3D Transformations (I)

Computer graphics deals primarily with transformations happening in $\mathbb{R}^3$. Most notably, the families of *linear*, *affine*, and *projective* transformations.

Linear transformations in $\mathbb{R}^3$

Any linear transformation $\mathcal{T} : \mathbb{R}^3 \to \mathbb{R}^3$ can be uniquely represented as a 3x3 real matrix $\mathbf{M}$ whose columns can be found by tracking the canonical basis vectors after the transformation:

$$\mathbf{M} = [\,\mathcal{T}(\hat{\mathbf{x}}) \quad \mathcal{T}(\hat{\mathbf{y}}) \quad \mathcal{T}(\hat{\mathbf{z}})\,] \tag{1}$$

for which

$$\mathcal{T}(\mathbf{v}) = \mathbf{M}\mathbf{v} \quad \forall \mathbf{v} \in \mathbb{R}^3 \tag{2}$$

Important linear transforms

- ***Change of basis***

  $$\mathbf{v}' = \mathbf{M}\mathbf{v} = [\,\mathbf{i} \quad \mathbf{j} \quad \mathbf{k}\,]\,\mathbf{v} \tag{3}$$

  A linear transformation that takes the canonical basis $\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}$ to vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$, respectively.

  This transformation can also be thought of as a change of basis from the $\mathbf{i}, \mathbf{j}, \mathbf{k}$ coordinate system to the $\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}$ system.

  Conversely, to express a (canonical-basis) vector in the $\mathbf{i}, \mathbf{j}, \mathbf{k}$ system, one applies the inverse transform represented by the inverse matrix $\mathbf{M}^{-1} = [\,\mathbf{i} \quad \mathbf{j} \quad \mathbf{k}\,]^{-1}$.

- ***Scaling***
  - **Along the x, y, z axes, each by $\mathbf{s} = s_x, s_y, s_z$**

    $$\mathbf{v}' = \mathbf{M}\mathbf{v} = \begin{bmatrix} s_x \cdot v_x \\ s_y \cdot v_y \\ s_z \cdot v_z \end{bmatrix} = \operatorname{diag}(\mathbf{s}) \cdot \mathbf{v} \tag{4}$$

  - **Along three (linearly independent) axes $\mathbf{i}, \mathbf{j}, \mathbf{k}$, each by $\mathbf{s} = s_i, s_j, s_k$**

    This can be achieved by

    1. linearly transforming $\mathbf{i}, \mathbf{j}, \mathbf{k}$ to the canonical basis vectors by $\mathbf{B}^{-1} = [\,\mathbf{i} \quad \mathbf{j} \quad \mathbf{k}\,]^{-1}$

       (Equivalently, a change of basis from the canonical to $\mathbf{i}, \mathbf{j}, \mathbf{k}$)
    2. applying the canonical transform (4) above
    3. linearly transforming the canonical basis vectors back to $\mathbf{i}, \mathbf{j}, \mathbf{k}$ by $\mathbf{B} = [\,\mathbf{i} \quad \mathbf{j} \quad \mathbf{k}\,]$.

       (Equivalently, a change of basis from $\mathbf{i}, \mathbf{j}, \mathbf{k}$ to the canonical)

    Combining the steps:

    $$\mathbf{v}' = \mathbf{M}\mathbf{v} = (\mathbf{B} \cdot \operatorname{diag}(\mathbf{s}) \cdot \mathbf{B}^{-1})\mathbf{v} \tag{5}$$

    We will see this general change-of-basis technique being frequently used.

  - Finally, any negative scale factor yields a ***reflection***.

- ***Rotation***
  - **About the $x, y,$ or z axis by θ**
    - x axis: Keep $\hat{\mathbf{x}}$ still and rotate the y-z plane.

$$\mathbf{v}' = \mathbf{R}_x(\theta)\mathbf{v} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \mathbf{v} \tag{6}$$

- y axis: Keep $\hat{\mathbf{y}}$ still and rotate the z-x (note the order) plane.

$$\mathbf{v}' = \mathbf{R}_y(\theta)\mathbf{v} = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \mathbf{v} \tag{7}$$

  We see $-\theta$ in place of the expected θ as a consequence of the right-handedness of the basis ($\hat{\mathbf{x}} \times \hat{\mathbf{z}} = -\hat{\mathbf{y}}$ instead of $\hat{\mathbf{y}}$).

- z axis: Keep $\hat{\mathbf{z}}$ still and rotate the x-y plane.

$$\mathbf{v}' = \mathbf{R}_z(\theta)\mathbf{v} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{v} \tag{8}$$

- **About an arbitrary axis $\hat{\mathbf{n}}$ by θ (Rodrigues' Formula)**

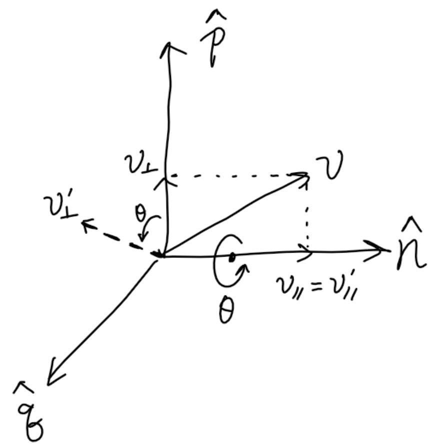


Fig: Rotation on $\mathbf{v}$ about (unit) vector $\hat{\mathbf{n}}$

A rotation on $\mathbf{v}$ about an arbitrary axis $\hat{\mathbf{n}}$ can be decomposed into two steps:

1. Express $\mathbf{v}$ as a combination of parallel and perpendicular components of $\hat{\mathbf{n}}$

   (Effectively writing $\mathbf{v}$ in a new orthonormal $\hat{\mathbf{n}}, \hat{\mathbf{p}}, \hat{\mathbf{q}}$ basis)

2. Rotate the perpendicular component while keeping the parallel component unchanged

   (Rotation about the $\hat{\mathbf{n}}$ axis, can be expressed neatly as a matrix)

First, compute components of $\mathbf{v}$:

$$\begin{aligned} \mathbf{v}_{/\!/} &= \hat{\mathbf{n}} \cdot \hat{\mathbf{n}}^\top \mathbf{v} \\ \mathbf{v}_\perp &= \mathbf{v} - \mathbf{v}_{/\!/} = (\mathbf{I} - \hat{\mathbf{n}}\hat{\mathbf{n}}^\top)\mathbf{v} \end{aligned}$$

Take the basis vectors

$$\begin{aligned} \hat{\mathbf{p}} &= \frac{\mathbf{v}_\perp}{\|\mathbf{v}_\perp\|} \\ \hat{\mathbf{q}} &= \hat{\mathbf{n}} \times \hat{\mathbf{p}} = \frac{\hat{\mathbf{n}} \times \mathbf{v}}{\|\mathbf{v}_\perp\|} \end{aligned} \tag{9}$$

so that

$$\mathbf{v} = [\hat{\mathbf{n}} \quad \hat{\mathbf{p}} \quad \hat{\mathbf{q}}] \begin{bmatrix} \|\mathbf{v}_{/\!/}\| \\ \|\mathbf{v}_\perp\| \\ 0 \end{bmatrix} \tag{10}$$

Rotating about the $\hat{\mathbf{n}}$ axis then becomes easy:

$$
\begin{aligned}
\mathbf{v}' &= [\hat{\mathbf{n}} \quad \hat{\mathbf{p}} \quad \hat{\mathbf{q}}] \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \|\mathbf{v}_{/\!/}\| \\ \|\mathbf{v}_\perp\| \\ 0 \end{bmatrix} \\
&= \|\mathbf{v}_{/\!/}\|\hat{\mathbf{n}} + \|\mathbf{v}_\perp\|(\cos\theta \cdot \hat{\mathbf{p}} + \sin\theta \cdot \hat{\mathbf{q}}) \\
&= \mathbf{v}_{/\!/} + \cos\theta \cdot \mathbf{v}_\perp + \sin\theta \cdot (\hat{\mathbf{n}} \times \mathbf{v}) \\
&= \hat{\mathbf{n}}\hat{\mathbf{n}}^\top \mathbf{v} + \cos\theta \cdot (\mathbf{I} - \hat{\mathbf{n}}\hat{\mathbf{n}}^\top)\mathbf{v} + \sin\theta \cdot \mathbf{N}\mathbf{v} \\
&= [(1-\cos\theta)\hat{\mathbf{n}}\hat{\mathbf{n}}^\top + \cos\theta \cdot \mathbf{I} + \sin\theta \cdot \mathbf{N}]\mathbf{v}
\end{aligned}
$$

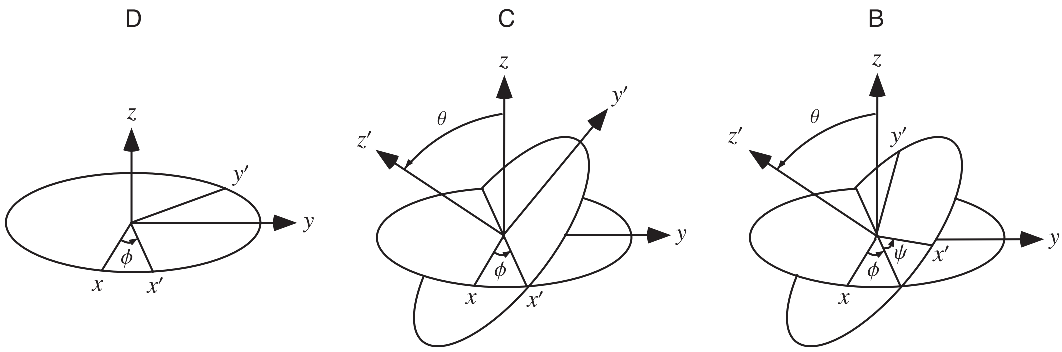
where

$$
\mathbf{N} = \begin{bmatrix} 0 & -\hat{n}_z & \hat{n}_y \\ \hat{n}_z & 0 & -\hat{n}_x \\ -\hat{n}_y & \hat{n}_x & 0 \end{bmatrix} \tag{11}
$$

This is **Rodrigues' Formula**.

- **As Euler angles**

  Euler's rotation theorem says any rotation can be expressed as three consecutive axis-aligned sub-rotations. For example, the figure below illustrates a z-x'-z' rotation with Euler angles (ϕ, θ, ψ).



  We shall next derive an expression of the rotation matrix $\mathbf{M} = \mathbf{R}_{BCD}$ in terms of (ϕ, θ, ψ).

  The first rotation D is given by Eq. (8)

$$
\mathbf{R}_D = \mathbf{R}_z(\phi) \tag{12}
$$

  To apply a second rotation C on top we resort to the change-of-basis technique: First undo D, then apply rotation about the x-axis, and finally apply D again:

$$
\mathbf{R}_{CD} = \mathbf{R}_D \mathbf{R}_x(\theta) \mathbf{R}_D^{-1} \mathbf{R}_D = \mathbf{R}_z(\phi)\mathbf{R}_x(\theta) \tag{13}
$$

  Adding the final rotation B is similar: Undo the previous rotations, apply the rotation about the z-axis, and bring the previous rotations back in the end:

$$
\begin{aligned}
&\mathbf{R}_{BCD} = \mathbf{R}_{CD}\mathbf{R}_z(\psi)\mathbf{R}_{CD}^{-1}\mathbf{R}_{CD} = \mathbf{R}_z(\phi)\mathbf{R}_x(\theta)\mathbf{R}_z(\psi) \\
&= \begin{bmatrix} \cos\phi\cos\psi - \sin\phi\cos\theta\sin\psi & -\cos\phi\sin\psi - \sin\phi\cos\theta\cos\psi & \sin\phi\sin\theta \\ \sin\phi\cos\psi + \cos\phi\cos\theta\sin\psi & -\sin\phi\sin\psi + \cos\phi\cos\theta\cos\psi & -\cos\phi\sin\theta \\ \sin\theta\sin\psi & \sin\theta\cos\psi & \cos\theta \end{bmatrix}
\end{aligned} \tag{14}
$$

  Note the reversed order of matrix multiplications.

Affine transformations in $\mathbb{R}^3$

Homogeneous coordinates

We associate each point $(x, y, z) \in \mathbb{R}^3$ with a collection of points in the homogeneous (projective) space $\mathbb{P}^3$:

$$
\mathbb{R}^3 \ni (x, y, z) \quad \sim \quad (xw, yw, zw, w) \in \mathbb{P}^3, \qquad w \neq 0 \tag{15}
$$

Conversely, each point $(x, y, z, w), w \neq 0$ in the projective space $\mathbb{P}^3$ represents the 3D Cartesian *point* $(x/w, y/w, z/w)$:

$$\mathbb{P}^3 \ni (x, y, z, w) \quad \sim \quad (x/w, y/w, z/w) \in \mathbb{R}^3, \qquad w \neq 0 \tag{16}$$

In particular, $w = 1$ gives the "canonical" homogeneous coordinates $(x, y, z, 1)$ of a 3D point (x, y, z).

For graphics purposes it is convenient to define $(x, y, z, 0) \in \mathbb{P}^3$ as representing the 3D Cartesian *vector* (x, y, z). This definition has the desirable properties that points in $\mathbb{P}^3$ representing

- Vector + Vector = Vector
- Point + Vector (or Vector + Point) = Point

in $\mathbb{R}^3$.

Homogeneous coordinates make it possible to write any affine transformation $\mathcal{A} : \mathbb{R}^3 \to \mathbb{R}^3$

$$\mathcal{A}(\mathbf{v}) = \mathcal{T}(\mathbf{v}) + \mathbf{t} = \mathbf{M}\mathbf{v} + \mathbf{t} \tag{17}$$

as a linear transformation in $\mathbb{P}^3$. Such a transformation can be uniquely represented as a 4x4 real matrix

$$\mathbf{F} = \left[\begin{array}{c|c} \mathbf{M} & \mathbf{t} \\ \mathbf{0}^\top & 1 \end{array}\right] \tag{18}$$

that acts on the canonical homogeneous coordinates $\tilde{\mathbf{v}} = [\mathbf{v}; 1]$:

$$\mathbf{F}\tilde{\mathbf{v}} = \begin{bmatrix} \mathbf{M}\mathbf{v} + \mathbf{t} \\ 1 \end{bmatrix} \sim \mathcal{A}(\mathbf{v}) \tag{19}$$

Important affine transforms

- ***Pure translation***

$$\tilde{\mathbf{v}}' = \mathbf{F}\tilde{\mathbf{v}} = \begin{bmatrix} v_x + t_x \\ v_y + t_y \\ v_z + t_z \\ 1 \end{bmatrix} = \left[\begin{array}{c|c} \mathbf{I} & \mathbf{t} \\ \mathbf{0}^\top & 1 \end{array}\right] \tilde{\mathbf{v}} \tag{20}$$

- ***Pure linear transform***

$$\tilde{\mathbf{v}}' = \mathbf{F}\tilde{\mathbf{v}} = \begin{bmatrix} \mathbf{M}\mathbf{v} \\ 1 \end{bmatrix} = \left[\begin{array}{c|c} \mathbf{M} & \mathbf{0} \\ \mathbf{0}^\top & 1 \end{array}\right] \tilde{\mathbf{v}} \tag{21}$$

- ***Inverse of an affine transform***

  Suppose we are given an affine transform $\mathbf{v} \mapsto \mathbf{M}\mathbf{v} + \mathbf{t}$ represented by the matrix

$$\mathbf{F} = \left[\begin{array}{c|c} \mathbf{M} & \mathbf{t} \\ \mathbf{0}^\top & 1 \end{array}\right] \tag{22}$$

  The inverse matrix

$$\mathbf{F}^{-1} = \left[\begin{array}{c|c} \mathbf{M}^{-1} & -\mathbf{M}^{-1}\mathbf{t} \\ \mathbf{0}^\top & 1 \end{array}\right] \tag{23}$$

  represents another affine transform

$$\mathbf{v} \mapsto \mathbf{M}^{-1}\mathbf{v} - \mathbf{M}^{-1}\mathbf{t} = \mathbf{M}^{-1}(\mathbf{v} - \mathbf{t}) \tag{24}$$

  Note the order: The inverse transform first undoes the translation, and then reverses the linear transform.

- ***Rotation about an axis of direction $\hat{\mathbf{n}}$ centered at $\mathbf{c}$***

  We already knew how to rotate about an axis centered at zero. To rotate around one that is not, simply apply the change-of-basis technique:

  1. Translate the center $\mathbf{c}$ to $\mathbf{0}$ (Pure translation)
  2. Apply Rodrigues' Formula (Linear)
  3. Translate $\mathbf{0}$ back to $\mathbf{c}$ (Pure translation)

References

[1] https://mathworld.wolfram.com/EulerAngles.html

[2] https://sites.cs.ucsb.edu/~lingqi/teaching/games101.html

[3] https://mathfor3dgameprogramming.com/

Next: Model-View-Projection; Projective transforms

To be continued.