CSC3001 Practice 4

2

Base case. $1^3 = 1 = 1^2(1+1)^2/4$.

Induction step. Assume $\sum_{i=1}^{k} i^3 = k^2(k+1)^2/4$. Then $\sum_{i=1}^{k+1} = k^2(k+1)^2/4 + k + 1 = (k+1)^2(k+2)^2/4$. $\square$

3

Base case. $\frac{1}{1} = 1 \le 1 = 2 - \frac{1}{1}$.

Induction step. Assume $\sum_{i=1}^{k} 1/i^2 \le 2 - 1/k$. Then

$$\sum_{i=1}^{k+1} \frac{1}{i^2} \le 2 - \frac{1}{k} + \frac{1}{(k+1)^2} = 2 - \frac{(k^2+k+1)/(k^2+k)}{k+1} \le 2 - \frac{1}{k+1}. \quad \square$$

4

Base case. $1 - 1/2 = 1/2$.

Induction step. Assume $\prod_{i=2}^{k}(1-1/i) = 1/k$. Then $\prod_{i=2}^{k+1}(1-1/i) = 1/k \cdot k/(k+1) = 1/(k+1)$. $\square$

7

Base case. $\binom{0}{0} = 1 = F_{0+1}$.

(Strong) Induction step. Assume $\sum_{i=0}^{k} \binom{k-i}{i} = F_{k+1}$ for all $k \le n$. Then

$$\begin{aligned}
\sum_{i=0}^{n+1} \binom{n+1-i}{i} &= \binom{n+1}{0} + \sum_{i=1}^{n} \binom{n+1-i}{i} \\
&= \binom{n}{0} + \sum_{i=1}^{n} \binom{n+1-i}{i} \\
&= \sum_{i=1}^{n} \binom{n-i}{i-1} + \sum_{i=0}^{n} \binom{n-i}{i} \\
&= \sum_{i=0}^{n-1} \binom{n-1-i}{i} + \sum_{i=0}^{n} \binom{n-i}{i} \\
&= F_{(n-1)+1} + F_{(n)+1} \\
&= F_{(n+1)+1}. \quad \square
\end{aligned}$$

9

We prove a stronger result: $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}^n = \begin{bmatrix} 1 & 2n \\ 0 & 1 \end{bmatrix}$ for all positive integer n.

Base case. $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}^1 = \begin{bmatrix} 1 & 2 \cdot 1 \\ 0 & 1 \end{bmatrix}$

Induction step. Assume $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}^k = \begin{bmatrix} 1 & 2k \\ 0 & 1 \end{bmatrix}$. Then

$$\begin{aligned} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}^{k+1} &= \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}^k \\ &= \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2k \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 2(k+1) \\ 0 & 1 \end{bmatrix}. \quad \square \end{aligned}$$