MAT 3007 – Optimization

Exercise Sheet Nr.: 1

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In the creation of this solution sheet, I worked together with:

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For correction:

Exercise								$\sum$
Grading								

MAT3007 Assignment 1

A1.1

(a)

Define $n :=$ # floors aboveground, $m :=$ # floors underground, and $u :=$ uniform height of floors. The optimization formulation is as follows:

$$\begin{aligned}
\min_{l,w,h,d,n,m,u} \quad & lwd \\
\text{subject to} \quad & w \le l \le 2w \\
& un = h \\
& um = d \\
& l \le 40 \\
& l \le h \\
& 1/10 \le m/(n+m) \le 1/4 \\
& lw(n+m) \ge 10000 \\
& lw + 2h(l+w) \le 5000 \\
& l, w, h, u \in \mathbb{R}^+ \\
& n, m \in \mathbb{N}^+
\end{aligned}$$

(b)

A feasible point:

$$\begin{bmatrix} l & w & h & d & n & m & u \end{bmatrix} = \begin{bmatrix} 30 & 30 & 30 & 10 & 30 & 10 & 1 \end{bmatrix}$$

A1.2

(a)

Let x, y be the # product I,II respectively. Then the LP formulation is:

$$\begin{aligned} \max_{x,y} \quad & 8x + 7y \\ \text{subject to} \quad & x/3 + y/4 \leq 100 \\ & x/5 + y/4 \leq 70 \\ & x, y \geq 0 \end{aligned}$$

(b)

$$\begin{aligned} \min_{x,y,s_1,s_2} \quad & -8x - 7y \\ \text{subject to} \quad & 4x + 3y + s_1 = 1200 \\ & 4x + 5y + s_2 = 1400 \\ & x, y, s_1, s_2 \geq 0 \end{aligned}$$

(c)

Define $z :=$ # overtime assembly hours. LP formulation:

$$\begin{aligned} \max_{x,y,z} \quad & 8x + 7y - 7z \\ \text{subject to} \quad & x/3 + y/4 \leq 100 + z \\ & x/5 + y/4 \leq 70 \\ & z \leq 60 \\ & x, y, z \geq 0 \end{aligned}$$

(d)

```
>> cvx_begin
    variable x
    variable y
    maximize (8 * x + 7 * y)
    subject to
        x/3 + y/4 <= 100
        x/5 + y/4 <= 70
        x >= 0
        y >= 0
cvx_end
```

yielding

$$\text{max\_profit}(x^*, y^*) = 2500$$

with

$$\begin{bmatrix} x^* & y^* \end{bmatrix} = \begin{bmatrix} 255 & 100 \end{bmatrix}$$

A1.3

(a)

Let $A = [a_{ij}] \geq 0$ be the actual flow and $C = [c_{ij}] \geq 0$ the capacity, both from v_i to v_j. Then the LP formulation of the problem is:

$$\begin{aligned} \max_{A} \quad & \sum_i a_{in} \\ \text{subject to} \quad & 0 \leq A \leq C \\ & \sum_j a_{ji} = \sum_j a_{ij} \qquad \forall i, j \in \{1, \ldots, n\} \end{aligned}$$

*Note that $c_{ij} := 0$ if $(i, j) \notin E$.

(b)

```
% capacity values
C = [0    11     8     0     0     0
     0     0    10    12     0     0
     0     1     0     0    11     0
     0     0     4     0     0    15
     0     0     0     7     0     4
     0     0     0     0     0     0];

cvx_begin
  variable A(6,6)
  maximize (ones(1,6) * A(:,6)) % 6th col sum
  subject to
    for i = 1 : 6*6
      % flow capacity restriction
      0 <= A(i) <= C(i)
    end
    for i = 2 : 6-1
      % flow conservation, ith row sum == ith col sum
      A(i,:) * ones(6,1) == ones(1,6) * A(:,i)
    end
cvx_end
```

yielding

$$\text{max\_flow}(A^*) = 19$$

with

$$A^* = \begin{bmatrix} 0 & 11 & 8 & 0 & 0 & 0 \\ 0 & 0 & 1.5947 & 10.2690 & 0 & 0 \\ 0 & 0.8637 & 0 & 0 & 9.3653 & 0 \\ 0 & 0 & 0.6343 & 0 & 0 & 15 \\ 0 & 0 & 0 & 5.3653 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

A1.4

(a)

$$\min_{x\in\mathbb{R}^n} c^\top x \quad \text{s.t.} \quad -\delta \le Ax-b \le \delta$$

To show equivalence, it suffices to show equality of two feasible regions (since objective functions are the same).

Say we have $x \in \mathbb{R}^n$ with $-\delta \le Ax-b \le \delta,\ x \ge 0$. Then

$$\begin{aligned}
& -\delta \le (Ax-b)_i \le \delta,\ x \ge 0 \quad \forall i \in \{1,\cdots,m\} \\
\implies & |(Ax-b)_i| \le \delta, x \ge 0 \quad \forall i \in \{1,\cdots,m\} \\
\implies & \max_{1\le i\le m} |(Ax-b)_i| \le \delta, x \ge 0 \\
\implies & ||Ax-b||_\infty \le \delta, x \ge 0
\end{aligned}$$

Conversely, suppose $||Ax-b||_\infty \le \delta, x \ge 0$. We have

$$\begin{aligned}
& ||Ax-b||_\infty \le \delta, x \ge 0 \\
\implies & \max_{1\le i\le m} |(Ax-b)_i| \le \delta, x \ge 0 \\
\implies & |(Ax-b)_j| \le \max_{1\le i\le m} |(Ax-b)_i| \le \delta, x \ge 0 \quad \forall j \in \{1,\cdots,m\} \\
\implies & -\delta \le Ax-b \le \delta,\ x \ge 0
\end{aligned}$$

Therefore two feasible regions are indeed equal, hence the equivalence.

(b)

Let $x = [x_1 \quad x_2]^\top := [\#\ \text{salad A} \quad \#\ \text{salad B}]^\top$. The LP can be formulated in the standard form as follows:

$$\begin{aligned}
\max_x \quad & [10 \quad 20]\, x \\
\text{subject to} \quad & \begin{bmatrix} 1/4 & 1/2 \\ 1/8 & 1/4 \\ 5 & 1 \end{bmatrix} x = \begin{bmatrix} 25 \\ 10 \\ 120 \end{bmatrix} \\
& x \ge 0
\end{aligned}$$

(c)

Suppose x_0 is a solution to the constraint. Then

$$25 = [1/4 \quad 1/2]\, x_0 = 2\,[1/8 \quad 1/4]\, x_0 = 2 \cdot 10 = 20,$$

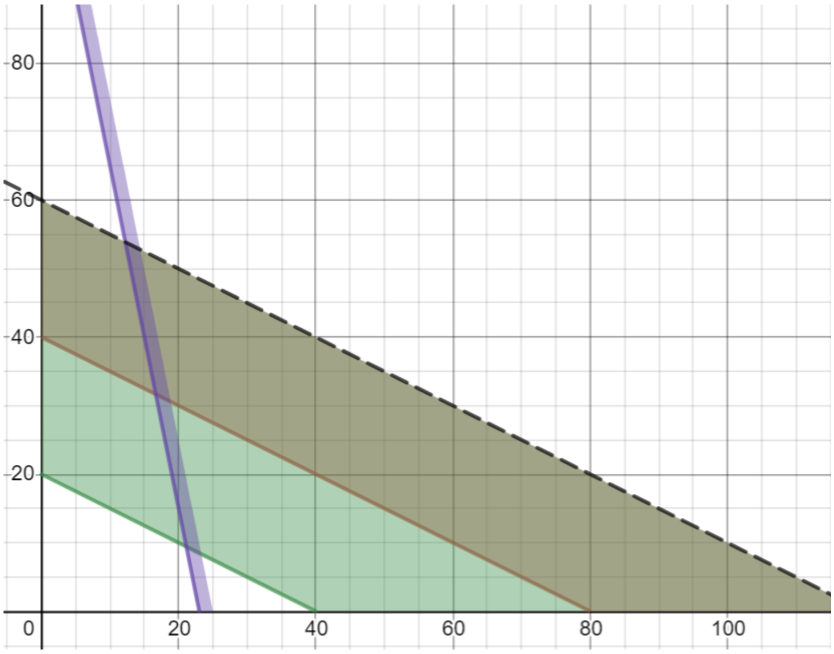
a contradiction. Hence the LP is infeasible.

(d)

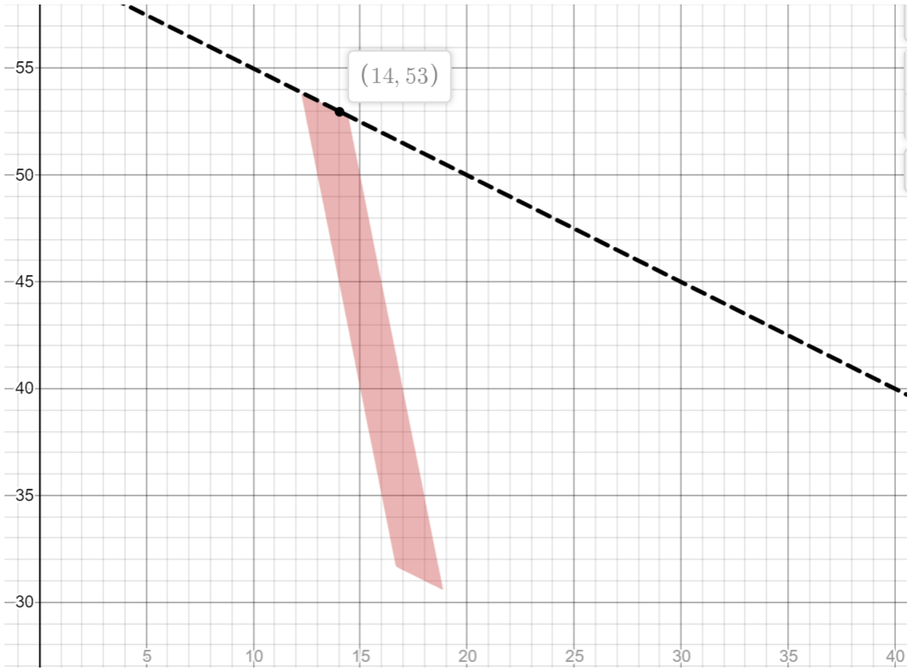
From (a) we have the equivalent formulation of the robust LP:

$$\begin{aligned}
\max_x \quad & & 10x_1 + 20x_2 \\
\text{subject to} \quad & & x \ge 0 \\
& \text{(mango)} & -5 \le x_1/4 + x_2/2 - 25 \le 5 \\
& \text{(pineapple)} & -5 \le x_1/8 + x_2/4 - 10 \le 5 \\
& \text{(strawberry)} & -5 \le x_1/8 + x_2/4 - 10 \le 5
\end{aligned}$$

The following graph shows the three constraints separately given $x \geq 0$. (horizontal: x_1, vertical: x_2)



The graph below shows the feasible set in red (overlap of three regions above)



where the black dotted line is the contour of the profit function at optimum, $10x_1 + 20x_2 = 1200$. We see there's an infinite number of solutions (the entire upper edge of the feasible region) yielding the optimal profit 1200 RMB. The active constraints are:

$$
\begin{aligned}
x_1/4 + x_2/2 - 25 \leq 5 \\
x_1/8 + x_2/4 - 10 \leq 5
\end{aligned}
$$

Nevertheless, $(x_1^*, x_2^*) = (14, 53)$ is the only lattice point among the solutions and will be our final production plan. Following this plan, the amounts of fruits used are $\#$ mango $= x_1^*/4 + x_2^*/2 = 30$;
$\#$ pineapple $= x_1^*/8 + x_2^*/4 = 15$; $\#$ strawberry $= 5x_1^* + x_2^* = 123$.