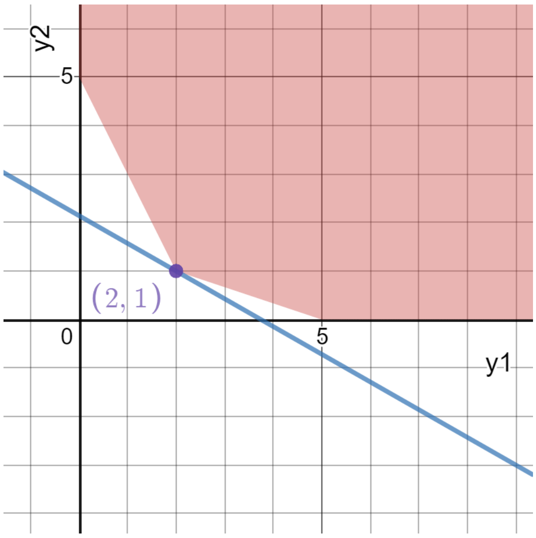
MAT3007 Assignment 3

A3.1

(a)

$$\begin{array}{ll} \text{min} & 4y_1 + 7y_2 \\ \text{subject to} & 2y_1 + y_2 \geq 5 \\ & 3y_1 + 2y_2 \geq 2 \\ & y_1 + 3y_2 \geq 5 \\ & y_1, y_2 \geq 0. \end{array}$$

(b)



The unique optimal solution is $y^* = [y_1; y_2] = [2; 1]$, with optimal value 15.

(c)

Let x^* denote the optimal solution for the primal. Using the Complementary Slackness Theorem and the fact that $y^* > 0$, the constraints in the dual must both be tight. Also, computing the slacks for the dual,

$$v = [5 - 5; 8 - 2; 5 - 5] = [0; 6; 0],$$

by the same theorem we know that $(x^*)_2$ must be 0.

These information transform the primal into a linear system of equation:

$$\begin{aligned} 2x_1 + x_3 &= 4 \\ x_1 + 3x_3 &= 7 \end{aligned}$$

which yields $x_1 = 1, x_3 = 2$.

Hence the optimal solution for the primal is $x^* = [1; 0; 2]$ with optimal value 15, coinciding with that of the dual.

A3.2

Rewriting the LP,

$$\min_x \; c^\top x \quad \text{s.t.} \quad \begin{bmatrix} -A \\ C \end{bmatrix} x \begin{matrix} \geq \\ = \end{matrix} \begin{bmatrix} b \\ d \end{bmatrix}. \tag{2'}$$

Its dual is then given by

$$\max_{y \in \mathbb{R}^m, z \in \mathbb{R}^p} \; [b^\top, d^\top] y \quad \text{s.t.} \quad [-A^\top, C^\top] y = c, \; y \geq 0.$$

The dual of the dual is then

$$\min_x \; c^\top x \quad \text{s.t.} \quad \begin{bmatrix} -A \\ C \end{bmatrix} x \begin{matrix} \geq \\ = \end{matrix} \begin{bmatrix} b \\ d \end{bmatrix}.$$

which is equivalent to (2'), hence to (2).

A3.3

(a)

The Strong Duality Theorem says there is no such an example.

(b)

Primal:

$$\min_{x\in\mathbb{R}} 0^\top x \quad \text{s.t.} \quad 0x = 0.$$

Dual:

$$\max_{y\in\mathbb{R}} 0^\top y \quad \text{s.t.} \quad 0^\top y = 0.$$

(c)

Primal:

$$\min_{x\in\mathbb{R}} 0^\top x \quad \text{s.t.} \quad 1x = 0.$$

Dual:

$$\max_{y\in\mathbb{R}} 0^\top y \quad \text{s.t.} \quad 1^\top y = 0.$$

(d)

Primal:

$$\min_{x\in\mathbb{R}^2} 0^\top x \quad \text{s.t.} \quad [1,-1]x = 0,\ x \geq 0.$$

Dual:

$$\max_{y\in\mathbb{R}} 0^\top y \quad \text{s.t.} \quad [1;-1]y \leq [0;0].$$

It can be easily checked that $x^* = [0;0]$ is a degenerate optimal BFS for the primal, and that $y^* = 0$ is the unique optimal solution for the dual.

A3.4

(a)

Let a_i denotes the i-th column of A. Then

$$a_i^\top x = \sum_{j=1}^{4} a_{ji}x_j = \mathbb{E}[\text{player I's winning}|\text{player II chooses } i], \quad i = 1:4.$$

Since t is a lower bound for $a_i^\top x$, finding the max of t is equivalent to maximizing the minimum of $a_i^\top x$, i.e., finding the optimal probabilistic strategy x for player I in the sense of maximizing his expected winning in the worst case.

MATLAB code:

```
cvx_begin
    variables x(4) t
    maximize(t)
    subject to
    A' * x >= t * ones(4, 1)
    ones(1, 4) * x == 1
    x >= zeros(4, 1)
cvx_end
```

yielding

$$p^* = t_{\max} = 0,$$

obtained at

$$x^* = [0.088; 0.338; 0.412; 0.162], \ t^* = 0.$$

(b)

Rewrite (3),

$$\begin{array}{ll} \max\limits_{x,t} & [0_{1\times 4}, 1][x; t] \\ \text{subject to} & [-A^\top, 1_{4\times 1}][x; t] \leq 0_{4\times 1} \\ & [1_{1\times 4}, 0][x; t] = 1 \\ & x \geq 0. \end{array}$$

Its dual is then given by

$$\begin{array}{ll} \min\limits_{y,s} & [0_{1\times 4}, 1][y; s] \\ \text{subject to} & [-A, 1_{4\times 1}][y; s] \geq 0_{4\times 1} \\ & [1_{1\times 4}, 0][y; s] = 1 \\ & y \geq 0, \end{array}$$

or equivalently

$$\begin{array}{ll} \min\limits_{y,s} & s \\ \text{subject to} & Ay \leq s \cdot 1 \\ & 1^\top y = 1 \\ & y \geq 0. \end{array}$$

This time we wish to minimize the upper bound s for $A_i^\top y, \ i = 1 : 4$, where $A_i^\top$ is the i-th row of A. If we interpret y as the probabilistic strategy for player II, then

$$A_i^\top y = \sum_{j=1}^{4} a_{ij} y_i = E[\text{player II's losses}|\text{player I chooses } i], \quad i = 1 : 4.$$

We see that the dual is to find the optimal probabilistic strategy y for player II in the sense of minimizing his expected loss in the worst case.

MATLAB code:

```
cvx_begin
    variables y(4) s
    maximize(s)
    subject to
    A * y <= s * ones(4, 1)
    ones(1, 4) * x == 1
    y >= zeros(4, 1)
cvx_end
```

which yields

$$d^* = s_{\min} = 0,$$

at

$$y^* = [0.25; 0.50; 0.25; 0.00], \ s^* = 0.$$

(c)

First note that

$$\begin{aligned}\max_t \ t \quad \text{s.t.} \quad Ax \geq t \cdot \mathbf{1} &= \max_t \ t \quad \text{s.t.} \quad \min_i \ A_i^\top x \geq t \\ &= \min_i \ A_i^\top x,\end{aligned}$$

where $A_i^\top$ denotes the i-th row of A, $i = 1:4$.

Now let $m := \operatorname{argmin}_i \ A_i^\top x$. We have

$$\begin{aligned}\min_{y \in P} \ y^\top Ax &= \min_{y \in P} \ \sum_i y_i A_i^\top x \\ &= \min_{y \in P} \ \left\{ y_m A_m^\top x + \sum_{i \neq m} y_i A_i^\top x \right\}.\end{aligned}$$

We claim that this minimum is exactly $A_m^\top x$, obtained at $\hat{y}$, the all-zero vector except $\hat{y}_m = 1$. Indeed, for any $y \in P$, we have

$$\begin{aligned}y^\top Ax - \hat{y}^\top Ax &= y^\top Ax - A_m^\top x \\ &= (y_m - 1)A_m^\top x + \sum_{i \neq m} y_i A_i^\top x \\ &\geq (y_m - 1)A_m^\top x + \sum_{i \neq m} y_i A_m^\top x \\ &= \left(-1 + \sum_i y_i \right) A_m^\top x \\ &= 0.\end{aligned}$$

Thus

$$\max_{Ax \geq t \cdot 1} \; t = \min_i \; A_i^\top x = A_m^\top x = \min_{y \in P} \; y^\top Ax$$
$$\implies \max_{x \in P} \; \max_{Ax \geq t \cdot 1} \; t = p^* = \max_{x \in P} \min_{y \in P} \; y^\top Ax.$$

Finally by Strong Duality,

$$p^* = \max_{x \in P} \min_{y \in P} \; y^\top Ax = d^*.$$

(d)

The game is fair in the sense that the expected winning for player I (or expected loss for player II) is zero in the worst case. Using only numbers one and two, the pay-off matrix becomes

$$B = \begin{bmatrix} -2 & 3 \\ 3 & -4 \end{bmatrix}.$$

Substituting matrix A with B in (3) and solving the problem in MATLAB with the code below,

```
cvx_begin
    variables x(2) t
    maximize(t)
    subject to
    B' * x >= t * ones(2, 1)
    ones(1, 2) * x == 1
    x >= zeros(2, 1)
cvx_end
```

we obtain $p^* = t_{\max} = 0.0833 > 0$ with strategy $x^* = [0.5833; 0.4167]$, which indicates a preference of the new game towards player I.

A3.5

(a)

$$\begin{array}{lrl} \min\limits_{x,t} & & t \\ \text{subject to} & t \cdot 1_{m\times 1} & \geq Ax - b \\ & t \cdot 1_{m\times 1} & \geq b - Ax \\ & t & \geq 0. \end{array} \tag{4'}$$

(b)

Rewrite (4'),

$$\begin{array}{ll} \min\limits_{x,t} & [0_{1\times n}, 1][x; t] \\ \text{subject to} & \left[\begin{array}{c|c} -A & 1_{m\times 1} \\ \hline A & 1_{m\times 1} \end{array}\right] \begin{bmatrix} x \\ t \end{bmatrix} \geq \begin{bmatrix} -b \\ b \end{bmatrix} \\ & t \geq 0. \end{array}$$

Thus the dual is given by

$$\begin{array}{lcr}
\max\limits_{z\in\mathbb{R}^{2m}} & [-b^\top, b^\top]z & \\
\text{subject to} & \left[\begin{array}{c|c} -A^\top & A^\top \\ \hline 1_{1\times m} & 1_{1\times m}\end{array}\right] z \begin{array}{c} = \\ \le \end{array} \begin{bmatrix} 0_{n\times 1} \\ 1 \end{bmatrix} & \text{(4'D)} \\
& z \ge 0. &
\end{array}$$

(c)

If we denote $z^- := z[1:m]$ and $z^+ := z[m+1:2m]$, we have

$$\begin{array}{ll}
\max\limits_{z^+,z^-\ge 0} & b^\top(z^+ - z^-) \\
\text{subject to} & A^\top(z^+ - z^-) = 0 \\
& 1^\top(z^+ + z^-) \le 1.
\end{array}$$

Setting $y := z^+ - z^-$, the problem further transforms into

$$\begin{array}{ll}
\max\limits_{z^+,z^-\ge 0,y} & b^\top y \\
\text{subject to} & y = z^+ - z^- \\
& A^\top y = 0 \\
& 1^\top(z^+ + z^-) \le 1.
\end{array} \tag{5}$$

Since the objective functions coincide now (and only depend on y), to prove the equivalence, it remains to show that ranges of y are equal in two problems:

$$||y||_1 \le 1 \iff \begin{array}{l} y = z^+ - z^- \\ 1^\top(z^+ + z^-) \le 1 \\ z^+, z^- \ge 0. \end{array}$$

Suppose we have $||y||_1 \le 1$, then we may define, for all i,

$$z_i^+ := \begin{cases} y_i, & \text{if } y_i \ge 0, \\ 0, & \text{otherwise;} \end{cases} \qquad z_i^- := \begin{cases} -y_i, & \text{if } y_i < 0, \\ 0, & \text{otherwise.} \end{cases}$$

Clearly $z^+, z^- \ge 0$, and $z^+ - z^- = y$. Also,

$$1^\top(z^+ + z^-) = \sum_{i=1}^m z_i^+ + z_i^- = \sum_{i=1}^m |y_i| = ||y||_1 \le 1.$$

Conversely, suppose there exists some (y, z^+, z^-) rendering RHS true. We have

$$||y||_1 = \sum_{i=1}^m |y_i| = \sum_{i=1}^m |z_i^+ - z_i^-| \le \sum_{i=1}^m |z_i^+| + |z_i^-| = \sum_{i=1}^m z_i^+ + z_i^- = 1^\top(z^+ + z^-) \le 1,$$

which completes the proof.

(d)

We already have $\text{RHS} \equiv (5) \equiv (4'\text{D})$. To show that the optimal value equals that of $\text{LHS} \equiv (4) \equiv (4')$, it suffices to show that $(4')$ and its dual $(4'\text{D})$ have the same optimal value.

First note that both problems are feasible: For $(4')$, $(x, t) = (0, ||b||_\infty)$ is a feasible solution; for $(4'\mathrm{D})$, $z = 0$ is feasible. Now $(4')$ must be bounded since otherwise $(4'\mathrm{D})$ would not be feasible due to the duality gap. Therefore $(4')$ attains a finite optimal value m, at some point p^*. By the Strong Duality, $(4'\mathrm{D})$ must have p^* as its optimal value as well, and we are done.

(e)

MATLAB code:

```
m = 100;
A = [ones(m), ones(m)];
b = (1:m)';

% original problem
tic;
cvx_begin quiet
    variable x(2 * m)
    minimize(norm(A * x - b, inf))
cvx_end
toc
Elapsed time is 0.441330 seconds.

% dual problem
tic;
cvx_begin quiet
    variable y(m)
    maximize(b' * y)
    subject to
    A' * y == zeros(2 * m, 1)
    norm(y, 1) <= 1
cvx_end
toc
Elapsed time is 0.249609 seconds.
```

Both methods obtain the optimal value 49.5, but solving dual is nearly twice as fast.