Homework

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0.1

Proof. (1). From the definition of measure, we have

$$\mu(\bigcup_{n=1}^{\infty} E_n) = \sum_{n=1}^{\infty} \mu(E_n) \tag{1}$$

if $E_i \bigcap E_j = \varnothing, i \neq j$. So for any $E \in \mathcal{F}$,

$$\mu(E) = \mu(\varnothing \bigcup E) = \mu(E) + \mu(\varnothing) \tag{2}$$

that is, $\mu(\varnothing) = 0$.

(2).Consider $E_1, E_2, ..., E_n, E_{n+1}, ... \in \mathcal{F}$, $E_i \bigcap E_j = \varnothing, i \neq j$, then from the definition we have

$$\mu(\bigcup_{k=1}^{\infty} E_k) = \sum_{k=1}^{\infty} \mu(E_k) \tag{3}$$

$$\mu(\bigcup_{k=n+1}^{\infty} E_k) = \sum_{k=n+1}^{\infty} \mu(E_k) \tag{4}$$

so that we have

$$\mu(\bigcup_{k=1}^{n} E_k) = \sum_{k=1}^{n} \mu(E_k) \tag{5}$$

(3). If $E_1 \subset E_2$, then $E_2 = E_1 \bigcup (E_2 - E_1)$, apparently $E_1 \bigcap (E_2 - E_1) = \varnothing$. Then from the definition we have

$$\mu(E_2) = \mu(E_1 \bigcup (E_2 - E_1)) = \mu(E_1) + \mu(E_2 - E_1) \geq \mu(E_1) \tag{6}$$

0.2

Proof.

$$P(A) = P(A \bigcap (\bigcup_{i=1}^{\infty} B_i)) = P(\bigcup_{i=1}^{\infty} (A \bigcap B_i)) \tag{7}$$

$$= \sum_{i=1}^{\infty} P(A \bigcap B_i) = \sum_{i=1}^{\infty} P(B_i) P(A|B_i) \tag{8}$$

0.3

Proof.

$$\int_0^\infty (1-F(t))dt = \int_0^\infty (1-P\{X \leq t\})dt \tag{9}$$
$$= \int_0^\infty P\{X > t\}dt \tag{10}$$
$$= \int_0^\infty \int_t^\infty f(s)dsdt \tag{11}$$
$$= \int_0^\infty \int_0^s f(s)dtds \tag{12}$$
$$= \int_0^\infty sf(s)ds \tag{13}$$
$$= E[X] \tag{14}$$

0.4

Proof. From property (2), we have

$$P\{N(s+t)-N(s)=k\} = P\{N(t)=k\} \tag{15}$$

Firstly consider $k=0$, denote $P_k(t) = P\{N(t)=k\}$, apparently for $h>0$

$$P\{N(t+h)=0\} = P\{N(t)=0, N(t+h)-N(t)=0\} \tag{16}$$
$$= P_0(t)P_0(h) \tag{17}$$

On the other hand, from property (3), (4), we have

$$P_0(h) = 1-(\lambda h + o(h)) \tag{18}$$

So that

$$\frac{P_0(t+h)-P_0(t)}{h} = -(\lambda P_0(t) + \frac{o(h)}{h}) \tag{19}$$

Let $h \to 0$,

$$P_0'(t) = -\lambda P_0(t) \tag{20}$$

By solving this differential equation with constraint $P_0(0)=1$, we can get

$$P_0(t) = e^{-\lambda t} \tag{21}$$

when $n>0$, similarly we have

$$P_n(t+h) = P_n(t)(1-\lambda h - o(h)) + P_{n-1}(t)(\lambda h + o(h)) + P_{n-2}(t)o(h) \tag{22}$$

that is

$$P'_n(t) = -\lambda P_n(t) + \lambda P_{n-1}(t) \tag{23}$$

$$\frac{d}{dt}[e^{\lambda t} P_n(t)] = e^{\lambda t} P_{n-1}(t) \tag{24}$$

Notice that $P_0(t) = e^{-\lambda t}$, so we have

$$P_1(t) = \lambda t e^{-\lambda t} \tag{25}$$

then apparently we have

$$P_n(t) = \frac{(\lambda t)^n}{n!} e^{-\lambda t} \tag{26}$$

0.5

Denote the transition matrix of Y, $Q = [Q_{ij}]$, the steady distribution of Y y. Then we have $Q_{ij} = p_i, i = j$, $Q_{ij} = (1 - p_i)P_{ij}, i \neq j$.

For steady distribution π, we have

$$\pi P = \pi \tag{27}$$

$$\sum_{i=1}^{K} \pi_i P_{ij} = \pi_j, \forall j \in \Phi \tag{28}$$

For steady distribution y, we have

$$yQ = y \tag{29}$$

$$\sum_{i=1}^{K} y_i Q_{ij} = y_j, \forall j \in \Phi \tag{30}$$

$$\sum_{i=1, i\neq j}^{K} y_i(1 - p_i)P_{ij} = (1 - p_j)y_j, \forall j \in \Phi \tag{31}$$

apparently $y' = (y_1(1-p_1), y_2(1-p_2)...y_K(1-p_K)$ is a solution of equation (28). so that we have

$$y_i = \frac{\pi_i}{(1 - p_i)} / \sum_{j=1}^{K} \frac{\pi_j}{(1 - p_j)} \tag{32}$$

0.6

Proof. (1) From Theory 1.15, we have

$$P\{X_{n+1} = j, T_{n+1} - T_n \leq t | X_0, ..., X_n = i; T_0, ..., T_n\} = P_{ij}(1 - e^{-\lambda(i)t}) \tag{33}$$

Let $t \to \infty$, we have

$$P\{X_{n+1} = j | X_0, ..., X_n = i\} = P_{ij} = P\{X_{n+1} = j | X_n = i\} \tag{34}$$

(2) Due to

$$P_{ij} \leq 0, P_{ii} = 0, \sum_{j \in \Phi} P_{ij} = \sum_{j \neq i, j \in \Phi} P_{ij} = 1 \tag{35}$$

we have

$$P\{X_{n+1} = j, T_{n+1} - T_n \leq t | X_n = i\} = P_{ij}(1 - e^{-\lambda(i)t}) \tag{36}$$

$$\sum_{j=1, j \neq i}^{K} P\{X_{n+1} = j, T_{n+1} - T_n \leq t | X_n = i\} = \sum_{j=1, j \neq i}^{K} P_{ij}(1 - e^{-\lambda(i)t}) \tag{37}$$

$$P\{X_{n+1} \neq i, T_{n+1} - T_n \leq t | X_n = i\} = (1 - e^{-\lambda(i)t}) \tag{38}$$

That is,

$$P\{X_{n+1} = i, T_{n+1} - T_n \leq t | X_n = i\} = e^{-\lambda(i)t} \tag{39}$$

0.7

Proof. Denote $P\{\alpha_n\} = P\{X_1 = i_1, ..., X_n = i_n, Y_1 = v_{j_1}, ..., Y_n = v_{j_n}\}$. Then we have

$$
\begin{aligned}
P_n &= P\{X_1 = i_1, ..., X_n = i_n, Y_1 = v_{j_1}, ..., Y_n = v_{j_n}\} &(40)\\
&= P\{X_n = i_n, Y_n = v_{j_n} | \alpha_{n-1}\} P\{\alpha_{n-1}\} &(41)\\
&= P\{Y_n = v_{j_n} | \alpha_{n-1}, X_n = i_n\} P\{X_n = i_n | \alpha_{n-1}\} P\{\alpha_{n-1}\} &(42)\\
&= P\{Y_n = v_{j_n} | X_n = i_n\} P\{X_n = i_n | X_{n-1} = i_{n-1}\} P\{\alpha_{n-1}\} &(43)\\
&= b_{i_n j_n} a_{i_{n-1} i_n} P\{\alpha_{n-1}\} &(44)
\end{aligned}
$$

Notice that $P\{\alpha_1\} = \pi_{i_1} b_{i_1 j_1}$, from above, we can get that

$$P\{\mathbf{X} = x, \mathbf{Y} = y\} = \pi_{i_1} b_{i_1 j_1} ... a_{i_{N-1} i_N} b_{i_N j_N} \tag{45}$$

0.8

Proof.

$$
\begin{aligned}
P\{\mathbf{Y} = y | \mathbf{X} = x\} &= \frac{P\{\mathbf{Y} = y, \mathbf{X} = x\}}{P\{\mathbf{X} = x\}} &(46)\\
&= b_{i_1 j_1} ... b_{i_N j_N} &(47)
\end{aligned}
$$

0.9

Proof.

$$\alpha_{n+1}(i) = P\{Y_1 = v_{j_1}, Y_n = v_{j_n}, Y_{n+1} = v_{j_{n+1}}, X_{n+1} = i|\lambda\} \tag{48}$$
$$= \sum_{k=1}^{K} P\{Y_1 = v_{j_1}, ..., Y_n = v_{j_n}, Y_{n+1} = v_{j_{n+1}}, X_n = k, X_{n+1} = i|\lambda\} \tag{49}$$
$$= \sum_{k=1}^{K} P_A P_B P_C \tag{50}$$

where

$$P_A = P\{Y_1 = v_{j_1}, ..., Y_n = v_{j_n}, X_n = k|\lambda\} \tag{51}$$
$$P_B = P\{X_{n+1} = i|Y_1 = v_{j_1}, ..., Y_n = v_{j_n}, X_n = k, \lambda\} \tag{52}$$
$$P_C = P\{Y_{n+1} = v_{j_{n+1}}|Y_1 = v_{j_1}, ..., Y_n = v_{j_n}, X_n = k, X_{n+1} = i, \lambda\} \tag{53}$$

Notice that for P_A, we have

$$P_A = P\{Y_1 = v_{j_1}, ..., Y_n = v_{j_n}, X_n = k|\lambda\} = \alpha_n(k) \tag{54}$$
$$\tag{55}$$

For P_B, we have

$$P_B = P\{X_{n+1} = i|Y_1 = v_{j_1}, ..., Y_n = v_{j_n}, X_n = k, \lambda\} \tag{56}$$
$$= P\{X_{n+1} = i|X_n = k, \lambda\} \tag{57}$$
$$= a_{ki} \tag{58}$$

For P_C, we have

$$P_C = P\{Y_{n+1} = v_{j_{n+1}}|Y_1 = v_{j_1}, ..., Y_n = v_{j_n}, X_n = k, X_{n+1} = i, \lambda\} \tag{59}$$
$$= P\{Y_{n+1} = v_{j_{n+1}}|X_{n+1} = i, \lambda\} \tag{60}$$
$$= b_{ij_{n+1}} \tag{61}$$

Combine the three equations together, we have

$$\alpha_{n+1}(i) = \sum_{k=1}^{K} P_A P_B P_C = \sum_{k=1}^{K} \alpha_n(k) a_{ki} b_{ij_{n+1}} \tag{62}$$

0.10

Proof.

$$\beta_n(i) = P\{Y_{n+1} = v_{j_{n+1}}, ..., Y_N = v_{j_N}|X_n = i, \lambda\} \tag{63}$$
$$= \sum_{k=1}^{K} P\{Y_{n+1} = v_{j_{n+1}}, ..., Y_N = v_{j_N}, X_{n+1} = k|X_n = i, \lambda\} \tag{64}$$
$$= \sum_{k=1}^{K} P_X P_Y P_Z \tag{65}$$

where

$$P_X = P\{X_{n+1} = k|X_n = i, \lambda\} \tag{66}$$
$$P_Y = P\{Y_{n+2} = v_{j_{n+1}}, ..., Y_N = v_{j_n}|X_{n+1} = k, X_n = i, \lambda\} \tag{67}$$
$$P_Z = P\{Y_{n+1} = v_{j_{n+1}}|X_{n+1} = k, X_n = i, Y_{n+2} = v_{j_{n+1}}, ..., Y_N = v_{j_n}, \lambda\} \tag{68}$$

For P_X, we have

$$P_X = P\{X_{n+1} = k|X_n = i, \lambda\} = a_{ik} \tag{69}$$

For P_Y, we have

$$\begin{aligned} P_Y &= P\{Y_{n+2} = v_{j_{n+1}}, ..., Y_N = v_{j_n}|X_{n+1} = k, X_n = i, \lambda\} && (70)\\ &= P\{Y_{n+2} = v_{j_{n+1}}, ..., Y_N = v_{j_n}|X_{n+1} = k, \lambda\} && (71)\\ &= \beta_{n+1}(k) && (72) \end{aligned}$$

For P_Z, we have

$$\begin{aligned} P_Z &= P\{Y_{n+1} = v_{j_{n+1}}|X_{n+1} = k, X_n = i, Y_{n+2} = v_{j_{n+1}}, ..., Y_N = v_{j_n}, \lambda\} && (73)\\ &= P\{Y_{n+1} = v_{j_{n+1}}|X_{n+1} = k, \lambda\} && (74)\\ &= b_{kj_{n+1}} && (75) \end{aligned}$$

Combine the three equations, we have

$$\beta_n(i) = \sum_{k=1}^{K} P_X P_Y P_Z = \sum_{k=1}^{K} \beta_{n+1}(k) a_{ik} b_{kj_{n+1}} \tag{76}$$

0.11

Proof.

$$\begin{aligned} &\delta_{n+1}(i) && (77)\\ &= \max_{i_1 \ldots i_n} P\{X_{n+1} = i, ..., X_1 = i_1; Y_{n+1} = v_{j_{n+1}}, ..., Y_1 = v_{j_1}|\lambda\} && (78)\\ &= b_{ij_{n+1}} \max_{i_1 \ldots i_n} P\{X_{n+1} = i, ..., X_1 = i_1; Y_n = v_{j_n}, ..., Y_1 = v_{j_1}|\lambda\} && (79)\\ &= b_{ij_{n+1}} \max_{i_n} \max_{i_1 \ldots i_{n-1}} [P\{X_n = i_n, ..., X_1 = i_1; Y_n = v_{j_n}, ..., Y_1 = v_{j_1}|\lambda\} && (80)\\ &\quad P\{X_{n+1} = i|X_n = i_n, ..., X_1 = i_1; Y_n = v_{j_n}, ..., Y_1 = v_{j_1}|\lambda\}] && (81)\\ &= b_{ij_{n+1}} \max_{i_n}[a_{i_n i} \delta_n(i_n)] && (82) \end{aligned}$$

0.12

Proof.

$$P\{X_n = i, X_{n+1} = j, \mathbf{Y} = y|\lambda\} \tag{83}$$
$$= P\{X_n = i, Y_1 = v_{j_1}, ..., Y_n = v_{j_n}|\lambda\} \tag{84}$$
$$P\{X_{n+1} = j, Y_{n+1} = v_{j_{n+1}}, ..., Y_N = v_{j_N}|X_n = i, Y_1 = v_{j_1}, ..., Y_n = v_{j_n}, \lambda\} \tag{85}$$
$$= \alpha_n(i)P\{X_{n+1} = j, Y_{n+1} = v_{j_{n+1}}, ..., Y_N = v_{j_N}|X_n = i, \lambda\} \tag{86}$$
$$= \alpha_n(i)a_{ij}P\{Y_{n+1} = v_{j_{n+1}}, ..., Y_N = v_{j_N}|X_n = i, X_{n+1} = j, \lambda\} \tag{87}$$
$$= \alpha_n(i)a_{ij}b_{jj_{n+1}}P\{Y_{n+2} = v_{j_{n+2}}, ..., Y_N = v_{j_N}|X_{n+1} = j, \lambda\} \tag{88}$$
$$= \alpha_n(i)a_{ij}b_{jj_{n+1}}\beta_{n+1}(j) \tag{89}$$

0.13

Proof.

$$d_{ij} = E[\sum_{n=0}^{\infty} f(X_n)|X_0 = j] - E[\sum_{n=0}^{\infty} f(X_n)|X_0 = i] \tag{90}$$
$$= \lim_{N\to\infty} \{E[\sum_{n=0}^{N} f(X_n)|X_0 = j] - E[\sum_{n=0}^{N} f(X_n)|X_0 = i]\} \tag{91}$$
$$= \lim_{N\to\infty} \sum_{n=0}^{N} \{E[f(X_n)|X_0 = j] - E[f(X_n)|X_0 = i]\} \tag{92}$$
$$= \lim_{N\to\infty} \sum_{n=0}^{N} \{(P^n)_j f - (P^n)_i f\} \tag{93}$$

So that we have

$$D = \lim_{N\to\infty} \sum_{n=0}^{N} \{ef^\tau(P^n)^\tau - P^n f e^\tau\} \tag{94}$$

Then,

$$
\begin{aligned}
&D - PDP^{\tau} && (95)\\
&= \lim_{N\to\infty} \sum_{n=0}^{N} \{ef^{\tau}(P^n)^{\tau} - P^n f e^{\tau} - (Pef^{\tau}(P^n)^{\tau}P^{\tau} - PP^n f e^{\tau} P^{\tau})\} && (96)\\
&= \lim_{N\to\infty} \sum_{n=0}^{N} \{ef^{\tau}((P^n)^{\tau} - (P^{n+1})^{\tau}) - (P^n - P^{n+1}) f e^{\tau}\} && (97)\\
&= \lim_{N\to\infty} ef^{\tau}((P^0)^{\tau} - (P^{n+1})^{\tau}) - (P^0 - P^{n+1}) f e^{\tau} && (98)\\
&= \lim_{N\to\infty} ef^{\tau}(I - (P^{n+1})^{\tau}) - (I - P^{n+1}) f e^{\tau} && (99)\\
&= ef^{\tau}(I - e\pi) - (I - pi^{\tau}e^{\tau}) f e^{\tau} && (100)\\
&= ef^{\tau} - fe^{\tau} = F && (101)
\end{aligned}
$$

0.14

Proof. If $(I - P + e\pi)$ is invertible, then there exists a vector $y \neq 0$ so that $y(I - P + e\pi) = 0$.

$$
\begin{aligned}
y(I - P + e\pi) &= 0 && (102)\\
y(I - P + e\pi)e &= 0 && (103)\\
ye - yPe + e\pi e &= 0 && (104)\\
ye - ye + e &= 0 && (105)
\end{aligned}
$$

Conflict!

0.15

Proof. (1) Sufficient condition:

If

$$f^{v^*} + A^{v^*} g^{v^*} \leq f^v + A^v g^{v^*}, v \in \Omega \tag{106}$$

then we have

$$
\begin{aligned}
f^{v^*} + A^{v^*} g^{v^*} - (f^v + A^v g^{v^*}) &\leq 0 && (107)\\
p^v[f^{v^*} + A^{v^*} g^{v^*} - (f^v + A^v g^{v^*})] &\leq 0 && (108)\\
\eta^{v^*} - \eta^v &\leq 0 && (109)
\end{aligned}
$$

that is, v^* is an optimal policy.

(2) Necessary condition: If v^* is an optimal policy, then we have $\eta^{v^*} - \eta^v \leq 0, \forall v \in \Omega_s$. If (106) does not stand, then there exist a policy u, so that at state i_0, we have

$$f^{v^*}(i_0) + A^{v^*}_{i_0} g^{v^*} > f^v(i_0) + A^v_{i_0} g^{v^*} \tag{110}$$

Then if we set policy $u'(i) = v^*(i), i \neq i_0$, at state i_0, $u'(i_0) = u(i_0)$. So apparently we have

$$f^{v^*} + A^{v^*} g^{v^*} > f^{u'} + A^{u'} g^{v^*} \tag{111}$$

From the sufficient condition, we can see that v^* is not an optimal policy which is conflict with our assumption.

0.16

Proof. (1) Sufficient condition: If

$$0 = \min_{v \in \Omega_s} \{f^v + A^v g^{v^*} - e\eta^{v^*}\} \tag{112}$$

we can get

$$f^{v^*} + A^{v^*} g^{v^*} \leq f^v + A^v g^{v^*}, v \in \Omega \tag{113}$$

So from Theory 3.2 we know that v^* is an optimal policy.

(2) Necessary condition: If v^* is an optimal policy, then from Theory 3.2 we have

$$f^{v^*} + A^{v^*} g^{v^*} \leq f^v + A^v g^{v^*}, v \in \Omega \tag{114}$$

that is

$$0 \leq f^v + A^v g^{v^*} - e\eta^{v^*}, v \in \Omega_s \tag{115}$$

set $v = v^*$ we have

$$0 = f^v + A^v g^{v^*} - e\eta^{v^*}, v \in \Omega_s \tag{116}$$

that is

$$0 = \min_{v \in \Omega_s} \{f^v + A^v g^{v^*} - e\eta^{v^*}\} \tag{117}$$

0.17

Proof. Notice that

$$U_\alpha = \int_0^\infty e^{-\alpha t} P(t) dt \tag{118}$$

exists, and $U_\alpha > 0$.

Due to

$$P_{ij}(t) = \int_0^t h(j, t-s) R_{ij}(ds) \tag{119}$$

We have

$$[U_\alpha]_{ij} = \int_0^\infty e^{-\alpha t} P_{ij}(t)dt \tag{120}$$
$$= \int_0^\infty e^{-\alpha t} \int_0^t h(j, t-s) R_{ij} ds dt \tag{121}$$
$$= \int_0^\infty \int_0^t e^{-\alpha t} h(j, t-s) R_{ij} ds dt \tag{122}$$
$$= \int_0^\infty \int_s^\infty e^{-\alpha t} h(j, t-s) R_{ij} dt ds \tag{123}$$
$$= \int_0^\infty e^{-\alpha s} \int_s^\infty e^{-\alpha(t-s)} h(j, t-s) R_{ij} dt ds \tag{124}$$
$$= \int_0^\infty e^{-\alpha s} \int_0^\infty e^{-\alpha \tau} h(j, \tau) R_{ij} d\tau ds \tag{125}$$
$$= \int_0^\infty e^{-\alpha s} h_\alpha(j) R_{ij} ds \tag{126}$$
$$= h_\alpha(j)[R_\alpha]_{ij} \tag{127}$$

So that we have

$$U_\alpha = R_\alpha H_\alpha = (I - Q_\alpha)^{-1} H_\alpha \tag{128}$$

Then we have

$$U_\alpha(\alpha I - A_\alpha) = (I - Q_\alpha)^{-1} H_\alpha H_\alpha^{-1}(I - Q_\alpha) = I \tag{129}$$

0.18

Proof. Notice that

$$[U_\alpha]_i e = \sum_{j=1}^{K} \int_0^\infty e^{-\alpha t} P_{ij}(t) dt = \int_0^\infty e^{-\alpha t} dt = \frac{e}{\alpha} \tag{130}$$

From 0.17, we have

$$\alpha I - A_\alpha = U_\alpha^{-1} \tag{131}$$

then we have for $\alpha > 0$,

$$(U_\alpha - \frac{ep_\alpha}{\alpha(1-\alpha)})(U_\alpha^{-1} + ep_\alpha) \tag{132}$$
$$= I - \frac{ep_\alpha U_\alpha^{-1}}{\alpha(1+\alpha)} + \frac{ep_\alpha}{\alpha} - \frac{ep_\alpha}{\alpha(1+\alpha)} \tag{133}$$
$$= I - \frac{ep_\alpha(\alpha I - A_\alpha) - \alpha ep_\alpha}{\alpha(1+\alpha)} \tag{134}$$
$$= I - \frac{ep_\alpha A_\alpha}{\alpha(1+\alpha)} \tag{135}$$
$$= I \tag{136}$$

when $\alpha = 0$, we need to prove that $A + ep$ is invertible.

If $A+ep$ is not invertible, then there exists a vector $y \neq 0$ so that $y(A+ep) = 0$.

$$y(A + ep) = 0 \tag{137}$$
$$yAe + yepe = 0 \tag{138}$$
$$ye = 0 \tag{139}$$

then we have $yA = 0$ as well. Notice that from equation $pe = 1, pA = 0$ we can get the only solution p, so that without equation $ye = 1$ we can get y must satisfies $y = cp, c \neq 0$, which is conflict with $ye = 0$.

0.19

Proof.

$$(\alpha I - A_\alpha)g_\alpha = f - \frac{ep_\alpha f}{1 + \alpha} \tag{140}$$
$$g_\alpha = U_\alpha(f - \frac{ep_\alpha f}{1 + \alpha}) \tag{141}$$
$$g_\alpha = \eta_\alpha - \frac{ep_\alpha f}{\alpha(1 + \alpha)} \tag{142}$$
$$\eta_\alpha = g_\alpha + \frac{ep_\alpha f}{\alpha(1 + \alpha)} \tag{143}$$

0.20