

3D Transformations (I)

Computer graphics deals primarily with transformations happening in \mathbb{R}^3 . Most notably, the families of *linear*, *affine*, and *projective* transformations.

Linear transformations in \mathbb{R}^3

Any linear transformation $\mathcal{T} : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ can be uniquely represented as a 3x3 real matrix \mathbf{M} whose columns can be found by tracking the canonical basis vectors after the transformation:

$$\mathbf{M} = [\mathcal{T}(\hat{\mathbf{x}}) \quad \mathcal{T}(\hat{\mathbf{y}}) \quad \mathcal{T}(\hat{\mathbf{z}})] \quad (1)$$

for which

$$\mathcal{T}(\mathbf{v}) = \mathbf{M}\mathbf{v} \quad \forall \mathbf{v} \in \mathbb{R}^3 \quad (2)$$

Important linear transforms

- **Change of basis**

$$\mathbf{v}' = \mathbf{M}\mathbf{v} = [\mathbf{i} \quad \mathbf{j} \quad \mathbf{k}] \mathbf{v} \quad (3)$$

A linear transformation that takes the canonical basis $\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}$ to vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$, respectively.

This transformation can also be thought of as a change of basis from the $\mathbf{i}, \mathbf{j}, \mathbf{k}$ coordinate system to the $\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}$ system.

Conversely, to express a (canonical-basis) vector in the $\mathbf{i}, \mathbf{j}, \mathbf{k}$ system, one applies the inverse transform represented by the inverse matrix $\mathbf{M}^{-1} = [\mathbf{i} \quad \mathbf{j} \quad \mathbf{k}]^{-1}$.

- **Scaling**

- **Along the x, y, z axes, each by $s = s_x, s_y, s_z$**

$$\mathbf{v}' = \mathbf{M}\mathbf{v} = \begin{bmatrix} s_x \cdot v_x \\ s_y \cdot v_y \\ s_z \cdot v_z \end{bmatrix} = \text{diag}(\mathbf{s}) \cdot \mathbf{v} \quad (4)$$

- **Along three (linearly independent) axes $\mathbf{i}, \mathbf{j}, \mathbf{k}$, each by $s = s_i, s_j, s_k$**

This can be achieved by

1. linearly transforming $\mathbf{i}, \mathbf{j}, \mathbf{k}$ to the canonical basis vectors by $\mathbf{B}^{-1} = [\mathbf{i} \quad \mathbf{j} \quad \mathbf{k}]^{-1}$
(Equivalently, a change of basis from the canonical to $\mathbf{i}, \mathbf{j}, \mathbf{k}$)
2. applying the canonical transform (4) above
3. linearly transforming the canonical basis vectors back to $\mathbf{i}, \mathbf{j}, \mathbf{k}$ by $\mathbf{B} = [\mathbf{i} \quad \mathbf{j} \quad \mathbf{k}]$.
(Equivalently, a change of basis from $\mathbf{i}, \mathbf{j}, \mathbf{k}$ to the canonical)

Combining the steps:

$$\mathbf{v}' = \mathbf{M}\mathbf{v} = (\mathbf{B} \cdot \text{diag}(\mathbf{s}) \cdot \mathbf{B}^{-1})\mathbf{v} \quad (5)$$

We will see this general change-of-basis technique being frequently used.

- Finally, any negative scale factor yields a **reflection**.

- **Rotation**

- **About the x, y , or z axis by θ**

- x axis: Keep $\hat{\mathbf{x}}$ still and rotate the y - z plane.

$$\mathbf{v}' = \mathbf{R}_x(\theta)\mathbf{v} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta \\ 0 & \sin \theta & \cos \theta \end{bmatrix} \mathbf{v} \quad (6)$$

- y axis: Keep $\hat{\mathbf{y}}$ still and rotate the z - x (note the order) plane.

$$\mathbf{v}' = \mathbf{R}_y(\theta)\mathbf{v} = \begin{bmatrix} \cos \theta & 0 & \sin \theta \\ 0 & 1 & 0 \\ -\sin \theta & 0 & \cos \theta \end{bmatrix} \mathbf{v} \quad (7)$$

We see $-\theta$ in place of the expected θ as a consequence of the right-handedness of the basis ($\hat{\mathbf{x}} \times \hat{\mathbf{z}} = -\hat{\mathbf{y}}$ instead of $\hat{\mathbf{y}}$).

- z axis: Keep $\hat{\mathbf{z}}$ still and rotate the x - y plane.

$$\mathbf{v}' = \mathbf{R}_z(\theta)\mathbf{v} = \begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{v} \quad (8)$$

- **About an arbitrary axis $\hat{\mathbf{n}}$ by θ (Rodrigues' Formula)**

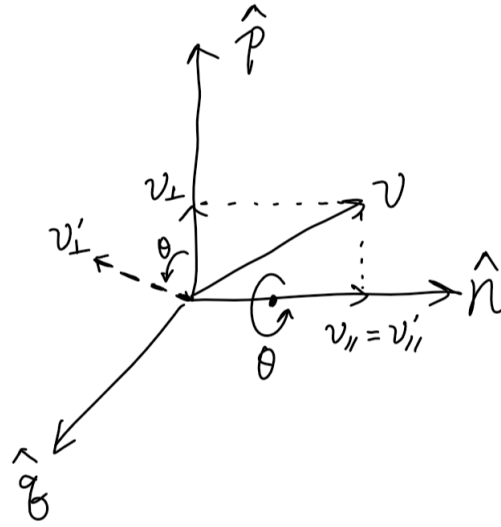


Fig: Rotation on \mathbf{v} about (unit) vector $\hat{\mathbf{n}}$

A rotation on \mathbf{v} about an arbitrary axis $\hat{\mathbf{n}}$ can be decomposed into two steps:

1. Express \mathbf{v} as a combination of parallel and perpendicular components of $\hat{\mathbf{n}}$
(Effectively writing \mathbf{v} in a new orthonormal $\hat{\mathbf{n}}, \hat{\mathbf{p}}, \hat{\mathbf{q}}$ basis)
2. Rotate the perpendicular component while keeping the parallel component unchanged
(Rotation about the $\hat{\mathbf{n}}$ axis, can be expressed neatly as a matrix)

First, compute components of \mathbf{v} :

$$\mathbf{v}_{\parallel} = \hat{\mathbf{n}} \cdot \hat{\mathbf{n}}^{\top} \mathbf{v}$$

$$\mathbf{v}_{\perp} = \mathbf{v} - \mathbf{v}_{\parallel} = (\mathbf{I} - \hat{\mathbf{n}}\hat{\mathbf{n}}^{\top})\mathbf{v}$$

Take the basis vectors

$$\hat{\mathbf{p}} = \frac{\mathbf{v}_{\perp}}{\|\mathbf{v}_{\perp}\|} \quad (9)$$

$$\hat{\mathbf{q}} = \hat{\mathbf{n}} \times \hat{\mathbf{p}} = \frac{\hat{\mathbf{n}} \times \mathbf{v}}{\|\mathbf{v}_{\perp}\|}$$

so that

$$\mathbf{v} = [\hat{\mathbf{n}} \quad \hat{\mathbf{p}} \quad \hat{\mathbf{q}}] \begin{bmatrix} \|\mathbf{v}_{\parallel}\| \\ \|\mathbf{v}_{\perp}\| \\ 0 \end{bmatrix} \quad (10)$$

Rotating about the $\hat{\mathbf{n}}$ axis then becomes easy:

$$\begin{aligned}
\mathbf{v}' &= [\hat{\mathbf{n}} \quad \hat{\mathbf{p}} \quad \hat{\mathbf{q}}] \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta \\ 0 & \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} \|\mathbf{v}_{\parallel}\| \\ \|\mathbf{v}_{\perp}\| \\ 0 \end{bmatrix} \\
&= \|\mathbf{v}_{\parallel}\| \hat{\mathbf{n}} + \|\mathbf{v}_{\perp}\| (\cos \theta \cdot \hat{\mathbf{p}} + \sin \theta \cdot \hat{\mathbf{q}}) \\
&= \mathbf{v}_{\parallel} + \cos \theta \cdot \mathbf{v}_{\perp} + \sin \theta \cdot (\hat{\mathbf{n}} \times \mathbf{v}) \\
&= \hat{\mathbf{n}} \hat{\mathbf{n}}^{\top} \mathbf{v} + \cos \theta \cdot (\mathbf{I} - \hat{\mathbf{n}} \hat{\mathbf{n}}^{\top}) \mathbf{v} + \sin \theta \cdot \mathbf{N} \mathbf{v} \\
&= [(1 - \cos \theta) \hat{\mathbf{n}} \hat{\mathbf{n}}^{\top} + \cos \theta \cdot \mathbf{I} + \sin \theta \cdot \mathbf{N}] \mathbf{v}
\end{aligned}$$

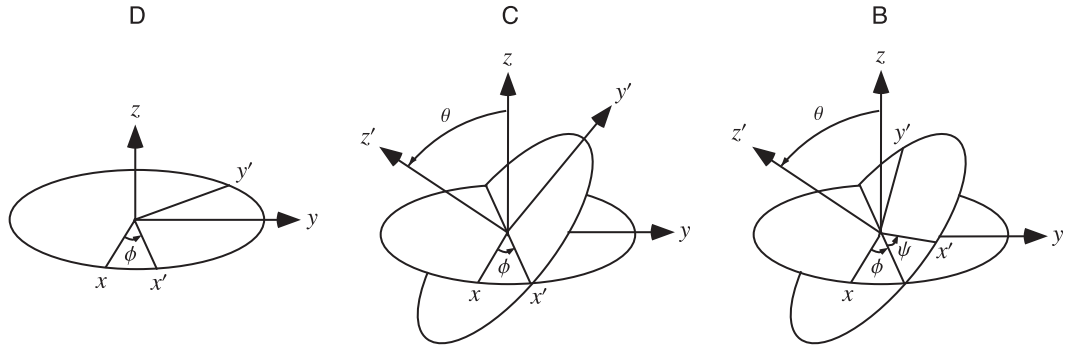
where

$$\mathbf{N} = \begin{bmatrix} 0 & -\hat{n}_z & \hat{n}_y \\ \hat{n}_z & 0 & -\hat{n}_x \\ -\hat{n}_y & \hat{n}_x & 0 \end{bmatrix} \quad (11)$$

This is **Rodrigues' Formula**.

◦ As Euler angles

Euler's rotation theorem says any rotation can be expressed as three consecutive axis-aligned sub-rotations. For example, the figure below illustrates a z - x' - z' rotation with Euler angles (ϕ, θ, ψ) .



We shall next derive an expression of the rotation matrix $\mathbf{M} = \mathbf{R}_{BCD}$ in terms of (ϕ, θ, ψ) .

The first rotation D is given by Eq. (8)

$$\mathbf{R}_D = \mathbf{R}_z(\phi) \quad (12)$$

To apply a second rotation C on top we resort to the change-of-basis technique: First undo D , then apply rotation about the x -axis, and finally apply D again:

$$\mathbf{R}_{CD} = \mathbf{R}_D \mathbf{R}_x(\theta) \mathbf{R}_D^{-1} \mathbf{R}_D = \mathbf{R}_z(\phi) \mathbf{R}_x(\theta) \quad (13)$$

Adding the final rotation B is similar: Undo the previous rotations, apply the rotation about the z -axis, and bring the previous rotations back in the end:

$$\begin{aligned}
\mathbf{R}_{BCD} &= \mathbf{R}_{CD} \mathbf{R}_z(\psi) \mathbf{R}_{CD}^{-1} \mathbf{R}_{CD} = \mathbf{R}_z(\phi) \mathbf{R}_x(\theta) \mathbf{R}_z(\psi) \\
&= \begin{bmatrix} \cos \phi \cos \psi - \sin \phi \cos \theta \sin \psi & -\cos \phi \sin \psi - \sin \phi \cos \theta \cos \psi & \sin \phi \sin \theta \\ \sin \phi \cos \psi + \cos \phi \cos \theta \sin \psi & -\sin \phi \sin \psi + \cos \phi \cos \theta \cos \psi & -\cos \phi \sin \theta \\ \sin \theta \sin \psi & \sin \theta \cos \psi & \cos \theta \end{bmatrix}
\end{aligned} \quad (14)$$

Note the reversed order of matrix multiplications.

Affine transformations in \mathbb{R}^3

Homogeneous coordinates

We associate each point $(x, y, z) \in \mathbb{R}^3$ with a collection of points in the homogeneous (projective) space \mathbb{P}^3 :

$$\mathbb{R}^3 \ni (x, y, z) \sim (xw, yw, zw, w) \in \mathbb{P}^3, \quad w \neq 0 \quad (15)$$

Conversely, each point (x, y, z, w) , $w \neq 0$ in the projective space \mathbb{P}^3 represents the 3D Cartesian point $(x/w, y/w, z/w)$:

$$\mathbb{P}^3 \ni (x, y, z, w) \sim (x/w, y/w, z/w) \in \mathbb{R}^3, \quad w \neq 0 \quad (16)$$

In particular, $w = 1$ gives the "canonical" homogeneous coordinates $(x, y, z, 1)$ of a 3D point (x, y, z) .

For graphics purposes it is convenient to define $(x, y, z, 0) \in \mathbb{P}^3$ as representing the 3D Cartesian vector (x, y, z) . This definition has the desirable properties that points in \mathbb{P}^3 representing

- Vector + Vector = Vector
- Point + Vector (or Vector + Point) = Point

in \mathbb{R}^3 .

Homogeneous coordinates make it possible to write any affine transformation $\mathcal{A} : \mathbb{R}^3 \rightarrow \mathbb{R}^3$

$$\mathcal{A}(\mathbf{v}) = \mathcal{T}(\mathbf{v}) + \mathbf{t} = \mathbf{M}\mathbf{v} + \mathbf{t} \quad (17)$$

as a linear transformation in \mathbb{P}^3 . Such a transformation can be uniquely represented as a 4x4 real matrix

$$\mathbf{F} = \left[\begin{array}{c|c} \mathbf{M} & \mathbf{t} \\ \mathbf{0}^\top & 1 \end{array} \right] \quad (18)$$

that acts on the canonical homogeneous coordinates $\tilde{\mathbf{v}} = [\mathbf{v}; 1]$:

$$\mathbf{F}\tilde{\mathbf{v}} = \left[\begin{array}{c|c} \mathbf{M}\mathbf{v} + \mathbf{t} \\ 1 \end{array} \right] \sim \mathcal{A}(\mathbf{v}) \quad (19)$$

Important affine transforms

- **Pure translation**

$$\tilde{\mathbf{v}}' = \mathbf{F}\tilde{\mathbf{v}} = \left[\begin{array}{c|c} v_x + t_x \\ v_y + t_y \\ v_z + t_z \\ 1 \end{array} \right] = \left[\begin{array}{c|c} \mathbf{I} & \mathbf{t} \\ \mathbf{0}^\top & 1 \end{array} \right] \tilde{\mathbf{v}} \quad (20)$$

- **Pure linear transform**

$$\tilde{\mathbf{v}}' = \mathbf{F}\tilde{\mathbf{v}} = \left[\begin{array}{c|c} \mathbf{M}\mathbf{v} \\ 1 \end{array} \right] = \left[\begin{array}{c|c} \mathbf{M} & \mathbf{0} \\ \mathbf{0}^\top & 1 \end{array} \right] \tilde{\mathbf{v}} \quad (21)$$

- **Inverse of an affine transform**

Suppose we are given an affine transform $\mathbf{v} \mapsto \mathbf{M}\mathbf{v} + \mathbf{t}$ represented by the matrix

$$\mathbf{F} = \left[\begin{array}{c|c} \mathbf{M} & \mathbf{t} \\ \mathbf{0}^\top & 1 \end{array} \right] \quad (22)$$

The inverse matrix

$$\mathbf{F}^{-1} = \left[\begin{array}{c|c} \mathbf{M}^{-1} & -\mathbf{M}^{-1}\mathbf{t} \\ \mathbf{0}^\top & 1 \end{array} \right] \quad (23)$$

represents another affine transform

$$\mathbf{v} \mapsto \mathbf{M}^{-1}\mathbf{v} - \mathbf{M}^{-1}\mathbf{t} = \mathbf{M}^{-1}(\mathbf{v} - \mathbf{t}) \quad (24)$$

Note the order: The inverse transform first undoes the translation, and then reverses the linear transform.

- **Rotation about an axis of direction \hat{n} centered at \mathbf{c}**

We already knew how to rotate about an axis centered at zero. To rotate around one that is not, simply apply the change-of-basis technique:

1. Translate the center \mathbf{c} to $\mathbf{0}$ (Pure translation)
2. Apply Rodrigues' Formula (Linear)
3. Translate $\mathbf{0}$ back to \mathbf{c} (Pure translation)

References

[1] <https://mathworld.wolfram.com/EulerAngles.html>

[2] <https://sites.cs.ucsb.edu/~lingqi/teaching/games101.html>

[3] <https://mathfor3dgameprogramming.com/>

Next: Model-View-Projection; Projective transforms

To be continued.