MAT3007 Assignment 2

A2.1

Writing the LP in standard form,

$$\begin{aligned}
\text{min} \quad & & z \\
\text{subject to} \quad & -x_1 - 2x_2 - 3x_3 - 8x_4 &= z \\
& x_1 - x_2 + x_3 + s_1 &= 2 \\
& x_3 - x_4 + s_2 &= 1 \\
& 2x_2 + 3x_3 + 4x_4 + s_3 &= 8 \\
& x, s &\geq 0.
\end{aligned}$$

Tableau:

	x_1	x_2	x_3	x_4	s_1	s_2	s_3	$-z_0$
$-z_0$	-1	-2	-3	-8	0	0	0	0
s_1	1	-2	1	0	1	0	0	2
s_2	0	0	1	-1	0	1	0	1
s_3	0	2	3	$\boxed{4}$	0	0	1	8
$-z_0$	-1	2	3	0	0	0	2	16
s_1	$\boxed{1}$	-2	1	0	1	0	0	2
s_2	0	$1/2$	$7/4$	0	0	1	$1/4$	3
x_4	0	$1/2$	$3/4$	1	0	0	$1/4$	2
$-z_0$	0	0	4	0	1	0	2	$\boxed{18}$
x_1	1	-2	1	0	1	0	0	2
s_2	0	$1/2$	$7/4$	0	0	1	$1/4$	3
x_4	0	$1/2$	$3/4$	1	0	0	$1/4$	2

Hence the optimal value is 18, obtained at

$$(x_1, x_2, x_3, x_4) = (2, 0, 0, 2).$$

A2.2

Rewriting the LP in standard form,

$$
\begin{aligned}
\text{min} \quad & & z \\
\text{subject to} \quad & & x_1 - x_2 + x_3 &= z \\
& & -2x_1 + x_2 - x_3 - s_1 &= 1 \\
& & x_1 - x_2 - x_3 + s_2 &= 4 \\
& & x_2 - x_4 &= 0 \\
& & x, s &\geq 0.
\end{aligned}
$$

The auxiliary problem is then:

$$
\begin{aligned}
\text{min} \quad & & \hat{z} \\
\text{subject to} \quad & & w_1 + w_2 + w_3 &= \hat{z} \\
& & x_1 - x_2 + x_3 &= z \\
& & -2x_1 + x_2 - x_3 - s_1 + w_1 &= 1 \\
& & x_1 - x_2 - x_3 + s_2 + w_2 &= 4 \\
& & x_2 - x_4 + w_3 &= 0 \\
& & x, s, w &\geq 0.
\end{aligned}
$$

Tableau I:

	x_1	x_2	x_3	x_4	s_1	s_2	w	
$-\hat{z}_0$	1	−1	2	1	1	−1	∗	−5
$-z_0$	1	−1	1	0	0	0	∗	0
w_1	−2	1	−1	0	−1	0	∗	1
w_2	1	−1	−1	0	0	$\boxed{1}$	∗	4
w_3	0	1	0	−1	0	0	∗	0
$-\hat{z}_0$	2	−2	1	1	1	0	∗	−1
$-z_0$	1	−1	1	0	0	0	∗	0
w_1	−2	1	−1	0	−1	0	∗	1
s_2	1	−1	−1	0	0	1	∗	4
w_3	0	$\boxed{1}$	0	−1	0	0	∗	0
$-\hat{z}_0$	2	0	1	−1	1	0	∗	−1
$-z_0$	1	0	1	−1	0	0	∗	0
w_1	−2	0	−1	$\boxed{1}$	−1	0	∗	1
s_2	1	0	−1	−1	0	1	∗	4
x_2	0	1	0	−1	0	0	∗	0
$-\hat{z}_0$	0	0	0	0	0	0	∗	0
$-z_0$	−1	0	0	0	−1	0	∗	1
x_4	−2	0	−1	1	−1	0	∗	1
s_2	−1	0	−2	0	−1	1	∗	5
x_2	−2	1	−1	0	−1	0	∗	1

Already we see by selecting x_1 as the entering variable, all the ratios will be non-positive. This means we may increase x_1 arbitrarily without ever violating the constraints (simply elevate the existing basic variables accordingly). It follows that $z_{\text{min}} = -\infty$. That is, z is unbounded from below.

A2.3

Notice that by subtracting the first two constraints, we have

$$7x_4 = 0,$$

which implies $x_4 = 0$. And we may eliminate x_4 from the LP, as in

$$
\begin{array}{ll}
\text{min} & z \\
\text{subject to} & 2x_1 + 3x_2 - 2x_5 = z \\
& x_1 + 3x_2 + x_5 = 2 \\
& -x_1 - 4x_2 + 3x_3 = 1 \\
& x \geq 0.
\end{array}
$$

The auxiliary problem:

$$
\begin{array}{ll}
\text{min} & \hat{z} \\
\text{subject to} & w_1 + w_2 = \hat{z} \\
& 2x_1 + 3x_2 - 2x_5 = z \\
& x_1 + 3x_2 + x_5 + w_1 = 2 \\
& -x_1 - 4x_2 + 3x_3 + w_2 = 1 \\
& x, w \geq 0.
\end{array}
$$

Tableau I:

	x_1	x_2	x_3	x_5	w	
$-\hat{z}_0$	0	1	-3	-1	$*$	-3
$-z_0$	2	3	0	-2	$*$	0
w_1	1	3	0	1	$*$	2
w_2	-1	-4	$\boxed{3}$	0	$*$	1
$-\hat{z}_0$	-1	-3	0	-1	$*$	-2
$-z_0$	2	3	0	-2	$*$	0
w_1	1	$\boxed{3}$	0	1	$*$	2
x_3	$-1/3$	$-4/3$	1	0	$*$	$1/3$
$-\hat{z}_0$	0	0	0	0	$*$	0
$-z_0$	1	0	0	-3	$*$	-2
x_2	$1/3$	1	0	$1/3$	$*$	$2/3$
x_3	$1/9$	0	1	$4/9$	$*$	$11/9$

Tableau II:

	x_1	x_2	x_3	x_5	
$-z_0$	1	0	0	-3	-2
x_2	$1/3$	1	0	$\boxed{1/3}$	$2/3$
x_3	$1/9$	0	1	$4/9$	$11/9$
$-z_0$	4	9	0	0	$\boxed{4}$
x_5	1	3	0	1	2
x_3	$-1/3$	$-4/3$	1	0	$1/3$

Hence the optimal value is -4, obtained at

$$(x_1, x_2, x_3, x_4, x_5) = (0, 0, 1/3, 0, 2).$$

A2.4

(a)

$$\beta > 0.$$

(b)

$$\delta < 0; \alpha \leq 0; \beta \geq 0.$$

(c)

$$\delta \geq 0; \beta \geq 10.$$

(d)

$$\delta = \eta = \beta = 10^{100}.$$

(e)

$$\beta = 0; \eta \geq 4; \delta + 2\alpha \geq 0.$$

A2.5

(a)

We start by writing the objective function as

$$f(x) = f(x^*) + \bar{c}_N^\top x_N$$

with $\bar{c}_N > 0$ and N being the non-basic indices. Let y be any feasible solution other than x^*. We claim that $y_N \neq 0$. For, if $y_N = 0$, then since y is feasible,

$$Ay = A_B y_B + A_N y_N = A_B y_B + A_N 0 = A_B y_B = b \implies y_B = A_B^{-1} b = x_B^*.$$

But then $y_N = x_N^* = 0$ forces $y = x^*$, a contradiction. Therefore $y_N \neq 0$. It follows from the feasibility that $y_n > 0$ for some non-basic index $i \in N$. Hence,

$$f(y) = f(x^*) + \bar{c}_N^\top y_N \geq f(x^*) + \bar{c}_i y_i > f(x^*),$$

from which we conclude x^* is the unique optimal solution.

(b)

Suppose otherwise. Then $\bar{c}_i \leq 0$ for some non-basic index i. Since x^* is nondegenerate, we may incorporate x_i into the basis by taking a sufficiently small step $\theta > 0$ in the i-th basic direction d_i, so that $\tilde{x} := x^* + \theta d_i$ is still feasible (Lecture 5; Slide 11). But then

$$f(\tilde{x}) = c^\top \tilde{x} = c^\top x^* + \theta c^\top d_i = f(x^*) + \theta \bar{c}_i \leq f(x^*),$$

whence x^* is not the unique optimal solution, the desired contradiction.