# Module 2

## Wave & Heat equation

### Chapter 1; Derivation of wave eqn & its soln;

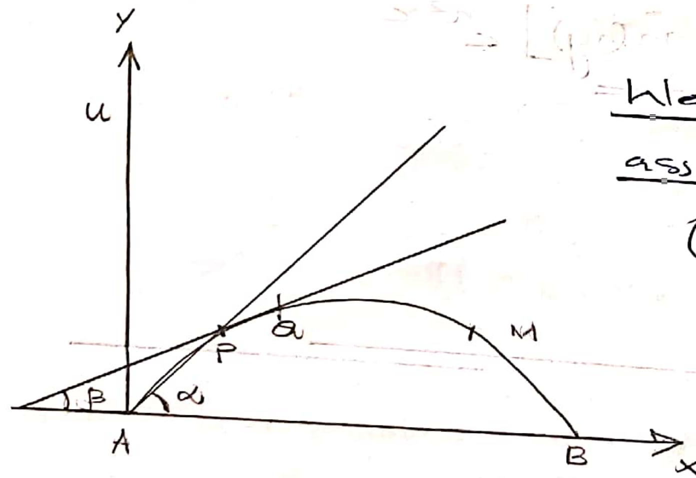
**Derive the one dimensional wave eqn:**

The one dimensional wave eqn is given by

$$\boxed{\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}}$$

Consider a tightly stretched elastic string of length 'l' with fixed ends A & B

Let the string be released from rest and allowed to vibrate. Each point of the string makes small vibrations. Taking A as the origin, AB as the x axis the line AU perpendicular to AB as the y axis we derive the one dimensional wave eqn:




For simplicity;

We make the following assumptions;

(i) The string is perfectly flexible and uniform.

(ii) The effect of gravity and air friction are negligible.

(iii) Let x axis be the line of eqn of the string.

(iv) The string makes only transverse vibrations, ie the motion of the string is perpendicular to the line of eqn of the string.

Scanned by CamScanner

(v) The tension, T of the string is constant.

(vi) The motion take place in the XY plane only.

Let the string be in the position at time t. Consider the motion of a small segment PQ, of the string at $P(x, y)$ and $Q(x + \Delta x, y + \Delta y)$

Let the tangent at P and Q make an angle $\alpha$ and $\beta$ with the x-axis respectively. The tension T at any point on the string is tangential to the curve

The vertical component of force along PQ is;

height at Q − height at P.

height at Q $= T\sin\beta$

height at P $= T\sin\alpha$

Height at Q − Height at P $= T(\sin\beta - \sin\alpha)$

Since $\alpha$ and $\beta$ are very small;

the vertical component of force, $F_u = T(\tan\beta - \tan\alpha)$

slope

Also; $$F_u = T\left[\left(\frac{\partial u}{\partial x}\right)_{x+\Delta x} - \left(\frac{\partial u}{\partial x}\right)_x\right]$$

$u \to$ fn of $x$ & $t$

$\partial u/\partial x$.

If m is the mass per unit length of the string, then By Newton's second Law of Motion;

$$F_u = m \times \Delta x \times \frac{\partial^2 u}{\partial t^2}$$

$$\therefore \quad m \times \Delta x \frac{\partial^2 u}{\partial t^2} = T\left[\left(\frac{\partial u}{\partial x}\right)_{x+\Delta x} - \left(\frac{\partial u}{\partial x}\right)_x\right]$$

$$\frac{\partial^2 u}{\partial t^2} = \frac{T}{m}\left[\frac{\left(\frac{\partial u}{\partial x}\right)_{x+\Delta x} - \left(\frac{\partial u}{\partial x}\right)_x}{\Delta x}\right]$$ $\to$ rate of change of $\partial u/\partial x$ wrt $\Delta x$

$$\frac{\partial^2 u}{\partial t^2} = \frac{T}{m}\frac{\partial^2 u}{\partial x^2} \quad \text{as } \Delta x \to 0.$$

put $T/m = a^2$ where a is a constant.

$$\frac{T}{m} = a^2$$

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}$$

Which is the one dimensional wave eqn.

<u>obtain the soln: of the wave eqn using the Method of separation of variables;</u>

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{a^2} \frac{\partial^2 u}{\partial t^2} \quad \text{———} \quad (1).$$

here u is a fn of x and t.

$$U = XT$$

X is a fn of x alone while T is fn of t alone

$$\frac{\partial^2 u}{\partial x^2} = X''T \quad ; \quad \frac{\partial^2 u}{\partial t^2} = XT''$$

$$X''T = \frac{1}{a^2} XT''$$

$$\frac{X''}{X} = \frac{1}{a^2} \frac{T''}{T} = k$$

$$\frac{X''}{X} = k$$

$$X'' - Xk = 0, \quad \text{———} \quad (2)$$

$$\frac{1}{a^2} \cdot \frac{T''}{T} = k.$$

$$T'' = Ta^2k$$

$$T'' - Ta^2k = 0 \quad \text{———} \quad (3)$$

Case 1 : $k > 0$

Let $k = \lambda^2$ — $\lambda^2$ since for any value of $\lambda$, $\lambda^2$ is positive.

Eqn ②;

$X'' - \lambda^2 X = 0.$

$m^2 - \lambda^2 = 0$

$m = \pm\lambda.$

$X = c_1 e^{\lambda x} + c_2 e^{-\lambda x}$

Eqn ③;

$T'' - Ta^2\lambda^2 = 0$

$m^2 = \lambda^2 a^2$

$m = \pm a\lambda.$

$T = c_3 e^{a\lambda t} + c_4 e^{-a\lambda t}$

So; $u = XT$

$= (c_1 e^{\lambda x} + c_2 e^{-\lambda x})(c_3 e^{a\lambda t} + c_4 e^{-a\lambda t})$

Case 2 : $k < 0$

Let $k = -\lambda^2$ — $-\lambda^2$ since $\lambda^2$ is +ve always. hence $-\lambda^2 \Rightarrow$ -ve.

Eqn ②.

$X'' + \lambda^2 X = 0$

$X'' = -\lambda^2 X$

$m^2 = -\lambda^2$

$m = \pm i\lambda$

$X = c_5 \cos\lambda x + c_6 \sin\lambda x.$

Eqn ③

$T'' + Ta^2\lambda^2 = 0$

$T'' = -Ta^2\lambda^2$

$m^2 = -a^2\lambda^2$

$m = \pm a\lambda i$

$T = c_7 \cos a\lambda t + c_8 \sin a\lambda t$

$u = XT$

$= (c_5 \cos \lambda x + c_6 \sin \lambda x)(c_7 \cos a\lambda t + c_8 \sin a\lambda t)$

<u>case:3 ; k = 0</u>

Eqn ②

$X'' = 0$

$m^2 = 0$

$m = 0, 0$

$X = c_9 + c_{10} x$

Eqn ③

$T'' = 0.$

$m^2 = 0, 0$

$T = c_{11} + c_{12} t.$

So; $u = XT = (c_9 + c_{10} x)(c_{11} + c_{12} t)$

In this problem of vibrating strings u is a periodic fn of x and t, hence we choose the soln. when k<0

ie $u = (c_5 \cos \lambda x + c_6 \sin \lambda x)(c_7 \cos a\lambda t + c_8 \sin a\lambda t)$

$\therefore \boxed{u = (A \cos \lambda x + B \sin \lambda x)(C \cos a\lambda t + D \sin a\lambda t)}$

Obtain the Fourier series soln. of wave eqn;

The wave eqn is given by;

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2} \quad \text{——— (1)}$$

The Gen. soln of eqn (1) is given by;

$$u(x,t) = (A\cos\lambda x + B\sin\lambda x)(C\cos a\lambda t + D\sin a\lambda t) \quad \text{——— (2)}$$

The boundary conditions are;

$$u(0,t) = 0 \quad \text{——— (a)}$$

$$u(L,t) = 0 \quad \text{——— (b)}$$

The initial conditions are

$$f(x) = u(x,0) \quad \text{——— (c)}$$

$$g(x) = \left[\frac{\partial u}{\partial t}\right]_{t=0} \quad \text{——— (d)}$$

Applying the boundary conditions (a) in (2).

$$0 = A \cdot (C\cos a\lambda t + D\sin a\lambda t)$$

$C\cos a\lambda t + D\sin a\lambda t \neq 0$ (else Gen. soln become zero)

$\Rightarrow A = 0.$

$\therefore$ (2) $\Rightarrow$ $u(x,t) = B\sin\lambda x\,(C\cos a\lambda t + D\sin a\lambda t)$ ——— (3)

Applying the boundary condition (b) in (3)

$$0 = B\sin\lambda L\,(C\cos a\lambda t + D\sin a\lambda t)$$

$B \neq 0$ Gen. soln $= 0$ (if $B = 0$) (since $A = 0$)

$\sin\lambda L = 0.$

$\Rightarrow \lambda L = n\pi$ or $\boxed{\lambda = \frac{n\pi}{L}}$

Applying the initial condition ⓒ in ③

$$u(x,0) = f(x) = B \sin \lambda x \cdot C = BC \sin \lambda x$$

$$f(x) = A_n \sin \lambda x. \qquad (\text{put } BC = A_n)$$

In Gen we can write;

$$\boxed{f(x) = \sum_{n=1}^{\infty} A_n \sin \lambda x}$$

Which is the ½ range fourier series (Sine series) in the interval $(0, L)$.

where;

$$A_n = \frac{2}{L} \int_0^L f(x) \sin \lambda x \, dx.$$

From ③ we have;

$$\frac{\partial u}{\partial t} = B \sin \lambda x \left[ -a\lambda C \sin a\lambda t + D a\lambda \cos a\lambda t \right] \quad ④$$

Applying the initial condition ⓓ in ④

$$g(x) = B \sin \lambda x \, [D a\lambda]$$

$$= BDa\lambda \sin \lambda x$$

$$g(x) = B_n a\lambda \sin \lambda x$$

In Gen we can write;

$$g(x) = \sum_{n=1}^{\infty} B_n a\lambda \sin \lambda x$$

$$\boxed{g(x) = a\lambda \sum_{n=1}^{\infty} B_n \sin \lambda x}$$

12

Which is the ½ range fourier sine series of $g(x)$ in the interval $(0, L)$

$$B_n = \frac{2}{a\lambda L}\int_0^L g(x) \sin \lambda x \, dx$$

ie;

$$\boxed{\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right)\left[A_n \cos \frac{n\pi at}{L} + B_n \sin \frac{n\pi at}{L}\right] \\ A_n &= \frac{2}{L}\int_0^L f(x) \sin \frac{n\pi x}{L} \, dx \\ B_n &= \frac{2}{a\lambda L}\int_0^L g(x) \sin \frac{n\pi x}{L} \, dx. \end{aligned}}$$

12/11/2020

1. A tightly stretched string with fixed end pts $x=0$ and $x=L$ is initially in a position given by $kx(L-x)$ if it is released from rest in this position find the displacement fn in terms of F. coefficient of $f(x)$

Released from rest ⇒ Initial velocity is zero.

Wave eqn is given by;

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2} \quad \text{——— ①}$$

A soln of ① is given by;

$$u(x,t) = (A \cos \lambda x + B \sin \lambda x)(C \cos \lambda at + D \sin \lambda at) \quad \text{——— ②}$$

The boundary conditions are given by;

$$u(0,t) = 0 \quad \text{——— ⓐ} \qquad u(L,t) = 0 \quad \text{——— ⓑ}$$

$$f(x) = u(x,0) = kx(L-x) \quad \text{——— ⓒ}$$

$$g(x) = \left(\frac{\partial u}{\partial t}\right)_{t=0} = 0 \quad \text{——— ⓓ}$$

Applying ⓐ in ②

$0 = A(C\cos\lambda at + D\sin\lambda at)$

$\Rightarrow A = 0$. | $C\cos\lambda at + D\sin\lambda at \neq 0$.
$\Rightarrow$ Gen eqn 0.

② $\Rightarrow$ $u(x,t) = B\sin\lambda x(C\cos a\lambda t + D\sin\lambda a t)$ ③

Applying ⓑ in ③

$0 = B\sin\lambda L\,(C\cos a\lambda t + D\sin a\lambda t)$

$\sin\lambda L = 0$.

$\lambda L = n\pi$

$$\boxed{\lambda = \frac{n\pi}{L}}$$

Applying initial condition ⓒ in ③

$f(x) = B\sin\lambda x \times C$

$f(x) = BC\sin\lambda x$

$= \sum_{n=1}^{\infty} A_n \sin\lambda x$

$A_n = \frac{2}{L}\int_0^L f(x)\sin\lambda x\,dx$.

$\partial u/\partial t = B\sin\lambda x\,[-a\lambda C\sin a\lambda t + a\lambda D\cos a\lambda t]$ ④.

Applying ⓓ in ④

At $t=0$;

$0 = B\sin\lambda x \times a\lambda \times D$

$0 = a\lambda BD\sin\lambda x$

$a \neq 0 \Rightarrow$ wave eqn $= 0$.
$\lambda \neq 0 \Rightarrow$ ③ $= 0$.
$x \neq 0 \Rightarrow$ ③ $= 0$

$\therefore \boxed{D = 0}$

$$u(x,t) = BC \sin \lambda x \cos a\lambda t$$

$$= \sum A_n \sin \lambda x \cos a\lambda t \quad \text{———— ⑤}$$

$$A_n = \frac{2}{L} \int_0^L f(x) \sin \lambda x \, dx$$

$$= \frac{2k}{L} \int_0^L (xL - x^2) \sin \lambda x \, dx$$

$$= \frac{2k}{L} \left[ (xL - x^2)\left(\frac{-\cos \lambda x}{\lambda}\right) - (L - 2x)\left(\frac{-\sin \lambda x}{\lambda^2}\right) - 2\left(\frac{\cos \lambda x}{\lambda^3}\right) \right]_0^L$$

$$= \frac{2k}{L} \left[ (xL - x^2)\left(\frac{-\cos \frac{n\pi x}{L}}{\frac{n\pi}{L}}\right) - (L - 2x) - \frac{\sin n\pi \frac{x}{L}}{(n\pi)^2 / L^2} - \frac{2 \cos \frac{n\pi x}{L}}{\frac{(n\pi)^3}{L^3}} \right]_0^L$$

$$= \frac{2k}{L} \left[ \left[ 0 - 0 - \frac{2 \cos n\pi}{\frac{n^3 \cdot \pi^3}{L^3}} + 0 - 0 \right] + \frac{2L^3}{n^3 \pi^3} \right]$$

$$= \frac{4k}{L} \left[ \frac{L^3}{n^3 \pi^3} (-1)^{n+1} + \frac{L^3}{n^3 \pi^3} \right]$$

$$= \frac{4kL^2}{n^3 \pi^3} \left[ (-1)^{n+1} + 1 \right]$$

$$u(x,t) = \sum_{n=1}^{\infty} \frac{4kL^2}{n^3 \pi^3} \left[ (-1)^{n+1} + 1 \right] \sin \frac{n\pi}{L} x \cos \frac{an\pi t}{L}$$

2. A uniform elastic string of length 60cm is subjected to a constant tension of 2kN. If the ends are fixed and the initial displacement is $60x - x^2$ and the initial velocity is zero find the displacement fn. $u(x,t)$

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2} \quad \text{—— (1)}$$

$$a^2 = T/m = 2/m$$

The soln of eqn (1) is given by;

$$u(x,t) = (A\cos\lambda x + B\sin\lambda x)(C\cos a\lambda t + D\sin a\lambda t) \quad \text{—— (2)}$$

The boundary conditions are given by;

$$u(0,t) = 0 \quad \text{—— (a)} \qquad u(60,t) = 0 \quad \text{—— (b)}$$

The initial conditions are given by;

$$f(x) = u(x,0) = 60x - x^2 \quad \text{—— (c)}$$

$$g(x) = \left(\frac{\partial u}{\partial t}\right)_{t=0} = 0 \quad \text{—— (d)}$$

(a) in (2)

$$\boxed{A = 0}$$

$$u(x,t) = B\sin\lambda x(C\cos a\lambda t + D\sin a\lambda t) \quad \text{—— (3)}$$

(b) in (3)

$$\Rightarrow \lambda = \frac{n\pi}{L} \Rightarrow \lambda = \frac{n\pi}{60}.$$

Applying (c) in (3)

$$f(x) = BC\sin\lambda x$$

$$f(x) = \sum_{n=1}^{\infty} A_n \sin\lambda x$$

$$A_n = \frac{2}{L}\int_0^L f(x)\sin\lambda x \, dx$$

$$\frac{\partial v}{\partial t} = B\sin\lambda x\left(-a\lambda(\sin a\lambda t + a\lambda D\cos a\lambda t\right) \quad (4)$$

Applying (d) in (4)

At $t=0$;

$0 = B\sin\lambda x \times a\lambda \times D$,

$0 = a\lambda BD\sin\lambda x$.

$\Rightarrow \boxed{D=0}$

$$u(x,t) = BC\sin\lambda x\cos a\lambda t.$$

$$= \sum_{n=1}^{\infty} A_n \sin\lambda x\cos a\lambda t \quad (5)$$

$$A_n = \frac{2}{60}\int_0^L f(x)\sin\lambda x\,dx.$$

$$= \frac{2}{L}\int_0^{60}(60x - x^2)\sin\lambda x\,dx.$$

$$= \frac{2}{60}\left[(60x - x^2)\times\left(\frac{-\cos\frac{n\pi x}{60}}{n\pi/60}\right) - (60-2x)\times\frac{-\sin\left(\frac{n\pi x}{60}\right)}{n^2\pi^2/60^2} - 2\frac{\cos\frac{n\pi x}{60}}{n^3\pi^3/60^3}\right]_0^{60}$$

$$= \frac{2}{60}\left[\frac{-2\cos\frac{n\pi\times 60}{60}}{n^3\pi^3/60^3} + \frac{2\times 60^3}{n^3\pi^3/60^3}\right]$$

$$= \frac{2}{\cancel{60}}\times\frac{2\times 60^{\cancel{3}\,2}}{n^3\pi^3}\left(1-(-1)^n\right)$$

$$= \frac{4\times 60^2}{n^3\pi^3}\left(1-(-1)^n\right)$$

$$= \frac{14400}{n^3\pi^3}\left(1-(-1)^n\right)$$

$$u(x,t) = \sum_{n=1}^{\infty}\frac{14400}{n^3\pi^3}\left(1-(-1)^n\right)\sin\frac{n\pi x}{60}\cos a\frac{n\pi t}{60}$$

$\frac{14400}{n^3\pi^3}[1-(-$

3. A tightly stretched string with end points $x=0$ & $x=L$ is initially in a position given by $\sin^3\left(\frac{\pi x}{L}\right)$. If it is released from rest in this position find the displacement for $u(x,t)$ in terms of Fourier co-efficient of $f(x)$.

13/11/2020

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2} \quad \text{——— } (1)$$

Soln of eqn (1) is given by;

$$u(x,t) = (A\cos\lambda x + B\sin\lambda x)(C\cos a\lambda t + D\sin a\lambda t)$$

——— (2)

The boundary conditions are;

$$u(0,t) = 0 \quad \text{——— } (a)$$

$$u(L,t) = 0 \quad \text{——— } (b)$$

The initial conditions are given by;

$$f(x) = \sin^3\left(\frac{\pi x}{L}\right) = u(x,0) \quad \text{——— } (c)$$

$$g(x) = \left(\frac{\partial u}{\partial t}\right)_{t=0} = 0 \quad \text{——— } (d)$$

(a) in (2) $\Rightarrow$ $A=0$.

$$u(x,t) = B\sin\lambda x\,(C\cos a\lambda t + D\sin a\lambda t)$$

——— (3)

(b) in (3) $\Rightarrow$ $\sin\lambda L = 0$

$$\lambda L = n\pi$$

$$\lambda L = n\pi$$

$$\boxed{\lambda = \frac{n\pi}{L}}$$

Scanned by CamScanner

Applying (C) in (3)

$f(x) = BC \sin \lambda x$

$f(x) = \sum_{n=1}^{\infty} A_n \sin \lambda x$

$A_n = \frac{2}{L} \int_0^L f(x) \sin \lambda x \, dx.$

$\frac{\partial u}{\partial t} = B \sin \lambda x \left( -a\lambda \sin a\lambda t + a\lambda D \cos a\lambda t \right)$ ——— (*)

Applying (d) in (4).

At $t = 0$;

$0 = B \sin \lambda x \times a\lambda \times D.$

$0 = B a\lambda D \sin \lambda x$

$\Rightarrow \boxed{D = 0}$

$u(x,t) = BC \sin \lambda x \cos a\lambda t$

$= \sum_{n=1}^{\infty} A_n \sin \lambda x \cos a\lambda t$ ——— (5)

$A_n = \frac{2}{L} \int_0^L f(x) \sin \lambda x \, dx$

When $f(x)$ or $g(x)$ are fns of sin or cos we use the following Method

In (5) $t = 0$

$u(x,0) = \sum_{n=1}^{\infty} A_n \sin \left( \frac{n\pi}{L} \right) x$

$\sin^3 \left( \frac{\pi x}{L} \right) = A_1 \sin \left( \frac{\pi}{L} \right) x + A_2 \sin \left( \frac{2\pi}{L} \right) x +$

$A_3 \sin \left( \frac{3\pi}{L} \right) x + A_4 \sin \left( \frac{4\pi}{L} \right) x + \dots$

$\sin 3x = 3 \sin x - 4 \sin^3 x.$

$$\sin^3 x = \frac{1}{4}(3\sin x - \sin 3x)$$

$$\sin^3\left(\frac{\pi x}{L}\right) = \frac{1}{4}\left[3\sin\frac{\pi x}{L} - \sin\left(\frac{3\pi x}{L}\right)\right]$$

$A_1 = 3/4$ $\qquad A_2 = 0$ $\qquad A_3 = -1/4$

$A_4 = A_5 = A_6 = \cdots\cdots = 0.$

$$u(x,t) = \frac{3}{4}\sin\frac{\pi x}{L}\cos\frac{a\pi t}{L} + -1/4\sin\frac{3\pi x}{L}\cos\frac{3a\pi t}{L}$$

$$u(x,t) = \frac{3}{4}\sin\frac{\pi x}{L}\cos\frac{a\pi t}{L} - \frac{1}{4}\sin\frac{3\pi x}{L}\cos\frac{3a\pi t}{L}$$

4. Solve the problem of vibrating string for the following conditions:

$y(0,t) = 0$

$y(20,t) = 0$

$y(x,0) = 0$

$$\left(\frac{dy}{dt}\right)_{t=0} = \begin{cases} x & 0 \le x < 10 \\ 20 - x & 10 \le x \le 20 \end{cases}$$

The wave eqn is given by;

$$\frac{\partial^2 y}{\partial t^2} = a^2\frac{\partial^2 y}{\partial x^2} \quad \text{———} \quad (1)$$

The soln of (1) is given by;

$$y(x,t) = (A\cos\lambda x + B\sin\lambda x)(C\cos a\lambda t + D\sin a\lambda t) \quad \text{———} \quad (2)$$

The boundary conditions are;

$$y(0,t) = 0 \quad \text{———} \quad (a)$$

Scanned by CamScanner

$y(20,t) = 0$ ——— (b)

Initial conditions:

$y(x,0) = 0$ ——— (c)

$$\left(\frac{\partial y}{\partial t}\right)_{t=0} = \begin{cases} x & 0 \le x < 10 \\ 20-x & 10 \le x \le 20 \end{cases} \quad \text{——— (d).}$$

Applying (a) in (2).

$\Rightarrow A = 0.$

$y(x,t) = B\sin\lambda x\,(C\cos a\lambda t + D\sin a\lambda t)$ ——— (3)

Applying (b) in (3)

$$B\sin\lambda \times 20\,(C\cos a\lambda t + D\sin a\lambda t) = 0$$

$$\sin 20\lambda = 0.$$

$$20\lambda = n\pi$$

$$\boxed{\lambda = n\pi/20}$$

Applying the initial conditions (c) in (3)

$$BC\sin\lambda x = 0$$

$\Rightarrow \boxed{C = 0}$  $B \neq 0$ (soln $= 0$) ($\sin\lambda x \neq 0$)

$y(x,t) = BD\sin\lambda x\,\sin a\lambda t$ ——— (4).

$$\frac{\partial y(x,t)}{\partial t} = BD\sin\lambda x \times \cos a\lambda t \times a\lambda.$$

$$= a\lambda BD\sin\lambda x\cos a\lambda t$$

put $t=0$. in (d) in the above eqn.

$$g(x) = a\lambda BD\sin\lambda x$$

$$= a\lambda \sum_{n=1}^{\infty} B_n \sin\lambda x \quad (\lambda = n\pi/20)$$

$$\Rightarrow B_n = \frac{2}{a\lambda L}\int_0^L g(x)\sin\lambda x\,dx.$$

∴ Soln of the given wave eqn is given by;

$$y(x,t) = \sum_{n=1}^{\infty} B_n \sin \lambda x \sin a \lambda t \quad \text{--- (5)}$$

$$B_n = \frac{2}{L a \lambda} \int_0^L g(x) \sin \lambda x \, dx$$

$$= \frac{1 \times 20}{10\, a \times n\pi} \int_0^{10} x \sin\left(\frac{n\pi x}{20}\right) dx + \int_{10}^{20} 20 x \sin \frac{n\pi x}{20} dx$$

$$= \frac{2}{a n\pi} \left\{ \left[ x \times \frac{-\cos \frac{n\pi x}{20}}{n\pi/20} + \frac{\sin \frac{n\pi x}{20}}{(n\pi/20)^2} \right]_0^{10} + \right.$$

$$\left. \left[ 20 x \times \frac{-\cos \frac{n\pi x}{20}}{n\pi/20} + \frac{20 \times \sin \frac{n\pi x}{20}}{(n\pi/20)^2} \right]_{10}^{20} \right\}$$

$$= \frac{2}{a n\pi} \left\{ \frac{-10 \cos(n\pi/2)}{n\pi/20} + \frac{400 \sin(n\pi/2)}{n^2\pi^2} - 0 + \right.$$

$$\left. - \frac{400 \times 20}{n\pi} \cos n\pi + \frac{20 \times 400}{n^2\pi^2} \sin n\pi + \frac{400 \times 10 \cos(n\pi/2)}{n\pi} - \frac{400 \times 20}{n^2\pi^2} \sin(n\pi/2) \right]$$

$$= \frac{2}{a n\pi} \left[ -\frac{200}{n\pi} \cos(n\pi/2) + \frac{400}{n^2\pi^2} \sin(n\pi/2) - \frac{400 \times 20}{n\pi} \cos n\pi + \frac{20 \times 400}{n^2\pi^2} \sin n\pi + \frac{4000 \cos(n\pi/2)}{n\pi} - \frac{8000}{n^2\pi^2} \sin(n\pi/2) \right]$$

$$= \frac{2}{an\pi}\left[\frac{3800}{n\pi}\cos(n\pi/2) - \frac{7600}{n^2\pi^2}\sin(n\pi/2) - \frac{8000}{n\pi}\cos n\pi + 0\right]$$

$$B_n = \frac{2}{an\pi}\left[\frac{3800}{n\pi}\cos(n\pi/2) - \frac{7600}{n^2\pi^2}\sin(n\pi/2) + (-1)^{n+1}\frac{8000}{n\pi}\right]$$

No 5. A string is stretched and fastened 2 pts L apart motion is started by displacing the string in the form $y = k\sin\frac{\pi x}{L}$ from which it is released at $t=0$ Show that $y(x,t) = k\sin\left(\frac{\pi x}{L}\right)\cos\left(\frac{\pi a t}{L}\right)$.

The wave eqn is given by;

$$\frac{\partial^2 y}{\partial t^2} = a^2\frac{\partial^2 y}{\partial x^2} \quad \text{——— ①}$$

The soln is given by;

$$y(x,t) = (A\cos\lambda x + B\sin\lambda x)(C\cos a\lambda t + D\sin a\lambda t) \quad \text{——— ②}$$

Boundary conditions;

$$y(0,t) = 0 \quad \text{——— ⓐ}$$

$$y(L,t) = 0 \quad \text{——— ⓑ}$$

Initial conditions;

$$f(x) = y(x,0) = 0 \quad \text{——— ⓒ}$$

$$g(x) = \left(\frac{\partial y}{\partial t}\right)_{t=0} = 0 \quad \text{——— ⓓ}$$

Applying ⓐ in ②

$y(x,t) =$ $\Rightarrow \boxed{A=0}$.

Scanned by CamScanner

$\therefore$ ② $\Rightarrow$ $y(x,t) = B\sin\lambda x\,(C\cos a\lambda t + D\sin a\lambda t)$ ———— ③.

Applying ⓑ in ③.

$\Rightarrow \quad \sin\lambda L = 0$

$\lambda L = 0$

$\Rightarrow \quad \lambda L = n\pi$

$$\boxed{\lambda = n\pi/L}$$

Applying ⓒ in ③.

$$f(x) = BC\sin\lambda x$$

$$= \sum_{n=1}^{\infty} A_n \sin\lambda x.$$

$$A_n = \frac{2}{L}\int_0^L f(x)\sin\lambda x\,dx$$

$\frac{\partial y}{\partial t} \Rightarrow B\sin\lambda x\,(-a\lambda C\sin a\lambda t + a\lambda D\cos a\lambda t)$

$\left[\partial y/\partial t\right]_{t=0} \Rightarrow a\lambda BD\sin\lambda x = 0$

$\Rightarrow \boxed{D=0}$

$y(x,t) = B\sin\lambda x \times C\cos a\lambda t$

$= BC\sin\lambda x\cos a\lambda t$

$$y(x,t) = \sum_{n=1}^{\infty} A_n \sin\lambda x\cos a\lambda t$$

$$A_n = \frac{2}{L}\int_0^L f(x)\sin\lambda x\,dx.$$

$$= \frac{2k}{L}\int_0^L \cancel{\sin\frac{\pi x}{L}}\,\sin\lambda x$$

$$y(x,0) = \sum_{n=1}^{\infty} A_n \sin \lambda x.$$

$$= A_1 \sin \frac{\pi x}{L} + A_2 \sin \frac{2\pi x}{L} + A_3 \sin \frac{3\pi x}{L} + \cdots$$

$$k \sin \frac{\pi x}{L} = A_1 \sin \frac{\pi x}{L} + A_2 \sin \frac{2\pi x}{L} + A_3 \sin \frac{3\pi x}{L} + \cdots$$

$$A_1 = k.$$

$$A_2 = A_3 = A_4 = \cdots = 0.$$

$$\Rightarrow \quad y(x,t) = k \sin\left(\frac{\pi x}{L}\right) \times \cos\left(\frac{a\pi t}{L}\right)$$

Scanned by CamScanner

$$y(x,0) = \sum_{n=1}^{\infty} A_n \sin \lambda x.$$

$$= A_1 \sin \frac{\pi x}{L} + A_2 \sin \frac{2\pi x}{L} + A_3 \sin \frac{3\pi x}{L} + \cdots$$

$$K \sin \frac{\pi x}{L} = A_1 \sin \frac{\pi x}{L} + A_2 \sin \frac{2\pi x}{L} + A_3 \sin \frac{3\pi x}{L} + \cdots$$

$A_1 = k.$

$A_2 = A_3 = A_4 = \cdots = 0.$

$$\Rightarrow \quad y(x,t) = k \sin\left(\frac{\pi x}{L}\right) \times \cos\left(\frac{a\pi t}{L}\right)$$

16/11/2020

6. If a string of length L is initially at rest in the equilibrium position. It is given at each point a velocity $y_0 \sin\left(\frac{3\pi x}{L}\right) \cos\left(\frac{2\pi x}{L}\right)$ find the displacement at any time t.

The wave eqn is;

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2} \text{ —— } (1)$$

Soln;

$$u(x,t) = (A\cos\lambda x + B\sin\lambda x)(C\cos a\lambda t + D\sin a\lambda t) \text{ —— } (2)$$

The boundary conditions are given by;

$u(0,t) = 0$ —— (a)

$u(L,t) = 0$ —— (b)

Initial conditions;

$f(x) = u(x,0) = 0$ —— (c)

$$g(x) = \left(\frac{\partial u}{\partial t}\right)_{t=0} = y_0 \sin\left(\frac{3\pi x}{L}\right) \cos\left(\frac{2\pi x}{L}\right) \text{ —— } (d)$$

Applying ⓐ in ②

$$\boxed{A = 0}$$

② ⇒ $u(x,t) = B \sin \lambda x \,(C \cos a\lambda t + D \sin \lambda x)$ ——— ③

Applying ⓑ in ③.

$\Rightarrow \quad \sin \lambda L = 0.$

$\lambda L = n\pi$

$$\boxed{\lambda = \frac{n\pi}{L}}$$

Applying ⓒ in ③

$0 = B \sin \lambda x \times C$

$BC \sin \lambda x = 0$

$\boxed{C = 0}$ $\quad B \neq 0 \quad \sin \lambda x \neq 0$

③ ⇒ $u(x,t) = B \sin \lambda x \times D \sin a\lambda t$

$= BD \sin \lambda x \sin a\lambda t$ ——— ④

$\partial u/\partial t = BD \sin \lambda x \times a\lambda \cos a\lambda t$

$(\partial u/\partial t)_{t=0} = BD \sin \lambda x \times a\lambda$

$g(x) = BD\, a\lambda \sin \lambda x.$

$g(x) = \sum_{n=1}^{\infty} B_n \, a\lambda \sin \lambda x$

$$B_n = \frac{2}{\lambda a L}\int_0^L g(x)\sin\lambda x\, dx$$

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\lambda x \sin a\lambda t \quad (\lambda = n\pi/L) \qquad \text{(5)}$$

$$\left[\frac{\partial u}{\partial t}\right]_{t=0} = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L} \times \cos\frac{n\pi a t}{L} \times \frac{n\pi a}{L}$$

$$= \sum_{n=1}^{\infty} B_n \times \frac{n\pi a}{L} \sin\frac{n\pi x}{L}$$

$$y_0 \sin\left(\frac{3\pi x}{L}\right)\cos\left(\frac{2\pi x}{L}\right) = \frac{a\pi}{L}\left[B_1 \sin\frac{\pi x}{L} + 2B_2 \sin\frac{2\pi x}{L} + 3B_3 \sin\frac{3\pi x}{L} + 4B_4 \sin\frac{4\pi x}{L} + 5B_5 \sin\frac{5\pi x}{L} + \cdots\right.$$

$$\frac{2}{2} y_0 \sin\left(\frac{3\pi x}{L}\right)\cos\left(\frac{2\pi x}{L}\right) = \frac{a\pi}{L}\left[B_1 \sin\frac{\pi x}{L} + 2B_2 \sin\frac{2\pi x}{L} + 3B_3 \sin\frac{3\pi x}{L} + 4B_4 \sin\frac{4\pi x}{L} + 5B_5 \sin\frac{5\pi x}{L} + \cdots\right]$$

$$y_0/2\left[\sin\left(\frac{5\pi x}{L}\right) + \sin\left(\frac{\pi x}{L}\right)\right] = \frac{a\pi}{L}\left[B_1 \sin\frac{\pi x}{L} + 2B_2 \sin\left(\frac{2\pi x}{L}\right) + 3B_3 \sin\left(\frac{3\pi x}{L}\right) + 4B_4 \sin\left(\frac{4\pi x}{L}\right) + 5B_5 \sin\left(\frac{5\pi x}{L}\right) + \cdots\right.$$

$$y_0/2 = \frac{a\pi}{L} B_1 \qquad y_0/2 = \frac{5a\pi}{L} B_5$$

$$B_1 = \frac{y_0 L}{2a\pi} \qquad B_5 = \frac{y_0 L}{10 a\pi}$$

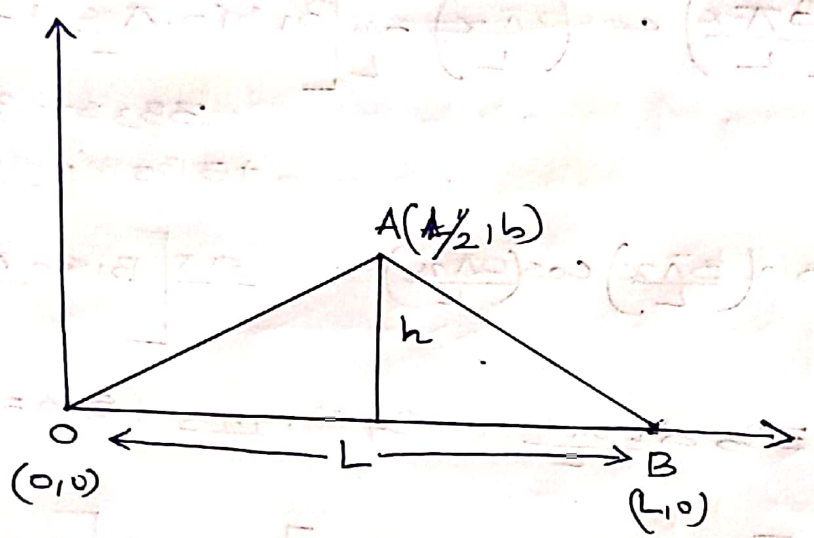
(5) $\Longrightarrow$

$$u(x,t) = \frac{y_0 L}{2a\pi}\sin\left(\frac{\pi x}{L}\right)\sin\left(\frac{a\pi t}{L}\right) + \frac{y_0 L}{10 a\pi}\sin\left(\frac{5\pi x}{L}\right)\sin\left(\frac{5a\pi t}{L}\right)$$

7. A tightly stretched string of length 'L' has its ends fastened at $x=0$ and $x=L$

3mks

The midpt of the string is then taken to a height h and then released from rest in that position. ~~Find the displacement of a point of the string at time t.~~ Find the initial & boundary conditions of the string at time t from the instant of release




$$\frac{y-y_1}{y_2-y_1} = \frac{x-x_1}{x_2-x_1}$$

released from rest $\Rightarrow g(x) = 0$

(OA)

$$\frac{y-0}{h-0} = \frac{x-0}{L/2-0}$$

$$\frac{y}{h} = \frac{2x}{L}$$

$$y = \frac{2hx}{L}, \quad 0 < x < L/2$$

(AB)

$$\frac{y-0}{h-0} = \frac{x-L}{L/2-L}$$

$\frac{L-2x}{2}$

$-\frac{L}{2}$

$$y = \frac{2h(L-x)}{L} \quad ; \quad L/2 < x < L$$

The boundary conditions are;

$$y(0,t) = 0 \quad y(L,t) = 0.$$

Initial conditions are;

$$f(x) = \begin{cases} \frac{2hx}{L} & 0 \le x < L/2 \\ \frac{2h(L-x)}{L} & L/2 \le x < L \end{cases}$$

$$g(x) = \left(\frac{\partial y}{\partial t}\right)_{t=0} = 0$$

Que 8. If a string of length $l$ is initially at rest in the equilibrium position & each of its points is given a velocity $v$ such that.

$$v = \begin{cases} ax, & 0 < x < l/2 \\ a(l-x), & l/2 < x < l \end{cases}$$

Determine the displacement $y(x,t)$ at any time $t$.

The wave eqn is;

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2} \quad \text{——— (1)}$$

Soln is;

$$u(x,t) = (A\cos \lambda x + B\sin \lambda x)(C\cos a\lambda t + D\sin a\lambda t) \quad \text{——— (2)}$$

Boundary conditions are given by;

$$u(0,t) = 0 \quad \text{——— (a)}$$

$$u(l,t) = 0 \quad \text{——— (b)}$$

Initial conditions

$$f(x) = u(x,0) = 0 \quad \text{——— (c)}$$

$$g(x) = \frac{\partial u}{\partial t} = V = \begin{cases} ax & ; \ 0 < x < l/2 \\ a(l-x) & ; \ l/2 < x < l \end{cases}$$

Applying (a) in (2).

$A = 0$.

(2) $\Rightarrow$ $u(x,t) = B \sin\lambda x \, (C\cos a\lambda t + D \sin a\lambda t)$ —— (3)

Applying (b) in (3)

$$\sin\lambda l = 0$$

$$\lambda l = n\pi$$

$$\lambda = \frac{n\pi}{l}$$

$$\boxed{\lambda = n\pi/l}$$

Applying (c) in (3)

$$0 = B \sin\lambda x \times C$$

$$BC \sin\lambda x = 0$$

$$\Rightarrow \boxed{C = 0}$$

$u(x,t) = BD \sin\lambda x \sin\lambda at$ —— (4)

$$\partial u/\partial t = BD \sin\lambda x \times a\lambda \times \cos a\lambda t$$

$$\partial u/\partial t = BDa\lambda \sin\lambda x \cos a\lambda t$$

$$= \sum_{n=1}^{\infty} B_n a\lambda \sin\lambda x \cos a\lambda t$$

$$(\partial u/\partial t)_{t=0} = \sum_{n=1}^{\infty} B_n a\lambda \sin\lambda x$$

$$\cancel{u(x,t) = \sum_{n=1}^{\infty}}$$

$$B_n = \frac{2}{\lambda a l}\left[\int_0^{l/2} ax \sin\lambda x \, dx + \int_{l/2}^{l} a(l-x)\sin\lambda x \, dx\right]$$

$$= \frac{2}{\lambda a l}\left[\int_0^{l/2} a \cdot x \sin\frac{n\pi x}{l}\,dx + \int_{l/2}^{l} a(l-x)\sin\frac{n\pi x}{l}\,dx\right]$$

$$= \frac{2}{\lambda a l} \times a\left\{\left[\frac{-l^2\cos(n\pi/2)}{2n\pi} + \frac{l^2\sin(n\pi/2)}{n^2\pi^2}\right] + \left[0-0+\frac{l^2\cos(n\pi/2)}{2n\pi} + \frac{l^2\sin(n\pi/2)}{n^2\pi^2}\right]\right\}$$

$$= \frac{2\times \cancel{a}}{n\pi \cancel{a}}\left[\cancel{-\frac{l^2\cos(n\pi/2)}{2n\pi}} + \frac{l^2\sin(n\pi/2)}{n^2\pi^2} + \cancel{\frac{l^2\cos(n\pi/2)}{2n\pi}} + \frac{l^2\sin(n\pi/2)}{n^2\pi^2}\right]$$

$$= \frac{2}{n\pi}\times\frac{2l^2\sin(n\pi/2)}{n^2\pi^2}$$

$$B_n = \frac{4l^2\sin(n\pi/2)}{n^3\pi^3}$$

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\lambda x \sin\lambda a t$$

$$u(x,t) = \sum_{n=1}^{\infty}\frac{4l^2\sin(n\pi/2)}{n^3\pi^3}\times\sin\left(\frac{n\pi x}{l}\right)\sin\left(\frac{an\pi t}{l}\right)$$

Module-6

## One dimensional ~~heat~~ heat equation

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$$

In a body heat will flow in a direction of decreasing temperature. Physical experiments show that the rate of flow is proportional to the gradient of the temperature so as to obtain the heat equation. We assume the following empirical laws-

(i) Heat flows from a higher to lower temperature.

(ii) The rate at which heat flows across any area is directly proportional to the area and to the temperature gradient normal to the area. This proportionality constant is called the "Thermal conductivity" ($K$) of the material

$$\frac{Q}{t} = -kA\left(\frac{\partial u}{\partial x}\right)_x$$

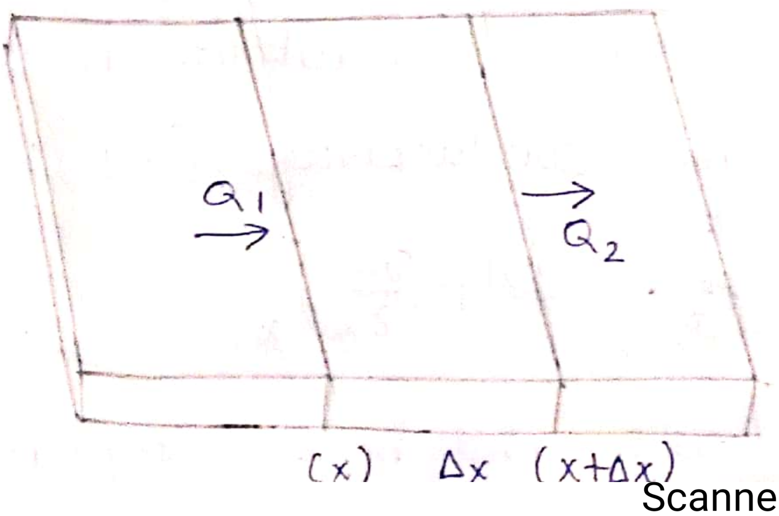
(iii) The rate of heat required to produce a given temperature change in a body is proportional to the mass of the body and the rate of change of temperature change. This proportionality constant is called the specific heat capacity (c) of the material.

$$\frac{Q}{t} \propto m \frac{\partial u}{\partial t}$$

$$\frac{Q}{t} = Cm \frac{\partial u}{\partial t} = Q'$$

Consider two cross section of the rod at point P and Q which are x and x + Δx distances from one end of the rod.

Let $Q_1$ and $Q_2$ be the quantity of heat which flows through the cross sections P and Q respectively.



The rate of heat flow through the cross section at P is

$$\frac{Q_1}{t} = -k\,A\left(\frac{\partial u}{\partial x}\right)_x = Q_1'$$

The rate of heat flow through the cross section at Q is

$$\frac{Q_2}{t} = -k\,A\left(\frac{\partial u}{\partial x}\right)_{x+\Delta x} = Q_2'$$

Hence the heat rate retained between the cross section at P and Q is

$$Q_1' - Q_2' = kA\left[\left(\frac{\partial u}{\partial x}\right)_{x+\Delta x} - \left(\frac{\partial u}{\partial x}\right)_x\right]$$

Also $Q_1' - Q_2' = C \times \text{mass} \times \dfrac{\Delta u}{\Delta t}$

$$= C \times \rho \times A \times \Delta x \, \frac{\Delta u}{\Delta t} \quad \boxed{\because \text{mass} = \rho V}$$

Where C is the specific heat capacity of the material and $\rho$ is the density of the material

$$C \rho A \Delta x \frac{\Delta u}{\Delta t} = KA\left[\left(\frac{\partial u}{\partial x}\right)_{x+\Delta x} - \left(\frac{\partial u}{\partial x}\right)_x\right]$$

$$\frac{\Delta u}{\Delta t} = \frac{k}{\rho c}\left[\frac{\left(\frac{\partial u}{\partial x}\right)_{x+\Delta x} - \left(\frac{\partial u}{\partial x}\right)_{x}}{\Delta x}\right]$$

Taking the limit $\Delta x \to 0$, $\Delta t \to 0$ we get $\frac{\partial u}{\partial t} = \frac{k}{\rho c}\frac{\partial^2 u}{\partial x^2} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$

where $\alpha^2 = \frac{k}{\rho c}$. is known as diffusivity of the material of the bar.

This PDE $\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$ is called one dimensional heat transfer equation (Diffusion equation).

Solution of heat equation using ~~method~~ method of separation of variables.

The heat equation is given by,

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2} \quad \text{——} ①$$

Here $u$ is a function of $x$ and $t$.
So $u = XT$ where $X$ is a function of $x$ alone and $T$ is a function of $t$ alone.

$$\frac{\partial u}{\partial t} = XT'$$

$$\frac{\partial^2 u}{\partial x^2} = X''T$$

Sub: in ①.

$$XT' = \alpha^2 X'' T$$

dividing throughout by $XT$

$$\frac{T'}{T} = \alpha^2 \frac{X''}{X}$$

$$\frac{X''}{X} = \frac{T'}{\alpha^2 T} = k$$

$$X'' - Xk = 0 \quad \text{——} ②$$

$$T' - \alpha^2 T k = 0 \quad \text{——} ③$$

Case 01 :- $k > 0$

Let $k = \lambda^2$

Eqn ② implies

$$X'' - X\lambda^2 = 0$$

The auxillary equation is

$$m^2 - \lambda^2 = 0$$

$$m^2 = \lambda^2$$

$$m = \pm \lambda$$

$$X = C_1 e^{\lambda x} + C_2 e^{-\lambda x}.$$

Eqn ③ implies

$$T' - \alpha^2 T \lambda^2 = 0.$$

The auxillary equatio is

$$m^3 - \alpha^2 \lambda^2 = 0$$

$$~~m = \pm\alpha\lambda~~. \quad m = \alpha^2\lambda^2$$

$$T = C_3 e^{\alpha^2\lambda^2 t}$$

So in this case

$$u = XT = (C_1 e^{\lambda x} + C_2 e^{-\lambda x})(C_3 e^{\alpha^2\lambda^2 t}$$

Case 02 :- $k = -\lambda^2 \quad k < 0.$

Eqn ② implies

$$X'' + X\lambda^2 = 0$$

The auxillary equation is

$$m^2 + \lambda^2 = 0$$

$$m^2 = -\lambda^2$$

$$m = \pm i\lambda.$$

$$X = C_4 \cos \lambda x + C_5 \sin \lambda x$$

Scanned by TapScanner

Eqn ③ implies

$$T' + \alpha^2\lambda^2 T = 0$$

auxillary equation is

$$m^2 + \alpha^2\lambda^2 = 0$$

$$m^2 = -\alpha^2\lambda^2$$

$$T = c_6\, e^{-\alpha^2\lambda^2 t}$$

So in this case

$$u = XT = (C_4 \cos\lambda x + C_5 \sin\lambda x)\, C_6 e^{-\alpha^2\lambda^2 t}$$

Case 03:- $k = 0$

$$X'' = 0 \qquad T' = 0$$

Auxillary equation is $m^2 = 0$

$$m = 0, 0$$

$$X = (C_7 + C_8 x)e^{0x}$$

$$T = C_9$$

So in this case,

$$u = (C_7 + C_8 x) C_9$$

In the problem,

$$u = (C_4 \cos\lambda x + C_5 \sin\lambda x) C_6 e^{-\alpha^2\lambda^2 t}$$

$$\therefore \quad u = (A\cos\lambda x + B\sin\lambda x)\, C e^{-\alpha^2\lambda^2 t}.$$

Chapter 2;

17/11/2020 **D'Alemberts' soln of Wave eqn:**

No Boundary conditions only initial conditions.

consider the wave eqn $\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$

Then; D'Alemberts soln of the wave eqn is given by;

$$y(x,t) = \frac{1}{2}\left[f(x+at) + f(x-at)\right]$$

**Using D'Alemberts' Method solve the following;**

q. Given a vibrating string of unit length having fixed ends with initial velocity zero & initial deflection $f(x) = k(\sin x - \sin 2x)$

$$y(x,t) = \frac{1}{2}\left[f(x+at) + f(x-at)\right]$$

$$= \frac{1}{2}\left[k[\sin(x+at) - \sin 2(x+at)] + k[\sin(x-at) - \sin 2(x-at)]\right]$$

$$y(x,t) = \frac{k}{2}\left[\sin x\cos at + \cancel{\cos x\sin at} - \sin 2x\cos 2at - \cancel{\cos 2x\sin 2at} + \sin x\cos at - \cancel{\cos x\sin at} - \sin 2x\cos 2at + \cancel{\cos 2x\sin 2at}\right]$$

$$= \frac{k}{2}\left[2\sin x\cos at - 2\sin 2x\cos 2at\right]$$

$$y(x,t) = \underline{k\left[\sin x\cos at - \sin 2x\cos 2at\right]}$$

$$y(x,t) = k[\sin x \cos at - \sin 2x \cos 2at] \quad \text{——— (1)}$$

$$\partial y/\partial t = k[\sin x \times -a \sin at - \sin 2x \times 2a \times -\sin 2at]$$

$$= k[-a \sin x \sin at + 2a \sin 2x \sin 2at]$$

$$y(x,0) = k[\sin x - \sin 2x] = f(x).$$

$$[\partial y/\partial t]_{t=0} = k \times 0 = 0.$$

10. Vibration of a string of unit length having fixed ends with initial velocity zero and initial deflection $f(x) = a(x-x^2)$.

$$y(x,t) = \tfrac{1}{2}[f(x+ct) + f(x+ct)]$$

$$= \tfrac{1}{2}\left[a[(x+ct) - (x-ct)^2] + a[(x-ct) - (x-ct)^2]\right]$$

$$= \frac{a}{2}\left[(x+ct) - (x+ct)^2 + (x-ct) - (x-ct)^2\right]$$

$$= \frac{a}{2}\left[x + \cancel{ct} - x^2 - c^2t^2 - \cancel{2xct} + x - \cancel{ct} - x^2 - c^2t^2 + \cancel{2xct}\right]$$

$$= a(x - x^2 - c^2t^2)$$

$$y(x,t) = a(x - x^2 - c^2t^2) \quad \text{——— (1)}$$

$$\partial y/\partial t = -ac^2 \times 2t$$

$$y(x,0) = a(x-x^2) = f(x)$$

$$(\partial y/\partial t)_{t=0} = 0.$$

ie Initial conditions are satisfied.

Scanned by CamScanner

Thus ① is the D' Alembert's' soln of the given wave eqn.

⑪. Vibrating string of unit length having fixed ends with initial velocity zero and initial deflection. $f(x) = c \sin^2 \pi x$.

$$y(x,t) = \frac{1}{2}\left[f(x+at) + f(x-at)\right]$$

$$= \frac{1}{2}\left[c \sin^2 \pi (x+at) + c \sin^2 \pi (x-at)\right]$$

$$= \frac{c}{2}\left[\sin^2 \pi (x+at) + \sin^2 \pi (x-at)\right]$$

$$= \frac{c}{4}\left[1 - \cos 2\pi(x+at) + 1 - \cos 2\pi (x-at)\right]$$

$$= \frac{c}{4}\left[2 - \cos 2\pi (x+at) - \cos 2\pi (x-at)\right]$$

$$= \frac{c}{4}\Big[2 - \cos 2\pi x \cos 2\pi at + \cancel{\sin 2\pi x \sin 2\pi at}$$

$$- \cos 2\pi x \cos 2\pi at - \cancel{\sin 2\pi x \sin 2\pi at}\Big]$$

$$= \frac{c}{4}\left[2 - 2\cos 2\pi x \cos 2\pi at\right]$$

$$= \frac{c}{2}\left[1 - \cos 2\pi x \cos 2\pi at\right]$$

$$y(x,t) = \frac{c}{2}\left(1 - \cos 2\pi x \cos 2\pi at\right] \quad \text{—①}$$

$$\frac{\partial y}{\partial t} = \frac{c}{2}\left[0 + \cos 2\pi x \times 2\pi a \times \sin 2\pi a t\right]$$

$$y(x,0) = \frac{c}{2}\left[1 - \cos 2\pi x\right]$$

$$= \frac{c \sin^2 \pi x \times \cancel{2}}{\cancel{2}}$$

$$= c \sin^2 \pi x = f(x)$$

$$\left(\partial y/\partial t\right)_{t=0} = 0.$$

Thus ① is the D'Alemberts' soln of the given wave eqn.

Scanned by CamScanner

1) Solve the heat equation $\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$ with initial temperature $u = \frac{Cx(L-x)}{L^2}$

Heat equation is given by

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2} \quad — (1)$$

General solution of equation (1) is given by

$$u(x,t) = (A\cos\lambda x + B\sin\lambda x)e^{-\alpha^2\lambda^2 t} \quad — (2)$$

The boundary conditions are given by

$$u(0,t) = 0 \quad — (a)$$

$$u(L,t) = 0 \quad — (b)$$

The initial conditions are given by

$$u(x,0) = f(x) = \frac{Cx(L-x)}{L^2} \quad — (c)$$

Sub (a) in equation (2) we get

$$0 = A e^{-\alpha^2\lambda^2 t}$$

$$A = 0$$

$$\therefore u(x,t) = B\sin\lambda x \, e^{-\alpha^2\lambda^2 t} \quad — (3)$$

Sub (b) in equation (3) we get

$$0 = B\sin\lambda L e^{-\alpha^2\lambda^2 t}$$

$$\sin \lambda L = 0$$

$$\lambda L = n\pi$$

$$\lambda = \frac{n\pi}{L}$$

Sub ⓒ in equation ③ we get

$$f(x) = B \sin \lambda x$$

$$f(x) = \sum_{n=1}^{\infty} B_n \sin \lambda x$$

Where $B_n = \frac{2}{L} \int_0^L f(x) \sin \lambda x \, dx$

∴ equation ③ becomes

$$u(x,t) = B \sin \lambda x \, e^{-\alpha^2 \lambda^2 t}$$

$$\sum_{n=1}^{\infty} B_n \sin \lambda x \, e^{-\alpha^2 \lambda^2 t}$$

$$\therefore B_n = \frac{2C}{L^3} \int_0^L x(L-x) \sin \lambda x \, dx$$

$$= \frac{2C}{L^3} \int_0^L x(L-x) \sin \frac{n\pi x}{L} \, dx$$

$$= \frac{2C}{L^3} \left[ x(L-x) \left[ \frac{-\cos \frac{n\pi x}{L}}{\frac{n\pi}{L}} \right] - (L-2x) \left[ \frac{-\sin \frac{n\pi x}{L}}{\frac{n^2\pi^2}{L^2}} \right] \right.$$

Scanned with CamScanner

$$\left. - \frac{2\cos\frac{n\pi x}{L}}{\frac{n^3\pi^3}{L^3}} \right]_0^L$$

$$= \frac{2C}{L^3}\left[0 - 0 - 2\frac{\cos n\pi}{n^3\pi^3} \times L^3 - \left(0 - 0 - \frac{2}{n^3\pi^3} L^3\right]\right.$$

$$= \frac{4C}{n^3\pi^3}\left[1 + (-1)^{n+1}\right]$$

$$\therefore u(x,t) = \frac{4C}{\pi^3} \sum_{n=1}^{\infty} \frac{1}{n^3}\left(1 + (-1)^{n+1}\right) \sin\frac{n\pi x}{L} \, e^{-\frac{\alpha^2 n^2\pi^2}{L^2}t}$$

$$u(x,t) = \frac{4c}{\pi^3} \sum_{n=1}^{\infty} 1/n^3 \left(1-(-1)^n\right) \sin \frac{n\pi x}{L} e^{-\frac{\alpha^2 n^2\pi^2}{L^2}t}$$

13. Solve the heat eqn with initial condition $u(x,0) = 3 \sin \pi x$ where $0 < x < 1$

23/11/2020

The heat eqn is given by;

$$\partial u/\partial t = \alpha^2 \frac{\partial^2 u}{\partial x^2} \quad \text{——— ①}$$

The soln of the heat eqn is;

$$u = (A \cos \lambda x + B \sin \lambda x) e^{-\alpha^2 \lambda^2 t} \quad \text{——— ②}$$

The Boundary conditions are;

$$u(0,t) = 0 \quad \text{——— ⓐ}$$

$$u(L,t) = 0 \quad \text{——— ⓑ}$$

$$[u(1,t) = 0]$$

$$u(x,0) = 3 \sin \pi x = f(x) \quad \text{——— ⓒ}$$

Applying ⓐ in ②.

$$0 = A e^{-\alpha^2 \lambda^2 t}$$

$$\Rightarrow \boxed{A = 0} \qquad e^{-\alpha^2 \lambda^2 t} \neq 0$$

$$② \Rightarrow u = B \sin \lambda x \, e^{-\alpha^2 \lambda^2 t} \quad \text{——— ③}$$

Applying ⓑ in ③

$$\sin \lambda = 0$$

$$\boxed{\lambda = n\pi}$$

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin n\pi x \, e^{\alpha^2 n^2 \pi^2 t} \quad \text{——— ④}$$

$$B_n = \frac{2}{L}\int_0^{L} f(x)\sin\lambda x\, dx.$$

By comparison test;

Ux: put $t = 0$ in (4).

$$u(x,0) = \sum_{n=1}^{\infty} B_n \sin\lambda x.$$

$$= \sum_{n=1}^{\infty} B_n \sin n\pi x \qquad [\lambda = n\pi]$$

$$u(x,0) = 3\sin\pi x = f(x) \text{ ——— (c).}$$

$$3\sin\pi x = B_1 \sin\pi x + B_2\sin 2\pi x + B_3 \sin 3\pi x + \dots$$

$$B_1 = 3;\ B_2 = B_3 = \dots = 0$$

$$\therefore u(x,t) = B_1 \sin\pi x\, e^{-\alpha^2\pi^2 t}$$

$$\underline{\underline{u(x,t) = 3\sin\pi x\, e^{-\alpha^2\pi^2 t}}}$$

14. Solve $\partial u/\partial t = \partial^2 u/\partial x^2$ subject to the condition

$\alpha^2 = 1$

Take $\alpha = +1$

$$u(x,0) = \begin{cases} x, & 0 < x < L/2 \\ L-x, & L/2 < x < L. \end{cases}$$

The heat eqn is given by;

$$\partial u/\partial t = \partial^2 u/\partial x^2 \text{ ——— (1)}$$

Soln of the heat eqn is;

$$u(x,t) = (A\cos\lambda x + B\sin\lambda x)\, e^{-\lambda^2 t} \text{ ——— (2)}$$

The Boundary conditions are;

$$u(0,t) = 0 \text{ ——— (a)}$$

$$u(L,t) = 0 \text{ ——— (b)}$$

$$u(x,0) = \begin{cases} x, & 0 < x < L/2 \\ L-x, & L/2 < x < L \end{cases} \text{ ——— (c)}$$

Applying (a) in 2

$$\Rightarrow \boxed{A = 0}$$

$$(2) \Rightarrow \quad u = B\sin\lambda x \, e^{-\lambda^2 t} \quad \text{———} \quad (3)$$

Applying (b) in (3)

$$\sin\lambda L = 0$$

$$\lambda L = n\pi$$

$$\boxed{\lambda = n\pi/L}$$

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L} \, e^{-\frac{n^2\pi^2 t}{L^2}} \quad \text{———} \quad (4)$$

$$B_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx.$$

$$= \frac{2}{L}\left[\int_0^{L/2} x\sin\frac{n\pi x}{L}\,dx + \int_{L/2}^{L}(L-x)\sin\frac{n\pi x}{L}\,dx\right]$$

$$= \frac{2}{L}\left\{\left[x\left(\frac{-\cos\frac{n\pi x}{L}}{n\pi/L}\right) + 1\left(\frac{\sin\frac{n\pi x}{L}}{n^2\pi^2/L^2}\right)\right]_0^{L/2} + \left[(L-x)\times\frac{-\cos\frac{n\pi x}{L}}{n\pi/L} - \frac{\sin\frac{n\pi x}{L}}{n^2\pi^2/L^2}\right]_{L/2}^{L}\right\}$$

$$= \frac{2}{L}\left\{\left[-\frac{L}{2}\cos\frac{n\pi}{2}\times\frac{L}{n\pi} + \frac{L^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right] - \left[\frac{L}{2}\left(-\cos\frac{n\pi}{2}\right)\times\frac{L}{n\pi} - \sin\frac{n\pi}{2}\times\frac{L^2}{n^2\pi^2}\right]\right\}$$

Scanned by CamScanner

$$\frac{2}{L} \times \frac{2L^2}{n^2\pi^2} \sin\frac{n\pi}{2}$$

$$B_n = \frac{4L}{n^2\pi^2} \sin(n\pi/2)$$

$$u(x,t) = \sum_{n=1}^{\infty} \frac{4L}{n^2\pi^2} \sin(n\pi/2) \sin\left(\frac{n\pi x}{L}\right) e^{-\frac{n^2\pi^2 t}{L^2}}$$

$$u(x,t) = \frac{4L}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \sin\left(\frac{n\pi}{2}\right) \sin\left(\frac{n\pi x}{L}\right) e^{-\frac{n^2\pi^2 t}{L^2}}$$

---

15. A homogeneous rod of conducting materials of length 100cm has its ends kept at zero temp. & the initial temp is

$$u(x,0) = \begin{cases} x, & 0 \le x < 50 \\ 100-x, & 50 \le x < 100 \end{cases}$$

Find the temp. distribution $u(x,t)$ at any time?

The heat eqn is given by;

$$\partial u/\partial t = \alpha^2 \, \partial^2 u/\partial x^2 \quad \text{—— (1)}$$

soln; $u(x,t) = (A\cos\lambda x + B\sin\lambda x)e^{-\alpha^2\lambda^2 t}$ —— (2)

Boundary condition;

$u(0,t) = 0$ —— (a)

$u(100,t) = 0$ —— (b)

$$u(x,0) = \begin{cases} x & 0 \le x < 50 \\ 100-x & 50 \le x < 100 \end{cases} \quad \text{—— (c).}$$

Applying (a) in (2).

$\boxed{A = 0}$

(2) $\Rightarrow$ $u(x,t) = B\sin\lambda x \, e^{-\alpha^2\lambda^2 t}$ —— (3)

Applying ⓑ in ③

$\sin \lambda 100 = 0$

$\lambda = \frac{n\pi}{100}$

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{100} e^{\frac{-\alpha^2 n^2 \pi^2 t}{(100)^2}} \quad \text{④}$$

$$B_n = \frac{2}{L}\int_0^L f(x) \sin \frac{n\pi x}{L} dx$$

$$= \frac{2}{100}\left[\int_0^{50} x \sin \frac{n\pi x}{100} dx + \int_{50}^{100} (100-x) \sin \frac{n\pi x}{100} dx\right]$$

$$= \frac{2}{100}\left\{\left[x \times \frac{-\cos \frac{n\pi x}{100}}{n\pi/100} + \frac{\sin \frac{n\pi x}{100}}{n^2\pi^2/(100)^2}\right]_0^{50} + \right.$$

$$\left.\left[(100-x) \times \frac{-\cos \frac{n\pi x}{100}}{n\pi/100} - \frac{\sin \frac{n\pi x}{100}}{n^2\pi^2/100^2}\right]_{50}^{100}\right\}$$

$$= \frac{2}{100}\left\{\left[50 \times -\cos \frac{n\pi}{2} \times \frac{100}{n\pi} + \sin \frac{n\pi}{2} \times \frac{100^2}{n^2\pi^2}\right]\right.$$

$$\left. - \left[50 \times -\cos \frac{n\pi}{2} \times \frac{100}{n\pi} - \sin \frac{n\pi}{2} \times \frac{100^2}{n^2\pi^2}\right]\right\}$$

$$= \frac{2}{\cancel{100}} \times \frac{2 \times 100^{\cancel{2}}}{n^2\pi^2} \times \sin(n\pi/2)$$

$$= \frac{400}{n^2\pi^2} \sin(n\pi/2)$$

$$B_n = \frac{400}{n^2\pi^2} \sin(n\pi/2)$$

$$u(x,t) = \frac{400}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \sin(n\pi/2) \sin \frac{n\pi x}{100} e^{\frac{-\alpha^2 n^2 \pi^2 t}{100^2}}$$

24/11/2020

16. A rod of length $l$ with insulated sides is initially at a uniform temperature. $u_0$ It is suddenly cooled to $0°c$ at kept at that temp. Find the temperature fn $u(x,t)$.

The heat eqn is given by;

$$\partial u/\partial t = \alpha^2 \partial^2 u/\partial x^2 \quad \text{——— ①}$$

The soln ;

$$u = (A\cos\lambda x + B\sin\lambda x) e^{-\alpha^2\lambda^2 t} \quad \text{——— ②}$$

The boundary conditions are;

$$u(0,t) = 0 \quad \text{——— ⓐ}$$

$$u(L,t) = 0 \quad \text{——— ⓑ}$$

$$u(x,0) = u_0 \quad \text{——— ⓒ}$$

Applying ⓐ in ②

$$\boxed{A = 0}$$

$$② \Rightarrow \quad u = B\sin\lambda x \, e^{-\alpha^2\lambda^2 t} \quad \text{——— ③}$$

Applying ⓑ in ③

$$\sin\lambda L = 0$$

$$\lambda L = n\pi$$

$$\boxed{\lambda = n\pi/L}$$

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L} e^{-\alpha^2 \frac{n^2\pi^2}{L^2} t} \quad \text{——— ④}$$

$$B_n = \frac{2}{L}\int_0^L u_0 \sin\frac{n\pi x}{L}\,dx$$

$$= \frac{2u_0}{L}\left[-\cos\frac{n\pi x}{L} \times \frac{L}{n\pi}\right]_0^L$$

$$= \frac{2u_0}{n\pi}\left(1-(-1)^n\right)$$

$$u(x,t) = 2u_0 \sum_{n=1}^{\infty} \frac{1-(-1)^n}{n\pi} \sin\frac{n\pi x}{L}\, e^{-\alpha^2 \frac{n^2\pi^2 t}{L^2}}$$

17. Find the temp. distribution of a rod of length 2m whose end points are maintained at 0° temp and the initial temp is $100(2x-x^2)$

The heat eqn is given by;

$$\partial u/\partial t = \alpha^2\, \partial^2 u/\partial x^2 \quad \text{———} \quad (1).$$

Soln is given by;

$$u = (A\cos\lambda x + B\sin\lambda x)\, e^{-\alpha^2\lambda^2 t} \quad \text{———} \quad (2)$$

The boundary conditions are;

$$u(0,t) = 0 \quad \text{———} \quad (a)$$

$$u(2,t) = 0 \quad \text{———} \quad (b)$$

$$u(x,0) = 100(2x - x^2) \quad \text{———} \quad (c)$$

Applying (a) in (2)

$A = 0$

$$(2) \Rightarrow u(x,t) = B\sin\lambda x\, e^{-\alpha^2\lambda^2 t} \quad \text{———} \quad (3)$$

Applying ⑤ in ③

$\Rightarrow \sin 2\lambda = 0$

$2\lambda = n\pi$

$$\boxed{\lambda = \frac{n\pi}{2}}$$

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{2} e^{-\frac{\alpha^2 n^2 \pi^2 t}{4}}$$

$$B_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{2} \, dx.$$

$$= \int_0^2 (200x - 100x^2) \sin \frac{n\pi x}{2} \, dx.$$

$$= \left[ (200x - 100x^2) \times \frac{-\cos \frac{n\pi x}{2}}{n\pi/2} - (200 - 200x) \frac{-\sin \frac{n\pi x}{2}}{n^2\pi^2/4} + \frac{-200 \cos n\pi/2 \, x}{n^3\pi^3/8} \right]_0^2$$

$$= \frac{-1600}{n^3\pi^3}(-1)^n + \frac{1600}{n^3\pi^3}.$$

$$= \frac{1600}{n^3\pi^3}\left(1 - (-1)^n\right)$$