Assignment 4

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1

(a)

$\forall A \in M_{n\times n}$,

$$A = I_n^{-1} A I_n = A. \tag{39}$$

(b)

Suppose $\exists T$ s.t.

$$B = T^{-1} A T. \tag{1}$$

Take $S := T^{-1}$. Then $S^{-1} = T$,

$$A = S^{-1} B S. \tag{2}$$

(c)

Suppose $\exists P, Q$ s.t.

$$\begin{aligned} A &= P^{-1} B P \\ B &= Q^{-1} C Q \end{aligned} \tag{3}$$

Take $S := QP$. Then $S^{-1} = P^{-1} Q^{-1}$,

$$A = S^{-1} C S. \tag{4}$$

2

The matrix representation of L w.r.t. standard basis E_2

$$\mathrm{Rep}_{\mathrm{E}_2}(L) = \begin{bmatrix} 3 & 0 \\ 1 & -1 \end{bmatrix}_{\mathrm{E}_2}. \tag{5}$$

Change-of-basis matrix from B to E_2

$$P := P_{B,\mathrm{E}_2} = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}_{B,\mathrm{E}_2}. \tag{6}$$

Then the change-of-basis matrix from E_2 to B is simply the inverse

$$P^{-1} = P_{\mathrm{E}_2,B} = \begin{bmatrix} -3 & 2 \\ 2 & -1 \end{bmatrix}_{\mathrm{E}_2,B}. \tag{7}$$

It follows that

$$\begin{aligned} \mathrm{Rep}_B(L) &= P^{-1} \mathrm{Rep}_{\mathrm{E}_2}(L) P \\ &= \begin{bmatrix} -3 & 2 \\ 2 & -1 \end{bmatrix}_{\mathrm{E}_2,B} \begin{bmatrix} 3 & 0 \\ 1 & -1 \end{bmatrix}_{\mathrm{E}_2} \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}_{B,\mathrm{E}_2} \\ &= \begin{bmatrix} -11 & -20 \\ 7 & 13 \end{bmatrix}_B. \end{aligned}$$

3

(a)

No.

Given a polynomial f of degree $\geq 3, 0 \cdot f = 0$ is of degree $0 < 3$.

(b)

No.

Given a polynomial f s.t. $f(1) + 2f(2) = 1$, $g := 0 \cdot f = 0$ is constant zero, implying $g(1) + 2g(2) = 0 \neq 1$.

(c)

Yes.

Given polynomials f_1, f_2 with $f(x) = f(1-x)$, let $g := \alpha f_1 + \beta f_2$. Then $g(x) = \alpha f_1(x) + \beta f_2(x) = \alpha f_1(1-x) + \beta f_2(1-x) = g(1-x)$.

4

(a)

Either a **parallelogram**, or a **straight line**, or a single **point**.

(b)

The region remains square after transformation A if and only if

$$A^T A = AA^T = cI_2,\ c \neq 0. \tag{8}$$

5

Denote ordered bases $\mathrm{E}_3 := \{e_1, e_2, e_3\}, B := \{b_1, b_2\}$.

The linear map

$$L : \mathbb{R}^3 \to \mathbb{R}^2 \tag{9}$$

is characterized by its action on the basis E_3:

$$\mathrm{E}_3 = \{e_1, e_2, e_3\} \xrightarrow{L} \{L(e_1), L(e_2), L(e_3)\} \xrightarrow{P_{\mathrm{E}_2,B}} \mathrm{Rep}_B\{L(e_1), L(e_2), L(e_3)\} \tag{10}$$

where $P_{\mathrm{E}_2,B}$ is the change of basis matrix from E_2 to B,

$$P_{\mathrm{E}_2,B} = P^{-1}_{B,\mathrm{E}_2} = [\,b_1, b_2\,]^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}. \tag{11}$$

Hence

$$\begin{aligned} A = \mathrm{Rep}_{\mathrm{E}_3,B}(L) &= P_{\mathrm{E}_2,B}[L(e_1), L(e_2), L(e_3)] \\ &= \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}_{\mathrm{E}_3,B}. \end{aligned}$$

6

Denote ordered bases $B = \{b_1, b_2, b_3\}, U = \{u_1, u_2\}$.

The linear map

$$L : \mathbb{R}^2 \to \mathbb{R}^3 \tag{12}$$

is characterized by its action on the basis U:

$$U = \{u_1, u_2\} \xrightarrow{L} \{L(u_1), L(u_2)\} \xrightarrow{P_{\mathrm{E}_3,B}} \mathrm{Rep}_B\{L(u_1), L(u_2)\} \tag{13}$$

where $P_{\mathrm{E}_3,B}$ is the change-of-basis matrix from E_3 to B,

$$P_{\mathrm{E}_3,B} = P^{-1}_{B,\mathrm{E}_3} = [\,b_1, b_2, b_3\,]^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}. \tag{14}$$

Hence

$$\begin{aligned} \mathrm{Rep}_{B,U}(L) &= P_{\mathrm{E}_3,B}[L(u_1), L(u_2)] \\ &= \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 3 & 4 \\ -1 & 2 \end{bmatrix} \\ &= \begin{bmatrix} -1 & -3 \\ 4 & 2 \\ -1 & 2 \end{bmatrix}_{B,U}. \end{aligned}$$

7

The linear map

$$T : \mathbb{P}^2 \to \mathbb{P}^2 \tag{15}$$

is characterized by its action on basis ϵ_2:

$$\epsilon_2 = \{1, x, x^2\} \xrightarrow{T} \{1, 3x-2, 9x^2-12x+4\} \xrightarrow{\mathrm{Rep}_{\epsilon_2}} \left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ -12 \\ 9 \end{bmatrix}\right\}_{\epsilon_2}. \tag{16}$$

Hence

$$\mathrm{Rep}_{\epsilon_2}(T) = \begin{bmatrix} 1 & -2 & 4 \\ 0 & 3 & -12 \\ 0 & 0 & 9 \end{bmatrix}_{\epsilon_2}. \tag{17}$$

8

Denote the basis $J := \{v_1, v_2, v_3\}$.

The change-of-basis matrix from J to E_3

$$V = [v_1, v_2, v_3] = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & -2 \\ 1 & 0 & 1 \end{bmatrix}_{J,\mathrm{E}_3}. \tag{18}$$

Note that the change-of-basis matrix from E_3 to J is simply the inverse,

$$V^{-1} = \begin{bmatrix} -2 & 1 & 2 \\ 3 & -1 & -2 \\ 2 & -1 & -1 \end{bmatrix}_{\mathrm{E}_3,J}. \tag{19}$$

Hence the matrix representation of L w.r.t. J

$$\begin{aligned} B = \mathrm{Rep}_J(L) &= V^{-1}AV \\ &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_J. \end{aligned}$$

9

Taking transpose on both sides of the first equation,

$$x^T A^T = 0. \tag{20}$$

Multiply both sides by y,

$$x^T A^T y = 0. \tag{21}$$

Now combine (21) with the second equation given,

$$x^T 2y = 2x^T y = 0 \implies x^T y = 0 \implies x \perp y. \tag{22}$$

10

(a)

True.

$$Q \text{ orthogonal } \implies QQ^T = I \implies (Q^T)^{-1}Q^{-1} = (Q^{-1})^T Q^{-1} = I \implies Q^{-1} \text{ orthogonal}. \tag{23}$$

Example.

$Q = I$ is orthogonal, $Q^{-1} = I$ is also orthogonal.

(b)

True.

$$\begin{aligned}
||Qx||^2 &= (Qx)^T Qx \\
&= x^T Q^T Qx \\
&= x^T \begin{bmatrix} q_1^T \\ q_2^T \\ \vdots \\ q_n^T \end{bmatrix} [q_1, q_2, \dots, q_n]x \\
&= x^T \begin{bmatrix} q_1^T q_1 & q_1^T q_2 & \cdots & q_1^T q_n \\ q_2^T q_1 & q_2^T q_2 & \cdots & q_2^T q_n \\ \vdots & \vdots & \ddots & \vdots \\ q_n^T q_1 & q_n^T q_2 & \cdots & q_n^T q_n \end{bmatrix} x \\
&= x^T I_n x \\
&= ||x||^2,
\end{aligned}$$

and hence the square roots remain equal.

Example.

$Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, Qx = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix}$. $||Qx|| = ||x|| = \sqrt{x_1^2 + x_2^2}$.

(c)

False.

Example.

For the same Q as in (b), $Q^T y = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}^T \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$. Clearly $||Q^T y|| = \sqrt{y_1^2 + y_2^2} \neq \sqrt{y_1^2 + y_2^2 + y_3^2} = ||y||$.

11

$\forall p \in P(\mathbb{R}) \forall q \in W_2$, their inner product

$$\begin{aligned}
\langle p, q\rangle &= \int_{-1}^{1} p(x)q(x)dx \\
&= \int_{-1}^{1} (a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots)(b_0 + b_2 x^2 + \dots)dx \\
&= \int_{-1}^{1} \sum_{n=0}^{\infty} c_n x^{2n+1} dx + \int_{-1}^{1} \sum_{n=0}^{\infty} d_n x^{2n} dx \\
&= \sum_{n=0}^{\infty} \int_{-1}^{1} c_n x^{2n+1} dx + \sum_{n=0}^{\infty} \int_{-1}^{1} d_n x^{2n} dx \\
&= \sum_{n=0}^{\infty} \int_{-1}^{1} d_n x^{2n} dx
\end{aligned}$$

where

$$d_n = \sum_{i+2j=2n} a_i b_{2j}. \tag{24}$$

For the inner product to be 0, it is equivalent to ask $d_n = 0$, i.e.

$$a_0 b_0 = a_0 b_2 + a_2 b_0 = a_0 b_4 + a_2 b_2 + a_4 b_0 = \dots = 0 \tag{25}$$

for any choice of $b_0, b_2, b_4 \dots$ Clearly this is true if and only if

$$a_0 = a_2 = a_4 = \dots = 0. \tag{26}$$

That is, $p \in W_1$. Thus we conclude in $P(\mathbb{R})$,

$$\forall q \in W_2, p \perp q \iff p \in W_1. \tag{27}$$

In other words,

$$W_1 = W_2^{\perp}. \tag{28}$$

12

To find the first basis vector $s := [s_1, s_2, s_3]^T$, we use the fact that

$$\langle s, [1, 2, -5]^T\rangle = 0. \tag{29}$$

Equivalently,

$$s_1 + 2s_2 - 5s_3 = 0. \tag{30}$$

A particular solution to (30) is

$$s = [s_1, s_2, s_3]^T = [1, 2, 1]^T. \tag{31}$$

To find the other basis vector u, we simply use the cross product

$$u = s \times [1, 2, -5]^T = [-12, 6, 0]. \tag{32}$$

Normalizing,

$$\begin{aligned}
\bar{s} &= \frac{1}{\sqrt{6}}[1, 2, 1]^T, \\
\bar{u} &= \frac{1}{\sqrt{5}}[-2, 1, 0]^T.
\end{aligned} \tag{33}$$

An orthonormal basis for U:

$$B_U = \{\bar{s}, \bar{u}\}. \tag{34}$$

13

We want:

$$\begin{aligned} 4 &= C + D(-2) \\ 2 &= C + D(-1) \\ -1 &= C + D(0) \\ 0 &= C + D(1) \\ 0 &= C + D(2) \end{aligned}$$

In matrix form,

$$b = \begin{bmatrix} 4 \\ 2 \\ -1 \\ 0 \\ 0 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & -2 \\ 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} C \\ D \end{bmatrix}}_{x}. \tag{35}$$

The normal equation is $A^T b = A^T Ax$:

$$\begin{bmatrix} 5 \\ -10 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 0 & 10 \end{bmatrix} x. \tag{36}$$

Solving (36),

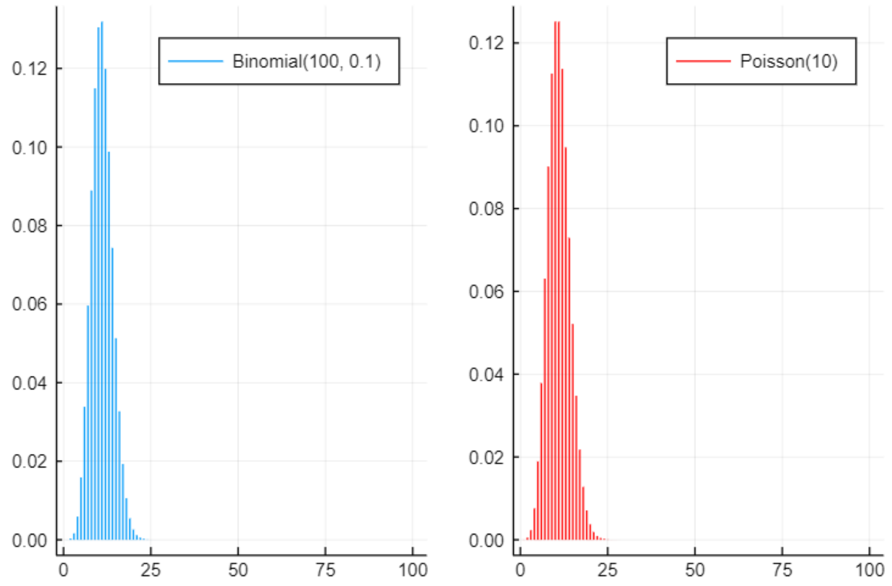
$$x = [1, -1]^T. \tag{37}$$

Hence the best fitting line in the sense of least-square is:

$$y = 1 - t. \tag{38}$$

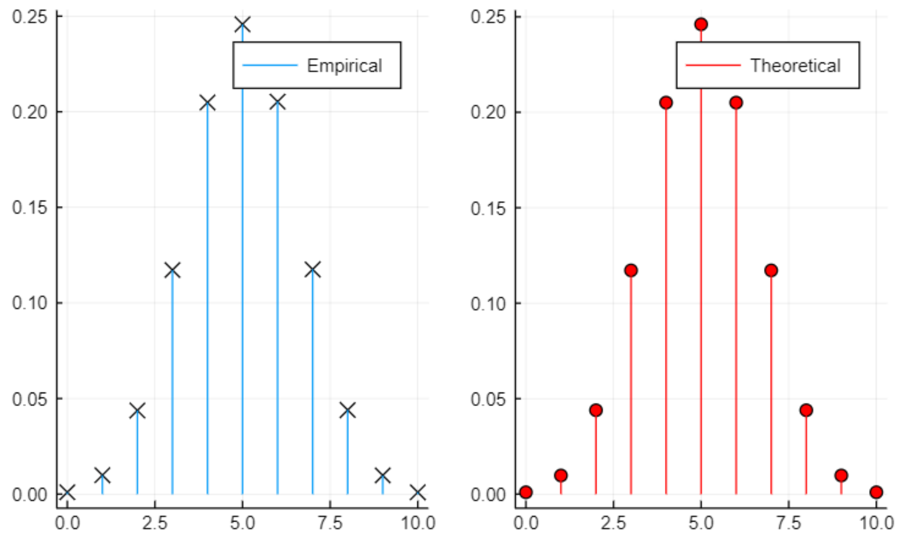
14

```
using Plots
n = BigInt(100); p = .1; λ=10; x = BigInt.(0:n);
bi(x) = binomial(n, x) * p^x * (1 - p)^(n - x)
poi(x) = λ^x / (e^λ * factorial(x))
x1 = plot(bi.(x), line=:stem, label="Binomial(100, 0.1)")
x2 = plot(poi.(x), line=:stem, label="Poisson(10)",
color=:red)
plot(x1,x2)
```



15

```
using Plots
n = 10; N = 1e6; p = .5
rslt = Dict([(x,0) for x=0:n])
for _ = 1:N
    c = count(x->(rand()<p), 1:n)
    rslt[c] += 1
end
f(x) = rslt[x] / N
bi(x) = binomial(n, x) * p^x * (1 - p)^(n - x)
emp = plot(f, 0:n, line=:stem, marker=:auto,
label="Empirical")
theo = plot(bi, 0:n, line=:stem, marker=:circle,
label="Theoretical", color=:red)
plot(emp, theo)
```



16

(a)

```
julia> using LinearAlgebra

julia> U=rand(4,4); V=rand(4,4); b=ones(4,1);

julia> V
4×4 Array{Float64,2}:
 0.583791  0.504311  0.33405   0.544164
 0.211525  0.545206  0.658239  0.913852
 0.514587  0.826214  0.768501  0.783236
 0.203812  0.794052  0.68816   0.697757

julia> rank(V)
4
```

(b)

```
julia> ϵ₄E = inv(U) # change-of-basis matrix from ϵ₄ to E
4×4 Array{Float64,2}:
  4.57283   5.16672  -1.57722  -4.54252
 -3.21203  -5.55121   2.9037    2.59196
  3.35517  11.4939   -2.21768  -7.08184
 -4.51906  -8.71237   1.52479   8.35272

julia> ϵ₄F = inv(V) # change-of-basis matirx from ϵ₄ to F
4×4 Array{Float64,2}:
  0.50853  -0.467216   2.65237  -2.76198
  2.55582  -2.06237   -4.19896   5.42121
 -5.31879   0.190387   7.94312  -5.01755
  2.18856   2.29569   -3.83018   1.01909
```

17

(a)

```
julia> using LinearAlgebra

julia> c = ϵ₄E * b # representation of b w.r.t. E
4×1 Array{Float64,2}:
  3.61980629652735
 -3.2675806363290505
  5.549578103410754
 -3.353928644644931

julia> d = ϵ₄F * b # representation of b w.r.t. F
4×1 Array{Float64,2}:
 -0.06829308417488722
  1.715707100063645
 -2.202830488545467
```

```
  1.6731589608599862

julia> norm(b - U*c)
1.041481514324134e-15

julia> norm(b - V*d)
4.965068306494546e-16
```

(b)

```
julia> S = ϵ₄F*U # change-of-basis matrix from E to F
4×4 Array{Float64,2}:
  1.08202    1.73673   0.276038  -0.0471195
  0.481655  -2.27235  -0.751135   0.979271
  0.446511   4.47906   1.831     -0.195379
 -0.555176  -2.02403  -0.527645   0.000801451

julia> T = ϵ₄E*V # change-of-basis matrix from F to E
4×4 Array{Float64,2}:
  2.02503   0.212938   0.590411   2.80508
 -1.0269   -0.189189  -0.711821  -2.73802
  1.80543   0.502986   2.10881    5.65114
 -1.99406   0.863238  -0.324607  -3.3985

julia> norm(d - S*c)
6.139584144267543e-15

julia> norm(c - T*d)
7.768388458966724e-15
```

18

```
julia> using LinearAlgebra

julia> A = [3 5;-2 -6;-5 -11]; b = [3; 3; 3];

julia> # A'Ax* = A'b ⟺ x* = inv(A'A)A'b

julia> x_ast = inv(A'*A)*A'*b
2-element Array{Float64,1}:
  3.9999999999999982
 -1.9999999999999991

julia> norm(A*x_ast - b)
1.7320508075688794
```