Module I — Introduction —

?/10/2020

Partial Differential Equations;

A differential eqn which involves partial derivatives is called a partial differential eqn.

eg: $$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 5$$

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = e^z$$

The order of a p.d.e is the order of the highest partial derivative in the eqn.

The degree of a p.d.e. is the degree of the highest order partial derivative occurring in the eqn.

eg: $$\left(\frac{\partial^2 z}{\partial x^2}\right)^3 + \frac{\partial z}{\partial y} = 0$$

order $\longrightarrow$ 2   2nd order 3rd degree pde

degree $\longrightarrow$ 3

Notations;

If z is a fn of 2 independent variables. x & y, we denote;

$$\frac{\partial z}{\partial x} = p \qquad \frac{\partial^2 z}{\partial y^2} = t$$

$$\frac{\partial z}{\partial y} = q \qquad \frac{\partial^2 z}{\partial x \partial y} = s$$

$$\frac{\partial^2 z}{\partial x^2} = r.$$

Chapter 1

Formation of P.D.E;

Method 1;

Method of elimination of arbitrary constants

Form a p.d.e from the following eqn by eliminating the arbitrary constants:

1. $z = ax + by + ab$

$$z = ax + by + ab \longrightarrow (1)$$

$$p = \frac{\partial z}{\partial x} = a \qquad p = a$$

$$q = \frac{\partial z}{\partial y} = b \qquad q = b.$$

here arbs obtained in 1st step coz; Independent variable Arbitrary const 2 hence p.d.e

$$z = px + qy + pq.$$

$$= x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} + \frac{\partial z}{\partial x} \times \frac{\partial z}{\partial y}$$

2. $z = ax + a^2y^2 + b$

$$p = \frac{\partial z}{\partial x} = a.$$

$$q = \frac{\partial z}{\partial y} = 2a^2y$$

$$\cancel{\frac{\partial^2 z}{\partial y^2}} \cancel{=} \cancel{2a^2}$$

$$q = 2p^2y \qquad p = a$$

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3. $z = (x^2+a)(y^2+b)$

$\frac{\partial z}{\partial x} = (y^2+b) \times 2x.$ (p)

$\frac{\partial z}{\partial y} = (x^2+a) \times 2y$ (q)

$p = 2x(y^2+b)$ —— ①

$q = 2y(x^2+a)$ —— ②.

$pq = 4xy(x^2+a)(y^2+b)$

$pq = 4xyz$

$z = \frac{pq}{4xy}.$

4. $2z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$

$2p = \frac{2x}{a^2} \Rightarrow p = x/a^2$

$2q = \frac{2y}{b^2} \Rightarrow q = y/b^2.$

$2z = xp + qy$

$2z = xp + yq$

5. $\frac{x^2}{a^2} + \frac{y^2}{c^2} + \frac{z^2}{c^2} = 1$

$\frac{z^2}{c^2} = 1 - \frac{x^2}{a^2} - \frac{y^2}{c^2}$

here the no: of arbitrary constants (3) is greater then the no: of independent variable (x, y) ②

∴ the req. p.d.e will be of order greater than 1.

$$\frac{2pz}{c^2} = -\frac{2x}{a^2} \qquad \frac{2x}{a^2} + \frac{2zp}{c^2} = 0 \quad \text{--- (2)}$$

$$zp = -\frac{xc^2}{a^2}$$

$$\frac{2zq}{c^2} = -\frac{2y}{b^2} \qquad \frac{2y}{b^2} + \frac{2zq}{c^2} = 0 \quad \text{--- (3)}$$

$$\frac{zq}{c^2} = -\frac{y}{b^2}$$

diff: eq (2) w.r.t. y.

$$0 + \frac{2}{c^2}\left[z \times \frac{\partial^2 z}{\partial x \partial y} + \frac{\partial z}{\partial x} \frac{\partial z}{\partial y} \right] = 0.$$

$$z\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial z}{\partial x} \times \frac{\partial z}{\partial y} = 0.$$

$$zs + pq = 0$$

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6. $z = ax^2 + bxy + cy^2$

$$p = \frac{\partial z}{\partial x} = 2ax + by$$

$$q = \frac{\partial z}{\partial y} = bx + 2cy$$

$$r = \frac{\partial^2 z}{\partial x^2} = 2a$$

$$s = \frac{\partial^2 z}{\partial x \partial y} = b$$

$$t = \frac{\partial^2 z}{\partial y^2} = 2c$$

$a = r/2 \qquad b = s \qquad c = t/2$

$$z = \frac{rx^2}{2} + sxy + \frac{t}{2}y^2.$$

Therefore the req. p.d.e is;

$$z = \frac{r}{2}x^2 + sxy + \frac{t}{2}y^2.$$

7. $4(1+a^2)z = (x+ay+b)^2$

No: of arbitrary constants : 2

Order $\longrightarrow 1$

diff: w.r.t x

$p = \frac{\partial z}{\partial x} \Rightarrow 4(1+a^2)p = 2(x+ay+b)$ — ②

diff: w.r.t y

$q = \frac{\partial z}{\partial y} \Rightarrow 4(1+a^2)q = 2(x+ay+b)a$ — ③

③ ÷ ②.

$$\frac{q}{p} = a \qquad a = q/p$$

$$4p\left(1+\frac{q^2}{p^2}\right) = 2\left(x+\frac{qy}{p}+b\right)$$

$$4p + \frac{4q^2}{p} = 2x + \frac{2qy}{p} + 2b.$$

$$4p + \frac{4q^2}{p} - 2x - \frac{2qy}{p} = 2b.$$

$$b = 2p + \frac{2q^2}{p} - x - \frac{qy}{p}$$

$$4\left(1+\frac{q^2}{p^2}\right)\times z = \left(\cancel{x} + \cancel{\frac{q}{p}y} + 2p + \frac{2q^2}{p} - \cancel{x} - \cancel{\frac{qy}{p}}\right)^2$$

$$z = \frac{\left(\frac{2p^2+2q^2}{p}\right)^2}{4\frac{(p^2+q^2)}{p^2}} \Rightarrow z = \frac{4(p^2+q^2)^2}{4(p^2+q^2)}$$

$$\underline{z = p^2+q^2}$$

(OR)→

(OR) $x+ay+b = 2(1+a^2)p$.

$$= 2\left(1+\frac{q^2}{p^2}\right)p.$$

$$= \frac{2}{p}\left[p^2+q^2\right]$$

$$4\left(1+\frac{q^2}{p^2}\right)z = \frac{4}{p^2}(p^2+q^2)^2$$

$$z = p^2+q^2$$

8. Find the p.d.e of all spheres of given radius 'a' and whose centres lie on the xy plane.

$(x-b)^2+(y-k)^2+z^2=a^2$ is the general eqn of the sphere in xy plane.

The no: of arbitrary constants $\longrightarrow$ 2 (b, k)

a is not an arbitrary constant
$\longrightarrow$ (fixed constant)
(given)

Order $\longrightarrow$ 1

$$(x-b)^2+(y-k)^2+z^2 = a^2. \quad \text{——— (1)}$$

diff: (1) w.r.t x

$$p = \frac{\partial z}{\partial x}$$

$$2(x-b) + 2z \times p = 0$$

$$2(x-b) + 2zp = 0 \quad \text{——— (2)}$$

diff: w.r.t to y.

$$2(y-k) + 2zq = 0 \quad \text{——— (3)}$$

From (2)

$$(x-h) = -zp.$$

From (3)

$$(y-k) = -zq.$$

$\therefore$

$$(-zp)^2 + (-zq)^2 + z^2 = a^2.$$

$$z^2p^2 + z^2q^2 + z^2 = a^2$$

$$z^2(p^2+q^2+1) = a^2$$

9. Form the p.d.e of all spheres ~~of given radius~~ and whose centre lie on the z-axis.

The eqn: of the sphere;

$$x^2 + y^2 + (z-k)^2 = r^2. \quad \text{——— (1)}$$

The no: of arbitrary constants $\longrightarrow$ 2 (k, r)

order $\longrightarrow$ 1

diff: (1) w.r.t x.

$$p = \frac{\partial z}{\partial x} \Rightarrow 2x + 2(z-k)p = 0 \quad \text{——— (2)}$$

diff: (1) w.r.t y

$$q = \frac{\partial z}{\partial y} \Rightarrow 2y + 2(z-k)q = 0 \quad \text{——— (3)}$$

(2) $\Longrightarrow$ $\cancel{2}x = -\cancel{2}(z-k)\cdot p$.

$$(z-k) = -x/p$$

(3) $\Longrightarrow$ $(z-k) = -y/q$

$$1 = \frac{x}{p} \times \frac{q}{y}$$

No: of arbitrary constant → 2
order → 1

$$\frac{q}{p} = \frac{y}{x}$$

No: of A.C → 3
order → 2

10. Family of planes having equal intercepts on the x, y axis

$$\frac{x}{a} + \frac{y}{a} + \frac{z}{c} = 1$$

$$\frac{1}{a} + \frac{p}{c} = 0 \longrightarrow \text{diff w.r.t } x. \longrightarrow ②$$

diff: w.r.t y.

$$\frac{1}{a} + \frac{q}{c} = 0 \longrightarrow ③$$

② − ③

$$\frac{p-q}{c} = 0$$

$$p = q$$

11. Find the p.d.e of all spheres whose radii are equal.

⇒ the radius is given. and is a fixed constant

The general eqn. of the sphere is;

$$(x-h)^2 + (y-k)^2 + (z-l)^2 = a^2 \quad \text{——} ①$$

The arbitrary constants : h, k, l ③

$a \longrightarrow$ fixed constant.

$p = \frac{\partial z}{\partial x}$ diff: ① w.r.t. x

$2(x-h) + 2(z-l)\,p = 0.$

$(x-h) + (z-l)\,p = 0$ ——— ②.

diff: ① w.r.t y

$q = \frac{\partial z}{\partial y}$

$(y-k) + (z-l)\,q = 0$ ——— ③

$\cancel{r = \frac{\partial^2 z}{\partial x^2}} \Rightarrow$

diff: ② w.r.t x.

$1 + (z-l) \times r + p \times p = 0$

$1 + (z-l)\,r + p^2 = 0$ ——— ④.

diff: ② w.r.t y.

$(z-l)\,s + pq = 0$ ——— ⑤

⑤ $\Rightarrow (z-l) = -pq/s.$

④ $\Rightarrow (z-l) = \frac{-(1+p^2)}{r}$

$$\frac{1+p^2}{r} = \frac{pq}{s}.$$

$$pqr = (1+p^2)s$$

12. ~~A.E-Lnar~~ $z = xy + y\sqrt{x+a} + b$

due

$z = xy + y\sqrt{x+a} + b$ ——— (1).

diff (1) w.r.t x;

$p = \frac{\partial z}{\partial x} = y + y \times \frac{1}{2\sqrt{x+a}} \Rightarrow y + \frac{y}{2\sqrt{x+a}} = p$ — (2)

diff: (1) w.r.t y;

$q = x + \sqrt{x+a}$ ——— (3)

(3) $\Rightarrow \sqrt{x+a} = q - x$.

Substituting (3) in (1).

$p = y + \frac{y}{2(q-x)}$

$p = \frac{(2(q-x)+1) \cdot y}{2(q-x)} \Rightarrow (p-y)(q-x) = y/2$

Method 2;

Method of elimination of arbitrary fns:

Form the p.d.e by eliminating the arbitrary fns: from the following eqns:

13. $z = f\left(\frac{xy}{z}\right)$

No: of arbitrary fn ⟶ (1)

hence we get a p.d.e of order 1

diff: ① w.r.t x.

$p = \frac{\partial z}{\partial x}$.

$p = f'\left(\frac{xy}{z}\right)\left[\frac{z[x \times 0 + y] - xy \times p}{z^2}\right]$ — ②.

$p = f'\left(\frac{xy}{z}\right)\left[\frac{zy - xyp}{z^2}\right]$ —— ②.

diff: ① w.r.t y.

$q = \frac{\partial z}{\partial y}$.

$q = f'\left(\frac{xy}{z}\right)\left[\frac{zx - xyq}{z^2}\right]$ —— ③.

② ÷ ③

$\frac{p}{q} = \frac{zy - xyp}{zx - xyq}$

$p \times zx - \cancel{xyqp} = qzy - q\cancel{xyp}$.

$px = qy$.

$\frac{x}{y} = q/p$

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14. $z = y^2 + 2f\left(\frac{1}{x} + \log y\right)$

Fn → ① p.d.e → order:1

$p = 2f'\left(\frac{1}{x} + \log y\right) \times -\frac{1}{x^2}$

$= \frac{-2}{x^2} f'\left(\frac{1}{x} + \log y\right)$ —— ②

$q = 2y + 2f'\left(\frac{1}{x} + \log y\right) \times \frac{1}{y}$ —— ③

② ÷ ③

$$\frac{p}{(q-2y)} = \frac{-\cancel{2} f'\cancel{(1/x + \log y)} \times 1/x^2}{\cancel{2} f'\cancel{(1/x + \log y)} \times 1/y}$$

$$\frac{p}{q-2y} = -\frac{y}{x^2}$$

$$x^2 p = -qy + 2y^2.$$

$$\underline{\underline{qy = 2y^2 - px^2}}$$

15. $$xyz = \phi(x+y+z)$$

$(xy)z$

$$xyp + yz = \phi'(x+y+z)(1+p)$$

$$xyq + xz = \phi'(x+y+z)(1+q)$$

diff: wrt $x \to (xp+z)y = \phi'(x+y+z)(1+p)$ — ②

diff: wrt $y \to (yq+z)x = \phi'(x+y+z)(1+q)$ — ③

$$②/③ \quad \frac{(xp+z)y}{(yq+z)x} = \frac{\phi'\cancel{(x+y+z)}(1+p)}{\phi\cancel{(x+y+z)}(1+q)}$$

$$y(xp+z)(1+q) = x(yq+z)(1+p)$$

$$(xyp + yz)(1+q) = (xyq + xz)(1+p)$$

$$xyp + \cancel{yz} + \cancel{xypq} + yzq = xyq + \cancel{xyqp} + \cancel{xz} + xzp$$

$$\cancel{xyp + xzq = xyq + xzp}$$

$$\cancel{x(yp+qz) = y(xq+pz)}.$$

$$xyp + yz + yzq - xyq - xz - xzp = 0.$$

$$xy(p-q) + zy(1+q) - xz(1+p) = 0$$

16. $z = f(2x+y) + g(3x-y)$

Ab: fn $\longrightarrow$ 2   p.d.e order $\longrightarrow$ 2

$$p = f'(2x+y)\times 2 + g'(3x-y)\times 3 \quad \text{——— (2)}.$$

$$q = f'(2x+y)\times 1 + g'(3x-y)\times -1$$

$$= f'(2x+y) - g'(3x-y) \quad \text{——— (3)}$$

$$r = 4f''(2x+y) + 9g''(3x-y) \quad \text{——— (4)}$$

$$s = 2f''(2x+y) - 3g''(3x-y) \quad \text{——— (5)}$$

$$t = f''(2x+y) + g''(3x-y) \quad \text{——— (6)}$$

~~r+s=6t~~ $r + s = 6t$.

17. $z = f(y) + e^x g(y)$

$$p = e^x g(y) \quad \text{(diff. w.r.t } x)$$

$$q = f'(y) + e^x g'(y) \quad \text{(diff. w.r.t } y)$$

$$r = e^x g(y)$$

$$s = e^x g'(y)$$

$$r = p$$

18. $\cancel{z = y^2 + 2f(1/x + \log y)}$

$\cancel{p = 2f'}$

18. $z = x f(2x-y) + g(2x-y)$ —— ①

diff ① wrt x $\quad p = x f'(2x-y)2 \quad f(2x-y) + 2g'(2x-y)$

$= 2x f'(2x-y) + f(2x-y) + 2g'(2x-y) \longrightarrow ②$

diff ① wrt y $\quad q = -x f'(2x-y) + -1 g'(2x-y)$

$= -x f'(2x-y) - g'(2x-y) \longrightarrow ③$

diff ② wrt x $\quad r = 4x f''(2x-y) + 2f'(2x-y) +$

$2f'(2x-y) + 4g'(2x-y) \longrightarrow ④$

$= 4x f''(2x-y) + 4f'(2x-y) + 4g''(2x-y)$

diff ② wrty $\quad s = -2x f''(2x-y) - f'(2x-y) - 2g''(2x-y) \longrightarrow ⑤$

diff ③ wrty $\quad t = x f''(2x-y) + g''(2x-y) \longrightarrow ⑥$

$r + 4s = -4t$

$r + 4s + 4t = 0$

19. $z = x f(y/x) + y\phi(x)$

$p = -\frac{\cancel{x}}{x^{\cancel{2}}} f'(y/x) * f(y/x) + y\phi'(x)$

$= \frac{-y f'(y/x)}{x} + f(y/x) + y\phi'(x)$ —— ①

$$q = f'(y/x) + y\phi'(x) + \phi(x) \quad \text{--- (3)}$$

$$r = -1 \times \frac{-y}{x} f''(y/x) \times \frac{-y}{x^2} + -1 \times \frac{-1}{x^2} f'(y/x) + f'(y/x) \times -\frac{1}{x^2} + y\phi''(x)$$

$$= \frac{y^2 f''(y/x)}{x^3} + \cancel{f'(y/x)}\; y\phi''(x) \quad \text{--- (4)}$$

$$s = \frac{-y}{x} \times f''(y/x) \times \frac{1}{x} + f'(y/x) \times \frac{1}{x} + \phi'(x).$$

$$= \frac{-y f''(y/x)}{x^2} \cancel{+ \frac{f'(y/x)}{x}} + \phi'(x) \longrightarrow \text{(5)}$$

$$t \Rightarrow \frac{1}{x} f''(y/x) \quad \text{--- (6)}$$

$$s = -\frac{y}{x}(t) + \phi'(x)$$

$$\phi'(x) = s + \frac{ty}{x} \quad \text{--- (7)}$$

$x \times p$

$$\Rightarrow -\frac{xy}{x} f'(y/x) + xf(y/x) + xy\,\phi'(x).$$

$$\Rightarrow -y f'(y/x) + xf(y/x) + xy\,\phi'(x).$$

yq ; $-y\,(f'(y/x)] + y\,\phi(x)$.

$$xp + yq \Rightarrow x f(y/x) - \cancel{yf'(y/x)} + xy\,\phi'(x) + \cancel{y f'(y/x)} + y\,\phi(x).$$

$$z \Rightarrow x f(y/x) + y\,\phi(x).$$

$$\Rightarrow xp + yq = z + xy\,\phi'(x).$$

$$xp + yq = z + xy\left(s + \frac{ty}{x}\right)$$

$$\underline{xp + yq = z + y(sx + ty)}$$

20 $z = x^2 f(y) + y^2 g(x)$ $\quad 2xp + yq - 2z = xy$

Sol:

$$z = x^2 f(y) + y^2 g(x) \text{ ——— } (1).$$

$R = 2x$

diff: (1) w.r.t x;

$$p = 2x f(y) + y^2 g'(x). \text{ ——— } (2)$$

diff: (1) w.r.t y;

$$q = x^2 f'(y) + 2y\, g(x). \text{ ——— } (3)$$

diff: (2) w.r.t x;

$$r = 2 f(y) + y^2 g''(x). \text{ ——— } (4)$$

diff (2) w.r.t y;

$$s = 2x f'(y) + 2y\, g'(x) \text{ ——— } (5).$$

diff (3) w.r.t y;

$$t = x^2 f''(y) + 2 g(x) \text{ ——— } (6).$$

Xing p with x;

$$xp = 2x^2 f(y) + xy^2 g'(x) \text{ ——— } (7).$$

Xing q with y;

$$yq = x^2 y f'(y) + 2y^2 g(x) \text{ ——— } (8).$$

(7) + (8)

$$xp+yq = 2x^2 f(y) + xy^2 g'(x) + x^2 y f'(y) + 2y^2 g(x).$$

$$= 2\left(x^2 f(y) + y^2 g(x)\right) + xy^2 g'(x) + x^2 y f'(y).$$

$$xp+yq = 2z + xy^2 g'(x) + x^2 y f'(y) \quad \text{—— (9)}$$

Xing (5) with xy.

$$xys = 2\left(x^2 y f'(y) + xy^2 g'(x)\right)$$

$$\frac{xys}{2} = x^2 y f'(y) + xy^2 g'(x) \quad \text{—— (10)}$$

Substituting (10) in (9).

$$xp + yq = 2z + \frac{xys}{2}.$$

$$2(xp+yq) = 4z + xys$$

$$2(xp+yq) - 4z = xys.$$

22/10/2020 **Type 3 / Method 3 :**

elimination of arbitrary fn 'f' from the relation

$f(u,v)=0$

Where u and v are fns of x,y,z

$u, v \ \phi(x,y,z)$

$f(u,v)=0$ where $u,v \ \phi(x,y,z)$

then;

diff: w.r.t x $\rightarrow$ $$\frac{\partial f}{\partial u}\left[\frac{\partial u}{\partial x}+\frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial x}\right]+\frac{\partial f}{\partial v}\left[\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}\times\frac{\partial z}{\partial x}\right]=0$$

diff: partially w.r.t y.

$$\frac{\partial f}{\partial u}\left[\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\times\frac{\partial z}{\partial y}\right]+\frac{\partial f}{\partial v}\left[\frac{\partial v}{\partial y}+\frac{\partial v}{\partial z}\times\frac{\partial z}{\partial y}\right]=0$$

Form the pde from the following fns:

21. $f(x+y+z,\ x^2+y^2+z^2) = 0$ ——— ①

Let $u = x+y+z$

$v = x^2+y^2+z^2$

diff: ① partially w.r.t x

$$\frac{\partial f}{\partial u}\left[1+p\right]+\frac{\partial f}{\partial v}\left[2x+2zp\right]=0 \quad \text{—②}$$

diff ① w.r.t y

$$\frac{\partial f}{\partial u}\left[1+q\right]+\frac{\partial f}{\partial v}\left[2y+2zq\right]=0 \quad \text{—③}$$

Eliminating $\partial f/\partial u$ & $\partial f/\partial v$ from ② & ③

we get.

$$\begin{vmatrix} 1+p & 2x+2zp \\ 1+q & 2y+2zq \end{vmatrix} = 0$$

$$(2y+2zq)(1+p)-(2x+2zp)(1+q)=0$$

$$2y+2zq+2yp+\cancel{2zqp}-2x-2xq-2zp-\cancel{2zpq}$$

$$y+zq+yp-x-xq-zp.$$

$$y(1+p)-x(1+q)+z(q-p)$$

$$(y-z)p+(z-x)q=x-y \qquad (Pp+Qq=R)$$

22. $f(x^2+y^2,\ z-xy)=0$ ——— ①

$u=x^2+y^2 \qquad v=z-xy.$

diff: ① w.r.t x.

$$\frac{\partial f}{\partial u}\left[2x\right]+\frac{\partial f}{\partial v}\left[-y+p\right]=0 \text{ ——— ②}$$

diff ① w.r.t y.

$$\frac{\partial f}{\partial u}\left[2y\right]+\frac{\partial f}{\partial v}\left[-x+q\right]=0 \text{ ——— ③}$$

$$\begin{vmatrix} 2x & -y+p \\ 2y & -x+q \end{vmatrix}=0$$

$$-2x^2+2xq+2y^2-2yp=0.$$

$$-x^2+xq+y^2-yp=0.$$

$$xq-yp=-y^2+x^2$$

$$yp-xq=y^2-x^2$$

23. $f(xy+z^2,\ x+y+z) = 0$ ——— ①

$u = xy+z^2$

$V = x+y+z$

diff ① w.r.t x.

$$\frac{\partial f}{\partial u}\left[y + 2zp\right] + \frac{\partial f}{\partial v}\left[1 + p\right] = 0 \quad \text{——— ②}$$

diff ① w.r.t y.

$$\frac{\partial f}{\partial u}\left[x + 2zq\right] + \frac{\partial f}{\partial v}\left[1 + q\right] = 0 \quad \text{——— ③}$$

$$\begin{vmatrix} y+2zp & 1+p \\ x+2zq & 1+q \end{vmatrix} = 0.$$

$$y + yq + 2zp - \cancel{2z}pq - x - xp - 2zq - \cancel{2z}pq = 0$$

$$y-x + (y-2z)q + (2z-x)p = 0.$$

$$(2z-x)p + (y-2z)q = x-y.$$

24. $f(x^2+y^2+z^2,\ z^2-2xy) = 0$ ——— ①

$u = x^2+y^2+z^2$

$V = z^2-2xy$

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diff ① w.r.t x

$$\frac{\partial f}{\partial u}\left[2x + 2zp\right] + \frac{\partial f}{\partial v}\left[-2y + 2zp\right] = 0 \quad \text{--- ②}$$

diff ① w.r.t y

$$\frac{\partial f}{\partial u}\left[2y + 2zq\right] + \frac{\partial f}{\partial v}\left[-2x + 2zq\right] = 0 \quad \text{--- ③}$$

$$\begin{vmatrix} 2x+2zp & -2y+2zp \\ 2y+2zq & -2x+2zq \end{vmatrix} = 0.$$

$$-4x^2 + 4xzq - 4xzp + \cancel{4z^2pq} + 4y^2 - 4yzp + 4zyq - \cancel{4z^2pq} = 0$$

$$4y^2 - 4x^2 + 4xzq + 4zyq - 4xzp - 4yzp = 0$$

$$(xz+yz)p - (xz+yz)q = y^2 - x^2$$

$$\Rightarrow \underline{\underline{zp - zq = y - x}}$$

25. $\phi(x^2+y^2+z^2, xyz) = 0$ --- ①

$u = x^2 + y^2 + z^2$

$v = xyz$

diff ① w.r.t x

$$\frac{\partial \phi}{\partial u}\left[2x + 2zp\right] + \frac{\partial \phi}{\partial v}\left[yz + xyp\right] = 0$$

diff : ① w.r.t y

$$\frac{\partial\phi}{\partial u}\left[2y + 2zq\right] + \frac{\partial\phi}{\partial v}\left[xz + xyq\right] = 0$$

$$\begin{vmatrix} 2x + 2zp & yz + xyp \\ 2y + 2zq & xz + xyq \end{vmatrix} = 0$$

$$x^2z + x^2yq + xz^2p - y^2z + xy^2p - yz^2q = 0$$

$$(x^2 - y^2)z + (x^2y - yz^2)q + (xz^2 + xy^2)p$$

$$\underline{(xz^2 + xy^2)p + (x^2y - yz^2)q = (y^2 - x^2)z}$$

26. $\phi(x^2 + y^2 + z^2,\ 2y + z - x) = 0$

$u = x^2 + y^2 + z^2$

$v = 2y + z - x.$

$$\frac{\partial\phi}{\partial u}\left[2x + 2zp\right] + \frac{\partial\phi}{\partial v}\left[-1 + p\right] = 0$$

$$\frac{\partial\phi}{\partial u}\left[2y + 2zq\right] + \frac{\partial\phi}{v}\left[2 + q\right] = 0.$$

$$\begin{vmatrix} 2x + 2zp & -1 + p \\ 2y + 2zq & 2 + q \end{vmatrix} = 0$$

$$4x + 4zp + 2xq + 2\cancel{z}qp + 2y - 2yp + 2zq - 2\cancel{z}qp = 0$$

$$p(z-y) + q(x+z) = -(2x+y)$$

Que 27

$$\phi(x^2-y^2,\ x^2-z^2) = 0 \quad \text{—— (1)}.$$

$$u = x^2 - y^2$$

$$v = x^2 - z^2$$

diff (1) w.r.t x

$$\frac{\partial \phi}{\partial u}\left[2x\right] + \frac{\partial \phi}{\partial v}\left[2x - 2zp\right] = 0 \quad \text{—— (2)}$$

diff (1) w.r.t y

$$\frac{\partial \phi}{\partial u}\left[-2y\right] + \frac{\partial \phi}{\partial v}\left[-2zq\right] = 0$$

$$\begin{vmatrix} 2x & 2x-2zp \\ -2y & -2zq \end{vmatrix} = 0$$

$$-4xzq + 4xy - 4yzp = 0$$

$$yzp + xzq = xy$$

Que 28.

$$f(x+y+z,\ x^2+2yz) = 0 \quad \text{—— (1)}$$

$$u = x+y+z \qquad v = x^2+2yz$$

diff: (1) w.r.t x;

$$\frac{\partial f}{\partial u}\left[1+p\right] + \frac{\partial f}{\partial v}\left[2x + 2yp\right] = 0$$

diff: (1) w.r.t y;

$$\frac{\partial f}{\partial u}\left[1+q\right] + \frac{\partial f}{\partial v}\left[2z + 2yq\right] = 0$$

$$\begin{vmatrix} 1+p & 2x+2yp \\ 1+q & 2z+2yq \end{vmatrix} = 0.$$

$$2z - 2yq + 2zp + \cancel{2yqp} - 2x - 2yp - 2qx$$
$$- \cancel{2yqp} = 0$$

$$z - x + (y-x)q + (z-y)p = 0.$$

$$(x-y)q + (y-z)p = z - x.$$

$$(y-z)p + (x-y)q = z - x$$

Chapter 2

23/10/2020 **Solutions of PDE:**

Method 1:

Method by direct integration

Solve the following p.d.e's;

29. $\frac{\partial^2 z}{\partial x \partial y} = \sin x$

Int. both sides w.r.t y.

$$\frac{\partial z}{\partial x} = y\sin x + f(x)$$

Int w.r.t x

$$z = -y\cos x + \int f(x)\, dx + g(y)$$

$$z = -y\cos x + \phi(x) + g(y)$$

30. $\frac{\partial^2 z}{\partial y^2} = \sin(xy)$

Int w.r.t y

$$\frac{\partial z}{\partial y} = \frac{-\cos xy}{x} + f(x)$$

Int w.r.t y

$$z = \frac{-\sin xy}{x^2} + y(f(x)) + g(x)$$

$$z = -\frac{\sin(xy)}{x^2} + y\, f(x) + g(x).$$

31. $\frac{\partial^2 u}{\partial y \partial x} = 4x \sin 3xy$

Int. w.r.t y

$$\frac{\partial u}{\partial x} = 4\cancel{x} \times -\cos 3xy / 3\cancel{x} + f(x)$$

$$\frac{\partial u}{\partial x} = -\frac{4}{3}\cos(3xy) + f(x).$$

~~Int~~ Int= wrt x

$$u = -\frac{4}{3}\,\frac{\sin 3xy}{3y} + \int f(x)\,dx + g(y)$$

$$u = -\frac{4\sin(3xy)}{9y} + \phi(x) + g(y)$$

32. $$\frac{\partial^2 z}{\partial x \partial y} = (\cos ax + by)$$

Int wrt y

$$\frac{\partial z}{\partial x} = \frac{\sin(ax+by)}{b} + f(x)$$

Int wrt x

$$z = -\frac{\cos(ax+by)}{ab} + \phi(x) + g(y)$$

33. $$\frac{\partial^2 z}{\partial x \partial y} = e^y \cos x$$

Int wrt y

$$\frac{\partial z}{\partial x} = \cos x\, e^y + f(x)$$

Int wrt x

$$z = e^y \times \sin x + \phi(x) + g(y)$$

$$z = e^y \sin x + \phi(x) * g(y)$$

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34. $\frac{\partial^2 z}{\partial y^2} = \cos(xy)$

Int wrt y;

$\frac{\partial z}{\partial y} = \frac{\sin(xy)}{x} + f(x)$

Int wrt y;

$z = \frac{-\cos(xy)}{x^2} + y f(x) + g(x)$

35. $\log\left[\frac{d^2 z}{\partial x^2}\right] = x+y$.

$\frac{\partial^2 z}{\partial x^2} = e^{x+y}$ or $(e^x \cdot e^y)$

Int wrt x

$\frac{\partial z}{\partial x} = e^y \cdot e^x + f(y)$

Int wrt x

$z = e^y \cdot e^x + x f(y) + g(y)$

$z = e^{x+y} + x f(y) + g(y)$

36. $\frac{\partial^3 z}{\partial x^2 \partial y} = \cos(2x+3y)$

Int: wrt y

$\frac{\partial^2 z}{\partial x^2} = \frac{\sin(2x+3y)}{3} + f(x)$.

Int : wrt x

$\frac{\partial z}{\partial x} = \frac{-\cos(2x+3y)}{6} + \int f(x)\,dx + g(y)$

Int wrt x

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$$z = -\frac{\sin(2x+3y)}{12} + \psi(x) + x g(y) + h(y).$$

37. $\frac{\partial z}{\partial x} = 6x+3y \qquad \frac{\partial z}{\partial y} = 3x-4y$

$$\frac{\partial z}{\partial x} = 6x+3y \quad \text{——— (1)}$$

$$\frac{\partial z}{\partial y} = 3x-4y \quad \text{——— (2)}.$$

Int (1) w.r.t x

$$z = \frac{6x^2}{2} + 3xy + f(y) \quad \text{——— (3)}$$

Int (2) w.r.t y

$$z = 3xy - \frac{4y^2}{2} + g(x) \quad \text{——— (4)}$$

$$z = 3x^2 + 3xy + f(y) \quad \text{——— (3)}$$

$$z = 3xy - 2y^2 + g(x) \longrightarrow (4).$$

diff = (3) w.r.t y

$$\frac{\partial z}{\partial y} = 0 + 3x + f'(y) \quad \text{——— (5)}.$$

comparing (5) & (2).

$$3x + f'(y) = 3x - 4y$$

$$f'(y) = -4y$$

$$f(y) = -2y^2 + c \quad \text{——— (6)}$$

Sub: ⑥ in ③.

$$z = 3x^2 + 3xy - 2y^2 + c$$

(OR)

By potential fn: from ③ & ④ dx

$$z = 3x^2 + 3xy - 2y^2 + c.$$

38. $\frac{\partial z}{\partial x} = 3x - y$ —— ①

$\frac{\partial z}{\partial y} = -x + \cos y$ —— ②

Int ① wrt x

$$z = \frac{3x^2}{2} - xy + f(y) \text{ —— ③}$$

Int ② wrt y

$$z = -xy + \sin y + g(x) \text{ —— ④}.$$

$$z = \frac{3x^2}{2} + \sin y - xy + c$$

(OR) diff ③ wrt y.

$$\frac{\partial z}{\partial y} = -x + f'(y) \text{ —— ⑤}$$

$f'(y) = \cos y$ (comparing ⑤ & ②)

$f(y) = \sin y + c$ —— ⑥

Sub: ⑥ in ③

$$z = \frac{3x^2}{2} - xy + \sin y + c$$

27/10/2020 Method 2

Solve the following pde's :

39. $\frac{\partial^2 z}{\partial x^2} + z = 0$ Given; when ; $x=0 \Rightarrow z = e^y$ & $\frac{\partial z}{\partial x} = 1$

If z is a fn: of x alone

$$\frac{\partial^2 z}{\partial x^2} + z = 0 \quad \text{--- (1)}$$

$$\Rightarrow \frac{d^2 z}{dx^2} + z = 0.$$

auxillary eqn:; $m^2+1=0$

$m = \pm i$

$z = c_1 \cos x + c_2 \sin x$

Since z is a fn of x and y. c_1 and c_2 are arbitary fn:s of y

$\therefore$ $z = f(y)\cos x + g(y)\sin x$ is the general soln of eqn (1)

$$z = f(y)\cos x + g(y)\sin x \quad \text{--- (2)}.$$

$$\frac{\partial z}{\partial x} = -f(y)\sin x + g(y)\cos x \quad \text{--- (3)}$$

put $x=0$ & $z=e^y$ in (2).

$e^y = f(y)$

put $x=0$ & $\partial z/\partial x = 1$ in (3)

$g(y) = 1$

$\therefore$ The particular soln or integral of ① is

$z = e^{y}\cos x + \sin x$

40. $\frac{\partial^2 z}{\partial y^2} = z$ —— ① given. when $y = 0$ $z = e^{x}$

$\frac{\partial z}{\partial y} = e^{-x}$.

If z is a fn of y alone, ① $\Rightarrow$

$\frac{d^2 z}{dy^2} - z = 0$.

auxillary eqn ; $m^2 - 1 = 0$

$m = \pm 1$

$z = c_1 e^{y} + c_2 e^{-y}$

Since z is a fn of x and y c_1 and c_2 are arbitrary fns of x.

$\therefore z = f(x) e^{y} + g(x) e^{-y}$ is the gen soln of ①

$z = f(x) e^{y} + g(x) e^{-(y)}$ —— ②.

$\frac{\partial z}{\partial y} = f(x) e^{y} - g(x) e^{-y}$ —— ③.

put $y = 0$ & $z = e^{x}$ in ②.

$e^{x} = f(x) + g(x)$

put $y = 0$ & $\frac{\partial z}{\partial y} = e^{-x}$ in ③.

$e^{-x} = f(x) - g(x)$

$f(x) = \frac{e^{x} + e^{-x}}{2} = \cosh x$

$g(x) = \sinh x = \left(\frac{e^{x} - e^{-x}}{2}\right)$.

$z = \cosh x \, e^{y} + \sinh x \, e^{-y}$ (p. soln)

41. $\frac{\partial^2 z}{\partial x^2} + a^2 z = 0$ — ① Given; $x=0 \Rightarrow z = e^y$

$\partial z/\partial y = a$

If z is a fn of x alone ①

$\Rightarrow \frac{d^2 z}{dx^2} + a^2 z = 0$

auxillary eqn; $m^2 + a^2 = 0$

$m^2 = -a^2$

$m = \pm ai$

$z = c_1 \cos ax + c_2 \sin ax$

Since z is the fn of x and y c_1 and c_2 are arbitrary fns of y.

$z = f(y) \cos ax + g(y) \sin ax$ — ②

$\frac{\partial z}{\partial x} = -f(y) a \sin ax + g(y) a \cos ax$ — ③.

put $x=0$ & $z = e^y$ in ②.

$e^y = f(y)$.

put $x=0$ & $\frac{\partial z}{\partial y} = a$. in ③

$a = g(y) a$ $\quad g(y) = 1$

Thus; particular soln is;

$z = e^y \cos ax + \sin ax$

42. $\frac{\partial^2 z}{\partial x^2} - a^2 z = 0$ ——— ①

When $x = 0 \Rightarrow \frac{\partial z}{\partial x} = a\sin y$ & $\frac{\partial z}{\partial y} = 0$

If z is a fn of x alone ①.

$\Rightarrow \frac{d^2 z}{dx^2} - a^2 z = 0.$

A. eqn ; $m^2 - a^2 = 0.$

$m^2 = a^2 \quad m = \pm a.$

$z = c_1 e^{ax} + c_2 e^{-ax}$ Since z is the fn of x and y. c_1 and c_2 are arbitary fns of y.

$z = f(y) e^{ax} + g(y) e^{-ax}$ ——— ②.

$\frac{\partial z}{\partial x} = f(y) e^{ax} \times a + g(y) \times e^{-ax} \times -a$

$= f(y) a e^{ax} - g(y) a e^{-ax}$ ——— ③.

$\frac{\partial z}{\partial y} = f'(y) e^{ax} + g'(y) e^{-ax}$ ——— ④.

put $x = 0$ & $\frac{\partial z}{\partial x} = a\sin y$ in ③.

$a\sin y = af(y) - ag(y)$

$f(y) - g(y) = \sin y$ ——— ⑤.

put $x = 0$ & $\frac{\partial z}{\partial y} = 0.$

$0 = f'(y) + g'(y)$ ——— ⑥.

$f(y) + g(y) = c_3$ ——— ⑦.

$2f(y) = \sin y + c_3$

$f(y) = \frac{\sin y + c_3}{2}. \quad g(y) = \frac{c_3 - \sin y}{2}.$

The soln is;

$$z = \left(\frac{\sin y + c_3}{2}\right) e^{ax} + \left(\frac{c_3 - \sin y}{2}\right) e^{-ax}$$

Method 3: *

Lagrange's Linear eqn of 1st order:

An eqn of the form;

$Pp + Qq = R$ where;

P, Q, R are $\longrightarrow$ fns of x, y & z

is known as

Lagrange's eqn.

Solution Method:

Step 1 $\longrightarrow$ Form the Lagrange's subsidiary eqn as follows

$$\boxed{\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}}$$

Step 2 $\longrightarrow$ Solve the above simultaneous eqn by the Method of grouping or Multiplier Method.

Step 3 $\longrightarrow$ Give; $u = a$ & $v = b$ are 2 independent solns of the subsidiary eqns then

$$\boxed{\begin{gathered}\phi(u, v) = 0 \\ \text{or} \\ u = \phi(v)\end{gathered}}$$

Where ϕ is an arbitrary fn.
is the Gen. Integral or Gen Soln
of the Lagrange's Eqn.

Method of Grouping :

This Method is Applicable only when one of the variables is absent from 2 terms in the subsidiary eqn.

Multiplier Method :

The subsidiary eqn is;

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

then we have;

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} = \frac{l\,dx + m\,dy + n\,dz}{lP + mQ + nR}.$$

where l, m, n be constants or fns of x, y, z

Choose l, m, n in such a way that;

$$lP + mQ + nR = 0$$

So that $l\,dx + m\,dy + n\,dz = 0$

Solve the following pde's :

43. $yq - xp = z$

$Pp + Qq = R \Rightarrow -xp + yq = z$

$P = -x \quad Q = y \quad R = z$

The subsidiary eqns are;

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\frac{dx}{-x} = \frac{dy}{y} = \frac{dz}{z}$$

Considering the 1st 2 eqns:

$$\frac{dx}{-x} = \frac{dy}{y}$$

$$-\log x = \log y - \log a$$

$$\log a = \log x + \log y$$

$$\log a = \log(xy)$$

$$xy = a$$

Considering the ~~2nd~~

$$\frac{dy}{y} = \frac{dz}{z}$$

$$\log y = \log z + \log b$$

$$\Rightarrow y/z = b$$

$u = xy \quad V = y/z$

The soln is; $\phi(xy, y/z) = 0$

44. $p\tan x + q\tan y = \tan z$

$Pp + Qq = R$

$P \longrightarrow \tan x \quad Q = \tan y \quad R = \tan z$

The subsidiary eqn is;

$$\frac{dx}{\tan x} = \frac{dy}{\tan y} = \frac{dz}{\tan z}$$

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$$\frac{dx}{\tan x} = \frac{dy}{\tan y}$$

$$\frac{\cos x\,dx}{\sin x} = \frac{\cos y\,dy}{\sin y}$$

$$\log(\sin x) = \log(\sin y) + \log a$$

$$a = \frac{\sin x}{\sin y}$$

$$\frac{dy}{\tan y} = \frac{dz}{\tan z}$$

$$\frac{\cos y\,dy}{\sin y} = \frac{\cos z\,dz}{\sin z}$$

$$\log(\sin y) = \log(\sin z) + \log b$$

$$b = \frac{\sin y}{\sin z}$$

$$u = \frac{\sin x}{\sin y} \qquad v = \frac{\sin y}{\sin z}$$

The soln is; $\phi\left(\sin x/\sin y,\ \sin y/\sin z\right) = 0$

45. $y^2p - xyq = x(z-2y)$

$Pp + Qq = R$

$P = y^2 \quad Q = -xy \quad R = xz - 2xy.$

The subsidiary eqn is

$$\frac{dx}{y^2} = \frac{dy}{-xy} = \frac{dz}{x(z-2y)}$$

Considering the 1st 2 eqns we have;

$$\frac{dx}{y^2} = \frac{dy}{-xy}$$

$$\frac{dx}{y} = \frac{dy}{-x}$$

$$-x\,dx = y\,dy.$$

$$\frac{x^2}{2} = -\frac{y^2}{2} + \frac{a}{2}$$

$$a = x^2+y^2$$

Considering the last 2 eqns.

$$\frac{-dy}{\cancel{x}y} = \frac{dz}{\cancel{x}(z-2y)}$$

$$-\frac{dy}{y} = \frac{dz}{z-2y}.$$

$$-z\,dy + 2y\,dy = y\,dz$$

$$2y\,dy = y\,dz + z\,dy$$

$$2y\,dy = d(yz)$$

$$y^2 = yz + b$$

$$b = y^2 - yz$$

$$u = x^2+y^2 \qquad v = y^2-yz$$

The soln is; $\phi(x^2+y^2,\ y^2-yz) = 0$

46 $\frac{y^2 z}{x} p + xzq = y^2$

$Pp + Qq = R.$

$P \to \frac{y^2 z}{x} \quad Q \to xz \quad R \to y^2.$

$$\frac{x\,dx}{y^2 z} = \frac{dy}{xz} = \frac{dz}{y^2}$$

$$\frac{x\,dx}{\cancel{y^2} z} = \frac{dz}{\cancel{y^2}}.$$

$x\,dx = z\,dz$

$x^2 = z^2 + a$

$a = x^2 - z^2$

$u = x^2 - z^2$

$$\frac{x\,dx}{y^2 z} = \frac{dy}{xz}$$

$x^2 dx = y^2 dy$

$x^3 = y^3 + b.$

$b = x^3 - y^3$

$V = x^3 - y^3$

The soln is $\phi(x^2 - z^2, x^3 - y^3) = 0$

47 $pz - qz = z^2 + (x+y)^2$

$Pp + Qq = R.$

$P = z \quad Q = -z \quad R = z^2 + (x+y)^2$

$$\frac{dx}{z} = \frac{dy}{-z} = \frac{dz}{z^2 + (x+y)^2}$$

$dx = -dy$

$x = -y + a \quad a = (x+y)$

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$$\frac{dx}{z} = \frac{dz}{z^2+a^2}$$

$$2dx = \frac{2z\,dz}{z^2+a^2}$$

$$2x = \log(z^2+a^2) + b.$$

$$b = 2x - \log(z^2+a^2)$$

The soln is $\phi\left((x+y),\ 2x - \log(z^2+a^2)\right) = 0$

48. $(y^2+z^2)p - xyq + xz = 0.$

$P = y^2+z^2 \qquad Q = -xy \qquad R = -xz.$

$$\frac{dx}{y^2+z^2} = \frac{dy}{-xy} = \frac{dz}{-xz}.$$

$$\frac{dy}{-\cancel{x}y} = \frac{dz}{-\cancel{x}z}$$

$$\frac{dy}{y} = \frac{dz}{z}$$

$$\log y = \log z + \log a.$$

$$a = y/z \longrightarrow (4)$$

By Multiplier Method;

Using the multipliers x, y, z,

$$\frac{dx}{y^2+z^2} = \frac{-dy}{xy} = \frac{-dz}{xz} = \frac{x\,dx + y\,dy + z\,dz}{x(y^2+z^2) + y(-xy) + z(-xz)}$$

$\Rightarrow xdx + ydy + zdz = 0.$

$$b = x^2 + y^2 + z^2 \longrightarrow (v)$$

The soln is; $\phi\left(y/z,\ x^2+y^2+z^2\right) = 0$

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47. $\frac{y-z}{yz} p + \frac{z-x}{zx} q = \frac{x-y}{xy}.$

$$P = \frac{y-z}{yz} \quad Q = \frac{z-x}{zx} \quad R = \frac{x-y}{xy}.$$

$$(y-z)x p + (z-x)y q = (x-y)z.$$

$$P = (y-z)x \quad Q = (z-x)y \quad R = (x-y)z.$$

$$\frac{dx}{(y-z)x} = \frac{dy}{(z-x)y} = \frac{dz}{(x-y)z}$$

Using the multipliers 1, 1, 1 we get;

$$\frac{dx}{(y-z)x} = \frac{dy}{(z-x)y} = \frac{dz}{(x-y)z} = \frac{dx+dy+dz}{(y-z)x+(z-x)y+(x-y)z}$$

$$dx + dy + dz = 0.$$

$$x+y+z = u.$$

Using the multipliers $1/x, 1/y, 1/z$

$$\frac{dx}{(y-z)x} = \frac{dy}{(z-x)y} = \frac{dz}{(x-y)z} = \frac{dx/x + dy/y + dz/z}{y-z+z-x+x-y}.$$

$$\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z} = 0$$

$$\log x + \log y + \log z = \log v$$

$xyz = V$

The soln is $z = \phi(x+y+z, xyz) = 0$

50. $x(y^2-z^2)p + y(z^2-x^2)q = z(x^2-y^2)$

$P = x(y^2-z^2)$ $\quad Q = y(z^2-x^2)$ $\quad R = z(x^2-y^2)$

$$\frac{dx}{x(y^2-z^2)} = \frac{dy}{y(z^2-x^2)} = \frac{dz}{z(x^2-y^2)} \quad \text{(S. eqn)}$$

Using the multipliers $1/x, 1/y, 1/z$

$$\frac{dx}{x(y^2-z^2)} = \frac{dy}{y(z^2-x^2)} = \frac{dz}{z(x^2-y^2)} = \frac{dx/x + dy/y + dz/z}{y^2-z^2+z^2-x^2+x^2-y^2}$$

$$\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z} = 0.$$

$$\log x + \log y + \log z = \log u$$

$$xyz = u$$

Using the multipliers x, y, z

$$\frac{dx}{x(y^2-z^2)} = \frac{dy}{y(z^2-x^2)} = \frac{dz}{z(x^2-y^2)} = \frac{x\,dx + y\,dy + z\,dz}{x^2(y^2-z^2)+y^2(z^2-x^2)+z^2(x^2-y^2)}$$

$$x\,dx + y\,dy + z\,dz = 0.$$

$$x^2 + y^2 + z^2 = V$$

The soln is;

$$z = \phi\left(xyz,\ x^2+y^2+z^2\right) = 0.$$

5. $(z^2 - 2yz - y^2)p + (xy + zx)q = xy - zx.$

$P = z^2 - 2yz - y^2 \qquad Q = xy + zx$

$R = xy - zx.$

$$\frac{dx}{z^2-2yz-y^2} = \frac{dy}{xy+zx} = \frac{dz}{xy-zx}.$$

$$\frac{dy}{y+z} = \frac{dz}{y-z}$$

$$ydy - zdy = ydz + zdz$$

$$ydy - zdz = ydz + zdy.$$

$$ydy - zdz = d(yz)$$

$$\frac{y^2}{2} - \frac{z^2}{2} = yz + \frac{u}{2}.$$

$$\frac{u}{2} = \frac{y^2 - z^2 - 2yz}{2}$$

$$u = y^2 - z^2 - 2yz$$

Using Multipliers x, y, z we get

$$\frac{dx}{z^2-2yz-y^2} = \frac{dy}{xy+zx} = \frac{dz}{xy-zx} = \frac{xdx + ydy + zdz}{xz^2 - 2yxz - xy^2 + xy^2 + xyz + xyz - xz^2}.$$

$$xdx + ydy + zdz = 0.$$

$$x^2 + y^2 + z^2 = v.$$

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The soln is ; $\phi\left(y^2-z^2-2yz,\ x^2+y^2+z^2\right)=0$

52. $x^2(y-z)p + y^2(z-x)q = z^2(x-y)$

$P = x^2(y-z) \quad Q = y^2(z-x) \quad R = z^2(x-y)$

$$\frac{dx}{x^2(y-z)} = \frac{dy}{y^2(z-x)} = \frac{dz}{z^2(x-y)}$$

Using Multipliers $1/x^2, 1/y^2, 1/z^2$.

$$\frac{dx}{x^2(y-z)} = \frac{dy}{y^2(z-x)} = \frac{dz}{z^2(x-y)} = \frac{dx/x^2 + dy/y^2 + dz/z^2}{(y-z)+(z-x)+(x-y)}$$

$$\frac{dx}{x^2} + \frac{dy}{y^2} + \frac{dz}{z^2} = 0.$$

$$\frac{-1}{x} + \frac{-1}{y} - 1/z = -u.$$

$$u = 1/x + 1/y + 1/z$$

Using the Multipliers $1/x, 1/y, 1/z$

$$\frac{dx}{x^2(y-z)} = \frac{dy}{y^2(z-x)} = \frac{dz}{z^2(x-y)} = \frac{dx/x + dy/y + dz/z}{x(y-z)+y(z-x)+z(x-y)}$$

$$\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z} = 0.$$

$$\log x + \log y + \log z = \log v$$

$$xyz = v.$$

The soln is;

$$\phi\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z},\ xyz\right)=0$$

53. $(2z-y)p+(x+z)q+2x+y=0.$

$$(2z-y)p+(x+z)q=-(2x+y)$$

$P=2z-y \quad Q=x+z \quad R=-(2x+y)$

$$\frac{dx}{2z-y}=\frac{dy}{x+z}=\frac{dz}{-(2x+y)}$$

Using Multipliers $1,-2,-1$

$$\frac{dx}{2z-y}=\frac{dy}{x+z}=\frac{dz}{-2x-y}=\frac{dx-2dy-dz}{\cancel{2z}-\cancel{y}-\cancel{2x}-\cancel{2z}+\cancel{2x}+\cancel{y}}$$

$$dx-2dy-dz=0$$

$$x-2y-z=u.$$

Using Multipliers x,y,z.

$$\frac{dx}{2z-y}=\frac{dy}{x+z}=\frac{dz}{-2x-y}=\frac{xdx+ydy+zdz}{x(2z-y)+y(x+z)+z(-2x-y)}$$

$$xdx+ydy+zdz=0.$$

$$x^2+y^2+z^2=v.$$

The soln is;

$$\phi\left(x-2y-z,\ x^2+y^2+z^2\right)=0$$

54. $(y+zx)p-(x+yz)q=x^2-y^2$.

$P=y+zx \quad Q=-x-yz \quad R=x^2-y^2$

$$\frac{dx}{y+zx}=\frac{dy}{-x-yz}=\frac{dz}{x^2-y^2}$$

Using Multipliers $y, x, 1$

$$\frac{dx}{y+zx}=\frac{dy}{-x-yz}=\frac{dz}{x^2-y^2}=\frac{y\,dx+x\,dy+dz}{y(y+zx)+x(-x-yz)+x^2-y^2}$$

$$y\,dx+x\,dy+dz=0.$$

$$d(xy)+dz=0 \qquad xy+z=v$$

Using Multipliers $x, y, -z$

$$\frac{dx}{y+zx}=\frac{dy}{-x-yz}=\frac{dz}{x^2-y^2}=\frac{x\,dx+y\,dy-z\,dz}{x(y+zx)+y(-x-yz)+z(x^2-y^2)}$$

$$x\,dx+y\,dy-z\,dz=0.$$

$$x^2+y^2-z^2=u.$$

Using Multipliers $y, x, 1$

$$\frac{dx}{y+zx}=\frac{dy}{-x-yz}=\frac{dz}{x^2-y^2}=\frac{y\,dx+x\,dy+dz}{y(y+zx)+x(-x-yz)+x^2-y^2}$$

$$y\,dx+x\,dy+dz$$

$$d(xy)+dz=0$$

$$xy+z=v$$

The soln is ; $\phi(x^2+y^2-z^2,\ xy+z)=0$

55.

Ans. $x^2p + y^2q = (x+y)z$

$P = x^2 \quad Q = y^2 \quad R = (x+y)z$

$$\frac{dx}{x^2} = \frac{dy}{y^2} = \frac{dz}{(x+y)z}$$

$$\frac{dx}{x^2} = \frac{dy}{y^2}.$$

$$\frac{-1}{x} = \frac{-1}{y} + a.$$

$$\frac{-1}{x} + \frac{1}{y} = a.$$

$$\frac{x-y}{xy} = a.$$

Multiplier $\quad 1/x, 1/y, -1/z$

$$\frac{dx}{x^2} = \frac{dy}{y^2} = \frac{dz}{(x+y)z} = \frac{\frac{dx}{x} + \frac{dy}{y} + \frac{-dz}{z}}{\cancel{x} + y + \cancel{x} - y}$$

$$\frac{dx}{x} + \frac{dy}{y} - \frac{dz}{z} = 0.$$

$$\log x + \log y - \log z = \log b.$$

$$\frac{xy}{z} = b.$$

The soln is $\phi\left(\frac{x-y}{xy}, \frac{xy}{z}\right) = 0$

Ans

56. $x(y^2 - z^2)p - y(z^2 + x^2)q = z(x^2 + y^2)$

$P = x(y^2 - z^2) \quad Q = -y(z^2 + x^2)$

$R = z(x^2 + y^2)$

$$\frac{dx}{x(y^2-z^2)} = \frac{dy}{-y(z^2+x^2)} = \frac{dz}{z(x^2+y^2)}$$

Multiplier x, y, z

$$\frac{dx}{x(y^2-z^2)} = \frac{dy}{-y(z^2+x^2)} = \frac{dz}{z(x^2+y^2)} = \frac{x\,dx + y\,dy + z\,dz}{x^2(y^2-z^2) - y^2(z^2+x^2) + z^2(x^2+y^2)}$$

$$x^2+y^2+z^2=a$$

Multiplier $1/x, -1/y, -1/z$

$$\frac{dx}{x(y^2-z^2)} = \frac{dy}{-y(z^2+x^2)} = \frac{dz}{z(x^2+y^2)} = \frac{\frac{dx}{x}-\frac{dy}{y}-\frac{dz}{z}}{y^2-z^2+z^2+x^2-x^2-y^2}$$

$$\frac{dx}{x}-\frac{dy}{y}-\frac{dz}{z}=0$$

$$\log x-\log y-\log z=\log a$$

$$\log y+\log z-\log x=\log b$$

$$b=yz/x$$

The soln is $\phi\left(x^2+y^2+z^2,\ yz/x\right)=0$

57. $xzp+yzq=xy.$

$P=xz \quad Q=yz \quad R=xy.$

$$\frac{dx}{xz}=\frac{dy}{yz}=\frac{dz}{xy}.$$

$$\frac{dx}{x\cancel{z}}=\frac{dy}{y\cancel{z}}$$

$$\log x=\log y+\log a.$$

$u=a=x/y$

$$\frac{dy}{\cancel{yz}} \neq \frac{dz}{\cancel{xy}}$$

~~Ans~~ The Multipliers are $y, x, 2z$.

$$\frac{dx}{xz} = \frac{dy}{yz} = \frac{dz}{xy} = \frac{y\,dx + x\,dy - 2z\,dz}{xyz + xyz - 2xyz}$$

$$y\,dx + x\,dy - 2z\,dz = 0.$$

$$d\,xy - 2z\,dz = 0.$$

$$xy - \frac{\cancel{2}z^2}{\cancel{2}} = V$$

$$V = xy - z^2.$$

The soln is $\phi\left(x/y,\ xy - z^2\right) = 0$.

Ans 58. $\dfrac{p}{x^2} + \dfrac{q}{y^2} = z.$

$P = 1/x^2 \quad Q = 1/y^2 \quad = R.$

$$\frac{dx}{1/x^2} = \frac{dy}{1/y^2} = \frac{dz}{z}.$$

$$x^2dx = y^2dy = \frac{dz}{z}.$$

$$x^2dx = y^2dy$$

$$\frac{x^3}{3} = \frac{y^3}{3} + \frac{a}{3}.$$

$$a = x^3 - y^3$$

$$y^2dy = \frac{dz}{z}$$

$$\frac{y^3}{3} = \log z + b.$$

$$b = y^3 - 3\log z.$$

The soln is $\phi\left(x^3 - y^3,\ y^3 - 3\log z\right) = 0$

59. $P/y + q/x = 1/z$

$P = 1/y \quad Q = 1/x \quad R = 1/z$

$$\frac{dx}{1/y} = \frac{dy}{1/x} = \frac{dz}{1/z}$$

$$y\,dx = x\,dy = z\,dz.$$

$$\frac{dx}{x} = \frac{dy}{y}.$$

$$\log x = \log y + \log a.$$

$$x/y = a$$

Multiplier $y, x, -2z$.

$$\frac{dx}{1/y} = \frac{dy}{1/x} = \frac{dz}{1/z} = \frac{y\,dx + x\,dy - 2z\,dz}{1+1-2.}$$

$$= d(yx) - 2z\,dz$$

$$= yx - z^2$$

The soln is $\phi(x/y, yx - z^2) = 0$

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60. $(x^2+y^2)p + 2xyq = (x+y)z$

$P = x^2+y^2 \quad Q = 2xy \quad R = (x+y)z$

$$\frac{dx}{x^2+y^2} = \frac{dy}{2xy} = \frac{dz}{(x+y)z}$$

$$\frac{dx}{x^2+y^2} = \frac{dy}{2xy}$$

$$2xy\,dx = x^2dy + y^2dy.$$

$$\frac{2xy\,dx - x^2dy}{y^2} = dy.$$

$$d\left(\frac{x^2}{y}\right) = dy.$$

$$\frac{x^2}{y} = y + u.$$

$$u = \frac{x^2}{y} - y$$

$$u = \frac{x^2-y^2}{y}$$

Multipliers $1, 1, \cancel{(x+y)/z}$

$$\frac{dx}{x^2+y^2} = \frac{dy}{2xy} = \frac{dz}{(x+y)z} = \frac{dx+dy-\frac{(x+y)}{z}dz}{(x^2+y^2)+2xy+z\frac{(x+y)}{}\times\frac{-(x+y)}{z}}$$

$$\Rightarrow dx+dy-\frac{x+y}{z}dz = 0$$

$$dx+dy = x+y\,\frac{dz}{z}.$$

$$\frac{dz}{z} = \frac{dx+dy}{x+y}$$

$$\log z = \log(x+y) + \log b.$$

$$b = z/x+y$$

The soln is $\phi\left((x^2-y^2)/y,\ z/x+y\right) = 0$

61. $(x-y)p + (y-x-z)q = z$

$P = x-y \quad Q = y-x-z \quad R = z$

$$\frac{dx}{x-y} = \frac{dy}{y-x-z} = \frac{dz}{z}$$

Multiplier 1, 1, 1

$$\frac{dx}{x-y} = \frac{dy}{y-x-z} = \frac{dz}{z} = \frac{dx+dy+dz}{x-y+y-x-z+z}$$

$$dx+dy+dz = 0.$$

$$\underline{\underline{x+y+z}} = a \; (u) \qquad x+z = u-y.$$

$$\frac{dy}{y-x-z} = \frac{dz}{z}$$

$$\frac{dy}{y+y-u} = \frac{dz}{z} \Rightarrow \frac{2dy}{2y-u} = \frac{2dz}{z}$$

$$\log(2y-u) = 2\log z + \log u.$$

$$2y-u = z^2 v$$

$$v = \frac{2y-u}{z^2}$$

The soln; $\underline{\underline{\phi\left(x+y+z,\ \frac{2y-u}{z^2}\right)=0}}$

$$\underline{\underline{\phi\left(x+y+z,\ \frac{y-x-z}{z^2}\right)=0}}$$

62. $px^2 - qy^2 = z(x-y)$

$P = x^2 \quad Q = -y^2 \quad R = (x-y)z$

$$\frac{dx}{x^2} = \frac{dy}{-y^2} = \frac{dz}{(x-y)z}$$

$$\frac{dx}{x^2} = \frac{dy}{-y^2}$$

$$\therefore \frac{-1}{x} = \frac{1}{y} - a.$$

$$-\frac{y-x}{xy} = -a.$$

$$u = \frac{y+x}{xy}$$

Multipliers are $1/x$, $1/y$, $-1/z$

$$\frac{dx}{x^2} = \frac{dy}{-y^2} = \frac{dz}{(x-y)z} = \frac{dx/x + dy/y - dz/z}{\frac{x^2}{x} + \frac{y^2}{y} - \frac{(x-y)z}{z}}.$$

$$\frac{dx}{x} + \frac{dy}{y} - \frac{dz}{z} = 0.$$

$$\log x + \log y - \log z = \log b.$$

$$xy/z = b$$

$$V = xy/z$$

The soln is $\phi\left(\frac{y+x}{xy}, xy/z\right) = 0$

63. $(x-2z)p + (2z-y)q = y-x.$

$P = x-2z \quad Q = 2z-y \quad R = y-x$

$$\frac{dx}{x-2z} = \frac{dy}{2z-y} = \frac{dz}{y-x}.$$

Multipliers 1, 1, 1

$$\frac{dx}{x-2z} = \frac{dy}{2z-y} = \frac{dz}{y-x} = \frac{dx+dy+dz}{x-2z+2z-y+y-x}$$

$$dx+dy+dz = 0.$$

$$\underline{\underline{x+y+z=u}}$$

Multiplier $y, x, 2z$

$$\frac{dx}{x-2z} = \frac{dy}{2z-y} = \frac{dz}{y-x} = \frac{y\,dx + x\,dy + 2z\,dz}{y(x-2z)+x(2z-y)+2z(y-x)}.$$

$$y\,dx + x\,dy + 2z\,dz = 0$$

$$xy + z^2 = v$$

The soln is $\underline{\underline{\phi(x+y+z,\ xy+z^2) = 0}}$

64. $(x^2-y^2-z^2)p + 2xyq = 2xz$

$P = x^2-y^2-z^2 \quad Q = 2xy \quad R = 2xz$

$$\frac{dx}{x^2-y^2-z^2} = \frac{dy}{2xy} = \frac{dz}{2xz}$$

$$\frac{dy}{\cancel{2x}y} = \frac{dz}{\cancel{2x}z}$$

$$\frac{dy}{y} = \frac{dz}{z}$$

$$\log y = \log z + \log u$$

$$u = y/z$$

Multipliers are x, y, z

$$\frac{dx}{x^2-y^2-z^2} = \frac{dy}{2xy} = \frac{dz}{2xz} = \frac{x\,dx + y\,dy + z\,dz}{x^3 - xy^2 - xz^2 + 2xy^2 + 2xz^2}$$

$$\frac{x\,dx + y\,dy + z\,dz}{x^3 + xy^2 + xz^2} = \frac{dz}{2xz}$$

$$\frac{2(x\,dx + y\,dy + z\,dz)}{x^2+y^2+z^2} = \frac{dz}{z}$$

$$\log(x^2+y^2+z^2) = \log z + \log v$$

$$v = \frac{x^2+y^2+z^2}{z}$$

The soln is $\phi\left(y/z,\ \frac{x^2+y^2+z^2}{z}\right) = 0$

$\longrightarrow$ Hint:

First Find:
f_p, f_q, f_x, f_y, f_z

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Method 4:

Charpit's Method

It is the general method for finding the complete integral of a nonlinear pde.

Consider the eqn $f(x, y, z, p, q) = 0$ ——— (1)

The subsidiary eqns in Charpit's Method are;

$$\frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{\frac{\partial f}{\partial q}} = \frac{dz}{-p\frac{\partial f}{\partial p} - q\frac{\partial f}{\partial q}} = \frac{dp}{\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}}$$

Find a relation from these eqns. involving x, y, z, p, q st $\phi(x, y, z, p, q) = 0$ ——— (2)

Since z depends on x and y we have

$dz = p\,dx + q\,dy \longrightarrow$ (3).

Solve ① and ② for p and q and substitute in ③

Solve the following using Charpits Method

65. $2xz + pq = 2xyq + x^2p$ ———— ①

$f = 2xz + pq - 2xyq - x^2p$.

$f_p = q - x^2$

$f_q = p - 2xy$

$f_x = 2z - 2yq - 2xp$

$\boxed{f_y = -2xq}$

$f_z = 2x$

The subsidiary eqns are given by;

$$\frac{dx}{-f_p} = \frac{dy}{-f_q} = \frac{dz}{-pf_p - qf_q} = \frac{dp}{f_x + pf_z} = \frac{dq}{f_y + qf_z}$$

$$\frac{dx}{-q + x^2} = \frac{dy}{-2x} = \frac{dz}{-p(q-x^2) - q(p-2xy)} = \frac{dp}{2z - 2yq - \cancel{2xp} + \cancel{2xp}} = \frac{dq}{\cancel{-2xq} + \cancel{2xq}}$$

$$\frac{dx}{-q+x^2} = \frac{dy}{-2x} = \frac{dz}{px^2 + 2xyq - 2pq} = \frac{dp}{2z - 2yq} = \frac{dq}{0}$$

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$\cancel{\frac{dp}{z}}\ dq = 0. \Rightarrow \underline{q = a}$

Substituting $q = a$ in ①.

$$2xz + ap = 2xya + x^2p.$$

$$(a - x^2)p = 2x(ya - z)$$

$$p = \frac{2x(ay - z)}{(a - x^2)}$$

$$\boxed{dz = p\,dx + q\,dy}$$

$$dz = \frac{2x(ay - z)}{a - x^2}\,dx + a\,dy.$$

$$\frac{dz - a\,dy}{ay - z} = \frac{2x}{a - x^2}\,dx$$

$$\frac{dz - a\,dy}{z - ay} = \frac{2x\,dx}{x^2 - a}.$$

$$\log(z - ay) = \log(x^2 - a) + \log b$$

$$\underline{\underline{\frac{z - ay}{x^2 - a} = b}}$$

$z = \underline{\underline{ay + (x^2 - a)b}}$ is the soln.

66. $p^2 + qy = z.$ —— ①

$$f = p^2 + qy - z$$

$$f_p = 2p$$

$$f_q = y$$

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$f_x = 0$

$f_y = q$

$f_z = -1$

The subsidiary eqns are :

$$\frac{dx}{-f_p} = \frac{dy}{-f_q} = \frac{dz}{-pf_p - qf_q} = \frac{dp}{f_x + pf_z} = \frac{dq}{f_y + qf_z}$$

$$\frac{dx}{-2p} = \frac{dy}{-y} = \frac{dz}{-2p^2 - qy} = \frac{dp}{-p} = \frac{dq}{0}.$$

$dq = 0 \Rightarrow q = a.$

Substituting $q = a$ in ①

$p^2 + ya = z$

$p = \sqrt{z - ay}$

$dz = p\,dx + q\,dy.$

$dz = \sqrt{z - ay}\,dx + a\,dy$

$$\frac{dz - a\,dy}{\sqrt{z - ay}} = dx$$

$2\sqrt{z - ay} = x + b$

$$\underline{\underline{4(z - ay) = (x + b)^2}}$$

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67. $p^2 - q^2 + (x+y)^2 = 0.$

$f = p^2 - q^2 + (x+y)^2$

$f_p = 2p$

$f_q = -2q$

$f_x = 2x + 2y.$

$f_y = 2x + 2y$

$f_z = 0.$

The S. eqns are;

$$\frac{dx}{-2p} = \frac{dy}{2q} = \frac{dz}{-2p^2+2q^2} = \frac{dp}{2x+2y} = \frac{dq}{2x+2y}$$

Considering the last 2 eqns we get;

$dp = dq$

$p = q + a.$

$(q+a)^2 - q^2 + (x+y)^2 = 0.$

$2qa + a^2 + (x+y)^2 = 0$

$$q = \frac{-a^2}{2a} - \frac{(x+y)^2}{2a}.$$

$$q = \frac{-a}{2} - \frac{(x+y)^2}{2a}$$

$dz = p\,dx + q\,dy.$

$$dz = \left(\frac{a}{2} - \frac{(x+y)^2}{2a}\right)dx - \left(\frac{(x+y)^2}{2a} + \frac{a}{2}\right)dy$$

68. $pxy + pq + qy = yz$

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$f = pxy + pq + qy - yz$

$f_p = xy + q$

$f_q = p + y$

$f_x = py.$

$f_y = px + q - z$

$f_z = -y.$

The subsidiary eqn is;

$$\frac{dx}{-(xy+q)} = \frac{dy}{-p-y} = \frac{dz}{-p(xy+q) - q(p+y)} = \frac{dp}{\cancel{py} + p\cancel{(-y)}} = \frac{dq}{px+q-z-yq}$$

$dp = 0$

$p = a$

$axy + aq + qy = yz.$

$$q = \frac{yz - axy}{(a+y)}$$

$dz = p\,dx + q\,dy.$

$$dz = a\,dx + \frac{(yz - axy)}{a+y}\,dy$$

$$\frac{dz - a\,dx}{z - ax} = \frac{y\,dy}{a+y}$$

$$d(z-ax) = \frac{a+y-a\,dy}{a+y}$$

$$= dy - \frac{a}{a+y}\,dy.$$

$$\log(z-ax) = y - a\log(a+y) + b.$$

$$\log(z-ax)(a+y)^a = y+b.$$

$$(z-ax)(a+y)^a = e^{y+b}$$

69. $(p^2+q^2)y = qz$

$$f = (p^2+q^2)y - qz$$

$$f_p = 2py$$

$$f_q = 2qy - z$$

$$f_x = 0.$$

$$f_y = p^2+q^2$$

$$f_z = -q$$

The subsidiary eqn;

$$\frac{dx}{-2py} = \frac{dy}{-2qy+z} = \frac{dz}{-p(2py)-q(2qy-z)} = \frac{dp}{0+-pq} = \frac{dq}{p^2+\cancel{q^2}\cancel{-q^2}}$$

$$\frac{dx}{-2py} = \frac{dy}{z-2qy} = \frac{dz}{-2p^2y-2q^2y+qz} = \frac{dp}{-pq} = \frac{dq}{p^2}$$

$$\frac{dp}{-pq} = \frac{dq}{p^2}$$

$$-p\,dp = q\,dq$$

$$-\frac{p^2}{2} = \frac{q^2}{2} - a/2 \Rightarrow \underline{\underline{a = p^2+q^2}}$$

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$$ay = qz$$

$$q = \frac{ay}{z}$$

$$dz = p\,dx + q\,dy.$$

$$dz = \sqrt{a-q^2}\,dx + \frac{ay}{z}\,dy.$$

$$dz = \sqrt{a - \frac{a^2y^2}{z^2}}\,dx + \frac{ay}{z}\,dy$$

$$dz = \sqrt{\frac{az^2 - a^2y^2}{z^2}}\,dx + \frac{ay}{z}\,dy.$$

$$dz = \frac{\sqrt{a}}{z}\sqrt{z^2 - ay^2}\,dx + \frac{ay}{z}\,dy$$

$$z\,dz = \sqrt{a}\sqrt{z^2-ay^2}\,dx + ay\,dy.$$

$$z\,dz - ay\,dy = \sqrt{a}\sqrt{z^2-ay^2}\,dx$$

$$\frac{z\,dz - ay\,dy}{\sqrt{z^2-ay^2}} = \sqrt{a}\,dx.$$

$$d\left(\sqrt{z^2-ay^2}\right) = \sqrt{a}\,dx$$

$$\sqrt{z^2-ay^2} = \sqrt{a}x + b$$

$$z^2 - ay^2 = (\sqrt{a}x + b)^2$$

10. $px + qy = pq.$

$f = px + qy - pq$

$f_p = x - q$

$f_q = y - p.$

$f_x = p$

$f_y = q$

$f_z = 0.$

The S. eqn is;

$$\frac{dx}{-x+q} = \frac{dy}{-y+p} = \frac{dz}{-p(x-q)-q(y-p)} = \frac{dp}{p+0} = \frac{dq}{q}$$

$$\frac{dp}{p} = \frac{dq}{q}$$

$$\log p = \log q + \log a.$$

$$a = \frac{p}{q}$$

$$p = aq$$

$$aqx + qy = aq^2.$$

$$ax + y = aq.$$

$$q = \frac{ax+y}{a.}$$

$$dz = aq\,dx + \frac{ax+y}{a}\,dy.$$

$$dz = \frac{a^2x + ay}{a}\,dx + \frac{ax+y}{a}\,dy$$

$$a\,dz = a^2x\,dx + y\,dy + a\,d(xy)$$

$$az = \frac{(ax+y)^2}{2} + b \qquad az = \frac{a^2x^2 + y^2 + 2axy}{2}.$$

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$$2(px+qy) = z(1-q^2)$$

\#

$$f = 2px + 2qy - z + zq^2$$

$$f_p = 2x$$

$$f_q = 2y + 2zq$$

$$f_x = 2p$$

$$f_y = 2q$$

$$f_z = -1 + q^2.$$

$$\frac{dx}{-2x} = \frac{dy}{-2y-2zq} = \frac{dz}{-2px-2yq-2zq^2} = \frac{dp}{2p+-p+q^2p} = \frac{dq}{2q-q+q^3}$$

$$\frac{dp}{p\cancel{(1+q^2)}} = \frac{dq}{q\cancel{(1+q^2)}}$$

$$\frac{dp}{p} = \frac{dq}{q}$$

$$\log p = \log q + \log a.$$

$$p/q = a.$$

$$\underline{\underline{p = aq}}$$

$$2aqx + 2qy = z(1-q^2).$$

$$(2ax + 2y)q = z - zq^2$$

$$\frac{dz}{-2px-2yq-2zq^2} = \frac{dz}{-2(px+yq+zq^2)}$$

$$= \frac{dz}{-2\left[\frac{z(1-q^2)}{2} + zq^2\right]} = \frac{dz}{-z(1+q^2)}$$

$$\frac{dz}{-z(\cancel{1+q^2})} = \frac{dp}{p(\cancel{1+q^2})}$$

$$-\log z = \log p - \log a$$

$$\log a = \log p + \log z$$

$$\underline{\underline{a = zp.}}$$

$$\frac{dz}{-z(\cancel{1+q^2})} = \frac{dq}{q(\cancel{1+q^2})}$$

$$-\log z = \log q - \log b.$$

$$\underline{\underline{b = zq}}$$

The soln is;

$$2\left(\frac{ax}{z} + \frac{b}{z}y\right) = z\left(1 - \frac{b^2}{z^2}\right)$$

$$2ax + 2by = z^2 - b^2$$

$$\underline{\underline{z^2 = b^2 + 2(ax+by)}}$$

Imp

73. $z^2 = pqxy$

$f = pqxy - z^2$

$f_p = qxy$

$f_q = pxy$

$f_x = pqy$

$f_y = pqx$

$f_z = -2z$

$$\frac{dx}{-qxy} = \frac{dy}{-pxy} = \frac{dz}{-pqxy-qpxy} = \frac{dp}{pqy+-2zp} = \frac{dq}{pqx-2zq}$$

$$\frac{dx}{qxy} = \frac{dy}{pxy} = \frac{dz}{2pqxy} = \frac{dp}{2zp-pqy} = \frac{dq}{2zq-pqx}.$$

$$\frac{p\,dx + x\,dp}{\cancel{pqxy} + 2pzx - \cancel{pqxy}} = \frac{q\,dy + y\,dq}{-\cancel{pqxy} + 2qzy + \cancel{pqxy}}$$

$$\frac{p\,dx + x\,dp}{\cancel{2}p\cancel{z}x} = \frac{q\,dy + y\,dq}{\cancel{2}q\cancel{z}y}$$

$$\log(px) = \log(qy) + \log a.$$

$$a = \frac{px}{qy}$$

$$p = \frac{aqy}{x}$$

$$z^2 = pqxy.$$

$$z^2 = \frac{aq^2y^2x}{x}$$

$$z^2 = aq^2y^2$$

$$\underline{\underline{q = \frac{z}{y\sqrt{a}}}} \qquad p = \frac{ay}{x} \times \frac{z}{y\sqrt{a}} \Rightarrow \underline{\underline{\frac{\sqrt{a}\,z}{x}}}$$

$$p\,dx + q\,dy = dz$$

$$dz = \frac{\sqrt{a}\,z}{x}dx + \frac{z}{y\sqrt{a}}dy$$

$$\frac{dz}{z} = \sqrt{a}\frac{dx}{x} + \frac{dy}{y\sqrt{a}}$$

$$\log z = \sqrt{a}\log x + \frac{1}{\sqrt{a}}\log y + \log b$$

$$\sqrt{a}\log z = a\log x + \log y + \sqrt{a}\log b$$

$$\log z^{\sqrt{a}} = \log x^{a} + \log y + \log b^{\sqrt{a}}$$

$$z^{\sqrt{a}} = x^{a} y b^{\sqrt{a}}$$

$$\left(\frac{z}{b}\right)^{\sqrt{a}} = x^{a} y.$$

74. $p^2x + q^2y = z$

$$f = p^2x + q^2y - z$$

$$f_p = 2px$$

$$f_q = 2qy$$

$$f_x = p^2$$

$$f_y = q^2$$

$$f_z = -1$$

$$\frac{dx}{-2px} = \frac{dy}{-2qy} = \frac{dz}{-2p^2x - 2q^2y} = \frac{dp}{p^2 - p} = \frac{dq}{q^2 - q}$$

$$\frac{dx}{-2\cancel{p}x} = \frac{dp}{\cancel{p}(p-1)}$$

$$\frac{-1}{2}\log x = \log(p-1) - \log a.$$

$$a^2 = (p-1)^2 x. \qquad p = \frac{a}{\sqrt{x}} + 1$$

$$\frac{dy}{-2\cancel{q}y} = \frac{dq}{\cancel{q}(q-1)}$$

$$b^2 = (q-1)^2 y.$$

$$q = \frac{b+\sqrt{y}}{\sqrt{y}}$$

$dz = p\,dx + q\,dy.$

$dz = \frac{a\,dx}{\sqrt{x}} + \frac{\sqrt{x}\,dx}{\sqrt{x}} + \frac{b}{\sqrt{y}}dy + \frac{\sqrt{y}}{\sqrt{y}}dy.$

$\frac{1}{\sqrt{x}}$

$\frac{\sqrt{x}}{1} = \frac{1}{2\sqrt{x}}$

$dz = \frac{a}{\sqrt{x}}dx + \frac{b}{\sqrt{y}}dy + dx + dy.$

$\therefore z = 2a\sqrt{x} + 2b\sqrt{y} + x + y + c$

6/11/2020

Module 1

Chapter 3

Application of PDE

Method of separation of variables.

Using the Method of separation of variables solve the following.

75. $\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial t} + u$, $u(x,0) = 6e^{-3x}$

$$\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial t} + u \quad \text{——} (1).$$

here u is a fn of x & t.

Assume $u = XT$

where X is a fn of x alone

T is a fn of t alone as a trial soln of eqn (1).

Now; $u = XT$

$$\Rightarrow \frac{\partial u}{\partial x} = X'T$$

$$\frac{\partial u}{\partial t} = XT'$$

$$(1) \Rightarrow X'T = 2XT' + XT$$

$$X'T = X(2T' + T)$$

$$\frac{X'}{X} = \frac{(2T'+T)}{T}$$

In the above eqn LHS is a fn of x and RHS a fn of t alone.

$\therefore$ We have; $\frac{X'}{X} = \frac{2T'+T}{T} = k$

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$$\frac{X'}{X} = k$$

$$\log X = kx + \log a.$$

$$\frac{X}{a} = e^{kx}.$$

$$X = ae^{kx}$$

$$\frac{2T'}{T} = k-1$$

$$\frac{T'}{T} = \frac{k-1}{2}$$

$$\log T = \left(\frac{k-1}{2}\right)t + \log b$$

$$T = be^{(k-1)t/2}$$

$$X = ae^{kx}, \quad T = be^{(k-1)t/2}$$

$$u = XT$$

$$= abe^{kx + (k-1)t/2}$$

$$u(x,0) = 6e^{-3x}$$

$$abe^{kx} = 6e^{-3x}$$

$$ab = 6$$

$$k = -3$$

$$u = 6e^{(-3x - 2t)}$$

is the particular soln.

76. $4u_x + u_y = 3u \qquad u = e^{-5y}, \; x = 0$

$$4\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 3u \quad \text{——— (1)}$$

here u is a fn of x and y

Assume $u = XY$

where X is a fn of x alone

Y is a fn of y alone, as a trial soln of (1)

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Now ; $u = XY$

$$\frac{\partial u}{\partial x} = X'Y$$

$$\frac{\partial u}{\partial y} = XY'$$

$$4X'Y + XY' = 3XY$$

$$4X'Y = X(3Y - Y')$$

$$\frac{X'}{X} = \frac{3Y - Y'}{4Y} = k$$

$$\log X = kx + \log a.$$

$$\frac{X}{a} = e^{kx}.$$

$$\underline{\underline{X = ae^{kx}}}$$

$$\frac{3Y - Y'}{4Y} = k$$

$$\frac{3}{4} - \frac{Y'}{4Y} = k$$

$$\frac{Y'}{4Y} = \frac{3}{4} - k$$

$$\frac{Y'}{Y} = 3 - 4k$$

$$\log Y = (3 - 4k)y + \log b.$$

$$\underline{\underline{Y = be^{(3-4k)y}}}$$

$$\therefore \; u = XY = ae^{kx} \times be^{(3-4k)y}$$

$$\underline{\underline{u = abe^{kx + (3-4k)y}}}.$$ gen. soln.

$x = 0 \Rightarrow u = e^{-5y}$

$$e^{-5y} = abe^{(3-4k)y}$$

$ab = 1$ $\qquad$ $3 - 4k = -5$

$-4k = -8$ $\qquad$ $k = 2$

$$u = e^{2x-5y} \quad \text{(P. soln)}$$

71. $3u_x + 2u_y = 0$

$$u(x,0) = 4e^{-x}$$

$$3\frac{\partial u}{\partial x} + 2\frac{\partial u}{\partial y} = 0. \quad \text{——— (1)}$$

here u is a fn of x & y.

Assume $U = XY$

where ; X is a fn of x alone

Y is a fn of y alone

$$\frac{\partial u}{\partial x} = X'Y \qquad \frac{\partial u}{\partial y} = XY'$$

$$3X'Y + 2XY' = 0$$

$$3X'Y = -2XY'$$

$$\frac{3X'}{X} = -\frac{2Y'}{Y} = k$$

$$\log X = \frac{kx}{3} + \log a.$$

$$\frac{X}{a} = e^{kx/3}$$

$$X = ae^{kx/3}$$

$$-\frac{2Y'}{Y} = k$$

$$\log Y = -ky/2 + \log b.$$

$$\frac{Y}{b} = e^{-ky/2}$$

$$u = abe^{k(x/3 - y/2)}$$

$$4e^{-x} = abe^{k/3}$$

$ab = 4 \quad k = -3.$

$$u = 4e^{-x + 3y/2}$$

75. $$\frac{\partial^2 z}{\partial x^2} + 4\frac{\partial^2 z}{\partial y^2} = 0$$

here ; z is a fn of x & y.

$Z = XY.$ X is a fn of x alone
Y is a fn of y alone.

$$\frac{\partial^2 z}{\partial x^2} = X''Y$$

$$\frac{\partial^2 z}{\partial y^2} = XY''$$

$$X''Y + 4XY'' = 0$$

$$X''Y = -4XY''$$

$$\frac{X''}{X} = -4\frac{Y''}{Y} = k^2$$

$$\frac{X''}{X} = k^2$$

$$X'' - k^2X = 0.$$

which is an ODE

Thus; $m^2 - k^2 = 0.$

$m^2 = k^2$

$m = \pm k$

$$X = c_1e^{kx} + c_2e^{-kx}$$

$$\frac{Y''}{Y} = \frac{-k^2}{4}$$

$$4Y'' + k^2Y = 0$$

$$4m^2 + k^2 = 0.$$

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$m^2 = -\frac{k^2}{4}$ $\quad m = \pm i k/2$

$Y = c_3 \cos(k/2)\, y + c_4 \sin(k/2)\, y$

$Z = XY$

$= \left(c_1 e^{kx} + c_2 e^{-kx}\right)\left(c_3 \cos(k/2) y + c_4 \sin(k/2) y\right)$

due

79. $\frac{\partial u}{\partial x} - \frac{2\partial u}{\partial t} = u$

$u(x,0) = 4e^{-3x}$ $\quad 4e^{-3x-2t}$

$\frac{\partial u}{\partial x} - \frac{2\partial u}{\partial t} = u$ —— ①.

here u is a fn of x and t

$u = XT$. $\quad$ X is a fn of x alone

T is a fn of t alone

trial soln of ①

Now, $u = XT$

$\frac{\partial u}{\partial x} = X'T$ $\quad \frac{\partial u}{\partial t} = XT'$

$X'T - 2XT' = \cancel{X}T$

$X'T = XT + 2XT'$

$X'T = X(2T' + T)$

$\frac{X'}{X} = \frac{2T' + T}{T}$

$\frac{X'}{X} = k$

$\log X = kx + \log a$

$\frac{X}{a} = e^{kx}$ $\quad X = ae^{kx}$

$$\frac{2T'+T}{T} = k$$

$$\frac{2T'}{T} + \frac{T}{T} = k.$$

$$\frac{2T'}{T} = k-1$$

$$2\frac{T'}{T} = k-1$$

$$\frac{T'}{T} = \frac{k-1}{2}$$

$$\log T = \frac{(k-1)}{2}t + \log b.$$

$$\frac{T}{b} = e^{(k-1)t/2}$$

$$T = be^{(k-1)t/2}$$

$u = XT$

$$u = abe^{kx+(k-1)t/2}$$

$$4e^{-3x} = abe^{kx}$$

$ab = 4 \qquad k = -3.$

$$\underline{\underline{u = 4e^{-3x-2t}}}$$

due
So. $x u_x - 2y u_y = 0$

$$x\frac{\partial u}{\partial x} - 2y\frac{\partial u}{\partial y} = 0$$

here u is a fn of x and y

$u = XY \qquad$ X is a fn of x alone
Y is a fn of y alone as a trial soln of (1)

Now; $u = XY$

$$\frac{\partial u}{\partial x} = X'Y \qquad \frac{\partial u}{\partial y} = XY'$$

$$xX'Y - 2yXY' = 0.$$

$$xX'Y = 2yXY'$$

$$x X' Y = \cancel{2x}y X Y'$$

$$x \frac{X'}{X} = 2y \frac{Y'}{Y}$$

$$\frac{X'}{X} = \frac{k}{x}.$$

$$\log X = k \log x + \log a.$$

$$\log X - \log a = k \log x.$$

$$\frac{X}{a} = x^k$$

$$\underline{\underline{X = a x^k}}$$

$$2y \frac{Y'}{Y} = k.$$

$$\frac{Y'}{Y} = \frac{k}{2y}.$$

$$\log Y = \frac{k}{2} \log y + \log b.$$

$$\frac{Y}{b} = y^{k/2}$$

$$\underline{\underline{Y = b y^{k/2}}}$$

$$\underline{\underline{u = a b x^k y^{k/2}}}$$