Assignment 7

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$$Av_1 = [-1, 0, 1]^T = v_1 . \tag{41}$$

$$Av_2 = [4, -4, 4]^T = 4v_1 . \tag{1}$$

Hence v_1, v_2 are indeed eigenvectors of A with eigenvalues $\lambda_1 = 1, \lambda_2 = 4$.

Normalizing v_1 and v_2,

$$q_1 := \tilde{v}_1 = [-1, 0, 1]^T/\sqrt{2}. \tag{2}$$

$$q_2 := \tilde{v}_2 = [1, -1, 1]^T/\sqrt{3}. \tag{3}$$

Now let

$$q_3 := q_1 \times q_2 = [1, 2, 1]^T/\sqrt{6}. \tag{4}$$

We have

$$Aq_3 = [1, 2, 1]^T/\sqrt{6} = q_3. \tag{5}$$

q_3 is also an eigenvector of A with eigenvalue $\lambda_3 = 1$.

Now, since $\{q_1, q_2, q_3\}$ forms an orthonormal basis, by the spectral theorem,

$$A = Q\Gamma Q^T, \tag{6}$$

where $Q = [q_1, q_2, q_3]$, and $\Gamma = \text{diag}(\lambda_1, \lambda_2, \lambda_3) = \text{diag}(1, 4, 1)$.

2

$$(B^T AB)^T = B^T A^T B = B^T AB. \tag{7}$$

$$(B^T B)^T = B^T (B^T)^T = B^T B. \tag{8}$$

$$(BB^T)^T = (B^T)^T B^T = BB^T. \tag{9}$$

3

(a)

False.

(b)

True.

(c)

False.

(d)

True.

4

$$x^T Ax = \sum_{i,j} a_{ij} x_i x_j = \sum_i a_{ii} x_i^2 + 2 \sum_{i<j} a_{ij} x_i x_j. \tag{10}$$

(a)

By (10),

$$x^T A x = 3x_1^2 + 2x_2^2 + 2(2x_1x_2 + x_2x_3). \tag{11}$$

(b)

Plugging in (11),

$$\begin{aligned} x^T A x &= 12 + 2 + 2(2 \cdot 2 + (-5)) \\ &= 14 + 2(-1) \\ &= 12. \end{aligned}$$

(c)

Plugging in (11),

$$\begin{aligned} x^T A x &= (3 + 2 + 2(2 + 1))/2 \\ &= (5 + 6)/2 \\ &= \frac{11}{2}. \end{aligned}$$

5

$$A^T A = \begin{bmatrix} 9 & -9 \\ -9 & 9 \end{bmatrix}. \tag{12}$$

Characteristic polynomial of $A^T A$ is

$$p_{A^T A}(\lambda) = (\lambda - 9)^2 - 81 = \lambda(\lambda - 18) \implies \lambda_1 = 18, \lambda_2 = 0. \tag{13}$$

Hence the singular values of A are $\sigma_1 = 3\sqrt{2}, \sigma_2 = 0$.

To find the right singular vectors, set

$$(\lambda I - A^T A)v = 0, \tag{14}$$

yielding

$$v_{1,2} = [1, \mp 1]^T/\sqrt{2}. \tag{15}$$

Let

$$\begin{aligned} u_1 &:= Av_1/\sigma_1 = [1, -2, -2]^T/3, \\ u_2 &:= [2, 1, 0]^T/\sqrt{5}, \\ u_3 &:= u_1 \times u_2 = [2, -4, 5]^T/\sqrt{45}. \end{aligned}$$

Then

$$A = U \Sigma V^T \tag{16}$$

where

$$
\begin{aligned}
U &:= [u_1, u_2, u_3] = \begin{bmatrix} 1/3 & 2/\sqrt{5} & 2/\sqrt{45} \\ -2/3 & 1/\sqrt{5} & -4/\sqrt{45} \\ 2/3 & 0 & 5/\sqrt{45} \end{bmatrix}, \\
V^T &:= [v_1, v_2]^T = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}, \\
\Sigma &:= \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 3\sqrt{2} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.
\end{aligned}
$$

6

Recall that

$$M \text{ has positive eigenvalues } \iff x^T M x > 0 \; \forall x. \tag{17}$$

Therefore $x^T A x, x^T B x > 0 \; \forall x$. It immediately follows that

$$x^T (A + B) x = x^T A x + x^T B x > 0 + 0 = 0 \; \forall x, \tag{18}$$

which is equivalent to say that $A + B$ has positive eigenvalues.

7

Suppose A has eigenvalue decomposition (since A is assumed to be symmetric)

$$A = Q \Lambda Q^T. \tag{19}$$

$Q = [q_1, \ldots, q_n]$ has q_i = normalized eigenvectors as columns, and

$\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_n)$, where λ_i are eigenvalues corresponding q_i.

Now consider the SVD

$$A = U \Sigma V^T. \tag{20}$$

$V = [v_1, \ldots v_n]$ where v_i are eigenvectors of $A^T A$, in this case, A^2. But we already know A and A^2 always shares the same eigenvectors. Hence $V = Q$.

$\Sigma = \text{diag}(\sqrt{\tilde{\lambda}_1}, \ldots \sqrt{\tilde{\lambda}_n})$, where $\tilde{\lambda}_i$ are the eigenvalues of $A^T A = A^2$, which we knows to be λ_i^2 :

$$\tilde{\lambda}_i = \lambda_i^2 \; \forall i. \tag{21}$$

If $A \succeq 0$, all eigenvalues of A are greater than zero. Taking square roots on both sides of (21) yields

$$\sqrt{\tilde{\lambda}_i} = \lambda_i \; \forall i. \tag{22}$$

Therefore

$$\Sigma = \text{diag}(\lambda_1, \ldots \lambda_n) = \Lambda, \tag{23}$$

which means SVD for a positive-definite matrix is identical to the eigenvalue decomposition:

$$A = U\Sigma V^T = Q\Lambda Q. \tag{24}$$

In the case when $A \preceq 0$, (22) becomes

$$\sqrt{\tilde{\lambda}_i} = -\lambda_i \ \forall i. \tag{25}$$

$$\Sigma = \operatorname{diag}(-\lambda_1, \ldots -\lambda_n) = -\Lambda. \tag{26}$$

Hence SVD becomes

$$A = U\Sigma V^T = (-Q)(-\Lambda)Q^T, \tag{27}$$

or

$$A = U\Sigma V^T = Q(-\Lambda)(-Q^T). \tag{28}$$

8

Let $\tilde{v} = v/\lambda$, then

$$\lambda v = Av \implies v = A\tilde{v} \implies v \in \text{Col } A. \tag{29}$$

9

Using SVD,

$$A = U\Sigma V^T. \tag{30}$$

$$A^T = V\Sigma U^T. \tag{31}$$

Then

$$AA^T = U\Sigma V^T V\Sigma U^T = U\Sigma^2 U^T. \tag{32}$$

$$A^T A = V\Sigma U^T U\Sigma V^T = V\Sigma^2 V^T. \tag{33}$$

Let $Q := VU^T$, which is orthogonal. (32) and (33) combined yields

$$AA^T = U(V^T V)\Sigma^2(V^T V)U^T = UV^T(A^T A)VU^T = Q^T(A^T A)Q, \tag{34}$$

as desired.

10

```
julia> using LinearAlgebra

julia> A = [
    1 0 2+2im 0 3-3im
    0 4 0 5 0
    6-6im 0 7 0 8+8im
    0 9 0 1 0
    2+2im 0 3-3im 0 4
    ];
```

(a)

```
julia> H_u = Hermitian(A)
5×5 Hermitian{Complex{Int64},Array{Complex{Int64},2}}:
   1+0im  0+0im  2+2im  0+0im  3-3im
   0+0im  4+0im  0+0im  5+0im  0+0im
   2-2im  0+0im  7+0im  0+0im  8+8im
   0+0im  5+0im  0+0im  1+0im  0+0im
   3+3im  0+0im  8-8im  0+0im  4+0im

julia> H_l = Hermitian(A, :L)
5×5 Hermitian{Complex{Int64},Array{Complex{Int64},2}}:
   1+0im  0+0im  6+6im  0+0im  2-2im
   0+0im  4+0im  0+0im  9+0im  0+0im
   6-6im  0+0im  7+0im  0+0im  3+3im
   0+0im  9+0im  0+0im  1+0im  0+0im
   2+2im  0+0im  3-3im  0+0im  4+0im
```

observation.

$$A \neq A^H \implies H_u(A) \neq H_l(A). \tag{35}$$

(b)

```
julia> A = Array(Hermitian(rand(Complex{Float64},5,5)));

julia> H_u = Hermitian(A)
5×5 Hermitian{Complex{Float64},Array{Complex{Float64},2}}:
  0.95988+0.0im       0.998716+0.361859im  …
 0.772618+0.487604im
 0.998716-0.361859im  0.650059+0.0im
 0.661267+0.159806im
 0.579079-0.613569im  0.473588-0.759278im
0.224817+0.274035im
 0.992797-0.45108im    0.12611-0.987278im
 0.12303+0.650897im
 0.772618-0.487604im  0.661267-0.159806im
0.386485+0.0im

julia> H_l = Hermitian(A, :L)
```

```
5×5 Hermitian{Complex{Float64},Array{Complex{Float64},2}}:
  0.95988+0.0im       0.998716+0.361859im  …
 0.772618+0.487604im
 0.998716-0.361859im  0.650059+0.0im
 0.661267+0.159806im
 0.579079-0.613569im  0.473588-0.759278im
0.224817+0.274035im
 0.992797-0.45108im    0.12611-0.987278im
 0.12303+0.650897im
 0.772618-0.487604im  0.661267-0.159806im
0.386485+0.0im

julia> H_u == H_l
true
```

observation.

$$A = A^H \implies H_u(A) = H_l(A). \tag{36}$$

11

```
julia> A=rand(5,5);

julia> U,S,V=svd(A);

julia> A-U*Diagonal(S)*V'
5×5 Array{Float64,2}:
 -5.55112e-16  -6.66134e-16  …  -3.33067e-16  -4.996e-16
  0.0          -6.245e-16       -3.88578e-16   0.0
 -3.33067e-16  -5.55112e-16     -1.66533e-16  -1.04083e-16
 -3.33067e-16  -2.22045e-16     -1.11022e-16  -1.11022e-16
  0.0          -3.05311e-16     -8.32667e-17   0.0
```

observation.

$$A = USV^T. \tag{37}$$

12

```
julia> m = 100; n = 80; p = 120;

julia> A = rand(m, n); B = rand(p, n);

julia> U, V, Q, D1, D2, R0 = svd(A, B);

julia> norm(A - U * D1 * R0 * Q')
7.243315372697871e-13

julia> norm(B - V * D2 * R0 * Q')
8.25995256920901e-13
```

observation.

$$A = UD_1R_0Q^T. \tag{38}$$

$$B = VD_2R_0Q^T. \tag{39}$$

13

code.

```
using Random, StatsBase
Random.seed!(1);

A = Set(['a','e','i','o','u']);
B = Set(['x','y','z']);
omega = 'a':'z';
N = 1e6;

println("mcEst1 \t \tmcEst2")
for _ in 1:5
    mcEst1 = sum([in(sample(omega),A) ||
in(sample(omega),B) for _ in 1:N])/N
    mcEst2 = sum([in(sample(omega),union(A,B)) for _ in
1:N])/N
    println(mcEst1,"\t",mcEst2)
end
```

output.

```
mcEst1          mcEst2
0.285158        0.307668
0.285686        0.307815
0.285022        0.308132
0.285357        0.307261
0.285175        0.306606
```

analysis.

The estimation given by mcEst2 is correct, as

$$P(A \cup B) = \frac{5+3}{26} = \frac{4}{13} \approx 0.3077. \tag{40}$$

mcEst1 is a faulty estimator because it provokes *sample* function twice as supposed to once. This means after checking one sample's belongness to A a new sample is drawn for checking belongness to B. mcEst1 really estimates the probability that two i.i.d. discrete RV's, $X_1, X_2 \sim \mathcal{U}(1, 26)$ take value $x_1 \leq 5$ or $x_2 \leq 3$, namely,

$$\begin{aligned} P(x_1 \leq 5 \cup x_2 \leq 3) &= P(x_1 \leq 5) + P(x_2 \leq 3) - P(x_1 \leq 5 \cap x_2 \leq 3) \\ &= F_{X_1}(5) + F_{X_2}(3) - F_{X_1}(5) \cdot F_{X_2}(3) \\ &= \frac{5}{26} + \frac{3}{26} - \frac{5 \cdot 3}{26 \cdot 26} \\ &= \frac{4}{13} - \frac{15}{676} \\ &\approx 0.2855. \end{aligned}$$

This coincides with the numerical result.

14

code.

```
N = 1e7;
a = b = ab = 0;
# a counts for A; b counts for B; ab counts for AB (13)

for _ in 1:N
    tens, ones = divrem(rand(10:25),10)
    if (tens, ones) == (1,3)
        global a += 1; global b += 1; global ab += 1;
    elseif tens == 1
        global a += 1
    elseif ones == 3
        global b += 1
    end
end

fA = a/N; fB = b/N;
fAB = ab/N
fAfB = fA*fB
```

output.

```
julia> fAB = ab/N
0.0625051

julia> fAfB = fA*fB
0.07811757925852
```

Indeed, the numerical result suggests that A, B are dependent events.