Diagonalization of Matrices.

If a square matrix A of order n has n linearly independent eigen vectors, then an invertible matrix P can be found such that $P^{-1}AP = D$, where D is a diagonal matrix.

Note:-

Let A be a square matrix of order 3. Let $\lambda_1, \lambda_2, \lambda_3$ be its eigen values & $X_1 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$, $X_2 = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$, $X_3 = \begin{bmatrix} x_3 \\ y_3 \\ z_3 \end{bmatrix}$ be the corresponding three independent eigen vectors of A.

Let $P = [X_1 \;\; X_2 \;\; X_3] = \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix}$

Since eigen vectors are the non-trivial solution of the matrix equation $AX = \lambda X$, then we have $AX_1 = \lambda_1 X_1$, $AX_2 = \lambda_2 X_2$, $AX_3 = \lambda_3 X_3$.

Now, $AP = A[X_1 \;\; X_2 \;\; X_3] = [AX_1 \;\; AX_2 \;\; AX_3]$

$= [\lambda_1 X_1 \;\; \lambda_2 X_2 \;\; \lambda_3 X_3]$

$$= \begin{bmatrix} \lambda_1 x_1 & \lambda_2 x_2 & \lambda_3 x_3 \\ \lambda_1 y_1 & \lambda_2 y_2 & \lambda_3 y_3 \\ \lambda_1 z_1 & \lambda_2 z_2 & \lambda_3 z_3 \end{bmatrix}$$

$$= \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

$= PD$

i.e., $AP = PD$

$\therefore P^{-1}AP = P^{-1}PD$

$\Rightarrow P^{-1}AP = D$, which proves the result for a square matrix of order 3.

<u>Note:-</u>

(1) When D and P are found for the given matrix A, we say that A has been diagonalized.

(2) D is the diagonal matrix ~~with the eigen values~~ whose diagonal elements are the eigen values of A and P is the ~~matrix~~ invertible matrix whose columns are the respective eigen vectors of A.

(3) The matrix P which diagonalize A is called the modal matrix of A and the resulting diagonal matrix D is known as the spectral matrix of A.

(4) Diagonalization of a matrix is quite useful for obtaining the powers of a matrix given by

$$A^n = PD^nP^{-1}$$

Problems:-

1) Diagonalize $A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$

Answer:-

The characteristic equation of A is given by $\lambda^3 - C_1\lambda^2 + C_2\lambda - C_3 = 0$

$\Rightarrow \lambda^3 - (-1)\lambda^2 + (-21)\lambda - 45 = 0$

$\Rightarrow \lambda^3 + \lambda^2 - 21\lambda - 45 = 0$. $\longrightarrow (1)$

$C_1 = -1$

$C_2 = \begin{vmatrix} 1 & -6 \\ -2 & 0 \end{vmatrix} + \begin{vmatrix} -2 & -3 \\ -1 & 0 \end{vmatrix} + \begin{vmatrix} -2 & 2 \\ 2 & 1 \end{vmatrix}$

$= -12 + (-3) + (-6)$

$= \underline{\underline{-21}}$

$C_3 = \begin{vmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{vmatrix}$

$= -2(-12) - 2(-6) - 3(-3)$

$= 24 + 12 + 9$

$= \underline{\underline{45}}$

By solving (1), we get the Eigen values of A.

Now, the factors of -45 are $\pm 1, \pm 3, \pm 5, \pm 9, \pm 15, \pm 45$.

$\lambda = -1$, $-1 + 1 + 21 - 45 \neq 0$

$\lambda = 1$, $1 + 1 - 21 - 45 \neq 0$

$\lambda = -3$ $-27 + 9 + 63 - 45 = 0$

Thus $\lambda = -3$ is one eigen value of A.

Now, Consider

$$\begin{array}{r|rrrr} -3 & 1 & 1 & -21 & -45 \\ & 0 & -3 & 6 & 45 \\ \hline & 1 & -2 & -15 & \boxed{0} \end{array}$$

$\Rightarrow \lambda^2 - 2\lambda - 15 = 0$

$\Rightarrow (\lambda + 3)(\lambda - 5) = 0$.

$\Rightarrow \lambda = -3, 5.$

Thus the Eigen values of A are

~~AAA~~ $\lambda = 5, -3, -3$ [-3 is repeated twice]

Now, to find the Eigen vectors of A corresponding to each λ, we have to solve the non-trivial solution X of the homogeneous system $(A - \lambda I) X = 0$.

Now, $(A - \lambda I) X = 0$

$$\Rightarrow \begin{bmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & -\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \longrightarrow (2)$$

Case 1: $\lambda = 5$

Put $\lambda = 5$ in (2), we get

$$\begin{bmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Consider the coefficient matrix & reduce it to Echelon form

ie $\begin{bmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{bmatrix}$

$$\sim \begin{bmatrix} -1 & -2 & -5 \\ 2 & -4 & -6 \\ -7 & 2 & -3 \end{bmatrix} \quad R_1 \leftrightarrow R_3$$

$$\sim \begin{bmatrix} -1 & -2 & -5 \\ 0 & -8 & -16 \\ 0 & 16 & 32 \end{bmatrix} \quad \begin{array}{l} R_2 \to R_2 + 2R_1 \\ R_3 \to R_3 - 7R_1 \end{array}$$

$$\sim \begin{bmatrix} -1 & -2 & -5 \\ 0 & -8 & -16 \\ 0 & 0 & 0 \end{bmatrix} \quad R_3 \to R_3 + 2R_2$$

which is in Echelon form.

Now, consider the Equivalent System

$$\begin{bmatrix} -1 & -2 & -5 \\ 0 & -8 & -16 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow -x_1 - 2x_2 - 5x_3 = 0 \longrightarrow (E_1)$$

$$-8x_2 - 16x_3 = 0 \longrightarrow (E_2)$$

Now, x_1 & x_2 are leading variables & x_3 is chosen as free variable.

Let $x_3 = a$

From (E_2), $-8x_2 = 16x_3$

$$\Rightarrow x_2 = -2x_3 = \underline{\underline{-2a}}$$

From (E_1), $-x_1 = 2x_2 + 5x_3$

$$= \cancel{2(2a)}\ 2(-2a) + 5a$$

$$= -4a + 5a = a$$

Thus, we have $x_1 = -a$

$x_2 = -2a$

$x_3 = a$

Let $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -a \\ -2a \\ a \end{bmatrix}$

Now, choose $a = 1$, then $X_1 = \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}$

is an eigen vector of A corresponding to $\lambda = 5$.

Case 2 : $\lambda = -3$

Put $\lambda = -3$ in (2), we get

$$\begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & -2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Consider the coefficient matrix & reduce it into Echelon form

i.e $\begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & -2 & 3 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 2 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $R_2 \to R_2 - 2R_1$, $R_3 \to R_3 + R_1$

which is in Echelon form.

Now, consider the Equivalent System

$$\begin{bmatrix} 1 & 2 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow x_1 + 2x_2 - 3x_3 = 0 \longrightarrow (E_1)$$

So x_1 is a leading variable & x_2, x_3 are free variables.

Let $x_2 = a$, $x_3 = b$.

From (E_1), $x_1 = -2x_2 + 3x_3$

$= -2a + 3b$

Thus, $x_1 = -2a + 3b$, $x_2 = a$, $x_3 = b$.

$$\text{Let } X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2a+3b \\ a \\ b \end{bmatrix}$$

$$= \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} a + \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} b$$

Now we can find two independent eigen vectors of A corresponding to the repeated eigen value $\lambda = -3$ by choosing $a=1, b=0$ & $a=0, b=1$

So when $a=1, b=0$, we get $X_2=\begin{bmatrix}-2\\1\\0\end{bmatrix}$

Also, when $a=0, b=1$, we get

$$X_3=\begin{bmatrix}3\\0\\1\end{bmatrix}$$

Thus, we found that corresponding to the eigen values $\lambda=5,-3,3$ of A, the three independent eigen vectors of A are

$$X_1=\begin{bmatrix}-1\\-2\\1\end{bmatrix},\quad X_2=\begin{bmatrix}-2\\1\\0\end{bmatrix},\quad X_3=\begin{bmatrix}3\\0\\1\end{bmatrix}$$

Now, we can set the modal matrix

$$P=\begin{bmatrix}X_1 & X_2 & X_3\end{bmatrix}$$

$$=\begin{bmatrix}-1 & -2 & 3\\-2 & 1 & 0\\1 & 0 & 1\end{bmatrix}$$

Then $P^{-1}AP=D$

$$=\begin{bmatrix}5 & 0 & 0\\0 & -3 & 0\\0 & 0 & -3\end{bmatrix}$$

(Verify yourself)

Hence A is Diagonalized.

2) Find a matrix P which transforms the matrix $A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$ to diagonal form. Hence evaluate A^4.

Solution:

Given $A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$

To find P, we compute the Eigen values and Eigen vectors of A.

Now, the characteristic equation of A is $|A - \lambda I| = 0$. It can be written as

$\lambda^3 - c_1\lambda^2 + c_2\lambda - c_3 = 0$

$\Rightarrow \lambda^3 - 7\lambda^2 + 0\lambda - (-36) = 0$

$\Rightarrow \lambda^3 - 7\lambda^2 + 36 = 0 \longrightarrow (1)$

By solving (1) we get the Eigen values of A.

Consider the factors of 36 is $\pm1, \pm2, \pm3, \pm4, \pm6, \pm9, \pm12, \pm36$.

$\lambda = -1$, $-1 - 7 + 36 \neq 0$

$\lambda = 1$, $1 - 7 + 36 \neq 0$

$c_1 = 7$

$c_2 = \begin{vmatrix} 5 & 1 \\ 1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 3 \\ 3 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ 1 & 5 \end{vmatrix}$

$= 4 + (-8) + 4$

$= 0$

$c_3 = \begin{vmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{vmatrix}$

$= 4 - 1(-2) + 3(-14)$

$= 4 + 2 - 42$

$= -36$

$\lambda = -2$, $-8 - 28 + 36 = 0$

$\Rightarrow \lambda = -2$ is a root.

Now, consider

$$\begin{array}{r|rrrr} -2 & 1 & -7 & 0 & 36 \\ & 0 & -2 & 18 & -36 \\ \hline & 1 & -9 & 18 & \boxed{0} \end{array}$$

$\Rightarrow \lambda^2 - 9\lambda + 18 = 0$

$\Rightarrow (\lambda - 3)(\lambda - 6) = 0$

$\Rightarrow \lambda = 3, 6$

Thus the Eigen values of A are

$\lambda = -2, 3, 6.$

Let us compute the Eigen vectors of A by considering the homogeneous system $(A - \lambda I)X = 0$ & solving for the non-trivial solution X.

$$\Rightarrow \begin{bmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \longrightarrow (2)$$

Case 1: $\lambda = -2$

Put $\lambda = -2$ in (2), we get

$$\begin{bmatrix} 3 & 1 & 3 \\ 1 & 7 & 1 \\ 3 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Reduce the coefficient matrix into Echelon form

iv) $\begin{bmatrix} 3 & 1 & 3 \\ 1 & 7 & 1 \\ 3 & 1 & 3 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 7 & 1 \\ 3 & 1 & 3 \\ 3 & 1 & 3 \end{bmatrix} \quad R_1 \leftrightarrow R_2$

$\sim \begin{bmatrix} 1 & 7 & 1 \\ 0 & -20 & 0 \\ 0 & -20 & 0 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 - 3R_1 \\ R_3 \to R_3 - 3R_1 \end{matrix}$

$\sim \begin{bmatrix} 1 & 7 & 1 \\ 0 & -20 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad R_3 \to R_3 - R_2$

which is in Echelon form.

Now, consider the Equivalent System

$$\begin{bmatrix} 1 & 7 & 1 \\ 0 & -20 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow x_1 + 7x_2 + x_3 = 0 \longrightarrow (E_1)$

$\quad -20x_2 = 0 \longrightarrow (E_2)$

~~From (E_2), $x_2 \neq 0$ ($\because -20 \neq 0$)~~

Now, x_1 & x_2 are leading variables & x_3 is taken to be free variable.

Let $x_3 = a$

From (E_2), $x_2 = 0$ $(\because -20 \neq 0)$

From (E_1), $x_1 = -7x_2 - x_3$

$= 0 - x_3$

$\Rightarrow x_1 = -x_3 = -a$

Thus, we have the solution of the system as $x_1 = -a,\ x_2 = 0,\ x_3 = a$.

Let $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -a \\ 0 \\ a \end{bmatrix}$

Now, choose $a = 1$ so that

$X_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ is an eigen vector of A corresponding to $\lambda = -2$.

Case 2: $\lambda = 3$

Put $\lambda = 3$ in (2), we get

$$\begin{bmatrix} -2 & 1 & 3 \\ 1 & 2 & 1 \\ 3 & 1 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Reduce the coefficient matrix into Echelon form

i.e. $\begin{bmatrix} -2 & 1 & 3 \\ 1 & 2 & 1 \\ 3 & 1 & -2 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 2 & 1 \\ -2 & 1 & 3 \\ 3 & 1 & -2 \end{bmatrix} \quad R_1 \leftrightarrow R_2$

$$\sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & 5 & 5 \\ 0 & -5 & -5 \end{bmatrix} \begin{matrix} R_2 \to R_2 + 2R_1 \\ R_3 \to R_3 - 3R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & 5 & 5 \\ 0 & 0 & 0 \end{bmatrix} \quad R_3 \to R_3 + R_2$$

which is in Echelon form.

Now, the Equivalent systems of Equation is

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 5 & 5 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} m_1 \\ m_2 \\ m_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow m_1 + 2m_2 + m_3 = 0 \longrightarrow (E_1)$$

$$5m_2 + 5m_3 = 0 \longrightarrow (E_2)$$

Thus, m_1 & m_2 are leading variable & m_3 is chosen to be free variable.

Let $m_3 = a$

From (E_2), $5m_2 = -5m_3$

$$\Rightarrow m_2 = -m_3$$
$$= -a$$

From (E_1), $m_1 = -2m_2 - m_3$

$$= -2(-a) - a$$
$$= 2a - a$$
$$= a$$

Hence we have $m_1 = a$, $m_2 = -a$, $m_3 = a$

Let $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} a \\ -a \\ a \end{bmatrix}$

Choose $a=-1$, then we have

$X_2 = \begin{bmatrix} -1 \\ +1 \\ -1 \end{bmatrix}$ as an eigen vector of A corresponding to the eigen value $\lambda = 3$

Case 3: $\lambda = 6$

Put $\lambda = 6$ in (2), we get

$$\begin{bmatrix} -5 & 1 & 3 \\ 1 & -1 & 1 \\ 3 & 1 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Reduce the coefficient matrix into Echelon form.

i.e, $\begin{bmatrix} -5 & 1 & 3 \\ 1 & -1 & 1 \\ 3 & 1 & -5 \end{bmatrix}$

$$\sim \begin{bmatrix} 1 & -1 & 1 \\ -5 & 1 & 3 \\ 3 & 1 & -5 \end{bmatrix} \quad R_2 \leftrightarrow R_1$$

$$\sim \begin{bmatrix} 1 & -1 & 1 \\ 0 & -4 & 8 \\ 0 & 4 & -8 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 + 5R_1 \\ R_3 \to R_3 - 3R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & -1 & 1 \\ 0 & -4 & 8 \\ 0 & 0 & 0 \end{bmatrix} \quad R_3 \to R_3 + R_2$$

which is in Echelon form.

Now, consider the ~~coefficient matrix~~ equivalent systems of equations as

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & -4 & 8 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow x_1 - x_2 + x_3 = 0 \longrightarrow (E_1)$$

$$-4x_2 + 8x_3 = 0 \longrightarrow (E_2)$$

Now, x_1 & x_2 are leading variables & x_3 is taken to be a free variable.

Let $x_3 = a$

Now, from (E_2), $-4x_2 = -8x_3$

$$\Rightarrow x_2 = 2x_3 = 2a$$

From (E_1), $x_1 = x_2 - x_3$

$$= 2a - a$$

$$= a$$

Thus, the solution of the system is

$x_1 = a$, $x_2 = 2a$, $x_3 = a$.

Let $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} a \\ 2a \\ a \end{bmatrix}$

Choose $a=1$, so that $X_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

as an eigen vector of A corresponding to the eigen value $\lambda = 6$.

Thus, we found that corresponding to the Eigen values $\lambda = -2, 3, 6$ of A, the three independent Eigen vectors of A are

$$X_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \quad X_2 = \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}, \quad X_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

Now, we have the Modal matrix

$$P = \begin{bmatrix} -1 & -1 & 1 \\ 0 & 1 & 2 \\ 1 & -1 & 1 \end{bmatrix}$$

Then $P^{-1}AP = D$

$$= \begin{bmatrix} -2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

<u>Verification of $P^{-1}AP = D$</u>

We have, $P^{-1} = \dfrac{\operatorname{Adj} P}{|P|}$

To find Adj P.

$\alpha_{11} = (-1)^2 3 = 3$, $\alpha_{12} = (-1)^3(-2) = 2$, $\alpha_{13} = (-1)^4(-1) = -1$

$\alpha_{21} = (-1)^3(0) = 0$, $\alpha_{22} = (-1)^4(-2) = -2$, $\alpha_{23} = (-1)^5(2) = -2$

$\alpha_{31} = (-1)^4(-3) = -3$, $\alpha_{32} = (-1)^5(-2) = 2$, $\alpha_{33} = (-1)^6(-1) = -1$

$$\therefore \text{Adj } P = \begin{bmatrix} \alpha_{11} & \alpha_{21} & \alpha_{31} \\ \alpha_{12} & \alpha_{22} & \alpha_{32} \\ \alpha_{13} & \alpha_{23} & \alpha_{33} \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 0 & -3 \\ 2 & -2 & 2 \\ -1 & -2 & -1 \end{bmatrix}$$

Also, $$|P| = \begin{vmatrix} -1 & -1 & 1 \\ 0 & 1 & 2 \\ 1 & -1 & 1 \end{vmatrix} = -6$$

$$\therefore P^{-1} = \frac{\text{Adj } P}{|P|} = \frac{-1}{6}\begin{bmatrix} 3 & 0 & -3 \\ 2 & -2 & 2 \\ -1 & -2 & -1 \end{bmatrix}$$

Now, $$P^{-1}AP = -\frac{1}{6}\begin{bmatrix} 3 & 0 & -3 \\ 2 & -2 & 2 \\ -1 & -2 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}\begin{bmatrix} -1 & -1 & 1 \\ 0 & 1 & 2 \\ 1 & -1 & 1 \end{bmatrix}$$

$$= -\frac{1}{6}\begin{bmatrix} -6 & 0 & 6 \\ 6 & -6 & 6 \\ -6 & -12 & -6 \end{bmatrix}\begin{bmatrix} -1 & -1 & 1 \\ 0 & 1 & 2 \\ 1 & -1 & 1 \end{bmatrix}$$

$$= -\frac{1}{6}\begin{bmatrix} 12 & 0 & 0 \\ 0 & -18 & 0 \\ 0 & 0 & -36 \end{bmatrix} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

$$= D$$

Thus A is diagonalized.

Now, to find A^4.

We know that $A^n = P D^n P^{-1}$

$$\therefore A^4 = P D^4 P^{-1}$$

$$= \begin{bmatrix} -1 & -1 & 1 \\ 0 & 1 & 2 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} (-2)^4 & 0 & 0 \\ 0 & 3^4 & 0 \\ 0 & 0 & 6^4 \end{bmatrix} \times -\frac{1}{6} \begin{bmatrix} 3 & 0 & -3 \\ 2 & -2 & 2 \\ -1 & -2 & -1 \end{bmatrix}$$

$$= -\frac{1}{6} \begin{bmatrix} -1 & -1 & 1 \\ 0 & 1 & 2 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 16 & 0 & 0 \\ 0 & 81 & 0 \\ 0 & 0 & 1296 \end{bmatrix} \begin{bmatrix} 3 & 0 & -3 \\ 2 & -2 & 2 \\ -1 & -2 & -1 \end{bmatrix}$$

$$= -\frac{1}{6} \begin{bmatrix} -1 & -1 & 1 \\ 0 & 1 & 2 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 48 & 0 & -48 \\ 162 & -162 & 162 \\ -1296 & -2592 & -1296 \end{bmatrix}$$

$$= -\frac{1}{6} \begin{bmatrix} -1506 & -2430 & -1410 \\ -2430 & -5346 & -2430 \\ -1410 & -2430 & -1506 \end{bmatrix}$$

$$= \begin{bmatrix} 251 & 405 & 235 \\ 405 & 891 & 405 \\ 235 & 405 & 251 \end{bmatrix}$$

Note:-

(1) Let A be a 2×2 matrix. If we could not find two independent eigen vectors corresponding to the eigen values of A, then A is NOT diagonalizable.

(2) Let A be a 2×2 matrix and let A has 2 distinct eigen values. Then there exist two independent eigen vectors and therefore A is certainly diagonalizable.

i.e. If A is a square matrix of order 2 & it has two distinct eigen values, then A is diagonalizable.

(3) Let A be a 3×3 matrix. If we could not find three independent eigen vectors corresponding to the eigen values of A, then A is NOT diagonalizable.

(4) Let A be a 3×3 matrix & Suppose A has 3 distinct eigen values. Then there exist three independent eigen vectors & hence A is Diagonalizable.

i.e. If A is a square matrix of order 3 & it has three distinct eigen values, then A is Diagonalizable.

(5) If A is a square matrix having repeated eigen values, then A may or may not be diagonalizable depending upon the existence of no. of independent eigen vectors equal to the order of A.