Assignment [redacted] 6

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Characteristic polynomial of A is

$$p_A(\lambda) = \lambda^2 + \lambda - 6 = (\lambda + 3)(\lambda - 2). \tag{1}$$

Hence the eigenvalues for A are $\lambda_1 = -3, \lambda_2 = 2$.

Let

$$(\lambda I - A)x = 0, \tag{2}$$

yielding eigenvectors for A :

$$x_1 = [-3, 2]^T, x_2 = [1, 1]^T. \tag{3}$$

Characteristic polynomial of A^2 is

$$p_{A^2}(\lambda) = \lambda^2 - 13\lambda + 36 = (\lambda - 9)(\lambda - 4). \tag{4}$$

Hence the eigenvalues for B are $\lambda_1' = 9, \lambda_2' = 4$.

Let

$$(\lambda' I - A^2)x = 0, \tag{5}$$

yielding eigenvectors for A^2 :

$$x_1 = [-3, 2]^T, x_2 = [1, 1]^T. \tag{6}$$

A^2 has the same ***eigenvectors*** as A. When A has eigenvalues λ_1 and λ_2, A^2 has eigenvalues λ_1^2, λ_2^2. In this example, $\lambda_1^2 + \lambda_2^2 = 13 = \text{tr}(A^2)$.

2

Characteristic polynomial of A :

$$\begin{aligned}
p_A(\lambda) &= \begin{vmatrix} \lambda I - B & -C \\ 0 & \lambda I - D \end{vmatrix} \\
&= \left| \begin{bmatrix} I & 0 \\ 0 & \lambda I - D \end{bmatrix} \begin{bmatrix} \lambda I - B & -C \\ 0 & I \end{bmatrix} \right| \\
&= \begin{vmatrix} I & 0 \\ 0 & \lambda I - D \end{vmatrix} \cdot \begin{vmatrix} I & -C \\ 0 & \lambda I - B \end{vmatrix} \\
&= |\lambda I - D| \cdot |\lambda I - B| \\
&= (p_D \cdot p_B)(\lambda).
\end{aligned}$$

Hence the eigenvalues of A are exactly those of B and D, namely
$\lambda_A = 1, 2, 5, 7$.

3

$1 \neq n = 2$.

4

$$p_A(\lambda) = \lambda^2 - 25\lambda = \lambda(\lambda - 25). \tag{7}$$

The eigenvalues of A are $\lambda_1 = 0, \lambda_2 = 25$.

Let $(\lambda I - A)x = 0$. A specific solution is $x_1 = [-4/5, 3/5], x_2 = [3/5, 4/5]^T$.

Hence there are exactly 8 orthogonal matrices that diagonalize A :

$$Q_{1,2} = \begin{bmatrix} \pm 4/5 & 3/5 \\ \mp 3/5 & 4/5 \end{bmatrix}, Q_{3,4} = \begin{bmatrix} \pm 4/5 & -3/5 \\ \mp 3/5 & -4/5 \end{bmatrix}, \tag{8}$$
$$Q_{5,6} = \begin{bmatrix} 3/5 & \pm 4/5 \\ 4/5 & \mp 3/5 \end{bmatrix}, Q_{7,8} = \begin{bmatrix} -3/5 & \pm 4/5 \\ -4/5 & \mp 3/5 \end{bmatrix}.$$

5

If $\lambda = a + ib$ is an eigenvalue of a real matrix A, then $Ax = \lambda x$ for some $x \neq 0$. Then

$$A\overline{x} = \overline{(Ax)} = \overline{(\lambda x)} = \overline{\lambda}\overline{x}. \tag{9}$$

Thus $\overline{\lambda}$ is also an eigenvalue of A, corresponding to the eigenvector $\overline{x}$.

From the previous proven proposition, any real 3×3 matrix A has characteristic polynomial of the form

$$p_A(\lambda) = (\lambda - \lambda_1)(\lambda - \overline{\lambda}_1)(\lambda - \lambda_2). \tag{10}$$

Note that the constant term in the polynomial is $r \cdot \lambda_2$, where $r = -\lambda_1 \overline{\lambda}_1 \in \mathbb{R}$. But $p_A(\lambda) = |\lambda I - A_{\mathbb{R}^{3\times 3}}|$, hence the constant term must be real, forcing λ_2 also to be real.

6

$$B = S^{-1}AS \implies SA = BS \implies S(A - \lambda I) = (B - \lambda I)S. \tag{11}$$

Since $|S| \neq 0$, it follows that A and B has the same characteristic polynomial:

$$p_A(\lambda) = p_B(\lambda). \tag{12}$$

Thus A and B has the same eigenvalues and hence same diagonalization.

$$Q_A^T A Q_A = \Lambda = Q_B^T B Q_B \implies B = (Q_B Q_A^T) A (Q_A Q_B^T). \tag{13}$$

The proof is done by noting $M := Q_B Q_A^T$ as an orthonormal matrix, and from (13) we have

$$B = MAM^T. \tag{14}$$

7

The characteristic polynomial of A :

$$p_A(\lambda) = (\lambda - \cos\theta)^2 + \sin^2\theta = \lambda^2 - 2\cos\theta \cdot \lambda + 1. \tag{15}$$

has complex roots $e^{\pm i\theta}$, also being eigenvalues of A corresponding to the eigenvectors $[1, \pm i]^T$, whenever $\theta \neq k\pi, k \in \mathbb{Z}$.

The geometric interpretation of the result is that A corresponds to an anticlockwise rotation (denoted as R) in $\mathbb{C}^2$ by an angle of θ, i.e. in the basis $\{[1, \pm i]^T\}$,

$$R([x, y]^T) = [e^{i\theta}x, e^{-i\theta}y]^T. \tag{16}$$

8

(a)

$$x^Tx = x^TQ^TQx = (Qx)^TQx = (\lambda x^T)(\lambda x) = \lambda^2(x^Tx) \implies \lambda^2 = 1. \tag{17}$$

Equivalently,

$$|\lambda| = 1. \tag{18}$$

(b)

$$QQ^T = I \implies |QQ^T| = |Q||Q^T| = |Q|^2 = 1. \tag{19}$$

Equivalently

$$|\det(Q)| = 1. \tag{20}$$

9

$$A(Sx) = ASx = (SBS^{-1}S)x = SBx = S(\lambda x) = \lambda(Sx). \tag{21}$$

10

(a)

$$< z_1, z_2 >= z_2^H z_1 = \frac{1-i}{2\sqrt{2}} + \frac{-(1-i)}{2\sqrt{2}} = 0. \tag{22}$$

$$< z_1, z_1 > = (2+2)/4 = 1 \tag{23}$$

$$< z_2, z_2 > = (1+1)/2 = 1. \tag{24}$$

(b)

$$z = 4z_1 + 2\sqrt{2}z_2. \tag{25}$$

11

(a)

$$\begin{gathered} u_1^H z = (4+2i)u_1^H u_1 = 4+2i \\ z^H u_1 = (u_1^H z)^H = 4-2i. \end{gathered} \tag{26}$$

$$\begin{gathered} u_2^H z = (6-5i)u_2^H u_2 = 6-5i \\ z^H u_2 = (u_2^H z)^H = 6+5i. \end{gathered} \tag{27}$$

(b)

$$
\begin{aligned}
||z|| &= \sqrt{z^H z} \\
&= \sqrt{[4-2i, 6+5i]\begin{bmatrix}4+2i\\6-5i\end{bmatrix}} \\
&= \sqrt{20+61} \\
&= 9.
\end{aligned}
$$

12

(a); (c).

13

$$U^H = \overline{(I-2uu^H)^T} = \overline{I-2\bar{u}u^T} = I-2uu^H = U. \tag{28}$$

Therefore U is Hermitian. Further,

$$UU^H = U^H U = (I-2uu^H)^2 = I - 4uu^H + 4(uu^H)^2 = I. \tag{29}$$

Hence U is also unitary and, consequently,

$$U^{-1} = U^H = U. \tag{30}$$

14

```
julia> using LinearAlgebra

julia> A = [1 2;3 4];
```

(a)

Suppose $A_{m\times n}$. Then

$$||A||_F^2 = \sum_{j=1}^{n}\sum_{i=1}^{m} a_{ij}^2 = \underbrace{\sum_{j=1}^{n}(a_j)^T a_j}_{\substack{a_j \text{ is the } j^{\text{th}} \\ \text{column of } A}} = \text{tr}(A^T A). \tag{31}$$

Taking the square root on both sides yields

$$||A||_F = \sqrt{\text{tr}(A^T A)} \tag{32}$$

as desired.

```
julia> tr(A'A) == tr(A*A') == sum((x -> x^2).(A))
true
```

(b)

```
julia> eigen(A'A).values
2-element Array{Float64,1}:
  0.13393125268149486
 29.866068747318508
```

15

```
julia> using LinearAlgebra
```

(a)

$$x_{k+1} = \begin{bmatrix} g_{k+2} \\ g_{k+1} \end{bmatrix} = \underbrace{\begin{bmatrix} 1-w & w \\ 1 & 0 \end{bmatrix}}_{A} \begin{bmatrix} g_{k+1} \\ g_k \end{bmatrix} = Ax_k. \tag{33}$$

(b)

$$p_A(\lambda) = \lambda^2 + (w-1)\lambda - w = (\lambda + w)(\lambda - 1). \tag{34}$$

Hence the eigenvalues of A are $\lambda_1 = -w, \lambda_2 = 1$.

Let $(A - \lambda I)x = 0$, we have corresponding eigenvectors
$x_1 = 1/\sqrt{w^2+1} \cdot [-w, 1]^T, x_2 = 1/\sqrt{2} \cdot [1, 1]^T$.

```
julia> w = .1;

julia> A = [1 - w w
            1 0];

julia> v1, v2 = [1;1] / sqrt(2), [-w;1] / sqrt(w^2 + 1);

julia> vals = [1; -w]; vecs = [v1 v2];

julia> (eigen(A).values, eigen(A).vectors) == (vals, vecs)
true
```

(c)

$\lambda_{1,2} \to \mp 1$ respectively; $x_{1,2} \to 1/\sqrt{2} \cdot [\pm 1, 1]^T$ respectively.

$\{x_1, x_2\}$ forms an orthonormal basis in the limit.

For $w = -1, x_1 = x_2 = 1/\sqrt{2} \cdot [1, 1]^T$. $\{x_1, x_2\}$ is linearly dependent and therefore does not form a basis. For the same reason, $[x_1, x_2]$ is non-invertible. Hence by definition A is no longer diagonalizable.

(d)

$$x_k = A^k x_0 = S\Lambda^k S^{-1} x_0. \tag{35}$$

The columns of S are the eigenvectors of A. Now as $k \to \infty, \Lambda^k = \mathrm{diag}[(-w)^k, 1] \to \mathrm{diag}(0, 1)$ given $0 < w < 1$. Hence g_k always converges to some constant (possibly zero.) In fact, as to be shown in (37), $g_k \to (1+w)^{-1}(wg_0 + g_1)$.

(e)

$$A^k = S\Lambda^k S^{-1} \to \frac{1}{1+w}\begin{bmatrix} 1 & w \\ 1 & w \end{bmatrix} =: B. \tag{36}$$

(f)

Using (35) and (36),

$$g_k \to (Bx_0)_{11} = \frac{1}{1+w}(wg_0 + g_1). \tag{37}$$

Plugging in the initial condition,

$$g_k \to \frac{1}{1+0.5}(0.5 \cdot 0 + 1) = \frac{2}{3}. \tag{38}$$

(g)

By (35) and the initial condition,

$$g_k = \frac{2}{3}[(-1)^{k+1}2^{-k} + 1], \tag{39}$$

and thus

$$\left|\frac{g_{k+1} - 2/3}{g_k - 2/3}\right| = \frac{2^{-k-1}}{2^{-k}} = \frac{1}{2}. \tag{40}$$

In other words,

$$|g_k - 2/3| \sim (1/2)^k. \tag{41}$$

This is verified by the numerical computation:

```
julia> n = 0:24;

julia> error(x) = abs(2/3*((-1)^(x+1)*.5^x)); # |gₖ - 2/3|

julia> expo(x) = .5^x;

julia> error.(n) ./ expo.(n) # |gₖ - 2/3| ∝ (1/2) ^ k
25-element Array{Float64,1}:
 0.6666666666666666
 0.6666666666666666
 0.6666666666666666
 0.6666666666666666
```

```
0.6666666666666666
⋮
0.6666666666666666
0.6666666666666666
0.6666666666666666
0.6666666666666666
0.6666666666666666
```

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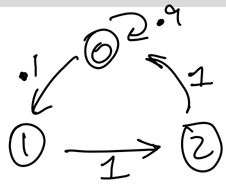
Sketch: 

The (post-multiplying) transition matrix:

$$\mathbf{P} = \begin{bmatrix} .9 & .1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}. \tag{42}$$

The steady-state probabilities are given numerically by

```
julia> P=[.9 .1 0; 0 0 1; 1 0 0];

julia> P^100
3×3 Array{Float64,2}:
 0.833333  0.0833333  0.0833333
 0.833333  0.0833333  0.0833333
 0.833333  0.0833333  0.0833333
```

That is, with any initial probabilities $\mathbf{x}^{(0)} = \begin{bmatrix} p_0^{(0)} & p_1^{(0)} & p_2^{(0)} \end{bmatrix}$,

$$\lim_{n\to\infty} \mathbf{x}^{(n)} = \mathbf{x}^{(0)} \lim_{n\to\infty} \mathbf{P}^n = [\,5/6 \quad .5/6 \quad .5/6\,]. \tag{43}$$

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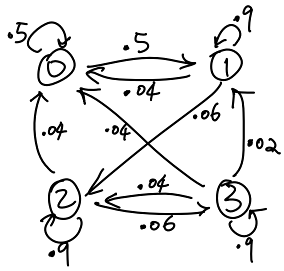
Sketch: 

The (post-multiplying) transition matrix:

$$\mathbf{P} = \begin{bmatrix} .5 & .5 & 0 & 0 \\ .04 & .9 & .06 & 0 \\ .04 & 0 & .9 & .06 \\ .04 & .02 & .04 & .9 \end{bmatrix}. \tag{44}$$

```
julia> P=[.5 .5 0 0
          .04 .9 .06 0
          .04 0 .9 .06
          .04 .02 .04 .9];

julia> P^1000
4×4 Array{Float64,2}:
 0.0740741  0.40913  0.322997  0.193798
 0.0740741  0.40913  0.322997  0.193798
 0.0740741  0.40913  0.322997  0.193798
 0.0740741  0.40913  0.322997  0.193798
```

Hence given any initial probabilities $\mathbf{x}^{(0)} = \begin{bmatrix} p_0^{(0)} & p_1^{(0)} & p_2^{(0)} \end{bmatrix}$,

$$\lim_{n\to\infty} \mathbf{x}^{(n)} = \mathbf{x}^{(0)} \lim_{n\to\infty} \mathbf{P}^n \approx [\,.074 \quad .409 \quad .323 \quad .194\,]\,. \tag{45}$$