Assignment 5

Table of contents

1

(a)

By the fact that

$$\text{rank } AA^T = \text{rank } A^T A \tag{1}$$

and the rank-nullity theorem, one has

$$\text{rank } A - \text{rank } AA^T = \text{nullity } A^T A - \text{nullity } A. \tag{2}$$

Now from the proven fact that

$$\text{Null } A = \text{Null } A^T A \implies \text{nullity } A = \text{nullity } A^T A, \tag{3}$$

one obtains

$$\text{rank } A = \text{rank } AA^T. \tag{4}$$

(b)

$$AA^T = [a_1, a_2, \ldots, a_n]A^T \tag{5}$$

where a_i is the i-th column of A. Note that every column of AA^T is a linear combination of a_i's. Hence

$$\text{Col } AA^T \subset \text{Col } A. \tag{6}$$

But from (4) we know that the dimensions of two column space are equal, so it must be the case that

$$\text{Col } AA^T = \text{Col } A. \tag{7}$$

2

(a)

The normal equation $A^TAx = A^Tb$ is:

$$\begin{bmatrix} 7 & 1 \\ 1 & 6 \end{bmatrix} x = \begin{bmatrix} 8 \\ 7 \end{bmatrix}, \tag{8}$$

which has a unique solution since A^TA is invertible.

(b)

$$x = \begin{bmatrix} 7 & 1 \\ 1 & 6 \end{bmatrix}^{-1} \begin{bmatrix} 8 \\ 7 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \tag{9}$$

(c)

$$\hat{b} = A(A^TA)^{-1}A^Tb = [3, 1, 2, -1]^T. \tag{10}$$

(d)

Let

$$A^Tx = 0. \tag{11}$$

Solving the system,

$$x = s[3, 1, -5, 0]^T + t[1, 2, 0, 5]^T \quad s, t \in \mathbb{R}. \tag{12}$$

Define

$$B := \begin{bmatrix} 3 & 1 \\ 1 & 2 \\ -5 & 0 \\ 0 & 5 \end{bmatrix}. \tag{13}$$

Then $\mathrm{Col}\, B = \mathrm{Null}\, A^T$, and

$$\tilde{b} = B(B^TB)^{-1}B^Tb = [1, 0, -2, -1]^T. \tag{14}$$

3

(a)

$$[p_1\ p_2\ p_3\ p_4] \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{bmatrix} = \begin{bmatrix} 8 & 2 & 4 \\ 2 & 2 & 2 \\ 4 & 2 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{15}$$

Hence $\det(\cdot) = \det(I_3) = 1 \neq 0$, implying $(\cdot)$ is invertible.

(b)

$$G = AP \implies GP^T = APP^T. \tag{16}$$

From (a) we know PP^T is invertible, thus (16) implies

$$A = GP^T(PP^T)^{-1} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{17}$$

4

$$\begin{aligned} < \sin(x), \cos(x) > &= \int_0^{\pi} \sin(x)\cos(x)dx \\ &= \frac{1}{2}\int_0^{\pi} \sin(2x)dx \\ &= -\cos(2x)|_0^{\pi} \\ &= 0. \end{aligned}$$

5

(a)

$$\{[1,0]^T, [0,1]^T\} \tag{18}$$

(b)

$$\{[1,0,0,0]^T, [0,1,0,0]^T, [0,0,1,0]^T, [0,0,0,1]^T\} \tag{19}$$

6

(a)

$$\begin{aligned} \frac{2}{\pi}\int_0^{\frac{\pi}{2}} r(t)\sqrt{2}\cos(1000t)\,dt &= \frac{4}{\pi}\int_0^{\frac{\pi}{2}} A\cos^2(1000t) + B\sin(1000t)\cos(1000t)\,dt \\ &= \frac{4}{\pi}\int_0^{\frac{\pi}{2}} A\cos^2(1000t)\,dt \\ &= \frac{2A}{\pi}\int_0^{\frac{\pi}{2}} 1 + \cos(2000t)\,dt \\ &= \frac{2A}{\pi}\cdot\frac{\pi}{2} \\ &= A. \end{aligned}$$

(b)

$$
\begin{aligned}
\frac{2}{\pi}\int_0^{\frac{\pi}{2}} r(t)\sqrt{2}\sin(1000t)\ dt &= \frac{4}{\pi}\int_0^{\frac{\pi}{2}} B\sin^2(1000t) + A\sin(1000t)\cos(1000t)\ dt \\
&= \frac{4}{\pi}\int_0^{\frac{\pi}{2}} B\sin^2(1000t)\ dt \\
&= \frac{2B}{\pi}\int_0^{\frac{\pi}{2}} 1-\cos(2000t)\ dt \\
&= \frac{2B}{\pi}\cdot\frac{\pi}{2} \\
&= B.
\end{aligned}
$$

7

Fourier coefficients:

$$
a_n = \frac{2}{P}\int_P f(x)\cdot\cos(2\pi x\frac{n}{P})\ dx \tag{20}
$$

$$
b_n = \frac{2}{P}\int_P f(x)\cdot\sin(2\pi x\frac{n}{P})\ dx. \tag{21}
$$

(a)

$P = 2\pi.$

$$
\begin{aligned}
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi}\cos(x)\cdot\cos(nx)\ dx \\
&= \begin{cases} x/2+\sin(2x)|_{-\pi}^{\pi} = 1, & n = 1 \\ \frac{2n\sin(\pi n)}{\pi(n^2-1)}\Big|_{\pi}^{-\pi} = 0, & n \neq 1. \end{cases}
\end{aligned}
$$

$$
\begin{aligned}
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi}\cos(x)\cdot\sin(nx)\ dx \\
&= 0.
\end{aligned}
$$

Hence the Fourier Series for $f(x) = \cos(x)$ is simply $f(x)$.

(b)

$P = 2\pi.$

$$
\begin{aligned}
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi}[2\cos(x)+4\cos(2x)]\cdot\cos(nx)\ dx \\
&= \begin{cases} 2, & n = 1 \\ 4, & n = 2 \\ 0, & \text{otherwise.} \end{cases}
\end{aligned}
$$

$$
\begin{aligned}
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi}[2\cos(x)+4\cos(2x)]\cdot\sin(nx)\ dx \\
&= 0.
\end{aligned}
$$

The Fourier Series for $f(x) = 2\cos(x)+4\cos(2x)$ is given by $f(x)$.

(c)

$P = 2\pi.$

$$
\begin{aligned}
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cdot\cos(nx)\,dx \\
&= \frac{A}{\pi}\int_{0}^{\pi} \cos(nx)\,dx \\
&= \begin{cases} A, & n=0 \\ \frac{A}{\pi n}\sin(nx)\Big|_0^\pi = 0, & n\neq 0. \end{cases}
\end{aligned}
$$

$$
\begin{aligned}
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cdot\sin(nx)\,dx \\
&= \frac{A}{\pi}\int_{0}^{\pi} \sin(nx)\,dx \\
&= \begin{cases} 0, & n=0 \\ \frac{A}{\pi n}[1-\cos(\pi n)], & n\neq 0. \end{cases} \\
&= \begin{cases} 0, & n \text{ is even} \\ \frac{2A}{\pi n}, & n \text{ is odd.} \end{cases}
\end{aligned}
$$

The Fourier Series for $f(x)$ is given by

$$
\frac{A}{2} + \sum_{n=1}^{\infty} b_n \sin(nx) \tag{22}
$$

where

$$
b_n = \begin{cases} 0, & n \text{ is even} \\ \frac{2A}{\pi n}, & n \text{ is odd.} \end{cases} \tag{23}
$$

8

(a)

$$
\begin{aligned}
F[n] &= \sum_{k=0}^{5} f[k] e^{-\frac{i\pi}{3}kn}, n = 0:5 \\
&= [0,0,0,6,0,0].
\end{aligned}
$$

The transformation matrix

$$
\begin{aligned}
\mathbf{F} &= [e^{-\frac{i\pi}{3}j\cdot k}],\ j,k = 0:5 \\
&= \begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 \\
1 & e^{-i\pi/3} & e^{-2i\pi/3} & e^{-3i\pi/3} & e^{-4i\pi/3} & e^{-5i\pi/3} \\
1 & e^{-2i\pi/3} & e^{-4i\pi/3} & e^{-6i\pi/3} & e^{-8i\pi/3} & e^{-10i\pi/3} \\
1 & e^{-3i\pi/3} & e^{-6i\pi/3} & e^{-9i\pi/3} & e^{-12i\pi/3} & e^{-15i\pi/3} \\
1 & e^{-4i\pi/3} & e^{-8i\pi/3} & e^{-12i\pi/3} & e^{-16i\pi/3} & e^{-20i\pi/3} \\
1 & e^{-5i\pi/3} & e^{-10i\pi/3} & e^{-15i\pi/3} & e^{-20i\pi/3} & e^{-25i\pi/3}
\end{bmatrix}.
\end{aligned}
$$

(b)

The inverse transformation matrix

$$
\mathbf{F}^{-1} = \frac{1}{6}\mathbf{F}^{*} = [e^{\frac{i\pi}{3}j\cdot k}],\ j,k = 0:5. \tag{24}
$$

Conducting on $F[k]$,

$$
\mathbf{F^{-1}}F[k]^T = [1,-1,1,-1,1,-1]^T. \tag{25}
$$

9

(a)

$$a_1 = [2, 1, 2]^T \tag{26}$$

$$a_2 = [1, 1, 1]^T. \tag{27}$$

Normalizing a_1,

$$n_1 := \frac{a_1}{||a_1||} = \frac{1}{3}[2, 1, 2]^T. \tag{28}$$

Then let

$$\begin{aligned} a_2' &= a_2 - < a_2, n_1 > n_1 \\ &= [1, 1, 1]^T - 5/9 \cdot [2, 1, 2]^T \\ &= [-1/9, 4/9, -1/9]^T. \end{aligned}$$

Normalizing a_2',

$$n_2 := \frac{a_2}{||a_2||} = \frac{1}{\sqrt{18}}[-1, 4, -1]^T. \tag{29}$$

Then $B := \{n_1, n_2\}$ is an orthonormal basis for Col A.

(b)

$$A = \underbrace{\begin{bmatrix} 2/3 & -1/\sqrt{18} \\ 1/3 & 4/\sqrt{18} \\ 2/3 & -1/\sqrt{18} \end{bmatrix}}_{Q} \underbrace{\begin{bmatrix} 3 & 5/3 \\ 0 & \sqrt{2}/3 \end{bmatrix}}_{R}. \tag{30}$$

(c)

Rewriting normal equation $A^T Ax = A^T b$ using (30),

$$R^T (Q^T Q)Rx = R^T Rx = R^T Q^T b. \tag{31}$$

Hence $Rx = Q^T b$,

$$x = R^{-1} Q^T b = [9, -3]^T. \tag{32}$$

10

(a)

$$h_1 = [0.3, 0.4]^T \tag{33}$$

$$h_2 = [0.8, 0.6]^T \tag{34}$$

First normalizing h_1,

$$n_1 := \frac{h_1}{||h_1||} = \frac{1}{5}[3,4]^T. \tag{35}$$

Let

$$\begin{aligned} h_2' &:= h_2 - < h_2, n_1 > n_1 \\ &= [0.8, 0.6]^T - \frac{4.8}{25}[3,4]^T \\ &= [0.224, -0.168]. \end{aligned}$$

Normalizing h_2',

$$n_2 := \frac{h_2'}{||h_2'||} = \frac{1}{5}[4,-3]^T. \tag{36}$$

Hence $B := \{n_1, n_2\}$ is an orthonormal basis for Col H.

(b)

$$H = \underbrace{\begin{bmatrix} 3/5 & 4/5 \\ 4/5 & -3/5 \end{bmatrix}}_{Q} \underbrace{\begin{bmatrix} 1/2 & 24/25 \\ 0 & 7/25 \end{bmatrix}}_{R}. \tag{37}$$

(c)

Rewriting the equation $y = Hx + n$ using (37),

$$[x_1, x_2]^T = R^{-1}Q^T(y - n) = [1, -1]^T. \tag{43}$$

11

(a)

$\det(AB) = \det(A) \cdot \det(B) = 12.$

(b)

$\det(5A) = 5^3 \det(A) = -375.$

(c)

$\det(B^T) = \det(B) = -4.$

(d)

$\det(A^{-1}) = \det^{-1}(A) = -\frac{1}{3}.$

(e)

$\det(A^3) = \det^3(A) = -27.$

12

(a)

$\det(A) = 3C_{11} + 1C_{12} + (-2)C_{13} = 0 + 6 + 0 = 6.$

(b)

$\det(A^4) = \det^4(A) = 6^4 = 1296.$

(c)

$$\begin{aligned}\operatorname{adj}(A) &= [c_{ij}]^T \\ &= \begin{bmatrix} 0 & 6 & 0 \\ -6 & 28 & 5 \\ 0 & 2 & 1 \end{bmatrix}^T \\ &= \begin{bmatrix} 0 & -6 & 0 \\ 6 & 28 & 2 \\ 0 & 5 & 1 \end{bmatrix}.\end{aligned}$$

(d)

$$x = A^{-1}[16, -2, -8]^T = [2, 4, -3]^T. \tag{39}$$

(e)

By Cramer's Rule,

$$x = \frac{\operatorname{adj}(A)}{\det(A)}[16, -2, -8]^T = [2, 4, -3]^T. \tag{40}$$

13

```julia
julia> using LinearAlgebra
julia> A, B = rand(5,5), rand(5,5);
```

(i)

```julia
julia> det(A*B)-det(A)det(B)
1.8431436932253575e-18
```

(ii)

```julia
julia> det(A)-det(A')
-3.469446951953614e-18
```

(iii)

```
julia> C = A;

julia> for i = 1:5
           C[2,i],C[4,i]=C[4,i],C[2,i]
           end

julia> det(C)-det(A)
0.0
```

14

```
julia> using LinearAlgebra

julia> A = rand(4,4); L,U,p = lu(A);P = zeros(4,4);

julia> for i = 1:4
           P[i,p[i]] = 1
           end
```

(i)

```
julia> det(A)
-0.16431584484935427

julia> det(P)
-1.0

julia> det(U)
0.16431584484935427

julia> det(L)
1.0
```

(ii)

$$\det(P) \cdot \det(A) = \det(U). \tag{41}$$

(iii)

```
julia> A[1,1]=A[2,1]=0; L,U,p = lu(A);P = zeros(4,4);

julia> for i = 1:4
           P[i,p[i]]=1
           end

julia> det(A)
-0.19272624085598167

julia> det(P)
1.0
```

```
julia> det(U)
-0.19272624085598167

julia> det(L)
1.0
```

15

```
julia> using LinearAlgebra

julia> A = rand(4,4); b = rand(4,1);

julia> B(j) = hcat([i==j ? b : A[:,i] for i = 1:4]...)
B (generic function with 1 method)

julia> x(j) = det(B(j)) / det(A)
x (generic function with 1 method)

julia> x_julia = A\b; x_cramer = [x(i) for i = 1:4];

julia> norm(x_julia-x_cramer)
3.3766115072321297e-16
```

16

```
julia> using LinearAlgebra

julia> n = 4; A = rand(n,n); val,vect = eigen(A);
```

(i)

```
julia> prod(val)-det(A)
3.122502256758253e-17
```

(ii)

```
julia> sum(val)-tr(A)
-2.4424906541753444e-15
```

(iii)

```
julia> A=Symmetric(A);

julia> val,B=eigen(A);

julia> C = inv(B)*A*B
4×4 Array{Float64,2}:
 -0.392068      1.03273e-16   1.49643e-16   3.13481e-16
  6.75254e-17  -0.00776449    2.61887e-16  -4.62468e-16
  1.54665e-16   4.0726e-16    0.307035     -2.15576e-16
  1.47826e-17  -6.11226e-16  -4.63173e-16   1.91908

julia> norm(inv(B)*A*B - diagm(0 => val))
1.9392390827813702e-15
```

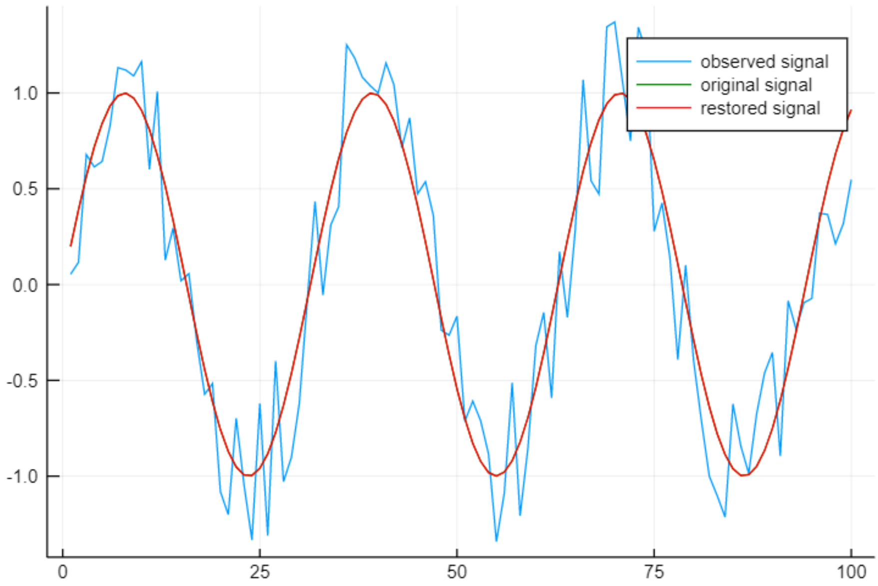
observation.

$$B^{-1}AB = \mathrm{diag}(\lambda_1, \ldots \lambda_n) \tag{42}$$

where λ_i is the eigenvalue of A corresponding to the eigenvector b_i, also being the i-th column of B.

17

```
julia> using Plots, FFTW

julia> n = 100; # number of samples

julia> N = rand(n) .- .5; # noise

julia> X = sin.((1:n)*.2); # original signal

julia> Y = N + X; # observed signal

julia> H_f = fft(X) ./ (fft(X) .+ fft(N)) # weiner filter
in the freq. domain
julia> Y_f = fft(Y);

julia> E = real(ifft(Y_f .* H_f)) # restored signal

julia> plot(Y,label="observed signal");

julia> plot!(X,label="original signal",color=:green);

julia> plot!(E,label="restored signal",color=:red)
```


observed signal
original signal
restored signal
1.0
0.5
0.0
-0.5
-1.0
0
25
50
75
100