MAT3007 Assignment 1

A1.1

(a)

Define $n :=$ # floors aboveground, $m :=$ # floors underground, and $u :=$ uniform height of floors. So $h = un, d = um$.

The optimization formulation is as follows:

$$\begin{aligned}
\min_{l,w,h,d,n,m,u} \quad & lwd \\
\text{subject to} \quad & w \le l \le 2w \\
& un = h \\
& um = d \\
& u \ge 7/2 \\
& l \le 40 \\
& l \le h \\
& 1/10 \le m/(n+m) \le 1/4 \\
& lw(n+m) \ge 10000 \\
& lw + 2h(l+w) \le 5000 \\
& l, w, h, u \in \mathbb{R}^+ \\
& n, m \in \mathbb{N}^+.
\end{aligned}$$

(b)

A feasible point:

$$\begin{bmatrix} l & w & h & d & n & m & u \end{bmatrix} = \begin{bmatrix} 27 & 27 & 38.5 & 10.5 & 11 & 3 & 3.5 \end{bmatrix}.$$

A1.2

(a)

Let x_1, x_2 be the # product I,II respectively. Then the LP formulation is:

$$\begin{aligned} \max_{x_1,x_2} \quad & 8x_1 + 7x_2 \\ \text{subject to} \quad & x_1/3 + x_2/4 \le 100 \\ & x_1/5 + x_2/4 \le 70 \\ & x_1, x_2 \ge 0. \end{aligned}$$

(b)

$$\begin{aligned} \min_{x_1,x_2,s_1,s_2} \quad & -8x_1 - 7x_2 \\ \text{subject to} \quad & x_1/3 + x_2/4 + s_1 = 100 \\ & x_1/5 + x_2/4 + s_2 = 70 \\ & x_1, x_2, s_1, s_2 \ge 0, \end{aligned}$$

or equivalently in matrix form,

$$\begin{aligned} \min_{x=(x_1,x_2,s_1,s_2)^\top} \quad & \begin{bmatrix} -8 & -7 & 0 & 0 \end{bmatrix} x \\ \text{subject to} \quad & \begin{bmatrix} 1/3 & 1/4 & 1 & 0 \\ 1/5 & 1/4 & 0 & 1 \end{bmatrix} x = \begin{bmatrix} 100 \\ 70 \end{bmatrix} \\ & x \ge 0. \end{aligned}$$

(c)

Define $x_3 :=$ # overtime assembly hours. LP formulation:

$$\begin{aligned} \max_{x_1,x_2,x_3} \quad & 8x_1 + 7x_2 - 7x_3 \\ \text{subject to} \quad & x_1/3 + x_2/4 \le 100 + x_3 \\ & x_1/5 + x_2/4 \le 70 \\ & x_3 \le 60 \\ & x_1, x_2, x_3 \ge 0. \end{aligned}$$

(d)

```
>> cvx_begin
    variable x
    variable y
    maximize (8 * x + 7 * y)
    subject to
        x/3 + y/4 <= 100
        x/5 + y/4 <= 70
        x >= 0
        y >= 0
cvx_end
```

yielding

$$\text{max\_profit}(x^*, y^*) = 2500$$

with

$$[\,x^* \quad y^*\,] = [\,225 \quad 100\,]$$

A1.3

(a)

Let $A = [a_{ij}] \geq 0$ be the actual flow and $C = [c_{ij}] \geq 0$ the capacity, both from v_i to v_j. Then the LP formulation is:

$$\begin{array}{lcr}
\max\limits_{A} & \sum\limits_i a_{in} & \\
\text{subject to} & 0 \leq A \leq C & \\
& \sum\limits_j a_{ji} = \sum\limits_j a_{ij} & \forall i, j \in \{1, \ldots, n\}
\end{array}$$

*Note that we define $c_{ij} := 0$ if $(i, j) \notin E$.

(b)

```
% capacity values
C = [0    11     8     0     0     0
     0     0    10    12     0     0
     0     1     0     0    11     0
     0     0     4     0     0    15
     0     0     0     7     0     4
     0     0     0     0     0     0];

cvx_begin
  variable A(6,6)
  maximize (ones(1,6) * A(:,6)) % 6th col sum
  subject to
    for i = 1 : 6*6
      % flow capacity restriction
      0 <= A(i) <= C(i)
    end
    for i = 2 : 6-1
      % flow conservation, ith row sum == ith col sum
      A(i,:) * ones(6,1) == ones(1,6) * A(:,i)
    end
cvx_end
```

yielding

$$\text{max\_flow}(A^*) = 19$$

with

$$A^* = \begin{bmatrix} 0 & 11 & 8 & 0 & 0 & 0 \\ 0 & 0 & 1.5947 & 10.2690 & 0 & 0 \\ 0 & 0.8637 & 0 & 0 & 9.3653 & 0 \\ 0 & 0 & 0.6343 & 0 & 0 & 15 \\ 0 & 0 & 0 & 5.3653 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

A1.4

(a)

$$\min_{x \in \mathbb{R}^n} c^\top x \quad \text{s.t.} \quad -\delta \le (Ax-b)_i \le \delta \quad \forall i.$$

To show equivalence, it suffices to show equality of two feasible regions (as objective functions are the same).

Say we have $x \in \mathbb{R}^n$ with $-\delta \le (Ax-b)_i \le \delta,\ x \ge 0 \quad \forall i \in \{1, \cdots, m\}$. Then

$$\begin{aligned} & -\delta \le (Ax-b)_i \le \delta,\ x \ge 0 \quad \forall i \in \{1, \cdots, m\} \\ \implies & |(Ax-b)_i| \le \delta, x \ge 0 \quad \forall i \in \{1, \cdots, m\} \\ \implies & \max_{1 \le i \le m} |(Ax-b)_i| \le \delta, x \ge 0 \\ \implies & ||Ax-b||_\infty \le \delta, x \ge 0. \end{aligned}$$

Conversely, suppose $||Ax-b||_\infty \le \delta, x \ge 0$. We have

$$\begin{aligned} & ||Ax-b||_\infty \le \delta, x \ge 0 \\ \implies & \max_{1 \le i \le m} |(Ax-b)_i| \le \delta, x \ge 0 \\ \implies & |(Ax-b)_j| \le \max_{1 \le i \le m} |(Ax-b)_i| \le \delta, x \ge 0 \quad \forall j \in \{1, \cdots, m\} \\ \implies & -\delta \le (Ax-b)_i \le \delta,\ x \ge 0 \quad \forall i \in \{1, \cdots, m\}. \end{aligned}$$

Therefore two feasible regions are indeed equal, hence the equivalence.

(b)

Let $x = [x_1 \quad x_2]^\top := [\#\text{ salad A} \quad \#\text{ salad B}]^\top$. The LP can be formulated in the standard form as follows:

$$\begin{aligned} \min_x \quad & [-10 \quad -20]\, x \\ \text{subject to} \quad & \begin{bmatrix} 1/4 & 1/2 \\ 1/8 & 1/4 \\ 5 & 1 \end{bmatrix} x = \begin{bmatrix} 25 \\ 10 \\ 120 \end{bmatrix} \\ & x \ge 0. \end{aligned}$$

(c)

Suppose $\hat{x}$ is a solution to the constraint. Then

$$25 = [1/4 \quad 1/2]\, \hat{x} = 2\, [1/8 \quad 1/4]\, \hat{x} = 2 \cdot 10 = 20,$$

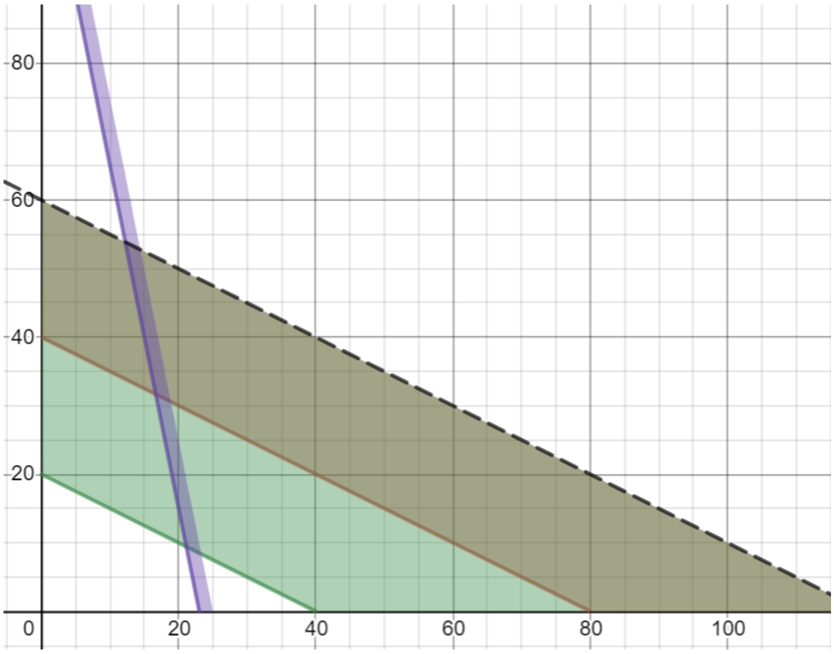
a contradiction. Hence the LP is infeasible.

(d)

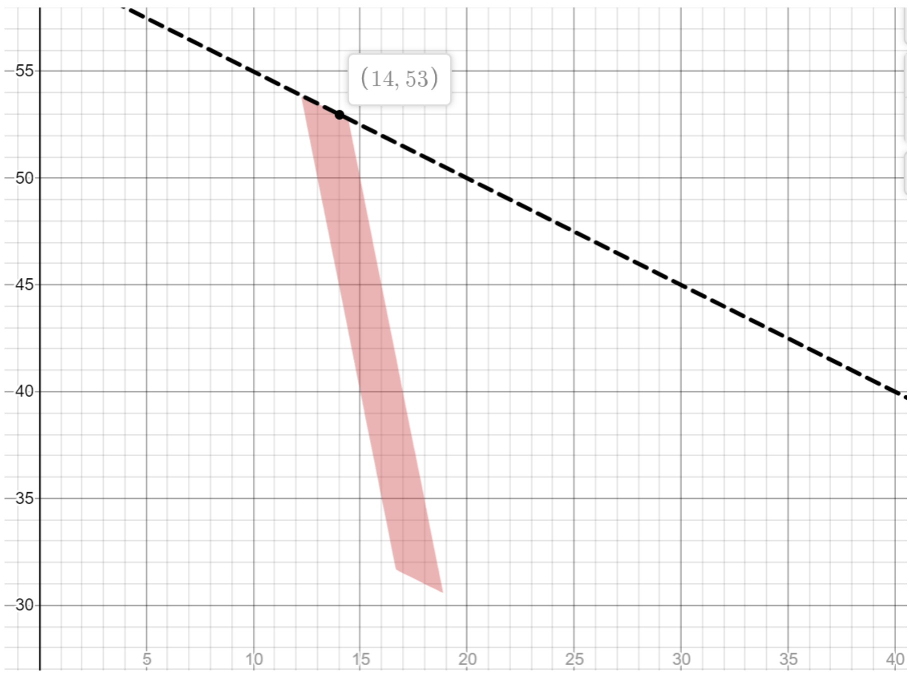
From (a) we have the equivalent formulation of the robust LP:

$$
\begin{aligned}
\max_{x} \quad & & 10x_1 + 20x_2 \\
\text{subject to} \quad & & x \geq 0 \\
& \text{(mango)} & -5 \leq x_1/4 + x_2/2 - 25 \leq 5 \\
& \text{(pineapple)} & -5 \leq x_1/8 + x_2/4 - 10 \leq 5 \\
& \text{(strawberry)} & -5 \leq x_1/8 + x_2/4 - 10 \leq 5
\end{aligned}
$$

The following graph shows the three constraints separately given $x \geq 0$ (horizontal: x_1, vertical: x_2).



The graph below shows the feasible set in red (overlap of three regions above).



where the black dotted line is the contour of the profit function at maximum, $10x_1 + 20x_2 = 1200$. We see there's an infinite number of solutions (the entire upper edge of the feasible region) yielding the optimal profit 1200 RMB with active constraints

$$x_1/4 + x_2/2 - 25 \leq 5,$$
$$x_1/8 + x_2/4 - 10 \leq 5.$$

Nevertheless, $(x_1^*, x_2^*) = (14, 53)$ is the only lattice point among the solutions and will be our final production plan. Following this plan, the amounts of fruits used are $\#$ mango $= x_1^*/4 + x_2^*/2 = 30$;
$\#$ pineapple $= x_1^*/8 + x_2^*/4 = 15$; $\#$ strawberry $= 5x_1^* + x_2^* = 123$.