Assignment 02

1

$$(AB)_{m\times p}C_{p\times q} = D_{m\times q}$$
$$A_{m\times n}(BC)_{n\times q} = D'_{m\times q}$$

Moreover,

$$D_{ij} = \sum_{k=1}^{p}(AB)_{ik}C_{kj} = \sum_{k=1}^{p}\sum_{q=1}^{n}A_{iq}B_{qk}C_{kj} = \sum_{q=1}^{n}A_{iq}\sum_{k=1}^{p}B_{qk}C_{kj} = \sum_{q=1}^{n}A_{iq}(BC)_{qj} = D'_{ij}$$

Hence

$$D = D'. \qquad \square$$

2

$$[A(B+C)]_{ij} = \sum_{k=1}^{n}A_{ik}(B+C)_{kj} = \sum_{k=1}^{n}A_{ik}(B_{kj}+C_{kj}) = \sum_{k=1}^{n}A_{ik}B_{kj} + \sum_{k=1}^{n}A_{ik}C_{kj} = (AB)_{ij} + (AC)_{ij} = (AB+AC)_{ij}.$$

Therefore

$$A(B+C) = AB + AC. \qquad \square$$

3

(a)

$$A^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$A^4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(b)

$$A^k = \begin{cases} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} & \text{if } k \text{ is odd,} \\ \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} & \text{if } k \text{ is even.} \end{cases}$$

4

Suppose $A_{n\times n}, B_{n\times n}$.
If A, B both invertible, then A^{-1}, B^{-1} both exist, and that

$$(B^{-1}A^{-1})AB = AB(B^{-1}A^{-1}) = I.$$

Therefore

$$A, B \text{ both invertible} \implies AB \text{ invertible.}$$

On the other hand, if AB is invertible, suppose first B is not invertible. Then $\exists x_B \neq 0$ s.t.

$$Bx_B = 0.$$

Choosing $x = x_B \neq 0$,

$$(AB)x = A(Bx) = A \cdot 0 = 0,$$

which implies AB is not invertible, a contradiction. Hence B is invertible.
Next, suppose A is not invertible. Then $\exists x_A \neq 0$ s.t.

$$Ax_A = 0.$$

Since B is invertible, choosing $x = B^{-1}x^A \neq 0$,

$$(AB)x = A(BB^{-1})x_A = Ax_A = 0,$$

a contradiction. Hence A must also be invertible. We proved

$$AB \text{ invertible} \iff A, B \text{ both invertible.}$$

Then, if $M = A_{n\times n}B_{n\times n}C_{n\times n}$,

$$M \text{ invertible} \iff A(BC) \text{ invertible} \iff A, BC \text{ both invertible} \iff A, B, C \text{ all invertible.} \qquad \square$$

5

(a)

$$\begin{aligned} 1 + ab &= 3 \\ 4 + a &= 5 \end{aligned} \implies a = 1, b = 2.$$

(b)

$$\begin{aligned} 1 + ab &= 1 \\ a + b &= b \end{aligned} \implies a = 0, b \text{ is free.}$$

6

(i)

$$\left[\begin{array}{c|cc} 1 & 2 & 3 \\ \hline 4 & 5 & 6 \\ 7 & 8 & 9 \end{array}\right] \quad \left[\begin{array}{c|cc} 1 & 2 & 3 \\ 4 & 5 & 6 \\ \hline 7 & 8 & 9 \end{array}\right] \quad \left[\begin{array}{cc|c} 1 & 2 & 3 \\ \hline 4 & 5 & 6 \\ 7 & 8 & 9 \end{array}\right] \quad \left[\begin{array}{cc|c} 1 & 2 & 3 \\ 4 & 5 & 6 \\ \hline 7 & 8 & 9 \end{array}\right]$$

(ii)

$$\left[\begin{array}{c|c|c} 1 & 0 & 5 \\ 4 & 6 & 1 \\ 7 & 9 & 3 \end{array}\right]$$

(iii)

$$\left[\begin{array}{ccc} 1 & 0 & 5 \\ \hline 4 & 6 & 1 \\ \hline 7 & 9 & 3 \end{array}\right]$$

7

(a)

True. Suppose the i-th row of $A_{4\times 4}$ is of zeros. Then $\vec{y} = [y_1\ y_2\ y_3\ y_4]$ with $y_i \neq 0$ and $y_j = 0\ \forall j \neq i$ satisfies

$$\vec{y}A = \sum_{n=1}^{4} y_n \vec{a}_n = \vec{0},$$

where $\vec{a}_n$ is the n-th row of A. Hence A is not invertible.

(b)

True. By definition if A is invertible, $\exists A^{-1}$ s.t.

$$AA^{-1} = A^{-1}A = I,$$

which, by definition implies $A = (A^{-1})^{-1}$. Hence A^{-1} is invertible.

(c)

True. Suppose A is not invertible, then $\exists x \neq 0$ s.t.

$$Ax = 0.$$

Taking the transpose,

$$\vec{y}A^T = \vec{0}$$

where $\vec{y} = x^T \neq \vec{0}$, which implies A^T is not invertible, a contradiction.

8

$$\begin{bmatrix} 2 & c & c \\ c & c & c \\ 8 & 7 & c \end{bmatrix} \xrightarrow{E_{12}E_{31}} \begin{bmatrix} 2-c & 0 & 0 \\ c & c & c \\ 0 & 7-4c & -3c \end{bmatrix} \xrightarrow[c \notin \{0,2\}]{E_{32}E_{21}} \begin{bmatrix} 2-c & 0 & 0 \\ 0 & c & c \\ 0 & 0 & c-7 \end{bmatrix}$$

If and only if $c \in \{0, 2, 7\}$, $\exists$ row of the reduced matrix consisting solely of zeros, and the matrix is not invertible.

9

$$\begin{bmatrix} 1 & \lambda & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{E_{23}E_{21}} \begin{bmatrix} 1 & \lambda & 0 \\ 0 & 1-\lambda & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

If and only if $\lambda = 1, \exists$ row of the reduced matrix consisting solely of zeros, and the matrix is not invertible. Hence the matrix is invertible if and only if $\lambda \neq 1$.

10

(a)

$$\left[\begin{array}{ccc|c} 1 & 2 & 2 & \\ 3 & 7 & 9 & I \\ -1 & -4 & -7 & \end{array}\right] \xrightarrow{E_{32}E_{31}E_{21}} \left[\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & 1 & 3 & -3 & 1 & 0 \\ 0 & 0 & 1 & -5 & 2 & 1 \end{array}\right] \xrightarrow{E_{12}E_{13}E_{23}} \left[\begin{array}{c|ccc} & -13 & 6 & 4 \\ I & 12 & -5 & -3 \\ & -5 & 2 & 1 \end{array}\right]$$

(b)

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & \\ 3 & 5 & 4 & I \\ 3 & 6 & 5 & \end{array}\right] \xrightarrow{E_{32}E_{31}E_{21}} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 1 & -3 & 1 & 0 \\ 0 & 0 & 1/2 & 3/2 & -3/2 & 1 \end{array}\right] \xrightarrow{E_{12}E_{13}E_{23}} \left[\begin{array}{c|ccc} & 1 & 1 & -1 \\ I & -3 & 2 & -1 \\ & 3 & -3 & 2 \end{array}\right]$$

11

The system is non-singular if and only if $a \neq 0, a \neq b, b \neq c, c \neq d$.

$$A \xrightarrow{L} \begin{bmatrix} a & a & a & a \\ & b-a & b-a & b-a \\ & & c-b & c-b \\ 0 & & & d-c \end{bmatrix}$$

where

$$L = \begin{bmatrix} 1 & & & 0 \\ 0 & 1 & & \\ 0 & 0 & 1 & \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & & & 0 \\ 0 & 1 & & \\ 0 & -1 & 1 & \\ 0 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & & & 0 \\ -1 & 1 & & \\ -1 & 0 & 1 & \\ -1 & 0 & 0 & 1 \end{bmatrix}.$$

Hence

$$A = \underbrace{\begin{bmatrix} 1 & & & 0 \\ 1 & 1 & & \\ 1 & 1 & 1 & \\ 1 & 1 & 1 & 1 \end{bmatrix}}_{\tilde{L} = L^{-1}} \underbrace{\begin{bmatrix} a & a & a & a \\ & b-a & b-a & b-a \\ & & c-b & c-b \\ 0 & & & d-c \end{bmatrix}}_{U}.$$

12

$$L = \begin{bmatrix} 1 & & & 0 \\ 1 & 1 & & \\ -3 & 3 & 1 & \\ 5 & -1 & -3 & 1 \end{bmatrix}, U = \begin{bmatrix} 3 & 3 & 1 & -4 \\ & 2 & -2 & 1 \\ & & 5 & 1 \\ 0 & & & 3 \end{bmatrix}.$$

Let $y = Ux, Ly = b$.
First solve for y using forward substitution.

$$\left[\begin{array}{cccc|c} 1 & & & 0 & 5 \\ 1 & 1 & & & 5 \\ -3 & 3 & 1 & & -5 \\ 5 & -1 & -3 & 1 & -5 \end{array}\right] \implies y = \begin{bmatrix} 5 \\ 0 \\ 10 \\ 0 \end{bmatrix}.$$

Then solve for x using backward substitution.

$$\left[\begin{array}{cccc|c} 3 & 3 & 1 & -4 & 5 \\ & 2 & -2 & 1 & 0 \\ & & 5 & 1 & 10 \\ 0 & & & 3 & 0 \end{array}\right] \implies x = \begin{bmatrix} -1 \\ 2 \\ 2 \\ 0 \end{bmatrix}.$$

13

(a)

$$\begin{aligned} (I - uv^T)(I + \frac{uv^T}{1 - v^T u}) &= I + \frac{uv^T}{1 - v^T u} - uv^T - \frac{(uv^T)^2}{1 - v^T u} \\ &= I + \frac{\cancel{uv^T} - (uv^T)^2 \cancel{- uv^T} + v^T u \cdot uv^T}{1 - v^T u} \\ &= I + \frac{v^T u \cdot uv^T - (uv^T)^2}{1 - v^T u} \end{aligned} \tag{$*$}$$

Note that $(v^T u \cdot uv^T)_{ij} = \sum_{x=1}^n v_x u_x u_i v_j = \sum_{x=1}^n (uv^T)_{ix}(uv^T)_{xj} = [(uv^T)^2]_{ij}$.
Hence

$$v^T u \cdot uv^T - (uv^T)^2 = O, \tag{$**$}$$

and

$$(*) = I \implies M^{-1} = I + \frac{uv^T}{1 - v^T u}.$$

(b)

$$\begin{aligned} (A - uv^T)(A^{-1} + \frac{A^{-1}uv^T A^{-1}}{1 - v^T A^{-1} u}) &= I + \frac{uv^T A^{-1}}{1 - v^T A^{-1} u} - uv^T A^{-1} - \frac{(uv^T A^{-1})^2}{1 - v^T A^{-1} u} \\ &= I + \frac{\tilde{v}^T u \cdot u\tilde{v}^T - (u\tilde{v}^T)^2}{1 - \tilde{v}^T u} \end{aligned}$$

where $\tilde{v} = (A^{-1})^T v$. Using $(**)$, a similar argument to (b) implies that

$$M^{-1} = A^{-1} + \frac{A^{-1}uv^T A^{-1}}{1 - v^T A^{-1} u}.$$

(c)

$$\begin{aligned} (I_n - UV)[I_n + U(I_m - VU)^{-1}V] &= I_n - UV + U(I_m - VU)^{-1}V - UVU(I_m - VU)^{-1}V \\ &= I_n + U[(I_m - VU)^{-1} - VU(I_m - VU)^{-1} - I_m]V \\ &= I_n + U[(I_m - VU)(I_m - VU)^{-1} - I_m]V \\ &= I_n + U[I_m - I_m]V \\ &= I_n. \end{aligned}$$

Hence

$$M^{-1} = I_n + U(I_m - VU)^{-1}V.$$

(d)

$$\begin{aligned}
&(A - UW^{-1}V)[A^{-1} + A^{-1}U(W - VA^{-1}U)^{-1}VA^{-1}] \\
= &I_n - UW^{-1}VA^{-1} + U(W - VA^{-1}U)^{-1}VA^{-1} - UW^{-1}VA^{-1}U(W - VA^{-1}U)^{-1}VA^{-1} \\
= &I_n + U[(W - VA^{-1}U)^{-1} - W^{-1}VA^{-1}U(W - VA^{-1}U)^{-1} - W^{-1}]VA^{-1} \\
= &I_n + U[W^{-1}W(I_m - W^{-1}VA^{-1}U)(W - VA^{-1}U)^{-1} - W^{-1}]VA^{-1} \\
= &I_n + U[W^{-1} - W^{-1}]VA^{-1} \\
= &I_n.
\end{aligned}$$

Hence

$$M^{-1} = A^{-1} + A^{-1}U(W - VA^{-1}U)^{-1}VA^{-1}.$$

14

$(a), (c)$.

15

$$\begin{aligned}
(AB)^* &= [(AB)^T]^{-1} \\
&= (B^T A^T)^{-1} \\
&= (A^T)^{-1}(B^T)^{-1} \\
&= A^* B^*.
\end{aligned}$$

16

```
julia> using LinearAlgebra
```

(a)

```
julia> res_norm_1 = norm(A * x1 - b)
2.541731896800047e-14
```

(b)

```
julia> res_norm_2 = norm(A * x2 - b)
3.617300165427831e-14
```

17

(a)

```
julia> @time A\b
  1.722943 seconds (3.48 M allocations: 197.092 MiB, 3.80% gc time)
```

(b)

```
julia> @time L\b
  0.005796 seconds (5 allocations: 15.906 KiB)
```

(c)

The matrix L is lower triangular while entries of A are all non-zero.

Hypothesis: Julia is solving the system using LU-decomposition.

18

```
julia> using LinearAlgebra
```

(a)

```
julia> norm(A * (B * C) - (A * B) * C) # Associativity
2.493235635302362e-14
```

(b)

```
julia> norm(A * (B + C) - (A * B + A * C)) # Distributivity
7.32410687763558e-15
```

(c)

```
julia> norm(A * B - B * A) # Noncommutativity
7.827281228408319
```

(d)

```
julia> Ã = [A1 A2; A3 A4] # Blocks of A

julia> B̃ = [B1 B2; B3 B4] # Blocks of B

julia> norm(A * B - Ã * B̃) # Block Mult.
0.0
```

(e)

```
julia> norm(inv(A * B) - inv(B) * inv(A)) # Properties of Inverse
3.5961261080715424e-13
```

(19)

(a)

```
julia> L, U, p = lu(A)
LU{Float64,Array{Float64,2}}
L factor:
10×10 Array{Float64,2}:
 1.0        0.0        0.0        …   0.0        0.0        0.0
 0.467841   1.0        0.0            0.0        0.0        0.0
 0.812289  -0.909545   1.0            0.0        0.0        0.0
 ⋮                                ⋱
 0.236137   0.640994   0.0606518     -0.554904   1.0        0.0
 0.647666  -0.162007   0.104964      -0.639265   0.978578   1.0
U factor:
10×10 Array{Float64,2}:
 0.805707  0.706994  0.656959  …  0.571168  0.104836   0.683771
 0.0       0.606948  0.517501     0.118659  0.817633   0.35018
 0.0       0.0       0.745954     0.580221  1.41523    0.278695
 ⋮                             ⋱
 0.0       0.0       0.0          0.0       1.53085   -0.324748
 0.0       0.0       0.0          0.0       0.0        0.171293
```

(b)

i.

Checked.

ii.

```
julia> Ã = A[p,:] # Ã = PA

julia> norm(Ã - L * U)
5.933264665155888e-16
```

(c)

```
function ldu(A)
    L, U, p = lu(A)
    D = diagm(0 => diag(U))
    U = inv(D) * U
    return L, D, U, p
end
```

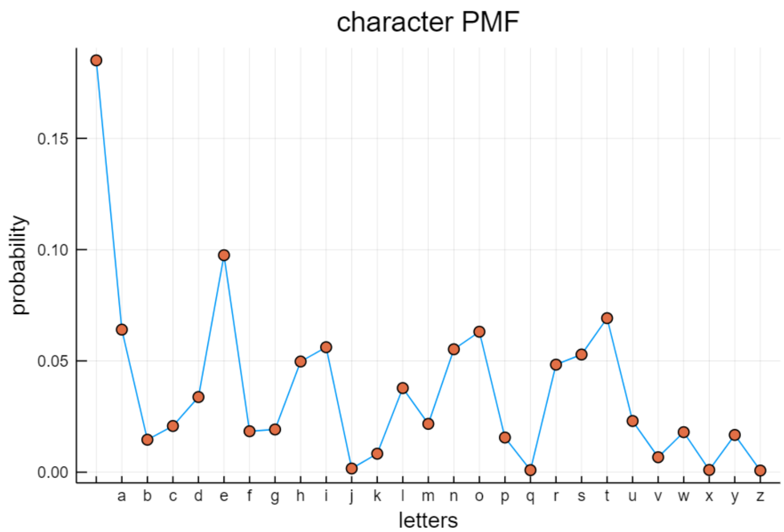
(d)

```
julia> L, D, U, p = ldu(A)

julia> norm(L * D * U - Ã)
8.266316859814843e-16
```

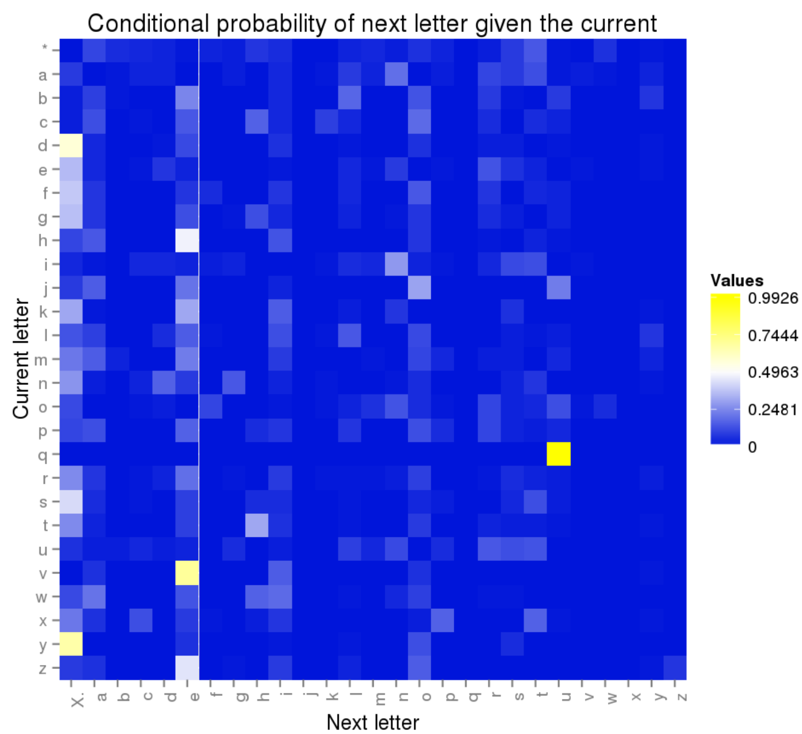
20

(a)



The space character has the highest probability. This is reasonable because the space constantly occurs between each word.

(b)



Some peaks in the conditional PMF:

- The letter "q" is almost always followed by the letter "u," as in words "**qu**een" and "ac**qu**ire."
- The letter "v" is often succeeded by the letter "e," as in words "fi**ve**" and "**ve**st."
- The letter "y" usually ends a word, followed by a space, as in words "da**y**," "keenl**y**," etc.

(c)

```
julia> randomSentence(ng, alphabet, 100, 0)
"s od keev ihdmluecu sr c eb b h u edrctilehia  shm pgetsfue  alcntod ett t daaysdfes oitecelinmkmfba"

julia> randomSentence(ng, alphabet, 100, 1)
" sizve t mes sevipere y phas f a dalonggeancororinus veary harvesllofouliny o tyod by rof alagamery "

julia> randomSentence(ng, alphabet, 100, 2)
"n topy ll ob saiddy yourawaidech askintentarieub the me mut what she we i fack has badodis ine of no"
```

Comments: The result is unsatisfactory. One main reason could be that the algorithm does not distinguish "characters" and "words." One way of improving the performance could be to include a dictionary of words allowed.