Laurent's Series

Let c_1 and c_2 be 2 concentric circles with centre z_0 and radius r_1 and r_2 with $r_1 > r_2$.

If $f(z)$ is analytic on and inside the ring shaped region, bounded b/w c_1 and c_2, then to each point z on this region, $f(z)$ can be expressed as

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} b_n (z-z_0)^{-n} \text{ where}$$

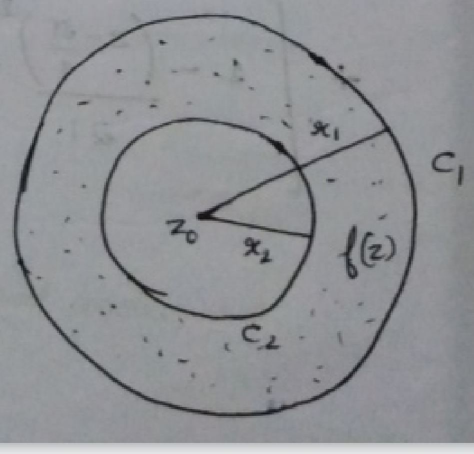
$$a_n = \frac{1}{2\pi i} \int_{C_1} \frac{f(z)\, dz}{(z-z_0)^{n+1}}$$

$$b_n = \frac{1}{2\pi i} \int_{C_2} \frac{f(z)\, dz}{(z-z_0)^{-n+1}}$$

where $\sum_{n=0}^{\infty} a_n (z-z_0)^n$ is called the analytic part

and $\sum_{n=1}^{\infty} b_n (z-z_0)^{-n}$ is called the principle part of

the Laurent's series



Q. Find the Laurent's series expansion of the following functions. & Identify the analytic and principal part.

1. $f(z) = \dfrac{1}{(z-1)(z-2)}$ in (a) $|z| < 1$ (b) $1 < |z| < 2$

(c) $|z| > 2$ (d) $0 < |z-1| < 1$

a) $f(z) = \dfrac{1}{(z-1)(z-2)}$

$$\frac{1}{(z-1)(z-2)} = \frac{A}{(z-1)} + \frac{B}{(z-2)}$$

$1 = A(z-2) + B(z-1)$

$z = 2 \Rightarrow$

$B = 1$

$z = 1 \Rightarrow$

$A = -1$

$$\therefore f(z) = \frac{-1}{(z-1)} + \frac{1}{(z-2)}$$

$$= \frac{-1}{-(1-z)} + \frac{1}{-2(1-\frac{z}{2})}$$

$$= \frac{1}{1-z} - \frac{1}{2(1-\frac{z}{2})}$$

$$= (1-z)^{-1} - \frac{1}{2}(1-\frac{z}{2})^{-1}$$

$$= (1 + z + z^2 + \cdots) - \frac{1}{2}\left(1 + \frac{z}{2} + \frac{z^2}{4} + \left(\frac{z^3}{2^3}\right) + \cdots\right).$$

This Laurent's series expansion has no term in the principal part. ie. this Laurent's series is analytic.

b)

$1 < |z| < 2$

$1 < |z|$ $\quad$ $|z| < 2$

$\frac{1}{|z|} < 1$ $\quad$ $|\frac{z}{2}| < 1$

$$f(z) = \frac{-1}{(z-1)} + \frac{1}{(z-2)}$$

$$= \frac{-1}{z(1-\frac{1}{z})} + \frac{1}{-2(1-\frac{z}{2})}$$

$$= -\frac{1}{z}(1-\frac{1}{z})^{-1} - \frac{1}{2}(1-\frac{z}{2})^{-1}$$

$$= \frac{-1}{z}(1+\frac{1}{z}+\frac{1}{z^2}+\dots) - \frac{1}{2}(1+\frac{z}{2}+(\frac{z}{2})^2+\dots).$$

Analytic part is

$$\frac{-1}{2}(1+\frac{z}{2}+(\frac{z}{2})^2+\dots).$$

principle part is

$$\frac{-1}{z} - (\frac{1}{z})^2 - \frac{1}{z^3} - \dots$$

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c) $|z| > 2$

$\left|\frac{2}{z}\right| < 1$

$$f(z) = \frac{-1}{(z-1)} + \frac{1}{(z-2)}$$

$$= \frac{-1}{+z(1-\frac{1}{z})} + \frac{1}{+z(1-\frac{2}{z})}$$

$$= -\frac{1}{z}(1-\frac{1}{z})^{-1} + \frac{1}{z}(1-\frac{2}{z})^{-1}$$

$$= -\frac{1}{z}\left[1+\frac{1}{z}+\frac{1}{z^2}+\dots\right] + \frac{1}{z}\left[1+\frac{2}{z}+(\frac{2}{z})^2+\dots\right]$$

$$= -(\frac{1}{z})^2 - (\frac{1}{z})^3 - \dots + \frac{2}{z^2} + \frac{2^2}{z^3} + \frac{2^3}{z^4} + \dots$$

This series has only principle part

d) $|z-1| < 1$

$$f(z) = -\frac{1}{(z-1)} + \frac{1}{(z-2)}$$

$$= \frac{-1}{z-1} + \frac{1}{(z-1)-1}$$

$$= -\frac{1}{z-1} - (1-(z-1))^{-1}$$

$$= \underbrace{-\frac{1}{z-1}}_{PP} - \underbrace{\left[1+(z-1)+(z-1)^2+\dots\right]}_{AP}$$

2* $f(z) = \frac{1}{(z-1)(z-3)}$ in $0 < |z-1| < 1$

$$\frac{1}{(z-1)(z-3)} = \frac{A}{(z-1)} + \frac{B}{(z-3)}$$

$$1 = A(z-3) + B(z-1)$$

$B = \frac{1}{2}$

$A = -\frac{1}{2}$

$$f(z) = -\frac{1}{2(z-1)} + \frac{1}{2(z-3)}$$

$$= -\frac{1}{2(z-1)} + \frac{1}{2((z-1)-2)}$$

$$= -\frac{1}{2(z-1)} - \frac{1}{4\left(1-\left(\frac{z-1}{2}\right)\right)}$$

$$= -\frac{1}{2(z-1)} - \frac{1}{4}\left[1 + \frac{z-1}{2} + \left(\frac{z-1}{2}\right)^2 + \cdots\right]$$

principle part is $-\frac{1}{2(z-1)}$

analytic part is $-\frac{1}{4}\left[1 + \frac{z-1}{2} + \frac{(z-1)^2}{4} + \cdots\right]$

3* $f(z) = \dfrac{1}{z - z^3}$ $1 < |z+1| < 2$

$$f(z) = \frac{1}{z(1+z)(1-z)} = \frac{A}{z} + \frac{B}{1+z} + \frac{C}{1-z}$$

$$1 = A(1+z)(1-z) + B(z)(1-z) + C(z)(1+z)$$

$z = 0 \Rightarrow$

$\underline{\underline{1 = A}}$

$z = 1 \Rightarrow$

$1 = 2C$

$C = \underline{\underline{\dfrac{1}{2}}}$

$z = -1 \Rightarrow$

$1 = -2B$ $(z-1)$

$B = \underline{\underline{\dfrac{-1}{2}}}$ $z+1-1-1$

$$f(z) = \frac{1}{z} - \frac{1}{2(1+z)} + \frac{1}{2(1-z)}$$

$1 < |z+1|$ $|z+1| < 2$

$\dfrac{1}{|z+1|} < 1$ $\dfrac{|z+1|}{2} < 1$

$$f(z) = \frac{1}{((z+1)-1)} - \frac{1}{2(z+1)} + \frac{1}{-2((z+1)-2)}$$

$$= \frac{1}{(z+1)\left(1-\left(\frac{1}{z+1}\right)\right)} - \frac{1}{2(z+1)} + \frac{1}{4\left(1-\frac{z+1}{2}\right)}$$

$$= +\left(1 - \frac{1}{(z+1)}\right)^{-1} - \frac{1}{2(z+1)} + \frac{1}{4}\left(1 - \frac{z+1}{2}\right)^{-1}$$

$$= -\frac{1}{2(z+1)} + \frac{1}{z+1}\left(1 + \frac{1}{(z+1)} + \frac{1}{(z+1)^2} + \cdots\right) + \frac{1}{4}\left(1 + \frac{z+1}{2} + \left(\frac{z+1}{2}\right)^2 + \cdots\right)$$

analytic part is $\frac{1}{4}\left(1 + \frac{z+1}{2} + \left(\frac{z+1}{2}\right)^2 + \cdots\right)$.

principle part is $-\frac{1}{2(z+1)} + \frac{1}{(z+1)}\left(1 + \frac{1}{(z+1)} + \frac{1}{(z+1)^2} + \cdots\right)$.

4)* $f(z) = \frac{z}{(z-1)(z-2)}$

a) $0 < |z-2| < 1$

b) $|z-1| > 1$

$$\frac{z}{(z-1)(z-2)} = \frac{A}{(z-1)} + \frac{B}{(z-2)}$$

$$z = A(z-2) + B(z-1)$$

$z = 2 \Rightarrow$

$2 = B$

$z = 1 \Rightarrow$

$-1 = A$

$$f(z) = \frac{-1}{z-1} + \frac{2}{z-2}$$

$|z-2| < 1$

$$f(z) = \frac{2}{z-2} - \frac{1}{(z-2+1)}$$

$$= \frac{2}{z-2} - (1 + (z-2))^{-1}$$

$$= \frac{2}{z-2} - (1 - z-2 + (z-2)^2 - (z-2)^3 + \dots).$$

analytic part is $-(1 - (z-2) + (z-2)^2 - (z-2)^3 + \dots)$

principle part is $\frac{2}{z-2}$

b) $|z-1| > 1$

$$\frac{1}{|z-1|} < 1$$

$$f(z) = \frac{-1}{(z-1)} + \frac{2}{[(z-1)-1]}$$

$$= \frac{-1}{z-1} + \frac{(+2)}{(z-1)}\left[1 - \frac{1}{(z-1)}\right]^{-1}$$

$$= -\frac{1}{z-1} + \frac{2}{z-1}\left[1 + \left(\frac{1}{z-1}\right) + \frac{1}{(z-1)^2} + \frac{1}{(z-1)^3} + \dots\right]$$

$$= \frac{-1}{z-1} + \frac{2}{z-1} + \frac{2}{(z-1)^2} + \frac{2}{(z-1)^3} + \dots$$

$$= \frac{1}{z-1} + \frac{2}{(z-1)^2} + \frac{2}{(z-1)^3} + \dots$$

5. HW

$f(z) = \frac{1}{z^3 - z^4}$ in (a) $0 < |z| < 1$

(b) $|z| > 1$

$f(z) = \frac{1}{z^3(1-z)}$

$\frac{1}{z^3(1-z)} = \frac{A}{z^3} + \frac{B}{z^2} + \frac{C}{z} + \frac{D}{1-z}$

$1 = A(1-z) + B(1-z)z + C(1-z)z^2 + Dz^3$

$z = 0 \Rightarrow$

$1 = A$

$z = 1 \Rightarrow$

$1 = D$

for $z = -1 \Rightarrow$

$1 = 2A - 2B + 2C - D$

$1 = 2 - 2B + 2C - 1$

$2C - 2B = 0$

$C - B = 0$ ——— (1)

$C = B$

for $z = 2$

$1 = -A + (-2B) + (-4C) + 8$

$1 = -1 - 2B - 4B + 8$

$-6B = -6$

$B = C = \underline{\underline{1}}$

$\therefore$

$$f(z) = \frac{1}{z^3} + \frac{1}{z^2} + \frac{1}{z} + \frac{1}{1-z}$$

a) $|z| < 1$

$\therefore$

$$f(z) = \frac{1}{z^3} + \frac{1}{z^2} + \frac{1}{z} + (1-z)^{-1}$$

$$= \frac{1}{z^3} + \frac{1}{z^2} + \frac{1}{z} + 1 + z + z^2 + z^3 + \cdots$$

Principle part is $\left[\frac{1}{z^3} + \frac{1}{z^2} + \frac{1}{z}\right]$

analytic part is $[1 + z + z^2 + \cdots]$.

b) $|z| > 1$

$\frac{1}{|z|} < 1$

$\therefore$

$$f(z) = \frac{1}{z^3} + \frac{1}{z^2} + \frac{1}{z} - \frac{1}{z}\left(1 - \frac{1}{z}\right)^{-1}$$

$$= \frac{1}{z^3} + \frac{1}{z^2} + \frac{1}{z} - \frac{1}{z} - \frac{1}{z^2} - \frac{1}{z^3} - \frac{1}{z^4} - \frac{1}{z^5} - \cdots$$

$$= \underline{\underline{-\frac{1}{z^4} - \frac{1}{z^5} - \frac{1}{z^6} - \cdots\cdots}}$$

This series has only principle part.

$$f(z) = \frac{1}{z^3} + \frac{1}{z^2} + \frac{1}{z} + \frac{1}{1-z}$$

$$= \frac{1}{z^3} + \frac{1}{z^2} + \frac{1}{z} + (1-z)^{-1}$$

$$= \frac{1}{z^3} + \frac{1}{z^2} + \frac{1}{z} + 1 + z^2 + z^3 + z^4 + \dots$$

P. part $\longrightarrow$ $1/z + 1/z^2 + 1/z^3$.

A. part $\longrightarrow$ $1 + z^2 + z^3 + z^4 + \dots$.

b) $|z| > 1$

$1/|z| < 1$

$$f(z) = 1/z^3 + 1/z^2 + 1/z - \frac{1}{z}(1 - 1/z)^{-1}$$

$$= 1/z^3 + 1/z^2 + 1/z - 1/z\left(1 + 1/z + 1/z^2 + \dots\right)$$

$$= \cancel{1/z^3} + \cancel{1/z^2} + \cancel{1/z} - \cancel{1/z} - \cancel{1/z^2} - \cancel{1/z^3} - 1/z^4 - \dots$$

$$= -1/z^4 - 1/z^5 - 1/z^6 - \dots$$

$\Rightarrow$ only principle part.

28/9/2020

(6) $\operatorname{Log}(z/(z-1))$ $|z| > 1$ (3mks)

about $z = 0$.

$$-\operatorname{Log}\left(\frac{z-1}{z}\right) \Rightarrow -\operatorname{Log}(1 - 1/z)$$

$$\operatorname{Log}(1-x) = -\left[x + x^2/2 + x^3/3 + \dots\right]$$

$$= 1/z + 1/2z^2 + 1/3z^3 + \dots$$

No analytic part.
Infinitely many elements are in the P. part
(Essential singularity)

(3mks)

(7) $\dfrac{e^{-1/z^2}}{z^2}$ with centre 0 (about $z=0$)

$\left(e^{-x} = 1 - x + x^2/2! - x^3/3! + \cdots\right)$

$$\frac{1}{z^2}\left[1 - \frac{1}{z^2} + \frac{1}{z^4\,2!} - \frac{1}{z^6\,3!} + \cdots\right]$$

$$= \frac{1}{z^2} - \frac{1}{z^4} + \frac{1}{z^6\,2!} - \frac{1}{6z^8} + \cdots$$

$$= \frac{1}{z^2} - \frac{1}{z^4} + \frac{1}{2z^6} - \frac{1}{6z^8} + \cdots$$

(8) $\frac{1}{z^4}\cos z$

$$\frac{1}{z^4}\left[1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots\right]$$

$$\frac{1}{z^4} - \frac{1}{2z^2} + \frac{1}{24} - \frac{z^2}{6!} + \cdots$$

P. part $\longrightarrow$ $\frac{1}{z^4} - \frac{1}{2z^2}$ $\cancel{+\ \frac{1}{2}}$

([illegible]) (pole)

A part $\longrightarrow$ $\frac{1}{4!} - \frac{z^2}{6!} + \cdots$

(9) $z^3 \cosh 1/z$

$$z^3\left(1 + \frac{1}{z^2\,2!} + \frac{1}{z^4\,4!} + \frac{1}{z^6\,6!}\right)$$

$$z^3 + \frac{z}{2!} + \frac{1}{z\,4!} + \frac{1}{z^3 6!} + \ldots\ldots$$

A part $\longrightarrow z^3 + z/2!$

P. part $\longrightarrow 1/z4! + 1/z^3 6! + \ldots\ldots$

(Infinite) essential singularity

10. $e^z/z^2 z^3$ $\quad |z| < 1$

$$\frac{e^z}{z^2(1-z)}.$$

$$\frac{e^z}{z^2}(1-z)^{-1} \Rightarrow \frac{e^z}{z^2}\left(1+z+z^2+z^3+\ldots\right)$$

$$\Rightarrow \frac{e^z}{z^2} + \frac{e^z}{z} + e^z + e^z \times z + e^z \times z^2 + \ldots\ldots$$

P part $\longrightarrow e^z/z^2 + e^z/z$ (Pole)

A part $\longrightarrow e^z + ze^z + z^2e^z + \ldots\ldots$

11) $z^{-5} \sin z.$

$$z^{-5}\left(z/1! - z^3/3! + z^5/5! - z^7/7! + z^9/9! - \ldots\right)$$

$$\frac{1}{z5}\left[\frac{z}{1!} - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \frac{z^9}{9!} - \ldots\right)$$

$$1/z^4 - 1/z^6 3! + 1/5! - z^2/7! + z^4/9! + \ldots\ldots$$

P part $\longrightarrow 1/z^4 - 1/3!z^6.$

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A part $\longrightarrow$ $1/5! - z^2/7! + \cdots$

12. $\dfrac{e^z}{(z-1)^2}$ about $z=1$

$$\frac{e^{(z-1)+1}}{(z-1)^2} \Rightarrow \frac{e^1 \cdot e^{z-1}}{(z-1)^2}$$

$$\frac{e}{(z-1)^2}\left[1+(z-1)+\frac{(z-1)^2}{2!}+\frac{(z-1)^3}{3!}+\frac{(z-1)^4}{4!}+\cdots\right.$$

$$\frac{e}{(z-1)^2}+\frac{e}{(z-1)}+\frac{e}{2!}+\frac{(z-1)e}{3!}+\frac{(z-1)^2e}{4!}+\cdots$$

P part $\longrightarrow$ $e/(z-1)^2 + e/(z-1)$

A part $\longrightarrow$ $e/2! + (z-1)e/3! + \dfrac{(z-1)^2e}{4!} + \cdots$

13. $\dfrac{-2z+3}{z^2-3z+2}$ (a) $|z|<1$ (c) $1<|z|<2$
(b) $|z|<2$ (d) $|z|>2$

$$\frac{-2z+3}{(z-1)(z-2)} = \frac{A}{(z-1)} + \frac{B}{(z-2)}$$

$$-2z+3 = A(z-2) + B(z-1)$$

put $z=2$ put $z=1$

$-1 = B$ $-A = 1$ $\underline{\underline{= -1}}$

$$\frac{-2z+3}{(z-1)(z-2)} = \frac{-1}{(z-1)} + \frac{-1}{z-2}$$

ⓐ $|z| < 1$

$$\frac{-1}{z-1} - \frac{1}{z-1-1} \qquad \frac{-1}{-1(1-z)} - \frac{1}{-2(1-z/2)}$$

$$\frac{1}{(1-z)} + \cancel{\frac{1}{(1-(1-z))}} \qquad (1-z)^{-1} + \frac{(1-z/2)^{-1}}{2}.$$

$$1+z+z^2+z^3+\ldots\ldots + \frac{1}{2}\left[1+z/2+(z/2)^2+(z/2)^3+\ldots\right]$$

$$\frac{3}{2} + \frac{5}{4}z + \frac{9z^2}{8} + \frac{17z^3}{16} + \ldots\ldots$$

No principle part. only Analytic part.

ⓑ $|z| < 2 \qquad \frac{|z|}{2} < 1.$

$$\frac{-1}{(z-1)} - \frac{1}{z-2}$$

$$\frac{-1}{-1(1-z)} - \frac{1}{-2(1-z/2)}$$

$$\frac{-1}{(z-1)} + \frac{1}{2}(1-z/2)^{-1}$$

$$\frac{-1}{(z-1)} + \frac{1}{2}\left[1+z/2+(z/2)^2+(z/2)^3+\ldots\right]$$

ⓒ $1 < |z| < 2.$

$$1/|z| < 1 \qquad \frac{|z|}{2} < 1$$

$$\frac{-1}{z(1-1/z)} + \frac{1}{2(1-z/2)}.$$

$$\frac{-1}{z}\left[1+\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\cdots\right]+\frac{1}{2}\left[1+z/2+(z/2)^2+\cdots\right]$$

P part $\longrightarrow -1/z\left(1+1/z+1/z^2+\cdots\right)$

A part $\longrightarrow 1/2\left(1+z/2+(z/2)^2+\cdots\right)$

(d) $|z|>2$.

$$\frac{2}{|z|}<1$$

$$\frac{-1}{z(1-2/z)} - \frac{1}{z(1-1/z)}$$

$$\Rightarrow \frac{-1}{z}(1-2/z)^{-1} - \frac{1}{z}(1-1/z)^{-1}$$

$$\Rightarrow -1/z\left(1+2/z+(2/z)^2+(2/z)^3+\cdots\right)-\frac{1}{z}\left(1+\frac{1}{z}+\frac{1}{z^2}\right.$$

$$\Rightarrow \frac{-1}{z}-\frac{2}{z^2}-\frac{4}{z^3}-\cdots-\frac{1}{z}-\frac{1}{z^2}-\frac{1}{z^3}-\cdots$$

$$\Rightarrow \frac{-2}{z}-\frac{3}{z^3}-\frac{5}{z^3}-9/z^4-\cdots$$

No Analytic part only P. part.

(Essential Singularity)

14. Find the Taylor series & Laurent's series

due

for : 1) $f(z) = 1/z^2$ about $z = i$

2) $f(z) = z^8/1-z^4$ about $z = 0$.

1) $f(z) = 1/z^2$ about $z = i$.

Taylor Series;

$$f(z) = \frac{1}{(z-i+i)^2}.$$

$$= \frac{1}{i^2\left(1+\frac{(z-i)}{i}\right)^2} = -1 \times \left(1+\frac{z-i}{i}\right)^{-2}$$

$$-1 \times \left[1 - \frac{2(z-i)}{i} + \frac{3(z-i)^2}{i^2} - \dots\right]$$

$$\Rightarrow -1\left[1 + 2(z-i)i + \frac{3(z-i)^2}{-1} - \dots\right]$$

$$= -1 - 2(z-i)i + 3(z-i)^2 + \dots \quad |z-i|<1.$$

Laurent's Series;

$|z-i| > 1$ $\qquad \frac{1}{|z-i|} < 1$

$1/z^2.$

$$\frac{1}{(z-i+i)^2} \Rightarrow \frac{1}{(z-i)\left(1+i/z-i\right)^2}.$$

$$= \frac{1}{(z-i)^2}\left(1+i/z-i\right)^{-2}$$

$$= 1/(z-i)^2\left[1 - \frac{2i}{z-i} + \frac{3i^2}{(z-i)^2} - \dots\right]$$

$$= \frac{1}{(z-i)^2} - \frac{2i}{(z-i)^3} - \frac{3}{(z-i)^4} + \cdots$$

No Analytic part only P. part.
(Essential singularity)

Que 2. $f(z) = \dfrac{z^8}{1-z^4}$ about $z=0$.

Taylor series;

$$\frac{z^8}{1-z^4} \Rightarrow z^8(1-z^4)^{-1}$$

$$\Rightarrow z^8(1+z^4+z^8+\cdots)$$

$$\Rightarrow z^8+z^{12}+z^{16}+\cdots \quad |z|<$$

Laurents Series;

About $|z|>1 \Rightarrow \frac{1}{|z|}<1$

$$\frac{z^8}{1-z^4} \Rightarrow \frac{\cancel{z^8}\, z^4}{-\cancel{z^4}(1-1/z^4)}$$

$$\Rightarrow \frac{-z^4}{(1-1/z^4)} \Rightarrow -z^4(1-1/z^4)^{-1}$$

$$\Rightarrow -z^4\left(1+\frac{1}{z^4}+\frac{1}{z^8}+\frac{1}{z^{12}}+\right.$$

$$-z^4 - 1 - 1/z^4 - 1/z^8 - \cdots$$

$\Rightarrow$ P.part $\longrightarrow$ $-\left(\frac{1}{z^4} + \frac{1}{z^8} + \cdots\right)$

A part $\longrightarrow$ $-z^4 - 1$

Module. 5

chapter. 2

Zeros and Singularities.

let $f(z)$ be an analytic fn then the value of z for which $f(z)=0$ is called a 'zero' of $f(z)$.

Q* Determine the zeros and their orders

1) $f(z) = (1-z)^4$

$f(z) = 0 \Rightarrow$

$(1-z)^4 = 0 \Rightarrow$

$z=1$ is a zero of order 4

2* $f(z) = 1+z^2$

$f(z) = 0 \Rightarrow$

$1+z^2 = 0$

$z^2 = -1$

$z = \pm i$

$z = \pm i$ are the roots each of order 1

3* $f(z) = (z^4-1)^4$.

$f(z)=0 \Longrightarrow$

$(z^4-1)^4 = 0$

$\therefore \quad (z^2+1)(z^2-1) = 0$

$z^2 = -1 \qquad z^2 = 1$

$z = \pm i \qquad z = \pm 1$

$z = i, -i, 1, -1$ are the zeros each of order 4

4* $f(z) = \sin z$

$f(z) = 0 \Rightarrow$

$\sin z = 0$.

$z = n\pi$, where $n = 0, \pm 1, \pm 2, \pm 3 \dots$

These are infinately many zeros each of order 1

5) $f(z) = (\sin z)^5$

$z = n\pi$, $n = 0, \pm 1, \pm 2 \dots$

These are infinately many zeros each of order 5

6) $f(z) = \tan^2 z$

$f(z) = 0 \Rightarrow$

$\tan^2 z = 0 \Rightarrow$

$\tan z = 0 \Rightarrow$

$\frac{\sin z}{\cos z} = 0 \Rightarrow$

$z = n\pi$ are the zeros each of order 2

7* $f(z) = \cot^3 z$

$\cot^3 z = \frac{\cos^3 z}{\sin^3 z} = \left[\frac{\cos z}{\sin z}\right]^3$

$\frac{\cos z}{\sin z} = 0 \Rightarrow$

ie $z = (2n+1\frac{\pi}{2})$ are the zeros each of order 3

Here $n = 0, \pm 1, \pm 2, \pm 3 \dots$

8* $(z^4-z^2-6)^{100}$

$f(z)=0 \Rightarrow$

$z^4-z^2-6=0$

Put $z^2=t$

$t^2-t-6=0$

$t=\frac{1\pm\sqrt{25}}{2}=\underline{\underline{3,-2}}$

$z^2=3, -2$

$\therefore z=\pm\sqrt{3}, \pm\sqrt{2}i$

$z=+\sqrt{3}, -\sqrt{3}, +\sqrt{2}i, -\sqrt{2}i$ are the zeros each of order 100

9* $f(z)=(z^2+1)(e^z-1)$

$f(z)=0 \Rightarrow$

$(z^2+1)(e^z-1)=0$

$(z^2+1)=0 \qquad (e^z-1)=0$

$z^2=-1 \qquad e^z-1=0$

$z=\pm i \qquad z=0$

$z=\pm i, 0$ are the zeros each of order 1.

10* $\sin^4(z/2)$

$f(z)=0 \Rightarrow$

$[\sin(z/2)]^4=0$

$\sin(z/2)=0$

$\frac{z}{2}= n\pi \quad , \quad n=0,\pm1,\pm2\ldots$

$z= 2n\pi \quad , \quad n=0,\pm1,\pm2\ldots$ are zeros each of order 4.

11* $f(z)= z\tan z$

$f(z)= 0 \Rightarrow$

$z\tan z=0$

$z=0$

$\tan z=0 \Rightarrow$

$z= n\pi \; , \; n= 0,\pm1,\pm2\ldots.$

$z=0, \; 0, \; n\pi \; , \; n= \pm1,\pm2\ldots$ ~~are the zeros of order 1~~

Here $z=0$ is a zero of order 2 while $z=n\pi \; , n=\pm1,\pm2\ldots$ are zeros of order 1. or simple zeros.

12* $f(z) = \frac{z-2}{z^2} \sin\left(\frac{1}{z-1}\right)$

$f(z) = 0 \Rightarrow$

$\frac{z-2}{z^2} \sin\left(\frac{1}{z-1}\right) = 0$

$\frac{z-2}{z^2} = 0$ or $\sin\left(\frac{1}{z-1}\right) = 0$

$z = 2$ or $\frac{1}{z-1} = n\pi$

$n\pi z - n\pi = 1$

$z = \frac{1+n\pi}{n\pi} = 1 + \frac{1}{n\pi}, n = \pm 1, \pm 2 \ldots$

$z = 2, 1+\frac{1}{n\pi}, n = \pm 1, \pm 2 \ldots$ are zeros each of order 1 or simple zeros.

$\frac{e^{-x} + e^{x}}{2}$

13* $f(z) = \cosh^4 z$

$f(z) = 0 \Rightarrow$

$\sinh z$

$(\cosh z)^4 = 0 \Rightarrow$

$\cosh z = 0$

$\cos iz = 0$

$iz = (2n+1)\frac{\pi}{2}$

$z = -i(2n+1)\frac{\pi}{2}, n = 0, \pm 1, \pm 2 \ldots$ are the zeros each of order 4.

14* $f(z) = \sinh^2 z$

$f(z) = 0 \Rightarrow$

$\sin(iz) = i\sinh z$
$\cos(iz) = \cosh z$

$\sinh z = 0$

$-i \sin iz = 0$

$iz = n\pi$

$z = -in\pi$, $n = 0, \pm 1, \pm 2 \ldots$ are the zeros each of order 2.

~

Singularities

* The points at which a complex function $f(z)$ fails to be analytic are called singular points.

Types of Singularities.

1. Removable singularity.

If the principle part of the laurents series expansion of $f(z)$ has no terms then the singular point z_0 is called a removable singularity of $f(z)$. In this case

$\lim_{z \to z_0} f(z)$ exists and is finate

2. Pole.

If the principle part of the laurents series expansion of $f(z)$ has finate no.of term then z_0 is called a pole. In the case

$\lim_{z \to z_0} f(z)$ in infinity.

3. Essential singularity.

If the principle part of the laurents series expansion of $f(z)$ has infinite no. of terms then z_0 is an essential singularity of $f(z)$. In this case $\lim_{z \to z_0} f(z)$ does not exist

REMARK If $f(z)$ is analytic and has a zero of order n at z_0 then $\frac{1}{f(z)}$ has a pole of order 'n'.

eg: $f(z) = (z-5)^4$

$z=5$ is a zero of order 4

$g(z) = \frac{1}{f(z)} = \frac{1}{(z-5)^4}$, $z=5$ is a pole of order 4

3. Essential singularity.

If the principle part of the laurents series expansion of $f(z)$ has infinite no. of terms then z_0 is an essential singularity of $f(z)$. In this case. $\lim_{z \to z_0} f(z)$ does not exist

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Q* Identify the singularities and their type

1* $\frac{\sin z}{z}$

$z=0$ is the singularity

$$\frac{\sin z}{z} = \frac{1}{z}\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \dots\right)$$

$$= 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \frac{z^6}{7!} + \dots$$

There is no element in the principal part

$\therefore$ it is a case of Removable singularity.

or

$\lim_{z \to z_0} f(z) = \lim_{z \to 0} \frac{\sin z}{z} = 1$ is finate

$\therefore$ it is a case of removable singularity.

2* $f(z) = \frac{\sin^2 z}{z^2}$

$z=0$ is the singularity.

$\lim_{z \to 0} \left(\frac{\sin z}{z}\right)^2 = 1$, which is finate

$\therefore$ $z=0$ is a removable singularity.

3* $f(z) = \frac{\sin(z-1)}{(z-1)^3}$

$z=1$ is the singularity

$$f(z) = \frac{1}{(z-1)^3}\left[(z-1) - \frac{(z-1)^3}{3!} + \frac{(z-1)^5}{5!} - \frac{(z-1)^7}{7!} + \cdots\right]$$

$$= \frac{1}{(z-1)^2} - \frac{1}{3!} + \frac{(z-1)^2}{5!} - \frac{(z-1)^4}{7!} + \cdots$$

No. of elements in the principal part

is finate

$\therefore$ $z=1$ is a pole of order 2 or double

pole.

Or

$$\lim_{z\to z_0} f(z) = \lim_{z\to 1} \frac{\sin(z-1)}{(z-1)^3}$$

$$= \lim_{z\to 0} \frac{\cos(z-1)}{3(z-1)^2} = \frac{1}{0} = \infty$$

ie $z=1$ is a pole of order 2.

4* $f(z) = z^2 - \frac{1}{z^2}$

$z=0$ is the singularity.

$f(z) = z^2 - \frac{1}{z^2}$

there is one element in the principal part.

$\therefore$ it is a pole of order 2. or double pole

5* $f(z) = e^{1/z}$

$z=0$ is the singularity.

$$e^{(1/z)} = 1 + \frac{1}{z} + \frac{1}{(z^2)2!} + \frac{1}{(z^3)3!} + \cdots$$

There are infinate no. of elements in the principal part $\therefore$ $z=0$ is an essential singularity

6* $f(z) = \tan\frac{\pi}{2}z$

$$f(z) = \frac{\sin\frac{\pi}{2}z}{\cos\frac{\pi}{2}z}$$

$\Rightarrow \frac{\pi}{2}z = (2n+1)\frac{\pi}{2}$

$Z = (2n+1)$, $n = 0, \pm 1, \pm 2 \cdots$

$$\lim_{Z \to \cancel{2n+1}\; z_0} f(z) =$$

for ~~z0~~ $n = 1$

$$\lim_{Z \to 1} \frac{\sin \frac{\pi}{2} z}{\cos \frac{\pi}{2} z} = \frac{1}{0} = \infty$$

$\therefore$ $Z = (2n+1)$, where $n = 0, \pm 1, \pm 2 \ldots$ are

~~exist~~ poles of order 1

7* $$f(z) = \frac{1}{\cos z - \sin z}$$

$\cos z = \sin z$

$\tan z = 1$

$Z = n\pi + \frac{\pi}{4}$, $n = 0, \pm 1, \pm 2 \ldots$ are poles of orders one.

8* $$f(z) = (z+i) e^{1/z+i}$$

$Z = i$ is the singular point.

$$f(z) = (z+i)\left[1 + \frac{1}{(z+i)} + \frac{1}{2!(z+i)^2} + \frac{1}{3!(z+i)^3} + \cdots\right]$$

$$= (z+i) + 1 + \frac{1}{2!(z+i)!} + \frac{1}{3!(z+i)^2} + \cdots$$

There are infinite no. of elements in principal part. $\therefore$ $z=-i$ is an essential singularity

9) $$f(z) = \frac{1}{(z+2i)^2} - \frac{2}{(z-i)} + \frac{z+1}{(z-i)^2}$$

$z=-2i$ is a pole of order 2

$$\frac{-2(z-i)+z+1}{(z-i)^2} = \frac{-z+2i+1}{(z-i)^2} = \frac{2i+1-z}{(z-i)^2}$$

$$\lim_{z\to 0} \frac{2i+1-z}{(z-i)^2} = \frac{i+1}{0} = \infty$$

$\therefore$ $z=i$ is a pole of order 2

10) $$f(z) = (z-\pi)^{-1} \sin z$$

$$\frac{\sin z}{z-\pi}$$

$z=\pi$ is the singular point

$$\lim_{z\to\pi} \frac{\sin z}{z-\pi} \cdot \frac{\sin(\pi-z)}{-(\pi-z)} = \underline{\underline{-1}}$$

$\therefore$ $z=\pi$ is a removable singularity.

11* $f(z) = \frac{1}{z^4-1}$

$z^4 - 1 = (z^2)^2 - 1^2 = (z^2+1)(z^2-1) = 0$

$= z^2 = -1, \quad z = \pm i$

$z^2 = 1, \quad z = \pm 1$

$z = \pm i, \pm 1$ are simple poles.

12* $\frac{z - \sin z}{z^3}$

$\lim_{z \to 0} \frac{z - \sin z}{z^3} = \lim_{z \to 0} 1 - \cos z$

$f(z) = \frac{1}{z^3}\left[z - z + \frac{z^3}{3!} - \frac{z^5}{5!} + \cdots\right]$

$= \frac{1}{3!} - \frac{z^2}{5!} + \frac{z^4}{7!} - \cdots$

There is no terms in the principal part.

$z=0$ is a removable singularity.

13* $\frac{e^{3z}}{(z-3)^3}$

$z = 3$ is the singular point.

$$\lim_{z \to 3} \frac{e^{3z}}{(z-3)^3} = \frac{e^9}{0} = \infty$$

$\therefore$ $z = 3$ is a pole of order 3.

$$\frac{e^{3z}}{(z-3)^3} = \frac{e^{3(z-3)+9}}{(z-3)^3}$$

$$= e^9 \cdot \frac{e^{3(z-3)}}{(z-3)^3}$$

$$= \frac{e^9}{(z-3)^3}\left[1 + \frac{3(z-3)}{1} + \frac{3^2(z-3)^2}{2!} + \cdots\right]$$

Here principal part has finite no. of elements $\therefore$ it is a pole of order 3.

$z \to 0$

14* $\frac{1}{z\sin z}$

$$\lim_{z \to 0, \infty} \frac{1}{z\sin z} = \infty$$

$\therefore$ ~~$z=0, n\pi, 0, \pm 1 \ldots$ are poles of order~~

$z=0$ is a double pole

$z=n\pi$, $n=\pm1, \pm2 \ldots$ are simple pole

1.5* $f(z) = \cosh(z^2+4)^{-1}$

$$f(z) = \cosh\left(\frac{1}{z^2+4}\right)$$

$$= 1 + \frac{1}{(z^2+4)^2 2!} + \frac{1}{(z^2+4)^4 4!} + \ldots$$

$z = \pm 2i$ are essential singularity

Module. 5 ⇒ chapter. 3

consider the laurent series expansion of $f(z)$:

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=0}^{\infty} b_n (z-z_0)^{-n}$$

$b_1 =$ Coefficient of $\frac{1}{z-z_0}$ is called the residue of $f(z)$ about $z=z_0$ and is denoted by $\underset{z=z_0}{\text{Res}} f(z)$

Q. Find the Residue of the following fn at their singularity.

1. $f(z) = e^{1/z}$

$z = 0$ is the singular point.

$$e^{1/z} = 1 + \frac{1}{z} + \frac{1}{z^2 2!} + \cdots$$

$$\underset{z=0}{\text{Res}}\, f(z) = \text{coeff } \frac{1}{z} = \underline{\underline{1}}$$

2. $f(z) = \frac{\sin z}{z^4}$

$z = 0$ is the singular point

$$\frac{\sin z}{z^4} = \frac{1}{z^4}\left[z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots\right]$$

$$= \frac{1}{z^3} - \frac{1}{z\,3!} + \frac{z}{5!} - \frac{z^3}{7!} + \cdots$$

$$\underset{z=0}{\text{Res}}\, f(z) = \text{coeff } \frac{1}{(z-0)} = \underline{\underline{\frac{-1}{3!}}} = \underline{\underline{\frac{-1}{6}}}$$

3. $e^{\frac{1}{1-z}}$

$z = 1$ is the singular point.

$$e^{\cancel{\frac{1}{1-z}}} = \cancel{1 + \frac{1}{1-z} + \frac{1}{(1-z)^2} + \frac{1}{(1-z)^3} + \cdots}$$

$$e^{-\frac{1}{z-1}} = 1 - \frac{1}{z-1} + \frac{1}{(z-1)^2} + \cdots$$

$$\underset{z=1}{\text{Res}}\, e^{\frac{1}{1-z}} = \text{coeff}\left(\frac{1}{z-1}\right) = \underline{\underline{-1}}$$

4* $\dfrac{\sin 2z}{z^6}$

$z=0$ is a singular point

$$\frac{\sin 2z}{z^6} = \frac{1}{z^6}\left[2z - \frac{2^3 z^3}{3!} + \frac{2^5 z^5}{5!} - \frac{2^7 z^7}{7!} + \dots\right]$$

$$= \frac{2}{z^5} - \frac{2^3}{3!\,z^3} + \frac{2^5}{5!\cdot z} - \frac{2^7}{7!}z + \dots$$

$$\operatorname*{Res}_{z=0} \frac{\sin 2z}{z^6} = \text{coeff } \frac{1}{z} = \frac{2^5}{5!} = \frac{32}{120} = \underline{\underline{\frac{4}{15}}}$$

5* $\dfrac{1-e^{2z}}{z^4}$

$z=0$ is the singular point

$$\frac{1-e^{2z}}{z^4} = \frac{1}{z^4} - \frac{1}{z^4}\left[1 + 2z + \frac{4z^2}{2!} + \frac{2^3 z^3}{3!} + \frac{2^4 z^4}{4!} + \frac{2^5 z^5}{5!}\right.$$

$$= \cancel{\frac{1}{z^4}} - \cancel{\frac{1}{z^4}} - \frac{2}{z^3} - \frac{4}{2z^2} - \frac{2^3}{3!\,z} - \frac{2^4}{4!} - \frac{2^5 z}{5!} + \dots$$

$$\operatorname*{Res}_{z=0} \frac{1-e^{2z}}{z^4} = \text{coeff } \frac{1}{z} = -\frac{2^3}{3!} = \cancel{\frac{4}{3}} = \underline{\underline{-\frac{4}{3}}}$$

6* $\dfrac{\sinh z}{z^4}$

$z = 0$ is the singular point

$$\frac{\sinh z}{z^4} = \frac{1}{z^4}\left[z + \frac{z^3}{3!} + \frac{z^5}{5!} + \frac{z^7}{7!} + \cdots\right]$$

$$= \frac{1}{z^3} + \frac{1}{3!\,z} + \frac{z}{5!} + \frac{z^3}{7!} + \cdots$$

$$\operatorname*{Res}_{z=0} f(z) = \frac{1}{6}$$

→ Important Formulas

① Residue at a simple pole is given by,

$$\operatorname*{Res}_{z=z_0} f(z) = \lim_{z\to z_0} \left[(z-z_0) f(z)\right]$$

(OR)

$$\operatorname*{Res}_{z=z_0} \frac{h(z)}{g(z)} = \lim_{z\to z_0} \frac{h(z)}{g'(z)}$$

② If $f(z)$ has a pole of order m at z_0 then,

$$\operatorname*{Res}_{z=z_0} f(z) = \frac{1}{(m-1)!} \lim_{z\to z_0} \left[\frac{d^{m-1}}{dz^{m-1}} (z-z_0)^m f(z)\right]$$

Q. Find the singularities and corresponding residues of the following functions.

(1) $f(z) = \dfrac{1}{z^4 - 1}$

Ans: $(z^2+1)(z^2-1) = 0$

$z^2 = -1$ or $z^2 = 1$

$\Rightarrow z = \pm i$, $z = \pm 1$ (each are simple poles)

$\boxed{At\ z = 1}$

$$\underset{z=z_0}{Res}\ f(z) = \lim_{z\to z_0} [(z-z_0) f(z)].$$

$$\underset{z=1}{Res}\ f(z) = \lim_{z\to 1} \left[(z-1)\frac{1}{z^4-1}\right]$$

$$= \lim_{z\to 1} \left[\frac{1}{(z+1)(z^2+1)}\right]$$

$$= \frac{1}{(1+1)(1+1)} = \frac{1}{4}.$$

(or) $$\underset{z=z_0}{Res}\ \frac{h(z)}{g(z)} = \lim_{z\to z_0} \frac{h(z)}{g'(z)}$$

$$\underset{z=1}{Res}\left(\frac{1}{z^4-1}\right) = \lim_{z\to 1} \frac{1}{4z^3}$$

$$= \frac{1}{4}$$

$\boxed{At\ z = -1,}$

$$\underset{z=-1}{Res}\left(\frac{1}{z^4-1}\right) = \lim_{z\to -1} \left((z+1)\left(\frac{1}{z^4-1}\right)\right)$$

$$= \lim_{z\to -1} \left[\frac{1}{(z^2+1)(z-1)}\right]$$

$$= \frac{1}{(2)(-2)} = \frac{-1}{4}$$

(or)

$$\underset{z=-1}{Res}\left(\frac{1}{z^4-1}\right) = \lim_{z\to -1} \frac{1}{4z^3}$$

$$= \frac{-1}{4}.$$

$\boxed{\text{At } z = i}$

$$\operatorname{Res}_{z=i}\left(\frac{1}{z^4-1}\right) = \lim_{z\to i} \frac{1}{4z^3} = \frac{1}{4i^3} = \frac{i}{4}$$

$\boxed{\text{At } z = -i}$

$$\operatorname{Res}_{z=-i}\left(\frac{1}{z^4-1}\right) = \lim_{z\to -i} \frac{1}{4z^3} = \frac{1}{-4i^3} = \frac{-i}{4}$$

(2) $f(z) = \dfrac{9z+i}{z^3+z}$

A. $z^3+z=0$

$\Rightarrow z(z^2+1)=0$

$\Rightarrow z=0, \pm i$

(all are simple pole)

$\boxed{z=0}$

$$\operatorname{Res}_{z=0} \frac{9z+i}{z^3+z} = \lim_{z\to 0}\left[\frac{9z+i}{3z^2+1}\right]$$

$$= \frac{i}{1} = i.$$

$\boxed{z=i}$

$$\operatorname{Res}_{z=i} \frac{9z+i}{z^3+z} = \lim_{z\to i}\left(\frac{9z+i}{3z^2+1}\right) = \frac{9i+i}{3(i^2)+1}$$

$$= \frac{10i}{-2} = -5i$$

$\boxed{z=-i}$

$$\operatorname{Res}_{z=-i} \frac{9z+i}{z^3+z} = \lim_{z\to -i}\left(\frac{9z+i}{3z^2+1}\right)$$

$$= \frac{-9i+i}{(3)(-i)^2+1} = \frac{-8i}{-3+1} = \frac{-8i}{-2}$$

$$= 4i$$

(3) $\dfrac{8}{1+z^2}$

A: $1+z^2=0$

$\Rightarrow z=\pm i$. (simple poles)

$\boxed{z=i}$

$$\operatorname*{Res}_{z=z_0} f(z) = \lim_{z\to z_0} \frac{h(z)}{g'(z)}$$

$$\operatorname*{Res}_{z=i} \frac{8}{1+z^2} = \lim_{z\to i} \frac{8}{2z} = \frac{8}{2i} = \underline{\underline{-4i}}.$$

$\boxed{z=-i}$

$$\operatorname*{Res}_{z=-i} f(z) = \lim_{z\to -i} \frac{8}{2z} = \frac{8}{2(-i)} = \frac{-4}{i} = \underline{\underline{4i}}$$

(4) $\dfrac{\cot \pi z}{(z-a)^2}$

A: $(z-a)^2=0$

$\Rightarrow z=a$ <u>double pole</u>.

$$\operatorname*{Res}_{z=z_0} f(z) = \frac{1}{(m-1)!} \lim_{z\to z_0} \left[\frac{d^{m-1}(z-z_0)^m}{dz^{m-1}} f(z)\right]$$

Since it is a <u>double pole</u>, $m=2$.

$$\Rightarrow \operatorname*{Res}_{z=a} \frac{\cot \pi z}{(z-a)^2} = \frac{1}{(2-1)!} \lim_{z\to a} \left[\frac{d(z-a)^2}{dz} \frac{\cot \pi z}{\cancel{(z-a)^2}}\right]$$

$$= \frac{1}{1!} \lim_{z\to a} \left[\frac{d \cot \pi z}{dz}\right]$$

$$= \lim_{z\to 0} \left[-\pi \operatorname{cosec}^2 \pi z\right] = \underline{\underline{-\pi \operatorname{cosec}^2 \pi a}}$$

(5) $\dfrac{50z}{z^3+2z^2-7z+4}$

A:

$z^3+2z^2-7z+4=0$.

$z=1$ is a root

$(z-1)(z^2+3z-4)=0$

$$\begin{array}{r|rrrr} 1 & 1 & 2 & -7 & 4 \\ & 0 & 1 & 3 & -4 \\ \hline & 1 & 3 & -4 & \boxed{0} \end{array}$$

$(z-1)(z+4)(z-1)=0$

$\Rightarrow (z-1)^2(z+4)=0$

$\Rightarrow z=1,1,-4$

$z=1$ is double pole
$z=-4$ is simple pole

$\boxed{z=1}$ is a double pole,

$$\Rightarrow \operatorname{Res}_{z=z_0} f(z)=\frac{1}{(m-1)!}\lim_{z\to z_0}\left[\frac{d^{m-1}(z-z_0)^m f(z)}{dz^{m-1}}\right]$$

$$=\frac{1}{1!}\lim_{z\to 1}\left[\frac{d(z-1)^2}{dz}\frac{50z}{(z-1)^2(z+4)}\right]$$

$$=\lim_{z\to 1}\left(\frac{d}{dz}\left(\frac{50z}{z+4}\right)\right)$$

$$=\lim_{z\to 1}\left[\frac{(z+4)(50)-50z(1)}{(z+4)^2}\right]$$

$$=\frac{250-50}{5^2}=\frac{200}{25}=\underline{\underline{8}}$$

$\boxed{z=-4}$ is simple pole

$$\operatorname{Res}_{z=-4} f(z)=\lim_{z\to -4}\left(\frac{50z}{3z^2+4z-7}\right)$$

$$=\frac{50(-4)}{3(16)+4(-4)-7}=\frac{-200}{25}=\underline{\underline{-8}}$$

(6) Find the residue at $z=0$ for the fn. $f(z) = \dfrac{1+e^z}{\sin z + z\cos z}$

$z_0 = 0$ is a simple pole.

Ans: $\operatorname{Res}_{z=0} f(z) = \lim_{z \to 0} \left[\dfrac{1+e^z}{\cos z + (-z\sin z) + \cos z} \right]$

$= \dfrac{1+1}{1+1} = \underline{\underline{1}}$

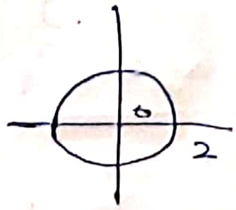
Chapter - 4 Residue Theorem

Let $f(z)$ be analytic in a simple closed curve C, except for finitely many singularities z_1, z_2, z_3 etc... z_k inside C. Then integral of $f(z)$ taken in the anti-clockwise direction around C equals $(2\pi i)$ times sum of the residues of $f(z)$ at $z = z_1, z_2, z \ldots\ldots z_k$.

ie, $\oint_C f(z)\,dz = 2\pi i$ (sum of residues at $z_1, z_2 \cdots z_k$)

61. $\oint \tan \pi z \, dz$

10/10/2020 $c : |z| = 2$.



$\frac{\sin \pi z}{\cos \pi z}$

$\cos \pi z = 0$.

$\pi z = (2n+1)\pi/2$

$z = \frac{(2n+1)}{2}$ $\quad n = 0, \pm 1, \pm 2$.

$n = 0$. $\quad z = 1/2$ (Inside)

$n = 1$ $\quad z = 3/2$ (Inside)

$n = -1$ $\quad z = -1/2$ (Inside)

$n = 2$ $\quad z = 5/2$ (Outside)

$n = -2$ $\quad z = -3/2$ (Inside)

The singular pts $z = 1/2, 3/2, -1/2, -3/2$ lies inside

$$\operatorname*{Res}_{z=1/2} f(z) = \lim_{z \to 1/2} \frac{\sin \pi z}{-\sin \pi z \times \pi} = -1/\pi$$

$$\operatorname*{Res}_{z=3/2} f(z) = \lim_{z \to 3/2} \frac{\sin \pi z}{-\sin \pi z \times \pi} = -1/\pi$$

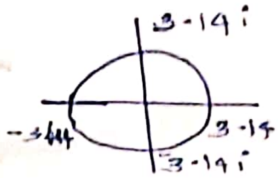
$$\operatorname*{Res}_{z=-1/2} f(z) = \lim_{z \to -1/2} \frac{\sin \pi z}{-\sin \pi z \, \pi} = -1/\pi$$

$$\operatorname*{Res}_{z=-3/2} f(z) = \lim_{z \to -3/2} \frac{\sin \pi z}{-\sin \pi z \times \pi} = -1/\pi$$

$$\oint_{c:|z|=2} \tan \pi z \, dz = 2\pi i \times 4 \times -1/\pi$$

$$= \underline{-8i}$$

62. $\oint_C \frac{z\cosh \pi z}{z^4+13z^2+36}\, dz \qquad c : |z| = \pi.$



$z^4 + 13z^2 + 36.$

put $z^2 = t$.

$t^2 + 13t + 36 = 0.$

$$t = \frac{-13 \pm \sqrt{169 - 144}}{2}$$

$$= -\frac{13 \pm 5}{2} = -8/2 \text{ or } -18/2$$

$$= -4 \text{ or } -9.$$

$t = -4, -9.$

$z = \pm 2i, \pm 3i$

~~Re~~ All the four singular points are ~~outside~~ inside c.

$$\operatorname*{Res}_{z=3i} f(z) = \lim_{z \to 3i} \frac{z\cosh \pi z}{4z^3 + 26z}.$$

$$= \frac{3i \cosh \pi 3i}{4 \times 27 \times i + 26 \times 3i}$$

$$= \frac{3i \cosh \pi 3i}{78i - 108i}$$

$$= \frac{\cancel{3i} \cosh \pi 3i}{-\cancel{30i}\, 10} = \frac{-\cosh \pi 3i}{10}$$

$$= \frac{-\left[e^{3i\pi} + e^{3i\pi}\right]}{20}$$

$$= \frac{-(\cos 3\pi + i\sin 3\pi + \cos 3\pi - i\sin 3\pi)}{20}$$

$$= \frac{-\cos 3\pi}{10}$$

$$= 1/10$$

$$\operatorname*{Res}_{z=-3i} f(z) = \lim_{z \to -3i} \frac{z\cosh \pi z}{4z^3 + 26z}$$

$$= \frac{-3i \cosh \pi (3i)}{4 \times (-3i)^3 - 26 \times 3i}$$

$$= \frac{-3i \cosh (3\pi i)}{108i - 78i}$$

$$= \frac{-\cosh (3\pi i)}{10}$$

$$= -\frac{\left[e^{-3\pi i} + e^{3\pi i}\right]}{20}$$

$$= -\frac{(\cos 3\pi - i\cancel{\sin 3\pi} + \cos 3\pi + i\cancel{\sin 3\pi})}{20}$$

$$= -\frac{\cos 3\pi}{10} = \underline{\underline{1/10}}$$

$$\underset{z=2i}{\text{Res}}\, f(z) = \lim_{z \to 2i} \frac{z \cosh \pi z}{4z^3 + 26z}$$

$$= \frac{2i \cosh 2\pi i}{-32i + 52i}$$

$$= \frac{2i \cosh 2\pi i}{20i}$$

$$= 1/10 \cosh 2\pi i$$

$$= 1/20 \left[e^{2\pi i} + e^{-2\pi i}\right]$$

$$= \frac{1}{20} (\cos 2\pi + i\cancel{\sin 2\pi} + \cos 2\pi - i\cancel{\sin 2\pi})$$

$$= \underline{\underline{1/10}}$$

$$\underset{z=-2i}{\text{Res}}\, f(z) = \lim_{z \to -2i} \frac{z \cosh \pi z}{4z^3 + 26z}$$

$$= \frac{-2i \cosh(-2\pi i)}{+4 \times 8i - 52i}$$

$$= \frac{2i \cosh(-2\pi i)}{20i}$$

$$= \underline{\underline{1/10}}$$

$$\oint_c \frac{z\cosh \pi z}{z^4+13z^2+36}\,dz = 2\pi i \times 4 \times \frac{1}{10}$$

$$= \frac{4\pi i}{5}$$

Imp 63.

$$\oint \frac{ze^{\pi z}}{z^4-16} + ze^{\pi/z}\,dz \qquad c: 9x^2+y^2=9.$$

$z^4-16 \rightarrow$ simplepole

$ze^{\pi/z} \rightarrow$ essential singularity

$z^4-16=0$ $\qquad z=0.$

$(z^2-4)(z^2-4)=0$

$z = \pm 2, \ \pm 2i$

$z=\pm 2, \pm 2i$ are simplepoles hence residue found using Cauchy's Residue Theorem. and for $z=0$ coefficient of $1/z$ is L-S expansion.

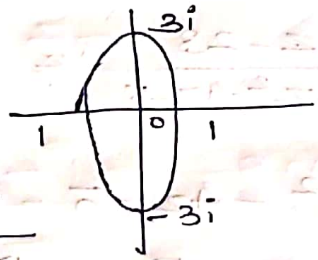
$9x^2+y^2=9 \qquad \frac{x^2}{1^2}+\frac{y^2}{3^2}=1$

ellipse with major axis $\rightarrow$ y axis

minor axis $\rightarrow$ x axis

$-2 \ \& \ 2 \rightarrow$ outside

$0, 2i, -2i \rightarrow$ Inside



$$\underset{z=2i}{\text{Res}}\, f(z) = \lim_{z\to 2i} \frac{ze^{\pi z}}{4z^3}$$

$$= \frac{2ie^{\pi \times 2i}}{-32i}$$

$$= \frac{-e^{2\pi i}}{16} = \frac{-(\cos 2\pi + i\sin 2\pi)}{16}$$

$$= \frac{-1}{16}$$

$$\underset{z=-2i}{\text{Res}}\, f(z) = \lim_{z\to -2i} \frac{ze^{\pi z}}{4z^3}$$

$$= \frac{-2ie^{-2\pi i}}{32i} = \frac{-1}{16}$$

$$ze^{\pi/2}$$

$$= \left[1 + \frac{\pi}{z \times 1!} + \left(\frac{\pi}{z}\right)^2 \times \frac{1}{2!} + \frac{\pi^3}{z^3 \times 3!} + \cdots\right]$$

$$\left[z + \pi + \frac{\pi^2}{2!\,z} + \frac{\pi^3}{z^2 3!} + \cdots\right]$$

$$z + \pi + \frac{\pi^2}{2z} + \frac{\pi^3}{6z^2} + \cdots$$

co-efficient of $1/z$ is $\pi^2/2$.

$$\underset{z=0}{\text{Res}}\ \underset{(ze^{\pi/z})}{f(z)} = \pi^2/2.$$

$$\oint_C \frac{ze^{\pi z}}{z^4-16} + ze^{\pi/z}\, dz = 2\pi i\left(-1/8 + \pi^2/2\right)$$

$$= \frac{\pi i.\,(4\pi^2 - 1)}{4}$$

J. Imp

64. $\oint_C \frac{z-23}{z^2-4z-5}\, dz \qquad C: |z-2-i| = 3.2$

$$z^2 - 4z - 5 = 0$$

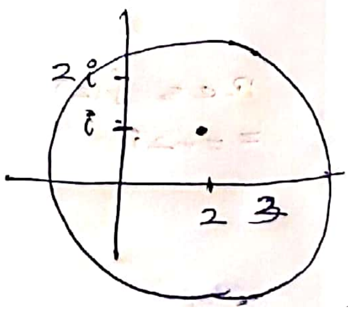
$z = 5$ and -1 are the singular points.

$$|z - 2+i| = 3.2$$

$$|z - (i+2)| = 3.2$$

centre : $i+2$ $(2+i)$

radius : 3.2



$z=5$

$$|5-2-i| = 3.2$$

$$|3-i| = 3.2$$

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$\sqrt{10} = 3.16$.

lies inside the circle.

$z = -1$.

$|-1-2+i| = \sqrt{10} = \underline{\underline{3.16}}$ inside c.

$$\operatorname*{Res}_{z=5} f(z) = \lim_{z \to 5} \frac{z-23}{2z-4}$$

$$= \frac{5-23}{10-4} = -18/6 = \underline{\underline{-3}}$$

$$\operatorname*{Res}_{z=-1} f(z) = \lim_{z \to -1} \frac{z-23}{2z-4}$$

$$= \frac{-24}{-6} = \underline{\underline{4}}$$

$$\oint_c \frac{z-23}{z^2-4z-5}\,dz = 2\pi i(-3+4)$$

$$= \underline{\underline{2\pi i}}$$

Chapter 5

Residue Integration :

Type I :

Integrals of Rational fn's of $\cos\theta$ and $\sin\theta$;

Consider the integral of the type,

$$I = \int_0^{2\pi} F[\cos\theta, \sin\theta]\, d\theta. \quad \text{——— (1)}$$

$$\text{or } I = \int_{-\pi}^{\pi} F(\cos\theta, \sin\theta)\, d\theta. \quad \text{——— (1)}.$$

put $z = e^{i\theta}$

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} = \frac{z + 1/z}{2}.$$

$$\boxed{\cos\theta = \frac{z^2+1}{2z}}$$

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2} = \frac{z^2-1}{2iz}$$

$$\boxed{\sin\theta = \frac{z^2-1}{2zi}}$$

$z = e^{i\theta}$

$dz = ie^{i\theta}\, d\theta$

$dz = iz\, d\theta$

$$\boxed{d\theta = \frac{dz}{iz}}$$

$$\oint_C f(z)\,\frac{dz}{iz}.$$

where $c : |z| = 1$ c is the unit circle $|z|=1$ in the anticlockwise direction.

$$z = e^{i\theta}$$

$$\sin 2\theta = 2\cos\theta\sin\theta$$

$$= 2 \times \frac{z^2+1}{2z} \times \frac{(z^2-1)}{2zi}$$

$$\boxed{\sin 2\theta = \frac{z^4-1}{2z^2 i}}$$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$

$$= \left(\frac{z^2+1}{2z}\right)^2 - \left(\frac{z^2-1}{2zi}\right)^2$$

$$= \frac{2z^4+2}{4z^2}$$

$$\boxed{\cos 2\theta = \frac{z^4+1}{2z^2}}$$

$$\oint_C f(z)\,\frac{dz}{iz}.$$

where $c: |z| = 1$ — c is the unit circle $|z|=1$ in the anticlockwise direction.

$z = e^{i\theta}$

$\sin 2\theta = 2\cos\theta\sin\theta$

$= 2 \times \frac{z^2+1}{2z} \times \frac{(z^2-1)}{2zi}$

$$\boxed{\sin 2\theta = \frac{z^4-1}{2z^2 i}}$$

$\cos 2\theta = \cos^2\theta - \sin^2\theta$

$= \left(\frac{z^2+1}{2z}\right)^2 - \left(\frac{z^2-1}{2zi}\right)^2$

$= \frac{2z^4+2}{4z^2}$

$$\boxed{\cos 2\theta = \frac{z^4+1}{2z^2}}$$

13/10/2020 Evaluate the following integrals.

Type I

65. $\displaystyle\int_0^{2\pi} \frac{d\theta}{\sqrt{2}-\cos\theta}$

put $z = e^{i\theta}$ $\qquad \cos\theta = \frac{z^2+1}{2z}$

$dz = i e^{i\theta} d\theta$

$dz = iz\,d\theta$

$d\theta = dz/iz$

$$\int_{|z|=1} \frac{1}{\sqrt{2}-\left(\frac{z^2+1}{2z}\right)} \frac{dz}{iz}$$

$$\frac{1}{i}\int_{|z|=1} \frac{2}{2\sqrt{2}z - z^2 - 1}\, dz$$

$$\frac{-1}{i}\int_{|z|=1} \frac{dz}{z^2 - 2\sqrt{2}z + 1}$$

– Numerator constant hence pole.

$$z^2 - 2\sqrt{2}z + 1 = 0$$

Let $I = \int_{|z=1} \frac{dz}{z^2 - 2\sqrt{2}z + 1}$

$$z^2 - 2\sqrt{2}z + 1 = 0.$$

$$\frac{2\sqrt{2} \pm \sqrt{8-4}}{2}$$

Singularpts:-

$\sqrt{2} \pm 1$

$$\frac{2\sqrt{2} \pm \sqrt{4}}{2}.$$

$\sqrt{2}+1 \approx 2.414$

$\sqrt{2}-1 \approx .414$

The singular pt $\sqrt{2}+1$ lies outside $|z|=1$

where $\sqrt{2}-1$ lies inside. $|z|=1$

$$\operatorname{Res}_{z=\sqrt{2}-1} f(z) = \lim_{z \to \sqrt{2}-1} \frac{1}{2z - 2\sqrt{2}}$$

$$= \frac{1}{2(\sqrt{2}-1) - 2\sqrt{2}}$$

$$= \frac{1}{\cancel{2\sqrt{2}} - 2 - \cancel{2\sqrt{2}}} = \underline{\underline{-\tfrac{1}{2}}}$$

$$\therefore \int_0^{2\pi} \frac{d\theta}{\sqrt{2}-\cos\theta} = \frac{-2}{i}\int_{|z|=1} \frac{dz}{z^2-2\sqrt{2}z+1}$$

$$\Rightarrow \frac{-2}{i}\int_{|z|=1} \frac{dz}{z^2-2\sqrt{2}z+1} = \frac{-2}{i}\left(2\pi i \times \frac{-1}{2}\right)$$

$$= \underline{\underline{2\pi}}.$$

66. $\int_0^{2\pi} \frac{\cos\theta}{13-12\cos 2\theta}\, d\theta.$ Type I

put $z = e^{i\theta}$

$d\theta = \frac{dz}{iz}$

$$\cos\theta \longrightarrow \frac{z^2+1}{2z}.$$

$$\cos 2\theta \longrightarrow \frac{z^4+1}{2z^2}$$

$$\int_{|z|=1} \frac{\frac{z^2+1}{2z}}{13-12\left(\frac{z^4+1}{2z^2}\right)} \frac{dz}{iz}.$$

$$\frac{1}{i}\int_{|z|=1} \frac{\frac{z^2+1}{2z}}{\frac{26z^2-12z^4-12}{2z^2}} \frac{dz}{z}$$

$$\frac{1}{i}\int_{|z|=1} \frac{(z^2+1)\cancel{2z^2}}{(26z^2-12z^4-12)\cancel{2z^2}}\, dz$$

$$\frac{1}{i}\int_{|z|=1} \frac{z^2+1}{26z^2-12z^4-12}\, dz$$

$$\frac{-1}{i}\int_{|z|=1} \frac{z^2+1}{12z^4-26z^2+12}\, dz$$

$$I = \int_{|z|=1} \frac{z^2+1}{12z^4-26z^2+12}\, dz$$

$12z^4 - 26z^2 + 12 = 0$. (singular pts)

put $z^2 = t$.

$12t^2 - 26t + 12 = 0$.

$$\frac{26 \pm \sqrt{676-576}}{24}$$

$$\frac{26 \pm 10}{24} = \frac{36}{24} \text{ or } \frac{16}{24}$$

$= 1.5$ or $.66$.

$z = \pm 1.2$ and $\pm .812$

$= \pm\sqrt{3/2}$ or $\pm\sqrt{2/3}$

The singular pts are poles.

The singular pt inside c are $z = \pm\sqrt{2/3}$

The pt $\pm\sqrt{3/2}$ are outside $|z| = 1$

$$\underset{z=\sqrt{2/3}}{\text{Res}} f(z) = \lim_{z \to \sqrt{2/3}} \frac{z^2+1}{48z^3 - 52z}$$

$$= \frac{\frac{2+3}{3}}{48 \times \frac{2}{3} \times \sqrt{\frac{2}{3}} - 52\sqrt{\frac{2}{3}}}$$

$$= \frac{\frac{5}{3}}{-\frac{20}{4}\sqrt{2/3}} = \frac{-\sqrt{3}}{12\sqrt{2}}$$

$$\underset{z=-\sqrt{2/3}}{\text{Res}} f(z) = \lim_{z \to -\sqrt{2/3}} \frac{z^2+1}{48z^3-52z}$$

$$= \frac{\frac{2}{3}+1}{-48 \times \frac{2}{3}\sqrt{\frac{2}{3}} + 52\sqrt{\frac{2}{3}}}$$

$$= \frac{5/3}{\cancel{20}\,4\sqrt{2/3}} = \frac{\sqrt{3}}{12\sqrt{2}}$$

$$\int_0^{2\pi} \frac{\cos\theta}{13-12\cos 2\theta}\, d\theta = \frac{-1}{i} \int_{|z|=1} \frac{z^2+1}{12z^4-26z^2+12}$$

$$= \frac{-1}{i}\left[2\pi i \times \frac{-\sqrt{3}}{12\sqrt{2}} + 2\pi i \times \frac{\sqrt{3}}{12\sqrt{2}}\right]$$

$$= \underline{\underline{0}}$$

67. $$\int_0^{2\pi} \frac{\sin^2\theta}{5-4\cos\theta}\, d\theta$$ (Type I)

put $z = e^{i\theta}$

$d\theta = dz/iz$

$$\sin^2\theta = \left(\frac{z^2-1}{2zi}\right)^2$$

$$\cos\theta = \frac{z^2+1}{2z}$$

$$\sin^2\theta = \frac{z^4-2z^2+1}{-4z^2}$$

$$\frac{1}{i}\int_{|z|=1} \frac{\frac{z^4-2z^2+1}{-4z^2}}{5-4\frac{(z^2+1)}{2z}} \frac{dz}{z}$$

$$\frac{1}{i}\int_{|z|=1} \frac{\frac{z^4-2z^2+1}{-4z^2}}{\frac{10z-4z^2-4}{2z}} \frac{dz}{z}$$

$$\frac{1}{i}\int_{|z|=1} \frac{(z^4-2z^2+1)\,\cancel{2z}}{\frac{(10z-4z^2)\times -4z^2}{2}} \times \frac{dz}{z}$$

$$-\frac{1}{i}\int_{|z|=1} \frac{z^4-2z^2+1}{20z^3-8z^4-8z^2}\,dz$$

$$\frac{1}{i}\int_{|z|=1} \frac{z^4-2z^2+1}{8z^4-20z^3+8z^2}\,dz$$

$$\frac{1}{2i}\int_{|z|=1} \frac{z^4-2z^2+1}{4z^4-10z^3+4z^2}\,dz.$$

$$I=\int_{|z|=1} \frac{z^4-2z^2+1}{4z^4-10z^3+4z^2}\,dz$$

Singular pts;

$$4z^4-10z^3+4z^2=0.$$

$$z^2(4z^2-10z+4)=0.$$

$$z^2=0 \qquad 4z^2-10z+4=0.$$

$$z=0,0 \qquad z=2,\ 1/2$$

(Double pole) (Simple pole)

$\frac{10\pm\sqrt{100}}{}$ $\frac{10\pm 6}{8}$ $\frac{16}{8}^{2}$ $\frac{4}{8}^{1}_{2}$

The singular points $z=0$ & $1/2$ are inside c while $z=2$ is outside c.

$$\underset{z=1/2}{\text{Res}}\ f(z) = \lim_{z\to 1/2} \frac{z^4-2z^2+1}{16z^3-30z^2+8z}$$

$$= \frac{1/16 - 2\times 1/4 + 1}{16\times 1/8 - 30\times 1/4 + 8/2}.$$

$$\frac{1/16 - 1/2 + 1}{2 - 15/2 + 4} = \frac{9/16}{-3/2}$$

$$= \underline{\underline{-3/8}}$$

$1/16 + 1/2$.

$\frac{1+8}{16}$.

$6 - 15/2$.

$\frac{-3}{2}$.

$\frac{\cancel{9}^3}{16\ 8} \times \frac{2}{\cancel{3}}$

For $z = 0$;

$$\operatorname*{Res}_{z=0} f(z) = \frac{1}{(m-1)!} \lim_{z \to 0} \left[\frac{d^{m-1}}{dz^{m-1}} (z-0)^m f(z)\right]$$

$$= \frac{1}{1!} \lim_{z \to 0} \left[d/dz \; z^2 f(z)\right]$$

$$= \lim_{z \to 0} \left[\frac{d}{dz} \left(\cancel{z}^2 \times \frac{(z^4 - 2z^2 + 1)}{\cancel{z}^2 (4z^2 - 10z + 4)}\right)\right]$$

$$= \lim_{z \to 0} \left[\frac{d}{dz} \left(\frac{z^4 - 2z^2 + 1}{4z^2 - 10z + 4}\right)\right]$$

$$= \lim_{z \to 0} \frac{(4z^2 - 10z + 4)(4z^3 - 4z) - (z^4 - 2z^2 + 1)(8z - 10)}{(4z^2 - 10z + 4)^2}$$

$$= 10/16 = \underline{\underline{5/8}}$$

$$\int_0^{2\pi} \frac{\sin^2\theta}{5 - 4\cos\theta}\, d\theta = \frac{1}{2i} \int_{|z|=1} \frac{z^4 - 2z^2 + 1}{4z^4 - 10z^3 + 4z^2}$$

$$= \frac{1}{2i} \left[2\pi i \left(-3/8 + 5/8\right)\right]$$

$$= \frac{1}{\cancel{2i}} \times \cancel{2}\pi \cancel{i} \times \frac{\cancel{2}}{\cancel{8}\,4} = \underline{\underline{\pi/4}}$$

Q.
68.

$$\int_0^{\pi} \frac{2\,d\theta}{k-\cos\theta}$$

put $d\theta = dz/iz$

$$\frac{1}{2} \times \int_{-\pi}^{\pi} \frac{2\,d\theta}{k-\cos\theta}$$

$\cos\theta = \frac{z^2+1}{2z}$

$$\frac{1}{2i} \int_{|z|=1} \frac{2}{k-\frac{(z^2+1)}{2z}} \frac{dz}{z}$$

$$\frac{1}{2i} \int_{|z|=1} \frac{4\cancel{z}}{(2kz-z^2-1)\cancel{z}}\, dz$$

$$\frac{\cancel{4}2}{\cancel{2}i} \int_{|z|=1} \frac{dz}{2kz-z^2-1}$$

$$-2/i \int_{|z|=1} \frac{dz}{z^2-2kz+1}$$

singular pts are obtained from;

$$I = \int_{|z|=1} \frac{dz}{z^2-2kz+1}$$

$$z^2-2kz+1=0$$

$$z = \frac{2k \pm \sqrt{4k^2-4}}{2}$$

$$z = \frac{2k \pm 2\sqrt{k^2-1}}{2}$$

$$= k \pm \sqrt{k^2-1}$$

$z = k + \sqrt{k^2-1}$

$z = k - \sqrt{k^2-1}$

The singular pt $k-\sqrt{k^2-1}$ lies inside the region i.e

$$\operatorname*{Res}_{z=k-\sqrt{k^2-1}} f(z) = \lim_{z \to k-\sqrt{k^2-1}} \frac{1}{2z-2k}$$

$$= \frac{1}{\cancel{2k} - 2\sqrt{k^2-1} - \cancel{2k}} = \underline{\underline{-1/2\sqrt{k^2-1}}}.$$

$$\therefore \int_0^{\pi} \frac{2\,d\theta}{k-\cos\theta} = -\frac{2}{i} \times 2\pi i \times \left(\frac{-1}{2\sqrt{k^2-1}}\right)$$

$$= \cancel{2} \times 2\pi \times \frac{1}{\cancel{2}\sqrt{k^2-1}}$$

$$= \underline{\underline{\frac{2\pi}{\sqrt{k^2-1}}}}$$

$$z = \frac{2k \pm 2\sqrt{k^2-1}}{2.}$$

$$= k \pm \sqrt{k^2-1}$$

$z = k+\sqrt{k^2-1}$

$z = k-\sqrt{k^2-1}$

The singular pt $k-\sqrt{k^2-1}$ lies inside the region. C

$$\underset{z=k-\sqrt{k^2-1}}{\text{Res}} f(z) = \lim_{z \to k-\sqrt{k^2-1}} \frac{1}{2z-2k}$$

$$= \frac{1}{\cancel{2k}-2\sqrt{k^2-1}-\cancel{2k}} = \underline{\underline{-1/2\sqrt{k^2-1}}}.$$

$$\therefore \int_0^{\pi} \frac{2\,d\theta}{k-\cos\theta} = -\frac{2}{i} \times 2\pi i \times \left(\frac{-1}{2\sqrt{k^2-1}}\right)$$

$$= \cancel{2} \times 2\pi \times \frac{1}{\cancel{2}\sqrt{k^2-1}}$$

$$= \underline{\underline{\frac{2\pi}{\sqrt{k^2-1}}}}$$

15/10/2020

Q9. $\displaystyle\int_0^{2\pi} \frac{d\theta}{2+\sin\theta}$

$$= \frac{1}{i} \int_{|z|=1} \frac{1}{2+\frac{z^2-1}{2iz}} \frac{dz}{z}$$

$$= \frac{1}{i} \int_{|z|=1} \frac{2i}{4iz+z^2-1}\,dz$$

$$= \frac{2\cancel{i}}{\cancel{i}} \int_{|z|=1} \frac{dz}{z^2+4iz-1}$$

$\frac{-4i \pm \sqrt{-16+4}}{2}$

$z^2+4iz-1 = 0$

$\underline{\underline{z = (-2 \pm \sqrt{3})i}}$

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$z = (-2+\sqrt{3})i$ lies inside c $|z|=1$

$$\underset{z=(-2+\sqrt{3})i}{\text{Res}} f(z) = \lim_{z \to (-2+\sqrt{3})i} \frac{1}{2z+4i}$$

$$= \frac{1}{2(-2i+\sqrt{3}i)+4i}$$

$$= \frac{1}{2\sqrt{3}i}$$

$$\int_0^{2\pi} \frac{d\theta}{2+\sin\theta} = 2 \times 2\pi i \times \frac{1}{2\sqrt{3}i}$$

$$= \frac{2\pi}{\sqrt{3}}$$

70. $$\int_0^{2\pi} \frac{d\theta}{37-12\cos\theta}$$

$$\cos\theta = \frac{z^2+1}{2z}$$

$$d\theta = \frac{dz}{iz}$$

$$\frac{1}{i}\int_{|z|=1} \frac{dz}{37-\frac{12z^2+12}{2z}} \times \frac{1}{z}$$

$$\frac{1}{i}\int_{|z|=1} \frac{2dz}{74z-12z^2-12}$$

$$-\frac{2}{i}\int_{|z|=1} \frac{dz}{12z^2-74z+12}$$

$$\frac{-1}{i}\int_{|z|=1} \frac{dz}{6z^2-37z+6}$$

$$6z^2 - 37z + 6 = 0.$$

$$z = \frac{37 \pm \sqrt{1369 - 144}}{12}$$

$$= \frac{37 \pm 35}{12} \Rightarrow 72/12 \text{ or } 2/12$$

$$\Rightarrow 6 \text{ or } 1/6 \ (\cdot 166)$$

The singular point; $z = 1/6$ lies inside c while $z=6$ lies outside c.

$$\underset{z=1/6}{\text{Res}} f(z) = \lim_{z \to 1/6} \frac{1}{12z - 37}$$

$$= \frac{1}{12 \times \frac{1}{6} - 37} = \underline{\underline{-1/35}}$$

$$\int_0^{2\pi} \frac{d\theta}{37 - 12\cos\theta} = \frac{-1}{i} \int_{|z|=1} \frac{dz}{6z^2 - 37z + 6}$$

$$= \frac{-1}{i} \times 2\pi i \times \frac{-1}{35}$$

$$= \underline{\underline{2\pi/35}}$$

<u>Type 2: Integral;</u>

Integral of the form

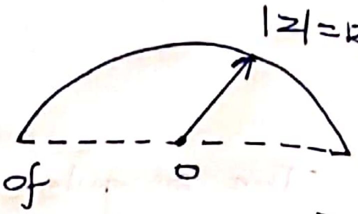
$$\int_{-\infty}^{\infty} \frac{h(x)}{g(x)}\,dx$$ where $h(x)$ and $g(x)$ are polynomials with degree of $g(x)$ exceeds degree of $h(x)$ by atleast 2

In this case; $$\int_{-\infty}^{\infty} \frac{h(x)}{g(x)}\,dx = \lim_{R \to \infty} \int_{-R}^{R} \frac{h(x)}{g(x)}\,dx.$$

$$= \int_C \frac{h(z)}{g(z)}\, dz$$

where; C is the upper half of the semicircle

$|z| = R$



$= 2\pi i$ (Sum of the residues of singular pts above the real axis)

Evaluate the following integrals :

71. $$\int_0^\infty \frac{x^2+2}{(x^2+1)(x^2+4)}\, dx$$

$x^2 \rightarrow$ even fn.

$f(x) \rightarrow$ even hence; $\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)$

$$\frac{1}{2}\int_{-\infty}^{\infty} \frac{x^2+2}{(x^2+1)(x^2+4)}\, dx$$

Num deg $\rightarrow 2$

Den deg $\rightarrow 4$

Diff $\rightarrow 2$

Den deg > Num deg

put $x = z$

$$\frac{1}{2}\int_C \frac{z^2+2}{(z^2+1)(z^2+4)}\, dz$$

$C \longrightarrow$ upper ½ of the semicircle $|z| = R$

Singular pts are given by,:

$(z^2+1) = 0$; $z^2+4 = 0$

$z = \pm i$; $z = \pm 2i$

The singular pts are;

$z = -i, i, 2i, -2i$

The singular points ; $z = i, 2i$ lies inside C

$$\underset{z=i}{\text{Res}}\ f(z) = \lim_{z \to i} \frac{z^2+2}{4z^3+10z}$$

$$= \frac{-1+2}{-4i+10i} = \frac{1}{6i}$$

$$\underset{z=2i}{\text{Res}}\ f(z) = \lim_{z \to 2i} \frac{z^2+2}{4z^3+10z} = \frac{(2i)^2+2}{4(2i)^3+10\times 2i} = \frac{4\times -1+2}{4\times 8\times -i+20i}$$

$$= \frac{-2}{-36i+20i} = \frac{-2}{-12i}$$

$$= \frac{1}{6i}$$

$$\int_0^\infty \frac{x^2+2}{(x^2+1)(x^2+4)}\,dx = \frac{1}{\cancel{2}}\left(\cancel{2}\pi i \times \frac{\cancel{2}}{3\cancel{6}i}\right)$$

$$= \frac{\pi}{3}$$

72. $$\int_{-\infty}^{\infty} \frac{x^2-x+2}{x^4+10x^2+9}\,dx.$$

$$= \int_C \frac{z^2-z+2}{z^4+10z^2+9}\,dz$$

$z^4+10z^2+9 = 0$

put $z^2 = t$

$t^2+10t+9 = 0.$

$(t+1)(t+9) = 0$

$t = -1, t = -9.$

$\Rightarrow z = \pm i, \pm 3i$

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$z = i$ & $3i$ lies in the region c

$$\operatorname*{Res}_{z=i} f(z) = \lim_{z\to i} \frac{z^2-z+2}{4z^3+20z}$$

$$= \frac{i^2 - i + 2}{4\times -1 i + 20i}$$

$$= \frac{1 - i}{16 i}$$

$$= \underline{\underline{\frac{1-i}{16i}}}$$

$$\operatorname*{Res}_{z=3i} f(z) = \lim_{z\to 3i} \frac{z^2-z+2}{4z^3+20z}$$

$$= \frac{7+3i}{48i}$$

$x^4+1=0$

$$\int_{-\infty}^{\infty} \frac{x^2-x+2}{x^4+10x^2+9}\,dx = 2\pi i\left[\frac{1-i}{16i} + \frac{7+3i}{48i}\right]$$

$$= \frac{\cancel{2}\pi \cancel{i}\left(3-\cancel{3i}+7+\cancel{3i}\right)}{\cancel{48i}\ 24}$$

$$= \frac{10\pi}{24} = \underline{\underline{\frac{5\pi}{12}}}$$

due

73.
$$\int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)(x^2+4)}\,dx$$

put $x = z$

$$\int_C \frac{z^2}{(z^2+1)(z^2+4)}\,dz.$$

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The singular pts are given by;

$z^2+1=0$; $z^2+4=0$

$z=\pm i$; $z=\pm 2i$

The singular pts $z=i, 2i$ lies inside the region c.

$$\underset{z=i}{\text{Res}}\ f(z) = \lim_{z\to i} \frac{z^2}{4z^3+10z}$$

$$= \frac{i^2}{-4i+10i} = \underline{\underline{-1/6i}}$$

$$\underset{z=2i}{\text{Res}}\ f(z) = \lim_{z\to 2i} \frac{z^2}{4z^3+10z}$$

$$= \frac{4i^2}{-4\times 8i+20i} = \frac{-4}{-32i+20i}$$

$$= \frac{-4}{-12i} = \underline{\underline{1/3i}}$$

$$\int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)(x^2+4)}\,dx = \int_c \frac{z^2}{(z^2+1)(z^2+4)}\,dz$$

$$= 2\pi i \left[\frac{-1}{6i} + \frac{1}{3i}\right]$$

$$= \frac{2\pi \cancel{i} \times 3\cancel{i}}{18\,\cancel{i^2}}$$

$$= \frac{\cancel{2}\pi}{\cancel{18}\ 9} = \underline{\underline{\pi/9}}$$

The singular pts are given by;

$z^2+1=0$; $z^2+4=0$

$z=\pm i$; $z=\pm 2i$

The singular pts $z = i, 2i$ lies inside the region c.

$$\operatorname{Res}_{z=i} f(z) = \lim_{z\to i} \frac{z^2}{4z^3+10z}$$

$$= \frac{i^2}{-4i+10i} = \frac{-1}{6i}$$

$$\operatorname{Res}_{z=2i} f(z) = \lim_{z\to 2i} \frac{z^2}{4z^3+10z}$$

$$= \frac{4i^2}{-4\times 8i+20i} = \frac{-4}{-32i+20i}$$

$$= \frac{-4}{-12i} = \frac{1}{3i}$$

$$\int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)(x^2+4)}\,dx = \int_c \frac{z^2}{(z^2+1)(z^2+4)}\,dz$$

$$= 2\pi i\left[\frac{-1}{6i}+\frac{1}{3i}\right]$$

$$= \frac{2\pi \cancel{i}\times 3\cancel{i}}{18\cancel{i^2}}$$

$$= \frac{\cancel{2}\pi\times 3}{\cancel{18}\ 9} = \pi/\cancel{9}\,3$$

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74. $\displaystyle\int_{-\infty}^{\infty} 1/1+x^4\,dx$

$\displaystyle\int_c \frac{1}{1+z^4}\,dz$ where c is the upper semicircle

Singular p5;

$-1 = \cos(2n+1)\pi + i\sin(2n+1)\pi$

$z^4 = -1$

$\cancel{\cos\pi + i\sin\pi}$

$= \cos(2n+1)\pi + i\sin(2n+1)\pi$

$n = 0,1,2,3$ (Since 4 roots has to be found)

$z = \cos(2n+1)\pi/4 + i\sin(2n+1)\pi/4$ — ①.

$n=0;$

$z = \cos\pi/4 + i\sin\pi/4$

$= 1/\sqrt{2} + i \times 1/\sqrt{2}$

$= \dfrac{1+i}{\sqrt{2}}$ (which is inside c) 1st Quadrant

$n=1$

$z = \cos 3\pi/4 + i\sin 3\pi/4$

$= -\cos\pi/4 + i\sin\pi/4$

$= \dfrac{-1+i}{\sqrt{2}}$ (which is inside c) 2nd Quadrant.

$n=2$

$z = \cos 5\pi/4 + i\sin 5\pi/4$

$= \cos(180+45) + i\sin(180+45)$

$= -\cos 45 + i \times -\sin 45$

$= \dfrac{-1-i}{\sqrt{2}}$ 3rd Quadrant (which is outside c)

$n=3$

$z = \cos 7\pi/4 + i\sin 7\pi/4$

$= \cos(360-45) + i\sin(360-45)$

$= 1/\sqrt{2} - i/\sqrt{2}$ 4th Quadrant which is outside c

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$$\operatorname*{Res}_{z=\frac{1+i}{\sqrt{2}}} f(z) = \lim_{z \to \frac{(1+i)}{\sqrt{2}}} \frac{1}{4z^3}$$

$$= \frac{1}{2\cancel{4}\frac{(1+i)^3}{\cancel{2}\sqrt{2}}} = \frac{\sqrt{2}}{2(1+i^3+3i^2+3i)}$$

$$= \frac{1}{\sqrt{2}(1-i-3+3i)}$$

$$= \underline{\underline{\frac{1}{\sqrt{2}(2i-2)}}} = \frac{1}{2\sqrt{2}(i-1)}$$

$$\operatorname*{Res}_{z=\frac{-1+i}{\sqrt{2}}} f(z) = \lim_{z \to \frac{-1+i}{\sqrt{2}}} \frac{1}{4z^3}$$

$$= \frac{1}{2\cancel{4}\frac{(-1+i)^3}{\cancel{2}\sqrt{2}}} = \frac{1}{\sqrt{2}(i-1)^3}$$

$$= \frac{1}{\sqrt{2}(-i-1+3+3i)}$$

$$= \frac{1}{\sqrt{2}(2i+2)} = \frac{1}{2\sqrt{2}(i+1)}$$

$$\int_{-\infty}^{\infty} \frac{1}{1+x^4}\,dx = \frac{2\pi i}{2\sqrt{2}}\left[\frac{i\cancel{+1}+i\cancel{-1}}{(i+1)(i-1)}\right]$$

$$= \frac{\cancel{2}\pi\cancel{i}}{\cancel{2}\sqrt{2}} \times \frac{\cancel{2}\cancel{i}}{\cancel{-2}} = \underline{\underline{\pi/\sqrt{2}}}$$

75. $$\int_{-\infty}^{\infty} \frac{x^2\,dx}{(x^2+a^2)^2}$$

put $x = z$

$$\int_C \frac{z^2\,dz}{(z^2+a^2)^2}$$ where C is the upper semicircle

$(z^2+a^2)^2 = 0.$ $\qquad z = \pm ai$

$z^2 = -a^2$

$z = \pm ai$ each of order 2

$z = ai$ are the singular point which lies inside C

(double pole)

$$\underset{z=ai}{\text{Res}}\ f(z) = \frac{1}{1!} \lim_{z\to ai} \frac{d}{dz} (z-ai)^2 \times \frac{z^2}{(z^2+a^2)^2}$$

$$= \lim_{z\to ai} \frac{d}{dz} (z-ai)^2 \times \frac{z^2}{(z^2+a^2)^2}$$

$$= \lim_{z\to ai} \frac{d}{dz} \frac{\cancel{(z-ai)^2} \times z^2}{(z+ai)^2 \cancel{(z-ai)^2}}$$

$$= \cancel{\lim_{z\to ai}} \ \cancel{z-ai}$$

$$= \lim_{z\to ai} \frac{d}{dz} \frac{z^2}{(z+ai)^2}$$

$$= \lim_{z\to ai} \frac{(z+ai)^2 \times 2z - z^2 \times 2(z+}{(z+ai)^4}$$

$$= \frac{(ai+ai)^2 \times 2ai - (ai)^2 \times 2(ai+ai)}{(ai+ai)^4}$$

$$= \frac{2ai \times 2ai - (ai)^2 \times 2}{(2ai)^3} \qquad 4a^2i^2$$

$$= \frac{4a^2i^2 - 2a^2i^2}{8a^3i^3}$$

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$$\frac{-4a^2+2a^2}{-8a^3i}$$

$$\frac{-2a^2}{-8a^3i} = \frac{1}{4ai}$$

$$\therefore \int_{-\infty}^{\infty} \frac{x^2\,dx}{(x^2+a^2)^2} = 2\pi i \times \frac{1}{4ai} = \pi/2a$$

76. $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{x^2+9}.$

$$\int_C \frac{dz}{z^2+9}.$$

$z^2 = -9$

$z = \pm 3i$

$z = 3i$ lies inside C.

$$\operatorname*{Res}_{z=3i} f(z) = \lim_{z\to 3i} \frac{1}{2z}$$

$$= \frac{1}{6i}$$

$$\int_{-\infty}^{\infty} \frac{dx}{x^2+9} = 2\pi \cancel{i} \times \frac{1}{6\cancel{i}} = \pi/3$$

77. $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{2x^2+2x+5}$

$$\int_C \frac{dz}{2z^2+2z+5}$$

$\frac{-2\pm 6i}{4}$

$-\frac{1\pm 3i}{2}$ $\sqrt{-36}$

$4\cdot 2\times 5$

$\frac{-2\pm\sqrt{4-40}}{4}$

$2z^2+2z+5=0.$

$z = -\frac{1\pm 3i}{2}$ $\quad z = \frac{-1+3i}{2}, \frac{-1-3i}{2}$ $\quad \frac{-2\pm 6i}{4}$

$z = \frac{-1+3i}{2}$ lies inside c.

$$\underset{z=\frac{-1+3i}{2}}{\text{Res}} f(z) = \lim_{z \to \frac{-1+3i}{2}} \frac{1}{4z+2}.$$

$$= \frac{1}{\cancel{4}^{2}\frac{(-1+3i)}{\cancel{2}} + 2}$$

$$= \frac{1}{\cancel{-2}+6i+\cancel{2}} = \frac{1}{6i}$$

$$\int_{-\infty}^{\infty} \frac{dx}{x^2+9} = 2\pi i \times \frac{1}{6i} = \pi/3$$

78. $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)(x^2+b^2)}$

put $x = z$

$$\int_c \frac{dz}{(z^2+a^2)(z^2+b^2)}$$

$z^4+b^2z^2+a^2z^2+a^2b^2$

$z^2+a^2 = 0 \qquad z^2+b^2 = 0$

$z^2 = -a^2 \qquad z^2 = -b^2$

$z = \pm ai; \qquad z = \pm bi$

$z = ai, bi$ lies inside the region c

$$\underset{z=ai}{\text{Res}} f(z) = \lim_{z \to ai} \frac{1}{4z^3+2b^2z+2za^2}$$

$$= \frac{1}{-4a^3i+2b^2ai+2a^2ai}$$

$$= \frac{1}{-4a^3i + 2a^3i + 2b^2ai}$$

$$= \frac{1}{-2a^3i + 2b^2ai}$$

$$= \frac{1}{2(b^2ai - a^3i)}$$

$$= \frac{1}{2i(b^2a - a^3)} \qquad = \frac{1}{2ai(b^2 - a^2)}$$

$$\operatorname*{Res}_{z=bi} f(z) = \lim_{z \to bi} \frac{1}{4z^3 + 2b^2z + 2a^2z}$$

$$= \frac{1}{-4b^3i + 2b^2 \times bi + 2a^2 \times bi}$$

$$= \frac{1}{-2b^3i + 2a^2bi}$$

$$= \frac{1}{2i(a^2b - b^3)} \qquad = \frac{1}{2bi(a^2 - b^2)}$$

$$\int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)(x^2+b^2)} = 2\pi i \times \left[\frac{1}{2i(b^2a - a^3)} + \frac{1}{2i[a^2b - b^3]} \right]$$

$$= \frac{\cancel{2\pi i}}{\cancel{2i}} \left[\frac{a^2b - b^3 + b^2a - a^3}{(ab^2 - a^3)(a^2b - b^3)} \right]$$

$$= \pi \left[\frac{1}{a(b^2 - a^2)} - \frac{1}{b(b^2 - a^2)} \right]$$

$$= \pi \left[\frac{b(b^2 - a^2) - a(b^2 - a^2)}{ab(b^2 - a^2)^2} \right]$$

$$= \pi \frac{\cancel{(b-a)}}{ab\cancel{(b-a)}(b+a)} = \frac{\pi}{ab(b+a)}$$

(79) due

$$\int_{-\infty}^{\infty} \frac{x^2+1}{x^4+1}\, dx \qquad\qquad \int_{-\infty}^{\infty} \frac{x}{(x^2-2x+2)^2}\, dx \qquad \pi\sqrt{2} \qquad \pi/2$$

$$\int_{-\infty}^{\infty} \frac{x^2+1}{x^4+1}\, dx \Rightarrow \int_C \frac{z^2+1}{z^4+1}\, dz.$$

$$z^4+1=0$$

$$z^4=-1 \qquad z = \cos(2n+1)\pi/4 + i\sin(2n+1)\pi/4$$
$$(n=0,1,2,3)$$

$z = \frac{1+i}{\sqrt{2}}$ & $\frac{-1+i}{\sqrt{2}}$ lies inside C.
(Ref Q: 74)

$$\operatorname*{Res}_{z=\frac{1+i}{\sqrt{2}}} f(z) = \lim_{z\to\frac{1+i}{\sqrt{2}}} \frac{z^2+1}{4z^3}.$$

$$= \frac{\left(\frac{1+i}{\sqrt{2}}\right)^2+1}{4\left(\frac{1+i}{\sqrt{2}}\right)^3}$$

$$= \frac{(1+i)^2+2}{2\sqrt{2}\,(1+i)^3} = \frac{2i+2}{2\sqrt{2}\,(2i-2)}$$

$$= \frac{(i+1)}{2\sqrt{2}\,(i-1)}$$

$$\operatorname*{Res}_{z=\frac{-1+i}{\sqrt{2}}} f(z) = \lim_{z\to\frac{-1+i}{\sqrt{2}}} \frac{z^2+1}{4z^3}.$$

$$= \frac{\left(\frac{-1+i}{\sqrt{2}}\right)^2+1}{4\left(\frac{-1+i}{\sqrt{2}}\right)^3}.$$

$$= \frac{(-1+i)^2+2}{2\sqrt{2}\,(-1+i)^3} = \frac{2-2i}{2\sqrt{2}\,(2+2i)}$$

$$= \frac{(1-i)}{2\sqrt{2}\,(1+i)}$$

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$$\int_{-\infty}^{\infty} \frac{x^2+1}{x^4+1}\,dx = 2\pi i\left[\frac{i+1}{2\sqrt{2}(i-1)} + \frac{(1-i)}{2\sqrt{2}(1+i)}\right]$$

$$= \frac{2\pi i}{2\sqrt{2}}\left[\frac{i+1}{i-1} + \frac{1-i}{1+i}\right]$$

$$= \frac{\pi i}{\sqrt{2}}\left[\frac{(1+i)^2 + (1-i)(i-1)}{(i-1)(i+1)}\right]$$

$$= \frac{\pi i}{\sqrt{2}} \times \frac{4i}{i^2-1} \Rightarrow \frac{\pi i}{\sqrt{2}} \times \frac{4i}{-2}.$$

$$= \frac{\pi \cancel{i} \times 4\cancel{i} \times \cancel{-1}}{2\sqrt{2}} = \frac{2\pi}{\sqrt{2}}$$

$$= \sqrt{2}\pi.$$

due

80. $\displaystyle\int_{-\infty}^{\infty} \frac{x}{(x^2-2x+2)^2}\,dx.$

$$\int_C \frac{z}{(z^2-2z+2)^2}\,dz$$

$z^2-2z+2=0$

$z = 1+i,\ 1-i.$

$z = 1+i$ lies inside c.

$$\operatorname*{Res}_{z=1+i} f(z) = \frac{1}{1!}\lim_{z\to 1+i} \frac{d}{dz}\,(\cancel{z-(1+i)})^2 \times \frac{z}{\left[(\cancel{z-(1+i)})(z-(1-i))\right]^2}$$

$$= \lim_{z\to 1+i} \frac{d}{dz}\left(\frac{z}{(z-1+i)^2}\right)$$

$$= \lim_{z\to 1+i} \frac{(z-1+i)^{\cancel{2}}\times 1 - 2\times(\cancel{z-1+i})\times z}{(z-1+i)^{\cancel{4}\,3}}$$

$$= \frac{1}{4i}$$

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$$\int_{-\infty}^{\infty} \frac{x}{(x^2-2x+2)^2}\, dx$$

$$= \cancel{2}\pi \cancel{i} \times \frac{1}{\cancel{4}\cancel{i}\,2}$$

$$= \pi/2$$

Module I — Introduction —

1/10/2020

Partial Differential Equations;

A differential eqn which involves partial derivatives is called a partial differential eqn.

eg: $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 5$

$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = e^z$

The order of a p.d.e is the order of the highest partial derivative in the eqn.

The degree of a p.d.e is the degree of the highest order partial derivative occurring in the eqn.

eg: $\left(\frac{\partial^2 z}{\partial x^2}\right)^3 + \frac{\partial z}{\partial y} = 0$

order $\longrightarrow$ 2 2nd order 3rd degree pde

degree $\longrightarrow$ 3

Notations;

If z is a fn of 2 independent variables x & y, we denote;

$\frac{\partial z}{\partial x} = p$

$\frac{\partial z}{\partial y} = q$

$\frac{\partial^2 z}{\partial x^2} = r.$

$\frac{\partial^2 z}{\partial y^2} = t$

$\frac{\partial^2 z}{\partial x \partial y} = s$

Chapter 1

Formation of P.D.E;

Method 1;

Method of elimination of arbitrary constants

Form a p.d.e from the following eqn by eliminating the arbitrary constants :

1. $z = ax + by + ab$

$z = ax + by + ab$ —— (1)

$p = \frac{\partial z}{\partial x} = a$

$p = a$

$q = \frac{\partial z}{\partial y} = b$

$q = b.$

here a & b obtained in 1st step coz; Independent variables Arbitrary const = hence p.d.e

$$z = px + qy + pq.$$

$$= x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} + \frac{\partial z}{\partial x} \times \frac{\partial z}{\partial y}$$

2. $z = ax + a^2y^2 + b$

$p = \frac{\partial z}{\partial x} = a.$

$q = \frac{\partial z}{\partial y} = 2a^2y$

$\cancel{\frac{\partial^2 z}{\partial y^2}} \cancel{=} \cancel{2a^2}$

$q = 2p^2y \qquad p = a$

3. $z = (x^2+a)(y^2+b)$

$$\frac{\partial z}{\partial x} = (y^2+b) \times 2x. \quad (p)$$

$$\frac{\partial z}{\partial y} = (x^2+a) \times 2y \quad (q)$$

$$p = 2x(y^2+b) \text{ —— } ①$$

$$q = 2y(x^2+a) \text{ —— } ②.$$

$$pq = 4xy(x^2+a)(y^2+b)$$

$$pq = 4xyz$$

$$z = \frac{pq}{4xy}.$$

4. $2z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$

$$2p = \frac{2x}{a^2} \Rightarrow p = x/a^2$$

$$2q = \frac{2y}{b^2} \Rightarrow q = y/b^2.$$

$$2z = xp + qy$$

$$2z = xp + yq$$

5. $\frac{x^2}{a^2} + \frac{y^2}{c^2} + \frac{z^2}{c^2} = 1$

$$\frac{z^2}{c^2} = 1 - \frac{x^2}{a^2} - \frac{y^2}{c^2}$$

here the no. of arbitrary constants (3) is greater then the no. of independent variable (x, y) ②

∴ the req. p.d.e. will be of order greater than 1.

$$\frac{\cancel{2}pz}{c^2} = -\frac{\cancel{2}x}{a^2} \qquad \frac{2x}{a^2} + \frac{2zp}{c^2} = 0 \quad —(2)$$

$$zp = -\frac{xc^2}{a^2}$$

$$\frac{\cancel{2}zq}{c^2} = -\frac{\cancel{2}y}{b^2} \qquad \frac{2y}{b^2} + \frac{2zq}{c^2} = 0 \quad —(3)$$

$$\frac{zq}{c^2} = -\frac{y}{b^2}$$

diff: eq (2) w.r.t. y.

$$0 + \frac{2}{c^2}\left[z \times \frac{\partial^2 z}{\partial x \partial y} + \frac{\partial z}{\partial x} \cdot \frac{\partial z}{\partial y}\right] = 0.$$

$$z\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial z}{\partial x} \times \frac{\partial z}{\partial y} = 0.$$

$$zs + pq = 0$$

$$\frac{2pz}{c^2} = -\frac{2x}{a^2} \qquad \frac{2x}{a^2} + \frac{2zp}{c^2} = 0 \quad \text{(2)}$$

$$zp = -\frac{xc^2}{a^2}$$

$$\frac{2zq}{c^2} = -\frac{2y}{b^2} \qquad \frac{2y}{b^2} + \frac{2zq}{c^2} = 0 \quad \text{(3)}$$

$$\frac{zq}{c^2} = -\frac{y}{b^2}$$

diff: eq (2) w.r.t. y.

$$0 + \frac{2}{c^2}\left[z \times \frac{\partial^2 z}{\partial x \partial y} + \frac{\partial z}{\partial x}\frac{\partial z}{\partial y}\right] = 0.$$

$$z\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial z}{\partial x} \times \frac{\partial z}{\partial y} = 0.$$

$$zs + pq = 0$$

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6. $z = ax^2 + bxy + cy^2$

$$p = \frac{\partial z}{\partial x} = 2ax + by$$

$$q = \frac{\partial z}{\partial y} = bx + 2cy$$

$$r = \frac{\partial^2 z}{\partial x^2} = 2a$$

$$s = \frac{\partial^2 z}{\partial x \partial y} = b$$

$$t = \frac{\partial^2 z}{\partial y^2} = 2c$$

$a = r/2 \quad b = s \quad c = t/2$

$$z = \frac{rx^2}{2} + sxy + \frac{t}{2}y^2.$$

Therefore the req. p.d.e is;

$$z = \frac{r}{2}x^2 + sxy + \frac{t}{2}y^2.$$

7. $4(1+a^2)z = (x+ay+b)^2$

No: of arbitary constants : 2

Order $\longrightarrow$ 1

diff: w.r.t x

$p = \frac{\partial z}{\partial x} \Rightarrow 4(1+a^2)p = 2(x+ay+b)$ —— ①

diff: w.r.t y

$q = \frac{\partial z}{\partial y} \Rightarrow 4(1+a^2)q = 2(x+ay+b)a$ —— ②

③ ÷ ②.

$\frac{q}{p} = a \qquad a = q/p$

$$4p\left(1+\frac{q^2}{p^2}\right) = 2\left(x + \frac{qy}{p} + b\right)$$

$$4p + \frac{4q^2}{p} = 2x + \frac{2qy}{p} + 2b.$$

$$4p + \frac{4q^2}{p} - 2x - \frac{2qy}{p} = 2b.$$

$$b = 2p + \frac{2q^2}{p} - x - \frac{qy}{p}$$

$$4\left(1+\frac{q^2}{p^2}\right)\times z = \left(\cancel{x} + \cancel{\frac{q}{p}y} + 2p + \frac{2q^2}{p} - \cancel{x} - \cancel{\frac{qy}{p}}\right)^2$$

$$z = \frac{\left(\frac{2p^2+2q^2}{p}\right)^2}{4\frac{(p^2+q^2)}{p^2}} \Rightarrow z = \frac{4(p^2+q^2)^2}{4(p^2+q^2)}$$

$$\underline{z = p^2+q^2}$$

(OR) $\longrightarrow$

(OR) $x + ay + b = 2(1+a^2)p$.

$$= 2\left(1 + \frac{q^2}{p^2}\right)p.$$

$$= \frac{2}{p}\left[p^2 + q^2\right]$$

$$4\left(1 + \frac{q^2}{p^2}\right)z = \frac{4}{p^2}(p^2+q^2)^2$$

$$z = p^2 + q^2$$

8. Find the p.d.e of all spheres of given radius 'a' and whose centres lie on the xy plane.

$(x-b)^2 + (y-k)^2 + z^2 = a^2$ is the general eqn of the sphere in xy plane.

The no: of arbitrary constants $\longrightarrow$ 2 (b, k)

a is not an arbitrary constant $\longrightarrow$ (fixed constant) (given)

Order $\longrightarrow$ 1

$$(x-b)^2 + (y-k)^2 + z^2 = a^2. \quad \text{——— (1)}$$

diff: (1) w.r.t x

$$p = \frac{\partial z}{\partial x}$$

$$2(x-b) + 2z \times p = 0$$

$$2(x-b) + 2zp = 0 \quad \text{——— (2)}$$

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diff: w.r.t to y.

$$2(y-k) + 2zq = 0 \quad \text{③}$$

From ②

$$(x-h) = -zp.$$

From ③

$$(y-k) = -zq.$$

$$\therefore (-zp)^2 + (-zq)^2 + z^2 = a^2.$$

$$z^2p^2 + z^2q^2 + z^2 = a^2$$

$$z^2(p^2+q^2+1) = a^2$$

9. Form the p.d.e of all spheres ~~of given radius~~ and whose centre lies on the z-axis.

The eqn: of the sphere;

$$x^2+y^2+(z-k)^2 = r^2. \quad \text{①}$$

The no: of arbitary constants $\longrightarrow$ 2 (k, r)

order $\longrightarrow$ 1

diff: ① w.r.t x.

$$p = \frac{\partial z}{\partial x} \Rightarrow 2x + 2(z-k)p = 0 \quad \text{②}$$

diff: ① w.r.t y

$$q = \frac{\partial z}{\partial y} \Rightarrow 2y + 2(z-k)q = 0 \quad \text{③}$$

$$\text{②} \Longrightarrow \cancel{2}x = -\cancel{2}(z-k).p.$$

$$(z-k) = -x/p$$

$$\text{③} \Longrightarrow (z-k) = -y/q$$

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$$1 = \frac{x}{p} \times \frac{q}{y}$$

No: of arbitrary constant → 2
order → 1

No: of A·C → 3
order → 2

$$\frac{q}{p} = \frac{y}{x}$$

10. Family of planes having equal intercepts on the x, y axis

$$\frac{x}{a} + \frac{y}{a} + \frac{z}{c} = 1$$

$$\frac{1}{a} + \frac{p}{c} = 0 \longrightarrow \text{diff w.r.t } x. \quad \to (2)$$

diff: w.r.t y.

$$\frac{1}{a} + \frac{q}{c} = 0 \longrightarrow (3)$$

(2) − (3)

$$\frac{p-q}{c} = 0$$

$$\underline{p = q}$$

11. Find the p.d.e of all spheres whose radii are equal.

⇒ the radius is given. and is a fixed constant

The general eqn. of the sphere is;

$$(x-h)^2 + (y-k)^2 + (z-l)^2 = a^2 \text{ ——— } (1)$$

The arbitrary constants : h, k, l (3)

$a \longrightarrow$ fixed constant.

$p = \frac{\partial z}{\partial x}$ diff: ① w.r.t. x

$2(x-h) + 2(z-l)p = 0.$

$(x-h) + (z-l)p = 0$ ——— ②.

diff: ① w.r.t y

$q = \frac{\partial z}{\partial y}$

$(y-k) + (z-l)q = 0$ ——— ③

$\cancel{r = \frac{\partial^2 z}{\partial x^2}} \Rightarrow$

diff: ② w.r.t x.

$1 + (z-l) \times r + p \times p = 0$

$1 + (z-l)r + p^2 = 0$ ——— ④.

diff: ② w.r.t y.

$(z-l)s + pq = 0$ ——— ⑤

⑤ $\Rightarrow (z-l) = -pq/s.$

④ $\Rightarrow (z-l) = \frac{-(1+p^2)}{r}$

$\frac{1+p^2}{r} = \frac{pq}{s}.$

$pqr = (1+p^2)s$

12. ~~Extra~~ $z = xy + y\sqrt{x+a} + b$

due

$$z = xy + y\sqrt{x+a} + b \quad \text{——— (1)}.$$

diff (1) w.r.t x;

$$p = \frac{\partial z}{\partial x} = y + y \times \frac{1}{2\sqrt{x+a}} \Rightarrow y + \frac{y}{2\sqrt{x+a}} = p \text{ — (2)}$$

diff: (1) wrt y;

$$q = x + \sqrt{x+a} \quad \text{——— (3)}$$

$$(3) \Rightarrow \sqrt{x+a} = q - x.$$

Substituting (3) in (1).

$$p = y + \frac{y}{2(q-x)}$$

$$p = \frac{(2(q-x)+1) \cdot y}{2(q-x)} \Rightarrow (p-y)(q-x) = y/2$$

Method 2;

Method of elimination of arbitrary fns:

Form the p.d.e by eliminating the arbitrary fns: from the following eqns:

13. $z = f\left(\frac{xy}{z}\right)$

No: of arbitrary fn $\longrightarrow$ (1)

hence we get a p.d.e of order 1

diff: ① w.r.t x.

$$p = \frac{\partial z}{\partial x}.$$

$$p = f'\left(\frac{xy}{z}\right)\left[\frac{z[x \times 0 + y] - xy \times p}{z^2}\right] \text{ --- ②}.$$

$$p = f'\left(\frac{xy}{z}\right)\left[\frac{zy - xyp}{z^2}\right] \text{ ——— ②}.$$

diff: ① w.r.t y.

$$q = \frac{\partial z}{\partial y}.$$

$$q = f'\left(\frac{xy}{z}\right)\left[\frac{zx - xyq}{z^2}\right] \text{ ——— ③}.$$

② ÷ ③

$$\frac{p}{q} = \frac{zy - xyp}{zx - xyq}$$

$$p \times zx - \cancel{xyqp} = qzy - q\cancel{xyp}.$$

$$px = qy.$$

$$\frac{x}{y} = q/p$$

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14. $z = y^2 + 2f(1/x + \log y)$

Fn → ① p.d.e — ordar: 1

$$p = 2f'(1/x + \log y) \times -1/x^2$$

$$= \frac{-2}{x^2} f'(1/x + \log y) \text{ ——— ②}$$

$$q = 2y + 2f'(1/x + \log y) \times 1/y \text{ ——— ③}$$

② ÷ ③

$$\frac{p}{(q-2y)} = \frac{-\cancel{2} f'\left(\cancel{1/x + \log y}\right) \times 1/x^2}{\cancel{2} f'\left(\cancel{1/x + \log y}\right) \times 1/y}$$

$$\frac{p}{q-2y} = \frac{-y}{x^2}$$

$$x^2 p = -qy + 2y^2.$$

$$\underline{\underline{qy = 2y^2 - px^2}}$$

15. $xyz = \phi(x+y+z)$

$xyp + \cancel{y}z = \phi'(x+y+z)(1+p)$ xy z

$xyq + xz = \phi'(x+y+z)(1+q)$

diff: wrt $x \rightarrow (xp+z)y = \phi'(x+y+z)(1+p)$ — ②

diff: wrt $y \rightarrow (yq+z)x = \phi'(x+y+z)(1+q)$ — ③

$$\text{②/③} \quad \frac{(xp+z)y}{(yq+z)x} = \frac{\phi'(\cancel{x+y}+z)(1+p)}{\phi(x+\cancel{y+z})(1+q)}$$

$$y(xp+z)(1+q) = x(yq+z)(1+p)$$

$$(xyp + \cancel{y}z)(1+q) = (xyq + xz)(1+p)$$

$$xyp + \cancel{y}z + \cancel{xypq} + \cancel{y}zq = xyq + \cancel{xyqp} + \cancel{x}z + xzp.$$

$\cancel{xyp + xzq = xyq + xzp}$

$\cancel{x(yp + qz) = y(xq + pz)}$.

$xyp + yz + yzq - xyq - xz - xzp = 0$.

$xy(p-q) + zy(1+q) - xz(1+p) = 0$

16. $z = f(2x+y) + g(3x-y)$

Ab: fn $\longrightarrow$ 2 p.d.e order $\longrightarrow$ 2

$p = f'(2x+y) \times 2 + g'(3x-y) \times 3$. —— (2).

$q = f'(2x+y) \times 1 + g'(3x-y) \times -1$

$= f'(2x+y) - g'(3x-y)$ —— (3)

$r = 4f''(2x+y) + 9g''(3x-y)$ —— (4)

$s = 2f''(2x+y) - 3g''(3x-y)$ —— (5)

$t = f''(2x+y) + g''(3x-y)$ —— (6)

$\cancel{r+s=6t}$ $r + s = 6t$.

17. $z = f(y) + e^x g(y)$

$p = e^x g(y)$ (diff: w.r.t x)

$q = f'(y) + e^x g'(y)$ (diff. w.r.t y)

$r = e^x g(y)$

$s = e^x g'(y)$

$r = p$

~~18. $z = y^2 + 2f(1/x + \log y)$~~

~~$p = 2f'$~~

18. $z = x f(2x-y) + g(2x-y) \longrightarrow (1)$

diff (1) w.r.t x $\quad p = x f'(2x-y)2 \quad f(2x-y) + 2g'(2x-y)$

$= 2x f'(2x-y) + f(2x-y) + 2g'(2x-y) \longrightarrow (2)$

diff (1) w.r.t y $\quad q = -x f'(2x-y) + -1 g'(2x-y)$

$= -x f'(2x-y) - g'(2x-y) \longrightarrow (3)$

diff (2) w.r.t x $\quad r = 4x f''(2x-y) + 2f'(2x-y) +$

$2f'(2x-y) + 4g'(2x-y) \longrightarrow (4)$

$= 4x f''(2x-y) + 4f'(2x-y) + 4g''(2x-y)$

diff (2) w.r.t y $\quad s = -2x f''(2x-y) - f'(2x-y) - 2g''(2x-y) \longrightarrow (5)$

diff (3) w.r.t y $\quad t = x f''(2x-y) + g''(2x-y) \longrightarrow (6)$

$r + 4s = -4t$

$\underline{\underline{r + 4s + 4t = 0}}$

19. $z = x f(y/x) + y \phi(x)$

$p = \frac{-\cancel{x}y}{x^{\cancel{2}}} f'(y/x) + f(y/x) + y \phi'(x)$

$= \frac{-y f'(y/x)}{x} + f(y/x) + y \phi'(x) \longrightarrow (1)$

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$q = \cancel{x} f'(y/x) + y\cancel{\phi'(x)} + \phi(x)$ —— ③

$r = +1 \times \frac{+y}{x} f''(y/x) \times \frac{+y}{x^2} + -1 \times \frac{-1}{x^2} f'(y/x) +$

$f'(y/x) \times -1/x^2 + y\phi''(x)$

$= \frac{y^2 f''(y/x)}{x^3} + \cancel{f'(y/x)}\; y\phi''(x)$ —— ④

$s = \frac{-y}{x} \times f''(y/x) \times \frac{1}{x} + f'(y/x) \times \frac{1}{x} + \cancel{y}\, \phi'(x).$

$= \frac{-y f''(y/x)}{x^2} + \cancel{\frac{f'(y/x)}{x}} + \phi'(x) \longrightarrow$ ⑤

$t \Rightarrow \frac{1}{x} f''(y/x)$ —— ⑥

$s = -\frac{y}{x}(t) + \phi'(x)$

$\phi'(x) = s + \frac{ty}{x}$ —— ⑦

$x \times p$

$\Rightarrow -\frac{xy}{\cancel{x}} f'(y/x) + x f(y/x) + xy\,\phi'(x).$

$\Rightarrow -y f'(y/x) + x f(y/x) + xy\,\phi'(x).$

yq ; $y\,[f'(y/x)] + y\,\phi(x)$.

$xp + yq \Rightarrow x f(y/x) - \cancel{y f'(y/x)} + xy\,\phi'(x) +$
$\cancel{y f'(y/x)} + y\,\phi(x).$

$z \Rightarrow x f(y/x) + y\,\phi(x).$

$\Rightarrow xp + yq = z + xy\,\phi'(x).$

$xp + yq = z + xy\left(s + \frac{ty}{x}\right)$

$xp + yq = z + y(sx + ty)$

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dw

$z = x^2 f(y) + y^2 g(x)$

$2xp + yq - 2z = ...$

$z = x^2 f(y) + y^2 (gx)$ ——— ①.

$R = 2x$

diff: ① w.r.t x;

$p = 2x f(y) + y^2 g'(x)$. ——— ②

diff: ① w.r.t y;

$q = x^2 f'(y) + 2y\, g(x)$. ——— ③

diff: ② w.r.t x;

$r = 2 f(y) + y^2 g''(x)$. ——— ④

diff ② w.r.t y;

$s = 2x f'(y) + 2y\, g'(x)$ ——— ⑤.

diff ③ w.r.t y;

$t = x^2 f''(y) + 2 g(x)$ ——— ⑥.

Xing p with x;

$xp = 2x^2 f(y) + xy^2 g'(x)$ ——— ⑦.

Xing q with y;

$yq = x^2 y f'(y) + 2y^2 g(x)$ ——— ⑧.

⑦ + ⑧

$$xp+yq = 2x^2 f(y) + xy^2 g'(x) + x^2 y f'(y) + 2y^2 g(x).$$

$$= 2\left(x^2 f(y) + y^2 g(x)\right) + xy^2 g'(x) + x^2 y f'(y).$$

$$xp+yq = 2z + xy^2 g'(x) + x^2 y f'(y) \quad \text{--- (9)}$$

Xing (5) with xy.

$$xys = 2(x^2 y f'(y) + xy^2 g'(x))$$

$$\frac{xys}{2} = x^2 y f'(y) + xy^2 g'(x) \quad \text{--- (10)}$$

Substituting (10) in (9).

$$xp+yq = 2z + \frac{xys}{2}.$$

$$2(xp+yq) = 4z + xys$$

$$2(xp+yq) - 4z = xys.$$

$$xp+yq = 2x^2 f(y) + xy^2 g'(x) + x^2y f'(y) + 2y^2 g(x).$$

$$= 2\left(x^2 f(y) + y^2 g(x)\right) + xy^2 g'(x) + x^2 y f'(y).$$

$$xp+yq = 2z + xy^2 g'(x) + x^2 y f'(y) \quad \text{—— (9)}$$

Xing (5) with xy.

$$xys = 2\left(x^2 y f'(y) + xy^2 g'(x)\right)$$

$$\frac{xys}{2} = x^2 y f'(y) + xy^2 g'(x) \quad \text{—— (10)}$$

Substituting (10) in (9).

$$xp+yq = 2z + \frac{xys}{2}.$$

$$2(xp+yq) = 4z + xys$$

$$\underline{\underline{2(xp+yq) - 4z = xys.}}$$

22/10/2020 **<u>Type 3 / Method 3 :</u>**

elimination of arbitrary fn 'f' from the relation

$f(u,v) = 0$

Where u and v are fns of x, y, z

$u, v \ \phi(x, y, z)$

$f(u,v) = 0$ where $u, v \ \phi(x, y, z)$

then;

diff: w.r.t x $\rightarrow$ $$\frac{\partial f}{\partial u}\left[\frac{\partial u}{\partial x}+\frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial x}\right]+\frac{\partial f}{\partial v}\left[\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}\times\frac{\partial z}{\partial x}\right]=0$$

diff: partially w.r.t y.

$$\frac{\partial f}{\partial u}\left[\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\times\frac{\partial z}{\partial y}\right]+\frac{\partial f}{\partial v}\left[\frac{\partial v}{\partial y}+\frac{\partial v}{\partial z}\times\frac{\partial z}{\partial y}\right]=0$$

Form the pde from the following fns:

21. $f(x+y+z,\ x^2+y^2+z^2) = 0$ —— ①

Let $u = x+y+z$

$v = x^2+y^2+z^2$

diff: ① partially w.r.t x

$$\frac{\partial f}{\partial u}\left[1+p\right]+\frac{\partial f}{\partial v}\left[2x+2zp\right]=0 \quad \text{—— ②}$$

diff ① w.r.t y

$$\frac{\partial f}{\partial u}\left[1+q\right]+\frac{\partial f}{\partial v}\left[2y+2zq\right]=0 \quad \text{—— ③}$$

Eliminating $\partial f/\partial u$ & $\partial f/\partial v$ from ② & ③

we get.

$$\begin{vmatrix} 1+p & 2x+2zp \\ 1+q & 2y+2zq \end{vmatrix} = 0$$

$$(2y+2zq)(1+p) - (2x+2zp)(1+q) = 0$$

$$2y+2zq+2yp+\cancel{2zqp} - 2x - 2xq - 2zp - \cancel{2zpq}$$

$$y+zq+yp-x-xq-zp.$$

$$y(1+p) - x(1+q) + z(q-p)$$

$$(y-z)p + (z-x)q = x-y \qquad (Pp+Qq=R)$$

22. $f(x^2+y^2, z-xy) = 0$. —— ①

$u = x^2+y^2 \qquad v = z-xy.$

diff: ① w.r.t x.

$$\frac{\partial f}{\partial u}\left[2x\right] + \frac{\partial f}{\partial v}\left[-y+p\right] = 0 \text{ —— ②}$$

diff ① w.r.t y.

$$\frac{\partial f}{\partial u}\left[2y\right] + \frac{\partial f}{\partial v}\left[-x+q\right] = 0 \text{ —— ③}$$

$$\begin{vmatrix} 2x & -y+p \\ 2y & -x+q \end{vmatrix} = 0$$

$$-2x^2+2xq+2y^2-2yp = 0.$$

$$-x^2+xq+y^2-yp = 0.$$

$$xq-yp = -y^2+x^2$$

$$yp-xq = y^2-x^2$$

23. $f(xy+z^2,\ x+y+z) = 0$ ——— ①

$u = xy+z^2$

$V = x+y+z$

diff ① w.r.t x.

$$\frac{\partial f}{\partial u}\left[y + 2zp\right] + \frac{\partial f}{\partial v}\left[1+p\right] = 0 \text{ ——— ②}$$

diff ① w.r.t y.

$$\frac{\partial f}{\partial u}\left[x + 2zq\right] + \frac{\partial f}{\partial v}\left[1+q\right] = 0 \text{ ——— ③}$$

$$\begin{vmatrix} y+2zp & 1+p \\ x+2zq & 1+q \end{vmatrix} = 0.$$

$$y + yq + 2zp - \cancel{2z}pq - x - xp - 2zq - \cancel{2z}pq = 0$$

$$y-x + (y-2z)q + (2z-x)p = 0.$$

$$(2z-x)p + (y-2z)q = x-y.$$

24. $f(x^2+y^2+z^2,\ z^2-2xy) = 0$ ——— ①

$u = x^2+y^2+z^2$

$V = z^2-2xy$

diff ① wrt x

$$\frac{\partial f}{\partial u}\left[2x + 2zp\right] + \frac{\partial f}{\partial v}\left[-2y + 2zp\right] = 0 \quad —②$$

diff ① wrt y

$$\frac{\partial f}{\partial u}\left[2y + 2zq\right] + \frac{\partial f}{\partial v}\left[-2x + 2zq\right] = 0 \quad —③$$

$$\begin{vmatrix} 2x + 2zp & -2y + 2zp \\ 2y + 2zq & -2x + 2zq \end{vmatrix} = 0.$$

$$-4x^2 + 4xzq - 4xzp + \cancel{4z^2pq} + 4y^2 - 4yzp + 4zyq - \cancel{4z^2pq} = 0$$

$$4y^2 - 4x^2 + 4xzq + 4zyq - 4xzp - 4yzp = 0$$

$$(xz + yz)p - (xz + yz)q = y^2 - x^2$$

$$\Rightarrow \underline{\underline{zp - zq = y - x}}$$

25. $\phi(x^2 + y^2 + z^2, xyz) = 0.$ —①

$u = x^2 + y^2 + z^2$

$v = xyz$

diff ① wrt x

$$\frac{\partial \phi}{\partial u}\left[2x + 2zp\right] + \frac{\partial \phi}{\partial v}\left[yz + xyp\right] = 0$$

diff : ① w.r.t y

$$\frac{\partial \phi}{\partial u}\left[2y + 2zq\right] + \frac{\partial \phi}{\partial v}\left[xz + xyq\right] = 0$$

$$\begin{vmatrix} 2x + 2zp & yz + xyp \\ 2y + 2zq & xz + xyq \end{vmatrix} = 0$$

$$x^2z + x^2yq + xz^2p - y^2z + xy^2p - yz^2q = 0$$

$$(x^2 - y^2)z + (x^2y - yz^2)q + (xz^2 + xy^2)p$$

$$\underline{(xz^2 + xy^2)p + (x^2y - yz^2)q = (y^2 - x^2)z}$$

26. $\phi(x^2 + y^2 + z^2,\ 2y + z - x) = 0$

$u = x^2 + y^2 + z^2$

$v = 2y + z - x.$

$$\frac{\partial \phi}{\partial u}\left[2x + 2zp\right] + \frac{\partial \phi}{\partial v}\left[-1 + p\right] = 0$$

$$\frac{\partial \phi}{\partial u}\left[2y + 2zq\right] + \frac{\partial \phi}{v}\left[2 + q\right] = 0.$$

$$\begin{vmatrix} 2x + 2zp & -1 + p \\ 2y + 2zq & 2 + q \end{vmatrix} = 0$$

$$4x + 4zp + 2xq + 2\cancel{zqp} + 2y - 2yp + 2zq - 2\cancel{zqp} = 0$$

$$p(2z-y)+q(x+z)=-(2x+y)$$

Qno 27.

$$\phi(x^2-y^2,\ x^2-z^2)=0 \quad \text{——} \ (1).$$

$$u=x^2-y^2$$

$$v=x^2-z^2$$

diff (1) w.r.t x

$$\frac{\partial\phi}{\partial u}\left[2x\right]+\frac{\partial\phi}{\partial v}\left[2x-2zp\right]=0 \quad \text{——} \ (2)$$

diff (1) w.r.t y

$$\frac{\partial\phi}{\partial u}\left[-2y\right]+\frac{\partial\phi}{\partial v}\left[-2zq\right]=0$$

$$\begin{vmatrix} 2x & 2x-2zp \\ -2y & -2zq \end{vmatrix}=0$$

$$-4xzq+4xy-4yzp=0$$

$$yzp+xzq=xy.$$

Qno 28.

$$f(x+y+z,\ x^2+2yz)=0 \quad \text{——} \ (1)$$

$$u=x+y+z \qquad v=x^2+2yz$$

diff: (1) w.r.t x;

$$\frac{\partial f}{\partial u}\left[1+p\right]+\frac{\partial f}{\partial v}\left[2x+2yp\right]=0$$

diff: (1) w.r.t y;

$$\frac{\partial f}{\partial u}\left[1+q\right]+\frac{\partial f}{\partial v}\left[2z+2yq\right]=0$$

$$\begin{vmatrix} 1+p & 2x+2yp \\ 1+q & 2z+2yq \end{vmatrix} = 0.$$

$$2z - 2yq + 2zp + \cancel{2yqp} - 2x - 2yp - 2qx - \cancel{2yqp} = 0$$

$$z - x + (y-x)q + (z-y)p = 0.$$

$$(x-y)q + (y-z)p = z - x.$$

$$(y-z)p + (x-y)q = z - x$$

Chapter 2

23/10/2020

Solutions of PDE:

Method 1:

Method by direct integration

Solve the following p.d.e's;

29. $$\frac{\partial^2 z}{\partial x \partial y} = \sin x$$

Int. both sides w.r.t y.

$$\frac{\partial z}{\partial x} = y\sin x + f(x)$$

Int w.r.t x

$$z = -y\cos x + \int f(x)\,dx + g(y)$$

$$z = -y\cos x + \phi(x) + g(y)$$

30. $$\frac{\partial^2 z}{\partial y^2} = \sin(xy)$$

Int w.r.t y

$$\frac{\partial z}{\partial y} = \frac{-\cos xy}{x} + f(x)$$

Int w.r.t y

$$z = \frac{-\sin xy}{x^2} + y(f(x)) + g(x)$$

$$z = -\frac{\sin(xy)}{x^2} + y\,f(x) + g(x).$$

31. $$\frac{\partial^2 u}{\partial y \partial x} = 4x\sin 3xy$$

Int. w.r.t y

$$\frac{\partial u}{\partial x} = 4\cancel{x}x \cdot -\cos 3xy \Big/ 3\cancel{x} + f(x)$$

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$$\frac{\partial u}{\partial x} = -\frac{4}{3}\cos(3xy) + f(x).$$

~~a~~ Int = wrt x

$$u = -\frac{4}{3}\frac{\sin 3xy}{3y} + \int f(x)\,dx + g(y)$$

$$u = -\frac{4\sin(3xy)}{9y} + \phi(x) + g(y)$$

32. $$\frac{\partial^2 z}{\partial x \partial y} = (\cos ax + by)$$

Int wrt y

$$\frac{\partial z}{\partial x} = \frac{\sin(ax+by)}{b} + f(x)$$

Int wrt x

$$z = -\frac{\cos(ax+by)}{ab} + \phi(x) + g(y)$$

33. $$\frac{\partial^2 z}{\partial x \partial y} = e^y \cos x$$

Int wrt y

$$\frac{\partial z}{\partial x} = \cos x\, e^y + f(x)$$

Int wrt x

$$z = e^y \times \sin x + \phi(x) + g(y)$$

$$z = e^y \sin x + \phi(x) * g(y)$$

34. $\frac{\partial^2 z}{\partial y^2} = \cos(xy)$

Int wrt y;

$\frac{\partial z}{\partial y} = \frac{\sin(xy)}{x} + f(x)$

Int wrt y;

$z = \frac{-\cos(xy)}{x^2} + y f(x) + g(x)$

35. $\log\left[\frac{d^2 z}{\partial x^2}\right] = x + y$.

$\frac{\partial^2 z}{\partial x^2} = e^{x+y}$ or $(e^x \cdot e^y)$

Int wrt x

$\frac{\partial z}{\partial x} = e^y \cdot e^x + f(y)$.

Int wrt x

$z = e^y . e^x + x f(y) + g(y)$

$z = e^{x+y} + x f(y) + g(y)$

36. $\frac{\partial^3 z}{\partial x^2 \partial y} = \cos(2x + 3y)$

Int : wrt y

$\frac{\partial^2 z}{\partial x^2} = \frac{\sin(2x + 3y)}{3} + f(x)$.

Int : wrt x

$\frac{\partial z}{\partial x} = \frac{-\cos(2x + 3y)}{6} + \int f(x)\,dx + g(y)$

Int wrt x

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$$z = -\frac{\sin(2x+3y)}{12} + \psi(x) + x g(y) + h(y).$$

37. $\frac{\partial z}{\partial x} = 6x+3y \qquad \frac{\partial z}{\partial y} = 3x - 4y$

$$\frac{\partial z}{\partial x} = 6x+3y \text{ ——— } (1)$$

$$\frac{\partial z}{\partial y} = 3x-4y \text{ ——— } (2).$$

Int (1) w.r.t x

$$z = \frac{6x^2}{2} + 3xy + f(y) \text{ —— } (3)$$

Int (2) w.r.t y

$$z = 3xy - \frac{4y^2}{2} + g(x) \text{ —— } (4)$$

$$z = 3x^2 + 3xy + f(y) \text{ ——— } (3)$$

$$z = 3xy - 2y^2 + g(x) \longrightarrow (4).$$

diff (3) w.r.t y

$$\frac{\partial z}{\partial y} = 0 + 3x + f'(y) \text{ ——— } (5).$$

comparing (5) & (2).

$$3x + f'(y) = 3x - 4y$$

$$f'(y) = -4y$$

$$f(y) = -2y^2 + c \text{ ——— } (6)$$

Sub: ⑥ in ③.

$$z = \underline{\underline{3x^2}} + 3xy - 2y^2 + c$$

(OR)

By potential fn: from ③ & ④ dx

$$z = \underline{\underline{3x^2}} + 3xy - 2y^2 + c.$$

38. $\frac{\partial z}{\partial x} = 3x - y$ —— ①

$\frac{\partial z}{\partial y} = -x + \cos y$ —— ②

Int ① wrt x

$$z = \frac{3x^2}{2} - xy + f(y) \text{ —— ③}$$

Int ② wrt y

$$z = -xy + \sin y + g(x) \text{ —— ④}.$$

$$z = \underline{\underline{\frac{3x^2}{2} + \sin y - xy + c}}$$

(OR) diff ③ wrt y.

$$\frac{\partial z}{\partial y} = -x + f'(y) \text{ —— ⑤}$$

$f'(y) = \cos y$ (comparing ⑤ & ②)

$f(y) = \sin y + c$ —— ⑥

Sub: ⑥ in ③

$$z = \underline{\underline{\frac{3x^2}{2} - xy + \sin y + c}}$$

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Solve the following pde's :

39. $\frac{\partial^2 z}{\partial x^2} + z = 0$ Given; when ; $x=0 \Rightarrow z = e^y$

$\& \frac{\partial z}{\partial x} = 1$

If z is a fn: of x alone

$$\frac{\partial^2 z}{\partial x^2} + z = 0 \quad \text{——— (1)}$$

$$\Rightarrow \frac{d^2 z}{dx^2} + z = 0.$$

auxillary eqn:; $m^2+1=0$

$m = \pm i$

$$z = c_1 \cos x + c_2 \sin x$$

Since z is a fn of x and y. c_1 and c_2 are arbitary fn:s of y

$\therefore$ $z = f(y)\cos x + g(y)\sin x$ is the general soln of eqn (1)

$$z = f(y)\cos x + g(y)\sin x \quad \text{——— (2)}.$$

$$\frac{\partial z}{\partial x} = -f(y)\sin x + g(y)\cos x \quad \text{——— (3)}$$

put $x=0$ $\&$ $z=e^y$ in (2).

$$e^y = f(y)$$

put $x=0$ $\&$ $\partial z/\partial x = 1$ in (3)

$$g(y) = 1$$

$\therefore$ The particular soln or integral of ① is

$z = e^{y}\cos x + \sin x$

40. $\frac{\partial^2 z}{\partial y^2} = z$ —— ① given. when $y=0$ $z = e^{x}$

$\frac{\partial z}{\partial y} = e^{-x}$.

If z is a fn of y alone, ① $\Rightarrow$

$$\frac{d^2 z}{dy^2} - z = 0.$$

auxillary eqn ; $m^2 - 1 = 0$

$m = \pm 1$

$$z = c_1 e^{y} + c_2 e^{-y}.$$

Since z is a fn of x and y c_1 and c_2 are arbitrary fns of x.

$\therefore$ $z = f(x)e^{y} + g(x)e^{-y}$ is the gen. soln of ①

$z = f(x)e^{y} + g(x)e^{-(y)}$ —— ②.

$\frac{\partial z}{\partial y} = f(x)e^{y} - g(x)e^{-y}$ —— ③.

put $y=0$ & $z = e^{x}$ in ②.

$e^{x} = f(x) + g(x)$

put $y=0$ & $\frac{\partial z}{\partial y} = e^{-x}$ in ③.

$e^{-x} = f(x) - g(x)$

$f(x) = \frac{e^{x} + e^{-x}}{2} = \cosh x$

$g(x) = \sinh x = \left(\frac{e^{x} - e^{-x}}{2}\right)$.

$z = \cosh x \, e^{y} + \sinh x \, e^{-y}$ (p. soln)

41. $\frac{\partial^2 z}{\partial x^2} + a^2 z = 0$ — ① Given; $x=0 \Rightarrow z = e^{y}$

$\partial z/\partial y = a$

If z is a fn of x alone ①

$\Rightarrow \frac{d^2 z}{dx^2} + a^2 z = 0$

auxillary eqn; $m^2 + a^2 = 0$

$m^2 = -a^2$

$m = \pm ai$

$z = c_1 \cos ax + c_2 \sin ax$

Since z is the fn of x and y, c_1 and c_2 are arbitary fns of y.

$z = f(y) \cos ax + g(y) \sin ax$ ← ②

$\frac{\partial z}{\partial x} = -f(y) a \sin ax + g(y) a \cos ax$ — ③.

put $x=0$ & $z = e^{y}$ in ②.

$e^{y} = f(y)$.

put $x=0$ & $\frac{\partial z}{\partial y} = a$. in ③

$a = g(y) a$ $g(y) = 1$

Thus; particular soln is;

$z = e^{y} \cos ax + \sin ax$

42. $\frac{\partial^2 z}{\partial x^2} - a^2 z = 0$ —— ①

When $x = 0 \Rightarrow \frac{\partial z}{\partial x} = a\sin y$ & $\frac{\partial z}{\partial y} = 0$

If z is a fn of x alone ①.

$\Rightarrow \frac{d^2 z}{dx^2} - a^2 z = 0$.

A. eqn ; $m^2 - a^2 = 0$.

$m^2 = a^2 \quad m = \pm a$.

$z = c_1 e^{ax} + c_2 e^{-ax}$ Since z is the fn of x and y. c_1 and c_2 are arbitary fns of y.

$z = f(y) e^{ax} + g(y) e^{-ax}$ —— ②.

$\frac{\partial z}{\partial x} = f(y) e^{ax} \times a + g(y) \times e^{-ax} \times -a$

$= f(y) a e^{ax} - g(y) a e^{-ax}$ —— ③.

$\frac{\partial z}{\partial y} = f'(y) e^{ax} + g'(y) e^{-ax}$ —— ④.

put $x = 0$ & $\frac{\partial z}{\partial x} = a\sin y$ in ③.

$a\sin y = af(y) - ag(y)$

$f(y) - g(y) = \sin y$ —— ⑤.

put $x = 0$ & $\frac{\partial z}{\partial y} = 0$.

$0 = f'(y) + g'(y)$ —— ⑥.

$f(y) + g(y) = c_3$ —— ⑦.

$2f(y) = \sin y + c_3$

$\boxed{f(y) = \frac{\sin y + c_3}{2}}$. $g(y) = \frac{c_3 - \sin y}{2}$.

The soln is;

$$z = \left(\frac{\sin y + c_3}{2}\right) e^{ax} + \left(\frac{c_3 - \sin y}{2}\right) e^{-ax}$$

Method 3: *

Lagrange's Linear eqn of 1st order:

An eqn of the form;

$Pp + Qq = R$ where;

P, Q, R are $\longrightarrow$ fns of x, y & z

is known as

Lagrange's eqn.

Solution Method:

Step 1 $\longrightarrow$ Form the Lagrange's subsidiary eqn as follows

$$\boxed{\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}}$$

Step 2 $\longrightarrow$ Solve the above simultaneous eqn by the Method of grouping or Multiplier Method.

Step 3 $\longrightarrow$ Give; $u = a$ & $v = b$ are 2 independent solns of the subsidiary eqns then

$$\boxed{\begin{array}{c}\phi(u, v) = 0 \\ \text{or} \\ u = \phi(v)\end{array}}$$

Where ϕ is an arbitrary fn. is the Gen. Integral or Gen. Soln of the Lagrange's Eqn.

Method of Grouping:

This Method is Applicable only when one of the variables is absent from 2 terms in the subsidiary eqn.

Multiplier Method:

The subsidiary eqn is;

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

then we have;

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} = \frac{l\,dx + m\,dy + n\,dz}{lP + mQ + nR}.$$

where l, m, n be constants or fns of x, y, z

Choose l, m, n in such a way that;

$$lP + mQ + nR = 0$$

So that $l\,dx + m\,dy + n\,dz = 0$

Solve the following pde's:

43. $yq - xp = z$

$Pp + Qq = R \Rightarrow -xp + yq = z$

$P = -x \quad Q = y \quad R = z$

The subsidiary eqns are;

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\frac{dx}{-x} = \frac{dy}{y} = \frac{dz}{z}$$

Considering the 1st 2 eqns:

$$\frac{dx}{-x} = \frac{dy}{y}$$

$$-\log x = \log y - \log a$$

$$\log a = \log x + \log y$$

$$\log a = \log(xy)$$

$$xy = a$$

Considering the ~~2nd~~

$$\frac{dy}{y} = \frac{dz}{z}$$

$$\log y = \log z + \log b$$

$$\Rightarrow y/z = b$$

$$u = xy \quad V = y/z$$

The soln is; $\phi\left(xy, y/z\right) = 0$

44. $p\tan x + q\tan y = \tan z$

$$Pp + Qq = R$$

$$P \to \tan x \quad Q = \tan y \quad R = \tan z$$

The subsidiary eqn is;

$$\frac{dx}{\tan x} = \frac{dy}{\tan y} = \frac{dz}{\tan z}$$

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$$\frac{dx}{\tan x} = \frac{dy}{\tan y}$$

$$\frac{\cos x\,dx}{\sin x} = \frac{\cos y\,dy}{\sin y}$$

$$\log(\sin x) = \log(\sin y) + \log a$$

$$a = \frac{\sin x}{\sin y}$$

$$\frac{dy}{\tan y} = \frac{dz}{\tan z}$$

$$\frac{\cos y\,dy}{\sin y} = \frac{\cos z\,dz}{\sin z}$$

$$\log(\sin y) = \log(\sin z) + \log b$$

$$b = \frac{\sin y}{\sin z}$$

$$u = \frac{\sin x}{\sin y} \qquad v = \frac{\sin y}{\sin z}$$

The soln is; $\phi\left(\sin x/\sin y, \sin y/\sin z\right) = 0$

45. $y^2p - xyq = x(z-2y)$

$Pp + Qq = R$

$P = y^2 \quad Q = -xy \quad R = xz - 2xy.$

The subsidiary eqn is

$$\frac{dx}{y^2} = \frac{dy}{-xy} = \frac{dz}{x(z-2y)}$$

Considering the 1st 2 eqns we have;

$$\frac{dx}{y^2} = \frac{dy}{-xy}$$

$$\frac{dx}{y} = \frac{dy}{-x}$$

$$-x\,dx = y\,dy.$$

$$\frac{x^2}{2} = -\frac{y^2}{2} + \frac{a}{2}$$

$$a = x^2+y^2$$

Considering the last 2 eqns.

$$\frac{-dy}{\cancel{x}y} = \frac{dz}{\cancel{x}(z-2y)}$$

$$-\frac{dy}{y} = \frac{dz}{z-2y}.$$

$$-z\,dy + 2y\,dy = y\,dz$$

$$2y\,dy = y\,dz + z\,dy$$

$$2y\,dy = d(yz)$$

$$y^2 = yz + b$$

$$b = y^2 - yz$$

$$u = x^2+y^2 \qquad v = y^2-yz$$

The soln is; $\phi(x^2+y^2,\ y^2-yz) = 0$

46 $\frac{y^2 z}{x} p + xzq = y^2$

$Pp + Qq = R.$

$P \rightarrow \frac{y^2 z}{x} \quad Q \rightarrow xz \quad R \rightarrow y^2.$

$$\frac{x\,dx}{y^2 z} = \frac{dy}{xz} = \frac{dz}{y^2}$$

$$\frac{x\,dx}{\cancel{y^2} z} = \frac{dz}{\cancel{y^2}}.$$

$x\,dx = z\,dz$

$x^2 = z^2 + a$

$a = x^2 - z^2$

$\underline{\underline{u = x^2 - z^2}}$

$$\frac{x\,dx}{y^2 z} = \frac{dy}{xz}$$

$x^2 dx = y^2 dy$

$x^3 = y^3 + b.$

$b = x^3 - y^3$

$\underline{\underline{V = x^3 - y^3}}$

The soln is $\underline{\underline{\phi(x^2 - z^2, x^3 - y^3) = 0}}$

47 $pz - qz = z^2 + (x+y)^2$

$Pp + Qq = R.$

$P = z \quad Q = -z \quad R = z^2 + (x+y)^2$

$$\frac{dx}{z} = \frac{dy}{-z} = \frac{dz}{z^2 + (x+y)^2}$$

$dx = -dy$

$x = -y + a \quad \underline{\underline{a = (x+y)}}$

$$\frac{dx}{z} = \frac{dz}{z^2+a^2}$$

$$2dx = \frac{2z\,dz}{z^2+a^2}$$

$$2x = \log(z^2+a^2) + b.$$

$$b = 2x - \log(z^2+a^2)$$

The soln is $\phi\left((x+y),\ 2x-\log(z^2+a^2)\right)=0$

48. $(y^2+z^2)p - xyq + xz = 0.$

$P = y^2+z^2 \quad Q = -xy \quad R = -xz.$

$$\frac{dx}{y^2+z^2} = \frac{dy}{-xy} = \frac{dz}{-xz}.$$

$$\frac{dy}{-\cancel{x}y} = \frac{dz}{-\cancel{x}z}$$

$$\frac{dy}{y} = \frac{dz}{z}$$

$$\log y = \log z + \log a.$$

$$a = y/z \longrightarrow (4)$$

By Multiplier Method;

Using the multipliers x, y, z,

$$\frac{dx}{y^2+z^2} = \frac{-dy}{xy} = \frac{-dz}{xz} = \frac{x\,dx + y\,dy + z\,dz}{x(y^2+z^2)+y(-xy)+z(-xz)}$$

$$\Rightarrow xdx + ydy + zdz = 0.$$

$$b = x^2 + y^2 + z^2 \longrightarrow (V)$$

The soln is; $\phi\left(y/z,\ x^2+y^2+z^2\right)=0$

$\Rightarrow xdx + ydy + zdz = 0.$

$$b = x^2 + y^2 + z^2 \longrightarrow (v)$$

The soln is; $\phi\left(y/z,\ x^2+y^2+z^2\right) = 0$

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47. $\frac{y-z}{yz} p + \frac{z-x}{zx} q = \frac{x-y}{xy}.$

$P = \frac{y-z}{yz} \quad Q = \frac{z-x}{zx} \quad R = \frac{x-y}{xy}.$

$(y-z)xp + (z-x)yq = (x-y)z.$

$P = (y-z)x \quad Q = (z-x)y \quad R = (x-y)z.$

$$\frac{dx}{(y-z)x} = \frac{dy}{(z-x)y} = \frac{dz}{(x-y)z}$$

Using the multipliers 1,1,1 we get;

$$\frac{dx}{(y-z)x} = \frac{dy}{(z-x)y} = \frac{dz}{(x-y)z} = \frac{dx+dy+dz}{(y-z)x+(z-x)y+(x-y)z}$$

$$dx + dy + dz = 0.$$

$$x+y+z = u.$$

Using the multipliers $1/x, 1/y, 1/z$

$$\frac{dx}{(y-z)x} = \frac{dy}{(z-x)y} = \frac{dz}{(x-y)z} = \frac{dx/x + dy/y + dz/z}{y-z+z-x+x-y}.$$

$$\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z} = 0$$

$$\log x + \log y + \log z = \log V$$

$xyz = v$

The soln is $z = \phi(x+y+z,\ xyz) = 0$

50. $x(y^2-z^2)p + y(z^2-x^2)q = z(x^2-y^2)$

$P = x(y^2-z^2)$ $\quad Q = y(z^2-x^2)$ $\quad R = z(x^2-y^2)$

$$\frac{dx}{x(y^2-z^2)} = \frac{dy}{y(z^2-x^2)} = \frac{dz}{z(x^2-y^2)} \quad \text{(S.eqn)}$$

Using the multipliers $1/x,\ 1/y,\ 1/z$

$$\frac{dx}{x(y^2-z^2)} = \frac{dy}{y(z^2-x^2)} = \frac{dz}{z(x^2-y^2)} = \frac{dx/x + dy/y + dz/z}{y^2-z^2+z^2-x^2+x^2-y^2}$$

$$\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z} = 0.$$

$$\log x + \log y + \log z = \log u$$

$$xyz = u$$

Using the multipliers x, y, z

$$\frac{dx}{x(y^2-z^2)} = \frac{dy}{y(z^2-x^2)} = \frac{dz}{z(x^2-y^2)} = \frac{x\,dx + y\,dy + z\,dz}{x^2(y^2-z^2)+y^2(z^2-x^2)+z^2(x^2-y^2)}$$

$$x\,dx + y\,dy + z\,dz = 0.$$

$$x^2+y^2+z^2 = v$$

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The soln is;

$$z = \phi\left(xyz,\ x^2+y^2+z^2\right) = 0.$$

5). $(z^2-2yz-y^2)p + (xy+zx)q = xy-zx.$

$P = z^2-2yz-y^2 \qquad Q = xy+zx$

$R = xy-zx.$

$$\frac{dx}{z^2-2yz-y^2} = \frac{dy}{xy+zx} = \frac{dz}{xy-zx}.$$

$$\frac{dy}{y+z} = \frac{dz}{y-z}$$

$$ydy - zdy = ydz + zdz$$

$$ydy - zdz = ydz + zdy.$$

$$ydy - zdz = d(yz)$$

$$\frac{y^2}{2} - \frac{z^2}{2} = yz + \frac{u}{2}.$$

$$\frac{u}{2} = \frac{y^2-z^2-2yz}{2}$$

$$u = y^2-z^2-2yz$$

using Multipliers x, y, z we get

$$\frac{dx}{z^2-2yz-y^2} = \frac{dy}{xy+zx} = \frac{dz}{xy-zx} = \frac{xdx+ydy+zdz}{xz^2-2yxz-xy^2+xy^2+xyz+xyz-xz^2}.$$

$$xdx+ydy+zdz = 0.$$

$$x^2+y^2+z^2 = v.$$

The soln is ; $\phi\left(y^2-z^2-2yz,\ x^2+y^2+z^2\right)=0$

52. $x^2(y-z)p + y^2(z-x)q = z^2(x-y)$

$P = x^2(y-z)$ $\quad Q = y^2(z-x)$ $\quad R = z^2(x-y)$

$$\frac{dx}{x^2(y-z)} = \frac{dy}{y^2(z-x)} = \frac{dz}{z^2(x-y)}$$

Using Multipliers $1/x^2$, $1/y^2$, $1/z^2$.

$$\frac{dx}{x^2(y-z)} = \frac{dy}{y^2(z-x)} = \frac{dz}{z^2(x-y)} = \frac{dx/x^2 + dy/y^2 + dz/z^2}{(y-z)+(z-x)+(x-y)}$$

$$\frac{dx}{x^2} + \frac{dy}{y^2} + \frac{dz}{z^2} = 0.$$

$$\frac{-1}{x} + \frac{-1}{y} - 1/z = -u.$$

$$u = 1/x + 1/y + 1/z$$

Using the Multipliers $1/x$, $1/y$, $1/z$

$$\frac{dx}{x^2(y-z)} = \frac{dy}{y^2(z-x)} = \frac{dz}{z^2(x-y)} = \frac{dx/x + dy/y + dz/z}{x(y-z)+y(z-x)+z(x-y)}$$

$$\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z} = 0.$$

$$\log x + \log y + \log z = \log v$$

$$xyz = v.$$

The soln is;

$$\phi\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z},\ xyz\right)=0$$

53. $(2z-y)p+(x+z)q+2x+y=0.$

$(2z-y)p+(x+z)q=-(2x+y)$

$P=2z-y \qquad Q=x+z \qquad R=-(2x+y)$

$$\frac{dx}{2z-y}=\frac{dy}{x+z}=\frac{dz}{-(2x+y)}$$

Using Multipliers $1,-2,-1$

$$\frac{dx}{2z-y}=\frac{dy}{x+z}=\frac{dz}{-2x-y}=\frac{dx-2dy-dz}{\cancel{2z}-\cancel{y}-\cancel{2x}-\cancel{2z}+\cancel{2x}+\cancel{y}}$$

$$dx-2dy-dz=0$$

$$x-2y-z=u.$$

Using Multipliers $x,y,z.$

$$\frac{dx}{2z-y}=\frac{dy}{x+z}=\frac{dz}{-2x-y}=\frac{xdx+ydy+zdz}{x(2z-y)+y(x+z)+z(-2x-y)}$$

$$xdx+ydy+zdz=0.$$

$$x^2+y^2+z^2=v.$$

The soln is;

$$\phi\left(x-2y-z,\ x^2+y^2+z^2\right)=0$$

54. $(y+zx)p-(x+yz)q=x^2-y^2$.

$P=y+zx \quad Q=-x-yz \quad R=x^2-y^2$

$$\frac{dx}{y+zx}=\frac{dy}{-x-yz}=\frac{dz}{x^2-y^2}$$

Using Multipliers $y, x, 1$

$$\frac{dx}{y+zx}=\frac{dy}{-x-yz}=\frac{dz}{x^2-y^2}=\frac{y\,dx+x\,dy+dz}{y(y+zx)+x(-x-yz)+x^2-y^2}.$$

$$y\,dx+x\,dy+dz=0.$$

$$d(xy)+dz=0 \qquad xy+z=v$$

Using Multipliers $x, y, -z$

$$\frac{dx}{y+zx}=\frac{dy}{-x-yz}=\frac{dz}{x^2-y^2}=\frac{x\,dx+y\,dy-z\,dz}{x(y+zx)+y(-x-yz)+z(x^2-y^2)}$$

$$x\,dx+y\,dy-z\,dz=0.$$

$$x^2+y^2-z^2=u.$$

Using Multipliers $y, x, 1$

$$\frac{dx}{y+zx}=\frac{dy}{-x-yz}=\frac{dz}{x^2-y^2}=\frac{y\,dx+x\,dy+dz}{y(y+zx)+x(-x-yz)+x^2-y^2.}$$

$$y\,dx+x\,dy+dz$$

$$d(xy)+dz=0$$

$$xy+z=v$$

The soln is ; $\phi(x^2+y^2-z^2,\ xy+z)=0$

No. 55. $x^2p + y^2q = (x+y)z$

$P = x^2 \quad Q = y^2 \quad R = (x+y)z$

$$\frac{dx}{x^2} = \frac{dy}{y^2} = \frac{dz}{(x+y)z}$$

$$\frac{dx}{x^2} = \frac{dy}{y^2}.$$

$$\frac{-1}{x} = \frac{-1}{y} + a.$$

$$\frac{-1}{x} + \frac{1}{y} = a.$$

$$\frac{x-y}{xy} = a.$$

Multiplier $1/x, 1/y, -1/z$

$$\frac{dx}{x^2} = \frac{dy}{y^2} = \frac{dz}{(x+y)z} = \frac{\frac{dx}{x} + \frac{dy}{y} + \frac{-dz}{z}}{\cancel{x} + \cancel{y} + \cancel{x} - \cancel{y}}$$

$$\frac{dx}{x} + \frac{dy}{y} - \frac{dz}{z} = 0.$$

$$\log x + \log y - \log z = \log b.$$

$$\frac{xy}{z} = b. = v$$

The soln $\phi\left(\frac{x-y}{xy}, \frac{xy}{z}\right) = 0$

No. 56. $x(y^2 - z^2)p - y(z^2 + x^2)q = z(x^2+y^2)$

$P = x(y^2 - z^2) \quad Q = -y(z^2+x^2)$

$R = z(x^2+y^2)$

$$\frac{dx}{x(y^2-z^2)} = \frac{dy}{-y(z^2+x^2)} = \frac{dz}{z(x^2+y^2)}$$

Multiplier x, y, z

$$\frac{dx}{x(y^2-z^2)} = \frac{dy}{-y(z^2+x^2)} = \frac{dz}{z(x^2+y^2)} = \frac{x\,dx + y\,dy + z\,dz}{x^2(y^2-z^2) - y^2(z^2+x^2) + z^2(x^2+y^2)}$$

$$x^2+y^2+z^2 = a$$

Multiplier $1/x, -1/y, -1/z$

$$\frac{dx}{x(y^2-z^2)} = \frac{dy}{-y(z^2+x^2)} = \frac{dz}{z(x^2+y^2)} = \frac{\frac{dx}{x} - \frac{dy}{y} - \frac{dz}{z}}{y^2-z^2+z^2+x^2-x^2-y^2}$$

$$\frac{dx}{x} - \frac{dy}{y} - \frac{dz}{z} = 0$$

$$\log x - \log y - \log z = \log a$$

$$\log y + \log z - \log x = \log b$$

$$b = yz/x$$

The soln is $\phi\left(x^2+y^2+z^2, \; yz/x\right) = 0$

57. $xzp + yzq = xy.$

$P = xz \quad Q = yz \quad R = xy.$

$$\frac{dx}{xz} = \frac{dy}{yz} = \frac{dz}{xy}.$$

$$\frac{dx}{x\cancel{z}} = \frac{dy}{y\cancel{z}}$$

$$\log x = \log y + \log a.$$

$$u = a = x/y$$

$$\cancel{\frac{dy}{yz} = \frac{dz}{xy}}$$

~~Ans~~ The Multipliers are $y, x, 2z$.

$$\frac{dx}{xz} = \frac{dy}{yz} = \frac{dz}{xy} = \frac{ydx + xdy - 2zdz}{xyz + xyz - 2xyz}$$

$$ydx + xdy - 2zdz = 0.$$

$$d\,xy - 2zdz = 0.$$

$$xy - \frac{2z^2}{2} = v$$

$$v = \underline{\underline{xy - z^2}}.$$

The soln is $\underline{\underline{\phi\left(x/y,\ xy - z^2\right) = 0}}$

Ano 58. $P/x^2 + q/y^2 = z$.

$$P = 1/x^2 \qquad Q = 1/y^2 = R.$$

$$\frac{dx}{1/x^2} = \frac{dy}{1/y^2} = \frac{dz}{z}.$$

$$x^2dx = y^2dy = \frac{dz}{z}.$$

$$x^2dx = y^2dy$$

$$\frac{x^3}{3} = \frac{y^3}{3} + \frac{a}{3}.$$

$$\underline{\underline{a = x^3 - y^3}}$$

$$y^2dy = \frac{dz}{z}$$

$$\frac{y^3}{3} = \log z + b.$$

$$b = y^3 - 3\log z.$$

The soln is $\underline{\underline{\phi\left(x^3 - y^3,\ y^3 - 3\log z\right) = 0}}$

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59. $P/y + Q/x = 1/z$

$P = 1/y \quad Q = 1/x \quad R = 1/z$

$$\frac{dx}{1/y} = \frac{dy}{1/x} = \frac{dz}{1/z}$$

$$y\,dx = x\,dy = z\,dz.$$

$$\frac{dx}{x} = \frac{dy}{y}.$$

$$\log x = \log y + \log a.$$

$$x/y = a$$

Multiplier $y, x, -2z$.

$$\frac{dx}{1/y} = \frac{dy}{1/x} = \frac{dz}{1/z} = \frac{y\,dx + x\,dy - 2z\,dz}{1+1-2}$$

$$= d(yx) - 2z\,dz$$

$$= yx - z^2$$

The soln is $\phi(x/y,\ yx - z^2) = 0$

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60. $(x^2+y^2)p + 2xyq = (x+y)z$

$P = x^2+y^2 \quad Q = 2xy \quad R = (x+y)z$

$$\frac{dx}{x^2+y^2} = \frac{dy}{2xy} = \frac{dz}{(x+y)z}$$

$$\frac{dx}{x^2+y^2} = \frac{dy}{2xy}$$

$$2xy\,dx = x^2 dy + y^2 dy.$$

$$\frac{2xy\,dx - x^2 dy}{y^2} = dy.$$

$$d\left(\frac{x^2}{y}\right) = dy.$$

$$\frac{x^2}{y} = y + u.$$

$$u = \frac{x^2}{y} - y$$

$$u = \frac{x^2-y^2}{y}$$

Multipliers $1, 1, \cancel{(x+y)/z}$

$$\frac{dx}{x^2+y^2} = \frac{dy}{2xy} = \frac{dz}{(x+y)z} = \frac{dx+dy-\frac{(x+y)}{z}dz}{(x^2+y^2)+2xy+\cancel{z}\frac{(x+y)}{}\times\frac{-(x+y)}{\cancel{z}}}$$

$$\Rightarrow dx+dy - \frac{x+y}{z}dz = 0$$

$$dx+dy = x+y\,\frac{dz}{z}.$$

$$\frac{dz}{z} = \frac{dx+dy}{x+y}$$

$$\log z = \log(x+y) + \log b.$$

$$b = z/x+y$$

The soln is $\phi\left((x^2-y^2)/y,\ z/x+y\right) = 0$

61. $(x-y)p + (y-x-z)q = z$

$P = x-y \quad Q = y-x-z \quad R = z$

$$\frac{dx}{x-y} = \frac{dy}{y-x-z} = \frac{dz}{z}$$

Multiplier 1, 1, 1

$$\frac{dx}{x-y} = \frac{dy}{y-x-z} = \frac{dz}{z} = \frac{dx+dy+dz}{x-y+y-x-z+z}$$

$$dx+dy+dz = 0.$$

$$\underline{\underline{x+y+z}} = a \;(u) \qquad x+z = u-y.$$

$$\frac{dy}{y-x-z} = \frac{dz}{z}$$

$$\frac{dy}{y+y-u} = \frac{dz}{z} \Rightarrow \frac{2\,dy}{2y-u} = \frac{2dz}{z}$$

$$\log(2\,y-u) = 2\log z + \log u.$$

$$2y-u = z^2 v$$

$$v = \frac{2y-u}{z^2}$$

The soln; $\underline{\underline{\phi\left(x+y,+z\,,\ \frac{2y-u}{z^2}\right)=0}}$

$$\underline{\underline{\phi\left(x+y+z\,,\ \frac{y-x-z}{z^2}\right)=0}}$$

62. $px^2 - qy^2 = z(x-y)$

$P = x^2 \quad Q = -y^2 \quad R = (x-y)z$

$$\frac{dx}{x^2} = \frac{dy}{-y^2} = \frac{dz}{(x-y)z}$$

$$\frac{dx}{x^2} = \frac{dy}{-y^2}$$

$$-\frac{1}{x} = \frac{1}{y} - a.$$

$$-\frac{y-x}{xy} = -a.$$

$$u = \frac{y+x}{xy}$$

Multipliers are $1/x$, $1/y$, $-1/z$

$$\frac{dx}{x^2} = \frac{dy}{-y^2} = \frac{dz}{(x-y)z} = \frac{dx/x + dy/y - dz/z}{\frac{x^2}{x} + \frac{y^2}{y} - \frac{(x-y)z}{z}}.$$

$$\frac{dx}{x} + \frac{dy}{y} - \frac{dz}{z} = 0.$$

$$\log x + \log y - \log z = \log b.$$

$$xy/z = b$$

$$V = xy/z$$

The soln is $\phi\left(\frac{y+x}{xy}, xy/z\right) = 0$

63. $(x-2z)p + (2z-y)q = y-x.$

$P = x-2z \quad Q = 2z-y \quad R = y-x$

$$\frac{dx}{x-2z} = \frac{dy}{2z-y} = \frac{dz}{y-x}.$$

Multipliers 1, 1, 1

$$\frac{dx}{x-2z} = \frac{dy}{2z-y} = \frac{dz}{y-x} = \frac{dx+dy+dz}{x-2z+2z-y+y-x}$$

$$dx+dy+dz = 0.$$

$$x+y+z = u$$

Multiplier $y, x, 2z$

$$\frac{dx}{x-2z} = \frac{dy}{2z-y} = \frac{dz}{y-x} = \frac{y\,dx + x\,dy + 2z\,dz}{y(x-2z)+x(2z-y)+2z(y-x)}.$$

$$y\,dx + x\,dy + 2z\,dz = 0$$

$$xy + z^2 = v$$

The soln is $\phi(x+y+z,\ xy+z^2) = 0$

64. $(x^2-y^2-z^2)p + 2xyq = 2xz$

$P = x^2-y^2-z^2 \quad Q = 2xy \quad R = 2xz$

$$\frac{dx}{x^2-y^2-z^2} = \frac{dy}{2xy} = \frac{dz}{2xz}$$

$$\frac{dy}{\cancel{2x}y} = \frac{dz}{\cancel{2x}z}$$

$$\frac{dy}{y} = \frac{dz}{z}$$

$$\log y = \log z + \log u$$

$$u = y/z$$

Multipliers are $x; y, z$

$$\frac{dx}{x^2-y^2-z^2} = \frac{dy}{2xy} = \frac{dz}{2xz} = \frac{x\,dx + y\,dy + z\,dz}{x^3 - xy^2 - xz^2 + 2xy^2 + 2xz^2}$$

$$\frac{x\,dx + y\,dy + z\,dz}{x^3 + xy^2 + xz^2} = \frac{dz}{2x\cancel{z}}$$

$$\frac{2(x\,dx + y\,dy + z\,dz)}{x^2+y^2+z^2} = \frac{dz}{z}$$

$$\log(x^2+y^2+z^2) = \log z + \log v$$

$$v = \frac{x^2+y^2+z^2}{z}$$

The soln is $\phi\left(y/z, \frac{x^2+y^2+z^2}{z}\right) = 0$

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Method 4
Chapter

Charpit's Method.

Hints:
First find
f_p, f_q, f_x, f_y, f_z

It is a general method for finding the complete integral of a non-linear pde.

Consider the equation $f(x,y,z,p,q)=0 \rightarrow (1)$

The subsidiary equns in charpit's method are

$$\frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{-\frac{\partial f}{\partial q}} = \frac{dz}{-p\frac{\partial f}{\partial p}-q\frac{\partial f}{\partial q}} = \frac{dp}{\frac{\partial f}{\partial x}+p\frac{\partial f}{\partial z}}$$

$$= \frac{dq}{\frac{\partial f}{\partial y}+q\frac{\partial f}{\partial z}}$$

Find a relation from these equations involving x,y,z,p,q st $\phi(x,y,z,p,q)=0 \rightarrow (2)$

Since z depends on x and y we have

$dz = p\,dx + q\,dy \rightarrow (3)$

Solve (1) & (2) for p and q and substitute in (3)

Multipliers are x, y, z

$$\frac{dx}{x^2-y^2-z^2} = \frac{dy}{2xy} = \frac{dz}{2xz} = \frac{x\,dx + y\,dy + z\,dz}{x^3 - xy^2 - xz^2 + 2xy^2 + 2xz^2}$$

$$\frac{x\,dx + y\,dy + z\,dz}{x^3 + xy^2 + xz^2} = \frac{dz}{2xz}$$

$$\frac{2(x\,dx + y\,dy + z\,dz)}{x^2+y^2+z^2} = \frac{dz}{z}$$

$$\log(x^2+y^2+z^2) = \log z + \log v$$

$$v = \frac{x^2+y^2+z^2}{z}$$

The soln is $\phi\left(y/z, \frac{x^2+y^2+z^2}{z}\right) = 0$

$\longrightarrow$ **Hint:**

First Find:
f_p, f_q, f_x, f_y, f_z

2/11/2020

Method 4:

Charpit's Method

It is the general method for finding the complete integral of a nonlinear pde.

Consider the eqn $f(x, y, z, p, q) = 0$ —— (1)

The subsidiary eqns in Charpit's Method are;

$$\frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{-\frac{\partial f}{\partial q}} = \frac{dz}{-p\frac{\partial f}{\partial p} - q\frac{\partial f}{\partial q}} = \frac{dp}{\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}}$$

Find a relation from these eqns. involving x, y, z, p, q st $\phi(x, y, z, p, q) = 0$ —— (2)

Since z depends on x and y we have

$dz = p\,dx + q\,dy \longrightarrow$ (3).

Solve ① and ② for p and q and substitute in ③

Solve the following using Charpits Method

65. $2xz + pq = 2xyq + x^2p$ ⟵ ①

$f = 2xz + pq - 2xyq - x^2p$.

$f_p = q - x^2$

$f_q = p - 2xy$

$f_x = 2z - 2yq - 2xp$

$\boxed{f_y = -2xq}$

$f_z = 2x$

The subsidiary eqns are given by;

$$\frac{dx}{-f_p} = \frac{dy}{-f_q} = \frac{dz}{-pf_p - qf_q} = \frac{dp}{f_x + pf_z} = \frac{dq}{f_y + qf_z}$$

$$\frac{dx}{-q + x^2} = \frac{dy}{-2x} = \frac{dz}{-p(q - x^2) - q(p - 2xy)} = \frac{dp}{2z - 2yq - \cancel{2xp} + \cancel{2xp}} = \frac{dq}{-\cancel{2xq} + \cancel{2xq}}$$

$$\frac{dx}{-q + x^2} = \frac{dy}{-2x} = \frac{dz}{px^2 + 2xyq - 2pq} = \frac{dp}{2z - 2yq} = \frac{dq}{0}$$

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$\cancel{\frac{dp}{2z}} \quad dq = 0 \quad \Rightarrow \quad \underline{\underline{q = a}}$

Substituting $q = a$ in ①.

$$2xz + ap = 2xya + x^2p$$

$$(a - x^2)p = 2x(ya - z)$$

$$\therefore p = \frac{2x(ay - z)}{(a - x^2)}$$

$$\boxed{dz = p\,dx + q\,dy}$$

$$dz = \frac{2x(ay - z)}{a - x^2}dx + a\,dy.$$

$$\frac{dz - a\,dy}{ay - z} = \frac{2x}{a - x^2}dx$$

$$\frac{dz - a\,dy}{z - ay} = \frac{2x\,dx}{x^2 - a}.$$

$$\log(z - ay) = \log(x^2 - a) + \log b$$

$$\underline{\underline{\frac{z - ay}{x^2 - a} = b}}$$

$z = \underline{\underline{ay + (x^2 - a)b}}$ is the soln.

66. $p^2 + qy = z$ ———— ①

$$f = p^2 + qy - z$$

$$f_p = 2p$$

$$f_q = y$$

$f_x = 0$

$f_y = q$

$f_z = -1$

The subsidiary eqns are :

$$\frac{dx}{-f_p} = \frac{dy}{-f_q} = \frac{dz}{-pf_p - qf_q} = \frac{dp}{f_x + pf_z} = \frac{dq}{f_y + qf_z}$$

$$\frac{dx}{-2p} = \frac{dy}{-y} = \frac{dz}{-2p^2 - qy} = \frac{dp}{-p} = \frac{dq}{0}.$$

$dq = 0 \Rightarrow q = a.$

Substituting $q = a$ in ①

$p^2 + ya = z$

$p = \sqrt{z - ay}$

$dz = p\,dx + q\,dy.$

$dz = \sqrt{z-ay}\,dx + a\,dy$

$$\frac{dz - a\,dy}{\sqrt{z-ay}} = dx$$

$2\sqrt{z-ay} = x + b$

$4(z-ay) = (x+b)^2$

67. $p^2 - q^2 + (x+y)^2 = 0.$

$f = p^2 - q^2 + (x+y)^2$

$f_p = 2p$

$f_q = -2q$

$f_x = 2x + 2y.$

$f_y = 2x + 2y$

$f_z = 0.$

The S. eqns are;

$$\frac{dx}{-2p} = \frac{dy}{2q} = \frac{dz}{-2p^2+2q^2} = \frac{dp}{2x+2y} = \frac{dq}{2x+2y}$$

Considering the last 2 eqns we get;

$dp = dq$

$p = q + a.$

$(q+a)^2 - q^2 + (x+y)^2 = 0.$

$2qa + a^2 + (x+y)^2 = 0$

$$q = \frac{-a^2}{2a} - \frac{(x+y)^2}{2a}.$$

$$q = \frac{-a}{2} - \frac{(x+y)^2}{2a}$$

$dz = p\,dx + q\,dy.$

$$dz = \left(\frac{a}{2} - \frac{(x+y)^2}{2a}\right)dx - \left(\frac{(x+y)^2}{2a} + \frac{a}{2}\right)dy$$

68. $pxy + pq + qy = yz$

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$f = pxy + pq + qy - yz$

$f_p = xy + q$

$f_q = p + y$

$f_x = py.$

$f_y = px + q - z$

$f_z = -y.$

The subsidiary eqn is;

$$\frac{dx}{-(xy+q)} = \frac{dy}{-p-y} = \frac{dz}{-p(xy+q)-q(p+y)} = \frac{dp}{\cancel{py}+p\cancel{(-y)}} = \frac{dq}{px+q-z-yq}$$

$dp = 0$

$p = a$

$axy + aq + qy = yz.$

$$q = \frac{yz - axy}{(a+y)}.$$

$dz = p\,dx + q\,dy.$

$$dz = a\,dx + \frac{(yz - axy)}{a+y}\,dy$$

$$\frac{dz - a\,dx}{z - ax} = \frac{y\,dy}{a+y}$$

$$d(z-ax) = \frac{a+y-a\,dy}{a+y}$$

$$= dy - \frac{a}{a+y}\,dy.$$

$$\log(z-ax) = y - a\log(a+y)+b.$$

$$\log(z-ax)(a+y)^a = y+b.$$

$$(z-ax)(a+y)^a = e^{y+b}$$

69. $(p^2+q^2)y = qz$

$$f = (p^2+q^2)y - qz$$

$$f_p = 2py$$

$$f_q = 2qy - z$$

$$f_x = 0.$$

$$f_y = p^2+q^2$$

$$f(z) = -q$$

The subsidiary eqn;

$$\frac{dx}{-2py} = \frac{dy}{-2qy+z} = \frac{dz}{-p(2py)-q(2qy-z)} = \frac{dp}{0+-pq} = \frac{dq}{p^2+\cancel{q^2}-\cancel{q^2}}$$

$$\frac{dx}{-2py} = \frac{dy}{z-2qy} = \frac{dz}{-2p^2y-2q^2y+qz} = \frac{dp}{-pq} = \frac{dq}{p^2}$$

$$\frac{dp}{-pq} = \frac{dq}{p^2}$$

$$-p\,dp = q\,dq$$

$$-\frac{p^2}{2} = \frac{q^2}{2} - a/2 \Rightarrow \underline{\underline{a = p^2+q^2}}$$

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$$ay = qz$$

$$q = \frac{ay}{z}$$

$$dz = p\,dx + q\,dy.$$

$$dz = \sqrt{a-q^2}\,dx + \frac{ay}{z}\,dy.$$

$$dz = \sqrt{a - \frac{a^2y^2}{z^2}}\,dx + \frac{ay}{z}\,dy$$

$$dz = \sqrt{\frac{az^2 - a^2y^2}{z^2}}\,dx + \frac{ay}{z}\,dy.$$

$$dz = \frac{\sqrt{a}}{z}\sqrt{z^2 - ay^2}\,dx + \frac{ay}{z}\,dy$$

$$z\,dz = \sqrt{a}\sqrt{z^2 - ay^2}\,dx + ay\,dy.$$

$$z\,dz - ay\,dy = \sqrt{a}\sqrt{z^2-ay^2}\,dx$$

$$\frac{z\,dz - ay\,dy}{\sqrt{z^2-ay^2}} = \sqrt{a}\,dx.$$

$$d\left(\sqrt{z^2-ay^2}\right) = \sqrt{a}\,dx$$

$$\sqrt{z^2-ay^2} = \sqrt{a}x + b$$

$$z^2 - ay^2 = (\sqrt{a}x + b)^2$$

10. $px + qy = pq.$

$f = px + qy - pq$

$f_p = x - q$

$f_q = y - p.$

$f_x = p$

$f_y = q$

$f_z = 0.$

The s. eqn is;

$$\frac{dx}{-x+q} = \frac{dy}{-y+p} = \frac{dz}{-p(x-q)-q(y-p)} = \frac{dp}{p+0} = \frac{dq}{q}$$

$$\frac{dp}{p} = \frac{dq}{q}$$

$$\log p = \log q + \log a.$$

$$a = \frac{p}{q}$$

$$\underline{\underline{p = aq}}$$

$$aqx + qy = aq^2.$$

$$ax + y = aq.$$

$$\underline{\underline{q = \frac{ax+y}{a}.}}$$

$$dz = aq\,dx + \frac{ax+y}{a}\,dy.$$

$$dz = \frac{a^2x + ay}{a}\,dx + \frac{ax+y}{a}\,dy$$

$$a\,dz = a^2x\,dx + y\,dy + a\,d(xy)$$

$$\underline{\underline{az = \frac{(ax+y)^2}{2} + b}}$$

$$az = \frac{a^2x^2 + y^2 + 2axy}{2}.$$

72.
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$$2(px+qy) = z(1-q^2)$$

#

$$f = 2px + 2qy - z + zq^2$$

$$f_p = 2x$$

$$f_q = 2y + 2zq$$

$$f_x = 2p$$

$$f_y = 2q$$

$$f_z = -1 + q^2.$$

$$\frac{dx}{-2x} = \frac{dy}{-2y-2zq} = \frac{dz}{-2px-2yq-2zq^2} = \frac{dp}{2p+-p+q^2p} = \frac{dq}{2q-q+q^3}$$

$$\frac{dp}{p\cancel{(1+q^2)}} = \frac{dq}{q\cancel{(1+q^2)}}$$

$$\frac{dp}{p} = \frac{dq}{q}$$

$$\log p = \log q + \log a.$$

$$p/q = a.$$

$$\underline{\underline{p = aq}}$$

$$2aqx + 2qy = z(1-q^2).$$

$$(2ax + 2y)q = z - zq^2$$

$$\frac{dz}{-2px-2yq-2zq^2} = \frac{dz}{-2(px+yq+zq^2)}$$

$$= \frac{dz}{-2\left[\frac{z(1-q^2)}{2} + zq^2\right]} = \frac{dz}{-z(1+q^2)}$$

$$\frac{dz}{-z(\cancel{1+q^2})} = \frac{dp}{p(\cancel{1+q^2})}$$

$$-\log z = \log p - \log a$$

$$\log a = \log p + \log z$$

$$\underline{\underline{a = zp}}.$$

$$\frac{dz}{-z(\cancel{1+q^2})} = \frac{dq}{q(\cancel{1+q^2})}$$

$$-\log z = \log q - \log b.$$

$$\underline{\underline{b = zq}}$$

The soln is;

$$2\left(\frac{ax}{z} + \frac{b}{z}y\right) = z\left(1 - \frac{b^2}{z^2}\right)$$

$$2ax + 2by = z^2 - b^2$$

$$\underline{\underline{z^2 = b^2 + 2(ax+by)}}$$

Imp

73. $z^2 = pqxy$

$f = pqxy - z^2$

$f_p = qxy$

$f_q = pxy$

$f_x = pqy$

$f_y = pqx$

$f_z = -2z$

$$\frac{dx}{-qxy} = \frac{dy}{-pxy} = \frac{dz}{-pqxy - qpxy} = \frac{dp}{pqy + -2zp} = \frac{dq}{pqx - 2zq}$$

$$\frac{dx}{qxy} = \frac{dy}{pxy} = \frac{dz}{2pqxy} = \frac{dp}{2zp - pqy} = \frac{dq}{2zq - pqx}.$$

$$\frac{p\,dx + x\,dp}{\cancel{pqxy} + 2pzx - \cancel{pqxy}} = \frac{q\,dy + y\,dq}{-\cancel{pqxy} + 2qzy + \cancel{pqxy}}$$

$$\frac{p\,dx + x\,dp}{\cancel{2}p\cancel{z}x} = \frac{q\,dy + y\,dq}{\cancel{2}q\cancel{z}y}$$

$$\log(px) = \log(qy) + \log a.$$

$$a = \frac{px}{qy}$$

$$p = \frac{aqy}{x}$$

$$z^2 = pqxy.$$

$$z^2 = \frac{aq^2y^2x}{x}$$

$$z^2 = aq^2y^2$$

$$q = \frac{z}{y\sqrt{a}} \qquad p = \frac{ay}{x} \times \frac{z}{y\sqrt{a}} \Rightarrow \frac{\sqrt{a}\,z}{x}$$

$$p\,dx + q\,dy = dz$$

$$dz = \frac{\sqrt{a}\,z}{x}dx + \frac{z}{y\sqrt{a}}dy$$

$$\frac{dz}{z} = \sqrt{a}\frac{dx}{x} + \frac{dy}{y\sqrt{a}}$$

$$\log z = \sqrt{a}\log x + \frac{1}{\sqrt{a}}\log y + \log b$$

$$\sqrt{a}\log z = a\log x + \log y + \sqrt{a}\log b$$

$$\log z^{\sqrt{a}} = \log x^{a} + \log y + \log b^{\sqrt{a}}$$

$$z^{\sqrt{a}} = x^{a} y b^{\sqrt{a}}$$

$$\left(\frac{z}{b}\right)^{\sqrt{a}} = x^{a} y.$$

74. $p^2x + q^2y = z$

$f = p^2x + q^2y - z$

$f_p = 2px$

$f_q = 2qy$

$f_x = p^2$

$f_y = q^2$

$f_z = -1$

$$\frac{dx}{-2px} = \frac{dy}{-2qy} = \frac{dz}{-2p^2x - 2q^2y} = \frac{dp}{p^2-p} = \frac{dq}{q^2-q}$$

$$\frac{dx}{-2\cancel{p}x} = \frac{dp}{\cancel{p}(p-1)}$$

$$\frac{-1}{2}\log x = \log(p-1) - \log a.$$

$$a^2 = (p-1)^2 x. \qquad p = \frac{a}{\sqrt{x}} + 1$$

$$\frac{dy}{-2\cancel{q}y} = \frac{dq}{\cancel{q}(q-1)}$$

$$b^2 = (q-1)^2 y.$$

$$q = \frac{b+\sqrt{y}}{\sqrt{y}}$$

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$dz = p\,dx + q\,dy.$

$$dz = \frac{a\,dx}{\sqrt{x}} + \frac{\sqrt{x}\,dx}{\sqrt{x}} + \frac{b}{\sqrt{y}}dy + \frac{\sqrt{y}}{\sqrt{y}}dy.$$

$\frac{1}{\sqrt{x}}$

$\frac{\sqrt{x}}{1} = \frac{1}{2\sqrt{x}}$

$$dz = \frac{a}{\sqrt{x}}dx + \frac{b}{\sqrt{y}}dy + dx + dy.$$

$$\therefore z = 2a\sqrt{x} + 2b\sqrt{y} + x + y + c$$

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Module 1

Chapter 3

Application of PDE

Method of separation of variables.

Using the Method of separation of variables solve the following.

75. $\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial t} + u$, $u(x,0) = 6e^{-3x}$

$$\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial t} + u \quad \text{--- (1)}.$$

here u is a fn of x & t.

Assume $u = XT$

where X is a fn of x alone

T is a fn of t alone as a trial soln of eqn (1).

Now; $u = XT$

$$\Rightarrow \frac{\partial u}{\partial x} = X'T$$

$$\frac{\partial u}{\partial t} = XT'$$

$$(1) \Rightarrow X'T = 2XT' + XT$$

$$X'T = X(2T' + T)$$

$$\frac{X'}{X} = \frac{(2T' + T)}{T}$$

In the above eqn LHS is a fn of x and RHS a fn of t alone.

$\therefore$ We have; $\frac{X'}{X} = \frac{2T'+T}{T} = k$

$$\frac{X'}{X} = k$$

$$\log X = kx + \log a.$$

$$\frac{X}{a} = e^{kx}.$$

$$X = ae^{kx}$$

$$\frac{2T'}{T} = k-1$$

$$\frac{T'}{T} = \frac{k-1}{2}$$

$$\log T = \left(\frac{k-1}{2}\right)t + \log b$$

$$T = be^{(k-1)t/2}$$

$$X = ae^{kx}, \quad T = be^{(k-1)t/2}$$

$$u = XT$$
$$= abe^{kx+(k-1)t/2}$$

$$u(x,0) = 6e^{-3x}$$

$$abe^{kx} = 6e^{-3x}$$
$$ab = 6$$
$$k = -3$$
$$u = 6e^{(-3x-2t)}$$

is the particular soln.

76. $4u_x + u_y = 3u \qquad u = e^{-5y}, \; x = 0$

$$4\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 3u \quad \text{——— (1)}$$

here u is a fn's x and y

Assume $u = XY$

where X is a fn of x alone
Y is a fn of y alone, as a trial soln of (1)

Now ; $u = XY$

$$\frac{\partial u}{\partial x} = X'Y$$

$$\frac{\partial u}{\partial y} = XY'$$

$$4X'Y + XY' = 3XY$$

$$4X'Y = X(3Y - Y')$$

$$\frac{X'}{X} = \frac{3Y - Y'}{4Y} = k$$

$$\log X = kx + \log a.$$

$$\frac{X}{a} = e^{kx}.$$

$$\underline{\underline{X = ae^{kx}}}$$

$$\frac{3Y - Y'}{4Y} = k$$

$$\frac{3}{4} - \frac{Y'}{4Y} = k$$

$$\frac{Y'}{4Y} = \frac{3}{4} - k$$

$$\frac{Y'}{Y} = 3 - 4k$$

$$\log Y = (3 - 4k)y + \log b.$$

$$\underline{\underline{Y = be^{(3-4k)y}}}$$

$$\therefore \; u = XY = ae^{kx} \times be^{(3-4k)y}$$

$$\underline{\underline{u = abe^{kx + (3-4k)y}}}. \quad \text{gen. soln.}$$

$$x = 0 \Rightarrow u = e^{-5y}$$

$$e^{-5y} = abe^{(3-4k)y}$$

$ab = 1 \qquad 3 - 4k = -5$

$-4k = -8 \qquad k = 2$

$$u = e^{2x-5y} \quad \text{(P. soln)}$$

71. $3u_x + 2u_y = 0$

$u(x,0) = 4e^{-x}$

$$3\frac{\partial u}{\partial x} + 2\frac{\partial u}{\partial y} = 0. \quad \text{——— (1)}$$

here u is a fn of x & y.

Assume $U = XY$

where ; X is a fn of x alone

Y is a fn of y alone

$$\frac{\partial u}{\partial x} = X'Y \qquad \frac{\partial u}{\partial y} = XY'$$

$$3X'Y + 2XY' = 0$$

$$3X'Y = -2XY'$$

$$\frac{3X'}{X} = -\frac{2Y'}{Y} = k$$

$$\log X = \frac{kx}{3} + \log a.$$

$$\frac{X}{a} = e^{kx/3}$$

$$X = ae^{kx/3}$$

$$-\frac{2Y'}{Y} = k$$

$$\log Y = -ky/2 + \log b.$$

$$\frac{Y}{b} = e^{-ky/2}$$

$$u = abe^{k(x/3 - y/2)}$$

$$4e^{-x} = abe^{k/3}$$

$ab = 4 \qquad k = -3.$

$$u = 4e^{-x+3y/2}$$

75. $$\frac{\partial^2 z}{\partial x^2} + 4\frac{\partial^2 z}{\partial y^2} = 0$$

here; z is a fn of x & y.

$Z = XY.$ X is a fn of x alone
  Y is a fn of y alone.

$$\frac{\partial^2 z}{\partial x^2} = X''Y$$

$$\frac{\partial^2 z}{\partial y^2} = XY''$$

$$X''Y + 4XY'' = 0$$

$$X''Y = -4XY''$$

$$\frac{X''}{X} = -4\frac{Y''}{Y} = k^2$$

$$\frac{X''}{X} = k^2$$

$$X'' - k^2X = 0.$$

which is an ODE

Thus; $m^2 - k^2 = 0.$

$m^2 = k^2$

$m = \pm k$

$$X = c_1e^{kx} + c_2e^{-kx}$$

$$\frac{Y''}{Y} = \frac{-k^2}{4}$$

$$4Y'' + k^2Y = 0$$

$$4m^2 + k^2 = 0.$$

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$m^2 = -\frac{k^2}{4}$ $\quad m = \pm i k/2$

$y = c_3 \cos(k/2) y + c_4 \sin(k/2) y$

$z = XY$

$= (c_1 e^{kx} + c_2 e^{-kx})(c_3 \cos(k/2) y + c_4 \sin(k/2) y)$

due

79. $\frac{\partial u}{\partial x} - \frac{2 \partial u}{\partial t} = u$

$u(x,0) = 4e^{-3x}$ $\quad 4e^{-3x-2t}$

$\frac{\partial u}{\partial x} - \frac{2 \partial u}{\partial t} = u$ —— (1)

here u is a fn of x and t

$u = XT$. $\quad$ X is a fn of x alone

T is a fn of t alone ...

trial soln of (1)

Now, $u = XT$

$\frac{\partial u}{\partial x} = X'T \qquad \frac{\partial u}{\partial t} = XT'$

$X'T - 2XT' = \cancel{X}T$

$X'T = XT + 2XT'$

$X'T = X(2T' + T)$

$\frac{X'}{X} = \frac{2T' + T}{T}$

$\frac{X'}{X} = k$

$\log X = kx + \log a$

$\frac{X}{a} = e^{kx} \qquad X = ae^{kx}$

$$\frac{2T'+T}{T} = k$$

$$\frac{2T'}{T} + \frac{T}{T} = k.$$

$$\frac{2T'}{T} = k-1$$

$$\frac{2T'}{T} = k-1$$

$$\frac{T'}{T} = \frac{k-1}{2}$$

$$\log T = \frac{(k-1)}{2}t + \log b.$$

$$\frac{T}{b} = e^{(k-1)t/2}$$

$$T = be^{(k-1)t/2}$$

$$u = XT$$

$$u = abe^{kx+(k-1)t/2}$$

$$4e^{-3x} = abe^{kx}$$

$$ab = 4 \qquad k = -3.$$

$$\underline{\underline{u = 4e^{-3x-2t}}}$$

dis

So. $xu_x - 2yu_y = 0$

$$x\frac{\partial u}{\partial x} - 2y\frac{\partial u}{\partial y} = 0$$

here u is a fn of x and y

$u = XY$ X is a fn of x alone

Y is a fn of y alone as a trial soln of (1)

Now; $u = XY$

$$\frac{\partial u}{\partial x} = X'Y \qquad \frac{\partial u}{\partial y} = XY'$$

$$xX'Y - 2yXY' = 0.$$

$$xX'Y = 2yXY'$$

$$xX'Y = 2yXY'$$

$$x\frac{X'}{X} = 2y\frac{Y'}{Y}$$

$$\frac{X'}{X} = \frac{k}{x}.$$

$$\log X = k\log x + \log a.$$

$$\log X - \log a = k\log x.$$

$$\frac{X}{a} = x^k$$

$$\underline{\underline{X = ax^k}}$$

$$2y\frac{Y'}{Y} = k.$$

$$\frac{Y'}{Y} = \frac{k}{2y}.$$

$$\log Y = \frac{k}{2}\log y + \log b.$$

$$\frac{Y}{b} = y^{k/2}$$

$$\underline{\underline{Y = by^{k/2}}}$$

$$\underline{\underline{u = abx^k y^{k/2}}}$$

72.
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$$2(px+qy) = z(1-q^2)$$

#

$$f = 2px + 2qy - z + zq^2$$

$$f_p = 2x$$

$$f_q = 2y + 2zq$$

$$f_x = 2p$$

$$f_y = 2q$$

$$f_z = -1+q^2.$$

$$\frac{dx}{-2x} = \frac{dy}{-2y-2zq} = \frac{dz}{-2px-2yq-2zq^2} = \frac{dp}{2p+-p+q^2p} = \frac{dq}{2q-q+q^3}$$

$$\frac{dp}{p\cancel{(1+q^2)}} = \frac{dq}{q\cancel{(1+q^2)}}$$

$$\frac{dp}{p} = \frac{dq}{q}$$

$$\log p = \log q + \log a.$$

$$p/q = a.$$

$$\underline{\underline{p = aq}}$$

$$2aqx + 2qy = z(1-q^2).$$

$$(2ax + 2y)q = z - zq^2$$

$$\frac{dz}{-2px-2yq-2zq^2} = \frac{dz}{-2(px+yq+zq^2)}$$

$$= \frac{dz}{-2\left[\frac{z(1-q^2)}{2} + zq^2\right]} = \frac{dz}{-z(1+q^2)}$$

$$\frac{dz}{-z(\cancel{1+q^2})} = \frac{dp}{p(\cancel{1+q^2})}$$

$$-\log z = \log p - \log a$$

$$\log a = \log p + \log z$$

$$\underline{\underline{a = zp.}}$$

$$\frac{dz}{-z(\cancel{1+q^2})} = \frac{dq}{q(\cancel{1+q^2})}$$

$$-\log z = \log q - \log b.$$

$$\underline{\underline{b = zq}}$$

The soln is;

$$2\left(\frac{ax}{z} + \frac{b}{z}y\right) = z\left(1 - \frac{b^2}{z^2}\right)$$

$$2ax + 2by = z^2 - b^2$$

$$\underline{\underline{z^2 = b^2 + 2(ax+by)}}$$

Imp

73. $z^2 = pqxy$

$$f = pqxy - z^2$$

$$f_p = qxy$$

$$f_q = pxy$$

$$f_x = pqy$$

$$f_y = pqx$$

$$f_z = -2z$$

$$\frac{dx}{-qxy} = \frac{dy}{-pxy} = \frac{dz}{-pqxy - qpxy} = \frac{dp}{pqy + -2zp} = \frac{dq}{pqx - 2zq}$$

$$\frac{dx}{qxy} = \frac{dy}{pxy} = \frac{dz}{2pqxy} = \frac{dp}{2zp - pqy} = \frac{dq}{2zq - pqx}.$$

$$\frac{p\,dx + x\,dp}{\cancel{pqxy} + 2pzx - \cancel{pqxy}} = \frac{q\,dy + y\,dq}{-\cancel{pqxy} + 2qzy + \cancel{pqxy}}$$

$$\frac{p\,dx + x\,dp}{\cancel{2}p\cancel{z}x} = \frac{q\,dy + y\,dq}{\cancel{2}q\cancel{z}y}$$

$$\log(px) = \log(qy) + \log a.$$

$$a = \frac{px}{qy}$$

$$p = \frac{aqy}{x}$$

$$z^2 = pqxy.$$

$$z^2 = \frac{aq^2y^2x}{x}$$

$$z^2 = aq^2y^2$$

$$q = \frac{z}{y\sqrt{a}} \qquad p = \frac{ay}{x} \times \frac{z}{y\sqrt{a}} \Rightarrow \frac{\sqrt{a}\,z}{x}$$

$$p\,dx + q\,dy = dz$$

$$dz = \frac{\sqrt{a}\,z}{x}\,dx + \frac{z}{y\sqrt{a}}\,dy$$

$$\frac{dz}{z} = \sqrt{a}\,\frac{dx}{x} + \frac{dy}{y\sqrt{a}}$$

$$\log z = \sqrt{a}\log x + \frac{1}{\sqrt{a}}\log y + \log b$$

$$\sqrt{a}\log z = a\log x + \log y + \sqrt{a}\log b$$

$$\log z^{\sqrt{a}} = \log x^{a} + \log y + \log b^{\sqrt{a}}$$

$$z^{\sqrt{a}} = x^{a} y b^{\sqrt{a}}$$

$$\left(\frac{z}{b}\right)^{\sqrt{a}} = x^{a} y.$$

74. $p^2x + q^2y = z$

$f = p^2x + q^2y - z$

$f_p = 2px$

$f_q = 2qy$

$f_x = p^2$

$f_y = q^2$

$f_z = -1$

$$\frac{dx}{-2px} = \frac{dy}{-2qy} = \frac{dz}{-2p^2x - 2q^2y} = \frac{dp}{p^2 - p} = \frac{dq}{q^2 - q}$$

$$\frac{dx}{-2\cancel{p}x} = \frac{dp}{\cancel{p}(p-1)}$$

$$\frac{-1}{2}\log x = \log(p-1) - \log a.$$

$$a^2 = (p-1)^2 x. \qquad p = \frac{a}{\sqrt{x}} + 1$$

$$\frac{dy}{-2\cancel{q}y} = \frac{dq}{\cancel{q}(q-1)}$$

$$b^2 = (q-1)^2 y.$$

$$q = \frac{b + \sqrt{y}}{\sqrt{y}}$$

$dz = p\,dx + q\,dy.$

$$dz = \frac{a\,dx}{\sqrt{x}} + \frac{\sqrt{x}\,dx}{\sqrt{x}} + \frac{b}{\sqrt{y}}dy + \frac{\sqrt{y}}{\sqrt{y}}dy.$$

$\frac{1}{\sqrt{x}}$

$\frac{\sqrt{x}}{1} = \frac{1}{2\sqrt{x}}$

$$dz = \frac{a}{\sqrt{x}}dx + \frac{b}{\sqrt{y}}dy + dx + dy.$$

$$\therefore z = \underline{\underline{2a\sqrt{x} + 2b\sqrt{y}}} + x + y + c$$

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Module 1

Chapter 3

Application of PDE

Method of separation of variables.

Using the Method of separation of variables solve the following.

75. $\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial t} + u$, $u(x,0) = 6e^{-3x}$

$$\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial t} + u \quad \text{—— (1)}.$$

Here u is a fn of x & t.

Assume $u = XT$

where X is a fn of x alone

T is a fn of t alone are a trial soln of eqn (1).

Now; $u = XT$

$$\Rightarrow \frac{\partial u}{\partial x} = X'T$$

$$\frac{\partial u}{\partial t} = XT'$$

$$(1) \Rightarrow X'T = 2XT' + XT$$

$$X'T = X(2T' + T)$$

$$\frac{X'}{X} = \frac{(2T' + T)}{T}$$

In the above eqn LHS is a fn of x and RHS a fn of t alone.

$\therefore$ We have; $\frac{X'}{X} = \frac{2T'+T}{T} = k$

$$\frac{X'}{X} = k$$

$$\log X = kx + \log a.$$

$$\frac{X}{a} = e^{kx}.$$

$$X = ae^{kx}$$

$$\frac{2T'}{T} = k-1$$

$$\frac{T'}{T} = \frac{k-1}{2}$$

$$\log T = \left(\frac{k-1}{2}\right)t + \log b$$

$$T = be^{(k-1)t/2}$$

$$X = ae^{kx}, \quad T = be^{(k-1)t/2}$$

$$u = XT$$
$$= abe^{kx + (k-1)t/2}$$

$$u(x,0) = 6e^{-3x}$$

$$abe^{kx} = 6e^{-3x}$$
$$ab = 6$$
$$k = -3$$

$$u = 6e^{(-3x - 2t)}$$

is the particular soln.

76. $4u_x + u_y = 3u \qquad u = e^{-5y}, \; x = 0$

$$4\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 3u \quad \text{——} \; (1)$$

here u is a fn of x and y

Assume $u = XY$

where X is a fn of x alone

Y is a fn of y alone, as a trial soln of (1)

Now ; $u = XY$

$$\frac{\partial u}{\partial x} = X'Y$$

$$\frac{\partial u}{\partial y} = XY'$$

$$4X'Y + XY' = 3XY$$

$$4X'Y = X(3Y - Y')$$

$$\frac{X'}{X} = \frac{3Y - Y'}{4Y} = k$$

$$\log X = kx + \log a.$$

$$\frac{X}{a} = e^{kx}.$$

$$\underline{\underline{X = ae^{kx}}}$$

$$\frac{3Y - Y'}{4Y} = k$$

$$\frac{3}{4} - \frac{Y'}{4Y} = k$$

$$\frac{Y'}{4Y} = \frac{3}{4} - k$$

$$\frac{Y'}{Y} = 3 - 4k$$

$$\log Y = (3 - 4k)y + \log b.$$

$$\underline{\underline{Y = be^{(3-4k)y}}}$$

$$\therefore\ u = XY = ae^{kx} \times be^{(3-4k)y}$$

$$\underline{\underline{u = abe^{kx + (3-4k)y}}}.$$ gen. soln.

$x = 0 \Rightarrow u = e^{-5y}$

$$e^{-5y} = abe^{(3-4k)y}$$

$ab = 1$ $\quad$ $3 - 4k = -5$

$-4k = -8$ $\quad$ $k = 2$

$$u = e^{2x-5y} \quad \text{(P. soln)}$$

77. $3u_x + 2u_y = 0$

$$u(x,0) = 4e^{-x}$$

$$3\frac{\partial u}{\partial x} + 2\frac{\partial u}{\partial y} = 0. \quad \text{——— (1)}$$

here u is a fn of x & y.

Assume $U = XY$

where ; X is a fn of x alone
Y is a fn of y alone

$$\frac{\partial u}{\partial x} = X'Y \qquad \frac{\partial u}{\partial y} = XY'$$

$$3X'Y + 2XY' = 0$$

$$3X'Y = -2XY'$$

$$\frac{3X'}{X} = -\frac{2Y'}{Y} = k$$

$$\log X = \frac{kx}{3} + \log a.$$

$$\frac{X}{a} = e^{kx/3}$$

$$X = ae^{kx/3}$$

$$-\frac{2Y'}{Y} = k$$

$$\log Y = -ky/2 + \log b.$$

$$\frac{Y}{b} = e^{-ky/2}$$

$$u = abe^{k(x/3 - y/2)}$$

$$4e^{-x} = abe^{k/3}$$

$ab = 4 \qquad k = -3.$

$$\underline{\underline{u = 4e^{-x+3y/2}}}$$

75. $$\frac{\partial^2 z}{\partial x^2} + 4\frac{\partial^2 z}{\partial y^2} = 0$$

here ; z is a fn of x & y.

$Z = XY.$ X is a fn of x alone
Y is a fn of y alone.

$$\frac{\partial^2 z}{\partial x^2} = X''Y$$

$$\frac{\partial^2 z}{\partial y^2} = XY''$$

$$X''Y + 4XY'' = 0$$

$$X''Y = -4XY''$$

$$\frac{X''}{X} = -4\frac{Y''}{Y} = k^2$$

$$\frac{X''}{X} = k^2$$

$$X'' - k^2X = 0.$$

which is an ODE

Thus; $m^2 - k^2 = 0.$

$m^2 = k^2$

$m = \pm k$

$$X = c_1 e^{kx} + c_2 e^{-kx}$$

$$\frac{Y''}{Y} = \frac{-k^2}{4}$$

$$4Y'' + k^2 Y = 0$$

$$4m^2 + k^2 = 0.$$

$m^2 = -\frac{k^2}{4}$ $\qquad m = \pm i k/2$

$Y = c_3 \cos(k/2)\, y + c_4 \sin(k/2)\, y$

$Z = XY$

$= \left(c_1 e^{kx} + c_2 e^{-kx}\right)\left(c_3 \cos(k/2) y + c_4 \sin(k/2) y\right)$

Due

79. $\frac{\partial u}{\partial x} - \frac{2\partial u}{\partial t} = u$

$u(x,0) = 4e^{-3x}$ $\qquad 4e^{-3x-2t}$

$\frac{\partial u}{\partial x} - \frac{2\partial u}{\partial t} = u$ —— (1)

here u is a fn of x and t

$u = XT$. $\qquad$ X is a fn of x alone

T is a fn of t alone is a

trial soln of (1)

Now, $u = XT$

$\frac{\partial u}{\partial x} = X'T \qquad \frac{\partial u}{\partial t} = XT'$

$X'T - 2XT' = XT$

$X'T = XT + 2XT'$

$X'T = X(2T' + T)$

$\frac{X'}{X} = \frac{2T' + T}{T}$

$\frac{X'}{X} = k$

$\log X = kx + \log a$

$\frac{X}{a} = e^{kx} \qquad X = ae^{kx}$

$$\frac{2T'+T}{T} = k$$

$$\frac{2T'}{T} + \frac{T}{T} = k.$$

$$\frac{2T'}{T} = k-1$$

$$\frac{2T'}{T} = k-1$$

$$\frac{T'}{T} = \frac{k-1}{2}$$

$$\log T = \frac{(k-1)}{2} t + \log b.$$

$$\frac{T}{b} = e^{(k-1)t/2}$$

$$T = b e^{(k-1)t/2}$$

$$u = XT$$

$$u = ab e^{kx+(k-1)t/2}$$

$$4e^{-3x} = abe^{kx}$$

$$ab = 4 \qquad k = -3.$$

$$\underline{\underline{u = 4e^{-3x-2t}}}$$

due
so.

$$x u_x - 2y u_y = 0$$

$$x\frac{\partial u}{\partial x} - 2y\frac{\partial u}{\partial y} = 0$$

here u is a fn of x and y

$u = XY$ X is a fn of x alone
Y is a fn of y alone as a trial soln of (1)

Now; $u = XY$

$$\frac{\partial u}{\partial x} = X'Y \qquad \frac{\partial u}{\partial y} = XY'$$

$$xX'Y - 2yXY' = 0.$$

$$xX'Y = 2yXY'$$

$$xX'Y = 2yXY'$$

$$x\frac{X'}{X} = 2y\frac{Y'}{Y}$$

$$\frac{X'}{X} = \frac{k}{x}.$$

$$\log X = k\log x + \log a.$$

$$\log X - \log a = k\log x.$$

$$\frac{X}{a} = x^k$$

$$\underline{\underline{X = ax^k}}$$

$$2y\frac{Y'}{Y} = k.$$

$$\frac{Y'}{Y} = \frac{k}{2y}.$$

$$\log Y = \frac{k}{2}\log y + \log b.$$

$$\frac{Y}{b} = y^{k/2}$$

$$\underline{\underline{Y = by^{k/2}}}$$

$$\underline{\underline{u = abx^k y^{k/2}}}$$

$$\frac{Y'}{Y} = \frac{k}{2y}.$$

$$\log Y = \frac{k}{2}\log y + \log b.$$

$$\frac{Y}{b} = y^{k/2}$$

$$Y = by^{k/2}$$

$$u = ab x^k y^{k/2}$$

9/11/2020

81 $2xp - 3yq = 0.$

$z = XY$ X is a fn of x alone & Y is a fn of y alone

$$2xX'Y - 3yXY' = 0$$

$$2xX'Y = 3yXY'$$

$$2x\frac{X'}{X} = k$$

$$\frac{X'}{X} = \frac{k}{2x}.$$

$$\log X = \frac{1}{2}k\log x + \log a.$$

$$X = a \cdot x^{k/2}$$

$$3y\frac{Y'}{Y} = k$$

$$\frac{Y'}{Y} = \frac{k}{3y}.$$

$$\log Y = \frac{1}{3}k\log y + \log b.$$

$$Y = by^{k/3}$$

$$z = XY = ab\,x^{k/2}y^{k/3}$$

$$= \underline{\underline{c\,x^{k/2}y^{k/3}}}$$

82. $\dfrac{\partial^2 u}{\partial x^2} - \dfrac{\partial u}{\partial t} = u$ Given $u(0,t) = 0 = u(\pi,t)$

here u's a fn of x and t.

$$u = XT$$

$$\frac{\partial^2 u}{\partial x^2} = X''T$$

$$\frac{\partial u}{\partial t} = XT'$$

$$X''T - XT' = XT$$

$$X''T = XT + XT'$$

$$X''T = X(T+T')$$

$$\frac{X''}{X} = \left(1+\frac{T'}{T}\right)$$

$$\frac{X''}{X} = k^2 \qquad m^2 - k^2 = 0. \quad m = \pm\sqrt{k^2} = \pm k$$

$$X = c_1e^{kx} + c_2e^{-kx}$$

$$1 + \frac{T'}{T} = k^2$$

$$\frac{T'}{T} = k^2 - 1$$

$$\log T = (k^2-1)t + \log b.$$

$$T = b e^{(k^2-1)t}$$

$$X = c_1 e^{kx} + c_2 e^{-kx}.$$

$$T = b e^{(k^2-1)t}.$$

$$u = XT.$$

$$= \left(c_1 e^{kx} + c_2 e^{-kx}\right) b e^{(k^2-1)t}$$

$$u(0,t) = 0$$

$$\Rightarrow (c_1 + c_2)\, b e^{(k^2-1)t} = 0$$

$$u(\pi, t)$$

$$\Rightarrow \left(c_1 e^{\pi k} + c_2 e^{-\pi k}\right) b e^{(k^2-1)t} = 0.$$

put;

$$c_1 b = A_1 \qquad c_2 b = A_2.$$

$$\Rightarrow (A_1 + A_2)\, e^{(k^2-1)t} = 0 \quad \text{——— (1)}$$

$$\Rightarrow \left(A_1 e^{k\pi} + A_2 e^{-k\pi}\right) e^{(k^2-1)t} = 0 \quad \text{——— (2)}$$

from (1) $\rightarrow e^{(k^2-1)t} \neq 0$ (in finite case)

$$\Rightarrow A_1 + A_2 = 0.$$

ie $A_1 = -A_2$.

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from ②.

$$\left(A_1 e^{k\pi} + A_2 e^{-k\pi}\right) e^{(k^2-1)t} = 0.$$

$$e^{(k^2-1)t} \neq 0.$$

$$\Rightarrow \quad A_1 e^{k\pi} - A_1 e^{-k\pi} = 0.$$

$$e^{k\pi} = e^{-k\pi}$$

$$e^{k\pi} = \frac{1}{e^{k\pi}}$$

$$e^{2k\pi} = 1 = e^{2n\pi i}$$

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

$k = in$

$k^2 = -n^2$

$$u = \left(A_1 e^{inx} - A_1 e^{-inx}\right) e^{-(n^2+1)t}$$

$$u = A_1 (2i\sin nx)\, e^{-(n^2+1)t}$$

83.

$$\frac{\partial z}{\partial x} + \frac{\partial^2 z}{\partial y^2} = 0$$

$$z(x, 0) = 0 = z(x, \pi)$$

$$z(0, y) = 4\sin 3y$$

$$\frac{\partial z}{\partial x} + \frac{\partial^2 z}{\partial y^2} = 0$$

here z is a fn of x & y

$$z = XY.$$

$$\frac{\partial z}{\partial x} = X'Y \qquad \frac{\partial^2 z}{\partial y^2} = XY''$$

$$X'Y + XY'' = 0.$$

$$\frac{X'}{X} = -\frac{Y''}{Y} = k^2$$

$$\frac{X'}{X} = k^2$$

$$\log X = xk^2 + \log a.$$

$$X = a e^{k^2 x}$$

$$\frac{Y''}{Y} = -k^2$$

$$m^2 = -k^2$$

$$m = \pm ki$$

$$\cancel{Y = c_1 e^{ky} + c_2 e^{-ky}}$$

$$\cancel{z = XY = (c_1 e^{ky} + c_2 e^{-ky})\, a e^{k^2 x}}$$

$$\cancel{= (A_1 e^{ky} + A_2 e^{-ky}) e^{k^2 x}}$$

$c_1 a = A_1$
$c_2 a = A_2$

$$m = \pm ki$$

$$Y = c_1 \cos ky + c_2 \sin ky$$

$c_1 a = A_1$
$c_2 a = A_2$

$$z = XY$$

$$= (c_1 \cos ky + c_2 \sin ky)\, a e^{k^2 x}$$

$$= (A_1 \cos ky + A_2 \sin ky)\, e^{k^2 x}$$

$$z(x, 0)$$

$$0 = A_1 e^{k^2 x} \qquad e^{k^2 x} \neq 0 \text{ (infinite case)}$$

$$A_1 = 0$$

$$z = (x, \pi) \qquad 0 = (A_1 \cos k\pi + A_2 \sin k\pi)\, e^{k^2 x}$$

$(A_2 \sin k\pi) e^{k^2 x} = 0.$

$e^{k^2 x} \neq 0.$

$\Rightarrow \quad A_2 \sin k\pi = 0$

$A_2 \neq 0$ since if $A_2 = 0. \Rightarrow z = 0.$

hence;

$\sin k\pi = 0.$

$k\pi = n\pi$

$k = n.$

$z(0,y) = 4 \sin 3\pi y$

$4 \sin 3y = A_2 \sin ny.$ $\quad$ $A_2 \to B.$

$$4 \sin 3y = \sum_{n=1}^{\infty} (a_2)_n \sin ny.$$

$$4 \sin 3y = (a_2)_1 \sin y + (a_2)_2 \sin 2y + (a_2)_3 \sin 3y + \ldots$$

$(a_2)_1 = 0 \quad (a_2)_2 = 0 \quad (a_2)_3 = 4 \quad (a_2)_4 = (a_2)_5 = 0$

$$z = [4 \sin(ny)]\, e^{n^2 x}$$

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