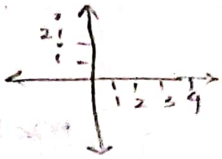
17/8/2020. Module 3

y axis imaginary



Complex Differenciation:

Chapter 1 Preliminaries

A complex number is of the form $z = x+iy$

where

x and y are real numbers

Complexplane ↓ Argent plane. ↓ $y \to Im$, $x \to Real$.

here; $x = Re(z)$ and $y = Im(z)$

Modules of z is denoted by $|z| = \sqrt{x^2+y^2}$

ie $|z|$ is the distance between origin and the point (x,y)

modulus is simply the distance formula

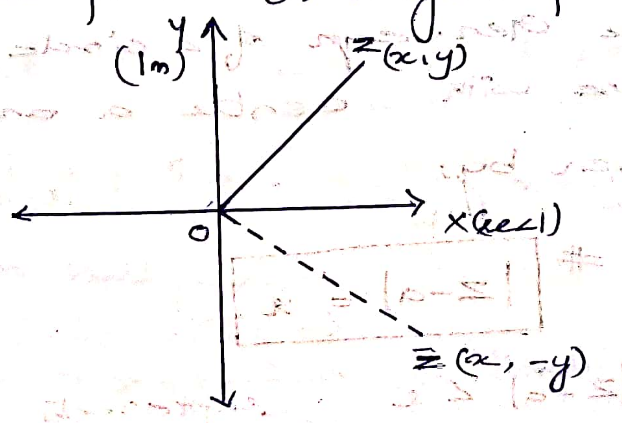
Co-ordinate plane $x, y \to$ Real

Conjugate of z denoted by $\bar{z} = x - iy$.

ie $\bar{z}$ is the mirror image of $x+iy$ wrt the +ve real axis

$$z\bar{z} = |z|^2$$

A complex number $z = x+iy$ can be represented in a complex plane or argent plane.



Complex Numbers in Polar Coordinates

Let the complex number $z = x+iy$ makes an angle θ w.r.t the +ve real axis in the anticlockwise direction.

θ is always consider in anticlockwise direction

Then;

$\cos\theta = x/r$.

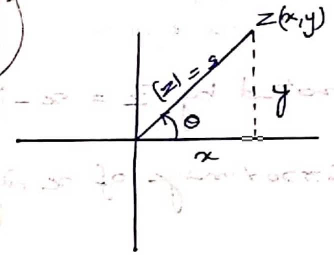
$\sin\theta = y/r$.

$c \to 2-i$, $r=4$, $|z-2+i|=4$

$\Rightarrow x = r\cos\theta$ $y = r\sin\theta$

So that $z = x+iy = r(\cos\theta + i\sin\theta)$

$= re^{i\theta}$



Where;

$r \longrightarrow |z|$

$\theta \longrightarrow$ argument or amplitude of z

θ lies b/w the interval 0 to 2π

* → The value of θ in the interval $(0,\pi)$ is principle argument of z

Example.

Circles & Discs

The Gen. eqn of a circle in the complex plane with centre a and radius r is given by:

$a = x+iy$.

\# $\boxed{|z-a| = r}$

$|z-a| < r$ represents the interior part of the circle. $|z-a| = r$.

~~$|z-a| = r$~~ It is also called an open disc or called r-neighbourhood of a.

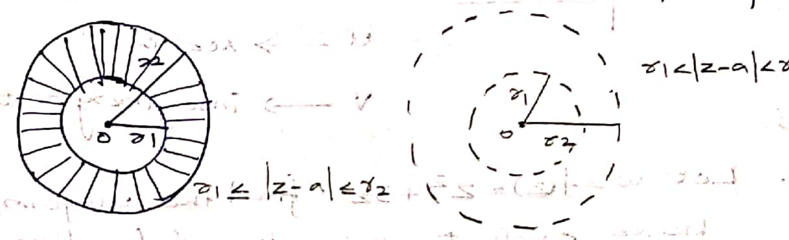
$|z-a| \le r$ represents the circle plus its interior and is called a closed disc

$|z-a| > r$ represents the ~~circle plus~~ exterior part of the circle and

$|z-a| \ge r$ represents the circle plus its ~~interior~~ exterior.

The inequality $r_1 < |z-a| < r_2$ represents an open circular ring or open annulus in the complex plane.

$r_1 \le |z-a| \le r_2$ represents closed circular ring or closed annulus in the complex plane.



Half planes

$x \ge 0 \to$ closed ½ plane
$x > 0 \to$ open ½ plane } Right ½ plane

Set of all z; $\{z = x+iy,\ x>0\}$ represents open right half plane.

$\{z = x+iy \ \ x \ge 0\} \longrightarrow$ closed right ½ plane.

$\{z = x+iy,\ x < 0\} \longrightarrow$ open ~~right~~ left half plane

$\{z = x+iy,\ x \le 0\} \longrightarrow$ closed left ½ plane

$\{z = x+iy,\ y > 0\} \longrightarrow$ open upper ½ plane.

$\{z = x+iy,\ y \ge 0\} \longrightarrow$ closed upper ½ plane

$\{z = x+iy,\ y < 0\} \longrightarrow$ open lower ½ plane

$\{z = x+iy,\ y \le 0\} \longrightarrow$ ~~open~~ closed lower ½ plane.

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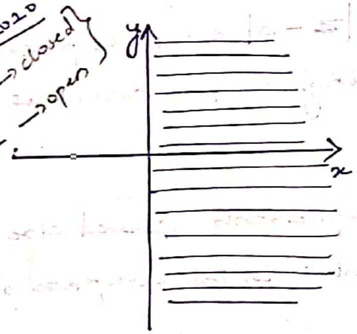
$\le \rightarrow$ closed

$<, > \rightarrow$ open



A complex function is ~~one~~ of the form;

$$\boxed{w = f(z) = u(x,y) + i\,v(x,y)}$$

Where;

$u \longrightarrow$ real part.

$v \longrightarrow$ imaginary part.

3mks

Q1. Let $w = f(z) = z^2 + 3z$. find the uv form and hence evaluate the value of function at $z = 1+3i$.

$$f(z) = z^2 + 3z$$

$$= (x+iy)^2 + 3(x+iy)$$

$$= x^2 + 2ixy - y^2 + 3x + 3iy$$

$$x^2 - y^2 + 3x + i(2xy + 3y).$$

$$f(1+3i) = (1^2 - 3^2 + 3) + i(2\times 3 + 3\times 3)$$

$$= \underline{\underline{-5+15i}}$$

Q2. Find the real and imaginary part of the functions $f(x) = 5z^2 - 12z + 3 + 2i$ and hence find its value at $4-3i$

$$z = x+iy.$$

$$f(x) = 5(x+iy)^2 - 12(x+iy) + 3 + 2i$$

$$= 5x^2 - 5y^2 + 10ixy - 12x - 12iy + 3 + 2$$

$$= 5x^2 - 5y^2 - 12x + 3 + i(10xy - 12y + 2)$$

$$f(4-3i) \Rightarrow 5\times 16 - 5\times 9 - 12\times 4 + 3 + i(10\times 4\times 3 - 12\times -3 + 2)$$

$$= \underline{\underline{-10 - 82i}}$$

※ Model Question

Q3. $f(z) = 1/1+z$ at $z = 1-i$

$$f(z) = \frac{1}{1+x+iy} = \frac{1+x-iy}{1+2x+x^2+y^2}$$

$$u = \frac{1+x}{1+2x+x^2+y^2}$$

$$v = \frac{-y}{1+2x+x^2+y^2}$$

$$f(1-i) = \frac{1+1\, i}{1+1+2+1}$$

$$= 2/5 + 1/5\, i$$

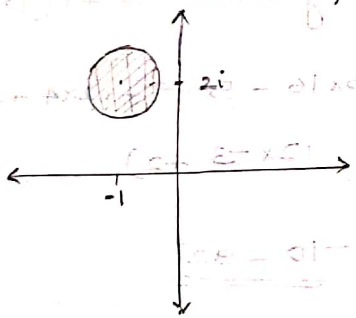
Q4) Determine the complex plane $|z+1-2i| \le 1/4$

Comparing with $|z-a| \le r$ we get

$a = -1+2i = 2i-1$

$r = 1/4$

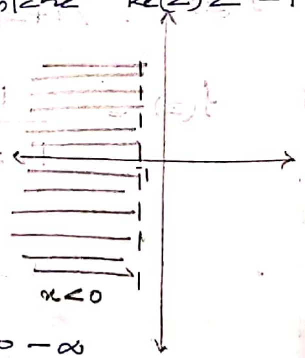
Which is a circle the interior with centre $2i-1$ and radius $1/4$ (Closed disc)



Q5) Determine the complex plane $Re(z) < -1$

$Re(z) < -1$

$\Rightarrow x < -1$



It is a left open plane extending from -1 to $-\infty$

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Chapter 2

Limit, Continuity & Derivative of a complex function:

Limit of a complex number:

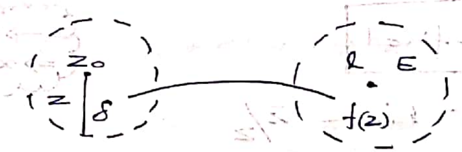
A function $f(z)$ is said to have a limit l as z approaches z_0 written as

$$\lim_{z \to z_0} f(z) = l$$

If for every small positive number ϵ we can find another small +ve number δ for $z \neq z_0$

$$|z - z_0| < \delta \Rightarrow |f(z) - l| < \epsilon$$

Geometrically for every $z \neq z_0$ in the delta disk $f(z)$ will be in the ϵ disk.



Remark:

* If a limit exist, it is unique
* z can approach z_0 from any direction in a complex plane.
* If $f(z)$ approaches more than 1 value, then the limit does not exist.

Continuity of a complex function:

A complex function $f(z)$ is said to be continuous at a point z_0 if : ① $f(z_0)$ is defined.

② $\lim_{z \to z_0} f(z) = f(z_0)$

A function $f(z)$ is continuous in a domain D if $f(z)$ is continuous at every point D

Note :-

$f(z) = u+iv$ is continuous if and only if both u and v are continuous.

7mks

Q.6) Check whether the following functions are continuous at $z=0$.

i) $f(z) = \begin{cases} \bar{z}/z & z \neq 0 \\ 0 & z = 0 \end{cases}$

Continuity at 0:

1mk $f(0)=0$ defined

6mk $\lim_{z \to 0} f(z) = 0$

$z \to 0$
$\Rightarrow x, y \to 0$

$\boxed{f(0) = 0 \text{ defined}}$

$\lim_{z \to z_0} f(z) = \lim_{z \to 0} \bar{z}/z$

$= \lim_{x+iy \to 0} \frac{x-iy}{x+iy}$

Assume that $z \to 0$ along $y = mx$.

$x+iy \to 0$

$x+imx \to 0$

$x(1+im) \to 0 \quad (1+im \neq 0)$

$\Rightarrow x \to 0$

$\lim_{x \to 0} \frac{x-imx}{x+imx}$

$\lim_{x \to 0} \frac{1-im}{1+im} = \frac{1-im}{1+im}$

Which is not unique since m can take any value $f(z)$ is not continuous at $z=0$.

Q7) $f(z) = \begin{cases} \text{Im}\left\{\frac{|z|^2}{z}\right\}, & z \neq 0 \\ 0 & z = 0 \end{cases}$

2mks

$\text{Im}\left(\frac{x^2+y^2}{x+iy}\right)$

$\text{Im}\left(\frac{(x^2+y^2)(x-iy)}{(x^2+y^2)}\right)$

$\text{Im}(x-iy)$

$\Rightarrow -y$.

$\boxed{f(0) = 0 \text{ is defined}}$

$\lim_{z \to 0} f(z) = \lim_{y \to 0} -y = 0$.

$\therefore f(z)$ is continuous at $z=0$

Q8) $f(z) = \begin{cases} \frac{\text{Re}(z)}{|z|} & z \neq 0 \\ 0 & z = 0 \end{cases}$

#

$z \to 0 \Rightarrow x+iy \to 0$

when $y = mx$

$z \to 0 \Rightarrow x+imx \to 0$

$\Rightarrow x(1+im) \to 0$

$1+im \neq 0$

hence

$x \to 0$.

$\frac{\text{Re}(z)}{|z|} = \frac{x}{\sqrt{x^2+y^2}}$

$\boxed{f(0) = 0 \text{ is defined}}$

$\lim_{z \to 0} f(z) = \lim_{x+iy \to 0} \frac{x}{\sqrt{x^2+y^2}}$

Assume $z \to 0$ along $y = mx$.

$\lim_{x \to 0} \frac{x}{\sqrt{x^2+m^2x^2}} = \frac{1}{\sqrt{1+m^2}}$

It is not unique as m can take any value.

Homework:

due Q.9. $f(z) = \begin{cases} \dfrac{(x+y)^2}{x^2+y^2} & x, y \neq 0 \\ 0 & x = y = 0 \end{cases}$

$\boxed{f(0) = 0 \text{ is defined}}$

$\lim_{z \to z_0} f(z) = \lim_{x \to 0} \dfrac{(x+y)^2}{x^2+y^2}.$

Assume that Let $z=0$ is along the line $y = mx$.

$\lim_{x \to 0} \dfrac{(x+mx)^2}{x^2+m^2x^2} = \lim_{x \to 0} \dfrac{(1+m)^2}{1+m^2}$

$= \dfrac{1+2m+m^2}{1+m^2}$

$= \dfrac{1+m^2+2m}{1+m^2} = 1 + \dfrac{2m}{1+m^2}$

Which is not unique as m can take any value $f(z)$ is not continuous.

due # Q.10 $f(z) = \begin{cases} Re\left\{\dfrac{z^2}{|z|}\right\} & z \neq 0 \\ 0 & z = 0 \end{cases}$ $Re\left(\dfrac{(x+iy)^2}{\sqrt{x^2+y^2}}\right) = \dfrac{x^2-y^2}{\sqrt{x^2+y^2}}$

#

$\boxed{f(0) = 0 \text{ is defined}}$

$\lim_{z \to 0} f(z) = \lim_{x+iy \to 0} \dfrac{x^2-y^2}{\sqrt{x^2+y^2}}$

Assume that $z=0$ is along $y = mx$.

$x+iy \to 0$

$x \to 0.$

$\lim_{x \to 0} \dfrac{x^2 - m^2x^2}{\sqrt{x^2+m^2x^2}}$

$\lim_{x \to 0} \dfrac{x^2(1-m^2)}{x\sqrt{1+m^2}} = 0$

$f(0) = 0$ is defined.

due Q.11. $f(z) = \begin{cases} |z|^2 \, Im(1/z) & z \neq 0 \\ 0 & z = 0 \end{cases}$

$\boxed{f_0 = 0 \text{ defined}}$

$\lim_{z \to 0} f(z).$

$\lim_{z \to 0} \dfrac{(x^2+y^2)(-y)}{x^2+y^2} = \lim_{x+iy \to 0} \dfrac{-x^2y-y^3}{x^2+y^2}$

Assume that $z=0$ $y = mx$.

$\lim_{x \to 0} \dfrac{-x^3m - m^3x^3}{x^2+m^2x^2} = \lim_{x \to 0} \dfrac{-x^3(m+m^3)}{x^2(1+m^2)}$

$= 0.$

$f(0)$ is continuous at $z = 0$

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Q.12. $f(z) = \begin{cases} \frac{Re(z)}{1-|z|} & z \neq 0 \\ 0 & z = 0 \end{cases}$

$\boxed{f(0) = 0 \text{ is defined}}$

$$\lim_{z \to 0} f(z) = \lim_{x+iy \to 0} \frac{x}{1-\sqrt{x^2+y^2}}$$

Assume $z=0$ is along $y=mx$

$$\lim_{x \to 0} \frac{x}{1-\sqrt{x^2+m^2x^2}} = 0$$

$f(z)$ is continuous at $z=0$

Derivatives of a complex numbers:

Derivative of a complex function f at z is given by $f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$

Note:

A differentiable function is always continuous but, a continuous function need not be differentiated.

Q. 13. Find the derivative of $f(z) = \frac{z-i}{z+i}$ at $z=i$

$$f'(z) = \frac{(z+i)\times 1 - (z-i)\times 1}{(z+i)^2}$$

$$\frac{\cancel{z}+i - \cancel{z}+i}{z^2-1+2zi}$$

$$= \frac{2i}{(z+i)^2}$$

$$f'(i) = \frac{2i}{(2i)^2} = \frac{1}{2i} = -\frac{i}{2}$$

$$\boxed{f'x = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}}$$

Q.14 Find the derivative of $(z-2i)^3$ at $z = 5+2i$

$f(z) = (z-2i)^3$

$f'(z) = 3(z-2i)^2$

$= 3(z^2 + 4i^2 - 4iz)$

$= 3(z^2 - 4zi - 4)$

$f'(5+2i) = 3\times 25$

$= 75$

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Chapter 1 Questions Contd:-

Hyperbolic funs:

$\sinh x = \frac{e^x - e^{-x}}{2}$ $\qquad \cosh x = \frac{e^x + e^{-x}}{2}$

Q.15. Seperate the real & imaginary part of the following functions.

1) $\sin(x+iy)$

$\sin(A+B)$

$\sin A\cos B + \sin B\cos A$

$\sin x \cos iy + \cos x \sin iy$

$\boxed{\begin{array}{l} \sin(i\theta) = i\sinh\theta \\ \cos i\theta = \cosh\theta \end{array}}$

$\sin x \cosh y + \cos x \times i \sinh y$

$\sin x \cosh y + i \cos x \sinh y$

$u = Re(z) = \cosh y \sin x$

$v = Im(z) = \cos x \sinh y$

Q.16 (2) $\cos(x+iy)$ $\qquad \cos(A+B) = \cos A \cos B - \sin A \sin B$

$\cos x \cos iy - \sin x \sin iy$.

$\cos x \cosh y - i \sin x \sinh y$.

$u = \cos x \cosh y \quad Re(z)$

$V = -\sin x \sinh y . \quad Im(z)$

Q.17) $\sinh(x+iy)$

$\sin(i\theta) = i \sinh\theta$.

$\sinh\theta = \frac{\sin(i\theta)}{i}$

$\boxed{\sinh\theta = -i \sin i\theta}$

$\theta = x+iy$.

$= -i \sin(i(x+iy))$

$= -i \sin(ix+i^2y)$

$= -i \sin(ix-y)$ $\qquad \sin(A-B)$

$= -i(\sin(ix)\cos y - \cos ix \sin y)$

$= -i(i\sinh x \cos y - \cosh x \sin y)$

$(-i^2 \sinh x \cos y + i \cosh x \sin y)$

$\sinh x \cos y + i \cosh x \sin y$.

$Re(z) = u = \sinh x \cos y$.

$Im(z) = V = \cosh x \sin y$.

\# Relation b/w Trigonometric & hyperbolic fxs:

$\sin i\theta = i \sinh\theta$

$\cos i\theta = \cosh\theta$.

Q18. $\cosh(x+iy)$

$\cos(i\theta) = \cosh\theta$.

$\cosh(x+iy) = \cos(ix-y)$

$\cos ix \cos y + \sin ix \sin y$.

$\cosh x \cos y + i \sinh x \sin y$.

$Re(z) = u = \cosh x \cos y$

$Im(z) = V = \sinh x \sin y$

Q19. $\sin(x-iy)$

$\sin x \cos iy - \cos x \sin iy$

$\sin x \cosh y - i \cos x \sinh y$.

$Re(z) = u = \sin x \cosh y$.

$Im(z) = V = -\cos x \sinh y$

Q20. $\cos(x-iy)$

$\cos x \cos iy + \sin x \sin iy$

$\cos x \cosh y + i \sinh x \sinh y$

$Re(z) = u = \cos x \cosh y$

$Im(z) = V = \sinh x \sinh y$

Due

Q21. $\sinh(x-iy)$

$\sin(i\theta) = i\sinh\theta.$

$\sinh\theta = \frac{\sin(i\theta)}{i}.$

$\frac{\sin(i(x-iy))}{i} \Rightarrow \frac{\sin(ix+y^2)}{i}$

$\sin(A+B) = \sin A\cos B + \cos A\sin B.$

$\frac{\sin ix\cos y + \cos ix\sin y}{i}$

$-i(i\sinh x\cos y + \cosh x\sin y)$

$-i^2\sinh x\cos y - i\cosh x\sin y.$

$\sinh x\cos y - i\cosh x\sin y.$

$Re(z) = u = \sinh x\cos y$

$Im(z) = v = -\cosh x\sin y.$

Due

Q22. $\cosh(x-iy)$

$\cos(i\theta) = \cosh\theta.$

$\cos(ix+y) = \cos ix\cos y - \sin ix\sin y$

$\cosh x\cos y - i\sinh x\sin y.$

$Re(z) = u = \cosh x\cos y.$

$Im(z) = v = -\sinh x\sin y$

20/8/2020 Chapter 2 Question contd:-

Q21. $f(z) = i(1-z)^n \quad z=0.$

$f'(z) = -in(1-z)^{n-1}$

$f'(0) = -in$

Q22. $\sinh 2z$ at $z=0$

$f'(z) = 2\cosh 2z$

$f'(0) = 2$

Q23. $(\cos 2z - z^2)^2$ at $z=0.$

$f'(z) = 2(\cos 2z - z^2)[-2\sin 2z - 2z]$

$f'(0) = 0$

21/5/2020 Chapter 3

"All entire fns are analytic
All analytic fns are not entire."

Analytic Functions & CR eqns:
(Holomorphic Functions)

A function $f(z)$ is said to be analytic in a domain D of the complex plane if $f(z)$ is defined and differentiated at all points of D.

eg: $f(z) = e^z$

* z = 0 is the only pt where 1/z is not analytic.

$f(z) = 1/z$ at $D: |z-3|<1$

analytic except at 0

* (Analytic fn is a differentiable fn:)

$f(z) = 2/z+1$ at $D: |z| = 1/2$

$f(z) = z^2$

* $\frac{1}{z-3}$ in $|z-2|<1$ analytic

$f(z) = \sin z, \cos z$

$\frac{1}{z-3}$ in $|z-2|\le 1$ not analytic

If a complex ~~fun~~ function $f(z)$ is analytic everywhere in the finite complex plane then the function ~~every~~ is called an entire function.

Partial derivatives:-

If $f(z) = u(x,y) + i\,v(x,y)$

$$f(z+\Delta z) = u(x+\Delta x, y+\Delta y) + i\,v(x+\Delta x, y+\Delta y)$$

$$\frac{\partial u}{\partial x} = u_x = \lim_{\Delta x \to 0} \frac{u(x+\Delta x, y) - u(x,y)}{\Delta x}$$

$$\frac{\partial u}{\partial y} = u_y = \lim_{\Delta y \to 0} \frac{u(x, y+\Delta y) - u(x,y)}{\Delta y}$$

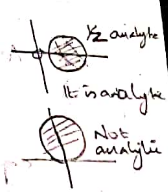
$$\frac{\partial v}{\partial x} = \lim_{\Delta x \to 0} \frac{v(x+\Delta x, y) - v(x,y)}{\Delta x}$$

$$\frac{\partial v}{\partial y} = \lim_{\Delta y \to 0} \frac{v(x, y+\Delta y) - v(x,y)}{\Delta y}$$

1/z-3 in |z|<1 analytic

$$(u_x)(0,0) = \lim_{x \to 0} \frac{u(x,0) - u(0,0)}{x}$$

$$(u_y)(0,0) = \lim_{y \to 0} \frac{u(0,y) - u(0,0)}{y}$$



1/z+3 this is analytic

CR eqns:

$$\boxed{\begin{aligned} u_x &= v_y \\ u_y &= -v_x \end{aligned}}$$ are called the Cauchy Reiman Equation.

Remark:

If $f(z) = u+iv$, then

$f'(z) = u_x + i v_x$.

* z^2 → continuously differentiable → analytic everywhere. → entire fun.

1/z not defined at z=0 at origin (z=0) is not defined. not analytic Not entire fn.

Exam

Theorem 1:

* Necessary condition for differentiability of a complex function:-

Let $f(z) = u(x,y) + i\,v(x,y)$ be a complex function. Then if, $f(z)$ is analytic or differentiable then, CR eqns are satisfied.

Assume that $f(z)$ is ~~differentiable~~ differentiable.

Proof:

Assume that $f(z) = u(x,y) + i\,v(x,y)$ is differentiable

Then;

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

$$f'(z) = \lim_{\Delta x + i\Delta y \to 0} \frac{u(x+\Delta x, y+\Delta y) + i\,v(x+\Delta x, y+\Delta y) - u(x,y) - i\,v(x,y)}{\Delta x + i\Delta y} \quad \text{——— (1)}$$

Assume $\Delta z \to 0$ along the real axis or x-axis

then $\Delta y = 0$.

eq (1) reduces to

put $\Delta y = 0$.

$$f'(z) = \lim_{\Delta x \to 0} \frac{u(x+\Delta x, y) + i\,v(x+\Delta x, y) - u(x,y) - i\,v(x,y)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{u(x+\Delta x, y) - u(x,y)}{\Delta x} + i \lim_{\Delta x \to 0} \frac{v(x+\Delta x, y) - v(x,y)}{\Delta x}$$

$$f'(z) = u_x + i v_x \quad \text{——— (2)}$$

Now, assume that $\Delta z \to 0$ along the y-axis

then $\Delta x = 0$

eq (1) reduces to: put $\Delta x = 0$

$$f'(z) = \lim_{\Delta y \to 0} \frac{u(x, y+\Delta y) + i\,v(x, y+\Delta y) - u(x,y) - i\,v(x,y)}{i\Delta y}$$

$$= \lim_{\Delta y \to 0} \frac{u(x, y+\Delta y) - u(x,y)}{i\Delta y} + i \lim_{\Delta y \to 0} \frac{v(x, y+\Delta y) - v(x,y)}{i\Delta y}$$

$$f'(z) = \cancel{u_y}; \cancel{v_y} \quad \frac{u_y}{i} + \frac{i\,v_y}{i}$$

$$f(z) = -i u_y + v_y \quad \text{——— (3)}$$

Comparing (2) & (3), we get.

$$u_x + i v_x = -i u_y + v_y.$$

$$u_x = v_y$$
$$u_y = -v_x.$$

which is the CR eqn.

$\therefore$ if $f(z)$ is analytic then u and v satisfy the CR eqn.

hence proved!

$\boxed{\text{All polynomials are analytic}}$

$\to$ All polynomials are continuous

Theorem 2: → converse of theorem 1 is not general.

Sufficient condition for differentiability of a complex function:

If a complex function $f(z) = u(x,y) + i\,v(x,y)$ is such that the partial derivatives are continuous and u and v satisfy CR equation then $f(z)$ is analytic

Remark:

(1) To prove that a function is not analytic it is enough to prove that u and v doesn't satisfy CR eqns.

(2) To prove that a fun: is analytic ~~it is~~ we have to prove (i) partial derivatives are continuous
(ii) u and v satisfy CR eqns.

Q-24 Check whether $f(z) = z^2$ is analytic

$$f(z) = z^2 \Rightarrow f(z) = (x+iy)^2$$

$= x^2 + 2ixy - y^2$

$u = x^2 - y^2 \qquad v = 2xy$

$u_x = 2x \qquad v_x = 2y$

$u_y = -2y \qquad v_y = 2x$

here;

$u_x = v_y$

$u_y = -v_x$

and the CR eqns are satisfied.

Also we can see that the partial derivatives are continuous being polynomials in x and y.

$\therefore f(z) = z^2$ is analytic.

Q.25 Show that e^z is an entire fn.

$f(z) = e^{x+iy} \quad \Rightarrow \boxed{e^{iy} = \cos y + i\sin y}$

$= e^x \cdot e^{iy}$

$\sin y \to$ continuous graph
$e^x \to$ continuous graph
$\cos y \to$ continuous graph
hence $e^x\cos y$ continuous, $e^x \sin y$

$= e^x(\cos y + i\sin y)$

$= e^x\cos y + i\sin y e^x$.

$u = e^x\cos y \qquad v = e^x \sin y$

$u_x = e^x\cos y \qquad v_x = e^x\sin y$

$u_y = -e^x\sin y \qquad v_y = e^x\cos y$

here; $u_x = v_y$

$u_y = -v_x$ hence the CR eqns are satisfied.

Also we can see that the partial derivatives are continuous i.e e^z is analytic everywhere in the finite complex plane.

$\therefore e^z$ is an entire function.

Q.26 Show that $\sin z$ is analytic everywhere.

$\sin z = \sin(x+iy)$

$= \sin x\cos iy + \sin iy\cos x$

$= \sin x\cosh y + i\sinh y\cos x$.

hyperbolic functions are continuous

$u = \sin x\cosh y \qquad v = \sinh y\cos x$.

$u_x = \cos x\cosh y \qquad v_x = -\sin x\sinh y$.

$u_y = \sin x\sinh y \qquad v_y = \cos x\cosh y$.

here; $u_x = v_y$

$u_y = -v_x$. hence the CR eqns are satisfied.

Also we can see that the partial derivatives are continuous.

$f(z) = \sin z$ is analytic everywhere

Q.27. Show that $\cos z$ is analytic.

$\cos z = \cos(x+iy)$

$= \cos x\cos iy - \sin x\sin iy$

$= \cos x\cosh y - i\sin x\sinh y$

$u = \cos x\cosh y \qquad v = -\sin x\sinh y$.

$u_x = -\sin x\cosh y \qquad v_x = -\cos x\sinh y$.

$u_y = \cos x\sinh y \qquad v_y = -\sin x\cosh y$

$U_x = V_y$
$U_y = -V_x$ hence the CR eqns are satisfied.

Also we know that partial derivatives are continuous

$f(z) = \cos z$ is analytic everywhere.

Q/28 $f(z) = \bar{z}$. analytic or not.

$\bar{z} = x - iy$.

$U = x$ $\quad V = -y$

$U_x = 1$ $\quad V_x = 0$

$U_y = 0$ $\quad V_y = -1$

$U_x \neq V_y$.

$U_y = -V_x$. CR eqns are not satisfied

hence

$f(z) = \bar{z}$ is not analytic everywhere.

Remark: Ent:-

* If a fn contains $\bar{z}$ the fn is not analytic.

* If a complex function contains the term $\bar{z}$ it is not analytic.

eg: $\dfrac{\sin z\bar{z}}{e^z\cos z + \sin z + z^k}$

Q.29. Give an example of a complex fn which is continuous but not differentiable. (analytic)

Imp Note

$\bar{z}$.

$f(z) = \bar{z} = x - iy$.

it is not differentiable (not analytic)
[proof: Q 28.]

But: $\bar{z} = x - iy$ is a continuous fn since the real part x and the imaginary part $-y$ are continuous being polynomials.

$\therefore$ $\bar{z}$ is not analytic but it is continuous.

Q30. Check whether $f(z) = \log z$ is analytic.

V. Imp

$u + iv = \log(x + iy)$

$e^u e^{iv} = x + iy$

$e^u(\cos v + i\sin v) = x + iy$

$x = e^u\cos v$ —— ①

$y = e^u\sin v$ —— ②

$①^2 + ②^2 \Rightarrow x^2 + y^2 = e^{2u}$

$x^2 + y^2 = e^{2u}$

$u = \frac{1}{2}\log(x^2 + y^2)$

② ÷ ① $\Rightarrow \tan v = y/x$

$v = \tan^{-1}(y/x)$

$u = \frac{1}{2}\log(x^2 + y^2)$ $\qquad v = \tan^{-1}(y/x)$

$U_x = \frac{1}{2(x^2+y^2)} \times 2x = \frac{x}{x^2+y^2}$ $\qquad V_x = \frac{-y}{x^2+y^2}$

$U_y = \frac{1}{2(x^2+y^2)} \times 2y = \frac{y}{x^2+y^2}$ $\qquad V_y = \frac{x}{x^2+y^2}$

$\frac{x^2}{x^2+y^2} \times \frac{-1}{x^2}$

$\frac{-1}{1+y^2/x^2}$

$\frac{1}{1+y^2/x^2}$

$\frac{y^2}{x^2+y^2}$

$U_x = V_y$

$U_y = -V_x$ hence the C.R eqns are satisfied.

At origin ~~these~~ ~~fox~~ partial derivatives are not continuous:

i.e $z = 0$.

Not continuous at $z = 0$.

$\log z$ is analytic except at $z=0$

※ Note:

Q:31 $f(z) = Re(z^2) - Im(z^2)$

$z^2 = x^2 - y^2 + 2ixy$.

$f(z) = x^2 - y^2 - 2xy$.

$U = x^2 - y^2 - 2xy$ $\quad V = 0$

$U_x = 2x - 2y$. $\quad V_x = 0$

$U_y = -2y - 2x$ $\quad V_y = 0$

$U_x = V_y$

$U_y = -V_x$ only if $x = y = 0$

is the above case partial derivatives are also continuous therefore the ~~f(z)~~ given fn is ~~conti~~ analytic only at origin $(z=0)$

Remark:

Analytic only at zero $\longrightarrow Re(z^2) - Im(z^2)$

Not Analytic only at zero $\longrightarrow 1/z$

Q.32 $f(z) = Re(z^2) - i\,Im(z^2)$

$f(z) = x^2 - y^2 - i2xy$.

$U = x^2 - y^2$ $\quad V = -2xy$

$U_x = 2x$ $\quad V_x = -2y$

$U_y = -2y$ $\quad V_y = -2x$.

Only at origin; $U_x = V_y$

$U_y = -V_x$.

if $x = y = 0$.

is the above case partial derivatives are also continuous therefore the given fn is analytic only at origin $(z=0)$.

Q33. Find the constant a and b for which $f(z) = (x+ay) + i(2x+by)$ is analytic.

$U = x + ay$ $\quad V = 2x + by$

$U_x = 1$ $\quad V_x = 2$

$U_y = a$ $\quad V_y = b$.

$U_x = V_y$

$U_y = -V_x$

~~thus~~ $a = -2$, $b = 1$

due

Q34 Check whether $\sinh z$ and $\cosh z$ are entire function.

$f z = \sinh z$.

$\sin i\theta = i \sinh\theta$.

$\sinh\theta = -i \sin i\theta$.

$-i \sin(i(x+iy))$

$= -i \sin(xi - y)$

$= -i(\sin ix \cos y - \cos ix \sin y)$

$= -i \sin ix \cos y + i \cos ix \sin y$

$= -i \times i \sinh x \cos y + i \cosh x \sin y$

$= \sinh x \cos y + i \cosh x \sin y$

$U = \sinh x \cos y$ $\quad V = \cosh x \sin y$

$U(x) = \cosh x \cos y$ $\quad V(x) = \sinh x \sin y$

$U(y) = -\sinh x \sin y$ $\quad V_y = \cosh x \cos y$

$\Rightarrow U(x) = V(y)$

$U(y) = -V(x)$ hence CR eqns are satisfied.

The partial derivatives are also continuous.

hence $f(z) = \sinh z$ is analytic function

due

Q.35. $f(z) = \cosh z$

$\cos i\theta = \cosh\theta$.

$\cosh z = \cos(i(x+iy))$

$= \cos(ix + i^2 y)$

$= \cos(ix - y)$

$\cos ix \cos y + \sin ix \sin y$

$\cosh x \cos y + i \sinh x \sin y$

$U = \cosh x \cos y$ $\quad V = \sinh x \sin y$

$U_x = \sinh x \cos y$ $\quad V_x = \cosh x \sin y$

$U_y = -\cosh x \sin$ $\quad V_y = \sinh x \cos y$.

$U_x = V_y$

$U_y = -V_x$ hence CR eqns are satisfied

The partial derivatives are also continuous

hence; $f(z) = \cosh z$ is an analytic function.

hence both are entire functions.

Remark:

If a function $f(z)$ fails to be analytic at a point z_0 then z_0 is called a singular point.

Q.36 Find the singular points.

i) $f z = 1/z$

$z = 0$ (singular pts)

ii) $f(z) = e^z$

no singular fn.

iii) $f(z) = \tan z$

$= \frac{\sin z}{\cos z}$

$z = (2n+1)\pi/2$.

(iv) $f(z) = \log z$

$z = 0$ (singular point)

v) $f(z) = \cot z$

$= \frac{\cos z}{\sin z} \Rightarrow \underline{\underline{z = n\pi}}$

* $\sinh z, \sin z, \cos z, \cosh z \longrightarrow$ no singular pt.

$\bar{z} \longrightarrow$ all pts are singular pts.

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Chapter - 4

$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$. → Laplace eqn.

Harmonic functions:

Laplace equations:

A real fn: u is said to follow laplace eqn: if

$$\boxed{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{or} \quad u_{xx} + u_{yy} = 0}$$

Harmonic fn:

A real fn: u is said to be harmonic if you follow laplace eqn.

Theorem:

If $f(z) = u + iv$ is analytic then both u and v are harmonic. v is called the harmonic conjugate of u.

u & v satisfies Laplace eqns.

$\boxed{\text{Im}(z) \text{ harmonic conjugate of } \text{Re}(z).}$

Example:

Consider $f(z) = z^2$.

$= x^2 - y^2 + 2ixy$.

We know that z^2 is an entire fn being a polynomial. All polynomials are entire fns:

$u = x^2 - y^2$ $\quad$ $v = 2xy$

$u_x = 2x$ $\quad$ $v_x = 2y$

$U_{xx} = 2$ $\qquad V_{xx} = 0$

$U_y = -2y$ $\qquad V_y = 2x$

$U_{yy} = -2$ $\qquad V_{yy} = 0$

$U_{xx} + U_{yy} = 0$. $\qquad V_{xx} + V_{yy} = 0$

$\Rightarrow$ U satisfies laplace eqn $\Rightarrow$ V satisfies laplace eqn

$\Rightarrow$ U is a harmonic fn. $\Rightarrow$ V is a harmonic fn.

ie $2xy$ is the harmonic conjugate of $x^2 - y^2$

Methods to find harmonic conjugate & corresponding analytic fn:

(*1) **Method 1: Using CR eqn**

Q.37 ST $U = x^2 - y^2 - y$ is harmonic & then find the harmonic conjugate & the corresponding analytic fn $f(z)$.

$U = x^2 - y^2 - y$

$U_x = 2x$

$U_{xx} = 2$

$U_y = -2y - 1$

$U_{yy} = -2$.

$U_{xx} + U_{yy} = 2 - 2 = 0$

$U_{xx} + U_{yy} = 0$

$\Rightarrow$ U satisfies laplace eqn

$\Rightarrow$ U is a harmonic fn.

Given; $f(z)$ is analytic.

Let $f(z) = u + iv$.

Since $f(z)$ is analytic

$U_x = V_y \qquad U_y = -V_x$.

$V_y = 2x$. $\qquad V_x = 2y+1$

$V_x = 2y+1$ —— ①

$V_y = 2x$ —— ②.

Int. ② wrt y.

$V = 2xy + f(x)$ —— ③.

diff: ③ wrt x.

$V_x = 2y + f'(x)$ —— ④.

Comp: ① & ④.

$2y + 1 = 2y + f'(x)$

$f'(x) = 1$

$f(x) = x + c$

$\therefore$ ③ $\Rightarrow$ $\boxed{V = 2xy + x + c}$ which is the harmonic conjugate of u.

Now we have to find;

$f(z) = U + iV$

$= x^2 - y^2 - y + i(2xy + x + c)$

$= x^2-y^2+2ixy - y + ix + ic$

$= z^2 + i(x+iy) + ic$

$= z^2 + iz + ic$

$= z^2 + i(z+c)$

(*2) Method 2: Milne – Thomson Method.

$$\boxed{f'(z) = U_x + iV_x}$$

$U_x = 2x$

$V_x = 2y+1$

$f'(z) = 2x + i(2y+1)$

In MT method we put $x=z$ and $y=0$

$f'(z) = 2z + i$

$f(z) = z^2 + iz + ic$

Put $z = x+iy$.

$f(x+iy) = x^2-y^2+2ixy + ix - y + ic$

$= x^2-y^2-y+i(2xy+x+c)$

Q.38 Are the following fns harmonic if so find the corresponding analytic fn, the real or Im part.

$U = x^3+y^3$.

$U_x = 3x^2$

$U_{xx} = 6x$.

$U_y = 3y^2$

$U_{yy} = 6y$.

$U_{xx} + U_{yy} = 6x+6y \neq 0$.

hence U does not satisfy Laplace eqn.

hence U is not harmonic.

Q39. $U = -2xy$.

$U_x = -2y$

$U_{xx} = 0$

$U_y = -2x$

$U_{yy} = 0$.

$U_{xx}+U_{yy} = 0$.

hence U satisfies Laplace eqn

U is harmonic.

$U_y = -V_x$

Given f z is analytic $U_x = -2y$

$V_x = 2x$.

$f(z) = U_x + iV_x$ using MTMethod.

$= -2y + i2x$.

put $x=z$ and $y=0$

$\Rightarrow f'(z) = 2iz$.

$f(z) = z^2 i + ic$

Put $z = x+iy$

$f(x+iy) = i(x^2+2ixy-y^2)+ic$

$= ix^2 - 2xy - iy^2 + ic$

$\Rightarrow -2xy + i(x^2-y^2+c)$

$V = x^2-y^2+c$ which is the harmonic conjugate of $U = -2xy$

Q.40 $V = xy$.

$V_x = y$.

$V_{xx} = 0$ $\qquad V_{xx}+V_{yy} = 0$.

$V_y = x$ $\qquad \Rightarrow$ V is satisfies Laplace eqn

$V_{yy} = 0$. $\qquad \Rightarrow$ V is a harmonic fn.

Given f_z is analytic

$U_x = V_y$

$U_y = -V_x$.

$\Rightarrow U_x = x$

$V_x = y$.

$f'(z) = U_x + iV_x$ using MT Method.

$= x+iy$.

Put $x = z$ $y=0$.

$f(z) = x+iy = z$.

$f(z) = z^2/2 + c$.

put $z = x+iy$.

$\Rightarrow \dfrac{x^2-y^2+i2xy+2c}{2}$.

$= \dfrac{x^2-y^2+2c}{2} + ixy$.

$\Rightarrow U = \dfrac{x^2-y^2+2c}{2}$

Sol
Q.41 $U = x^3-3xy^2$

$U_x = 3x^2-3y^2$

$U_{xx} = 6x$.

$U_y = -6xy$ $\qquad U_{xx}+U_{yy} = 6x-6x$

$U_{yy} = -6x$ $\qquad = 0$.

$\Rightarrow$ U satisfies Laplace eqn.

$\Rightarrow$ U is a harmonic fn.

F(z) is analytic

$U_x = V_y$ $\qquad U_x = 3x^2-3y^2$

$U_y = -V_x$. $\qquad V_x = 6xy$.

$f'(z) = U_x + iV_x$ using MT Method.

$= 3x^2-3y^2+i6xy$.

put $x = z$ & $y=0$.

$f'(z) = 3z^2$

$f(z) = \frac{3z^3}{3} = z^3 + ic$

Put $z = x+iy$..

$f(z) = (x+iy)^3 + ic$

$= x^3 + i^3y^3 + 3x^2iy + 3xi^2y^2 + ic$

$= x^3 - iy^3 + 3ix^2y - 3xy^2 + ic$

$= (x^3 - 3xy^2) + i(3x^2y - y^3 + c)$

$V = 3x^2y - y^3 + c$ which is the harmonic conjugate of $u = x^3 - 3xy^2$

Ans
Q.42 $u = e^{-x}(x\cos y + y\sin y)$

$u_x = e^{-x} \times x\cos y + e^{-x} y\sin y$.

$U_x = \cos y(xe^{-x} \times -1 + e^{-x}) + y\sin y e^{-x} \times -1$

$U_x = -xe^{-x}\cos y + e^{-x}\cos y - e^{-x} y\sin y$

~~U_y~~

$U_{xx} = -\cos y(-xe^{-x} + e^{-x}) + \cos y(-e^{-x}) - y\sin y \times -e^{-x}$

$= xe^{-x}\cos y - e^{-x}\cos y - e^{-x}\cos y + e^{-x}y\sin y$

$= xe^{-x}\cos y + e^{-x}y\sin y - 2e^{-x}\cos y$

$U_y = -xe^{-x} \times \sin y + e^{-x}(y\cos y + \sin y)$

$= -xe^{-x}\sin y + e^{-x}y\cos y + e^{-x}\sin y$.

$U_{yy} = -xe^{-x}\cos y + e^{-x}y \times -\sin y + e^{-x}\cos y + e^{-x}\cos y$.

$= -xe^{-x}\cos y - ye^{-x}\sin y + 2e^{-x}\cos y$

$U_{xx} + U_{yy} \Rightarrow$

$\cancel{xe^{-x}\cos y} + \cancel{e^{-x}y\sin y} - \cancel{2e^{-x}\cos y} + -\cancel{xe^{-x}\cos y} - \cancel{ye^{-x}\sin y} + \cancel{2e^{-x}\cos y} = 0$.

$\Rightarrow$ U satisfies Laplace eqn.

$\Rightarrow$ U is a harmonic fn.

f(z) is an by CR eqns:

$U_x = V_y$
$U_y = -V_x$

$U_x = -xe^{-x}\cos y + e^{-x}\cos y - ye^{-x}\sin y$

$V_x = xe^{-x}\sin y - e^{-x}y\cos y - e^{-x}\sin y$

$U_x + iV_x = f'(z)$ (M T Method)

$\Rightarrow -xe^{-x}\cos y + e^{-x}\cos y - ye^{-x}\sin y + i(xe^{-x}\sin y - e^{-x}y\cos y - e^{-x}\sin y)$

Put $x = z$ $y = 0$.

$f'(z) = 3z^2$

$f(z) = \frac{3z^3}{3} = z^3 + ic$

Put $z = x + iy$.

$f(z) = (x+iy)^3 + ic$

$= x^3 + i^3y^3 + 3x^2iy + 3xi^2y^2 + ic$

$= x^3 - iy^3 + 3ix^2y - 3xy^2 + ic$

$= (x^3 - 3xy^2) + i(3x^2y - y^3 + c)$

$V = 3x^2y - y^3 + c$ which is the harmonic conjugate of $u = x^3 - 3xy^2$

Ans
Q.42 $u = e^{-x}(x\cos y + y\sin y)$

$u = e^{-x} x\cos y + e^{-x} y\sin y$

$u_x = \cos y(xe^{-x} \times -1 + e^{-x}) + y\sin y e^{-x} \times -1$

$u_x = -xe^{-x}\cos y + e^{-x}\cos y - e^{-x} y\sin y$

$u_{xx} = -\cos y(-xe^{-x} + e^{-x}) + \cos y(-e^{-x}) - y\sin y \times -e^{-x}$

$= xe^{-x}\cos y - e^{-x}\cos y - e^{-x}\cos y + e^{-x} y\sin y$

$= xe^{-x}\cos y + e^{-x}y\sin y - 2e^{-x}\cos y$

$u_y = -xe^{-x} \times \sin y + e^{-x}(y\cos y + \sin y)$

$= -xe^{-x}\sin y + e^{-x} y\cos y + e^{-x}\sin y$.

$u_{yy} = -xe^{-x}\cos y + e^{-x} y \times -\sin y + e^{-x}\cos y + e^{-x}\cos y$.

$= -xe^{-x}\cos y - ye^{-x}\sin y + 2e^{-x}\cos y$

$u_{xx} + u_{yy} \Rightarrow$

$\cancel{xe^{-x}\cos y} + \cancel{e^{-x}y\sin y} - \cancel{2e^{-x}\cos y} + \cancel{-xe^{-x}\cos y}$
$\cancel{-ye^{-x}\sin y} + \cancel{2e^{-x}\cos y} = 0$.

$\Rightarrow$ u satisfies Laplace eqn.

$\Rightarrow$ u is a harmonic fn.

f(z) is an by CR eqns:

$u_x = v_y$ $\quad$ $u_x = -xe^{-x}\cos y + e^{-x}\cos y - ye^{-x}\sin y$

$u_y = -v_x$ $\quad$ $v_x = xe^{-x}\sin y - e^{-x} y\cos y - e^{-x}\sin y$

$u_x + iv_x = f'(z)$ (MT Method)

$\Rightarrow -xe^{-x}\cos y + e^{-x}\cos y - ye^{-x}\sin y + i(xe^{-x}\sin y - e^{-x}y\cos y - e^{-x}\sin y)$

Put $x = z$ $\quad$ $y = 0$.

$\Rightarrow f'(z) = -ze^{-z} + e^{-z}$

$f(z) = \frac{\cancel{-1}\,z\cdot e^{-z}}{\cancel{-1}} + \cancel{e^{-z}} - \cancel{e^{-z}} + ic$

$f(z) = ze^{-z} + ic$

put $z = x+iy$.

$(x+iy)\, e^{-(x+iy)} + ic$

$(x+iy)\left[e^{-x}\cdot e^{-iy}\right] + ic$

$(x+iy)\left[e^{-x}(\cos y - i\sin y)\right] + ic$

$(x+iy)\left[e^{-x}\cos y - i e^{-x}\sin y\right] + ic$

$xe^{-x}\cos y - ixe^{-x}\sin y + iye^{-x}\cos y - i^2 ye^{-x}\sin y + ic$

$(xe^{-x}\cos y + ye^{-x}\sin y) + i(ye^{-x}\cos y - xe^{-x}\sin y)$

$\text{Im}(z) = V = ye^{-x}\cos y - xe^{-x}\sin y + c$

$ye^{-x}\cos y - xe^{-x}\sin y$ is the harmonic conjugate of u

Que Q43. $u = e^x \cos y$.

$u_x = e^x \cos y$

$u_{xx} = e^x \cos y$

$u_y = -e^x \sin y$

$u_{yy} = -e^x \cos y$.

$u_{xx} + u_{yy} = e^x\cos y - e^x \cos y = 0$.

$\Rightarrow$ u satisfies the Laplace eqn.

$\Rightarrow$ u is a harmonic fun.

By CR eqns:

$u_x = v_y \qquad u_x = e^x\cos y$

$u_y = -v_x \qquad v_x = e^x \sin y$.

By MT Method

$f'(z) = u_x + iv_x$.

$= e^x \cos y + i e^x \sin y$.

$\cancel{f(z)=}$ put $x = z$, $y = 0$

$f'(z) = e^z$

$f(z) = e^z + ic$

put $z = x+iy$.

$f(z) = e^{x+iy} + ic \Rightarrow e^x \cdot e^{iy} + ic$

$= e^x(\cos y + i\sin y) + ic$

$= e^x \cos y + ie^x \sin y + ic$

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$= e^x\cos y + i(e^x\sin y + c)$

$V = e^x\sin y + c$ which is the harmonic conjugate of $u = e^x\cos y$

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Determine the constant k so that the following fns: are harmonic find the corresponding analytic fn and real or Im part.

Q.44. $U = x^3 + kxy^2$.

$U_x = 3x^2 + ky^2$

$U_{xx} = 6x$.

$U_y = 2kxy$.

$U_{yy} = 2kx$

Given U is harmonic

$\Rightarrow U_{xx} + U_{yy} = 0$.

$6x + 2kx = 0$.

$2kx = -6x$

$\underline{\underline{k = -3}}$

$U = x^3 - 3xy^2$.

$U_x = 3x^2 - 3y^2$

$U_y = -6xy$.

$U_y = -V_x$

$V_x = 6xy$.

$f(z) = U_x + iV_x$. [By MT Method]

$= 3x^2 - 3y^2 + i6xy$.

put $x = z$ $y = 0$

$f'(z) = 3z^2$

$f(z) = \frac{3z^3}{3} + ic = \underline{\underline{z^3 + ic}}$.

put $z = x + iy$.

$f(z) = (x+iy)^3 + ic$.

$= (x^3 - 3xy^2) + i(3x^2y - y^3 + c)$

$\Rightarrow V = 3x^2y - y^3 + c$ which is the harmonic conjugate of $u = x^3 - 3xy^2$.

$V = \mathrm{Im}(z) = 3x^2y - y^3 + c$

$u = \mathrm{Re}(z) = x^3 - 3xy^2$

Q.45. $V = xy + kx^2 - y^2$

V is harmonic

$\Rightarrow V_{xx} + V_{yy} = 0$.

$2k - 2 = 0$.

$2 = 2k$

$\underline{\underline{k = 1}}$

$V_x = y + 2xk$

$V_{xx} = 2k$

$V_y = x - 2y$

$V_{yy} = -2$

$V = xy + x^2 - y^2$

$V_x = y + 2x$

$u_x = v_y = x-2y$

$-v_x = u_y = -(y+2x)$

$= -2x-y$

$f'(z) = u_x + i v_x$. By MT Method.

$f'(z) = (x-2y) + i(-2x-y)$

put $x = z$ & $y = 0$

$f'(z) = z + -i2z$

$= \underline{\underline{z - i2z}}$

$f(z) = \frac{z^2}{2} + iz^2 + c$

put $z = x+iy$.

$f(z) = \frac{x^2-y^2+i2xy}{2} + i(x^2-y^2+i2xy) + c$

$= \frac{x^2-y^2}{2} - 2xy + c + i(xy + x^2 - y^2)$

$Re(z) = u = \underline{\underline{\frac{x^2-y^2}{2}}} - 2xy + c$

Q-46. $u = \cos kx \cosh 2y$.

Given u is a harmonic fn

$\Rightarrow u_{xx} + u_{yy} = 0$

$u_x = -k \sin kx \cosh 2y$.

$u_{xx} = -k^2 \cos kx \cosh 2y$.

$u_y = \cos kx \sinh 2y \times 2$.

$u_{yy} = 4 \cos kx \cosh 2y$.

$-k^2 \cos kx \cosh 2y + 4 \cos kx \cosh 2y = 0$.

$4 \cancel{\cos kx} \cosh 2y = k^2 \cos kx \cancel{\cosh 2y}$

$\underline{\underline{k = \pm 2}}$.

$u_x = -k \sin kx \cosh 2y$.

$v_x = -2 \cos kx \sinh 2y$

$f'(z) = -k \sin kx \cosh 2y \mp i\, 2 \cos kx \sinh 2y$

put $x = z$ $y = 0$. (By MT Method)

$f'(z) = -2 \sin 2z$

$f(z) = -2 \times -\frac{\cos 2z}{2} = \cos 2z + ic$

Put $z = x+iy$.

$\cos 2(x+iy) + ic$

$= \cos 2x \cos i2y - \sin 2x \sin i2y + ic$

$= \cos 2x \cosh 2y - i \sin 2x \sinh 2y + ic$

Scanned by CamScanner

$V = c - \sin 2x \sinh 2y$

Q.47. $U = kx^3 + 2xy$

$U_x = 3kx^2 + 2y$

$U_{xx} = 6kx$

$U_y = 2x$

$U_{yy} = 0$ Given U is a harmonic fn.

$U_{xx} + U_{yy} = 0$.

$\Rightarrow 6kx = 0$.

$k = 0$.

$U = 2xy$.

$U_x = \cancel{3k} 2y$.

$V_x = -2x$

$f'(z) = U_x + iV_x$ (By MT Method)

$f'(z) = 2y - i2x$.

Put $y = 0$ $x = z$.

$\Rightarrow f'(z) = -i2z$.

$f(z) = -i \frac{2z^2}{2} = -iz^2 + ic$

Put $z = x+iy$.

$f(z) = -i(x^2 - y^2 + i2xy) + ic$

$f(z) = -ix^2 + iy^2 \cancel{+} -i^2 2xy + ic$

$= -ix^2 + iy^2 + 2xy + ic$.

$= 2xy + i(y^2 - x^2 + c)$.

$V = y^2 - x^2 + c$ which is the harmonic conjugate of u

V.Imp

Q.48 Determine the analytic fn: $f(z) = u + iv$ $u - v = e^x[\cos y - \sin y]$ Also find the Real & Im. part

Given;

$U - V = e^x[\cos y - \sin y]$ ——— ①

diff: ① partially wrt x.

$U_x - V_x = e^x(\cos y - \sin y)$

$U_x - V_x = e^x \cos y - e^x \sin y$ ——— ②.

diff: ① partially wrt y

$U_y - V_y = e^x x - \sin y - e^x \cos y$

$= -e^x \sin y - e^x \cos y$ ——— ③.

③ $\Rightarrow$

$-V_x - U_x = -e^x \sin y - e^x \cos y$ ——— ④

② + ④.

$-2V_x = -2e^x \sin y$

$V_x = e^x \sin y$.

② − ④

$2U_x = 2e^x \cos y \Rightarrow U_x = e^x \cos y$

$f'(z) = U_x + i V_x$ By MT Method.

$= e^x \cos y + i e^x \sin y$

put $z = z$ $y = 0$.

$f'(z) = e^z$

$f(z) = e^z + c$.

put $z = x+iy$

$f(z) = e^x \cdot e^{iy} + c$

$= e^x (\cos y + i \sin y) + c$

$= e^x \cos y + c + i e^x \sin y$.

$Re(z) = U = e^x \cos y + c$

$Im(z) = V = e^x \sin y$

due

Q.49. Find k if $u = \cos y \cosh kx$ is harmonic.

Find $f(z)$ and V.

$u = \cos y \cosh kx$

$U_x = \cos y \sinh kx \times k$

$U_{xx} = \cos y \cosh kx \times k^2$

$u_y = -\sin y \cosh kx$

$U_{yy} = -\cos y \cosh kx$

Given U is harmonic

hence; $U_{xx} + U_{yy} = 0$

$k^2 \cos y \cosh kx = \cos y \cosh kx$

$k = \pm 1$.

$u = \cos y \cosh x$.

$u_x = \cos y \sinh x$.

$u_y = -V_x \Rightarrow V_x = \sin y \cosh x$.

$f'(z) = U_x + i V_x$ By MT Method.

$f'(z) = \cos y \sinh x + i \sin y \cosh x$.

put $x = z$ $y = 0$.

$f'(z) = \sinh z$.

$f z = \cosh z + ic$

put $z = x+iy$

$f(z) = \cosh(x+iy) + ic$

$\cosh(x+iy) = \cos i(x+iy)$

$= \cos(ix - y)$

$f(z) = \cos ix \cos y + \sin ix \sin y + ic$

$= \cos y \cosh x + i \sin y \sinh x + ic$

$= \cos y \cosh x + i(\sin y \sinh x + c)$

$V = \sin y \sinh x + c$ is the harmonic conjugate of u.

Q.50. Find k if $u = \cos yk \; e^{-\pi x}$ is harmonic. find $f(z)$ and v.

$u = \cos yk \; e^{-\pi x}$

$u_x = e^{-\pi x} \times -\pi \cos yk$

$u_{xx} = -\pi \cos yk \left(-\pi e^{-\pi x}\right)$

$u_y = e^{-\pi x} \times -\sin yk \times k$.

$u_{yy} = -e^{-\pi x} \cos yk \times k^2$.

u is harmonic

hence; $u_{xx} + u_{yy} = 0$

$\Rightarrow \pi^2 e^{-\pi x} \cancel{\cos yk} = e^{-\pi x} \cancel{\cos yk}\, k^2$

$\underline{\underline{k = \pi}}$

$u_x = -\pi e^{-\pi x} \cos y\pi$

$v_x = \pi e^{-\pi x} \sin y \pi$

$\Rightarrow f'(z) = u_x + i v_x$ By MTM method

put $x = z$ & $y = 0$

$f'(z) = -\pi e^{-\pi x} \cos y\pi + i \left(\pi e^{-\pi x} \sin y \pi \right)$

$= \underline{\underline{-\pi e^{-\pi z}}}$.

$f(z) = \frac{-\pi \, e^{-\pi z}}{-\pi} + ic$

$= e^{-\pi z} + ic$.

put $z = x + iy$.

$\Rightarrow f(z) = e^{-\pi(x+iy)} + ic$

$= e^{-\pi x} \cdot e^{-i\pi y} + ic$

$= e^{-\pi x} \left(\cos \pi y - i \sin \pi y \right) + ic$.

$= e^{-\pi x} \cos \pi y - i e^{-\pi x} \sin \pi y + ic$.

$= e^{-\pi x} \cos \pi y + i \left(c - e^{-\pi x} \sin \pi y \right)$.

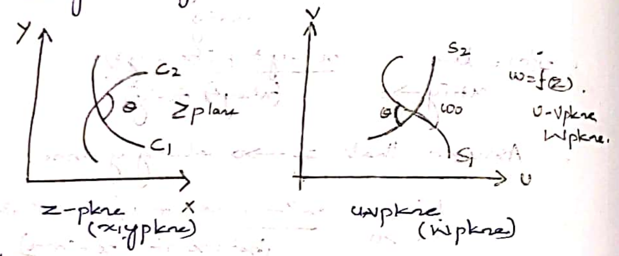
$\Rightarrow v = c - e^{-\pi x} \sin \pi y$ which is the harmonic $\underline{\underline{\text{conjugate}}}$ of u.

Chapter 5

7/9/2020 **Conformal Mapping:**

Consider the mapping $w = f(z)$

Let c_1 and c_2 be two curves in the z-plane making an angle θ b/w them at z_0



Now consider the images S_1 and S_2 of c_1 and c_2 in the w-plane and under the mapping $w = f(z)$. Now $w = f(z)$ conformal mapping If it preserves angles b/w oriented curves in magnitude as well as direction.

Remark:

→ Every analytic function $w = f(z)$ is conformal at z_0 if $f'(z_0) \neq 0$.

→ Constant functions are analytic everywhere but not conformal at any points.

→ conformal mapping preserves angles.

derivative of constant fns are zero.

eg: $2+i$

Critical Points:

Let $f(z)$ be a non-constant analytic fn. A point z_0 in a complete plane is called critical point of $f(z)$ if $f'(z_0) = 0$.

* Critical points are points at which an analytic fn is not conformal.

z^2 → analytic & conformal at every pt.

eg: of an analytic but not conformal.

$2+i$

51. Find the pts at which the following fns are not conformal. (find the critical pts)

$w = f(z) = z^2$.

$f'(z) = 2z$.

$f'(z) = 0$.

$\Rightarrow z = 0$.

hence zero is the critical pt.

At $z = 0$ the fn is not conformal.

z^2 is analytic everywhere but not conformal at the origin but conformal everywhere else.

52. $w = f(z) = \cos z$.

$f'(z) = -\sin z$

$f'(z) = 0$.

$\Rightarrow -\sin z = 0$.

$\sin z = 0$ $\qquad z = n\pi$ $(n = 0, \pm 1, \pm 2 \ldots)$

Analytic but not conformal (when $z = n\pi, \ldots$) at infinitely many pts.

53. $f(z) = e^{z}$.

$f'(z) = e^{z}$.

$f'(z) = 0 \Rightarrow e^{z} \neq 0$ for any value of z.

$\therefore$ e^{z} is analytic & conformal everywhere.

No critical pts. it is an entire fn.

Some elementary transformations:

I $W = z^2$

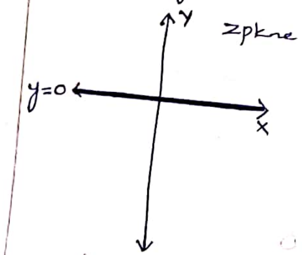
$$\boxed{\begin{aligned} u = R(z) &= x^2 - y^2 \\ v = Im(z) &= 2xy \end{aligned}}$$

$u + iv = x^2 - y^2 + i2xy$.

Find the images of the following curves under the mapping $w = z^2$:

54. The real axis in the zplane.

The real axis of the zplane is the x axis i.e $y = 0$



for $w = z^2$

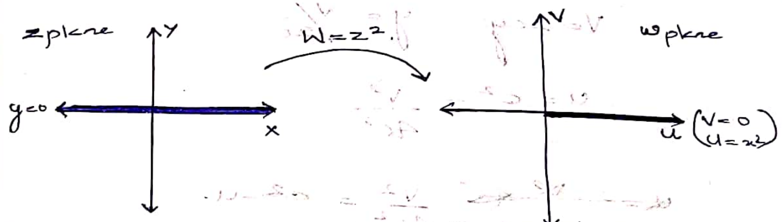
$u = x^2 - y^2$

$v = 2xy$.

$u = x^2$

$v = 0$ (u axis)

We have $v = 0 \Rightarrow u$ axis and it is mentioned $u = x^2 \Rightarrow$ the values are +ve. hence the image of x axis is the +ve u axis.



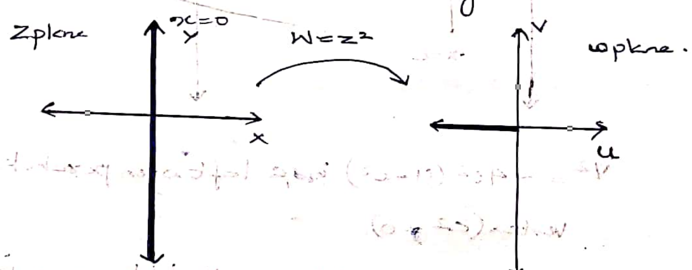
The image of x axis (real axis) in the zplane is +ve u axis of the w plane under the mapping $w = z^2$.

55. The imaginary axis of the zplane.

y axis. $\Rightarrow x = 0$.

for $w = z^2$ $\quad u = x^2 - y^2 \quad v = 2xy$.

$u = -y^2 \quad v = 0$.



The image of the imaginary axis in the zplane is the $-$ve real axis of the wplane under the mapping $w = z^2$.

56. Vertical line $x = c$ in the zplane.

$u = x^2 - y^2$ $\quad v = 2xy$

$u = c^2 - y^2$ $\quad v = 2cy$

to make ~~the~~ ~~terms~~ in terms of (u,v)

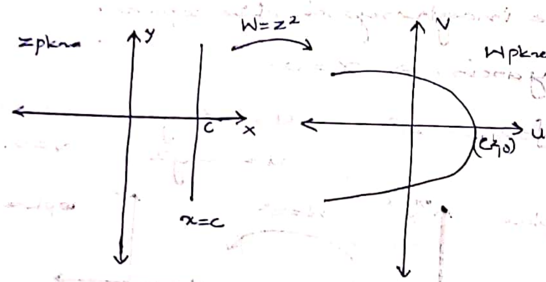
$v = 2cy \quad y = v/2c$.

$u = c^2 - \frac{v^2}{4c^2}$

$\cancel{u = \frac{v^2}{4c^2}}$ $\quad \frac{v^2}{4c^2} = c^2 - u$.

$v^2 = 4c^2(c^2 - u)$

$v^2 = -4c^2(u - c^2)$ $\quad$ Parabola of the form $y^2 = -4ax$.



$v^2 = -4c^2(u - c^2)$ is a left open parabola with vertex $(c^2, 0)$.

ie the image of the vertical line $x = c$ in the z plane is a left open parabola with vertex $(c^2, 0)$ in the w plane.

57 Horizontal line $y = c$, in the z plane.

$u = x^2 - y^2 \quad , v = 2xy$

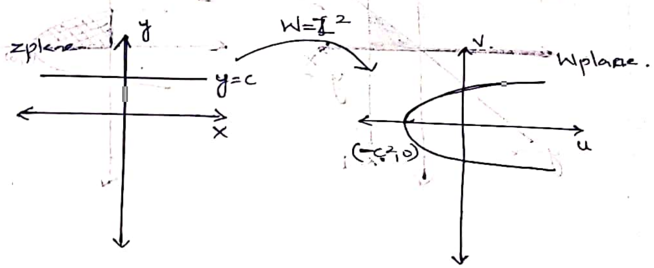
$u = x^2 - c^2 \quad v = 2cx$

$x = v/2c$.

$u = \frac{v^2}{4c^2} - c^2$.

$v^2 = 4c^2(u + c^2)$ $(y^2 = 4ax)$

$(y-b)^2 = 4(x-a)^2$
(a,b)



$v^2 = 4c^2(u + c^2)$ is a right open parabola with vertex $(-c^2, 0)$

ie the image of the horizontal line $y = c$ in the z plane is a right open parabola with vertex $(-c^2, 0)$ in the w plane.

due

58 Interior of triangle bounded by $x = 1, y = 0, y = x$.

$x = 1 \quad y = 0 \quad y = x$.

$u = x^2 - y^2 \quad y = 0$

$v = 2xy \quad \Rightarrow u = x^2$

$v = 2y \; (y = v/2)$ $\quad v = 0$.

hence it is the +ve real axis. (+u axis)

$u = \cancel{x} 1 - \frac{v^2}{4}$

$y^2 = -4ax$.

$v^2 = -4(u - 1)$.

Vertex $= (1, 0)$

$y = x$.

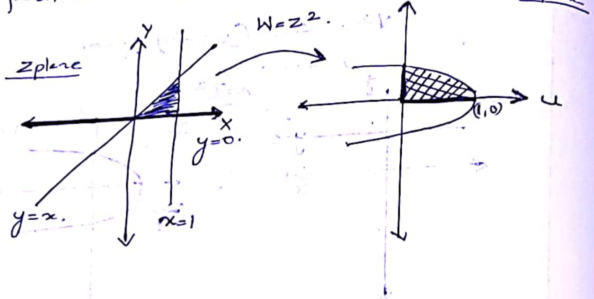
$u = x^2 - x^2 = 0$.

$v = \cancel{2y}\, 2x^2$.

$u = 0$ (v axis)

$v = 2x^2$ (+ve v axis)

hence the image will be a left open parabola with vertex (1,0) bounded by the positive u and v axis.



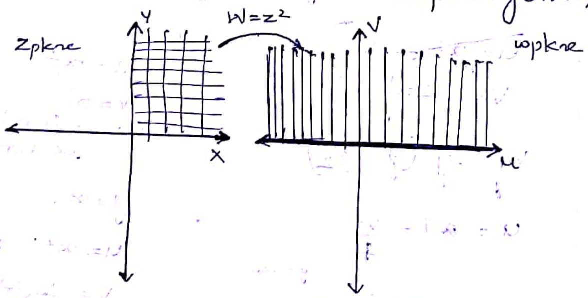
Q59. 1st Quadrant of z plane.

z plane 1st Quadrant $\Rightarrow x \geq 0 \; y \geq 0$.

$u = x^2 - y^2 \quad v = 2xy$.

$x \geq 0$ & $y \geq 0 \Rightarrow v \geq 0$.

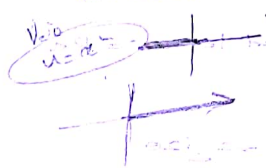
$u = x^2 - y^2$ (hence it can be +ve real as -ve real axis depending on x & y).



Q60 Triangle bounded by $x=1, y=1 \; x+y=1$.

$x=1$; $u = x^2 - y^2 = 1 - y^2$

$v = 2y \quad y = v/2$



$u = 1 - \frac{v^2}{4}$.

$v^2 = 4(1-u)$

$v^2 = -4(u-1)$.

vertex (1,0) . left open P.

$y=1$; $u = x^2 - y^2 = x^2 - 1$

$v = 2x$. $\quad x = v/2$.

$u = \frac{v^2}{4} - 1$

$\frac{v^2}{4} = u+1 \quad v^2 = 4(u+1)$.

vertex (-1,0) right open P.

$x+y=1$

$x = 1-y$. $\quad u = x^2 - y^2$

$u = (1-y)^2 - y^2$

$= 1 + \cancel{y^2} - \cancel{y^2} - 2y$.

$= 1 - 2y$

$v = 2xy = 2(1-y)y$

$= 2y - 2y^2$

$v = 2y - 2y^2$.

$y = \frac{(1-u)}{2}$ $\quad v = \frac{2(1-u)}{2} - \frac{\cancel{2}(1-u)^2}{\cancel{4}\,2}$

$v = \frac{1-u^2}{2}$

$u^2 = -2(v - 1/2)$

Q1. Sketch the region bounded by the images of

8/9/2021 $|z| \le 1/2$, $-\pi/8 < \arg z < \pi/8$

when $w = z^2$

Take $z = re^{i\theta}$ and let $w = Re^{i\phi}$

Mapping $w = z^2$

$Re^{i\phi} = r^2 e^{i2\theta}$

Comparing:

$R = r^2$

$\phi = 2\theta$.

$W = z^2$ in polar coordinate

$\Rightarrow$ R becomes r^2

ϕ become 2θ.

$|z| \le 1/2 \Rightarrow r \le 1/2$.

$r^2 \le 1/4$

But $r^2 = R$.

$\Rightarrow R \le 1/4$

here $\arg z = \theta$;

$\Rightarrow -\pi/8 < \theta < \pi/8$

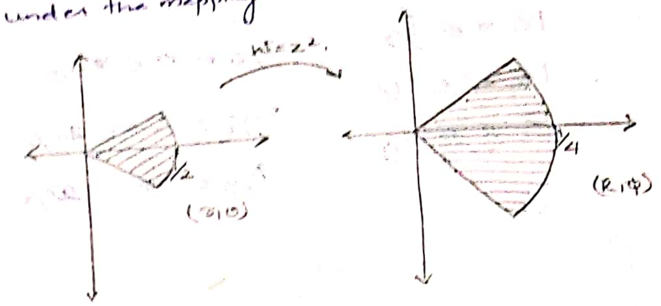
$-\pi/4 < 2\theta < \pi/4$

$-\pi/4 < \phi < \pi/4$

$|z| \le 1/2$ is mapped to $R \le 1/4$

$-\pi/8 < \arg z < \pi/8$ is mapped to $-\pi/4 < \phi < \pi/4$

under the mapping $w = z^2$.



Q2. $2 < |z| < 3$ & $|\arg z| < \pi/4$ with $w = z^2$.

Find the image of this region. $|\arg z| \Rightarrow \pm$ should add

$2 < r < 3$.

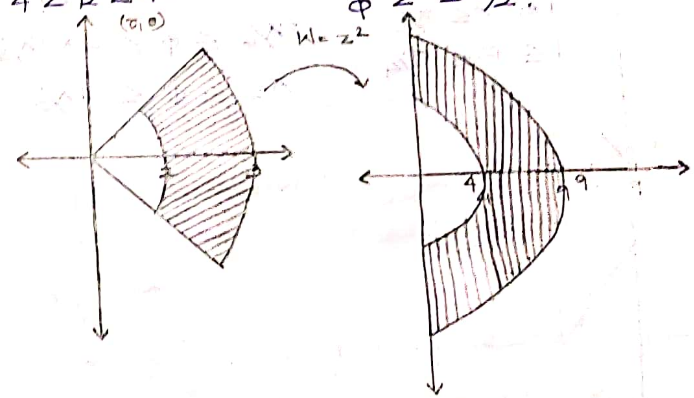
$4 < r^2 < 9$

$4 < R < 9$.

$\theta < \pm \pi/4$

$2\theta < \pm \pi/2$

$\phi < \pm \pi/2$.



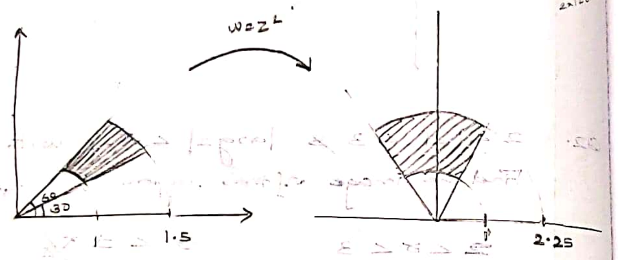
Q3. Sketch the region bounded by the images of $(\omega = z^2)$

$1 \le |z| \le 3/2 \quad , \quad \pi/6 \le \theta \le \cancel{2}\pi/3.$

$1 \le r \le 3/2 \qquad \pi/6 \le \theta \le \cancel{2}\pi/3.$

$1 \le r^2 \le 9/4 \qquad 2\pi/6 \le 2\theta \le \cancel{4}\pi/3.$

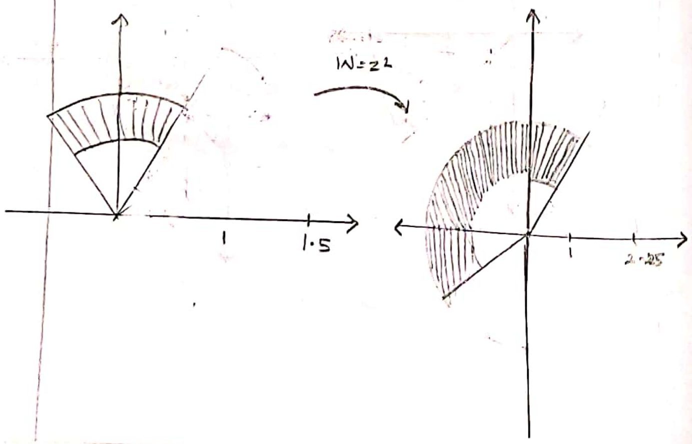
$1 \le R \le 9/4 \qquad \pi/3 \le \phi \le \cancel{2}\pi/3.$



Q4. $1 \le |z| \le 3/2 \qquad \pi/6 \le \theta \le 2\pi/3.$

$1 \le r \le 3/2 \qquad 2\pi/6 \le \phi \le 2 \times 2\pi/3$

$1 \le R \le 9/4 \qquad \pi/3 \le \phi \le 4\pi/3$



II

Transformation 2 : $\omega = 1/z$

$\omega = u + iv \qquad z = x + iy.$

$\omega = 1/z \qquad \Rightarrow z = 1/\omega = \dfrac{1}{u+iv} \times \dfrac{u-iv}{u-iv}$

$z = \dfrac{u - iv}{u^2+v^2}$

$$\boxed{x = \frac{u}{u^2+v^2} \qquad y = \frac{-v}{u^2+v^2}.}$$

Find the images of the following curves under the mapping $\omega = 1/z$:

Q5. $x = 1$:

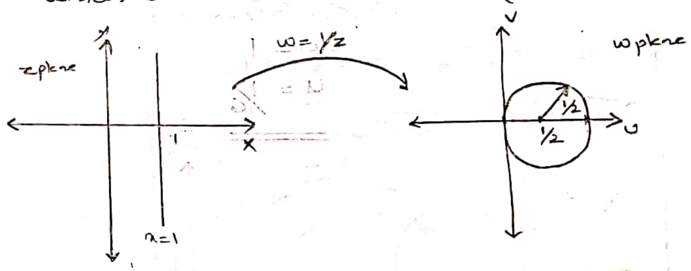
$x = \dfrac{u}{u^2+v^2} \Rightarrow 1 = \dfrac{u}{u^2+v^2} \qquad u^2+v^2-u = 0$

$u^2 - u + v^2 = 0. \qquad (x-h)^2 + (y-k)^2 = r^2.$

$u^2 - u + 1/4 + v^2 = 1/4$

$(u - 1/2)^2 + v^2 = 1/4$

which is a circle with centre $(1/2, 0)$ and $R = 1/2$



The image of the straight line $x = 1$ is a circle with centre $(1/2, 0)$ & radius $1/2$ under the mapping $\omega = 1/z$.

66. $x^2+y^2-6x=0.$

$x^2-6x+y^2=0$

$x^2-6x+9+y^2=9.$

$(x-3)^2+y^2=3^2.$

$\boxed{c\ (3,0)\quad R=3.}$

$\frac{u^2}{(u^2+v^2)^2}+\frac{v^2}{(u^2+v^2)^2}-\frac{6u}{u^2+v^2}=0.$

$\cancel{(u^2+v^2)-6}$

$(u^2+v^2)-6u(u^2+v^2)=0.$

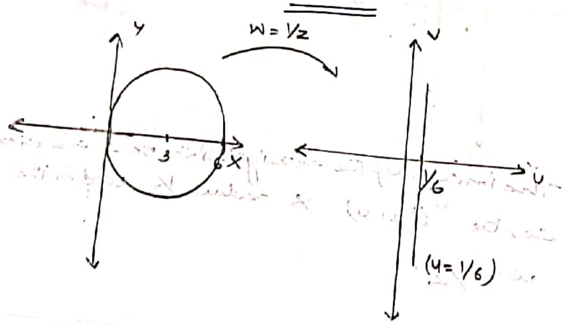
$(u^2+v^2)\ (1-6u)=0.$

$(u^2+v^2)\neq 0.$

hence; $1-6u=0.$

$6u=1$

$u=1/6$



67. $|z-2i|=2.$

Gen eq of a circle in complex plane $|z-a|=r.$

$z\to 2i\quad r\to 2.$

$|x+iy-2i|=2.$

$|x+i(y-2)|=2.$

$\sqrt{x^2+(y-2)^2}=2.$

$x^2+(y-2)^2=4.$

$x^2+y^2+4-4y=4.$

$x^2-4y+y^2=0.$

$x^2-4y+y^2=0.$

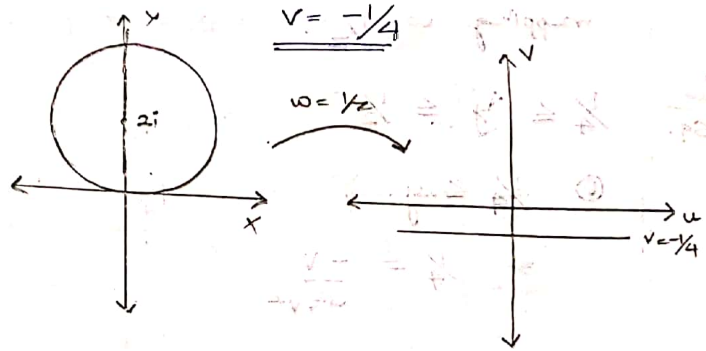
$\frac{u^2}{(u^2+v^2)^2}+\frac{4v}{u^2+v^2}+\frac{v^2}{(u^2+v^2)^2}=0$

$\frac{u^2+v^2}{(u^2+v^2)^2}+\frac{4v}{u^2+v^2}=0$

$1+4v\cancel{(u^2+v^2)}=0.$

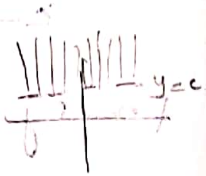
$1+4v=0$

$v=-1/4$



Ans
68. $y > c$

$y > c$



$\Rightarrow \dfrac{-v}{u^2+v^2} > c.$

$-v > c(u^2+v^2)$

$c(u^2+v^2) + v < 0.$

$u^2+v^2 + \cancel{v}/c < 0.$

$u^2 + v^2 + \dfrac{v}{c} + \dfrac{1}{4c^2} < \dfrac{1}{4c^2}.$

$u^2 + (v + 1/2c)^2 < 1/(2c)^2$

circle with centre $(0, -1/2c)$ radius $1/2c$

The image of $y>c$ will be a circle (interior), with centre $(0,-1/2c)$ and radius $1/2c$. under the mapping $w = 1/z$.

Ans
69. $1/4 \leq y \leq 1/2$

① $1/4 \leq y.$

$\Rightarrow 1/4 \leq \dfrac{-v}{u^2+v^2}.$

$u^2+v^2 \leq -4v.$

$u^2+v^2+4v \leq 0.$

$u^2+v^2+4v+4 \leq 4.$

$u^2+(v+2)^2 \leq 4.$

hence it is a circle with centre $(0,-2)$ and radius 2. (interior)

② $y \leq 1/2.$

$\dfrac{-v}{u^2+v^2} \leq 1/2.$

$-2v \leq u^2+v^2.$

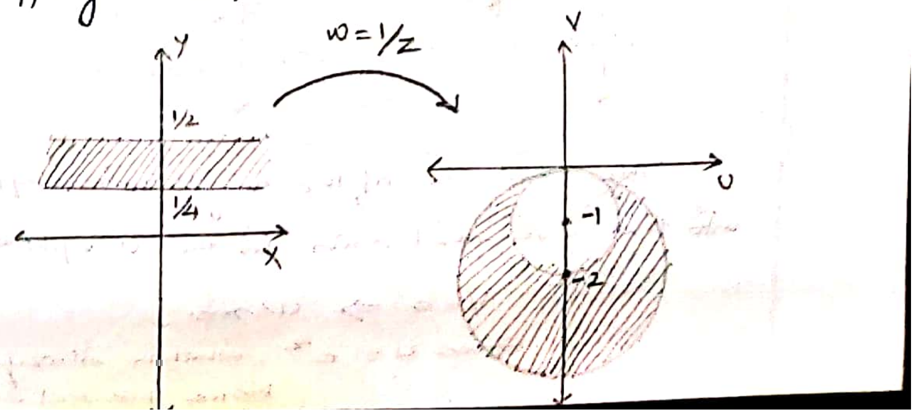
$u^2+v^2+2v \geq 0$

$u^2+v^2+2v+1 \geq 1.$

$u^2+(v+1)^2 \geq 1$

hence it is a circle interior with centre $(0,-1)$ and radius 1.

The image of the region between $1/4 \leq y \leq 1/2$ is the region between the circles $u^2+(v+2)^2 \leq 4$ & $u^2+(v+1)^2 \geq 1$ under the mapping $w = 1/z$.



III

11/9/20

$w = e^z$: Transformations 3

$w = u + iv \qquad z = x + iy$

$u + iv = e^{x+iy}$

$u = e^x \cos y$	$v = e^x \sin y$.

In polar form;

$w = Re^{i\phi} = e^{x+iy}$.

$\boxed{R = e^x, \; y = \phi}$

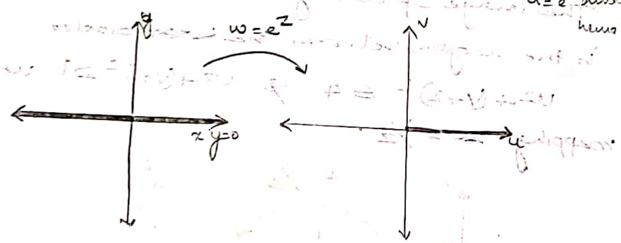
70. Find the image of the following curves under the mapping $w = e^z$:

Real axis $(y=0)$

$u = e^x \cos y = e^x$

$v = e^x \sin y = 0$.

$v = 0$ u axis
$u = e^x$ always +ve
hence



The real axis of the xy plane is mapped into the +ve real axis in the u-v plane

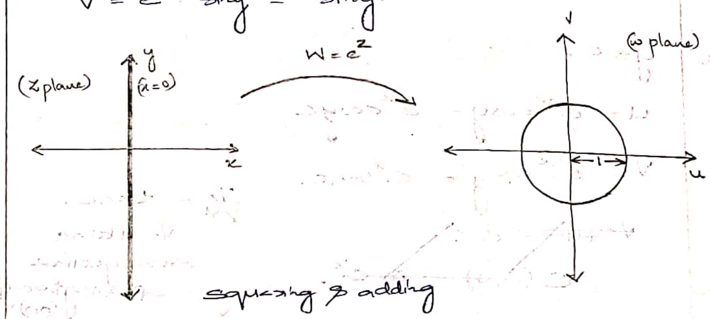
$v = 0 \Rightarrow u$ axis.

Given $u = e^x$ which is always +ve hence +ve real axis.

71. The imaginary axis $x = 0$.

$u = e^x \cos y = \cos y$

$v = e^x \sin y = \sin y$.



squaring & adding

$u^2 + v^2 = \cos^2 y + \sin^2 y$.

$u^2 + v^2 = 1$

Circle with centre origin & radius 1

The imaginary line $x = 0$ in the z plane is mapped to a unit circle centred at the origin the w plane under the mapping $w = e^z$.

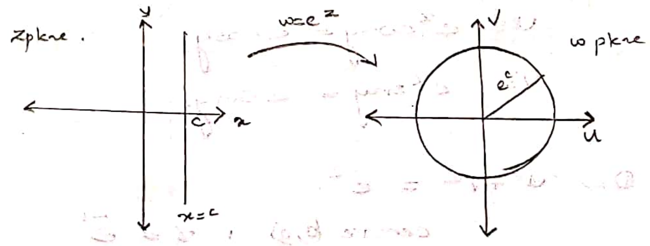
72. $x = c$.

$u = e^x \cos y = e^c \cos y$.

$v = e^x \sin y = e^c \sin y$

$u^2 + v^2 = e^{2c}$

Circle with centre $(0,0)$ radius e^c;



if $c=1$ radius e^c

$c=2$ radius e^{2c} .....

73. $y=c$.

$u = e^x \cos y = e^x \cos c$

$v = e^x \sin y = e^x \sin c$.

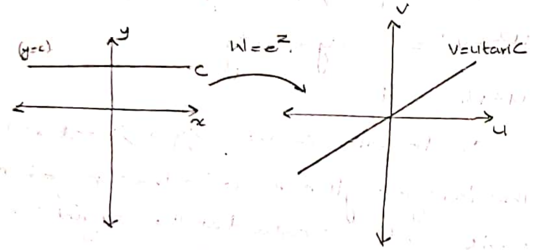
$v/u = \tan c$.

$v = u \tan c$.

straightline passingthrough $(0,0)$

~~$v^2+u^2 = e^{2x}$~~

~~$(0,0)$~~ $\cancel{\to}\ e^x$.



74. The infinite strip $-1 \le x \le 1$

① $x = -1$

$u = e^x \cos y = e^{-1} \cos y$.

$v = e^x \sin y = e^{-1} \sin y$.

② $x = 1$

$u = e^x \cos y = e \cos y$

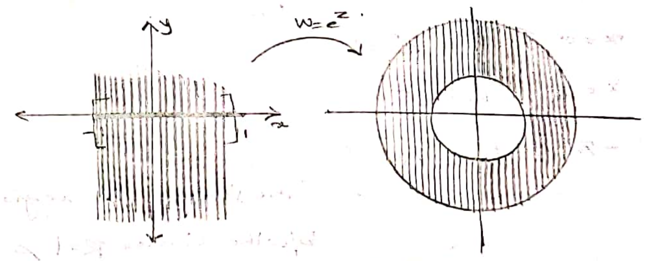
$v = e^x \sin y = e \sin y$.

① $\Rightarrow u^2+v^2 = e^{-2}$.

centre $(0,0)$, $r = e^{-1}$

② $\Rightarrow u^2+v^2 = e^2$

centre $(0,0)$ $r(e^1)$



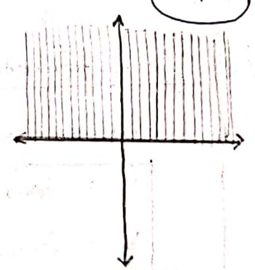
The image is the annular regions b/w the concentric circles.

75. The region bounded b/w the realaxis and $y=\pi$.

$y=0$ & $y=\pi$.

$y=0$	$y=\pi$
$u = e^x$.	$u = -e^x$
$v = 0$.	$v = 0$

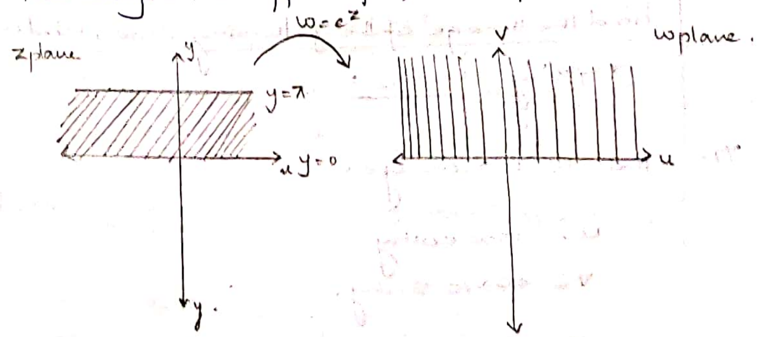
$R = |z|$



By Polar form;

$R = e^x$ $\phi = y$.

The image is the upperhalf of the w plane.



76. $0 \le x \le 2$, $-\pi \le y \le \pi$

$R = e^{x}$ $\quad y = \phi$.

$x = 0$; $R = e^{0} = 1$

$x = 2$; $R = e^{2}$.

$-\pi \le \phi \le \pi$.

The image is the region b/w the circles $R = 1$ & $R = e^{2}$ in the w-plane

IV **Transformation 4 : $w = \sin z$.**

$w = \sin x \cosh y + i \cos x \sinh y$

$$\boxed{\begin{aligned} u &= \sin x \cosh y \\ v &= \cos x \sinh y \end{aligned}}$$

Find the image of the following curves under the mapping $w = \sin z$;

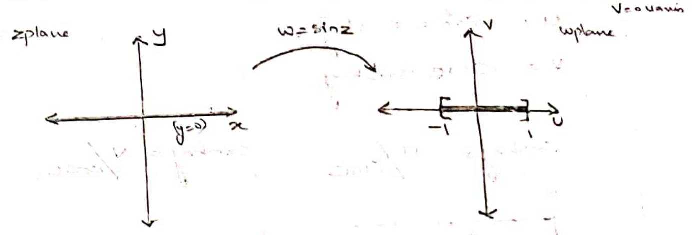
77. The real axis $y = 0$.

$u = \sin x \cosh y$

$v = \cos x \sinh y$.

$u = \sin x \quad v = 0$.

$v = 0$ u axis

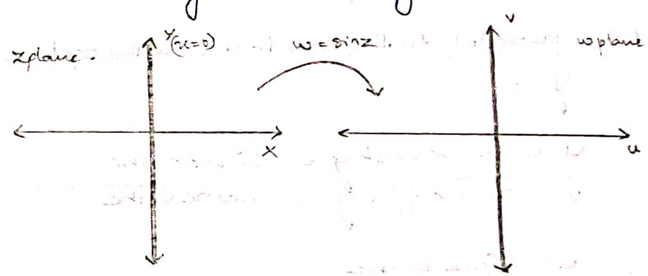


The image of real axis in z plane is the segment from -1 to 1 in the w plane under the mapping $w = \sin z$.

78. Imaginary axis $x = 0$.

$u = \cosh y \sin x = 0$ $\quad u = 0$ v axis

$v = \sinh y \cos x = \sinh y$. $\quad \sinh y = \frac{e^{y} - e^{-y}}{2}$.



The image of the imaginary axis in z plane is the imaginary axis itself in the w plane. Since the term $\sinh y = \frac{e^{y} - e^{-y}}{2}$ $\left.\begin{aligned} e^{y} &\to +ve \\ -e^{y} &\to -ve \end{aligned}\right\}$ hence both +ve & -ve values can be taken under the mapping $w = \sin z$.

79. Lines $\parallel^{el}$ to the Imaginary axis y in the z plane.

$x = c$. $\quad (c \ne 0)$

$u = \sin x \cosh y = \sin c \cosh y$

$v = \cos x \sinh y = \cos c \sinh y$.

$u = \sin c \cosh y$

$v = \cos c \sinh y$

$\cosh y = u/\sin c$ $\quad$ $\sinh y = v/\cos c$

$$\boxed{\cosh^2 y - \sinh^2 y = 1}$$

Focus $(\pm 1, 0)$

$$\frac{u^2}{\sin^2 c} - \frac{v^2}{\cos^2 c} = 1$$

which is a hyperbola.

The image is a hyperbola under the mapping $w = \sin z$

80. Line parallel to the real axis in the z plane.

$y = c$.

$u = \sin x \cosh y = \sin x \cosh c$.

$v = \cos x \sinh y = \cos x \sinh c$

$u = \sin x \cosh c$ $\quad$ $\tanh c \cot x = v/u$

$v = \cos x \sinh c$ $\quad$ $u = \sin x$

$u^2 = \sin^2 x \cosh^2 c$

$v^2 = \cos^2 x \sinh^2 c$

$$\frac{u^2}{\cosh^2 c} + \frac{v^2}{\sinh^2 c} = 1$$

which is an ellipse

The line ||el to real axis are mapped on to ellipse in the w plane under the mapping $w = \sin z$.

81. $x = \pi/2$.

$\frac{e^x + e^{-x}}{2}$.

$u = \sin x \cosh y$

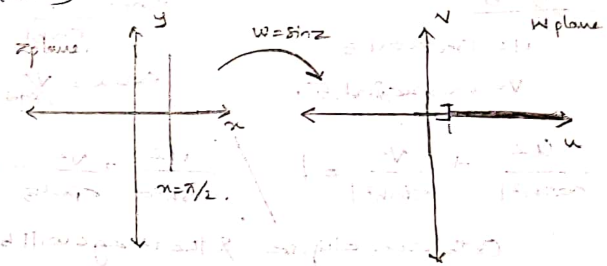
$= \cosh y$.

$v = \cos x \sinh y$. $\quad$ $\cosh y \geq 1$

$= 0$.

$v = 0$; $u = \cosh y$.

(u axis)

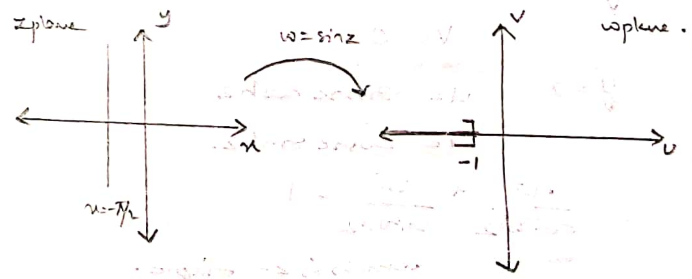


The image is u axis with $u \geq 1$.

82. $x = -\pi/2$.

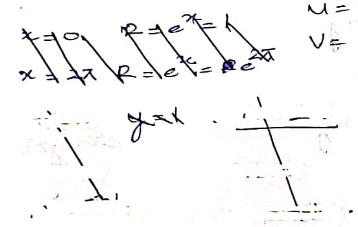
$u = \sin x \cosh y = -\cosh y$. ≤ -1

$v = 0$.



The image is u axis with $u \leq 1$

83. due $0 < x < 2\pi$ $1 < y < 5$.



$u = \sin x \cosh y$
$V = \cos x \sinh y$

$\underline{y=1}$

$u = \sin x \cosh 1$

$V = \cos x \sinh 1$

$\sin x = \frac{u}{\cosh 1}$ $\quad u/\cosh 5$

$\cos x = V/\sinh 1$ $\quad V/\sinh 5$

$\underline{y=5}$

$u = \sin x \cosh 5$

$V = \cos x \sinh 5$.

$$\frac{u^2}{\cosh^2 1} + \frac{V^2}{\sinh^2 1} = 1 \quad / \quad \frac{u^2}{\cosh^2 5} + \frac{V^2}{\sinh^2 5} = 1$$

Both are ellipse & the image will be the annulus portion b/w the 2 ellipse.

84. due $0 < x < \pi/2$ $\quad 0 < y < 2$.

$U = \sin x \cosh y$ $\quad V = \cos x \sinh y$.

$y = 0$; $\quad u = \sin x \cosh 0 = \sin x$

$V = 0$.

$y = 2$ $\quad u = \sin x \cosh 2$

$V = \cos x \sinh 2$.

$$\frac{u^2}{\cosh^2 2} + \frac{V^2}{\sinh^2 2} = 1$$

$\therefore$ which is an ellipse.

hence the image will be the inner portion of an ellipse in the 1st Quadrant.

<u>Invariant points / Fixed points :</u> (3mks)

A pt x in the complex plane is said to be invariant pts if ;

$$\boxed{f(z) = z}$$

<u>Find the invariant / fixed pts of the following mapping:</u> $w = f(z)$.

85. $W = z^2$.

Invariant pts are given by; $\quad z^2 - z = 0$

$f(z) = z$.

$\Rightarrow z = z^2$

$z^2 - z = 0$.

$= (z-1) = 0$

$\underline{\underline{z = 1, z = 0}}$.

86. $W = (4+i)z$.

Invariant pts are given by;

$f(z) = z$

$\Rightarrow z = (4+i)z$ $\quad z = 4z + iz$

$z(3+i) = 0$.. $\quad 3z + iz = 0$

$3 + i \neq 0$; $z = 0$. $\quad z(3+i) = 0$

87. $w = (z-i)^2$

Invariant pts are given by;

$f(z) = z$.

$z = z^2 + i^2 - 2zi$.

$z = z^2 - 1 - 2zi$

$z^2 - 2iz - 1 - z = 0$.

$z^2 - z(2i+1) - 1 = 0$.

$$z = \frac{(2i+1) \pm \sqrt{(2i+1)^2 + 4\times 1}}{2}$$

$$= \frac{(2i+1) \pm \sqrt{4i^2 + 1 + 4i + 4}}{2}$$

$$= \frac{(2i+1) \pm \sqrt{4i+1}}{2}$$

88. $w = \frac{z-1}{z+1}$

$f(z) = z$.

$z = \frac{z-1}{z+1}$

$z(z+1) = z-1$

$z^2+z = z-1$

$z^2+z-z+1 = 0$

$z^2+1 = 0$.

$z^2 = -1$

$z = \pm i$

89. $w = \frac{2iz-1}{z+2i}$

$f(z) = z$

$z = \frac{2iz-1}{z+2i}$

$z^2 + i2z = 2iz - 1$

$z^2 + \cancel{2iz} - \cancel{2iz} + 1 = 0$

$z^2+1 = 0$

$z^2 = -1$

$z = \pm i$

Inverse Mapping:

Find the Inverse Mapping of the following:

90. $w = \frac{3z}{2z-i}$

$2zw - iw = 3z$.

~~z 3 - 2zw = -iw~~.

$(2w-3)z = iw$.

~~z = iw~~

$z = iw/2w-3$.

91. $w = \frac{2z+5i}{4z}$.

$4zw = 2z + 5i$

$4zw - 2z = 5i$

$z(4w-2) = 5i$

$z = \frac{5i}{4w-2}$

92. $w = \frac{z-i}{z+i}$

$wz + iw = z - i$

$wz + iw = z - i$

$wz - z = -(i + iw)$

$z(w-1) = -i(1+w)$

$z = \dfrac{-i(1+w)}{(w-1)} = \dfrac{-iw-i}{w-1}$

$z = \dfrac{i(1+w)}{(1-w)}$

14/9/2020

Module 4:

Complex Integration:

Chapter 1 : Line integrals on a Complex Plain

Line Integrals or Contour Integrals.

Complex definite integrals are called line integrals and it is denoted by $\int_C f(z)\,dz$ where $dz = dx + i\,dy$

$C \longrightarrow$ path of integration.

Evaluation of line integrals:

Method 1: First evaluation Method (3mks)

Theorem:

Let $f(z)$ be analytic in a domain D, with $F'(z) = f(z)$ then;

$$\int_{z_0}^{z_1} f(z)\,dz = \Big[F(z)\Big]_{z_0}^{z_1} = F(z_1) - F(z_0)$$

1. $\displaystyle\int_0^{1+i} z^2\,dz$.

z^2 is analytic (contin. fn)

$$\Rightarrow \left(\frac{z^3}{3}\right)_0^{1+i} = \frac{(1+i)^3}{3} = \frac{1+3i-3-i}{3}$$

$$= \frac{2i-2}{3}$$

2. $\displaystyle\int_{-\pi i}^{\pi i} \cos z\,dz$

$\cos z$ is analytic. (contin. fn)