Assignment 03

1

Proof.

$$AB = I \implies (AB)A = A \implies A(BA) = A \implies BA = I. \qquad \square$$

2

Assume $f, g \in V$. Then

$$\int_0^1 [cf(t)]^2 \, dt = c^2 \int_0^1 f^2(t) \, dt < \infty \implies cf(t) \in V.$$

Also,

$$\begin{aligned}
\int_0^1 [f(t) + g(t)]^2 \, dt &= \int_0^1 f^2(t) \, dt + \int_0^1 g^2(t) \, dt + 2 \int_0^1 f(t)g(t) \, dt \\
&\leq \int_0^1 f^2(t) \, dt + \int_0^1 g^2(t) \, dt + 2 \int_0^1 |f(t)g(t)| \, dt \\
&\leq \int_0^1 f^2(t) \, dt + \int_0^1 g^2(t) \, dt + \int_0^1 f^2(t) + g^2(t) \, dt \\
&< \infty.
\end{aligned}$$

Hence

$$f(t) + g(t) \in V.$$

V is a vector space.

3

(a)

Linearly independent.

(b)

Linearly dependent.

$$\vec{v_1} - 3\vec{v_2} + 2\vec{v_3} + \vec{v_4} = \vec{0}.$$

4

$$\left\{\begin{bmatrix}1\\0\\-2\\1\end{bmatrix},\begin{bmatrix}-2\\3\\-3\\4\end{bmatrix},\begin{bmatrix}3\\-6\\1\\-3\end{bmatrix}\right\}$$

5

(a)

Yes. The set of all 2×2 diagonal matrices is a subset of $\mathbb{R}^{2\times 2}$ closed under linear combination.

(b)

No. $U_{2\times 2}+L_{2\times 2}=M_{2\times 2}$, which is no longer triangular.

(c)

Yes. The set of all 2×2 lower triangular matrices is a subset of $\mathbb{R}^{2\times 2}$ closed under linear combination.

(d)

No. The sum of two such matrices is a matrix B with $b_{12}=2$.

(e)

Yes. The set of all 2×2 matrices B with $b_{11}=0$ is a subset of $\mathbb{R}^{2\times 2}$ closed under linear combination.

(f)

Yes. The set of all symmetric 2×2 matrices is a subset of $\mathbb{R}^{2\times 2}$ closed under linear combination.

(g)

No. For example, $A=\begin{bmatrix}1&1\\0&0\end{bmatrix}, B=\begin{bmatrix}0&0\\1&2\end{bmatrix}$ are two singular matrices. But their sum $A+B=\begin{bmatrix}1&1\\1&2\end{bmatrix}$ is non-singular.

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(a)

$$\begin{bmatrix}18/5\\4/5\\7/5\end{bmatrix}$$

(b)

$$x_3\begin{bmatrix}2\\-1\\1\\0\end{bmatrix}+x_4\begin{bmatrix}-5\\1\\0\\1\end{bmatrix}+\begin{bmatrix}-5\\1\\0\\0\end{bmatrix},\ (x_3,x_4)\in\mathbb{R}^2.$$

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(a)

$$\begin{bmatrix}1&2&-3\\-2&-2&3\\2&4&6\end{bmatrix}\xrightarrow{E_{12}E_{13}E_{23}E_{31}E_{21}}\begin{bmatrix}1&0&0\\0&2&0\\0&0&12\end{bmatrix}.$$

Three column vectors are linearly independent. Hence

$$\dim(\cdot)=3.$$

(b)

Let

$$\alpha(x^2-4)+\beta x^2(x^4-2)+\theta(x^6-8)=0.$$

Simplifying the equation, we obtain

$$(\beta+\theta)x^6+(\alpha-2\beta)x^2-4(\alpha+2\theta)=0$$

which yields

$$\alpha=2\beta=-2\theta.$$

This shows that

$$x^2(x^4-2)\in\operatorname{span}(x^2-4,x^6-8).$$

Hence $\{x^2-4,x^6-8\}$ is a basis for the original span.

$$\dim(\cdot)=2.$$

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(a)

Proof. Let $S=\{\vec{x_1},...,\vec{x_n}\}$ be any set of n linearly independent vectors in V. We need to prove that every vector in V is in the span of vectors in S, by contradiction.
Suppose $\exists\vec{v}\in V$ s.t. $\vec{v}\notin\operatorname{span}(\vec{x_1},...,\vec{x_n})$, then $\vec{v},\vec{x_1},...,\vec{x_n}$ are $(n+1)$ L.I. vectors in V. We now use induction to show this is impossible to happen with $\dim(V)=n$, since any $(n+1)$ vectors in n-dimensional vector

space V must be L.D.

Base case. If $n=1$, a basis for V contains exactly one vector $\vec{b_1}$. Suppose $\vec{v_1}=c_1\vec{b_1}$ and $\vec{v_2}=c_2\vec{b_1}$ are two non-zero vectors in V (otherwise they are already L.D.). Clearly we may write $\vec{v_1}/c_1-\vec{v_2}/c_2=0$, showing $\vec{v_1},\vec{v_2}$ as L.D.

Inductive step. Suppose in any $(n-1)$-dimensional vector space V, any n vectors are L.D. We want to show this implies that in any n-dimensional V, any $(n+1)$ vectors $\vec{v_1},...,\vec{v_{n+1}}$ are L.D.
Let $B=\{\vec{b_1},...,\vec{b_n}\}$ be a basis for V. Then we may write

$$\vec{v_1}=a_{11}\vec{b_1}+a_{12}\vec{b_2}+...+a_{1n}\vec{b_n} \tag{1}$$

$$\vec{v_2}=a_{21}\vec{b_1}+a_{22}\vec{b_2}+...+a_{2n}\vec{b_n} \tag{2}$$

$$\vdots$$

$$\vec{v_n}=a_{n1}\vec{b_1}+a_{n2}\vec{b_2}+...+a_{nn}\vec{b_n} \tag{n}$$

$$\vec{v}_{n+1}=a_{n+1,1}\vec{b_1}+a_{n+1,2}\vec{b_2}+...+a_{n+1,n}\vec{b_n} \tag{n+1}$$

Note that if $a_{11},a_{21},...a_{n+1,1}$ are all 0, then $\vec{v_1},...,\vec{v}_{n+1}$ are L.D. by the hypothesis. Hence we may assume, WLOG, that $a_{11}\neq 0$. Multiply (1) by $\frac{a_{i1}}{a_{11}}$ and obtain

$$\frac{a_{i1}\vec{v_1}}{a_{11}}=a_{i1}\vec{b_1}+L_i(\vec{b_2},\vec{b_3},...,\vec{b_n}) \tag{1*}$$

where $L_i(\cdot)$ is some linear combination of $\vec{b_2},\vec{b_3},...,\vec{b_n}$. Now subtract (1^*) from equations (2), ..., (n+1):

$$\vec{v}_2^*:=\vec{v_2}-\frac{a_{21}\vec{v_1}}{a_{11}}=L_2^*(\vec{b_2},\vec{b_3},...,\vec{b_n}) \tag{2*}$$

$$\vdots$$

$$\vec{v}_n^*:=\vec{v_n}-\frac{a_{n1}\vec{v_1}}{a_{11}}=L_n^*(\vec{b_2},\vec{b_3},...,\vec{b_n}) \tag{n*}$$

$$\vec{v}_{n+1}^*:=\vec{v}_{n+1}-\frac{a_{n+1,1}\vec{v_1}}{a_{11}}=L_{n+1}^*(\vec{b_2},\vec{b_3},...,\vec{b_n}) \tag{n+1*}$$

Note that $\vec{v}_2^*,...,\vec{v}_n^*,\vec{v}_{n+1}^*$ are n vectors in $(n-1)$-dimensional vector space. By the hypothesis, they are L.D. i.e. $\exists$ non-trivial $\{\lambda_2,...,\lambda_n,\lambda_{n+1}\}$ s.t.

$$\lambda_2\vec{v}_2^*+...+\lambda_n\vec{v}_n^*+\lambda_{n+1}\vec{v}_{n+1}^*=\vec{0}.$$

Equivalently

$$\lambda_2(\vec{v_2}-\frac{a_{21}\vec{v_1}}{a_{11}})+...+\lambda_n(\vec{v_n}-\frac{a_{n1}\vec{v_1}}{a_{11}})+\lambda_{n+1}(\vec{v}_{n+1}-\frac{a_{n+1,1}\vec{v_1}}{a_{11}})=\vec{0}.$$

Rearranging, we obtain

$$-(\frac{\lambda_2 a_{21}}{a_{11}}+...+\frac{\lambda_n a_{n1}}{a_{11}}+\frac{\lambda_{n+1}a_{n+1,1}}{a_{11}})\vec{v_1}+\lambda_2\vec{v_2}+...+\lambda_n\vec{v_n}+\lambda_{n+1}\vec{v}_{n+1}=\vec{0}.$$

showing $\vec{v_1}, \ldots, \vec{v}_{n+1}$ as L.D. This completes the induction and our proof.

$\square$

(b)

Proof. Suppose $V = \text{span}(\vec{x_1}, \ldots, \vec{x_n})$. We want to show $S = \{\vec{x_1}, \ldots, \vec{x_n}\}$ are L.I. Then by **(a)** S forms a basis for V.

We shall prove by contradiction. Assume S is L.D. Further assume, WLOG, that $S' = \{\vec{x_1}, \ldots, \vec{x_j}\}$ is a maximal L.I. subset of S. Then V can be spanned solely by S'. But by **(a)** this creates a basis for V with only j vectors, implying $\dim(V) = j \neq n$, the desired contradiction.

$\square$

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$$\forall \vec{x} \in V, \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}.$$

Hence

$$V = \text{span}(\begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}). \tag{$*$}$$

The linear independence of S together with $(*)$ implies that S is a basis for V. Moreover,

$$\dim(V) = |S| = 3.$$

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(a)

From B, we observe the pivot variables at the first, second, and fifth columns. Hence a basis for $\text{Col}(A)$

$$S = \{a_1, a_2, a_5\} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 7 \\ 2 \end{bmatrix} \right\}$$

and

$$\text{rank}(A) = |S| = 3.$$

(b)

First note that

$$\text{N}(A) = \text{N}(B).$$

Solving $Bx = 0$,

$$x = \begin{bmatrix} -x_3 - x_4 + 14x_6 - 14x_7 \\ -x_3 - 2x_4 - 8x_6 + 5x_7 \\ x_3 \\ x_4 \\ -4x_6 + 2x_7 \\ x_6 \\ x_7 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ -2 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_6 \begin{bmatrix} 14 \\ -8 \\ 0 \\ 0 \\ -4 \\ 1 \\ 0 \end{bmatrix} + x_7 \begin{bmatrix} -14 \\ 5 \\ 0 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}.$$

A basis for $\mathrm{N}(B)$, therefore also a basis for $\mathrm{N}(A)$, is then

$$S = \left\{ \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 14 \\ -8 \\ 0 \\ 0 \\ -4 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -14 \\ 5 \\ 0 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Moreover,

$$\dim[\mathrm{N}(A)] = |S| = 4.$$

11

Let

$$\alpha y_1 + \beta y_2 + \theta y_3 = 0.$$

Equivalently

$$\alpha(x_1 + x_2) + \beta(x_2 + x_3) + \theta(x_3 + x_1) = 0.$$

Rearranging,

$$(\alpha + \theta)x_1 + (\alpha + \beta)x_2 + (\beta + \theta)x_3 = 0.$$

From L.I. of $\{x_i\}$ it follows that

$$\alpha + \theta = \alpha + \beta = \beta + \theta = 0,$$

which implies

$$\alpha = \beta = \theta = 0.$$

Hence $\{y_i\}$ is also L.I.

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(1)

$q=3$.

(2)

All $q\neq 3$ will do.

(3)

Impossible; $\operatorname{rank}(A)\leq 2$.

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Proof. Denote $\operatorname{rank}(A)=r\leq m$, which is the dimension of the row space of A. WLOG assume (through row permutation) $\{\vec{a_1},\vec{a_2},...\vec{a_r}\}$ is a maximal L.I. subset of $\{\vec{a_i}\}$, where $\vec{a_i}$ denotes the i-th row of A. Note that $AB=[\vec{a_i}B]$. Then since B is invertible, $\{\vec{a_1}B,\vec{a_2}B,...\vec{a_r}B\}$ remains a maximal L.I. subset of $\{(\vec{ab})_i\}$, where $(\vec{ab})_i$ denotes the i-th row of AB. Hence the dimension of the row space of AB remains to be r, i.e.

$$\operatorname{rank}(AB)=\operatorname{rank}(A).$$

□

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Proof. Denote $\operatorname{rank}(A)$ as α, $\operatorname{rank}(B)$ as β, the i-th row of a matrix M as $\vec{m_i}$. WLOG assume $\{\vec{a_i}\}_{i=1}^{\alpha}$ is a maximal L.I. subset of $\{\vec{a_i}\}_{i=1}^{m}$. Consider $\vec{a_1}$. Since $AB=O$,

$$\vec{a_1}B=[a_{11}\ a_{12}\ ...\ a_{1n}]\begin{bmatrix}\vec{b_1}\\ \vec{b_2}\\ \vdots\\ \vec{b_n}\end{bmatrix}=\vec{0}, \tag{1}$$

with at least one entry in $\vec{a_1}$ being non-zero. WLOG assume $a_{1n}\neq 0$, we may rewrite (1) as

$$\vec{b_n}=-\frac{1}{a_{1n}}(a_{11}\vec{b_1}+a_{12}\vec{b_2}+...+a_{1,n-1}\vec{b}_{n-1}).$$

Hence

$$\vec{b_n}\in\operatorname{span}(\vec{b_1},...,\vec{b}_{n-1})\implies\operatorname{Row}(B)=\operatorname{span}(\vec{b_1},...,\vec{b}_{n-1}). \tag{1*}$$

It follows that

$$\beta\leq|\{\vec{b_i}\}_{i=1}^{n-1}|=n-1.$$

Similarly, consider $\vec{a_2}$,

$$\vec{a_2}B = [a_{21}\ a_{22}\ ...\ a_{2n}] \begin{bmatrix} \vec{b_1} \\ \vec{b_2} \\ \vdots \\ \vec{b_n} \end{bmatrix} = \vec{0}.$$

Combined with (1) a little rearrangement yields

$$(a_{1n}a_{21} - a_{2n}a_{11})\vec{b_1} + ... + (a_{1n}a_{2,n-2} - a_{2n}a_{1,n-2})\vec{b}_{n-2} + (a_{1n}a_{2,n-1} - a_{2n}a_{1,n-1})\vec{b}_{n-1} = \vec{0} \qquad (2)$$

From L.I. of $\{\vec{a_i}\}_{i=1}^{\alpha}$, the coefficients $(a_{1n}a_{2i} - a_{2n}a_{1i})_{i=1}^{n-1}$ cannot all be zero. It follows that $\exists \vec{b_u} \in \{\vec{b_i}\}_{i=1}^{n-1}$ that we may take out without affecting the row space, thus

$$\beta \le (n-1) - 1 = n - 2.$$

Continuing the same process for every row in $\{\vec{a_i}\}_{i=1}^{\alpha}$. Each time $\vec{a_i}B = \vec{0}$ and the L.I. of $\{\vec{a_i}\}_{i=1}^{\alpha}$ together implies a new redundant row in $\{\vec{b_i}\}_{i=1}^{n}$ for spanning $\mathrm{Row}(B)$, reducing the upper bound for β by 1. We obtain at last

$$\beta \le n - |\{\vec{a_i}\}_{i=1}^{\alpha}| = n - \alpha. \qquad \square$$

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Proof. If $x \in \mathrm{N}(A)$, then

$$A^TAx = A^T(Ax) = A0 = 0 \implies x \in \mathrm{N}(A^TA).$$

If $x \in \mathrm{N}(A^TA)$, then

$$A^TAx = 0 \implies x^TA^TAx = 0 \iff (Ax)^TAx = 0 \implies Ax = 0 \implies x \in \mathrm{N}(A).$$

Therefore

$$\mathrm{N}(A) = \mathrm{N}(A^TA). \qquad \square$$

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Proof. Denote $\mathrm{rank}(A)$ as α, $\mathrm{rank}(B)$ as β, i-th column of a matrix M as m_i. WLOG assume $\{a_i\}_{i=1}^{\alpha}$ is a maximal L.I. subset of $\{a_i\}$, $\{b_i\}_{i=1}^{\beta}$ a maximal L.I. subset of $\{b_i\}$. Then $\{a_i\}_{i=1}^{\alpha}$ spans $\mathrm{Col}(A)$; $\{b_i\}_{i=1}^{\beta}$ spans $\mathrm{Col}(B)$. Consequently $\{a_i\}_{i=1}^{\alpha} \cup \{b_i\}_{i=1}^{\beta}$ spans $\mathrm{Col}(A+B)$, and

$$\mathrm{rank}(A+B) \le |\{a_i\}_{i=1}^{\alpha} \cup \{b_i\}_{i=1}^{\beta}| = \alpha + \beta. \qquad \square$$

17

(1)

$$T(A+B) = (A+B) + (A+B)^T = A + A^T + B + B^T = T(A) + T(B)$$

$$T(cA) = cA + (cA)^T = c(A + A^T) = c\,T(A).$$

(2)

Let

$$T(A) = A + A^T = O.$$

$$\begin{bmatrix} 2a_{11} & a_{12} + a_{21} \\ a_{21} + a_{12} & 2a_{22} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \implies A = \begin{bmatrix} 0 & a_{12} \\ -a_{12} & 0 \end{bmatrix}.$$

Therefore

$$\ker(T) = \left\{ A \mid A = \begin{bmatrix} 0 & x \\ -x & 0 \end{bmatrix} \right\}.$$

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Proof. Suppose L is one-to-one. Then

$$L(v_1) = L(v_2) \implies v_1 = v_2$$

$$\Downarrow$$

$$L(v_1) - L(v_2) = L(v_1 - v_2) = 0 \implies v_1 - v_2 = 0$$

$$\Downarrow$$

$$\ker(L) = 0.$$

Suppose $\ker(L) = 0$. Then

$$L(v) = 0 \implies v = 0$$

$$\Downarrow$$

$$L(v_1) - L(v_2) = L(v_1 - v_2) = 0 \implies v_1 - v_2 = 0$$

$$\Downarrow$$

$$L(v_1) = L(v_2) \implies v_1 = v_2.$$

L is one-to-one. Hence

$$L \text{ is one-to-one} \iff \ker(L) = 0.$$ □

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$$L : \mathbb{R}^3 \to \mathbb{R}^2.$$

This transformation is determined by

$$B = \left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 7 \\ 2 \end{bmatrix} \right\} \xrightarrow{L} \left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \end{bmatrix} \right\} \xrightarrow{\text{Rep}_{B'}} \left\{ \begin{bmatrix} -2 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}_{B'}.$$

Hence

$$\text{Rep}_{B,B'}(L) = \begin{bmatrix} -2 & -2 & 1 \\ 1 & 3 & 2 \end{bmatrix}_{B,B'}.$$

To find the $L(\vec{u})$ and $\text{Rep}_{B'}[L(\vec{u})]$, first use the fact that

$$\text{Rep}_B(\vec{u}) = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}_B.$$

Then

$$\begin{aligned} \text{Rep}_{B'}[L(\vec{u})] &= \text{Rep}_{B,B'}(L)\text{Rep}_B(\vec{u}) \\ &= \begin{bmatrix} -2 & -2 & 1 \\ 1 & 3 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}_B \\ &= \begin{bmatrix} -7 \\ 3 \end{bmatrix}_{B'} \end{aligned}$$

Converting back,

$$L(\vec{u}) = [\vec{u}_1' \mid \vec{u}_2'] \begin{bmatrix} -7 \\ 3 \end{bmatrix}_{B'} = \begin{bmatrix} 7 \\ 3 \end{bmatrix}.$$

20

```
julia> using LinearAlgebra

julia> # pairwise inner products are zero

julia> a1'a2
0.0

julia> a1'a3
0.0

julia> a2'a3
0.0

julia> # expansion of x in basis {a1, a2, a3}

julia> norm((a1'x)a1 + (a2'x)a2 + (a3'x)a3 - x)
4.577566798522237e-16
```

21

```
julia> using LinearAlgebra
```

(1)

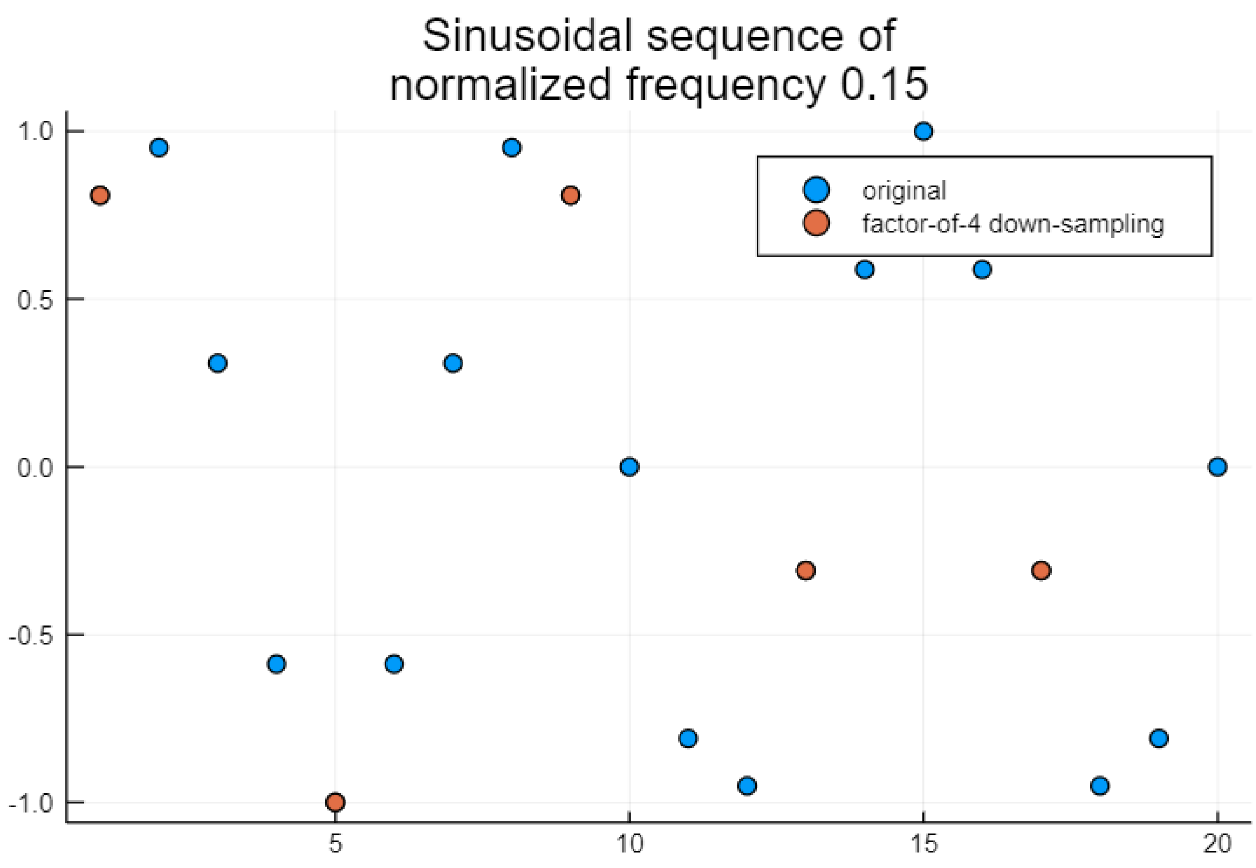
```
julia> rank(A * B) <= min(rank(A), rank(B))
true
```

(2)

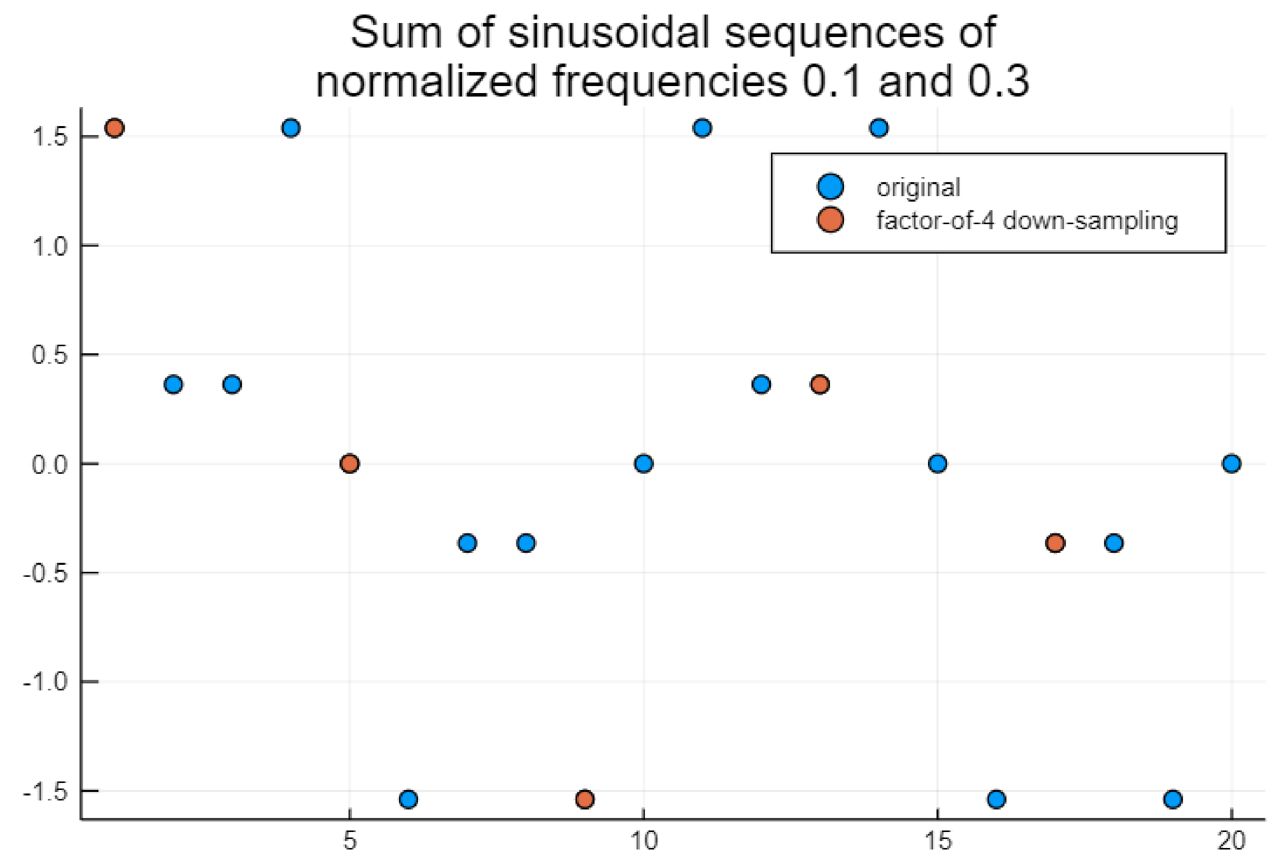
```
julia> rank(A + B) <= rank(A) + rank(B)
true
```

22

(i)



(ii)



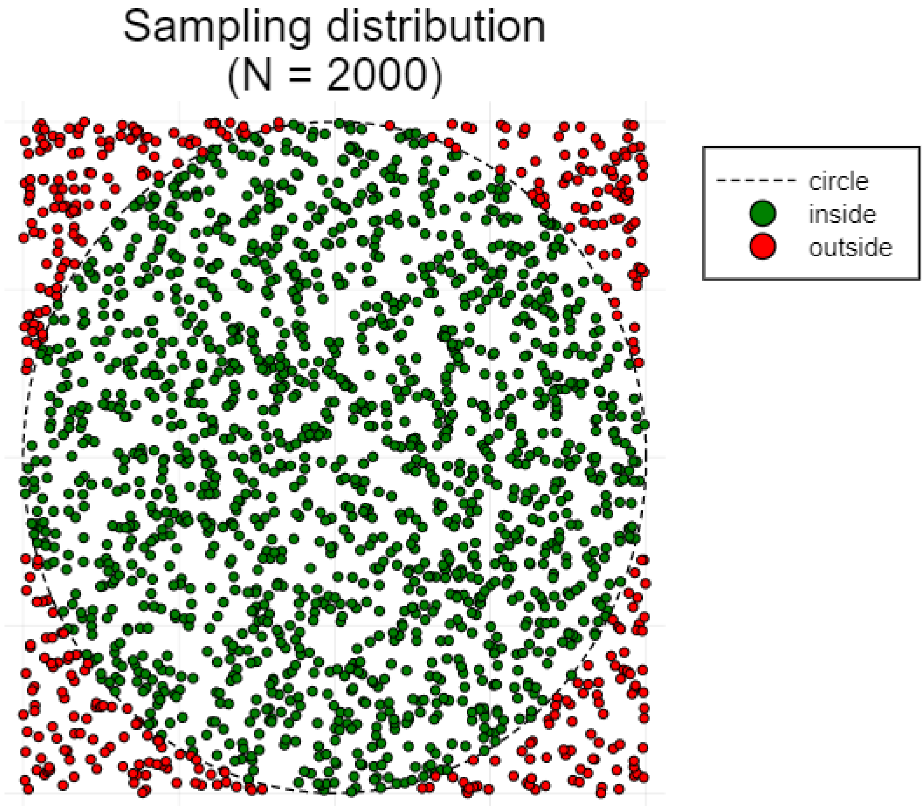
23

```
julia> X = count(x -> (rand() < p), 1:N)
4
```

24

```
julia> total = 1e5;

julia> est_PI = 4 * hit / total
3.13388
```


Sampling distribution
(N = 2000)
circle
inside
outside