CSC3001 Practice Exercise 3

Direct proof

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Denote the product by $\prod$. Any five consecutive integers $n+1, \ldots, n+5$ must contain one that's divisible by 5. Since, by division algorithm $n+5 = 5q + r$ with $0 \leq r < 5$. If $r = 0$ we are done. Otherwise $5q + r - k = 5q$ with $0 < k < 5;\ n + r = 5q$ with $1 \leq r \leq 4$. Similarly one must be divisible by 4, one by 3, one by 2. Therefore $3, 5 | \prod$. Since $(3, 5) = 1$ it follows that their product $15 | \prod$. Also note that if $4|a$, then $2|a$, then $2|a-2, a+2$. So two distinctive integers are divisible by 2 and by 4 respectively. Hence $8 | \prod$. But $(8, 15) = 1$. Therefore $120 | \prod$. $\square$

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If $n = 2k+1, n^2 - 1 = 4k(k+1)$. Note that either k or $k+1$ is even. Hence $8 | (n^2 - 1)$.

Proof by contrapositive

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Suppose n is composite. Then $n = ab$ with both $a, b > 1$. Then $2^n - 1 = (2^a)^b - 1 = (a-1)\sum_{i=0}^{b-1} 2^{ai}$. Hence $2^n - 1$ is composite. $\square$

Proof by contradiction

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Suppose $a^2 - 4b - 3 = 0$. Then $2|4|(a^2+1)$. Then $2|(a^2 - 1) = (a+1)(a-1)$. Therefore a is odd. Let $a = 2k+1$. From above $4|(a^2+1) = (4k^2 + 4k + 2)$, which implies $4|2$. Contradiction. $\square$

Proof by cases

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$$\max\{x, y\} + \min\{x, y\} = \begin{cases} x + y & \text{if } x \geq y \\ y + x & \text{if } x < y \end{cases} = x + y. \quad \square$$