

Homework

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0.1

Proof. (1). From the definition of measure, we have

$$\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \mu(E_n) \quad (1)$$

if $E_i \cap E_j = \emptyset, i \neq j$. So for any $E \in \mathcal{F}$,

$$\mu(E) = \mu(\emptyset \bigcup E) = \mu(\emptyset) + \mu(E) \quad (2)$$

that is, $\mu(\emptyset) = 0$.

(2). Consider $E_1, E_2, \dots, E_n, E_{n+1}, \dots \in \mathcal{F}$, $E_i \cap E_j = \emptyset, i \neq j$, then from the definition we have

$$\mu\left(\bigcup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} \mu(E_k) \quad (3)$$

$$\mu\left(\bigcup_{k=n+1}^{\infty} E_k\right) = \sum_{k=n+1}^{\infty} \mu(E_k) \quad (4)$$

so that we have

$$\mu\left(\bigcup_{k=1}^n E_k\right) = \sum_{k=1}^n \mu(E_k) \quad (5)$$

(3). If $E_1 \subset E_2$, then $E_2 = E_1 \bigcup (E_2 - E_1)$, apparently $E_1 \cap (E_2 - E_1) = \emptyset$. Then from the definition we have

$$\mu(E_2) = \mu(E_1 \bigcup (E_2 - E_1)) = \mu(E_1) + \mu(E_2 - E_1) \geq \mu(E_1) \quad (6)$$

0.2

Proof.

$$P(A) = P\left(A \cap \left(\bigcup_{i=1}^{\infty} B_i\right)\right) = P\left(\bigcup_{i=1}^{\infty} (A \cap B_i)\right) \quad (7)$$

$$= \sum_{i=1}^{\infty} P(A \cap B_i) = \sum_{i=1}^{\infty} P(B_i)P(A|B_i) \quad (8)$$

0.3

Proof.

$$\int_0^\infty (1 - F(t))dt = \int_0^\infty (1 - P\{X \leq t\})dt \quad (9)$$

$$= \int_0^\infty P\{X > t\}dt \quad (10)$$

$$= \int_0^\infty \int_t^\infty f(s)dsdt \quad (11)$$

$$= \int_0^\infty \int_0^s f(s)dt ds \quad (12)$$

$$= \int_0^\infty s f(s)ds \quad (13)$$

$$= E[X] \quad (14)$$

0.4

Proof. From property (2), we have

$$P\{N(s+t) - N(s) = k\} = P\{N(t) = k\} \quad (15)$$

Firstly consider $k = 0$, denote $P_k(t) = P\{N(t) = k\}$, apparently for $h > 0$

$$P\{N(t+h) = 0\} = P\{N(t) = 0, N(t+h) - N(t) = 0\} \quad (16)$$

$$= P_0(t)P_0(h) \quad (17)$$

On the other hand, from property (3), (4), we have

$$P_0(h) = 1 - (\lambda h + o(h)) \quad (18)$$

So that

$$\frac{P_0(t+h) - P_0(t)}{h} = -(\lambda P_0(t) + \frac{o(h)}{h}) \quad (19)$$

Let $h \rightarrow 0$,

$$P'_0(t) = -\lambda P_0(t) \quad (20)$$

By solving this differential equation with constraint $P_0(0) = 1$, we can get

$$P_0(t) = e^{-\lambda t} \quad (21)$$

when $n > 0$, similarly we have

$$P_n(t+h) = P_n(t)(1 - \lambda h - o(h)) + P_{n-1}(t)(\lambda h + o(h)) + P_{n-2}(t)o(h) \quad (22)$$

that is

$$P'_n(t) = -\lambda P_n(t) + \lambda P_{n-1}(t) \quad (23)$$

$$\frac{d}{dt}[e^{\lambda t} P_n(t)] = e^{\lambda t} P_{n-1}(t) \quad (24)$$

Notice that $P_0(t) = e^{-\lambda t}$, so we have

$$P_1(t) = \lambda t e^{-\lambda t} \quad (25)$$

then apparently we have

$$P_n(t) = \frac{(\lambda t)^n}{n!} e^{-\lambda t} \quad (26)$$

0.5

Denote the transition matrix of Y , $Q = [Q_{ij}]$, the steady distribution of Y y . Then we have $Q_{ij} = p_i, i = j$, $Q_{ij} = (1 - p_i)P_{ij}, i \neq j$.

For steady distribution π , we have

$$\pi P = \pi \quad (27)$$

$$\sum_{i=1}^K \pi_i P_{ij} = \pi_j, \forall j \in \Phi \quad (28)$$

For steady distribution y , we have

$$yQ = y \quad (29)$$

$$\sum_{i=1}^K y_i Q_{ij} = y_j, \forall j \in \Phi \quad (30)$$

$$\sum_{i=1, i \neq j}^K y_i (1 - p_i) P_{ij} = (1 - p_j) y_j, \forall j \in \Phi \quad (31)$$

apparently $y' = (y_1(1 - p_1), y_2(1 - p_2) \dots y_K(1 - p_K))$ is a solution of equation (28). so that we have

$$y_i = \frac{\pi_i}{(1 - p_i)} / \sum_{j=1}^K \frac{\pi_j}{(1 - p_j)} \quad (32)$$

0.6

Proof. (1) From Theory 1.15, we have

$$P\{X_{n+1} = j, T_{n+1} - T_n \leq t | X_0, \dots, X_n = i; T_0, \dots, T_n\} = P_{ij}(1 - e^{-\lambda(i)t}) \quad (33)$$

Let $t \rightarrow \infty$, we have

$$P\{X_{n+1} = j | X_0, \dots, X_n = i\} = P_{ij} = P\{X_{n+1} = j | X_n = i\} \quad (34)$$

(2) Due to

$$P_{ij} \leq 0, P_{ii} = 0, \sum_{j \in \Phi} P_{ij} = \sum_{j \neq i, j \in \Phi} P_{ij} = 1 \quad (35)$$

we have

$$P\{X_{n+1} = j, T_{n+1} - T_n \leq t | X_n = i\} = P_{ij}(1 - e^{-\lambda(i)t}) \quad (36)$$

$$\sum_{j=1, j \neq i}^K P\{X_{n+1} = j, T_{n+1} - T_n \leq t | X_n = i\} = \sum_{j=1, j \neq i}^K P_{ij}(1 - e^{-\lambda(i)t}) \quad (37)$$

$$P\{X_{n+1} \neq i, T_{n+1} - T_n \leq t | X_n = i\} = (1 - e^{-\lambda(i)t}) \quad (38)$$

That is,

$$P\{X_{n+1} = i, T_{n+1} - T_n \leq t | X_n = i\} = e^{-\lambda(i)t} \quad (39)$$

0.7

Proof. Denote $P\{\alpha_n\} = P\{X_1 = i_1, \dots, X_n = i_n, Y_1 = v_{j_1}, \dots, Y_n = v_{j_n}\}$. Then we have

$$P_n = P\{X_1 = i_1, \dots, X_n = i_n, Y_1 = v_{j_1}, \dots, Y_n = v_{j_n}\} \quad (40)$$

$$= P\{X_n = i_n, Y_n = v_{j_n} | \alpha_{n-1}\} P\{\alpha_{n-1}\} \quad (41)$$

$$= P\{Y_n = v_{j_n} | \alpha_{n-1}, X_n = i_n\} P\{X_n = i_n | \alpha_{n-1}\} P\{\alpha_{n-1}\} \quad (42)$$

$$= P\{Y_n = v_{j_n} | X_n = i_n\} P\{X_n = i_n | X_{n-1} = i_{n-1}\} P\{\alpha_{n-1}\} \quad (43)$$

$$= b_{i_n j_n} a_{i_{n-1} i_n} P\{\alpha_{n-1}\} \quad (44)$$

Notice that $P\{\alpha_1\} = \pi_{i_1} b_{i_1 j_1}$, from above, we can get that

$$P\{\mathbf{X} = x, \mathbf{Y} = y\} = \pi_{i_1} b_{i_1 j_1} \dots a_{i_{N-1} i_N} b_{i_N j_N} \quad (45)$$

0.8

Proof.

$$P\{\mathbf{Y} = y | \mathbf{X} = x\} = \frac{P\{\mathbf{Y} = y, \mathbf{X} = x\}}{P\{\mathbf{X} = x\}} \quad (46)$$

$$= b_{i_1 j_1} \dots b_{i_N j_N} \quad (47)$$

0.9

Proof.

$$\alpha_{n+1}(i) = P\{Y_1 = v_{j_1}, Y_n = v_{j_n}, Y_{n+1} = v_{j_{n+1}}, X_{n+1} = i | \lambda\} \quad (48)$$

$$= \sum_{k=1}^K P\{Y_1 = v_{j_1}, \dots, Y_n = v_{j_n}, Y_{n+1} = v_{j_{n+1}}, X_n = k, X_{n+1} = i | \lambda\} \quad (49)$$

$$= \sum_{k=1}^K P_A P_B P_C \quad (50)$$

where

$$P_A = P\{Y_1 = v_{j_1}, \dots, Y_n = v_{j_n}, X_n = k | \lambda\} \quad (51)$$

$$P_B = P\{X_{n+1} = i | Y_1 = v_{j_1}, \dots, Y_n = v_{j_n}, X_n = k, \lambda\} \quad (52)$$

$$P_C = P\{Y_{n+1} = v_{j_{n+1}} | Y_1 = v_{j_1}, \dots, Y_n = v_{j_n}, X_n = k, X_{n+1} = i, \lambda\} \quad (53)$$

Notice that for P_A , we have

$$P_A = P\{Y_1 = v_{j_1}, \dots, Y_n = v_{j_n}, X_n = k | \lambda\} = \alpha_n(k) \quad (54)$$

$$(55)$$

For P_B , we have

$$P_B = P\{X_{n+1} = i | Y_1 = v_{j_1}, \dots, Y_n = v_{j_n}, X_n = k, \lambda\} \quad (56)$$

$$= P\{X_{n+1} = i | X_n = k, \lambda\} \quad (57)$$

$$= a_{ki} \quad (58)$$

For P_C , we have

$$P_C = P\{Y_{n+1} = v_{j_{n+1}} | Y_1 = v_{j_1}, \dots, Y_n = v_{j_n}, X_n = k, X_{n+1} = i, \lambda\} \quad (59)$$

$$= P\{Y_{n+1} = v_{j_{n+1}} | X_{n+1} = i, \lambda\} \quad (60)$$

$$= b_{ij_{n+1}} \quad (61)$$

Combine the three equations together, we have

$$\alpha_{n+1}(i) = \sum_{k=1}^K P_A P_B P_C = \sum_{k=1}^K \alpha_n(k) a_{ki} b_{ij_{n+1}} \quad (62)$$

0.10

Proof.

$$\beta_n(i) = P\{Y_{n+1} = v_{j_{n+1}}, \dots, Y_N = v_{j_N} | X_n = i, \lambda\} \quad (63)$$

$$= \sum_{k=1}^K P\{Y_{n+1} = v_{j_{n+1}}, \dots, Y_N = v_{j_N}, X_{n+1} = k | X_n = i, \lambda\} \quad (64)$$

$$= \sum_{k=1}^K P_X P_Y P_Z \quad (65)$$

where

$$P_X = P\{X_{n+1} = k | X_n = i, \lambda\} \quad (66)$$

$$P_Y = P\{Y_{n+2} = v_{j_{n+1}}, \dots, Y_N = v_{j_n} | X_{n+1} = k, X_n = i, \lambda\} \quad (67)$$

$$P_Z = P\{Y_{n+1} = v_{j_{n+1}} | X_{n+1} = k, X_n = i, Y_{n+2} = v_{j_{n+1}}, \dots, Y_N = v_{j_n}, \lambda\} \quad (68)$$

For P_X , we have

$$P_X = P\{X_{n+1} = k | X_n = i, \lambda\} = a_{ik} \quad (69)$$

For P_Y , we have

$$P_Y = P\{Y_{n+2} = v_{j_{n+1}}, \dots, Y_N = v_{j_n} | X_{n+1} = k, X_n = i, \lambda\} \quad (70)$$

$$= P\{Y_{n+2} = v_{j_{n+1}}, \dots, Y_N = v_{j_n} | X_{n+1} = k, \lambda\} \quad (71)$$

$$= \beta_{n+1}(k) \quad (72)$$

For P_Z , we have

$$P_Z = P\{Y_{n+1} = v_{j_{n+1}} | X_{n+1} = k, X_n = i, Y_{n+2} = v_{j_{n+1}}, \dots, Y_N = v_{j_n}, \lambda\} \quad (73)$$

$$= P\{Y_{n+1} = v_{j_{n+1}} | X_{n+1} = k, \lambda\} \quad (74)$$

$$= b_{kj_{n+1}} \quad (75)$$

Combine the three equations, we have

$$\beta_n(i) = \sum_{k=1}^K P_X P_Y P_Z = \sum_{k=1}^K \beta_{n+1}(k) a_{ik} b_{kj_{n+1}} \quad (76)$$

0.11

Proof.

$$\delta_{n+1}(i) \quad (77)$$

$$= \max_{i_1 \dots i_n} P\{X_{n+1} = i, \dots, X_1 = i_1; Y_{n+1} = v_{j_{n+1}}, \dots, Y_1 = v_{j_1} | \lambda\} \quad (78)$$

$$= b_{ij_{n+1}} \max_{i_1 \dots i_n} P\{X_{n+1} = i, \dots, X_1 = i_1; Y_n = v_{j_n}, \dots, Y_1 = v_{j_1} | \lambda\} \quad (79)$$

$$= b_{ij_{n+1}} \max_{i_n} \max_{i_1 \dots i_{n-1}} [P\{X_n = i_n, \dots, X_1 = i_1; Y_n = v_{j_n}, \dots, Y_1 = v_{j_1} | \lambda\}] \quad (80)$$

$$P\{X_{n+1} = i | X_n = i_n, \dots, X_1 = i_1; Y_n = v_{j_n}, \dots, Y_1 = v_{j_1} | \lambda\} \quad (81)$$

$$= b_{ij_{n+1}} \max_{i_n} [a_{i_n i} \delta_n(i_n)] \quad (82)$$

0.12*Proof.*

$$P\{X_n = i, X_{n+1} = j, \mathbf{Y} = y | \lambda\} \quad (83)$$

$$= P\{X_n = i, Y_1 = v_{j_1}, \dots, Y_n = v_{j_n} | \lambda\} \quad (84)$$

$$P\{X_{n+1} = j, Y_{n+1} = v_{j_{n+1}}, \dots, Y_N = v_{j_N} | X_n = i, Y_1 = v_{j_1}, \dots, Y_n = v_{j_n}, \lambda\} \quad (85)$$

$$= \alpha_n(i) P\{X_{n+1} = j, Y_{n+1} = v_{j_{n+1}}, \dots, Y_N = v_{j_N} | X_n = i, \lambda\} \quad (86)$$

$$= \alpha_n(i) a_{ij} P\{Y_{n+1} = v_{j_{n+1}}, \dots, Y_N = v_{j_N} | X_n = i, X_{n+1} = j, \lambda\} \quad (87)$$

$$= \alpha_n(i) a_{ij} b_{jj_{n+1}} P\{Y_{n+2} = v_{j_{n+2}}, \dots, Y_N = v_{j_N} | X_{n+1} = j, \lambda\} \quad (88)$$

$$= \alpha_n(i) a_{ij} b_{jj_{n+1}} \beta_{n+1}(j) \quad (89)$$

0.13*Proof.*