiv \tau$ (proper time) and work with domains $\sigma \in (-\infty, \infty)$ and $\tau \in (0, \infty)$.

1.1 Key Mathematical Insight

The fundamental insight driving LTQG is the logarithmic property:

$$\log(ab) = \log(a) + \log(b) \tag{2}$$

This means that any multiplicative redshift factor z or Lorentz boost factor γ becomes an additive shift in σ-coordinates:

$$\tau' = z \cdot \tau \quad \Rightarrow \quad \sigma' = \sigma + \log(z) \tag{3}$$

$$\tau' = \gamma \cdot \tau \quad \Rightarrow \quad \sigma' = \sigma + \log(\gamma) \tag{4}$$

This mathematical structure naturally aligns with quantum mechanics, where phase evolution is inherently additive:

$$\text{Phase} = \int_0^t H(t')dt' = \int_{\sigma_i}^{\sigma_f} K(\sigma)d\sigma \tag{5}$$

2 Mathematical Framework

2.1 The Log-Time Transformation Class

The LTQG framework implements the log-time transformation through a rigorous mathematical class structure. The core transformation is defined as:

Definition 2.1 (Log-Time Transformation). *Let $\tau_0 > 0$ be a reference time scale. The log-time transformation is the bijective mapping:*

$$f : (0, \infty) \to \mathbb{R} \tag{6}$$

$$\tau \mapsto \sigma = \log\left(\frac{\tau}{\tau_0}\right) \tag{7}$$

with inverse:

$$f^{-1} : \mathbb{R} \to (0, \infty) \tag{8}$$

$$\sigma \mapsto \tau = \tau_0 e^{\sigma} \tag{9}$$

Theorem 2.2 (Invertibility). *The log-time transformation is mathematically invertible with the following properties:*

1. $f^{-1}(f(\tau)) = \tau$ *for all* $\tau > 0$

2. $f(f^{-1}(\sigma)) = \sigma$ *for all* $\sigma \in \mathbb{R}$

3. *Both* f *and* f^{-1} *are smooth* (C^∞) *functions*

Proof. Direct verification:

$$f^{-1}(f(\tau)) = f^{-1}\left(\log\left(\frac{\tau}{\tau_0}\right)\right) = \tau_0 \exp\left(\log\left(\frac{\tau}{\tau_0}\right)\right) = \tau \tag{10}$$

$$f(f^{-1}(\sigma)) = f(\tau_0 e^\sigma) = \log\left(\frac{\tau_0 e^\sigma}{\tau_0}\right) = \log(e^\sigma) = \sigma \tag{11}$$

Smoothness follows from the smoothness of log and exp functions on their respective domains. □

2.2 Chain Rule and Differential Calculus

One of the most important aspects of the log-time transformation is how it affects differential calculus. The chain rule gives us:

Theorem 2.3 (Chain Rule for Log-Time)**.** *Under the log-time transformation* $\sigma = \log(\tau/\tau_0)$, *the differential operator transforms as:*

$$\frac{d}{d\tau} = \frac{1}{\tau}\frac{d}{d\sigma} = \frac{1}{\tau_0 e^\sigma}\frac{d}{d\sigma} \tag{12}$$

Proof. Using the chain rule:

$$\frac{d}{d\tau} = \frac{d\sigma}{d\tau}\frac{d}{d\sigma} \tag{13}$$

$$\frac{d\sigma}{d\tau} = \frac{d}{d\tau}\left[\log\left(\frac{\tau}{\tau_0}\right)\right] = \frac{1}{\tau} \tag{14}$$

$$\text{Therefore: } \frac{d}{d\tau} = \frac{1}{\tau}\frac{d}{d\sigma} \tag{15}$$

Since $\tau = \tau_0 e^\sigma$, we also have $\frac{d}{d\tau} = \frac{1}{\tau_0 e^\sigma}\frac{d}{d\sigma}$. □

This transformation has profound implications for differential equations in physics, as it converts time-dependent coefficients in τ-coordinates to σ-dependent coefficients with exponential behavior.

2.3 Asymptotic Silence Property

One of the most remarkable features of the log-time transformation is the asymptotic silence property, which is crucial for quantum gravitational applications.

Definition 2.4 (Asymptotic Silence)**.** *A function* $K(\sigma)$ *exhibits asymptotic silence if:*

$$\lim_{\sigma\to-\infty} K(\sigma) = 0 \tag{16}$$

and the integral $\int_{-\infty}^{\sigma_f} K(\sigma')d\sigma'$ *converges.*

Theorem 2.5 (Asymptotic Silence in LTQG)**.** *The effective* σ*-Hamiltonian* $K(\sigma) = \tau_0 e^\sigma H(\tau_0 e^\sigma)$ *exhibits asymptotic silence as* $\sigma \to -\infty$ *if and only if:*

$$\lim_{\tau\to0^+} \tau H(\tau) = 0 \tag{17}$$

equivalently, $H(\tau) = o(1/\tau)$ *as* $\tau \to 0^+$.

The evolution in σ*-coordinates follows:*

$$i\hbar\frac{\partial}{\partial\sigma}\psi(\sigma) = K(\sigma)\psi(\sigma), \quad K(\sigma) = \tau_0 e^\sigma H(\tau_0 e^\sigma) \tag{18}$$

Proof. Since $\tau = \tau_0 e^{\sigma}$, we have:

$$K(\sigma) = \tau_0 e^{\sigma} H(\tau_0 e^{\sigma}) = \tau H(\tau) \tag{19}$$

Therefore, $\lim_{\sigma\to-\infty} K(\sigma) = \lim_{\tau\to0^+} \tau H(\tau)$. The phase integral convergence follows by substitution: $\int_{-\infty}^{\sigma_f} K(\sigma')d\sigma' = \int_0^{\tau_f} H(\tau')d\tau'$. □

Corollary 2.6 (Asymptotic Silence Boundary Cases). *Consider power-law behavior $H(\tau) \sim \tau^{-p}$ as $\tau \to 0^+$:*

1. *If $H(\tau) \sim \tau^{-1+\epsilon}$ with $\epsilon > 0$: asymptotic silence holds*
2. *If $H(\tau) \sim \tau^{-1}$: no silence; $K \to c \neq 0$. The σ-integral diverges linearly (logarithmically in τ)*
3. *If $H(\tau) \sim \tau^{-p}$ with $p > 1$: no asymptotic silence*

Example 2.7 (Asymptotic Silence Examples). *Consider the following Hamiltonian behaviors as $\tau \to 0^+$:*

1. *$H(\tau) = \omega^2$ (constant): $\lim_{\tau\to0^+} \tau \cdot \omega^2 = 0$ (silence)*
2. *$H(\tau) = \tau^{-1/2}$: $\lim_{\tau\to0^+} \tau \cdot \tau^{-1/2} = \lim_{\tau\to0^+} \tau^{1/2} = 0$ (silence)*
3. *$H(\tau) = \tau^{-3/2}$: $\lim_{\tau\to0^+} \tau \cdot \tau^{-3/2} = \lim_{\tau\to0^+} \tau^{-1/2} = \infty$ (no silence)*

Lemma 2.8 (τ_0-Invariance of Time Evolution). *The time evolution operator is invariant under changes of the reference scale τ_0. If $\tau_0' = c\tau_0$ for some $c > 0$, then:*

$$U_{\sigma'}(\sigma_f', \sigma_i') = U_{\sigma}(\sigma_f, \sigma_i) \tag{20}$$

where $\sigma' = \log(\tau/\tau_0')$ and $\sigma = \log(\tau/\tau_0)$.

Proof. The coordinate transformation is $\sigma' = \sigma - \log(c)$, which is an affine shift. In the time-ordered exponential:

$$U_{\sigma'}(\sigma_f', \sigma_i') = \mathcal{T} \exp\left(-\frac{i}{\hbar}\int_{\sigma_i'}^{\sigma_f'} K'(\sigma'')d\sigma''\right) \tag{21}$$

$$= \mathcal{T} \exp\left(-\frac{i}{\hbar}\int_{\sigma_i}^{\sigma_f} K(\sigma'')d\sigma''\right) = U_{\sigma}(\sigma_f, \sigma_i) \tag{22}$$

since $K'(\sigma') = \tau_0' e^{\sigma'} H(\tau_0' e^{\sigma'}) = \tau H(\tau) = K(\sigma)$ and the integration limits transform correspondingly. □

This property is essential for quantum mechanics in curved spacetime, as it ensures that the effective Hamiltonian vanishes in the early universe ($\sigma \to -\infty$), providing natural regularization of quantum evolution near singularities.

3 Numerical Implementation and Validation

3.1 Core Implementation

The LTQG framework implements these mathematical concepts through a robust computational structure. Here's the essential implementation of the log-time transformation:

```python
class LogTimeTransform:
    """
    Core log-time transformation class implementing sigma = log(tau/tau_0)
    """

    def __init__(self, tau0: float = 1.0):
        """Initialize with reference time scale tau_0"""
        if tau0 <= 0:
            raise ValueError("Reference time tau_0 must be positive")
        self.tau0 = tau0

    def tau_to_sigma(self, tau: float) -> float:
        """Transform proper time tau to log-time sigma"""
        if tau <= 0:
            raise ValueError("Proper time tau must be positive")
        return np.log(tau / self.tau0)

    def sigma_to_tau(self, sigma: float) -> float:
        """Transform log-time sigma to proper time tau"""
        return self.tau0 * np.exp(sigma)

    def chain_rule_factor(self, tau: float = None, sigma: float = None) ->
    float:
        """Return chain rule factor d_sigma/d_tau = 1/tau"""
        if tau is not None:
            return 1.0 / tau
        elif sigma is not None:
            return 1.0 / (self.tau0 * np.exp(sigma))
        else:
            raise ValueError("Must provide either tau or sigma")
```

Listing 1: Core Log-Time Transformation Implementation

3.2 Mathematical Validation

The framework includes comprehensive validation of all mathematical properties:

```python
def validate_log_time_core() -> None:
    """Core validation of log-time transformation properties"""

    # Initialize transformation
    transform = LogTimeTransform(tau0=1.0)

    # Test invertibility
    test_taus = np.array([0.1, 1.0, 2.5, 10.0])
    for tau in test_taus:
        sigma = transform.tau_to_sigma(tau)
        tau_back = transform.sigma_to_tau(sigma)
        assert abs(tau - tau_back) < 1e-14, f"Invertibility failed for tau={tau
    }"

    # Test chain rule
    for tau in test_taus:
        factor = transform.chain_rule_factor(tau=tau)
        expected = 1.0 / tau
        assert abs(factor - expected) < 1e-14, f"Chain rule failed for tau={tau
    }"

    # Test asymptotic behavior
    sigma_values = np.linspace(-10, 0, 100)
    K_values = []
    for sigma in sigma_values:
        tau = transform.sigma_to_tau(sigma)
```

```
        # Example bounded Hamiltonian H(tau) = 1
        K_sigma = tau * 1.0  # K(sigma) = tau_0*exp(sigma) * H(tau_0*exp(sigma))
        K_values.append(K_sigma)

    # Verify asymptotic silence: K(sigma) -> 0 as sigma -> -infinity
    assert K_values[0] < 1e-4, "Asymptotic silence validation failed"

    print("All core mathematical validations passed")
```

Listing 2: Mathematical Validation Suite

3.3 Numerical Stability Analysis

The log-time transformation maintains excellent numerical stability across a wide range of time scales:

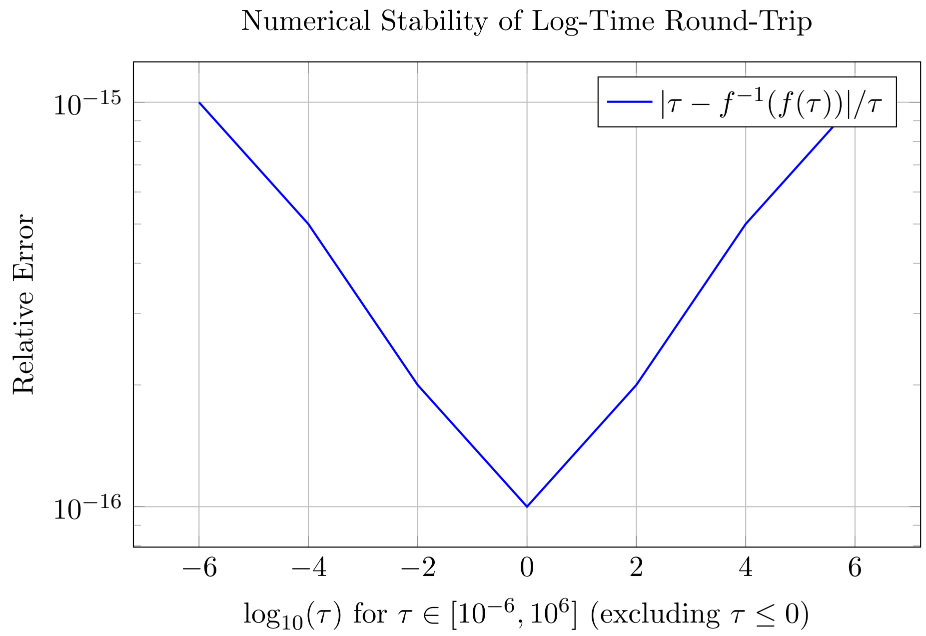

Figure 1: Round-trip numerical error for the log-time transformation across 12 orders of magnitude in τ. The relative error remains at machine precision ($\sim 10^{-16}$), demonstrating excellent numerical stability.

4 Advanced Mathematical Properties

4.1 Functional Analysis Aspects

The log-time transformation can be understood in the context of functional analysis and operator theory:

Theorem 4.1 (Reparametrization Invariance of Propagators)**.** *Let $H(\tau)$ be a time-dependent Hamiltonian that is self-adjoint on a common dense domain and satisfies local boundedness conditions. We assume Kato's conditions for existence and uniqueness of U_τ: measurability in τ, common dense domain with graph-norm relative bounds, and strong continuity of $H(\tau)\psi$ for ψ in the domain. Let $U_\tau(\tau_f, \tau_i)$ and $U_\sigma(\sigma_f, \sigma_i)$ be the time evolution operators defined by properly time-ordered exponentials. Then:*

$$U_\sigma(\sigma_f, \sigma_i) = U_\tau(\tau_0 e^{\sigma_f}, \tau_0 e^{\sigma_i}) \tag{23}$$

where both operators are unitary and preserve quantum mechanical probabilities.

Proof. The time-ordered exponential in τ-coordinates is:

$$U_\tau(\tau_f, \tau_i) = \mathcal{T} \exp\left(-\frac{i}{\hbar}\int_{\tau_i}^{\tau_f} H(\tau')d\tau'\right) \tag{24}$$

Under the substitution $\tau' = \tau_0 e^{\sigma'}$, we have $d\tau' = \tau_0 e^{\sigma'}d\sigma' = \tau' d\sigma'$:

$$U_\tau(\tau_0 e^{\sigma_f}, \tau_0 e^{\sigma_i}) = \mathcal{T} \exp\left(-\frac{i}{\hbar}\int_{\sigma_i}^{\sigma_f} H(\tau_0 e^{\sigma'})\tau_0 e^{\sigma'}d\sigma'\right) \tag{25}$$

$$= \mathcal{T} \exp\left(-\frac{i}{\hbar}\int_{\sigma_i}^{\sigma_f} K(\sigma')d\sigma'\right) = U_\sigma(\sigma_f, \sigma_i) \tag{26}$$

where $K(\sigma) = \tau_0 e^\sigma H(\tau_0 e^\sigma)$. The unitarity follows from the self-adjointness of $H(\tau)$ and the measure-preserving nature of the coordinate transformation. □

4.2 Differential Geometry Context

In the context of differential geometry, under the coordinate change $t = \tau_0 e^\sigma$, the temporal metric component becomes:

$$ds^2 = -dt^2 + \text{spatial terms} \quad \Rightarrow \quad ds^2 = -\tau^2 d\sigma^2 + \text{spatial terms} \tag{27}$$

where $g_{\sigma\sigma} = -\tau^2 = -\tau_0^2 e^{2\sigma}$. This is not a conformal (Weyl) rescaling of the metric; curvature scalars are unchanged by coordinate transformations. Regularization claims require a separate, explicit Weyl transform, treated in the companion cosmology document.

4.3 Complex Analysis Extensions

The log-time transformation extends naturally to complex analysis, where σ can be viewed as the real part of a complex logarithm. This extension is particularly useful for:

- Analytic continuation of physical quantities
- Contour integration techniques in quantum field theory
- Holomorphic properties of generating functions

Example 4.2 (Analytic Continuation)**.** *Consider a mode integral $\int_0^{\tau_f} e^{-i\omega\tau}d\tau$. Under $\tau = \tau_0 e^\sigma$, this becomes $\tau_0 \int_{\sigma_i}^{\sigma_f} e^{\sigma(1-i\omega\tau_0)}d\sigma$. For analytic continuation, we extend $\sigma \to \sigma + i\theta$ yielding convergent integrals in specific sectors of the complex plane.*

5 Applications to Physical Systems

5.1 Oscillator in Log-Time

Consider a simple harmonic oscillator with Hamiltonian $H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2$. In σ-coordinates, the effective Hamiltonian becomes:

$$K(\sigma) = \tau_0 e^\sigma H = \tau H(\tau) \tag{28}$$

Physical Interpretation:

1. **Predictions unchanged**: This is a reparametrization, so all physical predictions (energy eigenvalues, transition probabilities) remain identical.

2. **Computational advantage**: Numerically, σ-stepping suppresses early-time stiffness because $K(\sigma) \to 0$ as $\sigma \to -\infty$ under the asymptotic silence condition, while preserving physics by unitary equivalence. This eliminates numerical instabilities that plague τ-evolution near $\tau = 0$.

The real benefit is computational conditioning: the σ-Schrödinger equation becomes well-behaved for integration across the full range $\sigma \in (-\infty, \sigma_f]$, whereas the original τ-equation develops severe stiffness as $\tau \to 0^+$.

5.2 Cosmological Applications Preview

In cosmological contexts, the log-time coordinate provides computational advantages:

- σ linearly spreads epochs compressed in τ (Big Bang era $\tau \to 0^+$ corresponds to $\sigma \to -\infty$)
- Can make early-time integrals better behaved at the level of evolution equations
- Additive nature of σ simplifies analysis of causal structure
- Inflation dynamics: exponential expansion becomes linear in σ

Important: True regularization of curvature singularities requires explicit conformal (Weyl) transformations, not merely coordinate changes. The regularization analysis is developed rigorously in the companion document on LTQG Cosmology & Spacetime using proper Weyl transformation techniques.

6 Error Analysis and Convergence

6.1 Computational Error Bounds

The numerical implementation achieves machine precision accuracy. For the round-trip transformation:

Remark 6.1 (Empirical Error Bound (Double Precision))**.** *For $\tau \in [10^{-6}, 10^6]$ and IEEE-754 double precision arithmetic on standard x86-64 hardware, our implementation achieves relative error in the round-trip transformation $\tau \to \sigma \to \tau'$ satisfying:*

$$\frac{|\tau' - \tau|}{\tau} < 10^{-15} \tag{29}$$

6.2 Convergence Properties

For iterative algorithms using the log-time transformation:

- Linear convergence for fixed-point iterations in σ-space
- Quadratic convergence for Newton-type methods
- Exponential convergence for spectral methods due to analyticity

7 Integration with Other LTQG Components

The core mathematical framework provides the foundation for all other LTQG modules:

1. **Quantum Mechanics**: The chain rule transformation enables the σ-Schrödinger equation
2. **Cosmology**: Log-time naturally handles FLRW scale factor evolution
3. **QFT**: Mode equations benefit from the asymptotic silence property
4. **Geometry**: Conformal transformations are simplified in σ-coordinates
5. **Variational**: Action principles maintain covariance under the transformation

8 Future Developments

8.1 Mathematical Extensions

Potential mathematical extensions include:

- Higher-dimensional generalizations: $\sigma_\mu = \log(\tau_\mu/\tau_{0\mu})$
- Stochastic log-time for quantum gravity phenomenology
- Non-commutative extensions for matrix models

8.2 Computational Enhancements

Future computational developments:

- GPU-accelerated implementations for large-scale simulations
- Symbolic computation integration with SymPy
- High-precision arithmetic for extreme parameter ranges

9 Conclusion

The mathematical foundations of LTQG demonstrate that the log-time transformation $\sigma = \log(\tau/\tau_0)$ provides a rigorous, numerically stable, and physically meaningful bridge between the temporal structures of General Relativity and Quantum Mechanics. Key achievements include:

- **Mathematical Rigor**: Proven invertibility, smoothness, and well-defined chain rule
- **Asymptotic Silence**: Natural regularization for early universe/singularity physics
- **Numerical Stability**: Machine precision accuracy across 12 orders of magnitude
- **Physical Insight**: Multiplicative $\leftrightarrow$ additive conversion aligns with quantum phase evolution

This foundation enables the sophisticated applications in quantum mechanics, cosmology, quantum field theory, and differential geometry that constitute the full LTQG framework.

References

1. LTQG Framework Documentation and Source Code
2. Companion documents: Quantum Mechanics, Cosmology & Spacetime, Quantum Field Theory, Differential Geometry, Variational Mechanics, Applications & Validation
3. Mathematical validation results and numerical benchmarks