CSC3100 Assignment 2

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Growth of Functions

Problem 1

Writing $n = 2^k$, we apply induction on k to show $T(n) = n\log n = k \cdot 2^k$

Base case: For $k = 1$,

$$T(2^k) = T(2^1) = 2 = 1 \cdot 2^1$$

Induction hypothesis: Assume for $k = m \geq 1$,

$$T(2^k) = T(2^m) = m \cdot 2^m$$

Induction step: For $k = m + 1 > 1$, we have

$$T(2^k) = T(2^{m+1}) = 2T(2^m) + 2^{m+1}$$

due to the recurrence. But by the hypothesis this is nothing but

$$2m \cdot 2^m + 2^{m+1} = (m+1)2^{m+1}$$

which completes the induction.

Problem 2

No.

For a reversed list L of length n, although it only takes, using binary search, $\Theta(\log(j-1)) = \Theta(\log j)$ time to find the insertion index of $\text{key} = L[j], j \in \{2, 3, \cdots, n\}$ (in this case the index would always be 1 as $L[j] < L[i]$ for all $i < j$), we have to shift the sorted subarray $L[1 : j-1]$ up one index before inserting key at index 1. Thus the insertion of key for each j would consist of the following commands

```
key = L[j] /* Θ(1) */
insert_idx = binary_search(key, L[1..j - 1]) /* Θ(log j) */
 /* i = j downto 2, therefore Θ(j) in total */
for i = j downto insert_idx + 1
    L[i] = L[i - 1] /* Θ(1) */
L[j] = key /* Θ(1) */
```

which takes $t(j) = \Theta(\log(j)) + \Theta(j) + \Theta(1) = \Theta(j)$ time. That is, there exist $c_1, c_2, J > 0$ s.t.

$$c_1 j \leq t(j) \leq c_2 j, \quad \forall j \geq J \tag{1}$$

Now for $n \geq \lceil J \rceil$, the total time complexity is given by the sum

$$T(n) = \sum_{j=2}^{n} t(j) = \sum_{j=2}^{\lfloor J \rfloor} t(j) + \sum_{j=\lceil J \rceil}^{n} t(j) = S + \sum_{j=\lceil J \rceil}^{n} t(j)$$

where we denote $S := \sum_{j=2}^{\lfloor J \rfloor} t(j)$. Therefore by (1) for all $n \geq \lceil J \rceil$,

$$
\begin{aligned}
S + c_1 \sum_{j=\lceil J \rceil}^{n} j \leq T(n) \leq S + c_2 \sum_{j=\lceil J \rceil}^{n} j \\
\underbrace{S + c_1 \frac{(n + \lceil J \rceil)(n - \lceil J \rceil + 1)}{2}}_{f(n)} \leq T(n) \leq \underbrace{S + c_2 \frac{(n + \lceil J \rceil)(n - \lceil J \rceil + 1)}{2}}_{g(n)}
\end{aligned}
$$

Clearly $f(n), g(n) = \Theta(n^2)$. So there exist $f_1, f_2, F, g_1, g_2, G > 0$ s.t.

$$
\begin{aligned}
f_1 n^2 \leq f(n) \leq f_2 n^2, \quad \forall n \geq F \\
g_1 n^2 \leq g(n) \leq g_2 n^2, \quad \forall n \geq G
\end{aligned}
$$

Define $M := \max\{F, G, \lceil J \rceil\}$. Then,

$$
f_1 n^2 \leq f(n) \leq T(n) \leq g(n) \leq g_2 n^2, \quad \forall n \geq M
$$

In other words $T(n) = \Theta(n^2) \neq \Theta(n \log n)$.

Problem 3

a.

Insertion-sorting each of the n/k subarrays of length k takes $\Theta(k^2)$ time, the sum of which is then

$$
\sum_{i=1}^{n/k} \Theta(k^2) = \frac{n}{k} \Theta(k^2) = \Theta(nk)
$$

b.

Merging two sorted array of length m takes in the worst case $\Theta(m)$ time. At each level $i \in \{1, 2, \cdots \log(n/k)\}$ of the merging tree we have $n/2^i k$ pairs of subarrays of length $2^{i-1}k$ to merge, taking $n/2^i k \cdot \Theta(2^{i-1}k) = \Theta(n)$ time. Hence the total running time in the worst case is $\log(n/k) \cdot \Theta(n) = \Theta(n \log(n/k))$.

c.

View $k := k(n)$ as some fixed function of n. We set

$$
\begin{aligned}
\Theta(nk + n\log(n/k)) &= \Theta(n \log n) \\
\implies \Theta(n \log n + nk - \log k) &= \Theta(n \log n) \\
\implies nk - \log k &= O(n \log n) \\
\implies nk &= O(n \log n) \\
\implies k &= O(\log n)
\end{aligned}
$$

d.

Try the algorithm on samples with large n and find the best choice of k.

Problem 4

a.

$(1,5), (2,5), (3,4), (3,5), (4,5)$.

b.

$[n, n-1, \cdots, 1]$. Each possible (i,j) with $i<j$ is an inversion, in total $n-1+n-2+\cdots+2+1 = n(n-1)/2$ of which.

c.

Suppose the array A has length n and number of inversions $m \in \{0, 1, \cdots, n(n-1)/2\}$. Then if we look at the algorithm

```
INSERTION-SORT(A)
    for j = 2 to n
        key = A[j]
        i = j - 1
        while i > 0 and A[i] > key
            A[i + 1] = A[i]
            i = i - 1
        A[i + 1] = key
```

The while-loop

```
while i > 0 and A[i] > key
    A[i + 1] = A[i]
    i = i - 1
```

should perform $\Theta(m)$ instructions within the entire execution. The reason is that for any $j \in \{2, \cdots, n\}$, (i,j) is an inversion in the original array

$$A^{(1)} = [A[1..j-1], A[j], A[j..n]]$$

if and only if (i', j) is an inversion in the $j-$partially sorted array

$$A^{(j)} = [\text{sorted}(L[1..j-1]), L[j], L[j..n]]$$

where i' is the index of element $A^{(1)}[i]$ in $A^{(j)}$. Therefore it takes

$$\begin{aligned} m^{(j)} &:= \#(i', j) : i' < j, \text{sorted}(A[1..j-1])[i'] > A[j] \\ &= \#(i', j) : (i', j) \text{ is an inversion in } A^{(j)} \\ &= \#(i, j) : (i, j) \text{ is an inversion in } A \end{aligned}$$

shifting operations of the last $m^{(j)}$ elements of $\text{sorted}(A[1..j-1])$ up one to make room for the element $A[j]$ (in the while loop).

The time complexity of the j-th for-loop is therefore $\Theta(m^{(j)})$, since operations outside the while loop are all constant-time. Summing over all j, the total complexity is given by

$$T(m) = \sum_{j=2}^{n} \Theta(m^{(j)}) = \Theta\left(\sum_{j=2}^{n} m^{(j)}\right) = \Theta(m).$$

d.

```
/* Merge sort A[p..r], return number of inversions (i, j) in A
   with p <= i, j <= r */
MSORT-COUNT-INV(A, p, r)
    if p < r
        q = floor((p + r) / 2)
        left_inv = MSORT-COUNT-INV(A, p, q)
        right_inv = MSORT-COUNT-INV(A, q + 1, r)
        cross_inv = MERGE-COUNT-CROSS-INV(A, p, q, r)
        inv = left_inv + right_inv + cross_inv
        return inv
    return 0

/* Merge A[p..q] and A[q+1..r], return number of inversions (i, j) in A
   with p <= i <= q && q + 1 <= j <= r. We assume p <= q < r. */
MERGE-COUNT-CROSS-INV(A, p, q, r)
    L = A[p..q]
    R = A[q+1..r]
    i = j = 1
    inv = 0
    while true
        idx = p + i + j - 2
        if L[i] > R[j]
            inv += r - j - idx + 1 /* Θ(1) added */
            A[idx] = R[j]
            j += 1
        else
            A[idx] = L[i]
            i += 1
        if i > q - p + 1
            A[q+j..r] = R[j..r-q]
            break
        else if j > r - q
            A[p+i-1..q] = L[i..q-p+1]
            break
    return inv
```

The above pseudocode merge-sorts an array `A` and returns the number of inversions. The structure of the algorithm is almost identical to the vanilla merge sort. The only difference is that in the merge subroutine we add some constant-time operations. Therefore this modified merge sort has the same recurrence relation as merge sort:

$$T(n) = 2 \cdot T(n/2) + \Theta(n)$$

which yields the familiar complexity

$$T(n) = \Theta(n \log n)$$

Problem 5

Denote $h(n) := \max\{f(n), g(n)\}$. Then

$$\begin{matrix} h(n) \geq f(n) \\ h(n) \geq g(n) \end{matrix} \implies h(n) \geq \frac{f(n) + g(n)}{2}$$

On the other hand, since f and g are asymptotically nonnegative, for sufficiently large n,

$$h(n) = \begin{cases} f(n), & \text{if } f(n) > g(n) \\ g(n), & \text{otherwise} \end{cases} \le f(n) + g(n)$$

Therefore

$$h(n) = \max\{f(n), g(n)\} = \Theta\left(f(n) + g(n)\right)$$

Problem 6

a.

$$\begin{array}{l}
\text{order} \longrightarrow \\
\boxed{1 \overset{\Theta}{=} n^{1/\log n}} \\
\|_o \\
\boxed{\log(\log^* n)} \overset{o}{=} \boxed{\log^*(\log n) \overset{\Theta}{=} \log^* n} \overset{o}{=} \boxed{2^{\log^* n}} \overset{o}{=} \boxed{\ln\ln n} \overset{o}{=} \boxed{\sqrt{\log n}} \\
\|_o \\
\boxed{\ln n} \overset{o}{=} \boxed{\log^2 n} \overset{o}{=} \boxed{2^{\sqrt{2\log n}}} \overset{o}{=} \boxed{\sqrt{2}^{\log n}} \\
\|_o \\
\boxed{n \overset{\Theta}{=} 2^{\log n}} \overset{o}{=} \boxed{n\log n} \overset{o}{=} \boxed{\log(n!)} \overset{o}{=} \boxed{n^2 \overset{\Theta}{=} 4^{\log n}} \overset{o}{=} \boxed{n^3} \overset{o}{=} \boxed{(\log n)!} \overset{o}{=} \boxed{(\log n)^{\log n} \overset{\Theta}{=} n^{\log\log n}} \\
\|_o \\
\boxed{(3/2)^n} \overset{o}{=} \boxed{2^n} \overset{o}{=} \boxed{n \cdot 2^n} \overset{o}{=} \boxed{e^n} \\
\|_o \\
\boxed{n!} \overset{o}{=} \boxed{(n+1)!} \\
\|_o \\
\boxed{2^{2^n}} \overset{o}{=} \boxed{2^{2^{n+1}}}
\end{array} \tag{2}$$

Note: Each box contains all functions of the same order, i.e., f, g are in the same box if and only if $f = \Theta(g)$. The little o notations are read from left to right, top to bottom. That is, functions on the upper levels are of lower orders than functions on levels below. On a same horizontal level, functions in the left boxes are of lower orders than functions in the right ones.

b.

Denote $\text{big}(n) := 2^{2^{n+1}}$, the fastest growing function in (2). Then $\text{DADDY}(n) := \text{big}^2(n)$ grows even faster, that is, for any function g in (2),

$$g = o(\text{DADDY})$$

Now define

$$f(n) := \begin{cases} 0, & \text{if } n \text{ is prime} \\ \text{DADDY}(n), & \text{otherwise} \end{cases}$$

Then no function g in (2) can upper-bound f asymptotically by any constant factor $c > 0$, since

$$\begin{aligned} g = o(\text{DADDY}) \implies \limsup_{n\to\infty} \frac{f}{cg}(n) &= \limsup_{n\to\infty} \begin{cases} \dfrac{\text{DADDY}}{cg}(n), & \text{if } n \text{ is prime} \\ 0, & \text{otherwise} \end{cases} \\ &= \limsup_{m\to\infty} \frac{\text{DADDY}}{cg}(p_m) \end{aligned}$$

where p_m is the m-th prime. As $m \to \infty, p_m \to \infty$ due to infinitude of primes. Thus $(\text{DADDY}/cg)_{m=1}^{\infty}$ is a subsequence of $(f/cg)_{n=1}^{\infty}$, and

$$\lim_{m\to\infty} \frac{\text{DADDY}}{cg}(p_m) = \lim_{n\to\infty} \frac{f}{cg}(n) = \infty$$

Hence

$$\limsup_{n\to\infty} \frac{f}{cg}(n) = \limsup_{m\to\infty} \frac{\text{DADDY}}{cg}(p_m) = \infty$$

which implies

$$f \neq O(g)$$

On the other hand, no g in (2) can either lower-bound f asymptotically by any constant factor $c > 0$, since

$$\liminf_{n\to\infty} \frac{f}{cg}(n) = 0$$

and so

$$f \neq \Omega(g)$$

Divide-and-Conquer

Problem 7

We wish to construct positive constants c, N s.t. for all $n \geq N$,

$$T(n) \leq cn \log n$$

through strong induction on n.

Base cases: For $2 \leq n \leq N, T(n) \leq cn \log n$.

Induction hypothesis: Assume for $N \leq n < k$, we have $T(n) \leq cn \log n$.

Induction step: For $n = k$, we utilize the recurrence relation to see

$$\begin{aligned}
T(k) - ck\log k &= 2T(\lfloor k/2 \rfloor + 17) + k - ck\log k \\
&\leq 2c(\lfloor k/2 \rfloor + 17)\log(\lfloor k/2 \rfloor + 17) + k - ck\log k && (2 \leq \lfloor k/2 \rfloor + 17 < k) \\
&\leq c(k + 34)\log(k/2 + 17) + k - ck\log k \\
&= ck\log(1/2 + 17/k) + 34c\log(k/2 + 17) + k \\
&\leq -ck/2 + 34c\log(k/2 + 17) + k && (k \geq 83) \\
&\leq -ck/2 + 34c\log(k) + k && (k \geq 34) \\
&\leq (1 - c/2)k + 34c\sqrt{k} && (k \geq 16) \\
&\leq 0 && (c > 2, \sqrt{k} \geq 68/(1 - 2/c) > 68)
\end{aligned}$$

For the induction step and base cases to hold we need the intersection of all conditions previously imposed on constant k, c to be true:

$$k > 68^2, c > 2, T(n) \leq cn\log n, \quad 2 \leq n \leq N$$

which can be easily achieved by picking

$$N := 68^2, c := \max\left\{\frac{T(n)}{n\log n} : 2 \leq n \leq N\right\} + 3$$

One may verify that the above induction works under this set of constants. Hence,

$$T(n) = O(n\log n)$$

Elementary Data Structures

Problem 8

Using existing Python `Queue` class,

```
from queue import Queue
class QStack:
    def __init__(self):
        self.inQ = Queue()
        self.outQ = Queue()
        self.top = None

    # len: θ(1)
    def __len__(self):
        return self.inQ.qsize()

    # is_empty: θ(1)
    def is_empty(self):
        return len(self) == 0

    # push: θ(1)
    def push(self, item):
        self.inQ.put(item)
        self.top = item

    # pop: θ(n)
    def pop(self):
        assert not self.is_empty()
        if len(self) == 1: return self.inQ.get()
        while True:
            item = self.inQ.get()
            if len(self) == 1:
                self.top = item
                self.outQ.put(item)
                item = self.inQ.get()
                self.inQ, self.outQ = self.outQ, self.inQ
                return item
            self.outQ.put(item)

    # peek: θ(1)
    def peek(self):
        assert not self.is_empty()
        return self.top
```

- Both `__len__` and `is_empty` invoke `qsize` method from the `Queue` class to retrieve the queue length, which is $\Theta(1)$.
- `push` invokes `put` method from the `Queue` class once, which takes $\Theta(1)$ time; it then assign `item` to the `top` field, taking $\Theta(1)$ as well. So the total complexity of `push` is $\Theta(1)$.
- `pop` successively `get` all n items from the `inQ` and, for each item, performs constant-time operations on it. Since `get` method is $\Theta(1)$, the `pop` operation takes $n \cdot \Theta(1) = \Theta(n)$ time. Note in the end of the while loop, the swapping of `inQ` and `outQ` is essentially a swapping of two pointers and only takes constant time.
- `peek` takes $\Theta(1)$ as it only retrieves the `top` data field of the object.

Problem 9.

Using existing Python `sllist` class from `llist` module,

```python
from llist import sllist
class LLStack:
    def __init__(self):
        self.ll = sllist()

    # len: θ(1)
    def __len__(self):
        return self.ll.size

    # push: θ(1)
    def push(self, value):
        # Equivalent to L.LIST-INSERT(L.head) in the book
        return self.ll.appendleft(value)

    # pop: θ(1)
    def pop(self):
        assert len(self) > 0
        # Equivalent to L.LIST-DELETE(L.head) in the book
        return self.ll.popleft()

    # peek: θ(1)
    def peek(self):
        assert len(self) > 0
        return self.ll.first.value
```

Heap

Problem 10

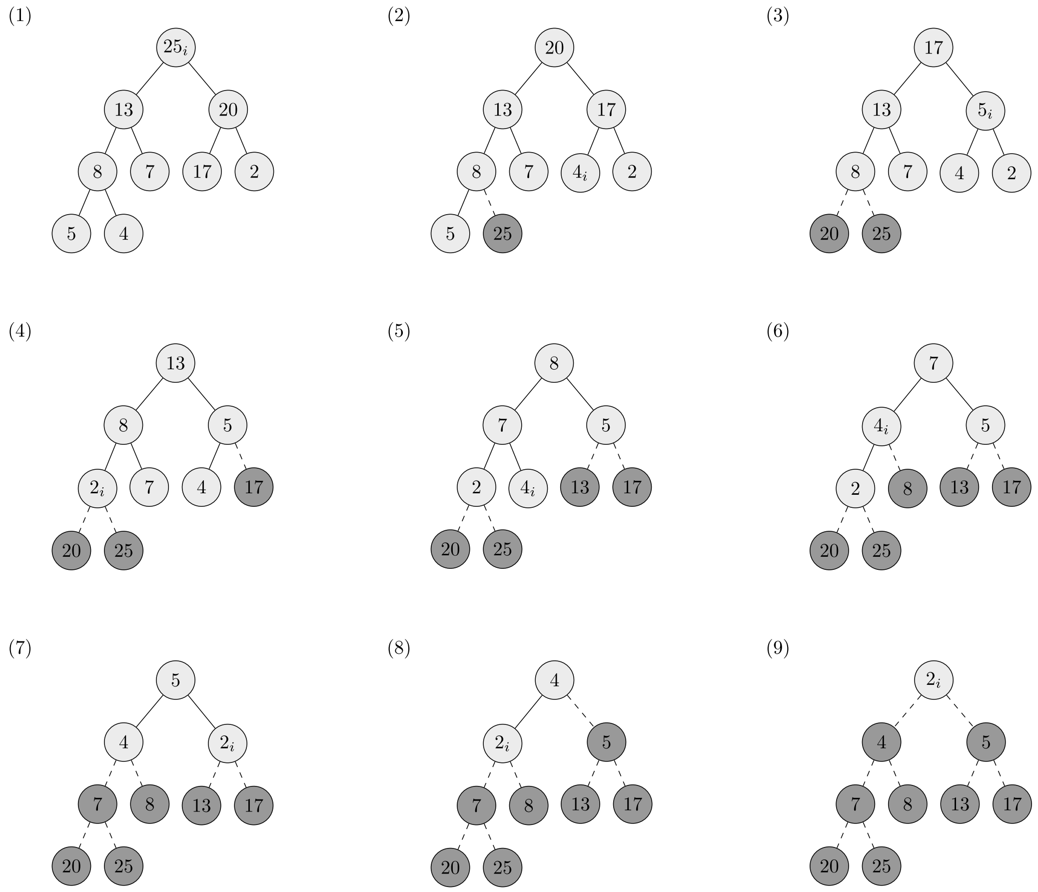
Suppose that the n-element heap is represented by array A, then for $1 \leq i \leq n$,

$$A[i] \text{ is a leave} \iff A[i] \text{ has no child} \iff 2i > n \iff i \geq \frac{n}{2}$$

which is equivalent to

$$i \in \left\{ \left\lfloor \frac{n}{2} \right\rfloor + 1, \cdots, n \right\}$$

Problem 11



A

2	4	5	7	8	13	17	20	25