STA2002 Assignment 2

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1

(a)

	x^D	x^S	y^D	y^S
Sample mean	54.26	54.27	47.83	47.84
Sample variance	281.07	281.20	725.52	725.24

(b)

$$\hat{\alpha} = 47.83, \hat{\beta} = -0.10$$
$$\hat{a} = 47.84, \hat{b} = -0.10$$

(c)

For Dinosaur, we have

$$\begin{aligned} \hat{\alpha} = 47.83, n^D = 142, t_{0.025}(140) \approx z_{0.025} = 1.96 \\ \hat{\beta} = -0.10, S_R^D = 35.68 \\ \sum_i (x_i^D - \bar{x}^D)^2 = 39630.87 \end{aligned}$$

Therefore a 95% C.I. for α is

$$\hat{\alpha} \pm t_{0.025}(n-2) S_R^D \sqrt{1/n^D} = \boxed{47.83 \pm 5.87}$$

For β it is

$$\hat{\beta} \pm t_{0.025}(n-2) S_R^D \sqrt{\frac{1}{\sum_i (x_i^D - \bar{x}^D)^2}} = \boxed{-0.10 \pm 0.35}$$

For Star, we have

$$\begin{aligned} \hat{a} = 47.84, n^S = 142, t_{0.025}(140) \approx z_{0.025} = 1.96 \\ \hat{\beta} = -0.10, S_R^S = 35.50 \\ \sum_j (x_j^S - \bar{x}^S)^2 = 39648.92 \end{aligned}$$

Therefore a 95% C.I. for a is

$$\hat{a} \pm t_{0.025}(n-2) S_R^S \sqrt{1/n^S} = \boxed{47.84 \pm 5.84}$$

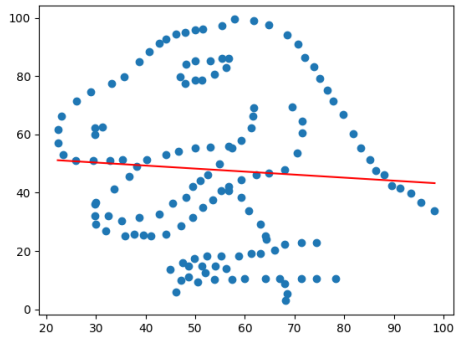
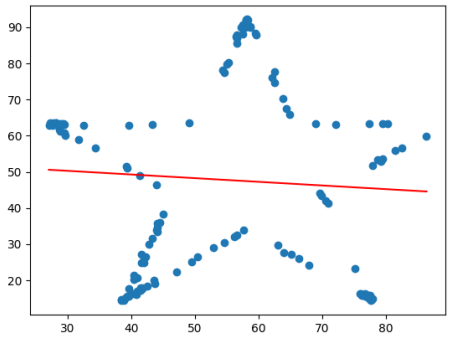
For b it is

$$\hat{b} \pm t_{0.025}(n-2)S_R^S\sqrt{\frac{1}{\sum_i (x_i^S - \bar{x}^S)^2}} = \boxed{-0.10 \pm 0.35}$$

(d)

They are almost identical in terms of those statistics.

(e)

Dinosaur	Star
	

(f)

Simple linear regression model is indeed simple in that it fails to capture the complicated geometries of two datasets (that is, it cannot distinguish a dinosaur from a star, or even from a genuine straight line). This is because the simple linear regression model is based on the a-prior assumption that the samples are linearly distributed with some gaussian noise - we didn't do anything to validate this assumption! The example tells us we should not blindly apply the regression. Rather, we should always look at the scatter plot first for a gut feeling of what the distribution might be (or carry out some statistical tests for linearity).

Python Code:

```
import csv
import os, sys
import matplotlib.pyplot as plt
import numpy as np

class LinReg:
    def __init__(self, name=None, file=None, reg=True):
        self.name = name
        self.xs = self.ys = []
        self.size = 0
        self.x_mean = self.y_mean = None
        self.x_S2 = self.y_S2 = None
        self.a = self.b = self.sigma2 = self.SR2 = None
        self.a_max_err = self.b_max_err = None
        if file:
            self.load_data(file)
            if reg: self.regress()
```

```
    def load_data(self, file):
        self.__init__(self.name)
        with open(file, newline='') as f:
            reader = csv.reader(f)
            data = list(reader)[1::]
            self.size = len(data)
            self.xs = [float(x) for [x, y] in data]
            self.ys = [float(y) for [x, y] in data]
            self.x_mean = sum(self.xs) / self.size
            self.y_mean = sum(self.ys) / self.size
            self.x_S2 = sum((x - self.x_mean) ** 2 for x in self.xs) /
(self.size - 1)
            self.y_S2 = sum((y - self.y_mean) ** 2 for y in self.ys) /
(self.size - 1)

    def regress(self, confidence=0.95):
        xy_sum = sum(x * y for x, y in zip(self.xs, self.ys))
        x2_sum = sum(x ** 2 for x in self.xs)
        y2_sum = sum(y ** 2 for y in self.ys)
        num = xy_sum - self.size * self.x_mean * self.y_mean
        den = x2_sum - self.size * self.x_mean ** 2
        x_SE = (self.size - 1) * self.x_S2
        t = 1.96 # should ideally be a t-quantile function, estimated by z_0.025

        self.a = self.y_mean
        self.b = num / den
        self.sigma2 = y2_sum / self.size - self.y_mean ** 2 - \
                      self.b * (xy_sum / self.size + self.x_mean * self.y_mean)
        self.SR2 = self.sigma2 * self.size / (self.size - 2)
        self.a_max_err = t * (self.SR2 / self.size) ** .5
        self.b_max_err = t * (self.SR2 / x_SE) ** .5

        self.print_stat()

    def plot(self):
        plt.scatter(self.xs, self.ys)
        if (self.a and self.b):
            X = np.arange(min(self.xs), max(self.xs), 0.01)
            Y = self.a + self.b * (X - self.x_mean)
            plt.plot(X, Y, 'r')
        plt.show()

    def print_stat(self):
        print("{} (n = {})".format(self.name, self.size))
        print("{:-^60}".format(" statistics"))
        print("{:<13}{:<13}{:<13}".format("", "x", "y"))
        print("{:<13}{:<13.2f}{:<13.2f}".format("mean", self.x_mean,
self.y_mean))
        print("{:<13}{:<13.2f}{:<13.2f}\n".format("sample var", self.x_S2,
self.y_S2))
        print("{:-^60}".format(" regression "))
        print("{:<13}{:<13}{:<13}{:<13}{:<13}".format("", "a^", "b^", "σ2^",
"SR2"))
        print("{:<13}{:<13.2f}{:<13.2f}{:<13.2f}{:<13.2f}".format("estimator",
self.a, self.b, self.sigma2, self.SR2))
        print("{:<13}{:<13.2f}{:<13.2f}{:<13.2}{:<13.2}\n".format("max error",
self.a_max_err, self.b_max_err, "", ""))
```

```
dinosaur = LinReg("Dinosaur", os.path.join(sys.path[0], 'D.csv'))
star = LinReg("Star", os.path.join(sys.path[0], 'S.csv'))
dinosaur.plot()
star.plot()
```

Output

```
Dinosaur (n = 142)
------------------------ statistics-------------------------
             x            y
mean         54.26        47.83
sample var   281.07       725.52

------------------------ regression ------------------------
             a^           b^           σ2^          SR2
estimator    47.83        -0.10        1255.12      1273.05
max error    5.87         0.35

Star (n = 142)
------------------------ statistics-------------------------
             x            y
mean         54.27        47.84
sample var   281.20       725.24

------------------------ regression ------------------------
             a^           b^           σ2^          SR2
estimator    47.84        -0.10        1242.28      1260.03
max error    5.84         0.35
```

2

Denote the outcome of each coin flip by $X_1, \cdots X_n \overset{\text{i.i.d.}}{\sim} \text{Bernoulli}(p)$.

(a)

If n is large enough by CLT we have

$$\hat{p} = \bar{X} = \frac{\sum_{i=1}^{n} X_i}{n} \overset{\text{approx}}{\sim} N\left(p, \frac{p(1-p)}{n}\right) \approx N\left(p, \frac{\hat{p}(1-\hat{p})}{n}\right)$$

Hence a 90% C.I. for p can be approximated by

$$\hat{p} \pm z_{0.05}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$

In our example

$$\hat{p} = \frac{159}{314} = 0.5064, n = 314$$

Plugging in $z_{0.05} = 1.645$, we obtain the approximated 90% C.I. for p :

$$0.5064 \pm 0.0464$$

(b)

Keeping $\hat{p} = 50.64\%$, we want the maximum error at

$$z_{0.05}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}} = 0.01$$

Therefore

$$\begin{aligned} n &= \hat{p}(1-\hat{p})\left(\frac{0.01}{z_{0.05}}\right)^{-2} \\ &= 6764 \end{aligned}$$

(c)

That is

$$\hat{p} \pm z_{\alpha/2}\sqrt{\frac{\hat{p}(1-\hat{p})}{314}} = 0.445 \pm 0.055$$

Therefore

$$\begin{aligned} \hat{p} &= 0.445 \\ z_{\alpha/2} &= 0.055\sqrt{\frac{314}{0.445(1-0.445)}} \\ &= 1.9411 \\ \implies \alpha &= 0.0524 \end{aligned}$$

Thus (s)he is using $100(1-\alpha)\% \approx 95\%$ confidence level.

3

(a)

We have

$$\begin{aligned} \bar{X} = \frac{\sum_{i=1}^{n} X_i}{n} &\sim N(\mu_X, \sigma_X^2/n) \\ \bar{Y} = \frac{\sum_{j=1}^{m} Y_j}{m} &\sim N(\mu_Y, \sigma_Y^2/m) \end{aligned}$$

which are independent. Hence

$$\begin{aligned} \bar{X} - \bar{Y} &\sim N(\mu_X - \mu_Y, \sigma_X^2/n + \sigma_Y^2/m) \\ \frac{\bar{X} - \bar{Y} - (\mu_X - \mu_Y)}{\sqrt{\sigma_X^2/n + \sigma_Y^2/m}} &\sim N(0,1) \end{aligned}$$

It follows that

$$\begin{aligned} 1-\alpha &= P\left(-z_{\alpha/2} \le \frac{\bar{X} - \bar{Y} - (\mu_X - \mu_Y)}{\sqrt{\sigma_X^2/n + \sigma_Y^2/m}} \le z_{\alpha/2}\right) \\ &= P\left(\bar{X} - \bar{Y} - \epsilon \le \mu_X - \mu_Y \le \bar{X} - \bar{Y} + \epsilon\right) \end{aligned}$$

where $\epsilon = z_{\alpha/2}\sqrt{\sigma_X^2/n + \sigma_Y^2/m}$.

Therefore a two-sided $100(1-\alpha)\%$ C.I. for $\mu_X - \mu_Y$ is given by

$$\bar{x} - \bar{y} \pm z_{\alpha/2}\sqrt{\sigma_X^2/n + \sigma_Y^2/m}$$

(b)

We need

$$\begin{aligned}
n &= \arg\min_{n,m\in\mathbb{N}^2} \epsilon \quad \text{s.t.} \quad n+m=6000 \\
&= \arg\min_{n,m\in\mathbb{N}^2} z_{\alpha/2}\sqrt{\sigma_X^2/n + \sigma_Y^2/m} \quad \text{s.t.} \quad n+m=6000 \\
&= \arg\min_{n,m\in\mathbb{N}^2} \sigma_X^2/n + \sigma_Y^2/m \quad \text{s.t.} \quad n+m=6000 \\
&= \arg\min_{n\in\{0,1,\cdots,6000\}} \sigma_X^2/n + \sigma_Y^2/(6000-n) \\
&= \arg\min_{n\in\{0,1,\cdots,6000\}} 2500/n + 900/(6000-n) \\
&= 3750
\end{aligned}$$

samples from Company X.

4

Sample size $n = 12$.

(a)

A $100(1-\alpha)\%$ C.I. for μ is given by

$$\begin{aligned}
\bar{x} \pm z_{\alpha/2}(\sigma/\sqrt{n}) &= 41.83 \pm z_{\alpha/2}(11/\sqrt{12}) \\
&= 41.83 \pm z_{\alpha/2}\cdot 3.175
\end{aligned}$$

To obtain a 90% C.I., we set $\alpha = 0.1$, $z_{\alpha/2} = 1.645$, which gives

$$41.83 \pm 5.22$$

(b)

Set $\alpha = 0.05, z_{\alpha/2} = 1.96$. Thus a 95% C.I. for μ is given by

$$41.83 \pm 6.22$$

Set instead $\alpha = 0.01, z_{\alpha/2} = 2.576$. This yields a 99% C.I. for μ :

$$41.83 \pm 8.18$$

(c)

Without the knowledge of σ^2, we estimate it by S^2 and reach a $100(1-\alpha)\%$ C.I.:

$$\begin{aligned}
\bar{x} \pm t_{\alpha/2}(n-1)(s/\sqrt{n}) &= 41.83 \pm t_{\alpha/2}(11)(11.8/\sqrt{12}) \\
&= 41.83 \pm t_{\alpha/2}(11)\cdot 3.406
\end{aligned}$$

Set $\alpha = 0.1, t_{\alpha/2}(11) = 1.796$. The $100(1-\alpha)\%$ C.I. is given by

$$41.83 \pm 6.12$$

5

(a)

Consider

$$U := \sum_{i=1}^{n} \left(\frac{X_i - \mu}{\sigma} \right)^2 = \sum_{i=1}^{n} Z_i^2 = \sum_{i=1}^{n} C_i$$

where $Z_i \overset{\text{i.i.d}}{\sim} N(0,1), C_i \overset{\text{i.i.d}}{\sim} \chi^2(1)$.

Due to additivity of i.i.d. Chi-square RVs,

$$U \sim \chi^2(n)$$

Hence

$$\begin{aligned}
1 - \alpha &= P\left(\chi^2_{1-\alpha/2}(n) \leq U \leq \chi^2_{\alpha/2}(n) \right) \\
&= P\left(\chi^2_{1-\alpha/2}(n) \leq \sum_{i=1}^{n} \left(\frac{X_i - \mu}{\sigma} \right)^2 \leq \chi^2_{\alpha/2}(n) \right) \\
&= P\left(\chi^2_{1-\alpha/2}(n) \leq \frac{1}{\sigma^2} \sum_{i=1}^{n} (X_i - \mu)^2 \leq \chi^2_{\alpha/2}(n) \right) \\
&= P\left(\frac{\sum_{i=1}^{n} (X_i - \mu)^2}{\chi^2_{\alpha/2}(n)} \leq \sigma^2 \leq \frac{\sum_{i=1}^{n} (X_i - \mu)^2}{\chi^2_{1-\alpha/2}(n)} \right)
\end{aligned}$$

(b)

Consider

$$W := \sum_{i=1}^{n} \left(\frac{X_i - \bar{X}}{\sigma} \right)^2 = \frac{(n-1)S^2}{\sigma^2} \sim \chi^2(n-1)$$

Similarly

$$\begin{aligned}
1 - \alpha &= P\left(\chi^2_{1-\alpha/2}(n-1) \leq W \leq \chi^2_{\alpha/2}(n-1) \right) \\
&= P\left(\chi^2_{1-\alpha/2}(n-1) \leq \sum_{i=1}^{n} \left(\frac{X_i - \bar{X}}{\sigma} \right)^2 \leq \chi^2_{\alpha/2}(n-1) \right) \\
&= P\left(\chi^2_{1-\alpha/2}(n-1) \leq \frac{1}{\sigma^2} \sum_{i=1}^{n} (X_i - \bar{X})^2 \leq \chi^2_{\alpha/2}(n-1) \right) \\
&= P\left(\frac{\sum_{i=1}^{n} (X_i - \bar{X})^2}{\chi^2_{\alpha/2}(n-1)} \leq \sigma^2 \leq \frac{\sum_{i=1}^{n} (X_i - \bar{X})^2}{\chi^2_{1-\alpha/2}(n-1)} \right)
\end{aligned}$$

(c)

By (b),

$$\begin{aligned}
1 - \alpha &= P\left(\frac{\sum_{i=1}^{n} (X_i - \mu)^2}{\chi^2_{\alpha/2}(n-1)} \leq \sigma^2 \leq \frac{\sum_{i=1}^{n} (X_i - \mu)^2}{\chi^2_{1-\alpha/2}(n-1)} \right) \\
&= P\left(\sqrt{\frac{\sum_{i=1}^{n} (X_i - \mu)^2}{\chi^2_{\alpha/2}(n-1)}} \leq \sigma \leq \sqrt{\frac{\sum_{i=1}^{n} (X_i - \mu)^2}{\chi^2_{1-\alpha/2}(n-1)}} \right)
\end{aligned}$$

6

Let

$$f(\alpha, \beta) := \sum_{i=1}^{n} w_i (y_i - \alpha - \beta x_i)^2$$

FONC:

$$\nabla_\alpha f(\alpha, \beta) = \sum_{i=1}^{n} 2w_i (y_i - \alpha - \beta x_i)(-1) = 0$$
$$\nabla_\beta f(\alpha, \beta) = \sum_{i=1}^{n} 2w_i (y_i - \alpha - \beta x_i)(-x_i) = 0$$

Hence

$$\sum_i w_i (y_i - \alpha - \beta x_i) = \sum_i w_i y_i - \alpha \sum_i w_i - \beta \sum_i w_i x_i = 0$$
$$\sum_i w_i x_i (y_i - \alpha - \beta x_i) = \sum_i w_i y_i x_i - \alpha \sum_i w_i x_i - \beta \sum_i w_i x_i^2 = 0$$

Denote $A := \sum_i w_i y_i, B := \sum_i w_i, C := \sum_i w_i x_i, D := \sum_i w_i y_i x_i, E := \sum_i w_i x_i^2$. We then have

$$A - B\alpha - C\beta = 0$$
$$D - C\alpha - E\beta = 0$$

which yields

$$\alpha = \alpha^* = \frac{A}{B} - \frac{C}{B}\beta$$
$$\beta = \beta^* = \frac{BD - AC}{BE - C^2},$$

the suggested answer. To validate optimality we further calculate the Hessian at (α^*, β^*)

$$H(\alpha^*, \beta^*) = 2 \begin{bmatrix} B & C \\ C & E \end{bmatrix}$$

with eigenvalues λ_1, λ_2 satisfying

$$\begin{aligned}
(\lambda_1 + \lambda_2)/2 &= B + E > 0 \\
\lambda_1 \lambda_2 / 4 &= BE - C^2 \\
&= \sum_i w_i \sum_j w_j x_j^2 - \left(\sum_k w_k x_k \right)^2 \quad \implies \lambda_1, \lambda_2 > 0 \implies H \succ 0 \\
&= \sum_{i \neq j} w_i w_j x_i (x_i - x_j) \\
&= \sum_{i<j} w_i w_j (x_i - x_j)^2 > 0
\end{aligned}$$

Hence by SOSC, (α^*, β^*) is the unique minimizer.

7

(a)

$$\bar{x} = 3055.91, \bar{y} = 3317.91$$

(b)

A $100(1-\alpha)\%$ C.I. for $\mu_X - \mu_Y$ is

$$\bar{X} - \bar{Y} \pm t_{\alpha/2}(n+m-2) S_p \sqrt{\frac{1}{n} + \frac{1}{m}}$$

where

$$S_p^2 = \frac{(n-1)S_X^2 + (m-1)S_Y^2}{n+m-2}$$

is the pooled estimator of σ^2.

Since $n+m$ is large, we may use the approximation

$$\bar{X} - \bar{Y} \pm z_{\alpha/2} S_p \sqrt{\frac{1}{n} + \frac{1}{m}}$$

With $\alpha = 0.05$, plug in

$$\begin{gathered} n = 13391, m = 5672 \\ \bar{X} = \bar{x} = 3055.91, \bar{Y} = \bar{y} = 3317.91 \\ z_{\alpha/2} = 1.96, S_p = 1985.65 \end{gathered}$$

we have the pooled t-interval

$$-262.00 \pm 61.66$$

(c)

A $100(1-\alpha)\%$ Welch's t-interval for $\mu_X - \mu_Y$ is

$$\bar{X} - \bar{Y} \pm t_{\alpha/2}(r) \sqrt{\frac{S_X^2}{n} + \frac{S_Y^2}{m}}$$

where

$$r = \left\lfloor \frac{\left(\frac{S_X^2}{n} + \frac{S_Y^2}{m}\right)^2}{\frac{1}{n-1}\left(\frac{S_X^2}{n}\right)^2 + \frac{1}{m-1}\left(\frac{S_Y^2}{m}\right)^2} \right\rfloor$$

Since n, m are both large, r is also large. Hence we may use the approximation

$$\bar{X} - \bar{Y} \pm z_{\alpha/2} \sqrt{\frac{S_X^2}{n} + \frac{S_Y^2}{m}}$$

Plugging in the data, we obtain

$$-262.00 \pm 61.60$$

(d)

No. Instead, the data shows at 95% confidence level that there are *more* cars when the weather is "Rain".

Python Code

```
import csv
import os, sys
import matplotlib.pyplot as plt
import numpy as np

class Traffic:
    def __init__(self, file=None):
        self.xs = []    # traffic when clear
        self.n = 0
        self.x_mean = None
        self.x_S2 = None

        self.ys = []    # traffic when rain
        self.m = 0
        self.y_mean = None
        self.y_S2 = None

        self.Sp2 = None
        self.pooled_max_err = None
        self.welch_max_err = None

        if file:
            self.load_data(file)

    def load_data(self, file):
        self.__init__()
        with open(file, newline='') as f:
            reader = csv.reader(f)
            data = list(reader)[1::]
            self.xs = [float(d[-1]) for d in data if d[5] == 'Clear']
            self.ys = [float(d[-1]) for d in data if d[5] == 'Rain']
            self.n, self.m = len(self.xs), len(self.ys)
            self.x_mean = sum(self.xs) / self.n
            self.y_mean = sum(self.ys) / self.m
            self.x_S2 = sum((x - self.x_mean) ** 2 for x in self.xs) / (self.n -
1)
            self.y_S2 = sum((y - self.y_mean) ** 2 for y in self.ys) / (self.m -
1)
            num = (self.n - 1) * self.x_S2 + (self.m - 1) * self.y_S2
            den = self.n + self.m - 2
            self.Sp2 = num / den
            self.pooled_max_err = 1.96 * self.Sp2 ** .5 * (1 / self.n + 1 /
self.m) ** .5
            self.welch_max_err = 1.96 * (self.x_S2 / self.n + self.y_S2 /
self.m) ** .5

    def plot(self):
        plt.scatter(self.xs, self.ys)
        if (self.a and self.b):
            X = np.arange(min(self.xs), max(self.xs), 0.01)
            Y = self.a + self.b * (X - self.x_mean)
            plt.plot(X, Y, 'r')
        plt.show()

    def print_stat(self):
        print("{:-^60}".format(" statistics"))
        print("{:<13}{:<13}{:<13}".format("", "x (clear)", "y (rain)"))
```

```
        print("{:<13}{:<13}{:<13}".format("size", self.n, self.m))
        print("{:<13}{:<13.2f}{:<13.2f}".format("mean", self.x_mean,
self.y_mean))
        print("{:<13}{:<13.2f}{:<13.2f}".format("sample var", self.x_S2,
self.y_S2))
        print("{:<13}{:<13.2f}({:<.2f})".format("Sp2 (Sp)", self.Sp2, self.Sp2
** .5))
        print("{:<13}{:<13.2f}".format("pooled error", self.pooled_max_err))
        print("{:<13}{:<13.2f}\n".format("welch error", self.welch_max_err))

traffic = Traffic(os.path.join(sys.path[0], 'traffic.csv'))
traffic.print_stat()
```

Output

```
------------------------ statistics-------------------------
             x (clear)    y (rain)
size         13391        5672
mean         3055.91      3317.91
sample var   3948572.02   3929230.64
Sp2 (Sp)     3942817.60   (1985.65)
pooled error 61.66
welch error  61.60
```