CSC3001 Assignment 1

1

Suppose not, then $S := \{2^{4n-1} : n \in \mathbb{Z}^+, 2^{4n-1} \not\equiv 8 \pmod{10}\}$ is a non-empty subset of $\mathbb{N}$. By **Well Ordering Principle** S has a minimum m, with $m = 2^{4k-1}$ for some integer $k \geq 2$. But then $m/16 = 2^{4(k-1)-1} \not\equiv 8 \pmod{10}$. For, if $m/16 \equiv 8 \pmod{10}$, $m \equiv 8 \cdot 16 = 128 \equiv 8 \pmod{10}$. It follows that $m/16 \in S$, contradicting the minimality of m. Therefore $2^{4n-1} \equiv 8 \pmod{10}$ for all $n \in \mathbb{Z}^+$. $\square$

2

For fixed $x \geq 2$, we proceed by induction on y. Clearly $x^2 \geq x + 2$. Assume $x^k \geq x + k$ for some $k \geq 2$. Then

$$x^{k+1} \geq (x+k)x = x^2 + kx \geq x + kx + 2 \geq x + k.$$

So

$$\forall y \geq 2 (x^y \geq x + y).$$

By Universal Generalization, $\forall x \geq 2 (\forall y \geq 2 (x^y \geq x + y))$. $\square$

3

We first prove $\cup_{r \in \mathbb{Z}} [r] = \mathbb{Z}[x]$.

If $y \in [r]$ for some integer r, then $y = (x-1)p(x) + r$ for some $p(x) \in \mathbb{Z}[x]$. Therefore

$$\begin{aligned} y &= r + (x-1)\sum_{i=0}^{n} a_i x^i \\ &= r + \sum_{i=0}^{n} a_i x^{i+1} - a_i x^i \\ &= (r - a_0)x^0 + \sum_{i=1}^{n} (a_{i-1} - a_i)x^i + a_n x^{n+1} \\ &= \sum_{i=0}^{n+1} b_i x^i \in \mathbb{Z}[x]. \end{aligned} \tag{1}$$

Conversely, if $y = \sum_{i=0}^{n} b_i x_i$, pick integers r and a_0 s.t. $b_0 = r - a_0$. Pick $a_1 = a_0 - b_1, a_2 = a_1 - b_2 = a_0 - (b_1 + b_2), \ldots, a_{n-1} = a_0 - \sum_{i=1}^{n-1} b_i$. Reverse the algebra in (1), we have $y = r + (x-1)\sum_{i=0}^{n-1} a_i x_i \in [r]$. So $\cup_{r \in \mathbb{Z}} [r] = \mathbb{Z}[x]$.

Next we prove $[s] \cap [r] = \emptyset$ whenever $s \neq r$.

Let $s \neq r$. Suppose there exists $y \in [s] \cap [r]$ with

$$y = (x-1)p(x) + s = (x-1)q(x) + r \tag{2}$$

for some $p(x), q(x) \in \mathbb{Z}[x]$. Let $n := \max\{\deg(p), \deg(q)\}$, where $\deg(\cdot)$ denotes the degree of a polynomial. From (2),

$$\begin{aligned} r - s &= \sum_{i=0}^{n} (p_i - q_i)(x-1)x^i \\ &= (p_0 - q_0)(-1) + \sum_{i=1}^{n} [(p_{i-1} - q_{i-1}) - (p_i - q_i)]x_i + (p_n - q_n)x^{n+1}. \end{aligned}$$

Therefore $p_n - q_n = p_{n-1} - q_{n-1} = \ldots = p_0 - q_0 = 0$. But then $r - s = 0$, a contradiction. Therefore $[s] \cap [r] = \emptyset$ given $s \neq r$. This completes the proof. $\square$

4

(i)

$$\forall a[\neg(a \in \mathbb{Z}^+) \vee (\forall k[\neg(k \in \mathbb{Z}^+) \vee n \neq (k+2)(a + \frac{k+1}{2})])].$$

(ii)

We prove two contrapositives.

Suppose $S(n)$ is false, that is, $n = (k+2)(a + \frac{k+1}{2})$ for some positive integers a, k. Then $(k+2)|n$, n is not prime.

Suppose n is not prime. By Prime Factorization $n = pq$ for some positive integers p, q, where p is a prime greater than 2 (since n is odd) and $q > 1$. (since n is not prime itself) Pick $k := p - 2 \geq 1$. Since p is odd, $k + 1 = p - 1$ is even. We may then pick integer $a := q - \frac{k+1}{2} \geq q - 1 \geq 1$. Now $(k+2)(a + \frac{k+1}{2}) = pq = n$ with both $a, k \in \mathbb{Z}^+$. $S(n)$ is false. $\square$

5

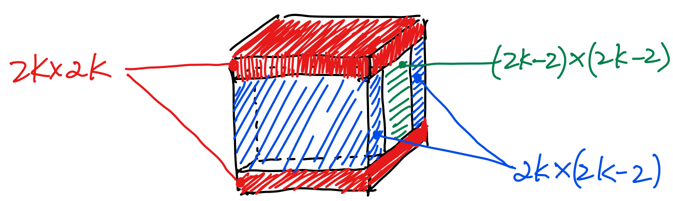
Claim: Assume $n \geq 2$. An n-hull can be tiled by dicubes if and only if n is even.

We first prove that an $2k$-hull can always be tiled.

Lemma 1. A $(2a \times b)$ layer (cube with height 1) can be tiled by dicubes.

Proof. Denote a dicube as (2×1) layer, placing a dicubes paraller to each other yields $(2a \times 1)$ layer. Stacking b $(2m \times 1)$ layers gives $(2m \times p)$.

To prove we can tile a $2k$-hull, note that we can break the hull as two $(2k \times 2k)$ layers, two $(2k \times 2(k-1))$ layers, two $(2(k-1) \times 2(k-1))$ layers, as shown below.

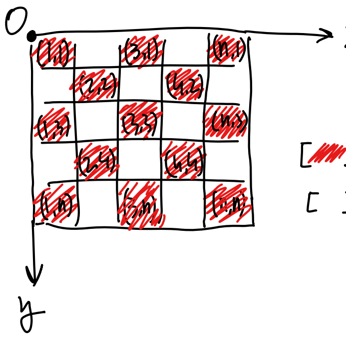


By Lemma 1, each of the six layers can be tiled by dicubes, It follows that the $2k$-hull can also be tiled.

- <u>Next we prove any n-hull with n odd cannot be tiled.</u>

To show this, we assign either number -1 or 1 to each unit cube in an n^3 cube, in such a way ~~that~~ no neiboring unit cubes share the same number. This is possible, as we may number first layer via

$$\#(x,y) = \begin{cases} 1 & \text{if } (x+y) \text{ even}, \\ -1 & \text{if } (x+y) \text{ odd}, \end{cases}$$



[red]: 1 assigned
[ ]: -1 assigned

and extend # to z-dimension. Let $\#(x,y,1) := \#(x,y)$, and

$$\text{for } z \geq 2, \quad \#(x,y,z) := -\#(x,y,z-1).$$

In other words, we number a unit cube 1 if one below it is -1, and -1 if one below is 1.

- It can be checked that $\#(x,y,z) = -\#(x-1,y,z)$, $x \geq 2$;
$= -\#(x,y-1,z)$, $y \geq 2$.

So indeed, # numbers the n^3 cube in a way s.t. neiboring unit cubes have different numbers assigned. Now if we remove the inner $(n-2)^3$ cube to obtain an n-hull, this property remains for all unit cubes still in the hull.

- We're now ready to prove the claim. Assume we're able to tile the hull with dicubes. We add up # everytime a dicube touches 2 unit cubes. They are neiboring. So the sum is $1+(-1) = 0$.

It follows that this tiling will eventually reach a sum of $(0+\cdots+0) = \underline{0}$.

But if we add up # for all unit cubes in the hull, we have

$$\Sigma_1 + \Sigma_2 + \cdots + \Sigma_{n-1} + \Sigma_n = 2\Sigma_1 + \Sigma_2$$

$$\left(\Sigma_k := \sum_{(x,y,k)\in \text{hull}} \#(x,y,k) \right)$$

$$= \left\{ \#(1,1,1) + \sum_{i=2}^{n-1} [\cancel{\#(i,1,1) + \#(i+1,1,1)}]^{0} \right\} \cdot 2$$

$$+ \Big[\underbrace{\sum_{i=1}^{n-1} \#(i,1,2)}_{(n-1)\text{ terms, even, sum is zero}} + \underbrace{\sum_{i=1}^{n-1} \#(n,i,2)}_{(n-1)\text{ terms, even, sum is zero}} + \underbrace{\sum_{i=1}^{n-1} \#(i,n,2)}_{(n-1)\text{ terms, even, sum is zero}} + \underbrace{\sum_{i=2}^{n} (1,i,2)}_{(n-3)\text{ terms, even, sum is zero}} \Big]$$

$$= 2\cdot\#(1,1,1) = \underline{2} \neq 0. \; ↯$$

6

Suppose there are n days. The total money spent is

$$(x+y+z)n = 15+17+19 = 51.$$

Note that 51 has 3 and 17 as its unique prime factors. Since $x<y<z$, we conclude that $x+y+z=17$ and $n=3$. Since Cony spent 17 in total, he must have bought distinctive snacks on each day (cannot all be the same item, as $3 \nmid 17$; cannot be that exactly two are the same, as $2a+b \neq x+y+z=17, \forall a,b \in \{x,y,z\}, a \neq b$). WLOG assume Brown spent y on day 1. Then Cony spent y either on day 2 or day 3. WLOG assume it was day 2. Then Brown couldn't have spent y on day 2. He couldn't have spent z either, as otherwise he would've spent at least $y+z+x=17$. So Brown spent x on day 2. Then Sally must've spent z on day 2. So far three possibilities remain:

	BROWN	**CONY**	**SALLY**	**BROWN**	**CONY**	**SALLY**
D1	y	x	z	y	z	x
D2	x	y	z	x	y	z
D3	x (y)	z	y (x)	y	x	z
Total	15	17	19	15	17	19

which correspond to two Diophantine systems:

$$\begin{cases} 2x+y=15 \\ x+y+z=17 \\ y+2z=19 \end{cases} \qquad \begin{cases} x+2y=15 \\ x+y+z=17 \\ x+2z=19 \end{cases}$$

each yielding

$$\begin{cases} x=t \\ y=15-2t \\ z=t+2 \end{cases} \qquad \begin{cases} x=15-2t \\ y=t \\ z=t+2 \end{cases}, t \in \mathbb{Z}.$$

Considering $0<x<y<z$, the first system yields no solution, as $15-2t \neq t+1$ for all integer t. From the second system, we have $0<15-2t<t<t+2$, equivalently $t=6$ or 7. All solutions are thus given by ordered pairs $(x,y,z)=(3,6,8),(1,7,9)$.