Homework

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0.1

Proof. (1). From the definition of measure, we have

$$\mu(\bigcup_{n=1}^{\infty} E_n) = \sum_{n=1}^{\infty} \mu(E_n) \tag{1}$$

if $E_i \bigcap E_j = \varnothing, i \neq j$. So for any $E \in \mathcal{F}$,

$$\mu(E) = \mu(\varnothing \bigcup E) = \mu(E) + \mu(\varnothing) \tag{2}$$

that is, $\mu(\varnothing) = 0$.

(2).Consider $E_1, E_2, ..., E_n, E_{n+1}, ... \in \mathcal{F}$, $E_i \bigcap E_j = \varnothing, i \neq j$, then from the definition we have

$$\mu(\bigcup_{k=1}^{\infty} E_k) = \sum_{k=1}^{\infty} \mu(E_k) \tag{3}$$

$$\mu(\bigcup_{k=n+1}^{\infty} E_k) = \sum_{k=n+1}^{\infty} \mu(E_k) \tag{4}$$

so that we have

$$\mu(\bigcup_{k=1}^{n} E_k) = \sum_{k=1}^{n} \mu(E_k) \tag{5}$$

(3). If $E_1 \subset E_2$, then $E_2 = E_1 \bigcup (E_2 - E_1)$, apparently $E_1 \bigcap (E_2 - E_1) = \varnothing$. Then from the definition we have

$$\mu(E_2) = \mu(E_1 \bigcup (E_2 - E_1)) = \mu(E_1) + \mu(E_2 - E_1) \geq \mu(E_1) \tag{6}$$

0.2

Proof.

$$P(A) = P(A \bigcap (\bigcup_{i=1}^{\infty} B_i)) = P(\bigcup_{i=1}^{\infty} (A \bigcap B_i)) \tag{7}$$

$$= \sum_{i=1}^{\infty} P(A \bigcap B_i) = \sum_{i=1}^{\infty} P(B_i) P(A|B_i) \tag{8}$$

0.3

Proof.

$$\begin{aligned}
\int_0^\infty (1-F(t))dt &= \int_0^\infty (1-P\{X \le t\})dt &(9)\\
&= \int_0^\infty P\{X > t\}dt &(10)\\
&= \int_0^\infty \int_t^\infty f(s)dsdt &(11)\\
&= \int_0^\infty \int_0^s f(s)dtds &(12)\\
&= \int_0^\infty sf(s)ds &(13)\\
&= E[X] &(14)
\end{aligned}$$

0.4

Proof. From property (2), we have

$$P\{N(s+t) - N(s) = k\} = P\{N(t) = k\} \tag{15}$$

Firstly consider $k = 0$, denote $P_k(t) = P\{N(t) = k\}$, apparently for $h > 0$

$$\begin{aligned}
P\{N(t+h) = 0\} &= P\{N(t) = 0, N(t+h) - N(t) = 0\} &(16)\\
&= P_0(t)P_0(h) &(17)
\end{aligned}$$

On the other hand, from property (3), (4), we have

$$P_0(h) = 1 - (\lambda h + o(h)) \tag{18}$$

So that

$$\frac{P_0(t+h) - P_0(t)}{h} = -(\lambda P_0(t) + \frac{o(h)}{h}) \tag{19}$$

Let $h \to 0$,

$$P_0'(t) = -\lambda P_0(t) \tag{20}$$

By solving this differential equation with constraint $P_0(0) = 1$, we can get

$$P_0(t) = e^{-\lambda t} \tag{21}$$