STA2002 Assignment 1

1

Denote $Y := \sum_{i=1}^{n} X_i$. Since X_i are independent RVs, the MGF of Y is given by the product of respective MGFs of X_i's:

$$M_Y(t) = \prod_{i=1}^{n} M_{X_i}(t) = \prod_{i=1}^{n} (1-2t)^{-r_i/2} = (1-2t)^{-\sum_{i=1}^{n} r_i/2}.$$

Therefore $Y \sim \chi^2(\sum_{i=1}^{n} r_i)$.

2

Let $X_i \sim_{\text{i.i.d.}} \text{Poisson}(1.8)$ be the number of floods within the i-th year. Then,

$$\begin{aligned} \mu_X &= 1.8 \\ \sigma_X^2 &= 1.8 \end{aligned}$$

Let also $Y_i \sim_{\text{i.i.d.}} \text{Exponential}(\lambda = 1/3)$ be the number of days during which ground is flooded, in the span of the i's flood. Then,

$$\begin{aligned} \mu_Y &= 3 \\ \sigma_Y^2 &= 9 \end{aligned}$$

(a)

By CLT,

$$\frac{\sum_{i=1}^{20} X_i - 20\mu_X}{\sqrt{20\sigma_X^2}} \approx Z$$

where $Z \sim N(0,1)$, the standard normal distribution. Whence

$$\begin{aligned} P\left(\sum_{i=1}^{20} X_i \geq 19\right) &\approx P\left(\sqrt{20\sigma_X^2} \cdot Z + 20\mu_X \geq 19\right) \\ &= P\left(Z \geq \frac{19 - 20\mu_X}{\sqrt{20\sigma_X^2}}\right) \\ &= P(Z \geq -2.83) \\ &= 99.767\% \end{aligned}$$

(b)

By CLT,

$$\frac{\sum_{i=1}^{120} Y_i - 120\mu_Y}{\sqrt{120\sigma_Y^2}} \approx Z$$

where $Z \sim N(0,1)$, the standard normal distribution. Then,

$$\begin{aligned}
P\left(\sum_{i=1}^{120} Y_i < 365\right) &\approx P\left(\sqrt{120\sigma_Y^2}\cdot Z + 120\mu_Y < 365\right) \\
&= P\left(Z < \frac{365 - 120\mu_Y}{\sqrt{120\sigma_Y^2}}\right) \\
&= P\left(Z < 0.152\right) \\
&= 56.041\%
\end{aligned}$$

3

(a)

Set sample moment at

$$\hat{\mu}_1 = \mu_1 = \frac{1}{\lambda} \implies \hat{\lambda}_{\text{mom}} = \frac{1}{\hat{\mu}_1}$$

(b)

The log-likelihood function is

$$l_X(\lambda) = \sum_{i=1}^{n} \ln\left(\lambda e^{-\lambda x_i}\right) = \sum_{i=1}^{n} [\ln(\lambda) - \lambda x_i] = n\ln(\lambda) - \lambda\sum_{i=1}^{n} x_i$$

Set the derivative at zero,

$$l'_X(\lambda) = \frac{n}{\lambda} - \sum_{i=1}^{n} x_i = 0 \implies \lambda = \frac{n}{\sum_i x_i} = \frac{1}{\bar{x}}$$

Since $l''_X(\lambda) = -n/\lambda^2 < 0$, the point is a global minimum and we conclude

$$\hat{\lambda}_{\text{mle}} = \frac{1}{\bar{X}}$$

which coincides with the method of moment estimator.

(c)

$$\hat{\lambda}_{\text{mom}} = \hat{\lambda}_{\text{mle}} = \frac{n}{\sum_{i=1}^{n} X_i} = \frac{6}{18.76} = 0.32$$

(d)

$$\mathbb{E}[\hat{\lambda}_{\text{mle}}] = \mathbb{E}[1/\bar{X}] = n \cdot \mathbb{E}\left[1/\sum_{i=1}^{n} X_i\right] = n \cdot \mathbb{E}[1/G]$$

where $G := \sum_{i=1}^n X_i \sim \text{Gamma}(n, \lambda)$. Hence for $n > 1$,

$$\begin{aligned}
\mathbb{E}[\hat{\lambda}_{\text{mle}}] &= n \int_0^\infty \frac{\lambda^n}{\Gamma(n)} \cdot x^{n-2} e^{-\lambda x} \, dx \\
&= \frac{n\lambda \cdot \Gamma(n-1)}{\Gamma(n)} \underbrace{\int_0^\infty \frac{\lambda^{n-1}}{\Gamma(n-1)} \cdot x^{n-2} e^{-\lambda x} \, dx}_{1} \\
&= \frac{n}{n-1} \lambda \neq \lambda
\end{aligned}$$

Therefore $\hat{\lambda}_{\text{mle}}$ is a biased estimator of λ. However, it is asymptotically unbiased since $\mathbb{E}[\hat{\lambda}_{\text{mle}}] \to \lambda$ as $n \to \infty$.

4

Set sample moments

$$\begin{aligned}
\hat{\mu}_1 &= \mu_1 = \frac{a+b}{2} \\
\hat{\mu}_2 &= \mu_2 = \frac{a^2+ab+b^2}{3}
\end{aligned} \implies \begin{aligned}
a^2 - 2\hat{\mu}_1 a + 4\hat{\mu}_1^2 - 3\hat{\mu}_2 &= 0 \\
a + b - 2\hat{\mu}_1 &= 0
\end{aligned}$$

Solving for a, b yields

$$\begin{aligned}
\hat{a}_{\text{mom}} &= \hat{\mu}_1 - \sqrt{3\left(\hat{\mu}_2 - \hat{\mu}_1^2\right)} \\
\hat{b}_{\text{mom}} &= \hat{\mu}_1 + \sqrt{3\left(\hat{\mu}_2 - \hat{\mu}_1^2\right)}
\end{aligned}$$

5

Denote $m := \min_{1 \leq i \leq n} \{x_i\}$. The likelihood function is given by

$$L_X(\theta) = \prod_{i=1}^n f_X(x;\theta) = \begin{cases} \exp\left[\sum_{i=1}^n (\theta - x_i)\right], & \text{if } m \geq \theta \\ 0, & \text{otherwise} \end{cases}$$

If $m \geq \theta$, we have $L_X(\theta) > 0$, and so the MLE can be found by maximizing the log-likelihood function

$$\begin{aligned}
\theta^* &= \arg\max_{\theta \leq m} L_X(\theta) \\
&= \arg\max_{\theta \leq m} l_X(\theta) \\
&= \arg\min_{\theta \leq m} \sum_{i=1}^n (x_i - \theta) \geq 0
\end{aligned}$$

since $\theta \leq m \leq x_i$ for all i. Moreover, $\theta^* = m$ brings the sum to zero and so must be the minimizer. Hence the MLE in this case is given by

$$\hat{\theta} = \min_{1 \leq i \leq n} \{X_i\}$$

Otherwise, the likelihood function is constant zero and we may choose $\hat{\theta}$ to be any RV. Therefore $\hat{\theta} = \min_{1 \leq i \leq n} \{X_i\}$ is the final MLE.

6

We first derive the CDF of $\hat{\theta}$ by noting that

$$\begin{aligned} F_{\hat{\theta}}(x) &= P\left(\hat{\theta} \leq x\right) \\ &= P\left(\max_i \{X_i\} \leq x\right) \\ &= P\left[\cap_i (X_i \leq x)\right] \end{aligned}$$

Since X_i are independently distributed this further equates to

$$\begin{aligned} F_{\hat{\theta}}(x) &= \prod_i P(X_i \leq x) \\ &= \prod_i F_{X_i}(x) \\ &= \prod_i \frac{x}{\theta} \\ &= \frac{x^n}{\theta^n} \end{aligned}$$

Differentiating $F_{\hat{\theta}}(x)$ yields the PDF

$$f_{\hat{\theta}}(x) = \frac{d}{dx} F_{\hat{\theta}}(x) = \frac{n}{\theta^n} x^{n-1}$$

Hence the mean of $\hat{\theta}$ is given by

$$\begin{aligned} \mathbb{E}[\hat{\theta}] &= \int_{x=0}^{\theta} x f_{\hat{\theta}}(x)\, dx \\ &= \int_{x=0}^{\theta} \frac{n}{\theta^n} x^n \\ &= \frac{n}{(n+1)\theta^n} x^{n+1} \bigg|_{x=0}^{\theta} \\ &= \frac{n}{n+1} \theta \neq \theta \end{aligned}$$

from which we see $\hat{\theta}$ is biased. However, $\Theta := (n+1)\hat{\theta}/n$ would be an unbiased estimator of θ, since

$$\mathbb{E}[\Theta] = \frac{n+1}{n} \mathbb{E}[\hat{\theta}] = \theta$$

7

(a)

We have the conditional probabilities

$$f_{X|K}(x|k) = \phi(x; \mu_k, \sigma_k^2)$$

Thus the joint PDF is given by

$$\begin{aligned} f_{X,K}(x,k) &= P(K=k) \cdot f_{X|K}(x|k) \\ &= \pi_k \phi\left(x; \mu_k, \sigma_k^2\right) \\ &= \frac{\pi_k}{\sqrt{2\pi\sigma_k^2}} \exp\left[-\frac{1}{2}\left(\frac{x-\mu_k}{\sigma_k}\right)^2\right] \end{aligned}$$

with support $(x,k) \in \mathbb{R} \times \{0,1\}$.

(b)

Denote the observed values

$$
\begin{aligned}
i_0 &:= \{i : k_i = 0\},\ i_1 := \{i : k_i = 1\} \\
n_0 &:= |i_0|,\ n_1 = |i_1| \\
s_0 &:= \sum_{i \in I_0} x_i,\ s_1 := \sum_{j \in I_1} x_j \\
q_0 &:= \sum_{i \in I_0} x_i^2,\ q_1 := \sum_{j \in I_1} x_j^2
\end{aligned}
$$

and the random variables

$$
\begin{aligned}
I_0 &:= \{i : K_i = 0\},\ I_1 := \{i : K_i = 1\} \\
N_0 &:= |I_0|,\ N_1 = |I_1| \\
S_0 &:= \sum_{i \in I_0} X_i,\ S_1 := \sum_{j \in I_1} X_j \\
Q_0 &:= \sum_{i \in I_0} X_i^2,\ Q_1 := \sum_{j \in I_1} X_j^2
\end{aligned}
$$

The log-likelihood function of (X,K) is given by

$$
\begin{aligned}
l_{X,K}(\pi_0, \mu_0, \sigma_0^2, \mu_1, \sigma_1^2) &= \sum_{i=1}^{n} \ln\Big[\pi_{k_i} \phi\left(x_n; \mu_{k_i}, \sigma_{k_i}^2\right)\Big] \\
&= \sum_{i=1}^{n} \left[-\ln\big(\sqrt{2\pi}\big) + \ln(\pi_{k_i}/\sigma_{k_i}) - \frac{1}{2}\left(\frac{x_i - \mu_{k_i}}{\sigma_{k_i}}\right)^2\right] \\
&= -n\ln\big(\sqrt{2\pi}\big) + n_0(\ln\pi_0 - \ln\sigma_0) - \frac{1}{2\sigma_0^2}(q_0 - 2\mu_0 s_0 + n_0\mu_0^2) \\
&\qquad + n_1(\ln\pi_1 - \ln\sigma_1) - \frac{1}{2\sigma_1^2}(q_1 - 2\mu_1 s_1 + n_1\mu_1^2)
\end{aligned}
$$

Setting gradient at zero,

$$
\nabla l_{X,K} = \begin{bmatrix}
n_0/\pi_0 - n_1/(1-\pi_0) \\
\frac{1}{\sigma_0^2}(s_0 - n_0\mu_0) \\
-\frac{1}{2}n_0/\sigma_0^2 + \frac{1}{2}\left(\sigma_0^2\right)^{-2}(q_0 - 2\mu_0 s_0 + n_0\mu_0^2) \\
\frac{1}{\sigma_1^2}(s_1 - n_1\mu_1) \\
-\frac{1}{2}n_1/\sigma_1^2 + \frac{1}{2}\left(\sigma_1^2\right)^{-2}(q_1 - 2\mu_1 s_1 + n_1\mu_1^2)
\end{bmatrix} = 0
$$

we obtain

$$
\begin{cases}
\pi_0 = n_0/(n_0+n_1) \\
\mu_0 = s_0/n_0 \\
\sigma_0^2 = q_0/n_0 - (s_0/n_0)^2 \\
\mu_1 = s_1/n_1 \\
\sigma_1^2 = q_1/n_1 - (s_1/n_1)^2
\end{cases}
$$

from which we deduce the MLE's

$$
\begin{cases}
\hat{\pi}_0 = N_0/(N_0+N_1) \\
\hat{\mu}_0 = S_0/N_0 \\
\hat{\sigma}_0^2 = Q_0/N_0 - (S_0/N_0)^2 \\
\hat{\mu}_1 = S_1/N_1 \\
\hat{\sigma}_1^2 = Q_1/N_1 - (S_1/N_1)^2
\end{cases}
$$

(c)

Python code:

```
import csv
import os, sys

with open(os.path.join(sys.path[0], 'GMM.csv'),
newline='') as f:
    reader = csv.reader(f)
    lst = list(reader)[1::]

n_0 = n_1 = s_0 = s_1 = q_0 = q_1 = 0
for r in lst:
    k, x = int(r[0]), float(r[1])
    if k:
        n_1 += 1
        s_1 += x
        q_1 += x**2
    else:
        n_0 += 1
        s_0 += x
        q_0 += x**2

pi_0  = n_0 / (n_0 + n_1)
mu_0  = s_0 / n_0
var_0 = q_0 / n_0 - (s_0 / n_0) ** 2
mu_1  = s_1 / n_1
var_1 = q_1 / n_1 - (s_1 / n_1) ** 2

print('pi_0  = ', pi_0, '\n',
        'mu_0  = ', mu_0, '\n',
        'var_0 = ', var_0, '\n',
        'mu_1  = ', mu_1, '\n',
        'var_1 = ', var_1, sep='')
```

Output:

```
pi_0  = 0.94914
mu_0  = 49.95766742434471
var_0 = 99.8605412030156
mu_1  = 60.81269474517893
var_1 = 101.61414449594167
```