CSC3001 PQ5

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Base Case. $a_0 = 1 = 2^0$.

Inductive Step. Suppose $a_k = 2^k$ for all $k \leq n$. Then

$$a_{n+1} = 1 + \sum_{i=0}^{n} a_i = 1 + 2^0 + 2^1 + \ldots + 2^n = 1 + \frac{1 - 2^{n+1}}{-1} = 2^{n+1}. \quad \square$$

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Induction on $n \geq 0$.

Base Case. $0 = d \cdot 0 + 0$.

Inductive Step. Suppose $n = dq + r$ with $d > 0$ and $0 \leq r < d$. Then

$$n + 1 = dq + (r + 1).$$

If $r + 1 < d$ we are done. If $r + 1 \geq d$, we have

$$n + 1 = d(q + 1) + (r + 1 - d)$$

with $0 \leq r + 1 - d < 1 < d$. $\square$

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Fix any m. Induction on n :

Base Case. For $m \times 1$, clearly we need exactly $m - 1$ splits. So

$$\#\text{splits}_1 = m - 1 \geq m \cdot 1 - 1.$$

Inductive Step. Suppose for $m \times n$ the number of splits is

$$\#\text{splits}_n \geq mn - 1.$$

Then in order to divide up $m \times (n + 1)$, we need at least $m + 1$ splits to separate the chocolate bar into one $m \times n$ bar and m unit squares. So

$$\#\text{splits}_{n+1} \geq m + 1 + \#\text{splits}_n = m(n + 1) \geq m(n + 1) - 1. \quad \square$$

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Assume n is the least non-zero integer s.t. $1+2+\ldots+n \neq n(n+1)/2$. Then $1+2+\ldots+(n-1) \neq n(n+1)/2 - n = (n-1)n/2$, contradicting the minimality of n.

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Proceed by strong induction on n.

Clearly $F_0 = 0 \leq \phi^{-1}$. Suppose $F_k \leq \phi^{k-1}$ for all $k \leq n$. Then

$$F_{n+1} = F_{n-1} + F_n \leq \phi^{n-2} + \phi^{n-1} = \phi^{n-2}(1+\phi) \leq \phi^n$$

where the last inequality follows from the fact that $1+\phi = \phi^2$. $\square$