Module 2

Time domain specifications

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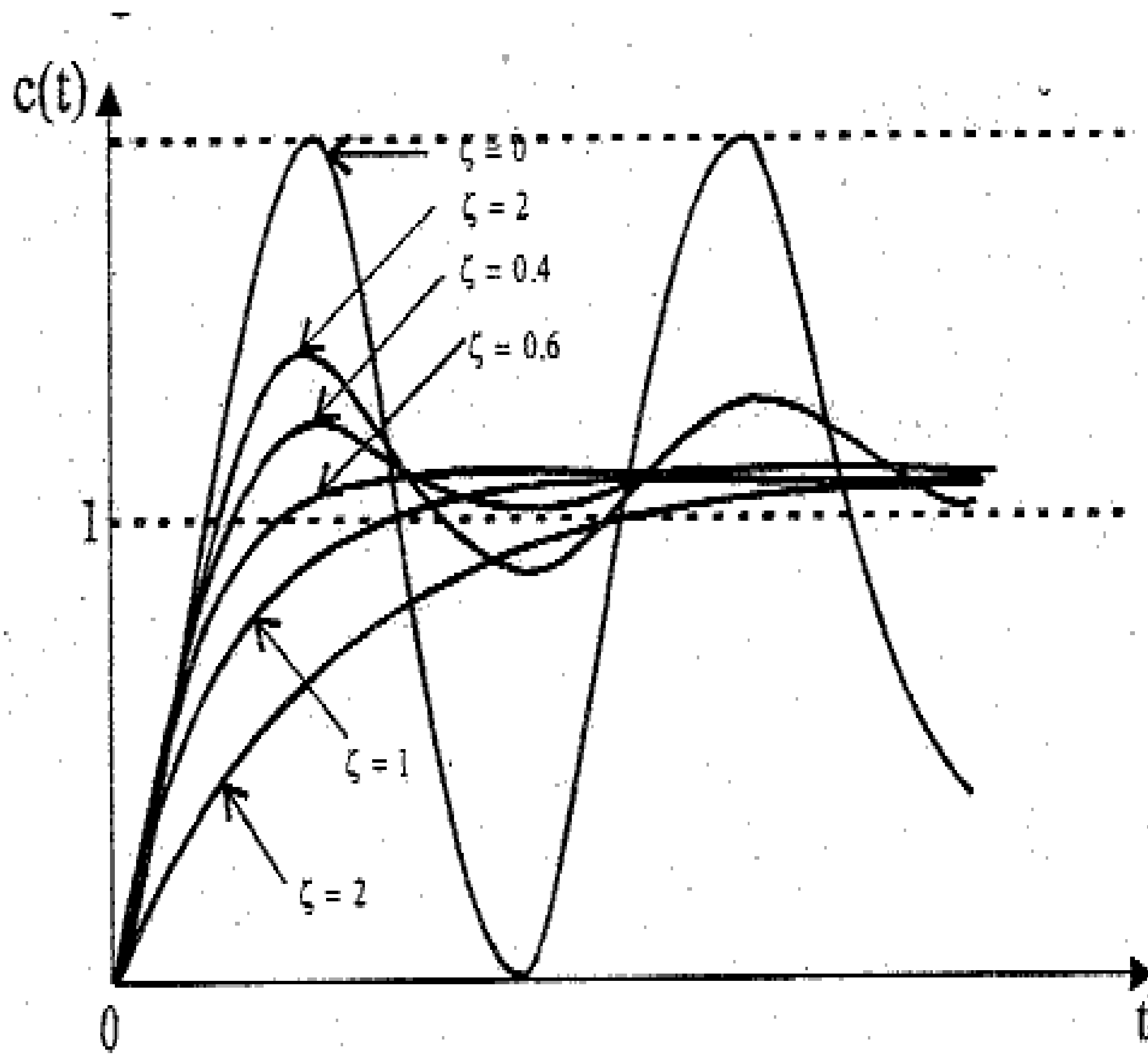
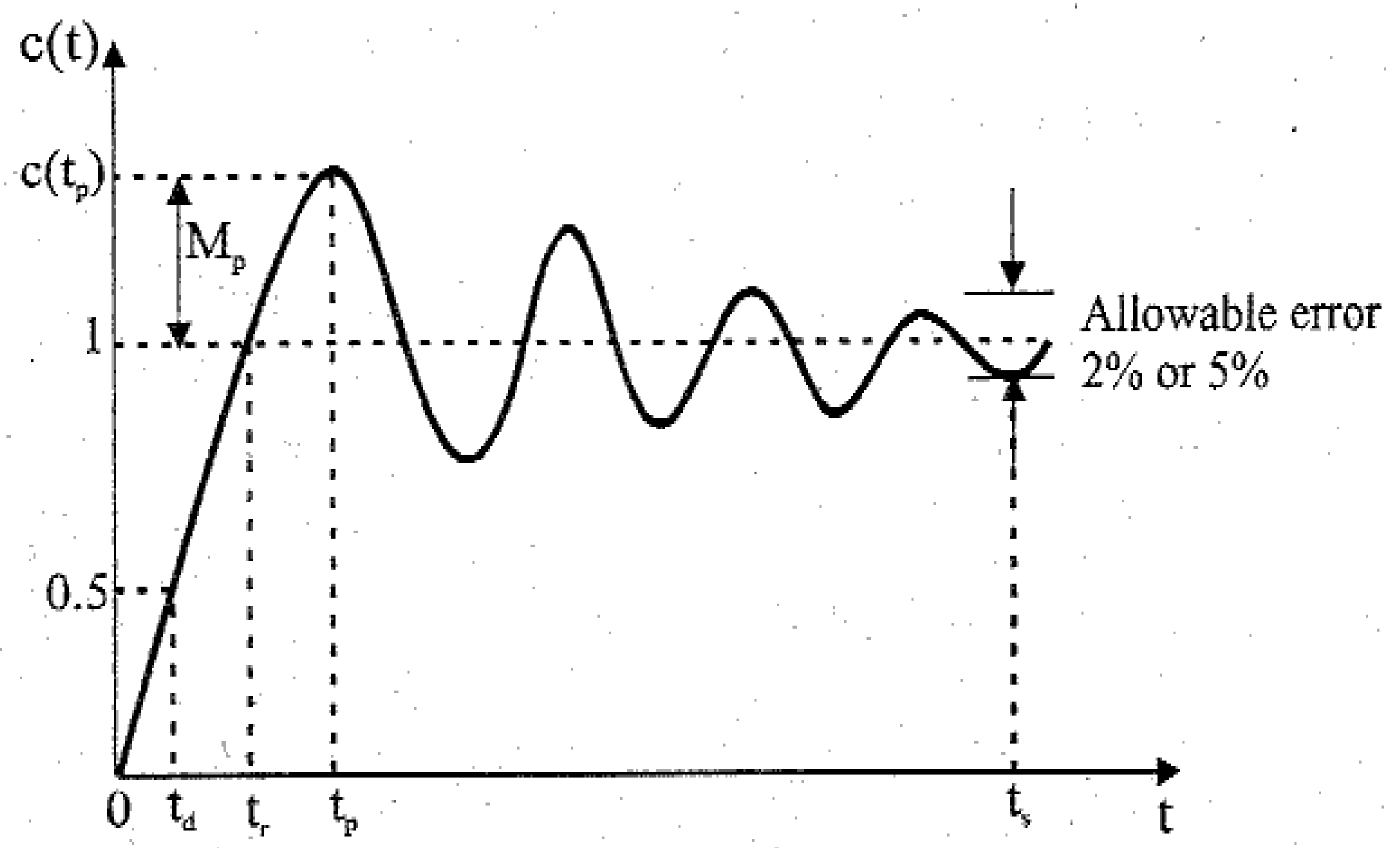


Fig 2.13.b : Response.

Time domain specifications

1. Delay time, t_d
2. Rise time, t_r
3. Peak time, t_p
4. Maximum overshoot, M_p
5. Settling time, t_s


c(t)
c(tp)
Mp
1
0.5
0
td
tr
tp
ts
t
Allowable error
2% or 5%

1. DELAY TIME (t_d) : It is the time taken for response to reach 50% of the final value, for the very first time.

2. RISE TIME (t_r) : It is the time taken for response to raise from 0 to 100% for the very first time. For underdamped system, the rise time is calculated from 0 to 100%. But for overdamped system it is the time taken by the response to raise from 10% to 90%. For critically damped system, it is the time taken for response to raise from 5% to 95%.

3. PEAK TIME (t_p) : It is the time taken for the response to reach the peak value the very first time. (or) It is the time taken for the response to reach the peak overshoot, M_p.

4. PEAK OVERSHOOT (M_p) : It is defined as the ratio of the maximum peak value to the final value, where the maximum peak value is measured from final value.

Let, $c(\infty)$ = Final value of c(t).

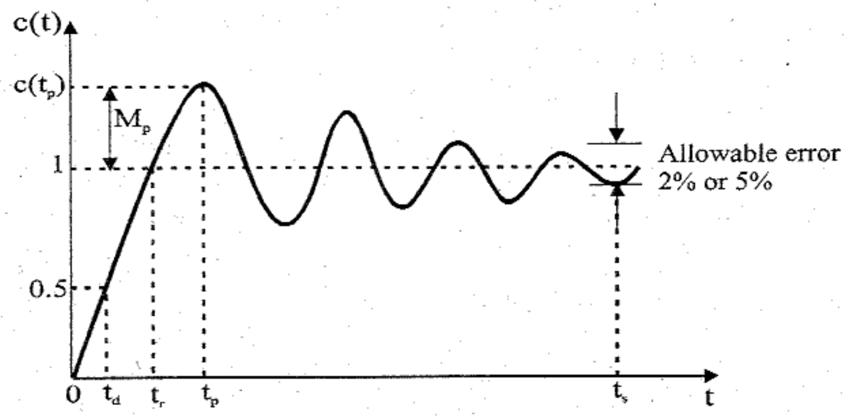
$c(t_p)$ = Maximum value of c(t).

$$\text{Now, Peak overshoot, } M_p = \frac{c(t_p) - c(\infty)}{c(\infty)} \qquad .....(2.37)$$

$$\text{\% Peak overshoot, } \%M_p = \frac{c(t_p) - c(\infty)}{c(\infty)} \times 100 \qquad .....(2.38)$$

5. SETTLING TIME (t_s) : It is defined as the time taken by the response to reach and stay within a specified error. It is usually expressed as % of final value. The usual tolerable error is 2 % or 5% of the final value.

Rise time (tr)



The unit step response of second order system for underdamped case is given by,

$$c(t) = 1 - \frac{e^{-\zeta\omega_n t_r}}{\sqrt{1-\zeta^2}} \sin(\omega_d t + \theta)$$

At $t = t_r$, $c(t) = c(t_r) = 1$ **(Refer fig 2.14).**

$$\therefore c(t_r) = 1 - \frac{e^{-\zeta\omega_n t_r}}{\sqrt{1-\zeta^2}} \sin(\omega_d t_r + \theta) = 1$$

$$\therefore \frac{-e^{-\zeta\omega_n t_r}}{\sqrt{1-\zeta^2}} \sin(\omega_d t_r + \theta) = 0$$

Since $-e^{-\zeta\omega_n t_r} \neq 0$, the term, $\sin(\omega_d t_r + \theta) = 0$

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When, $\phi = 0, \pi, 2\pi, 3\pi \ldots$, $\quad \sin\phi = 0$

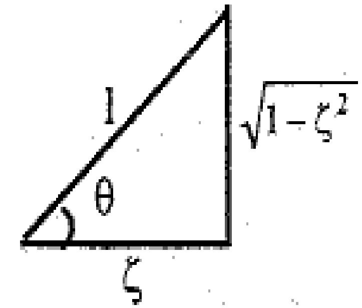
$$\therefore \omega_d t_r + \theta = \pi$$

$$\omega_d t_r = \pi - \theta$$

$$\boxed{\therefore \text{ Rise Time, } t_r = \frac{\pi - \theta}{\omega_d}}$$

we get

$$\tan\theta = \frac{\sqrt{1-\zeta^2}}{\zeta}$$



$$\boxed{\therefore\ \text{Rise Time, } t_r = \frac{\pi-\theta}{\omega_d}}$$

Here, $\theta = \tan^{-1}\dfrac{\sqrt{1-\zeta^2}}{\zeta}$; Damped frequency of oscillation, $\omega_d = \omega_n\sqrt{1-\zeta^2}$

$$\boxed{\therefore \text{Rise time, } t_r = \frac{\pi - \tan^{-1}\dfrac{\sqrt{1-\zeta^2}}{\zeta}}{\omega_n\sqrt{1-\zeta^2}} \text{ in sec}}$$

Peak time (tp)

PEAK TIME (t_p) : It is the time taken for the response to reach the peak value the very first time. (or) It is the time taken for the response to reach the peak overshoot, M_p.

To find the expression for peak time, t_p, differentiate c(t) with respect to t and equate to 0.

$$\text{i.e., } \frac{d}{dt}c(t)\Big|_{t=t_p} = 0$$

The unit step response of under damped second order system is given by,

$$c(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin(\omega_d t + \theta)$$

$$\frac{d}{dt}c(t) = \frac{-e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}(-\zeta\omega_n)\sin(\omega_d t+\theta) + \left(\frac{-e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\right)\cos(\omega_d t+\theta)\omega_d$$

Put, $\omega_d = \omega_n\sqrt{1-\zeta^2}$

$$\therefore \frac{d}{dt}c(t) = \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}(\zeta\omega_n)\sin(\omega_d t+\theta) - \frac{\omega_n\sqrt{1-\zeta^2}}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\cos(\omega_d t+\theta)$$

$$= \frac{\omega_n e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\left[\zeta\sin(\omega_d t+\theta) - \sqrt{1-\zeta^2}\cos(\omega_d t+\theta)\right]$$

$$= \frac{\omega_n}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\left[\cos\theta\sin(\omega_d t+\theta) - \sin\theta\cos(\omega_d t+\theta)\right] \quad \textit{(refer note)}$$

$$= \frac{\omega_n}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\left[\sin(\omega_d t+\theta)\cos\theta - \cos(\omega_d t+\theta)\sin\theta\right]$$

$$= \frac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t}[\sin((\omega_d t + \theta) - \theta)] = \frac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin(\omega_d t)$$

at $t = t_p$, $\frac{d}{dt}c(t) = 0$

$$\therefore \frac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t_p} \sin(\omega_d t_p) = 0$$

Since, $e^{-\zeta\omega_n t_p} \neq 0$, the term, $\sin(\omega_d t_p) = 0$

When $\phi = 0, \pi, 2\pi, 3\pi, \sin\phi = 0$

$\therefore \ \omega_d t_p = \pi$

$$\therefore \quad \omega_d t_p = \pi$$

$$\boxed{\therefore \text{Peak time, } t_p = \frac{\pi}{\omega_d}}$$

The damped frequency of oscillation, $\omega_d = \omega_n\sqrt{1-\zeta^2}$

$$\boxed{\therefore \text{Peak time, } t_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}}$$

Peak overshoot (Mp)

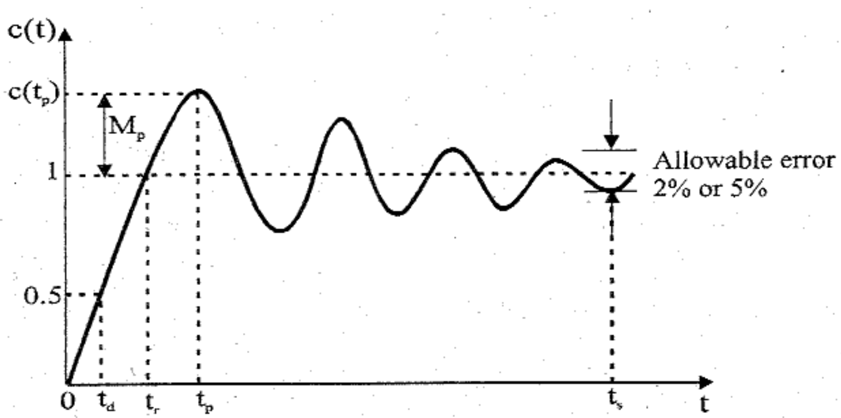
$$\boxed{\%\text{Peak overshoot, } \%M_p = \frac{c(t_p) - c(\infty)}{c(\infty)} \times 100}$$

where, $c(t_p)$ = Peak response at $t = t_p$.

$c(\infty)$ = Final steady state value.

The unit step response of second order system is given by,

$$c(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin(\omega_d t + \theta)$$



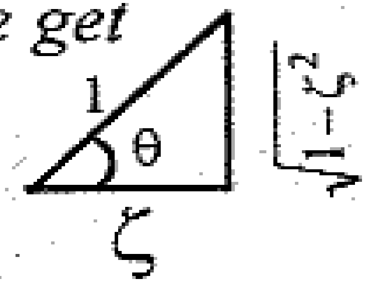
$$\text{At } t=\infty, \qquad c(t) = c(\infty) = 1-\frac{e^{-\infty}}{\sqrt{1-\zeta^2}}\sin(\omega_d t+\theta) = 1-0 =$$

$$\text{At } t=t_p, \quad c(t)=c(t_p)=1-\frac{e^{-\zeta\omega_n t_p}}{\sqrt{1-\zeta^2}}\sin(\omega_d t_p+\theta)$$

$$=1-\frac{e^{-\zeta\omega_n \frac{\pi}{\omega_d}}}{\sqrt{1-\zeta^2}}\sin\left(\omega_d\frac{\pi}{\omega_d}+\theta\right)$$

$$=1-\frac{e^{-\zeta\omega_n\frac{\pi}{\omega_n\sqrt{1-\zeta^2}}}\sin(\pi+\theta)}{\sqrt{1-\zeta^2}}$$

we get



$$=1+\frac{e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}}}{\sqrt{1-\zeta^2}}\sin\theta$$

$$=1+\frac{e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}}}{\sqrt{1-\zeta^2}}\sqrt{1-\zeta^2} = 1+e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}}$$

$$\text{Percentage Peak Overshoot, } \%M_p = \frac{c(t_p)-c(\infty)}{c(\infty)} \times 100 = \frac{1+e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}}-1}{1} \times 100$$

$$= e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}} \times 100$$

$$\boxed{\therefore \text{ Percentage Peak Overshoot, } \%M_p = e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}} \times 100}$$

Settling time (ts)

The response of second order system has two components. They are,

1. Decaying exponential component, $\dfrac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}$.
2. Sinusoidal component, $\sin(\omega_d t + \theta)$.

$$c(t) = 1 - \frac{e^{-\zeta\omega_n t_r}}{\sqrt{1-\zeta^2}}\sin(\omega_d t + \theta)$$

In this the decaying exponential term dampens (or) reduces the oscillations produced by sinusoidal component. Hence the settling time is decided by the exponential component. The settling time can be found out by equating exponential component to percentage tolerance errors.

For 2 % tolerance error band, at $t = t_s$, $\dfrac{e^{-\zeta\omega_n t_s}}{\sqrt{1-\zeta^2}} = 0.02$

For 2 % tolerance error band, at $t = t_s$, $\frac{e^{-\zeta\omega_n t_s}}{\sqrt{1-\zeta^2}} = 0.02$

For least values of ζ, $e^{-\zeta\omega_n t_s} = 0.02$.

On taking natural logarithm we get,

$$-\zeta\omega_n t_s = ln(0.02) \quad \Rightarrow \quad -\zeta\omega_n t_s = -4 \quad \Rightarrow \quad t_s = \frac{4}{\zeta\omega_n}$$

For the second order system, the time constant, $T = \frac{1}{\zeta\omega_n}$

$\therefore$ Settling time, $t_s = \frac{1}{\zeta\omega_n} = 4T$ (for 2% error)

For 5% error, $e^{-\zeta\omega_n t_s} = 0.05$

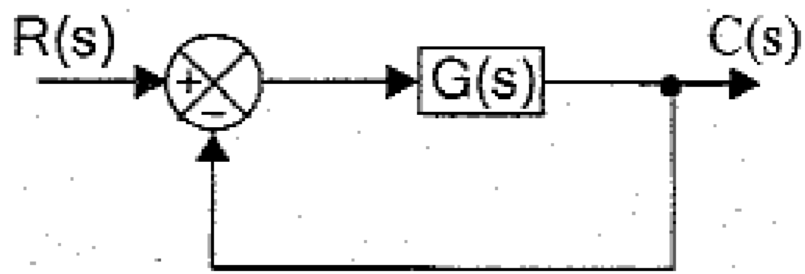
On taking natural logarithm we get,

$$-\zeta\omega_n t_s = ln(0.05) \quad \Rightarrow \quad -\zeta\omega_n t_s = -3 \quad \Rightarrow \quad t_s = \frac{3}{\zeta\omega_n}$$

$$\boxed{\therefore \text{ Settling time, } t_s = \frac{3}{\zeta\omega_n} = 3T \quad \text{(for 5\% error)}}$$

$$\boxed{\therefore \text{ Settling time, } t_s = \frac{ln(\%\text{ error})}{\zeta\omega_n} = \frac{ln(\%\text{ error})}{T}}$$

Obtain the response of unity feedback system whose open loop transfer function is $G(s) = \frac{4}{s(s+5)}$ and when the input is unit step.



$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)}$$

$$C(s) = \frac{4}{s\,(s+1)\,(s+4)}$$

$$= \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+4}$$

$$c(t) = \mathcal{L}^{-1}\{C(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{4}{3}\,\frac{1}{s+1} + \frac{1}{3}\,\frac{1}{s+4}\right\}$$

$$\underline{c(t) = 1 - \frac{1}{3}\left[4e^{-t} - e^{-4t}\right]}$$

The response of a servomechanism is, $c(t) = 1 + 0.2\,e^{-60t} - 1.2\,e^{-10t}$ when subject to a unit step input. Obtain an expression for closed loop transfer function. Determine the undamped natural frequency and damping ratio.

Given that, $c(t) = 1 + 0.2\,e^{-60t} - 1.2\,e^{-10t}$

On taking Laplace transform of c(t) we get,

$$C(s) = \frac{1}{s} + 0.2\,\frac{1}{(s+60)} - 1.2\,\frac{1}{(s+10)}$$

$$\therefore \; C(s) = R(s) \; \frac{60}{(s+60)\,(s+10)}$$

$$\therefore \; \frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{600}{s^2 + 70s + 600}$$

On comparing we get,

$$\omega_n^2 = 600$$

$$\therefore \; \omega_n = \sqrt{600} = 24.49 \text{ rad / sec}$$

$$2\zeta\omega_n = 70$$

$$\therefore \; \zeta = \frac{70}{2\omega_n} = \frac{70}{2 \times 24.49} = 1.43$$

The unity feedback system is characterized by an open loop transfer function $G(s) = K/s(s+10)$. Determine the gain K, so that the system will have a damping ratio of 0.5 for this value of K. Determine peak overshoot and time at peak overshoot for a unit step input.

The closed loop transfer function $\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)}$

, $G(s) = K/s\,(s+10)$

$$\frac{C(s)}{R(s)} = \frac{\dfrac{K}{s(s+10)}}{1+\dfrac{K}{s(s+10)}} = \frac{K}{s\,(s+10)+K} = \frac{K}{s^2+10s+K}$$

On comparing we get,

$\omega_n^2 = K$	$2\zeta\omega_n = 10$	$K = 100$
$\therefore \omega_n = \sqrt{K}$	Put $\zeta = 0.5$ and $\omega_n = \sqrt{K}$	$\omega_n = 10$ rad / sec
	$\therefore 2 \times 0.5 \times \sqrt{K} = 10$	
	$\sqrt{K} = 10$	

The value of gain, K =100.

The value of gain, K =100.

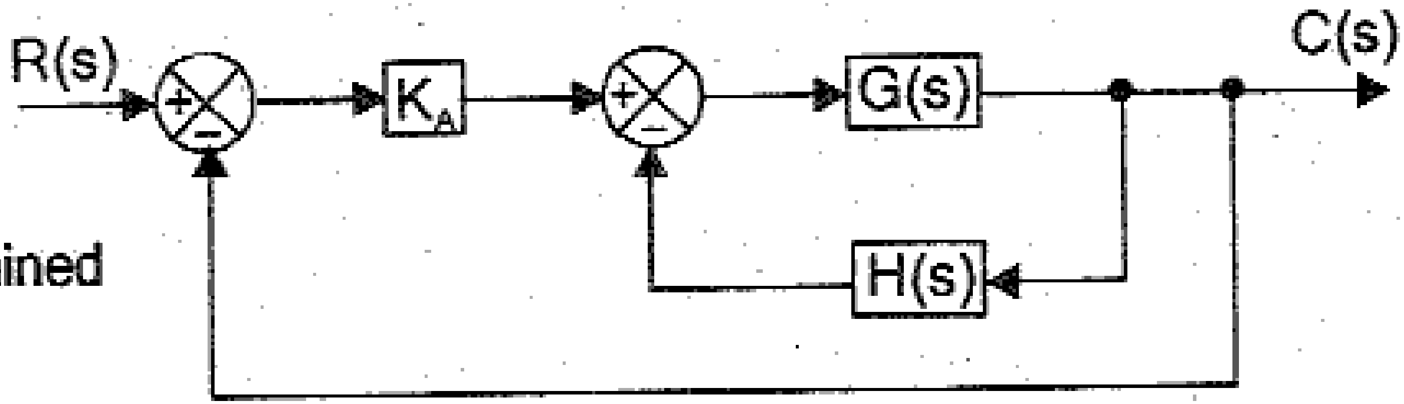
Percentage peak overshoot, $\%M_p = e^{-\zeta\pi/\sqrt{1-\zeta^2}} \times 100$

$$= e^{-0.5\pi/\sqrt{1-0.5^2}} \times 100 = 0.163 \times 100 = 16.3\%$$

Peak time, $t_p = \dfrac{\pi}{\omega_d} = \dfrac{\pi}{\omega_n\sqrt{1-\zeta^2}} = \dfrac{\pi}{10\sqrt{1-0.5^2}} = 0.363 \text{ sec}$

A unity feedback control system has an amplifier with gain $K_A = 10$ and gain ratio, $G(s) = 1/s(s+2)$ in the feed forward path. A derivative feedback, $H(s) = sK_0$ is introduced as a minor loop around G(s). Determine the derivative feedback constant, K_0 so that the system damping factor is 0.6.

ained



$$\frac{C(s)}{R(s)} = \frac{10}{s^2 + (2 + K_o)\,s + 10}$$

$$\left.\begin{array}{l}\text{Standard form of} \\ \text{Second order transfer function}\end{array}\right\} \frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

On comparing equation (1) & (2) we get,

$$\omega_n^2 = 10$$

$$\therefore\ \omega_n = \sqrt{10} = 3.162 \text{ rad / sec}$$

$$2 + K_o = 2\zeta\omega_n$$

$$\therefore\ K_o = 2\zeta\omega_n - 2$$

$$= 2 \times 0.6 \times 3.162 - 2 = 1.7944$$

A positional control system with velocity feedback is shown in fig 1. What is the response c(t) to the unit step input. Given that $\zeta = 0.5$. Also calculate rise time, peak time, maximum overshoot and settling time.

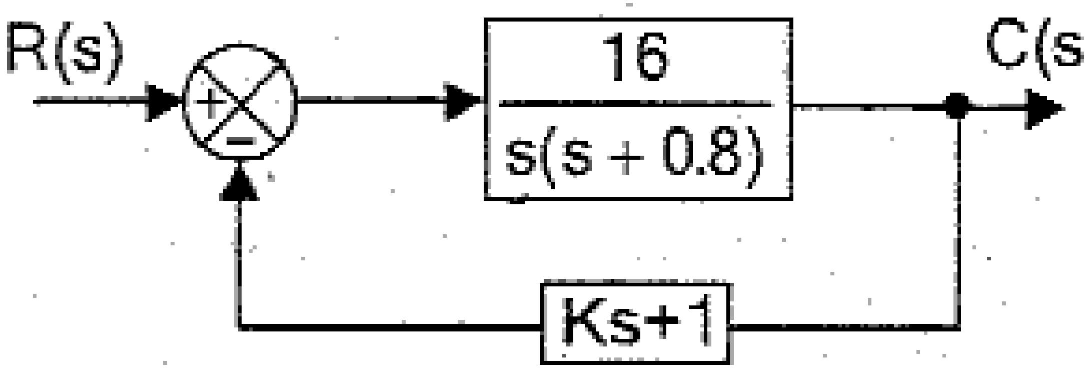


Fig 1

$$\therefore \frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{16}{s^2 + (0.8 + 16K)\, s + 16}$$

On comparing we get.

$\omega_n^2 = 16$	$0.8 + 16K = 2\zeta\omega_n$
$\therefore \omega_n = 4 \text{ rad / sec}$	$\therefore K = \frac{2\zeta\omega_n - 0.8}{16} = \frac{2 \times 0.5 \times 4 - 0.8}{16} = 0.2$

$$\therefore \frac{C(s)}{R(s)} = \frac{16}{s^2 + (0.8 + 16 \times 0.2)\, s + 16} = \frac{16}{s^2 + 4s + 16}$$

The response in s - domain, $C(s) = R(s)\dfrac{16}{s^2 + 4s + 16}$

For unit step input, $R(s) = 1/s$.

$$\therefore\ C(s) = \frac{1}{s}\,\frac{16}{s^2 + 4s + 16} = \frac{16}{s\,(s^2 + 4s + 16)}$$

By partial fraction expansion we can write,

$$C(s) = \frac{16}{s\,(s^2 + 4s + 16)} = \frac{A}{s} + \frac{Bs + C}{s^2 + 4s + 16}$$

$$c(t) = \mathcal{L}^{-1}\{C(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{s+2}{(s+2)^2+12} - \frac{2}{\sqrt{12}}\,\frac{\sqrt{12}}{(s+2)^2+12}\right\}$$

$$= 1 - e^{-2t}\cos\sqrt{12}\,t - \frac{2}{2\sqrt{3}}\,e^{-2t}\sin\sqrt{12}\,t$$

$$= 1 - e^{-2t}\left[\frac{1}{\sqrt{3}}\sin(\sqrt{12}\,t) + \cos(\sqrt{12}\,t)\right]$$

$$\left.\begin{array}{l}\text{Damped frequency}\\ \text{of oscillation}\end{array}\right\}\omega_d = \omega_n\sqrt{1-\zeta^2} = 4\sqrt{1-0.5^2} = 3.464 \text{ rad / sec}$$

$$\therefore \text{ Rise time, } t_r = \frac{\pi-\theta}{\omega_d} = \frac{\pi-1.047}{3.464} = 0.6046 \text{ sec}$$

$$\text{Peak time, } t_p = \frac{\pi}{\omega_d} = \frac{\pi}{3.464} = 0.907 \text{ sec}$$

$$\left.\begin{array}{l}\text{\% Maximum}\\ \text{overshoot}\end{array}\right\}\%M_p = e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}} \times 100 = e^{\frac{-0.5\times\pi}{\sqrt{1-0.5^2}}} \times 100 = 0.163 \times 100 = 16.3\%$$

$$\text{Time constant, } T = \frac{1}{\zeta\omega_n} = \frac{1}{0.5\times 4} = 0.5 \text{ sec}$$

For 5% error, Settling time, $t_s = 3T = 3 \times 0.5 = 1.5$ sec

For 2% error, Settling time, $t_s = 4T = 4 \times 0.5 = 2$ sec

A unity feedback control system is characterized by the following open loop transfer function $G(s) = (0.4s+1)/s(s+0.6)$. Determine its transient response for unit step input and sketch the response. Evaluate the maximum overshoot and the corresponding peak time.