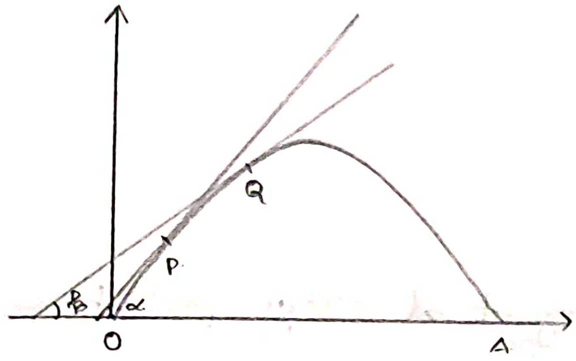
MODULE 2

Wave Equation.

Derivation of One dimensional Wave Equation

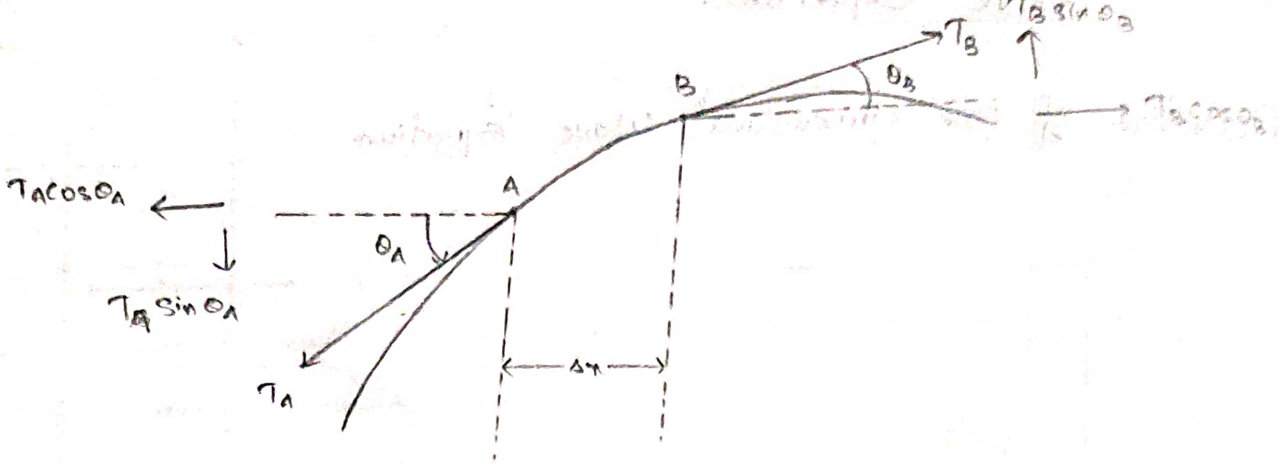


Consider an elastic string of length l tightly stretched between two points O and A. The string is then disturbed at time $t=0$ is released and allowed to vibrate.

We make the following assumptions.

1. The string is perfectly elastic; that is no resistance to bending.
2. The tension caused by stretching is large compared to the weight of the string. So the force due to gravity is negligible.
3. Mass per unit length is constant (m).
4. There is only transverse vibration, that is, the motion takes place entirely in one plane and each point on the string moves at right angles to the equilibrium position.
5. The displacement $y(x,t)$ is small compared to the length, so stretching of the string is negligible.

Consider the forces acting on a small portion (AB) of the string.



⇒ Tension is tangential to the string at every point because string is perfectly elastic.

T_A and T_B denotes the tension at end points A and B respectively. θ_A and θ_B are the angles between the string and x-axis at A and B. Because of transverse motion, net horizontal force due to tension is zero.

$$T_B\cos\theta_B - T_A\cos\theta_A = 0.$$

$$T_A\cos\theta_A = T_B\cos\theta_B = T$$

$$T_A = \frac{T}{\cos\theta_A} \qquad T_B = \frac{T}{\cos\theta_B} \quad \text{--- (1)}$$

Now assume that string is moving in upward direction

Vertical components of T_A and T_B are $T_A\sin\theta_A$ and $T_B\sin\theta_B$ respectively

Net vertical upward force on AB =

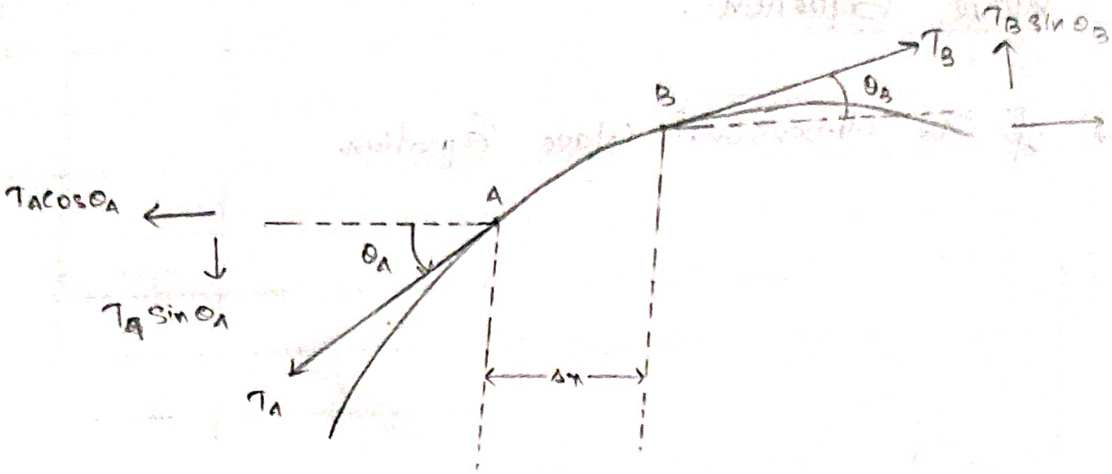
$$F = T_B\sin\theta_B - T_A\sin\theta_A$$

$$= \frac{T}{\cos\theta_B}\sin\theta_B - \frac{T}{\cos\theta_A}\sin\theta_A$$

$$= T(\tan\theta_B - \tan\theta_A).$$

$$= T\left(\frac{\partial y}{\partial x}\bigg|_{\Delta x + x} - \frac{\partial y}{\partial x}\bigg|_{x}\right) \text{--- (2)}$$

Consider the forces acting on a small portion (AB) of the string.



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$$= T(\tan\theta_B - \tan\theta_A).$$

$$= T\left(\frac{\partial y}{\partial x}\bigg|_{\Delta x + x} - \frac{\partial y}{\partial x}\bigg|_{x}\right) \quad \text{—— ②}$$

By applying Newton's second law, the net vertical force on AB is mass times acceleration.

$$F = m\Delta x \frac{\partial y^2}{\partial t^2} \quad \text{——} (3)$$

from (2) and (3)

$$m\Delta x \frac{\partial y^2}{\partial t^2} = T\left(\frac{\partial y}{\partial x}\Big|_{x+\Delta x} - \frac{\partial y}{\partial x}\Big|_{x}\right).$$

$$\frac{\partial y^2}{\partial t^2} = \frac{T}{m}\left[\frac{\frac{\partial y}{\partial x}\Big|_{x+\Delta x} - \frac{\partial y}{\partial x}\Big|_{x}}{\Delta x}\right]$$

If we let $\Delta x \longrightarrow 0$, then we get

$$\frac{\partial y^2}{\partial t^2} = \frac{T}{m}\frac{\partial^2 y}{\partial x^2}$$

Hence the de

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2} \quad \text{——} (4)$$

Where $a^2 = \frac{T}{m}$ is called the stiffness of the material.

(4) is known as the one dimensional wave equation.

Solution of One dimensional Wave Equation.

Solution using method of separation of variables.

One dimensional wave equation is

$$\frac{\partial y^2}{\partial t^2} = a^2 \frac{\partial y^2}{\partial x^2} \quad \text{—} (1).$$

Assume that

$$y(x,t) = X(x)T(t).$$

then

$$\frac{\partial^2 y}{\partial x^2} = X''T \qquad \frac{\partial^2 y}{\partial t^2} = XT''$$

Substituting these values in (1).

$$XT'' = a^2 X''T$$

$$\frac{X''}{X} = \frac{1}{a^2}\frac{T''}{T} = k.$$

$$\frac{X''}{X} = k. \qquad \frac{1}{a^2}\frac{T''}{T} = k.$$

$$\frac{1}{a^2} T'' = kT$$

$$X'' = kX \qquad T'' = ka^2T$$

$$T'' - ka^2T = 0$$

$$X'' - kX = 0. \qquad [D^2 - ka^2]T = 0$$

$$[D^2 - k]X = 0$$

The ~~above~~ solutions of the above equations depends on the value of k.

Depending on the value of the real constant k, we have the following three possible solutions.

Case 1:

When k is positive

Let $k = \lambda^2$

where λ is a real number

$$m^2 - \lambda^2 = 0 \qquad m^2 - a\lambda^2 = 0$$

$$\Rightarrow m = \pm\lambda \qquad m = \pm a\lambda$$

$$X = C_1 e^{\lambda x} + C_2 e^{-\lambda x} \qquad T = C_3 e^{a\lambda t} + C_4 e^{-a\lambda t}$$

$$y(x,t) = (C_1 e^{\lambda x} + C_2 e^{-\lambda x})(C_3 e^{a\lambda t} + C_4 e^{-a\lambda t}).$$

Case 2:

When k is negative

Let $k = -\lambda^2$

$$m^2 + \lambda^2 = 0 \qquad m^2 + a\lambda^2 = 0.$$

$$\Rightarrow m = \pm i\lambda \qquad m = \pm ia\lambda$$

$$X = C_5 \cos\lambda x + C_6 \sin\lambda x \qquad T = C_7 \cos\lambda at + C_8 \sin\lambda at$$

$$\therefore y(x,t) = (C_5 \cos\lambda x + C_6 \sin\lambda x)(C_7 \cos\lambda at + C_8 \sin\lambda at)$$

Case 3:

When $k = 0$

$$D^2 X = 0 \qquad D^2 T = 0.$$

$$m^2 = 0 \qquad m^2 = 0.$$

$$X = C_9 + C_{10} x \qquad T = C_{11} + C_{12} t.$$

$$y(x,t) = (C_9 + C_{10} x)(C_{11} + C_{12} t).$$

Of these three possible solution the most suitable solution is

$$y = (A\cos\lambda x + B\sin\lambda x)(C\cos\lambda at + D\sin\lambda at).$$

Problems.

1. A tightly stretched flexible string has its ends fixed at $x=0$ and $x=l$. At time $t=0$, the string is given a shape defined by $f(x) = \mu x(l-x)$, where μ is a constant and then released. Find the displacement of any point x of the string at any time.

Soln, Wave Equation is

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}.$$

Boundary conditions are.

i) $y(0,t) = 0$

ii) $y(l,t) = 0.$

iii) $\frac{\partial y}{\partial t}(x,0) = 0$

iv) $y(x,0) = \mu x(l-x)$
$= \mu(lx - x^2)$

Suitable solution of Wave Equation is

$$y(x,t) = [A\cos\lambda x + B\sin\lambda x][C\cos\lambda at + D\sin\lambda at] \quad \text{--- (1)}$$

Applying Boundary conditions in (1)

$y(0,t) = 0.$

$$[A\cos 0 + B\sin 0].[C\cos\lambda at + D\sin\lambda at] = 0$$

$$A[C\cos\lambda at + D\sin\lambda at] = 0.$$

$$C\cos\lambda at + D\sin\lambda at \neq 0$$

$\therefore A = 0.$

New suitable solution is

$$y(x,t) = B\sin\lambda x\,[C\cos\lambda at + D\sin\lambda at] \quad \text{--- (2)}.$$

Applying Boundary Condition (ii) in ②

$y(l,t) = 0.$

$B\sin\lambda l\,[C\cos\lambda at + D\sin\lambda at] = 0.$

$B\sin\lambda l = 0.$

$\lambda l = n\pi$

$\lambda = \frac{n\pi}{l}.$

$\therefore$ New Suitable Solution is

$$y(x,t) = B\sin\frac{n\pi}{l}x\left[C\cos\frac{n\pi}{l}at + D\sin\frac{n\pi}{l}at\right]$$

$$= \sin\frac{n\pi}{l}x\left[E\cos\frac{n\pi}{l}at + F\sin\frac{n\pi}{l}at\right]$$

Partially differentiating w.r.to t

$$\frac{\partial y}{\partial t}(x,t) = \sin\frac{n\pi}{l}x\left[-E\frac{n\pi}{l}a\sin\frac{n\pi}{l}at + F\frac{n\pi}{l}a\cos\frac{n\pi}{l}at\right] \quad ④$$

Applying Boundary conditions (iii) in ④.

$$\frac{\partial y}{\partial t}(x,0) = 0.$$

$$\sin\frac{n\pi x}{l}\left[0 + F\frac{n\pi a}{l}\right] = 0.$$

$F = 0.$

New Suitable Solution is

$$y(x,t) = \sin\frac{n\pi x}{l}\left[E\cos\frac{n\pi}{l}at\right] \quad ⑤.$$

Most general solution is

$$y(x,t) = \sum_{n=1}^{\infty} E_n\sin\frac{n\pi x}{l}\cos\frac{n\pi}{l}at.$$

Applying Boundary Condition (iv) in ⑤.

$y(x,0) = \mu(lx - x^2).$

$$\sum_{n=1}^{\infty} A_n\sin\frac{\pi n x}{l} = \mu(lx - x^2) \quad ⑥.$$

(6) is Half Range Fourier Sine Series

$$E_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{l}\int_0^l \mu(lx-x^2)\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{2\mu}{l}\left[(lx-x^2)\left[\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right] + (l-2x)\left[\frac{\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right] + -2\left[\frac{\cos\frac{n\pi x}{l}}{\frac{n^3\pi^3}{l^3}}\right]\right]_0^l$$

$$= \frac{2\mu}{l}\left[0+0+ -2\times\frac{l^3}{n^3\pi^3}\cos n\pi\right] - \left[0+0-2\times\frac{l^3}{n^3\pi^3}\right]$$

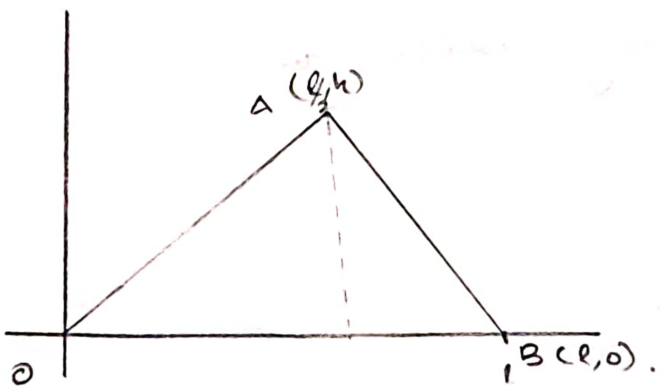
$$= \frac{2\mu}{l}\times\frac{2l^3}{n^3\pi^3}\left[-(-1)^n+1\right]$$

$$= \frac{4\mu l^2}{n^3\pi^3}\left[1-(-1)^n\right]$$

$$\therefore y(x,t) = \sum_{n=1}^{\infty}\frac{4\mu l^2}{n^3\pi^3}\left[1-(-1)^n\right]\sin\frac{n\pi x}{l}\cos\frac{n\pi}{l}at.$$

2. A string of length l is fastened at both ends. the midpoint of the string is taken to a height h and then released from rest in the position. Find the displacement function

Soln.



Equation of OA is.

$$\frac{y-0}{0-h} = \frac{x-0}{0-l/2}$$

$$y/h = \frac{x}{l/2}$$

$$y = \frac{2hx}{l}, \quad 0<x<l/2$$

the equation of AB is

$$\frac{y-h}{h-0} = \frac{x-l/2}{l/2-l}$$

$$\frac{y-h}{h} = \frac{x-l/2}{-l/2}$$

$$y-h = \frac{-2h}{l}(x-l/2)$$

$$y = -\frac{2hx}{l} + 2h$$

$$y = \frac{2h(l-x)}{l}, \quad l/2 \le x \le l.$$

∴ ~~the initial conditions are~~

$$y(x,0) = f(x) = \begin{cases} \frac{2hx}{l} & 0 \le x < l/2 \\ \frac{2h}{l}(l-x) & l/2 \le x \le l. \end{cases}$$

$$\frac{\partial y}{\partial t}\cancel{(x,0)} = \cancel{0}.$$

~~General soln of satisfying the above condition is.~~

~~$y(x,t) =$~~ Boundary Conditions are

i) $y(0,t)=0$

ii) $y(l,t)=0.$

iii) $\frac{\partial y}{\partial t}(x,0) = 0.$

iv) $y(x,0)=f(x).$

Suitable Solution is

$$y(x,t) = [C_1\cos\lambda x + C_2\sin\lambda x][C_3 \cdots \lambda ct + C_4 \cdots \lambda ct] \quad \text{—①}$$

Applying Boundary Condition ① in ①.

$$y(0,t)=0.$$

$$C_1[C_3\cos\lambda at + C_4\sin\lambda at]=0.$$

$$C_1=0$$

New suitable solution is.

$$y(x,t) = C_2 \sin\lambda x\,[C_3\cos\lambda at + C_4\sin\lambda at] \quad \text{--- (2)}$$

Applying Boundary condition (ii) in eq (2)

$$y(l,t) = 0.$$

$$C_2 \sin\lambda l\,[C_3\cos\lambda at + C_4\sin\lambda at]$$

$$\sin\lambda l = 0.$$

$$\lambda l = n\pi$$

$$\lambda = \frac{n\pi}{l}.$$

New suitable solution is

$$y(x,t) = C_2 \sin\frac{n\pi}{l}x\,[C_3\cos\lambda at + C_4\sin\lambda at]$$

$$= \sin\frac{n\pi}{l}x\,\left[A\cos\frac{n\pi at}{l} + B\sin\frac{n\pi at}{l}\right] \quad \text{--- (3).}$$

differentiating (3) partially w.r.t t

$$\frac{\partial y}{\partial t}(x,t) = \sin\frac{n\pi}{l}x\left[-A\frac{n\pi a}{l}\sin\frac{n\pi at}{l} + B\frac{n\pi a}{l}\cos\frac{n\pi at}{l}\right].$$

$$\frac{\partial y}{\partial t}(x,0) = 0.$$

$$\sin\frac{n\pi}{l}x\left[B\frac{n\pi a}{l}\right] = 0$$

$$B = 0.$$

$\therefore$ New suitable solution is

$$y(x,t) = A\sin\frac{n\pi x}{l}\cos\frac{n\pi at}{l}.$$

$\therefore$ Most general solution is

$$y(x,t) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{l}\cos\frac{n\pi at}{l} \quad \text{--- (5).}$$

Applying Boundary condition. iv in (5).

$$y(x,0) = f(x).$$

$$\sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{l} = f(x) \quad \text{--- (6).}$$

(6) is a Half range sine fourier series

$$A_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi}{l}x\,dx$$

$$= \frac{2}{l}\left[\int_0^{l/2}\frac{2h}{l}x\sin\frac{n\pi x}{l}dx + \int_{l/2}^{l}\frac{2h}{l}(l-x)\sin\frac{n\pi x}{l}dx\right]$$

$$= \frac{2}{l}\times\frac{2h}{l}\left[\int_0^{l/2}x\sin\frac{n\pi x}{l}dx + \int_{l/2}^{l}(l-x)\sin\frac{n\pi x}{l}dx\right]$$

$$= \frac{4h}{l^2}\left[x\left[\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right] + \left[\frac{\sin\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)^2}\right]\right]_0^{l/2} +$$

$$\left[(l-x)\left[\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right] - (-1)\left(-\frac{\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right)\right]_{l/2}^{l}$$

$$= \frac{4h}{l^2}\left[-\frac{l}{2}\times\frac{l}{n\pi}\cos\frac{n\pi}{2} + \frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right] - [0+0] +$$

$$[0+0] - \left[-\frac{l}{2}\times\frac{l}{n\pi}\times\cos\frac{n\pi}{2}\right] - \frac{l^2}{n^2\pi^2}\sin\frac{n\pi\, l/2}{l}\Bigg]$$

$$= \frac{4h}{l^2}\left[\frac{2\times l^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right]$$

$$= \frac{8h}{n^2\pi^2}\sin\frac{n\pi}{2}$$

$\therefore$ the soln is

$$y(x,t) = \sum_{n=1}^{\infty}\frac{8h}{n^2\pi^2}\sin\frac{n\pi}{2}\sin\frac{n\pi x}{2}\cos\frac{n\pi at}{l}.$$

3. A string of length l is initially at rest in equilibrium position and each of its points is given the velocity $kx(l-x)$, $0<x<l$. Determine the displacement function $y(x,t)$.

Sol^n The wave equation is

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$$

The boundary conditions are

i) $y(0,t) = 0$.

ii) $y(l,t) = 0$.

iii) $y(x,0) = 0$.

iv) $\frac{\partial y}{\partial t}(x,0) = V_0 \sin^3\left(\frac{\pi x}{l}\right)$.

Suitable solution is

$$y(x,t) = \left[C_1 \cos \lambda x + C_2 \sin \lambda x\right]\left[C_3 \cos \lambda at + C_4 \sin \lambda at\right] \quad \text{--- (1)}$$

Applying Boundary condition (i) in eq (1).

$$y(0,t) = 0.$$

$$C_1\left[C_3 \cos \lambda at + C_4 \sin \lambda at\right] = 0.$$

$$C_1 = 0.$$

Now suitable solution is

$$y(x,t) = C_2 \sin \lambda x\left[C_3 \cos \lambda at + C_4 \sin \lambda at\right] \quad \text{--- (2)}$$

Applying Boundary condition ii in (2).

$$y(l,t) = 0.$$

$$C_2 \sin \lambda l\left[C_3 \cos \lambda at + C_4 \sin \lambda at\right] = 0$$

$$\sin \lambda l = 0$$

$$\lambda l = n\pi$$

$$\lambda = \frac{n\pi}{l}.$$

y.f.s.m

Next Suitable Solution is

$$y(x,t) = C_2 \sin \frac{n\pi}{l} x \left[C_3 \cos \frac{n\pi}{l} at + C_4 \sin \frac{n\pi}{l} at \right]$$

$$= \sin \frac{n\pi}{l} x \left[A \cos \frac{n\pi}{l} at + B \sin \frac{n\pi}{l} at \right] \quad \text{--- (3)}$$

Applying Boundary Condition iii in (3).

$$y(x,0) = 0.$$

$$\sin \frac{n\pi}{l} x \times A = 0.$$

$$A = 0.$$

Next Suitable Soln is

$$y(x,t) = \sin \frac{n\pi x}{l} \, B \sin \frac{n\pi at}{l} \quad \text{--- (4)}.$$

Most general Solution is

$$y(x,t) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{l} \sin \frac{n\pi at}{l} \quad \text{--- (5)}.$$

differentiating (5) p.w.r.t to t

$$\frac{\partial y}{\partial t} = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{l} \times \frac{n\pi a}{l} \cos \frac{n\pi at}{l}.$$

$$\frac{\partial y}{\partial t}(x,t) = \sum_{n=1}^{\infty} B_n \frac{n\pi a}{l} \sin \frac{n\pi}{l} x \cos \frac{n\pi at}{l}$$

$$B_n \frac{n\pi a}{l} = A_n.$$

$$\frac{\partial y}{\partial t}(x,t) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{l} \cos \frac{n\pi at}{l}. \quad \text{--- (6)}.$$

Applying Boundary condition iv in (6).

$$\cancel{\frac{\partial y}{\partial t}(x,0) = V_0 \sin^3\left(\frac{\pi x}{l}\right)}.$$

$$\cancel{\sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{l} = V_0 \sin^3\left(\frac{\pi x}{l}\right)}. \quad \text{--- (7)}.$$

(7) ~~is Half range~~ ~~Fourier~~ ~~Sine~~ ~~Series.~~

$$\frac{\partial y}{\partial t}(x,0) = k\,(lx - x^2)$$

$$\sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{l} = k\,(lx - x^2) \quad \text{——— (7)}$$

(7) is Half Range fourier Sine Series.

$$A_n = \frac{2}{l}\int_0^l f(x)\,\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{l}\int_0^l k(lx - x^2)\,\sin\frac{n\pi x}{l}\,dx.$$

$$= \frac{2k}{l}\left[(lx - x^2)\left[\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right] + (l-2x)\left[\frac{\sin\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)^2}\right] + -2\,\frac{\cos\frac{n\pi x}{l}}{\frac{n^3\pi^3}{l^3}}\right]_0^l$$

$$= \frac{2k}{l}\left[\left[0 - 0 - 2\,\frac{l^3 (-1)^n}{n^3\pi^3}\right] - \left[0 + 0 - 2\,\frac{l^3}{n^3\pi^3}\right]\right]$$

$$= \cancel{\frac{2k}{l}}\Big[\cancel{-2}\ \frac{2k}{l} \times \frac{2l^3}{n^3\pi^3}\left[-(-1)^n + 1\right].$$

$$= \frac{4kl^2}{n^3\pi^3}\,(1 - (-1)^n).$$

$$\frac{n\pi a}{l}\,B_n = A_n$$

$$\frac{n\pi a}{l}\,B_n = \frac{4kl^2}{n^3\pi^3}\,(1 - (-1)^n).$$

$$B_n = \frac{4kl^3}{n^4\pi^4 a}\,(1 - (-1)^n).$$

D'Alembert's Solution of Wave Equation.

Wave Equation is given by

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$$

Let $u = x + at$ and $v = x - at$ are independent variables

$$y = f(u, v).$$

differentiating u and v w.r.t to x and t

$$\frac{\partial u}{\partial x} = 1. \qquad \frac{\partial v}{\partial x} = 1.$$

$$\frac{\partial u}{\partial t} = a \qquad \frac{\partial v}{\partial t} = -a$$

$$\frac{\partial y}{\partial x} = \frac{\partial y}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial y}{\partial v}\frac{\partial v}{\partial x}$$

$$= \frac{\partial y}{\partial u} \times 1 + \frac{\partial y}{\partial v}(1).$$

$$\frac{\partial y}{\partial x} = \frac{\partial y}{\partial u} + \frac{\partial y}{\partial v} \quad \text{--- (1).}$$

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial u} + \frac{\partial}{\partial v} \quad \text{--- (2).}$$

$$\frac{\partial^2 y}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial y}{\partial x}\right).$$

$$= \frac{\partial^2 y}{\partial u^2} + \frac{\partial^2 y}{\partial u \partial v} + \frac{\partial^2 y}{\partial v \partial u} + \frac{\partial^2 y}{\partial v^2}$$

$$\frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 y}{\partial u^2} + 2\frac{\partial^2 y}{\partial u \partial v} + \frac{\partial^2 y}{\partial v^2} \quad \text{--- (3).}$$

$$\frac{\partial y}{\partial t} = \frac{\partial y}{\partial u}\frac{\partial u}{\partial t} + \frac{\partial y}{\partial v}\frac{\partial v}{\partial t}$$

$$= a\frac{\partial y}{\partial u} - a\frac{\partial y}{\partial v}.$$

$$\frac{\partial y}{\partial t} = a\left[\frac{\partial y}{\partial u} - \frac{\partial y}{\partial v}\right].$$

$$\frac{\partial}{\partial t} = a\left[\frac{\partial}{\partial u} - \frac{\partial}{\partial v}\right]$$

$$\frac{\partial^2 y}{\partial t^2} = \frac{\partial}{\partial t}\left[\frac{\partial y}{\partial t}\right]$$

$$= a\left[\frac{\partial}{\partial u} - \frac{\partial}{\partial v}\right] a\left[\frac{\partial y}{\partial u} - \frac{\partial y}{\partial v}\right]$$

$$= a^2\left[\frac{\partial^2 y}{\partial u^2} - \frac{\partial^2 y}{\partial u \partial v} - \frac{\partial^2 y}{\partial u \partial v} + \frac{\partial^2 y}{\partial v^2}\right]$$

$$= a^2\left[\frac{\partial^2 y}{\partial u^2} - 2\frac{\partial^2 y}{\partial u \partial v} + \frac{\partial^2 y}{\partial v^2}\right].$$

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$$

$$\cancel{a^2}\left[\cancel{\frac{\partial^2 y}{\partial u^2}} - 2\frac{\partial^2 y}{\partial u \partial v} + \cancel{\frac{\partial^2 y}{\partial v^2}}\right] = \cancel{a^2}\left[\cancel{\frac{\partial^2 y}{\partial u^2}} + 2\frac{\partial^2 y}{\partial u \partial v} + \cancel{\frac{\partial^2 y}{\partial v^2}}\right]$$

$$4\frac{\partial^2 y}{\partial u \partial v} = 0.$$

$$\frac{\partial^2 y}{\partial u \partial v} = 0$$

integrating w.r.t v.

$$\frac{\partial y}{\partial u} = h(u).$$

integrating w.r.t u.

$$y = \int h(u)\, du,$$

$$y = H(u) + g(v), \quad \text{—— (I)}.$$

$$y = H(x + at) + g(x - at)$$

This is the general solution of above eqn.

$$y(x,t) = H(x+at) + g(x-at) \quad \text{--- (8)}$$

To determine H and g.

Suppose $y(x,0) = f(x)$

$$\frac{\partial y}{\partial t}(x,0) = 0.$$

differentiating (8) partially w.r.to t.

$$\frac{\partial y}{\partial t}(x,t) = aH'(x+at) - ag'(x-at).$$

Applying $\frac{\partial y}{\partial t}(x,0) = 0.$

$$\cancel{aH'(x)} \cdot \quad aH'(x) - ag'(x) = 0.$$

$$H'(x) = g'(x).$$

integrating w.r. to x

$$H(x) = g(x) + c$$

Applying $y(x,0) = f(x)$ --- (8).

$$H(x) + g(x) = f(x)$$

$$g(x) + c + g(x) = f(x).$$

$$2g(x) + c = f(x).$$

$$g(x) = \frac{1}{2}[f(x) - c]$$

$$\therefore \quad H(x) = g(x) + c$$

$$= \frac{1}{2}[f(x) - c] + c$$

$$= \frac{f(x)}{2} - \frac{c}{2} + c$$

$$= \frac{f(x)}{2} + \frac{c}{2}$$

$$= \frac{1}{2}[f(x) + c]$$

$$y(x,t) = H(x+ct) + g(x-ct).$$

$$= \frac{1}{2}[f(x+ct) - \cancel{c}] + \frac{1}{2}[f(x-ct) + \cancel{c}]$$

$$\boxed{y(x,t) = \frac{1}{2}[f(x+ct) + f(x-ct)]}$$

this is the d'Alembert's Solution of wave Equation.

Q. Find the deflection of a vibrating string of unit length having fixed ends with ~~init~~ initial velocity zero and initial deflection $f(x) = k(\sin x - \sin 2x)$ using d' Alembert's method.

Soln: By d' Alembert's method.

$$y(x,t) = \frac{1}{2}\left[f(x+ct) + f(x-ct)\right]$$

given

i) $\frac{\partial y}{\partial t}(x,0) = 0.$

ii) $y(x,0) = k(\sin x - \sin 2x). = f(x).$

$$y(x,t) = \frac{1}{2}\left[f(x+ct) + f(x-ct)\right]$$

$$= \frac{1}{2}\left[k\{\sin(x+ct) - \sin 2(x+ct)\} + k[\sin(x-ct) - \sin 2(x-ct)\right]$$

$$= \cancel{\frac{k}{2}}\left[\cancel{2\sin x \sin ct}\right] * \cancel{2\sin}$$

$$= \frac{k}{2}\left[\sin(x+ct) + \sin(x-ct) - (\sin 2(x+ct) + \sin 2(x-ct))\right]$$

$$= \frac{k}{2}\left[\cancel{2\sin x \sin ct - 2\sin 2x \sin 2ct}\right]$$

$$= k\left[\cancel{\sin x \sin ct - 2\sin 2x \sin ct}\right]$$

$$\cancel{y(x,0) = k[\sin x \sin 0 - \sin 2x \sin 0]}$$

$$= k\left[\cancel{\sin x - \sin 2x}\right]$$

$$= k\left[\cancel{\sin x - 2\sin 2x \cos x}\right.$$

$$= \frac{k}{2}\left[2\sin x \cos ct - 2\sin 2x \cos 2ct\right]$$

$$= k\left[\sin x \cos ct - \sin 2x \cos 2ct\right]$$

$$y(x,0) = k\left[\sin x \cos 0 - \sin 2x \cos 0\right].$$

$$= k\left[\sin x - \sin 2x\right]$$

$$= f(x,0).$$

$$\frac{\partial y}{\partial t}(x,t) = \frac{k}{2}\left[-2c \sin x \sin ct + 2\times 2c \sin 2x \sin 2ct\right].$$

$$\frac{\partial y}{\partial t}(x,0) = \frac{k}{2}\left[0+0\right]$$

$$= 0.$$

Conditions are satisfied.

Heat Equation

Derivation of One dimensional Heat Equation

Assumptions to obtain heat equation are:

i) heat flow from higher to lower temperature

ii) the rate at which heat flows any area is jointly proportional to the area and to the temperature gradient normal to the area.

this proportionality constant is called the thermal conductivity (k) of the material.

$$Q \propto A \frac{\partial u}{\partial x}$$

$$Q = kA \frac{\partial u}{\partial x}$$

iii) the quantity of heat required to produce a given temperature change In a body is proportional to mass of the body and the rate of change of temperature w.r.to time. this proportionality constant is called specific heat (c) of the material.

$$Q \propto m \, \partial u/\partial t$$

$$= c\, m \, \partial u/\partial t.$$

⇒ Choose a uniform bar of constant cross-sectional area A.

⇒ Sides of bar are insulated so that the loss of heat from the sides of bar by conduction or radiation is negligibly small.

⇒ Take one end of the bar as origin.

⇒ Direction of heat flow as +ve x-axis.

Amount of heat ~~energy~~ crossing any section of the bar per second depends on Area A and rate of change of temperature w.r.to distance and thermal conductivity k of the material.

Q_1 - amount of heat flowing through the section

$$Q_1 = -kA\left(\frac{\partial u}{\partial x}\right)_x.$$

Q_2 = amount of heat flowing out of the section

$$Q_2 = -kA\left(\frac{\partial u}{\partial x}\right)_{x+\Delta x}$$

Amount of heat retained by the slab with thickness Δx is

$$Q_1 - Q_2 = kA\left[\left(\frac{\partial u}{\partial x}\right)_{x+\Delta x} - \left(\frac{\partial u}{\partial x}\right)_x\right] \quad \text{——— (1).}$$

Rate of increase of heat in this section = $c\rho A\,\Delta x\left(\frac{\partial u}{\partial t}\right)$ ——— (2).

eq (1) & (2)

$$c\rho A\,\Delta x\,\frac{\partial u}{\partial t} = kA\left[\left(\frac{\partial u}{\partial x}\right)_{x+\Delta x} - \left(\frac{\partial u}{\partial x}\right)_x\right]$$

$$\frac{\partial u}{\partial t} = \frac{kA}{c\rho}\,\frac{\left[\left(\frac{\partial u}{\partial x}\right)_{x+\Delta x} - \left(\frac{\partial u}{\partial x}\right)_x\right]}{\Delta x}$$

if $\Delta x \longrightarrow 0$.

$$\boxed{\frac{\partial u}{\partial t} = \frac{k}{c\rho}\,\frac{\partial^2 u}{\partial x^2}}$$

$$\boxed{\frac{\partial u}{\partial t} = a^2\,\frac{\partial^2 u}{\partial x^2}} \quad \text{——— (3)}$$

where $c = \frac{k}{c\rho}$.

diffusivity of the material.

(3) is the heat equation.

Solution of heat Equation by separation of Variable method.

heat Equation is

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \quad \text{——— (1)}$$

Let solution is $U = XT$ ——— (2).

differentiating (2) partially w.r.to x and t

$$\frac{\partial u}{\partial x} = X'T \qquad \frac{\partial u}{\partial t} = XT'$$

differentiating w.r.to x

$$\frac{\partial^2 u}{\partial x^2} = X''T$$

$$XT' = c^2 X''T$$

$$\frac{1}{c^2}\frac{T'}{T} = \frac{X''}{X} = k$$

$$\frac{X''}{X} = k \qquad \frac{1}{c^2}\frac{T'}{T} = k$$

$$X'' = kX \qquad T' = kc^2T$$

$$X'' - kX = 0. \qquad T' - kc^2T = 0.$$

$$[D^2 - k]X = 0 \quad \text{——— (3)}. \qquad [D - kc^2]T = 0 \quad \text{——— (4)}.$$

Solution of (3) & (4) depends on value of k

Case (i)

when k is +ve $(k = p^2)$.

$$(D^2 - p^2)x = 0 \qquad [D - p^2c^2]t = 0.$$

Auxillary equation is

$$m^2 - p^2 = 0. \qquad m - p^2c^2 = 0.$$

$$m^2 = p^2 \qquad m = p^2c^2$$

$$m = \pm p.$$

$$X = C_1 e^{px} + C_2 e^{-px} \qquad T = C_3 e^{p^2c^2t}$$

Solution is

$$u(x,t) = [C_1 e^{px} + C_2 e^{-px}][C_3 e^{p^2c^2t}] \quad \text{——— (I)}.$$

Case (ii).

when k is -ve $(k = -p^2)$.

$[D^2+p^2]X = 0.$ $\qquad$ ~~$[D^2+p^2c^2]T =$~~

$m^2+p^2 = 0.$ $\qquad$ $[D + p^2c^2]T = 0$

$m^2 = -p^2$ $\qquad$ $m+p^2c^2 = 0.$

$m = \pm ip$ $\qquad$ $m = -p^2c^2$

$T = C_6 e^{-p^2c^2t}$

$X = C_4 \cos px + C_5 \sin px$

Solution is

$U(x,t) = [C_4 \cos px + C_5 \sin px][C_6 e^{-p^2c^2t}].$

Case (iii).

when $k = 0$.

$X'' = 0$ $\qquad$ $T' = 0.$

$D^2X = 0$ $\qquad$ $DT = 0.$

$m^2 = 0$ $\qquad$ $m = 0.$

$m = 0, 0.$

$T = C_9.$

$X = C_7 + C_8 x$

Solution is

$U(x,t) = [C_7 + C_8 x] C_9.$

The suitable solution for heat equation is

$U(x,t) = [C_1 \cos px + C_2 \sin px] e^{-p^2c^2t}$

Problems.

1. Find the temperature $u(x,t)$ in a homogeneous bar of heat conducting material of length l, whose ends are kept at temperature $0°c$ and whose initial temperature in (°c) is given by $\frac{ax(l-x)}{l^2}$.

Soln: Heat equation is given by

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$

Boundary conditions of Heat Equation is

i) $u(0,t) = 0°c$

ii) $u(l,t) = 0.$

iii) $u(x,0) = \frac{a}{l^2}(lx - x^2).$

Suitable Solution is

$$u(x,t) = [C_1 \cos px + C_2 \sin px] e^{-p^2c^2t} \quad \text{——— (1).}$$

Applying Boundary Condition i) in (1).

$u(0,t) = 0.$

$C_1 e^{-p^2c^2t} = 0.$

$C_1 = 0.$

Then Suitable Solution is

$$u(x,t) = C_2 e^{-p^2c^2t} \sin px \quad \text{——— (2).}$$

Applying Boundary condition ii) in (2).

$u(l,t) = 0.$

$\cancel{[C_1 \cos pl + C_2 \sin pl] e^{-p^2c^2t} = 0}.$

$C_2 e^{-p^2c^2t} \sin pl = 0.$

$\sin pl = 0.$

$pl = n\pi$

$p = n\pi/l.$

Now suitable solution is

$\therefore u(x,t) = C_2 \sin \frac{n\pi x}{l} \times e^{-p^2c^2t}$ —— ③

Most general solution is

$$u(x,t) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{l} e^{-\frac{n^2\pi^2c^2}{l^2}t} \quad \text{—— ④.}$$

Applying Boundary Condition (iii) in ④.

$u(x,0) = a/l^2 (lx - x^2)$.

$$\sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{l} = \frac{a}{l^2}(lx - x^2) \quad \text{—— ⑤.}$$

⑤ is Half range Fourier Sine Series.

$$\therefore A_n = \frac{2}{l}\int_0^l f(x) \sin \frac{n\pi x}{l} dx$$

$$= 2/l \int_0^l \frac{a}{l^2}(lx - x^2) \sin \frac{n\pi x}{l} dx$$

$$= \frac{2a}{l^3}\left[(lx - x^2)\left[\frac{-\cos \frac{n\pi x}{l}}{\frac{n\pi}{l}}\right] - (l - 2x)\left[\frac{-\sin \frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right] + \right.$$

$$\left. -2\left[\frac{\cos \frac{n\pi x}{l}}{\frac{n^3\pi^3}{l^3}}\right]\right]_0^l.$$

$$= \frac{2a}{l^3}\left[\left[0 + 0 - \frac{2l^3}{n^3\pi^3}(-1)^n\right] - \left[0 + 0 - \frac{2l^3}{n^3\pi^3}\right]\right].$$

$$= \frac{2a}{l^3}\left[-\frac{2l^3}{n^3\pi^3}(-1)^n + \frac{2l^3}{n^3\pi^3}\right].$$

$$= \frac{4a}{n^3\pi^3}\left[1 - (-1)^n\right]$$

Solution is.

$$u(x,t) = \sum_{n=1}^{\infty} \frac{4a}{n^3\pi^3}\left[1-(-1)^n\right] \sin\frac{n\pi x}{l}\, e^{-\frac{n^2\pi^2c^2}{l^2}t}$$

2. Solve the equation $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ with boundary conditions $u(x,0) = 3\sin n\pi x$, $u(0,t)=0$ and $u(1,t)=0$ where $0<x<1, t>0$

Soln:

Heat equation is given

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

Boundary conditions:

i) $u(0,t) = 0$.

ii) $u(1,t) = 0$.

iii) $u(x,0) = \cancel{3\sin\pi x}.\ 3\sin n\pi x$.

Suitable solution is

$$u(x,t) = [C_1\cos px + C_2\sin px]e^{-p^2c^2t} \qquad c=1, \text{given}$$

$$= [C_1\cos px + C_2\sin px]e^{-p^2t} \quad \text{——} (1).$$

Applying Boundary condition i) in (1).

$$u(0,t) = 0.$$

$$C_1 e^{-p^2t} = 0$$

$$C_1 = 0.$$

New suitable solution is

$$u(x,t) = C_2\sin px \cdot e^{-p^2t} \quad \text{——} (2).$$

Applying B.C ii) in (2).

$$u(1,t) = 0.$$

$C_2 \sin p \times e^{-p^2 t} = 0.$

$\sin p = 0.$

$p = n\pi$

$\therefore$ New suitable solution is

$u(x,t) = C_2 \sin n\pi x \, e^{-n^2\pi^2 t}$ ——— (3).

Most general solution

$u(x,t) = \sum_{n=1}^{\infty} A_n \sin n\pi x \, e^{-n^2\pi^2 t}$ ——— (4).

Applying B.C (iii) in (4).

$u(x,0) = 3 \sin n\pi x.$

$\sum_{n=1}^{\infty} A_n \sin n\pi x = 3 \sin n\pi x$

$A_n = 3.$

$\therefore$ Solution is.

$u(x,t) = \sum_{n=1}^{\infty} 3 \sin n\pi x \, e^{-n^2\pi^2 t}.$

Steady State Condition.

Steady state mean temperature is independent of time

ie, $\frac{\partial u}{\partial t} = 0.$

$\therefore$ Heat equation is

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$

$$c^2 \frac{\partial^2 u}{\partial x^2} = 0.$$

$$\frac{\partial^2 u}{\partial x^2} = 0.$$

$$\frac{d^2 u}{dx^2} = 0.$$

integrating w.r.to x.

$$\frac{du}{dx} = a.$$

Integrating w.r.to x.

$$u = \int a\,dx$$

$$u = ax + b.$$

$$\boxed{u(x,0) = ax + b.}$$

this is the initial condition for steady state condition.

Q. An insulated rod of length l has its ends A and B maintained at $0°c$ and $100°c$ respectively ~~unit~~ until steady state condition prevail. If B is suddenly reduced to $0°c$ and maintained at $0°c$. Find the temperature at a distance x from A at time t.

Soln:

$$u(x,0) = ax + b.$$

when $x = 0$.

$$u(0,0) = b.$$

$$0 = b.$$

when $x = l$.

$$u(l,0) = al + b.$$

$$100 = al + 0.$$

$$al = 100.$$

$$a = \frac{100}{l}.$$

$$u(x,0) = \frac{100x}{l}.$$

Boundary Conditions are

i) $u(0,t) = 0.$

ii) $u(l,t) = 0.$

iii) $u(x,0) = \frac{100x}{l}$

Suitable Solution

$$u(x,t) = \left[C_1 \cos px + C_2 \sin px\right] e^{-p^2c^2t} \quad \text{—— (1)}.$$

Applying B.C i) in (1).

$u(0,t) = 0$

$C_1 = 0.$

$\therefore$ New suitable solution is.

$$u(x,t) = C_2 \sin px \, e^{-p^2c^2t} \quad \text{—— (2)}.$$

Applying B.C ii) in (2).

$u(l,t) = 0.$

$\sin pl = 0.$

$pl = n\pi$

$p = \frac{n\pi}{l}.$

$\therefore$ New suitable solution is.

$$u(x,t) = C_2 \sin \frac{n\pi x}{l} \, e^{-\frac{n^2\pi^2}{l^2}c^2t} \quad \text{—— (3)}.$$

Most general solution is.

$$u(x,t) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{l} \, e^{-\frac{n^2\pi^2}{l^2}c^2t} \quad \text{—— (4)}$$

Applying Boundary Condition (iii) in (4).

$$u(x,0) = \frac{100}{l}x$$

$$\sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{l} = \frac{100x}{l}$$

$$A_n = \frac{2}{l}\int_0^l f(x) \sin \frac{n\pi x}{l}\, dx.$$

$$= \frac{2}{l}\int_0^l \frac{100x}{l} \sin \frac{n\pi x}{l}\, dx$$

$$= \frac{200}{l^2}\int_0^l x \sin \frac{n\pi x}{l}\, dx.$$

$$\frac{200}{l^2}\left[x\left[\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right] - 1\left[\frac{-\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right]\right]_0^l$$

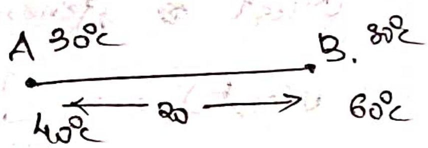
$$\frac{200}{l^2}\left[\left[\frac{-l^2}{n\pi}(-1)^n + 0\right] - \left[0+0\right]\right]$$

$$= \frac{200(-1)^n}{n\pi}$$

Solution.

$$u(x,t) = \sum_{n=1}^{\infty} \frac{-200(-1)^n}{n\pi} \sin\frac{n\pi}{l}x \; e^{-\frac{n^2\pi^2c^2}{l^2}t}.$$

Q. The ends A and B of a rod 20 cm ~~long~~ long have the temperature of 30°C and 80°C ~~unit~~ until steady-state prevails. The temperature of the ends are changed to 40°C and 60°C respectively. Find the temperature distribution in the rod at time t.



Soln: Steady state attained.

$u(x,0) = ax+b.$

when $x=0$

$u(0,0) = 0+b.$

$\underline{b=30.}$

when $x = 20.$

$u(20,0) = a\times 20 + b.$

$80 = 20a + 30.$

$20a = 50.$

$\underline{a = 5/2}$

$u(x,0) = \frac{5}{2}x + 30.$

Boundary conditions are

i) $u(0,t) = 40.$

ii) $u(20,t) = 60.$

iii) $u(x,0) = 5/2x + 30.$

Suitable solution is

$$u(x,t) = [c_1 \cos px + c_2 \sin px]\, e^{-p^2c^2t} \quad \text{—(1)}.$$

$$u(x,t) = u_t(x,t) + U_s(x,0) \quad \text{—(3)}.$$

$U_t(x,t) \rightarrow$ transcient state solution

$U_s(x,0) \rightarrow$ steady state solution.

To find $U_s(x,0)$.

$U_s(x,0) = a_1x + b_1$

when $x = 0$.

$U_s(0,0) = a \times 0 + b_1$

$40 = b_1.$

when $x = 20$

$U_s(20,0) = \cancel{20a} \cdot a \times 20 + 40.$

$60 = 20a + 40.$

$20 = 20a$

$\underline{\underline{a = 1.}}$

$\therefore U_s(x,0) = x + 40.$

To find $U_t(x,t)$.

$U_t(x,t) = u(x,t) - U_s(x,0);$

when $x = 0.$

$U_t(0,t) = u(0,t) - U_s(0,0).$

$= 40 - 40$

$\underline{\underline{= 0.}}$

When $x = 20$.

$u_t(20,t) = u(20,t) - u_s(20,0)$

$= 60 - 60$

$= \underline{\underline{0}}.$

When $t = 0$.

$u_t(x,0) = u(x,0) - u_s(x,0).$

$= 5/2\,x + 30 - [x + 40].$

$= 5/2\,x + 30 - x - 40$

$= \underline{\underline{\frac{3x}{2} - 10}}.$

i) $u_t(0,t) = 0.$

ii) $u_t(20,t) = 0.$

iii) $u_t(x,0) = \frac{3x}{2} - 10.$

Suitable solution to find $u_t(x,t)$ is.

$u_t(x,t) = [c_1 \cos px + c_2 \sin px]\, e^{-p^2c^2t}$ ——— (4)

Applying B.C i) in (4).

$c_1 = 0.$

New suitable solution is

$u_t(x,t) = c_2 \sin px \cdot e^{-p^2c^2t}$ ——— (5).

Applying B.C ii) in (5).

~~Step~~ $u_t(20,t) = 0.$

$\cancel{u_t(20,0)} = \sin 20p = 0.$

$20p = n\pi$

$p = \frac{n\pi}{20}.$

∴ New suitable solution is.

$$U_t(x,t) = C_2 \sin \frac{n\pi}{20} x \, e^{-\frac{n^2\pi^2 c^2 t}{20^2}} \quad \text{—— (6)}$$

Most general solution is.

$$U_t(x,t) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi}{20} x \, e^{-\frac{n^2\pi^2 c^2 t}{20^2}} \quad \text{—— (7)}.$$

Applying B.C (ii) in (7).

$$U_t(x,0) = \frac{3}{2}x - 10.$$

$$\sum_{n=1}^{\infty} A_n \sin \frac{n\pi}{20} x = \frac{3}{2}x - 10 \quad \text{—— (8)}$$

(8) is Half Range ~~Fori~~ Fourier Sine Series.

$$A_n = 2/l \int_0^l f(x) \sin \frac{n\pi}{l} x \, dx$$

$$= 2/20 \int_0^{20} (3/2 x - 10) \sin \frac{n\pi x}{l} \, dx.$$

$$= \frac{1}{10} \left[\left(\frac{3}{2}x - 10\right) \left[\frac{-\cos \frac{n\pi x}{l}}{\frac{n\pi}{l}} \right] - 3/2 \left[\frac{-\sin \frac{n\pi}{20} x}{\frac{n^2\pi^2}{20^2}} \right] \right]_0^{20}.$$

$$= \frac{1}{10} \left[\left[-20 \frac{20}{n\pi} (-1)^n \right] - \left[10 \frac{20}{n\pi} \times 1 \right] \right]$$

$$= \frac{20}{n\pi} \left[-2(-1)^n - 1 \right]$$

$$= \frac{20}{n\pi} \left[-2(-1)^n - 1 \right].$$

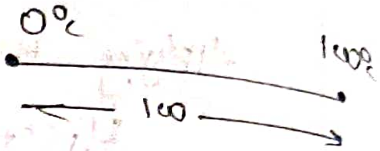
$$U_t(x,t) = \sum_{n=1}^{\infty} \frac{20}{n\pi} \left[-2(-1)^n - 1 \right] \sin \frac{n\pi}{20} x \cdot e^{-\frac{n^2\pi^2 c^2 t}{20^2}}$$

$$U(x,t) = U_t(x,t) + U_s(x,0)$$

$$U(x,t) = x + 40 + \sum_{n=1}^{\infty} \frac{20}{n\pi} \left[-2(-1)^n - 1 \right] \sin \frac{n\pi}{20} x \, e^{-\frac{n^2\pi^2 c^2 t}{20^2}}$$

Q. A bar 100 cm long with insulated sides, has its ends kept at 0° and 100° until steady state condition prevail. The two ends are then ~~such~~ suddenly insulated and kept so. Find the temperature distribution.

Soln. Since steady state condition attained.



$u(x,0) = ax+b.$

When $x=0$.

$u(0,0) = a\times 0+b.$

$\underline{0=b.}$

When $x=100$.

$u(100,0) = a\times 100+0.$

$100 = a\times 100.$

$\underline{a=1.}$

$u(x,0) = x$

Boundary conditions are

i) $\frac{\partial u}{\partial x}(0,t) = 0.$

ii) $\frac{\partial u}{\partial x}(100,t) = 0.$

iii) $u(x,0) = x$

Suitable solution is.

$$u(x,t) = [c_1 \cos px + c_2 \sin px]\, e^{-p^2c^2t} \quad \text{—— (1).}$$

differentiate (1) partially w.r.t to x.

$$\frac{\partial u}{\partial x}(x,t) = [-c_1 p \sin px + p c_2 \cos px]\, e^{-p^2c^2t} \quad \text{—— (2).}$$

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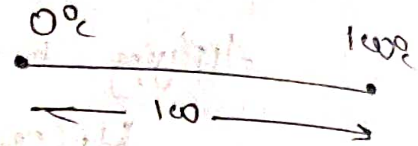

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$u(x,t) = [c_1\cos px + c_2\sin px]e^{-p^2c^2t}$ —— ①.

differentiate ① partially w.r.t to x.

$\frac{\partial u}{\partial x}(x,t) = [-c_1 p\sin px + p c_2\cos px]e^{-p^2c^2t}$ —— ②.

Applying B.C i) in (2)

$$\frac{\partial u}{\partial x}(0,t) = 0$$

$$\cdot p C_2 e^{-p^2c^2t} = 0.$$

$$C_2 = 0.$$

∴ Ideal suitable solution is

$$\frac{\partial u}{\partial x}(x,t) = -C_1 p \sin px \, e^{-p^2c^2t} \quad \text{—— (3).}$$

Applying B.C ii) in (3).

$$\frac{\partial u}{\partial x}(100,t) = 0.$$

$$-C_1 p \sin 100p \, e^{-p^2c^2t} = 0.$$

$$\sin 100p = 0.$$

$$100p = n\pi$$

$$p = \frac{n\pi}{100}.$$

∴ Ideal suitable solution is.

$$\frac{\partial u}{\partial x}(x,t) = -C_1 p \sin \frac{n\pi x}{100} \times e^{-p^2c^2t}$$

∴ the most ~~gs~~ general solution is.

$$\cancel{\frac{\partial u}{\partial x}(x,t)}$$

$$u(x,t) = \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{100} e^{-p^2c^2t}$$

$$u(x,t) = A_0 + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{100} \, e^{-\frac{n^2\pi^2c^2}{100}t} \quad \text{—— (4).}$$

Applying (iii) in (4).

$$u(x,0) = x$$

$$A_0 + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{100} = x \quad \text{—— (5).}$$

(5) is Half range Fourier ~~Sine~~ cos series.

$$A_0 = \frac{2}{l}\int_0^l f(x)\,dx$$

$$= \cancel{2/l}\ \frac{2}{100}\int_0^{100} x\,dx$$

$$= \frac{2}{100}\left[\frac{x^2}{2}\right]_0^{100}$$

$$= \frac{1}{100}\left[100^2 - 0\right]$$

$$\underline{\underline{= 100.}}$$

$$A_n = \frac{2}{l}\int_0^l f(x)\,\cancel{si}\cos\frac{n\pi}{100}x\,dx.$$

$$= \frac{2}{100}\int_0^{100} x\cos\frac{n\pi}{100}x\,dx$$

$$= \frac{1}{50}\left[x\left[\frac{-\sin\frac{n\pi}{100}x}{\frac{n\pi}{100}}\right] - 1\left[\frac{-\cos\frac{n\pi}{100}x}{\frac{n^2\pi^2}{100^2}}\right]\right]_0^{100}.$$

$$= \frac{1}{50}\left[\left[0 + \frac{100^2}{n^2\pi^2}(-1)^n\right] - \left[0 + \frac{100^2}{n^2\pi^2}\right]\right]$$

$$= \frac{1}{50}\,\frac{100^2}{n^2\pi^2}\left[(-1)^n - 1\right]$$

$$= \frac{200}{n^2\pi^2}\left[(-1)^n - 1\right]$$

$$\therefore \text{Sol}^n \text{ is } u(x,t) = 100 + \sum_{n=1}^{\infty}\frac{200}{n^2\pi^2}\left[(-1)^n - 1\right]\cos\frac{n\pi}{100}x\, e^{-\frac{n^2\pi^2c^2t}{100^2}}$$

Q Solve $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$ with conditions

i) $u(0,t) = 0 \ \forall t$

ii) $\frac{\partial u}{\partial x}(l,t) = 0.$

iii) $u(x,0) = x$

Sol^n^s. B.C are

i) $u(0,t) = 0.$

ii) $\frac{\partial u}{\partial x}(l,t) = 0.$

iii) $u(x,0) = x.$

Suitable solution is.

$$u(x,t) = [C_1 \cos px + C_2 \sin px] e^{-p^2c^2t} \quad \text{—— (1).}$$

Applying B.C i) in (1)

$u(0,t) = 0.$

$C_1 = 0.$

New suitable solution is.

$$\boxed{u(x,t) = C_2 \sin px \, e^{-p^2c^2t}} \quad \text{—— (2).}$$

differentiating partially w.r.to x

$$\frac{\partial u}{\partial x}(x,t) = C_2 \, p \cos px \, e^{-p^2c^2t} \quad \text{—— (3).}$$

Applying B.C ii) in (3).

$\frac{\partial u}{\partial x}(l,t) = 0.$

$C_2 \, p \cos pl \, e^{-p^2c^2t} = 0.$

$\cos pl = 0.$

$$pl = \frac{(2n-1)\pi}{2}$$

$$p = \frac{(2n-1)\pi}{2l}.$$

New suitable
~~Most~~ ~~general~~ solution is.

$$u(x,t) = c_2 \sin \frac{(2n-1)\pi}{2l} e^{-\frac{(2n-1)^2\pi^2c^2}{4l^2}t} \quad \text{--- (4)}$$

Most general solution.

$$u(x,t) = \sum A_n \sin \frac{(2n-1)\pi}{2l} e^{-\frac{(2n-1)^2\pi^2c^2t}{4l^2}} \quad \text{--- (5)}$$

Applying B.C (iii) in (5).

$$u(x,0) = x.$$

$$\sum_{n=1}^{\infty} A_n \sin \frac{(2n-1)\pi x}{2l} = x \quad \text{--- (6)}$$

(6) in Half Range Fourier Sine Series.

$$A_n = \frac{2}{l}\int_0^l f(x) \sin \frac{(2n-1)\pi x}{2l} dx$$

$$= 2/l \int_0^l x \sin \frac{(2n-1)\pi x}{2l} dx$$

$$= 2/l \left[x \left[\frac{-\cos \frac{(2n-1)\pi x}{2l}}{\frac{(2n-1)\pi}{2l}} \right] - \left[\frac{-\sin \frac{(2n-1)\pi x}{2l}}{\frac{(2n-1)^2\pi^2}{4l^2}} \right] \right]_0^l$$

$$= 2/l \left[\left[-l \frac{2l}{(2n-1)\pi} \cos \frac{(2n-1)\pi}{2} + \frac{4l^2}{(2n-1)^2\pi^2} \sin \frac{(2n-1)\pi}{2} \right] \right]$$

$$= 2 \frac{4l^2}{(2n-1)^2\pi^2} \sin \frac{(2n-1)\pi}{2}$$

Solution is

$$u(x,t) = \sum_{n=1}^{\infty} \frac{8l.}{(2n-1)^2\pi^2} \sin \frac{(2n-1)\pi}{2} \sin \frac{(2n-1)\pi x}{2l} e^{-\frac{(2n-1)^2\pi^2c^2t}{4l^2}}$$