PHY 1002: Exercise on Error Estimation

1

1234	123400	123.4	1001	1000.	10.10	0.0001010	100
4	4	4	4	4	4	4	4
1010	1.01×10^3	1.010×10^3	0.015	1.5×10^{-2}	1.50×10^{-2}		
3	3	4	2	2	3		

2

Expand f at point $(\bar{u}, \bar{v})$ to the first order:

$$f(u_i, v_i) = f(\bar{u}, \bar{v}) + f_u(\bar{u}) \cdot (u_i - \bar{u}) + f_v(\bar{v}) \cdot (v_i - \bar{v}) + \ldots$$

Therefore

$$\begin{aligned}
\sigma_x^2 &= \lim_{N \to \infty} \left[\frac{1}{N} \sum_{i=1}^{N} (x_i - \bar{x})^2 \right] \\
&= \lim_{N \to \infty} \left[\frac{1}{N} \sum_{i=1}^{N} \left(f(u_i, v_i) - f(\bar{u}, \bar{v}) \right)^2 \right] \\
&\approx \lim_{N \to \infty} \left[\frac{1}{N} \sum_{i=1}^{N} \left(f_u(\bar{u}, \bar{v}) \cdot (u_i - \bar{u}) + f_v(\bar{u}, \bar{v}) \cdot (v_i - \bar{v}) \right)^2 \right] \\
&= \lim_{N \to \infty} \left[\frac{1}{N} \sum_{i=1}^{N} f_u^2 \cdot (u_i - \bar{u})^2 + f_v^2 \cdot (v_i - \bar{v})^2 + 2 f_u f_v \cdot (u_i - \bar{u})(v_i - \bar{v}) \right]_{(\bar{u}, \bar{v})} \\
&= \lim_{N \to \infty} \left[\frac{1}{N} \sum_{i=1}^{N} (u_i - \bar{u})^2 \right] f_u^2(\bar{u}, \bar{v}) + \lim_{N \to \infty} \left[\frac{1}{N} \sum_{i=1}^{N} (v_i - \bar{v})^2 \right] f_v^2(\bar{u}, \bar{v}) + \lim_{N \to \infty} \left[\frac{1}{N} \sum_{i=1}^{N} (u_i - \bar{u})(v_i - \bar{v}) \right] f_u \cdot f_v(\bar{u}, \bar{v}) \\
&= \left[\sigma_u^2 f_u^2 + \sigma_v^2 f_v^2 + 2 \sigma_{uv}^2 f_u f_v \right]_{(\bar{u}, \bar{v})}.
\end{aligned}$$

If $\mathrm{Cov}(u, v) = 0$, then we have

$$\sigma_x^2 \approx \left[\sigma_u^2 f_u^2 + \sigma_v^2 f_v^2 \right]_{(\bar{u}, \bar{v})}.$$

As standard error is defined as

$$\delta = \frac{\sigma}{\sqrt{N}},$$

$$\sigma^2 = N \cdot \delta^2.$$

Hence

$$N \cdot \delta_x^2 \approx N \cdot (\delta_u^2 f_u^2 + \delta_v^2 f_v^2).$$

$$\delta_x^2 \approx \delta_u^2 f_u^2 + \delta_v^2 f_v^2.$$

where f_u, f_v are both evaluated at $(\bar{u}, \bar{v})$.

3

a)

$$\sigma_x \approx \sqrt{\frac{\sigma_u^2 + \sigma_v^2}{4(\bar{u} + \bar{v})^4}} = \frac{\sqrt{\sigma_u^2 + \sigma_v^2}}{2(\bar{u} + \bar{v})^2}$$

b)

$$\sigma_x \approx \sqrt{\frac{\sigma_u^2 + \sigma_v^2}{4(\bar{u} - \bar{v})^4}} = \frac{\sqrt{\sigma_u^2 + \sigma_v^2}}{2(\bar{u} - \bar{v})^2}$$

c)

$$\sigma_x \approx \sqrt{\frac{4\sigma_u^2}{\bar{u}^6}} = \frac{2\sigma_u}{|\bar{u}^3|}$$

d)

$$\sigma_x \approx \sqrt{\sigma_u^2 \bar{v}^4 + 4\sigma_v^2 \bar{u}^2 \bar{v}^2}$$

e)

$$\sigma_x \approx 2\sqrt{\sigma_u^2 \bar{u}^2 + \sigma_v^2 \bar{v}^2}$$

f)

$$\sigma_x \approx \frac{|ab\cos(b\bar{u}/\bar{v})|}{\bar{v}^2}\sqrt{\sigma_u^2 \bar{v}^2 + \sigma_v^2 \bar{u}^2}$$

g)

$$\sigma_x \approx \sqrt{\sigma_u^2 a^2/\bar{u}^2} = \sigma_u |a/\bar{u}|$$

h)

$$\sigma_x \approx e^{b\bar{u}+c\bar{v}}\sqrt{\sigma_u^2 b^2 + \sigma_v^2 c^2}$$

i)

$$\sigma_x = 0$$

4

In radians, $\theta_1 = 0.1224\pi \pm 0.001111\pi, \theta_2 = 0.08028\pi \pm 0.001111\pi$.

$$
\begin{aligned}
\bar{n}_2(\bar{\theta}_1, \bar{\theta}_2, n_1) &= \frac{\sin\bar{\theta}_1}{\sin\bar{\theta}_2} n_1 \\
&= \frac{\sin(0.1224\pi)}{\sin(0.08028\pi)} \cdot 1.0000 \\
&= 1.503.
\end{aligned}
$$

$$
\begin{aligned}
\delta_{n_2} &= \csc\bar{\theta}_2 \sqrt{\delta_{\theta_1}^2 \cos^2\bar{\theta}_1 + \delta_{\theta_2}^2 \sin^2\bar{\theta}_1 \cot^2\bar{\theta}_2} \\
&= 0.024.
\end{aligned}
$$

Therefore,

$$
n_2 = 1.503 \pm 0.024.
$$

5

$$
\begin{aligned}
\bar{g} &= \frac{4\pi^2 t}{\bar{T}^2} \\
&= 9.801 \text{ m} \cdot \text{s}^{-2}.
\end{aligned}
$$

$$
\begin{aligned}
\delta_g &= 4\pi^2 \sqrt{4\delta_T^2 \bar{t}^2 \bar{T}^{-6} + \delta_t^2 \bar{T}^{-4}} \\
&= 0.028 \text{ m} \cdot \text{s}^{-2}.
\end{aligned}
$$

Therefore,

$$
g = 9.801 \pm 0.028 \text{ m} \cdot \text{s}^{-2}.
$$

6

The mean speed of the disk is

$$
\omega_i(\theta_i, \theta_{i-1}, t_i, t_{i-1}) = \frac{\Delta\theta_i}{\Delta t_i} = 1000\ {}^\circ\text{s}^{-1} = 17.45 \text{ s}^{-1} = 20 \text{ s}^{-1}.
$$

If $\delta_\theta = 0.09^\circ = 0.00157$ rad, $\delta_t = 10\ \mu$s, then

$$
\begin{aligned}
\delta_{\omega_i} &= \sqrt{(\delta_{\theta_i}^2 + \delta_{\theta_{i-1}}^2)(\Delta t_i)^{-2} + (\delta_{t_i}^2 + \delta_{t_{i-1}}^2)(\Delta\theta_i)^2(\Delta t_i)^{-4}} \\
&= \sqrt{2}\sqrt{\delta_\theta^2(\Delta t_i)^{-2} + \delta_t^2(\Delta\theta_i)^2(\Delta t_i)^{-4}} \\
&= 2.235 \text{ s}^{-1} \\
&= 2 \text{ s}^{-1},
\end{aligned}
$$

and

$$
\begin{aligned}
\delta_v &= \sqrt{\delta_R^2\omega^2 + \delta_\omega^2 R^2} \\
&= \sqrt{(.1\text{mm})^2(17.45 \text{ s}^{-1})^2 + (2.235 \text{ s}^{-1})^2(20 \text{ mm})^2} \\
&= 0.0447 \text{ m} \cdot \text{s}^{-1} \\
&= 0.04 \text{ m} \cdot \text{s}^{-1}.
\end{aligned}
$$