Problems :-

1) Find the eigen values and eigen vectors of the matrix $A = \begin{bmatrix} 8 & -4 \\ 2 & 2 \end{bmatrix}$.

Answer :-

Given $A = \begin{bmatrix} 8 & -4 \\ 2 & 2 \end{bmatrix}$

The characteristic equation of A is given by $|A - \lambda I| = 0$.

$\Rightarrow \lambda^2 - B_1\lambda + B_2 = 0$, where B_1 = Sum of the diagonal elements of A $= 10$

$\Rightarrow \lambda^2 - 10\lambda + 24 = 0 \longrightarrow (1)$

$B_2 = |A| = 24$

By solving (1), We get the Eigen values of A.

Now, $\lambda^2 - 10\lambda + 24 = 0 \Rightarrow (\lambda-4)(\lambda-6) = 0$

$\Rightarrow \lambda = 4, 6.$

Thus, the Eigen values of A are ~~...~~ $\lambda = 4, 6$.

Now, to find the Eigen vectors of A corresponding to each λ, We ~~...~~ Consider the homogeneous system $(A - \lambda I)X = 0$ & solve it for a non-trivial solution.

Now, $(A - \lambda I)X = 0 \Rightarrow \begin{bmatrix} 8-\lambda & -4 \\ 2 & 2-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \longrightarrow (2)$

~~...~~ Case 1 : $\lambda = 4$

Put $\lambda = 4$ in (2), We get

$$\begin{bmatrix} 8-4 & -4 \\ 2 & 2-4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 4 & -4 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now, Consider the coefficient matrix

$$\begin{bmatrix} 4 & -4 \\ 2 & -2 \end{bmatrix} \sim \begin{bmatrix} 4 & -4 \\ 0 & 0 \end{bmatrix} \quad R_2 \to R_2 - \tfrac{1}{2}R_1$$

↳ Echelon form.

Consider the Equivalent System

$$\begin{bmatrix} 4 & -4 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow 4x_1 - 4x_2 = 0 \longrightarrow (E_1)$

Thus, x_1 is leading & x_2 is free.

Let $x_2 = a$.

~~...~~ From (E_1), $4x_1 = 4x_2$

$\Rightarrow x_1 = x_2$

$= a$

$\therefore \boxed{x_1 = a}$

Hence $\boxed{x_1 = a,\ x_2 = a}$

Let $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} a \\ a \end{bmatrix}$

Choose $a = 1$, then $X_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigen vector of A corresponding to the eigen value $\lambda = 4$.

<u>Case 2: $\lambda = 6$</u>

Put $\lambda = 6$ in (2), We get

$$\begin{bmatrix} 2 & -4 \\ 2 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now, consider the coefficient matrix $\begin{bmatrix} 2 & -4 \\ 2 & -4 \end{bmatrix} \sim \begin{bmatrix} 2 & -4 \\ 0 & 0 \end{bmatrix}$ $R_2 \to R_2 - R_1$

↳ Echelon form

Consider the Equivalent System

$$\begin{bmatrix} 2 & -4 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow 2x_1 - 4x_2 = 0 \longrightarrow (E_1)$$

Now, x_1 is leading & x_2 is free.

Let $x_2 = a$

from (E_1), $2x_1 = 4x_2 \Rightarrow x_1 = 2x_2$
$= 2a$.

Then $\boxed{x_1 = 2a}$

Hence $\boxed{x_1 = 2a,\ x_2 = a}$

Let $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2a \\ a \end{bmatrix}$

Choose $a = 1$, then $X_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is an eigen vector of A corresponding to the Eigen value $\lambda = 6$.

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2) Find the Eigen values & Eigen Vectors of the matrix $A = \begin{bmatrix} 7 & -1 \\ 4 & 3 \end{bmatrix}$

<u>Answer:-</u>

Given $A = \begin{bmatrix} 7 & -1 \\ 4 & 3 \end{bmatrix}$

Characteristic Eqn. of A is $|A - \lambda I| = 0$

$\Rightarrow \lambda^2 - B_1\lambda + B_2 = 0$

$\Rightarrow \lambda^2 - 10\lambda + 25 = 0$ $\left[\begin{array}{l} B_1 = 7+3 = 10 \\ B_2 = |A| = 21+4 = 25 \end{array}\right]$ $\rightarrow (1)$

By solving (1), we get the Eigen values of A.

Consider $\lambda^2 - 10\lambda + 25 = 0$

$\Rightarrow (\lambda-5)^2 = 0$

$\Rightarrow (\lambda-5)(\lambda-5) = 0$

$\Rightarrow \lambda = 5, 5$ (i.e. $\lambda = 5$ (a repeated root))

$\therefore$ The eigen values of A ~~are~~ is $\lambda = 5, 5$ ($\lambda = 5$ is a repeated root of the characteristic equation) or $\lambda = 5$ is a repeated eigen value of A.

Now, to find the Eigen vectors of A corresponding to $\lambda = 5$, we consider the homogeneous system $(A - \lambda I)X = 0$ & solve it for a non-trivial solution.

Consider $(A - \lambda I)X = 0$

$$\Rightarrow \begin{bmatrix} 7-\lambda & -1 \\ 4 & 3-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \longrightarrow (2)$$

~~Case I: $\lambda = 5$~~

Here we have to consider only one ~~case~~ case, ie, for $\lambda = 5$, for a $\lambda = 5$ is a repeated ~~root~~ eigen value of A.

Now, put $\lambda = 5$ in (2), we get

$$\begin{bmatrix} 2 & -1 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now, consider the coefficient matrix

$$\begin{bmatrix} 2 & -1 \\ 4 & -2 \end{bmatrix} \sim \begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix} R_2 \to R_2 - 2R_1$$

↳ which is in Echelon form

So consider the Equivalent system

$$\begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow 2x_1 - x_2 = 0 \longrightarrow (E_1)$

Now, x_1 is leading & x_2 is free.

Let $x_2 = a$.

From (E_1), $2x_1 = x_2$

$= a$

$\Rightarrow x_1 = \frac{a}{2}$

$\therefore \boxed{x_1 = \frac{a}{2}}$

Hence $\boxed{x_1 = \frac{a}{2},\ x_2 = a}$

Let $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} a/2 \\ a \end{bmatrix}$

Now, choose $a = 2$, then $X = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is an eigen vector of A corresponding to the eigen value $\lambda = 5$.

Now we can observe that $A = \begin{bmatrix} 7 & -1 \\ 4 & 3 \end{bmatrix}$ has only one Independent Eigen Vector corresponding to the eigen value $\lambda = 5$.

3) Find the Eigen Values & Eigen Vectors of the matrix $A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$.

Answer:-

Given $A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$

The characteristic Equation of A is $|A - \lambda I| = 0$

$\Rightarrow \lambda^3 - C_1\lambda^2 + C_2\lambda - C_3 = 0$

$\Rightarrow \lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$ $\longrightarrow (1)$

By solving (1), we get the Eigen Values of A.

Now, to solve (1), we look for integer valued solutions and let's try with $\pm 1, \pm 2, \pm 3$ & ± 6 [Consider ~~only~~ the factors of the Constant term of (1), i.e., -6]

$\lambda = -1 \Rightarrow (-1)^3 - 6(-1)^2 + 11(-1) - 6$
$= -1 - 6 - 11 - 6$
$\neq 0$

$\lambda = 1 \Rightarrow 1 - 6 + 11 - 6 = 0$

Thus $\lambda = 1$ is a root of (1).

Now, to obtain the remaining two roots of (1), let's either divide (1) by $\lambda - 1$ or ~~by Synthetic division~~

$C_1 = 1 + 2 + 3$
$= \underline{6}$

$C_2 = \begin{vmatrix} 2 & 1 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 1 & -1 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 1 & 2 \end{vmatrix}$

$= 4 + 5 + 2$
$= \underline{\underline{11}}$

$C_3 = \begin{vmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{vmatrix}$

$= 1(4) - 0(1) + -1(-2)$
$= 4 + 2$
$= \underline{\underline{6}}$

by applying the synthetic division as follows:

$$\begin{array}{c|cccc} 1 & 1 & -6 & 11 & -6 \\ & 0 & 1 & -5 & 6 \\ \hline & 1 & -5 & 6 & \boxed{0} \end{array}$$

→ Coefficients of $\lambda^3, \lambda^2, \lambda$ & constant term of (1)

The root we obtained from trial

$\Rightarrow \lambda^2 - 5\lambda + 6 = 0$

[Note:- 1, −5, 6 are taken to be the respective ~~coefficients~~ terms of the 2nd degree equation to be solved for the remaining two roots]

$\Rightarrow (\lambda-2)(\lambda-3) = 0$

$\Rightarrow \lambda = 2, 3$. (the remaining two roots of (1))

Hence the eigen values of A are

$\lambda = 1, 2, 3$.

Next, to find the Eigen vectors of A corresponding to each λ we consider the homogeneous system $(A - \lambda I)X = 0$ & solve it for a non-trivial solution.

Now, $(A - \lambda I)X = 0$

$$\Rightarrow \begin{bmatrix} 1-\lambda & 0 & -1 \\ 1 & 2-\lambda & 1 \\ 2 & 2 & 3-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \longrightarrow (2)$$

Case 1: $\lambda = 1$

Put $\lambda = 1$ in (2), we get

$$\begin{bmatrix} 0 & 0 & -1 \\ 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now for solving the system, consider the coefficient matrix $\begin{bmatrix} 0 & 0 & -1 \\ 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix}$ & reduce it into Echelon form

Now $\begin{bmatrix} 0 & 0 & -1 \\ 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & -1 \\ 2 & 2 & 2 \end{bmatrix}$ $R_1 \leftrightarrow R_2$

$\sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}$ $R_3 \to R_3 - 2R_1$

which is in Echelon form

Now, consider the equivalent system

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow x_1 + x_2 + x_3 = 0 \longrightarrow (E_1)$

$-x_3 = 0 \longrightarrow (E_2)$

Here x_1 & x_3 are leading variable & x_2 is a free variable.

Let $x_2 = a$.

From (E_2), we have $\cancel{x_3 = 0}$ $x_3 = 0$.

From (E_1), $x_1 = -x_2 - x_3$

$= -a - 0$

$= -a$

$\Rightarrow x_1 = -a$

Thus, we have $\boxed{x_1 = -a,\ x_2 = a,\ x_3 = 0}$

Let $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -a \\ a \\ 0 \end{bmatrix}$

Now, choose $a = 1$, then $X_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ is an eigen vector of A corresponding to the eigen value $\lambda = 1$.

Case 2: $\lambda = 2$

Put $\lambda = 2$ in (2), we get

$$\begin{bmatrix} -1 & 0 & -1 \\ 1 & 0 & 1 \\ 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now, for solving this system, consider the coefficient matrix $\begin{bmatrix} -1 & 0 & -1 \\ 1 & 0 & 1 \\ 2 & 2 & 1 \end{bmatrix}$ & reduce it into Echelon form

Now, $\begin{bmatrix} -1 & 0 & -1 \\ 1 & 0 & 1 \\ 2 & 2 & 1 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 0 & 1 \\ -1 & 0 & -1 \\ 2 & 2 & 1 \end{bmatrix} \quad R_1 \leftrightarrow R_2$

$$\sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 2 & -1 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 + R_1 \\ R_3 \to R_3 - 2R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{bmatrix} \quad R_2 \leftrightarrow R_3$$

↳ which is in Echelon form.

Now, consider the Equivalent System

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} m_1 \\ m_2 \\ m_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow m_1 + m_3 = 0 \longrightarrow (E_1)$

$2m_2 - m_3 = 0 \longrightarrow (E_2)$

Thus, the leading variables are m_1 & m_2 & m_3 is a ~~free~~ free variable.

Let $m_3 = a$

From (E_2), $2m_2 = m_3$

$$\Rightarrow m_2 = \frac{m_3}{2}$$

$$= \frac{a}{2}$$

From (E_1), $m_1 = -m_3$

$$= -a$$

Thus, we have $x_1 = -a$, $x_2 = \frac{a}{2}$, $x_3 = a$

Let $X_2 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -a \\ a/2 \\ a \end{bmatrix}$

Now, choose $a = 2$, then $X_2 = \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}$ is an eigen vector of A corresponding to the eigen value $\lambda = 2$.

Case 3: $\lambda = 3$

Put $\lambda = 3$ in (2), we get

$$\begin{bmatrix} -2 & 0 & -1 \\ 1 & -1 & 1 \\ 2 & 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now, for solving the system, consider the coefficient matrix $\begin{bmatrix} -2 & 0 & -1 \\ 1 & -1 & 1 \\ 2 & 2 & 0 \end{bmatrix}$ & reduce it into Echelon form.

Now, $\begin{bmatrix} -2 & 0 & -1 \\ 1 & -1 & 1 \\ 2 & 2 & 0 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & -1 & 1 \\ -2 & 0 & -1 \\ 2 & 2 & 0 \end{bmatrix}$ $R_1 \leftrightarrow R_2$

$\sim \begin{bmatrix} 1 & -1 & 1 \\ 0 & -2 & 1 \\ 0 & 4 & -2 \end{bmatrix}$ $R_2 \to R_2 + 2R_1$, $R_3 \to R_3 - 2R_1$

$$\sim \begin{bmatrix} 1 & -1 & 1 \\ 0 & -2 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad R_3 \to R_3 + 2R_2$$

which is in Echelon form.

Now, consider the Equivalent System,

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & -2 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow x_1 - x_2 + x_3 = 0 \longrightarrow (E_1)$$

$$-2x_2 + x_3 = 0 \longrightarrow (E_2)$$

Here x_1 & x_2 are leading variables & x_3 is a free variable.

Let $x_3 = a$

From (E_2), $-2x_2 = -x_3$

$$\Rightarrow x_2 = \frac{x_3}{2}$$

$$= \frac{a}{2}$$

From (E_1) $x_1 = x_2 - x_3$

$$= \frac{a}{2} - a$$

$$= -\frac{a}{2}$$

Thus, we have

$$x_1 = -\frac{a}{2}, \quad x_2 = \frac{a}{2}, \quad x_3 = a.$$

Let $X_3 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -a/2 \\ a/2 \\ a \end{bmatrix}$

Now, choose $a = 2$, then $X_3 = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$

is an eigen vector of A corresponding to the Eigen value $\lambda = 3$.

Hence the Eigen values of $A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$

are $\lambda = 1, 2, 3$ & the corresponding Eigen Vectors of A are $X_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$, $X_2 = \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}$

and $X_3 = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$

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4) Find the Eigen values and Eigen vectors of the matrix $A = \begin{bmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{bmatrix}$

Solution:-

Given $A = \begin{bmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{bmatrix}$

The characteristic equation of A is $|A - \lambda I| = 0$

$\Rightarrow \lambda^3 - c_1 \lambda^2 + c_2 \lambda - c_3 = 0$

(14) $\Rightarrow \lambda^3 - 0\lambda^2 + (-12)\lambda - 16 = 0$

$\Rightarrow \lambda^3 - 12\lambda - 16 = 0 \longrightarrow (1)$

By solving (1), we get the Eigen values of A.

Now, for solving, we consider $\pm 1, \pm 2, \pm 4, \pm 8, \pm 16$ as the factors of the constant term ~~is~~ (-16) in (1).

$\lambda = -1$, $(-1)^3 - 12(-1) - 16$
$= -1 + 12 - 16 \neq 0$

$\lambda = 1$, $1 - 12 - 16 \neq 0$

$\lambda = -2$, $(-2)^3 - 12(-2) - 16$
$= -8 + 24 - 16$
$= 0$

$\Rightarrow \lambda = -2$ is a root of (1)

Let us find the other two roots of (1) as follows:

$$\begin{array}{r|rrrr} -2 & 1 & 0 & -12 & -16 \\ & 0 & -2 & 4 & 16 \\ \hline & 1 & -2 & -8 & \boxed{0} \end{array}$$

$\Rightarrow \lambda^2 - 2\lambda - 8 = 0$

$\Rightarrow (\lambda + 2)(\lambda - 4) = 0$

$\Rightarrow \lambda = -2, 4$

$C_1 = 0 \quad (1 + -5 + 4 = 0)$

$C_2 = \begin{vmatrix} -5 & 3 \\ -6 & 4 \end{vmatrix} + \begin{vmatrix} 1 & 3 \\ 6 & 4 \end{vmatrix} + \begin{vmatrix} 1 & -3 \\ 3 & -5 \end{vmatrix}$

$= -2 + (-14) + 4$

$= -12$

$C_3 = \begin{vmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{vmatrix}$

$= 1(-2) - (-3)(-6) + 3(12)$

$= -2 - 18 + 36$

$= 16$

Note: Here $1, -2, -8$ are the respective terms of the 2^{nd} degree equation to be solved for the remaining two roots.

Hence the Eigen values of A are
$\lambda = 4, -2, -2$ ($\lambda = -2$ is a repeated eigen value of A)

Next, to find the Eigen Vectors of A corresponding to each λ, Consider the homogeneous system $(A - \lambda I)X = 0$ & solve it for a non-trivial solution.

Now, $(A - \lambda I)X = 0$

$$\Rightarrow \begin{bmatrix} 1-\lambda & -3 & 3 \\ 3 & -5-\lambda & 3 \\ 6 & -6 & 4-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \longrightarrow (2)$$

Case 1 : $\lambda = 4$

Put $\lambda = 4$ in (2), We get

$$\begin{bmatrix} -3 & -3 & 3 \\ 3 & -9 & 3 \\ 6 & -6 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now, for solving this system, Consider the Coefficient matrix & reduce it into the Echelon form.

i.e. $\begin{bmatrix} -3 & -3 & 3 \\ 3 & -9 & 3 \\ 6 & -6 & 0 \end{bmatrix}$

$$\sim \begin{bmatrix} -3 & -3 & 3 \\ 0 & -12 & 6 \\ 0 & -12 & 6 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 + R_3 \\ R_3 \to R_3 + 2R_3 \end{matrix}$$

$$\sim \begin{bmatrix} -3 & -3 & 3 \\ 0 & -12 & 6 \\ 0 & 0 & 0 \end{bmatrix} \quad R_3 \to R_3 - R_2$$

which is in Echelon form.

Now, consider the Equivalent ~~form~~ System

$$\begin{bmatrix} -3 & -3 & 3 \\ 0 & -12 & 6 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow -3x_1 - 3x_2 + 3x_3 = 0 \longrightarrow E_1$$

$$-12x_2 + 6x_3 = 0 \longrightarrow E_2$$

Here x_1 & x_2 are leading variables & x_3 is a free variable.

Let $x_3 = a$

From E_2, $-12x_2 = -6x_3$

$$\Rightarrow x_2 = \frac{1}{2}x_3$$

$$= \frac{a}{2}$$

From E_1, $-3x_1 = 3x_2 - 3x_3$

$$\Rightarrow x_1 = x_2 - x_3$$

$$= \frac{a}{2} - a$$

$$= -\frac{a}{2}$$

Thus, we have $x_1 = -\frac{a}{2}$, $x_2 = \frac{a}{2}$, $x_3 = a$

Let $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -a/2 \\ a/2 \\ a \end{bmatrix}$

Now, choose $a = 2$, then $X_1 = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$ is

an eigen vector of A corresponding to the eigen value $\lambda = 4$.

Case 2 : $\lambda = -2$

Put $\lambda = -2$ in (2), we get

$$\begin{bmatrix} 3 & -3 & 3 \\ 3 & -3 & 3 \\ 6 & -6 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now, for solving this system, consider the coefficient matrix & reduce it into the Echelon form

i.e, $$\begin{bmatrix} 3 & -3 & 3 \\ 3 & -3 & 3 \\ 6 & -6 & 6 \end{bmatrix}$$

$$\sim \begin{bmatrix} 3 & -3 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - 2R_1 \end{matrix}$$

which is in Echelon form.

Now consider the Equivalent ~~System~~ system,

$$\begin{bmatrix} 3 & -3 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Longrightarrow 3x_1 - 3x_2 + 3x_3 = 0 \longrightarrow (E_1)$

Here x_1 is a leading variable & x_2 & x_3 are free variables.

Let $x_2 = a$, $x_3 = b$.

From (E_1), $3x_1 = 3x_2 - 3x_3$

$\Longrightarrow x_1 = x_2 - x_3$

$= a - b$

Thus, we have, $x_1 = a-b$, $x_2 = a$, $x_3 = b$

Let $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} a-b \\ a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} a + \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} b$

~~Now choose val a=1, b=0~~

~~Now choose a=1 & b=0, we get~~

Now, we can find two independent eigen vectors of A corresponding to the repeated eigen value $\lambda = -2$ by choosing $a=1, b=0$ & $a=0, b=1$.

i.e. When $a=1, b=0$, we get $X_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

When $a=0, b=1$, we get $X_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

Thus the Eigen values of A are $-4, -2, -2$ & the corresponding Eigen vectors are

$$X_1 = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}, \quad X_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad \& \quad X_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

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5) Find the Eigen values & Eigen vectors of the matrix $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 2 & 4 \end{bmatrix}$.

Solution:-

Given $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 2 & 4 \end{bmatrix}$

The characteristic equation of A is $|A - \lambda I| = 0$.

$\Rightarrow \lambda^3 - C_1\lambda^2 + C_2\lambda - C_3 = 0$

$\Rightarrow \lambda^3 - 7\lambda^2 + 16\lambda - 12 = 0 \rightarrow (1)$

$C_1 = 2 + 1 + 4 = 7$

$C_2 = \begin{vmatrix} 1 & -1 \\ 2 & 4 \end{vmatrix} + \begin{vmatrix} 2 & 0 \\ 0 & 4 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix}$

$= 6 + 8 + 2$

$= 16$

$C_3 = \begin{vmatrix} 2 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 2 & 4 \end{vmatrix}$

$= 2(6) - 1(0) + 0(0)$

$= 12$

~~Hence~~ By solving (1) we get the Eigen values of A.

Now, for solving (1), we consider the factors of -12 as $\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12$

$\lambda = -1$, $(-1)^3 - 7(-1)^2 + 16(-1) - 12$
$= -1 - 7 - 16 - 12 \neq 0$

$\lambda = 1$, $1 - 7 + 16 - 12 \neq 0$

$\lambda=-2$, $-8-28-32-12\neq 0$

$\lambda=2$, $8-28+32-12=0$

Thus $\lambda=2$ is a root of (1).

The other two roots of (1) can be obtained as follows:

$$\begin{array}{r|rrrr} 2 & 1 & -7 & 16 & -12 \\ & 0 & 2 & -10 & 12 \\ \hline & 1 & -5 & 6 & \boxed{0} \end{array}$$

$\Rightarrow \lambda^2-5\lambda+6=0$

$\Rightarrow (\lambda-2)(\lambda-3)=0$

$\Rightarrow \lambda=2, 3.$

Hence the Eigen values of A are

$\lambda=3, 2, 2$ ($\lambda=2$ is a repeated eigen value of A)

Next, to find the Eigen vectors of A corresponding to each λ, consider the homogeneous system $(A-\lambda I)X=0$ & solve it for a non-trivial solution.

Now, $(A-\lambda I)X=0$

$$\Rightarrow \begin{bmatrix} 2-\lambda & 1 & 0 \\ 0 & 1-\lambda & -1 \\ 0 & 2 & 4-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \longrightarrow (2)$$

Case 1: $\lambda = 3$

Put $\lambda = 3$ in (2), we get

$$\begin{bmatrix} -1 & 1 & 0 \\ 0 & -2 & -1 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now, for solving this system, consider the coefficient matrix & reduce it into the Echelon form

ie, $\begin{bmatrix} -1 & 1 & 0 \\ 0 & -2 & -1 \\ 0 & 2 & 1 \end{bmatrix}$

$\sim \begin{bmatrix} -1 & 1 & 0 \\ 0 & -2 & -1 \\ 0 & 0 & 0 \end{bmatrix}$ $R_3 \to R_3 + R_2$

which is in Echelon form

Now, consider the Equivalent System

$$\begin{bmatrix} -1 & 1 & 0 \\ 0 & -2 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow -x_1 + x_2 = 0 \longrightarrow (E_1)$

$-2x_2 - x_3 = 0 \longrightarrow (E_2)$

Here x_1 & x_2 are leading variables & x_3 is a free variable.

Let $x_3 = a$

From (E_2), $-2x_2 = x_3$

$\Rightarrow x_2 = -\frac{1}{2} x_3$

$= -\frac{a}{2}$

From (E_1), $-m_1 = -m_2$

$$\Rightarrow m_1 = m_2$$

$$= -\frac{a}{2}$$

Thus, we have, $m_1 = -\frac{a}{2}$, $m_2 = -\frac{a}{2}$, $m_3 = a$.

Let $X = \begin{bmatrix} m_1 \\ m_2 \\ m_3 \end{bmatrix} = \begin{bmatrix} -a/2 \\ -a/2 \\ a \end{bmatrix}$

Now, choose $a = 2$, then $X_1 = \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}$

is an eigen vector of A corresponding to the eigen value $\lambda = 3$.

Case 2: $\lambda = 2$

Put $\lambda = 2$ in (2), we get

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & -1 & -1 \\ 0 & 2 & 2 \end{bmatrix} \begin{bmatrix} m_1 \\ m_2 \\ m_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now, for solving this system, consider the coefficient matrix & reduce it into the Echelon form.

i.e. $\begin{bmatrix} 0 & 1 & 0 \\ 0 & -1 & -1 \\ 0 & 2 & 2 \end{bmatrix}$

$$\sim \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 2 \end{bmatrix} \begin{matrix} R_2 \to R_2 + R_1 \\ R_3 \to R_3 - 2R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} R_3 \to R_3 + 2R_2$$

which is in Echelon form.

Now, Consider the Equivalent system

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow x_2 = 0 \to E_1$

$-x_3 = 0 \to E_2$

Here x_2 & x_3 are leading variables & x_1 is a free variable.

Let $x_1 = a$

From E_1, $x_2 = 0$

From E_2, $x_3 = 0$.

Thus, We have $x_1 = a$, $x_2 = 0$, $x_3 = 0$.

Let $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} a \\ 0 \\ 0 \end{bmatrix}$

Now, choose $a = 1$, then $X_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is an eigen vector of A corresponding to the eigen value $\lambda = 2$.

Here, we noticed that there is only one independent eigen vector of A corresponding to the repeated eigen value $\lambda = 2$.

Thus, we found that $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 2 & 4 \end{bmatrix}$

has only two independent eigen vectors

corresponding to the eigen values $\lambda = 3, 2, 2$

i.e. one is $X_1 = \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}$ corresponding to the

eigen value $\lambda = 3$ & the other is

$X_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ corresponding to the eigen

value $\lambda = 2$ ($\lambda = 2$ is a repeated eigen value)

6) Find the Eigen values & Eigen Vectors of the matrix $A = \begin{bmatrix} -3 & -7 & -5 \\ 2 & 4 & 3 \\ 1 & 2 & 2 \end{bmatrix}$

Solution:-

Given $A = \begin{bmatrix} -3 & -7 & -5 \\ 2 & 4 & 3 \\ 1 & 2 & 2 \end{bmatrix}$

The characteristic equation of A is

$|A - \lambda I| = 0$

$\Rightarrow \lambda^3 - C_1\lambda^2 + C_2\lambda - C_3 = 0$

$\Rightarrow \lambda^3 - 3\lambda^2 + 3\lambda - 1 = 0 \longrightarrow (1)$

$C_1 = -3 + 4 + 2$
$= 3$

$C_2 = \begin{vmatrix} 4 & 3 \\ 2 & 2 \end{vmatrix} + \begin{vmatrix} -3 & -5 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} -3 & -7 \\ 2 & 4 \end{vmatrix}$

$= 2 + (-1) + 2$

$= \underline{\underline{3}}$

$C_3 = \begin{vmatrix} -3 & -7 & -5 \\ 2 & 4 & 3 \\ 1 & 2 & 2 \end{vmatrix}$

$= -3(2) - (-7)(1) - 5(0)$

$= -6 + 7$

$= 1$

By solving (1), we get the Eigen values of A.

For solving (1), we consider the factors of the constant term, (i.e. -1) is ± 1.

$\lambda = -1$, $(-1)^3 - 3(-1)^2 + 3(-1) - 1$
$= -1 - 3 - 3 - 1$
$\neq 0$

$\lambda = 1$, $1 - 3 + 3 - 1 = 0$

Thus $\lambda = 1$ is a root of (1).

To find the other ~~or~~ two roots of (1) we proceed as follows:

1	1	-3	3	-1
	0	1	-2	1
	1	-2	1	0

$\Rightarrow \lambda^2 - 2\lambda + 1 = 0$

$\Rightarrow (\lambda - 1)^2 = 0$

$\Rightarrow (\lambda-1)(\lambda-1) = 0$

$\Rightarrow \lambda = 1, 1$

Thus, the Eigen values of A are $\lambda = 1, 1, 1$ ($\lambda = 1$ is a repeated eigen value of A)

Next, to find the eigen vectors of A corresponding to the eigen value λ, consider the homogeneous system $(A - \lambda I)X = 0$ & solve it for a non-trivial solution.

Now, $(A - \lambda I)X = 0$

$$\Rightarrow \begin{bmatrix} -3-\lambda & -7 & -5 \\ 2 & 4-\lambda & 3 \\ 1 & 2 & 2-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \longrightarrow (2)$$

Here we have only one case for $\lambda = 1$, since $\lambda = 1$ is repeated 3 times.

Now put $\lambda = 1$ in (2), we get

$$\begin{bmatrix} -4 & -7 & -5 \\ 2 & 3 & 3 \\ 1 & 2 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now, for solving this system, consider the coefficient matrix & reduce it into the Echelon form.

i.e $\begin{bmatrix} -4 & -7 & -5 \\ 2 & 3 & 3 \\ 1 & 2 & 1 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 3 \\ -4 & -7 & -5 \end{bmatrix}$ $R_1 \leftrightarrow R_3$

$\sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix}$ $R_2 \to R_2 - 2R_1$, $R_3 \to R_3 + 4R_1$

$\sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ $R_3 \to R_3 + R_2$

which is in Echelon form.

Now, consider the Equivalent system

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow x_1 + 2x_2 + x_3 = 0 \longrightarrow E_1$

$-x_2 + x_3 = 0 \longrightarrow E_2$

Here x_1 & x_2 are leading variables &

x_3 is a free variable.

Let $x_3 = a$

From (E_2), $-x_2 = -x_3$

$\Rightarrow x_2 = x_3$

$= a.$

Thus $x_2 = a.$

From (E_1), $x_1 = -2x_2 - x_3$

$= -2a - a$

$= -3a$

Thus, we have $x_1 = -3a$, $x_2 = a$, $x_3 = a$

Let $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -3a \\ a \\ a \end{bmatrix}$

Now choose $a=1$, then $X_1 = \begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix}$

is an eigen vector of A corresponding to the eigen value $\lambda = 1$.

Here, we noticed that $A = \begin{bmatrix} -3 & -7 & -5 \\ 2 & 4 & 3 \\ 1 & 2 & 2 \end{bmatrix}$

has only one independent eigen vector $X_1 = \begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix}$ corresponding to the repeated eigen value ~~...~~ $\lambda = 1$.

Note:-

1) If A is a 3×3 matrix and A has three distinct eigen values, then ~~A~~ we can always find three independent eigen vectors corresponding to the eigen values.

2) If A is a 3×3 matrix and A has repeated eigen values, then it is not always to find three independent eigen vectors corresponding to the eigen values.

(i.e, In some cases, we ~~cannot~~ ~~we~~ can find three independent eigen vectors (refer: problem 4), but in some ~~oth~~ cases, ~~if~~ we cannot find three independent eigen vectors corresponding to the eigen values [refer: problem 5) & refer: problem 6]