MAT 4003 Assignment 1

1

Let $S = \{s_1, s_2, \ldots s_{n+1}\}$ be an $(n+1)$-element subset of $\{1, 2, \ldots, 2n+1\}$, where $1 \le s_1 < s_2 < \ldots < s_{n+1} \le 2n+1$. If S contains two consecutive integers $s, s+1$, then $(s, s+1) = (s, 1) = 1$. If S does not contain two consecutive integers, then $1 \le s_1 \le s_2 - 2 \le \ldots \le s_{n+1} - 2n \le 1$. So $s_1 = 1$. $(s_1, s_k) = (1, s_k) = 1$. In either case, S contains a pair of relatively prime integers. $\square$

2

Using EEA:

i	r_i	q_i	x_i	y_i
-1	43	-	1	0
0	5	-	0	1
1	3	8	1	-8
2	2	1	-1	9
3	1	1	2	-17
4	0	-	-	-

$$43(2) + 5(-17) = (43, 5) = 1. \tag{1}$$

Hence a particular solution to the original equation is

$$x_0 = 500, y_0 = -4250. \tag{2}$$

And the complete solution is given by

$$\{(x, y)\} = \{(500 + 1250k, -4250 - 10750k) : k \in \mathbb{Z}\}. \tag{3}$$

3

Set $p := (a^m - 1, a^n - 1), d := (m, n), m = sd, n = td$. Then $(a^d - 1) | ((a^d)^s - 1) = a^m - 1$. Similarly $(a^d - 1) | (a^n - 1)$. Hence $(a^d - 1) | p$. Now by Bézout's lemma we may write $d = mx + ny$. WLOG assume $y \le 0 \le x$. Then $p | (a^{mx} - 1), (a^{-ny} - 1)$. It follows that $p | (a^{mx} - 1 - a^d(a^{-ny} - 1)) = a^d - 1$. Therefore $p = 2^d - 1$, or

$$(a^m - 1, a^n - 1) = a^{(m,n)} - 1. \tag{4}$$

This in turn shows the equivalence:

$$(a^m - 1) | (a^n - 1) \iff (a^m - 1, a^n - 1) = a^m - 1 = a^{(m,n)} - 1 \iff m | n. \tag{5}$$

4

Proceed by strong induction on n.

Clearly $F_3 = 2 > \phi$. Now suppose $\forall n$ with $3 \leq n \leq k$ we have $F_n > \phi^{n-2}$. Then $F_{k+1} = F_k + F_{k-1} > \phi^{k-1}(\phi^{-2} + \phi^{-1}) = \phi^{k-1}$. Therefore,

$$F_n > \phi^{n-2} \quad \forall n \geq 3. \square \tag{6}$$

5

Consider the Diophantine equation

$$ax + by = ab - a - b. \tag{7}$$

Note that $(x_0, y_0) = (b-1, -1)$ is a particular solution. Then by Theorem (1.12) the complete solution set of the equation is given by

$$\{(x, y)\} = \{((1-k)b - 1, ka - 1) : k \in \mathbb{Z}\}. \tag{8}$$

Now, for x, y to both be nonnegative integers, one must have $(1-k)b > 0$ and $ka > 0$. Equivalently $0 < k < 1$. Therefore x, y cannot both be nonnegative. That is, $ab - a - b$ is never a nonnegative integer linear combination of a and b. $\square$