MAT3007 Assignment 5

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A5.1

(a)

Rewrite

$$
\begin{aligned}
f_\beta(x) &= \frac{1}{2}(x-b)^\top(x-b) + \frac{\beta}{2}\left(\mathbf{1}^\top x\right)^2 \\
&= \frac{1}{2}\left(x^\top x - 2b^\top x + b^\top b\right) + \frac{\beta}{2}\left(\mathbf{1}^\top x\right)^2,
\end{aligned}
$$

where $\mathbf{1}$ denotes the all-one vector.

Then the gradient is given by

$$
\begin{aligned}
\nabla f_\beta(x) &= \frac{1}{2}(2x-2b) + \frac{\beta\cdot 2}{2}\left(\mathbf{1}^\top x\right)\mathbf{1} \\
&= x - b + \left(\beta\mathbf{1}^\top x\right)\mathbf{1}.
\end{aligned}
$$

The Hessian is then

$$
\begin{aligned}
\nabla^2 f_\beta(x) &= \begin{bmatrix} 1+\beta & \beta & \cdots & \beta \\ \beta & 1+\beta & \cdots & \beta \\ \vdots & \vdots & \ddots & \beta \\ \beta & \beta & \cdots & 1+\beta \end{bmatrix} \\
&= \beta\cdot\mathbf{1}\mathbf{1}^\top + I.
\end{aligned}
$$

(b)

Set $\nabla f_\beta(x) = 0$. Then $x_i + \beta\sum_j x_j = b_i$ for all i. In matrix form,

$$\begin{bmatrix} 1+\beta & \beta & \cdots & \beta \\ \beta & 1+\beta & \cdots & \beta \\ \vdots & \vdots & \ddots & \beta \\ \beta & \beta & \cdots & 1+\beta \end{bmatrix} x = b$$

After some gruesome calculation we obtain

$$\begin{aligned} x_\beta^* &= \frac{1}{1+n\beta} \begin{bmatrix} 1+(n-1)\beta & -\beta & \cdots & -\beta \\ -\beta & 1+(n-1)\beta & \cdots & -\beta \\ \vdots & \vdots & \ddots & -\beta \\ -\beta & -\beta & \cdots & 1+(n-1)\beta \end{bmatrix} b \\ &= \left(I - \frac{\beta}{1+n\beta}\mathbf{1}\mathbf{1}^\top\right) b. \end{aligned}$$

To determine whether x_β^* is a local minimizer, note that for all $x \neq 0$,

$$\begin{aligned} x^\top \nabla^2 f_\beta(x_\beta^*) x &= x^\top \left(\beta \mathbf{1}\mathbf{1}^\top + I\right) x \\ &= \beta x^\top \mathbf{1}\mathbf{1}^\top x + x^\top x \\ &= \beta\big(\mathbf{1}^\top x\big)^2 + ||x||^2 \\ &\geq ||x||^2 > 0. \end{aligned}$$

Thus by SOSC, x_β^* is always a local minimizer.

(c)

We have

$$x^* = \lim_{\beta\to\infty} x_\beta^* = \lim_{\beta\to\infty} \left(I - \frac{\beta}{1+n\beta}\mathbf{1}\mathbf{1}^\top\right) b = \left(I - \mathbf{1}\mathbf{1}^\top\right) b,$$

and

$$\mathbf{1}^\top x^* = \mathbf{1}^\top \left(I - \mathbf{1}\mathbf{1}^\top\right) b = \mathbf{1}^\top b - \mathbf{1}^\top \left(\mathbf{1}\mathbf{1}^\top\right) b = \mathbf{1}^\top b - \mathbf{1}^\top b = 0.$$

(d)

The set

$$\left\{\nabla \left(\mathbf{1}^\top x\right)\right\} = \{\mathbf{1}\}$$

is clearly linearly independent at all feasible points, i.e., LICQ is always satisfied.

Introduce μ, the dual multiplier for the equality constraint. The Lagrangian is then

$$\mathcal{L}(x,\mu) = \frac{1}{2}||x-b||^2 + \mu \cdot \mathbf{1}^\top x.$$

Setting

$$\nabla_x \mathcal{L}(x,\mu)\big|_{x=x^*} = x^* - b + \mu\cdot\mathbf{1} = 0 \implies \mu\cdot\mathbf{1} = \mathbf{1}\mathbf{1}^\top b \implies \mu = \mathbf{1}^\top b.$$

By (c), x^* is always a feasible solution. We have met all KKT conditions at $(x, \mu) = (x^*, \mathbf{1}^\top b)$. Thus x^* is a KKT point. It is also the only KKT point because if we let

$$\nabla_x \mathcal{L}(x, \mu) = 0 \implies x = b - \mu \cdot \mathbf{1},$$

the Primal Feasibility would impose

$$\mathbf{1}^\top x = \mathbf{1}^\top b - \mu = 0 \implies \mu = \mathbf{1}^\top b \implies x = b - \mathbf{1}^\top b \mathbf{1} = x^*.$$

Adding the LICQ, x^* must be the unique local solution.

A5.2

(a)

We have

$$\begin{array}{ll} \min & f(x) = x_1^2 + x_2^2 + x_3^2 + x_1 x_2 + x_2 x_3 - 2x_1 - 5x_2 - 6x_3 \\ \text{subject to} & g(x) = x_1 + x_2 + x_3 - 1 \le 0 \\ & h(x) = x_1 - x_2^2 = 0. \end{array}$$

The Lagrangian for the problem is

$$\mathcal{L}(x, \lambda, \mu) = f(x) + \lambda g(x) + \mu h(x).$$

The KKT conditions are:

- Main Condition

$$\nabla_x \mathcal{L}(x, \lambda, \mu) = \begin{bmatrix} 2x_1 + x_2 - 2 + \lambda + \mu \\ 2x_2 + x_1 + x_3 - 5 + \lambda - 2\mu x_2 \\ 2x_3 + x_2 - 6 + \lambda \end{bmatrix} = 0.$$

- Dual feasibility

$$\lambda \ge 0.$$

- Primal feasibility

$$g(x) \le 0,\ h(x) = 0.$$

- Complementarity

$$\lambda g(x) = 0.$$

(b)

Clearly x^* is a feasible point. Set $\nabla_x \mathcal{L}(x, \lambda, \mu)|_{x=x^*} = 0$. We have

$$[\lambda + \mu - 2; \lambda - 4; \lambda - 4] = 0 \implies \lambda = 4,\ \mu = -2.$$

So the dual feasibility is met. Also, $g(x^*) = 1 - 1 = 0$, meeting the complementarity condition. Thus x^* is a KKT point.

We then compute the Hessian at x^*

$$H := \nabla^2_{xx}\mathcal{L}(x^*, 4, -2) = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 6 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

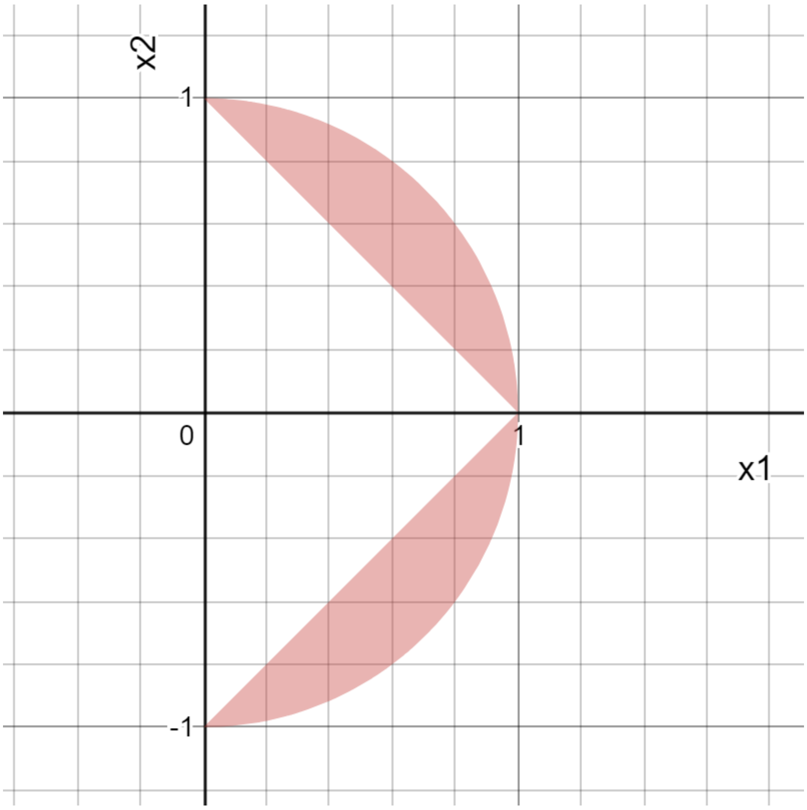
We see that the determinant of all leading principals of H :

$$\Delta_1 = 2,\ \Delta_2 = 11,\ \Delta_3 = 20,$$

are positive. Thus $H \succ 0$, whence the SOSC holds and x^* is a strict local minimizer.

A5.3

(a)



(b)

At $\bar{x} = [0; 1]$, we have

$$g_1(\bar{x}) = 1 - 1 = 0,\ g_2(\bar{x}) = 1 - 1 = 0.$$

Thus both constraints are active, i.e., $\mathcal{A}(\bar{x}) = \{1, 2\}$.

The gradients of the inequality constraints are

$$\nabla g_1(x) = [2x_1; 2x_2],\ \nabla g_2(x) = [2x_1 - 2; -2x_2].$$

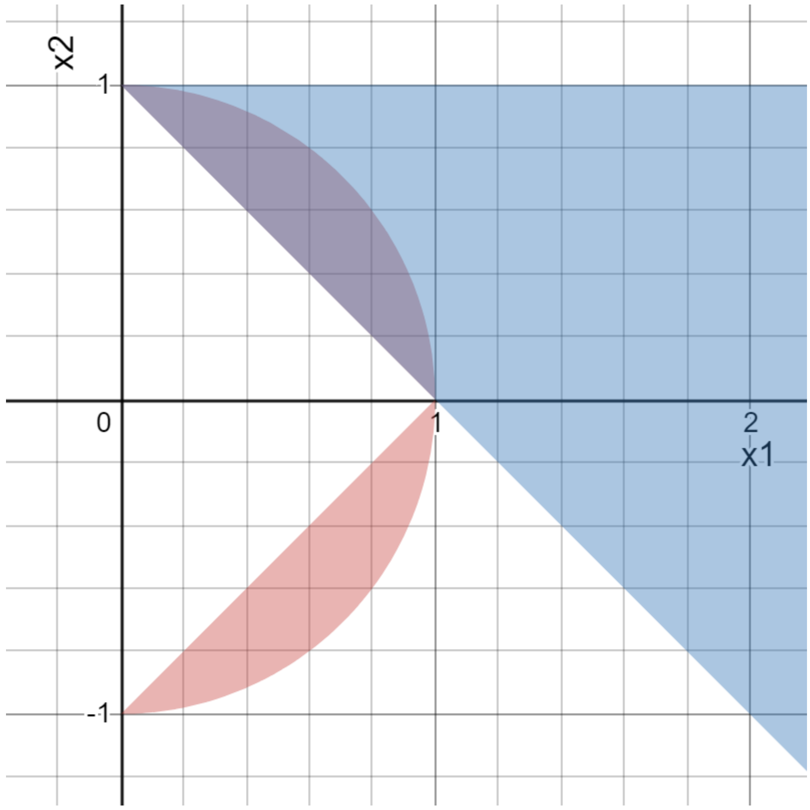
At $\bar{x}$, the gradients are

$$\nabla g_1(\bar{x}) = [0; 1],\ \nabla g_2(x) = [-2; -2].$$

The linearized tangent set at $\bar{x}$ is then given by

$$\mathcal{T}_\ell(\bar{x}) = \{[d_1; d_2] : d_2 \leq 0,\ d_1 + d_2 \geq 0\}.$$

We plot the set **after shifting the start of all direction vectors to** $\bar{x}$:



(c)

Since the objective function f is continuous and the feasible set Ω is compact, by the Extreme Value Theorem $f(\Omega)$ is also compact. Hence f must attain the minimum at some $x^* \in \Omega$.

(d)

The Lagrangian:

$$\mathcal{L}(x, \lambda) = x_2^2 - 2x_1 + \lambda_1(x_1^2 + x_2^2 - 1) + \lambda_2(x_1^2 - x_2^2 - 2x_1 + 1).$$

Main condition:

$$\nabla_x \mathcal{L}(x, \lambda) = [-2 + 2\lambda_1 x_1 + 2\lambda_2 x_1 - 2\lambda_2; 2x_2 + 2\lambda_1 x_2 - 2\lambda_2 x_2] = 0.$$

Dual feasibility:

$$\lambda \geq 0.$$

Complementarity:

$$\lambda_1(x_1^2 + x_2^2 - 1) = 0,\ \lambda_2(x_1^2 - x_2^2 - 2x_2 + 1) = 0.$$

yielding the KKT point

$$\boxed{x^* = [1; 0],\ \lambda = [1; t],\ t \geq 0.}$$

Compute the Hessian at x^*:

$$H := \nabla_{xx}\mathcal{L}(x^*, \lambda) = 2\begin{bmatrix} 1+t & 0 \\ 0 & 2-t \end{bmatrix}.$$

We may choose $t = 0 \implies H \succ 0$. By SOSC x^* is a strict local minimizer with $f(k_2) = -2$. But since all feasible points other than x^* are regular, we must attain global minimum at x^*, with optimal value $f(x^*) = -2$.