Problems (Homogeneous Systems)

1) Test whether the following equations possess non-trivial solutions. If so, find the non-trivial solutions:

$$3x + 2y + z = 0$$
$$2x + 3z = 0$$
$$x + 2y + 3z = 0.$$

Solution:-

Consider the matrix equation $AX = 0$ as

$$\begin{bmatrix} 3 & 2 & 1 \\ 2 & 0 & 3 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We know that the system possesses a non-trivial solutions if $\rho(A) < n$.

Consider $A = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 0 & 3 \\ 1 & 2 & 3 \end{bmatrix}$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & 3 \\ 3 & 2 & 1 \end{bmatrix} \quad R_1 \leftrightarrow R_3$$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -4 & -3 \\ 0 & -4 & -8 \end{bmatrix} \quad \begin{array}{l} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \end{array}$$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -4 & -3 \\ 0 & 0 & -5 \end{bmatrix} \quad R_3 \to R_3 - R_2$$

which is in Echelon form.

Now, $\rho(A) = 3$. (2)

Since $n = 3$, then we have $\rho(A) \nless n$. So the system does not possess non-trivial solution.

$\therefore$ $\rho(A) = 3 = n$, the system has only trivial solution. i.e. $x = 0, y = 0, z = 0$.

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2) Solve the homogeneous system

$$
\begin{aligned}
-2x_1 - 5x_2 + 8x_3 + 0x_4 - 17x_5 &= 0 \\
x_1 + 3x_2 - 5x_3 + x_4 + 5x_5 &= 0 \\
3x_1 + 11x_2 - 19x_3 + 7x_4 + x_5 &= 0 \\
x_1 + 7x_2 - 13x_3 + 5x_4 - 3x_5 &= 0.
\end{aligned}
$$

Solution:-

Consider the matrix equation $AX = 0$ as

$$
\begin{bmatrix} -2 & -5 & 8 & 0 & -17 \\ 1 & 3 & -5 & 1 & 5 \\ 3 & 11 & -19 & 7 & 1 \\ 1 & 7 & -13 & 5 & -3 \end{bmatrix}
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}
=
\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
$$

Now, consider

$$
A = \begin{bmatrix} -2 & -5 & 8 & 0 & -17 \\ 1 & 3 & -5 & 1 & 5 \\ 3 & 11 & -19 & 7 & 1 \\ 1 & 7 & -13 & 5 & -3 \end{bmatrix}
$$

$$
\sim \begin{bmatrix} 1 & 3 & -5 & 1 & 5 \\ -2 & -5 & 8 & 0 & -17 \\ 3 & 11 & -19 & 7 & 1 \\ 1 & 7 & -13 & 5 & -3 \end{bmatrix} \quad R_1 \leftrightarrow R_2
$$

$$
\sim \begin{bmatrix} 1 & 3 & -5 & 1 & 5 \\ 0 & 1 & -2 & 2 & -7 \\ 0 & 2 & -4 & 4 & -14 \\ 0 & 4 & -8 & 4 & -8 \end{bmatrix} \quad \begin{aligned} R_2 &\to R_2 + 2R_1 \\ R_3 &\to R_3 - 3R_1 \\ R_4 &\to R_4 - R_1 \end{aligned}
$$

$$\sim \begin{bmatrix} 1 & 3 & -5 & 1 & 5 \\ 0 & 1 & -2 & 2 & -7 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -4 & 20 \end{bmatrix} \quad \begin{array}{l} R_3 \to R_3 - 2R_2 \\ R_4 \to R_4 - 4R_2 \end{array}$$

$$\sim \begin{bmatrix} 1 & 3 & -5 & 1 & 5 \\ 0 & 1 & -2 & 2 & -7 \\ 0 & 0 & 0 & -4 & 20 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad R_3 \leftrightarrow R_4$$

which is in Echelon form.

Now, $\rho(A) = 3$ & $n = 5$ (the no. of unknown)

$\because \rho(A) < n$, then the system has infinitely many non-trivial solutions.

To solve the system, we consider the Equivalent system

$$\begin{bmatrix} 1 & 3 & -5 & 1 & 5 \\ 0 & 1 & -2 & 2 & -7 \\ 0 & 0 & 0 & -4 & 20 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow x_1 + 3x_2 - 5x_3 + x_4 + 5x_5 = 0 \longrightarrow (1)$

$x_2 - 2x_3 + 2x_4 - 7x_5 = 0 \longrightarrow (2)$

$-4x_4 + 20x_5 = 0 \longrightarrow (3)$

Thus, x_1, x_2, x_4 are leading variables & x_3, x_5 are free variables.

Choose $x_3 = a$, $x_5 = b$, where a & b are arbitrary real numbers.

From (3), $-4x_4 = -20x_5$

$\Rightarrow x_4 = 5x_5$

$= 5b$

$\therefore \boxed{x_4 = 5b}$

From (2), $x_2 = 2x_3 - 2x_4 + 7x_5$

$= 2a - 2(5b) + 7b$

$= 2a - 10b + 7b$

$= 2a - 3b$

$\therefore \boxed{x_2 = 2a - 3b}$

From (1), $x_1 = -3x_2 + 5x_3 - x_4 - 5x_5$

$= -3(2a - 3b) + 5a - 5b - 5b$

$= -6a + 9b + 5a - 10b$

$= -a - b$

$\therefore \boxed{x_1 = -a - b}$

Thus, the solution of the given system of Equations is $x_1 = -a - b$

$x_2 = 2a - 3b$

$x_3 = a$

$x_4 = 5b$

$x_5 = b$

where a & b are arbitrary real nos.

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3) Solve the homogeneous system

$x + 3y + 2z = 0$, $2x - y + 3z = 0$, $3x - 5y + 4z = 0$, $x + 17y + 4z = 0$.

Solution:-

consider the matrix Equation $AX=0$ is

$$\begin{bmatrix} 1 & 3 & 2 \\ 2 & -1 & 3 \\ 3 & -5 & 4 \\ 1 & 17 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Consider $A = \begin{bmatrix} 1 & 3 & 2 \\ 2 & -1 & 3 \\ 3 & -5 & 4 \\ 1 & 17 & 4 \end{bmatrix}$

$$\sim \begin{bmatrix} 1 & 3 & 2 \\ 0 & -7 & -1 \\ 0 & -14 & -2 \\ 0 & 14 & 2 \end{bmatrix} \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \\ R_4 \to R_4 - R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 3 & 2 \\ 0 & -7 & -1 \\ 0 & 7 & 1 \\ 0 & 7 & 1 \end{bmatrix} \begin{matrix} R_3 \to \frac{R_3}{-2} \\ R_4 \to \frac{R_4}{2} \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 3 & 2 \\ 0 & -7 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} R_3 \to R_3 + R_2 \\ R_4 \to R_4 + R_2 \end{matrix}$$

which is in Echelon form.

Now, $\rho(A) = 2$ & $n = 3$ (the no. of unknowns)

$\therefore$ $\rho(A) < n$, then we have non-trivial solution for the given system. It can be solved by considering the system

$$\begin{bmatrix} 1 & 3 & 2 \\ 0 & -7 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow x+3y+2z=0 \longrightarrow (1)$

$-7y-z=0 \longrightarrow (2)$

Here x & y are leading variables & z is a free variable.

Choose $z=k$, where k is an arbitrary real number.

From (2), $-7y=z$

$$\Rightarrow y=-\frac{1}{7}z$$

$$=-\frac{1}{7}k$$

$$\therefore \boxed{y=\frac{-k}{7}}$$

From (1), $x=-3y-2z$

$$=-3\left(\frac{-k}{7}\right)-2k$$

$$=\frac{3}{7}k-2k$$

$$=\frac{3k-14k}{7}$$

$$=\frac{-11}{7}k$$

$$\therefore \boxed{x=\frac{-11k}{7}}$$

Thus, the solution of the given system are

$x=\frac{-11k}{7}$, $y=\frac{-k}{7}$, $z=k$, where k is an arbitrary real number.

Practice Problems.

1) Check whether the following system of equations possesses non-trivial solution.

$$x - 3y - 8z = 0$$
$$3x + y = 0$$
$$2x + 5y + 6z = 0.$$

2) Show that the equations $x + 2y - z = 0$, $3x + y - z = 0$, $2x - y = 0$ have non-trivial solutions & find them.

3) Solve completely the system of equations

$$x + y - 2z + 3w = 0$$
$$x - 2y + z - w = 0$$
$$4x + y - 5z + 8w = 0$$
$$5x - 7y + 2z - w = 0$$

4) Find the value of k so that the equations $x + y + 3z = 0$, $4x + 3y + kz = 0$, $2x + y + 2z = 0$ have non-trivial solution.

5) Determine the values of λ for which the following set of equations may possess non-trivial solution:

$$3x_1 + x_2 - \lambda x_3 = 0$$
$$4x_1 - 2x_2 - 3x_3 = 0$$
$$2\lambda x_1 + 4x_2 + \lambda x_3 = 0$$

For each permissible value of λ, determine the general solution.