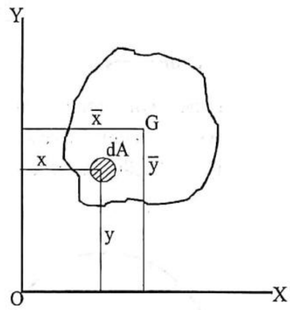
❖ Centroid

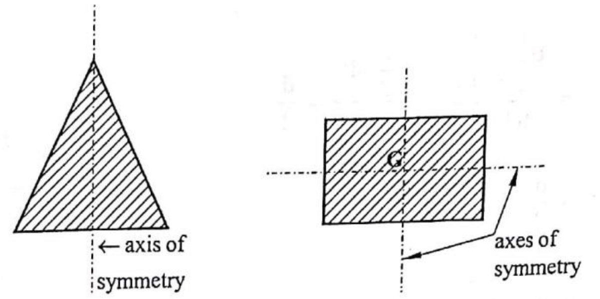
Centroid is the geometrical centre of an area, curve or an object. When the material of the body is homogeneous, the geometrical centre that is the centroid coincides with the centre of gravity of the body.

$$\bar{x} = \frac{\int (x\, dA)}{\int dA}$$

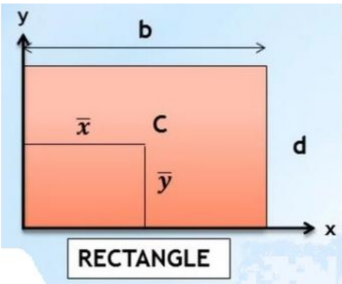
$$\bar{y} = \frac{\int (y\, dA)}{\int dA}$$



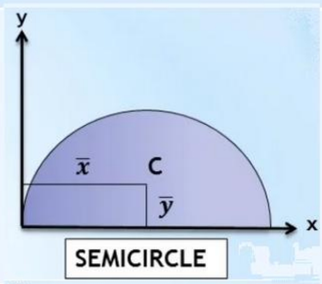
When a plane area has an axis of symmetry, the centroid will be on that axis. When there are two axes of symmetry, the point of intersection of the axes of symmetry will be the centroid of the area.



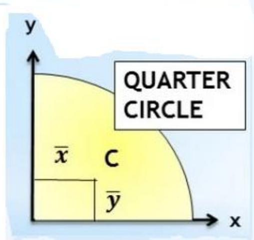
❖ Centroidal Distance of Standard Geometrical Shapes:



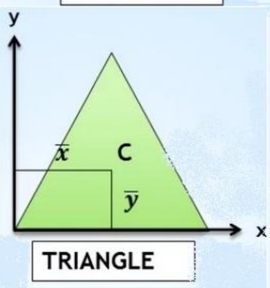
$\bar{x} = \frac{b}{2}$ $\bar{y} = \frac{d}{2}$



$\bar{x} = R$ $\bar{y} = \frac{4.R}{3.\pi}$

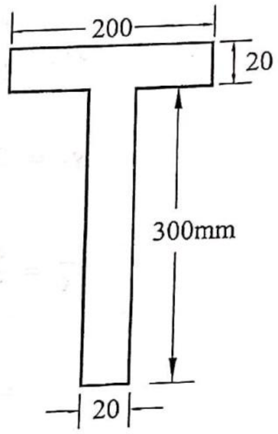


$\bar{x} = \frac{4.R}{3.\pi}$ $\bar{y} = \frac{4.R}{3.\pi}$



$\bar{x} = \frac{b}{3}$ $\bar{y} = \frac{h}{3}$

Q: Locate the centroid of the 'T' section shown



Solution

Since the section is symmetrical with respect to the Y axis, $\bar{x} = 0$.

$$\bar{y} = \frac{a_1 y_1 + a_2 y_2}{a_1 + a_2}$$

$$a_1 = 300 \times 20 = 6000 \text{ mm}^2$$

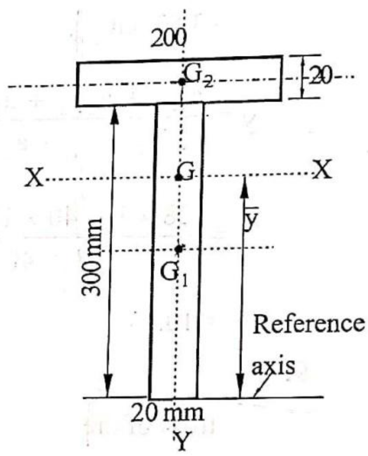
$$a_2 = 200 \times 20 = 4000 \text{ mm}^2$$

$$y_1 = \frac{300}{2} = 150 \text{ mm}$$

$$y_2 = 300 + \frac{20}{2} = 310 \text{ mm}$$

$$\therefore \bar{y} = \frac{6000 \times 150 + 4000 \times 310}{6000 + 4000}$$

$$= 214 \text{ mm}$$



Q: Locate the centroid of the area shown

Solution

$$a_1 = 14 \times 2 = 28 \text{ cm}^2;$$

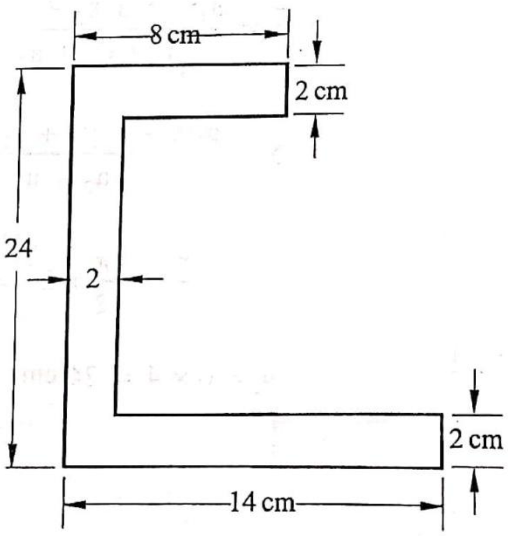
$$a_2 = 20 \times 2 = 40 \text{ cm}^2$$

$$a_3 = 8 \times 2 = 16 \text{ cm}^2$$

$$x_1 = \frac{14}{2} = 7 \text{ cm}$$

$$x_2 = \frac{2}{2} = 1 \text{ cm}$$

$$x_3 = \frac{8}{2} = 4 \text{cm}$$



$$y_1 = \frac{2}{2} = 1 \text{cm} \quad y_2 = 2 + \frac{20}{2} = 12 \text{ cm}$$

$$y_3 = 2 + 20 + \frac{2}{2} = 23 \text{cm}$$

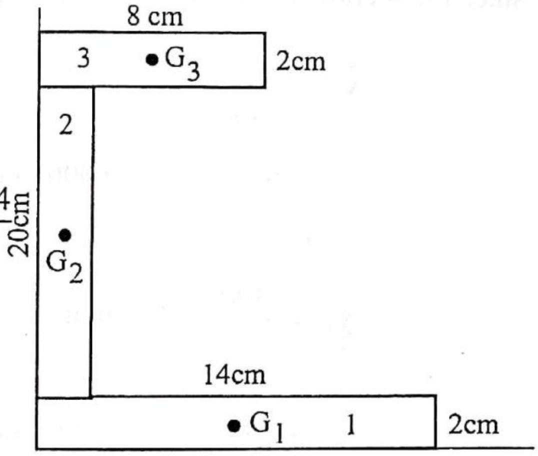


Fig.6.40

$$\bar{x} = \frac{a_1x_1 + a_2x_2 + a_3x_3}{a_1 + a_2 + a_3}$$

$$= \frac{28 \times 7 + 40 \times 1 + 16 \times 4}{28 + 40 + 16}$$

$$= 3.57 \text{ cm}$$

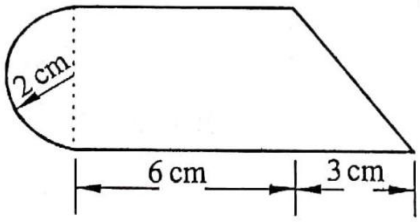
$$\bar{y} = \frac{a_1y_1 + a_2y_2 + a_3y_3}{a_1 + a_2 + a_3}$$

$$= \frac{28 \times 1 + 40 \times 12 + 16 \times 23}{28 + 40 + 16}$$

$$= 10.43 \text{ cm}$$

Q: Determine the centroid of the area shown

Solution.



$$\bar{x} = \frac{a_1x_1 + a_2x_2 + a_3x_3}{a_1 + a_2 + a_3}$$

$$\bar{y} = \frac{a_1y_1 + a_2y_2 + a_3y_3}{a_1 + a_2 + a_3}$$

$$a_1 = \frac{\pi r^2}{2} = \frac{\pi}{2} \times 2^2 = 6.28 \text{ cm}^2$$

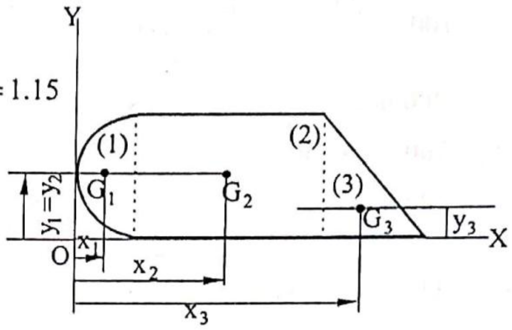
$$a_2 = 6 \times 4 = 24 \text{ cm}^2$$

$$a_3 = \frac{1}{2} \times 3 \times 4 = 6 \text{ cm}^2$$

$$x_1 = r - \frac{4r}{3\pi} = 2 - \frac{4 \times 2}{3\pi} = 1.15$$

$$x_2 = 2 + \frac{6}{2} = 5 \text{ cm}$$

$$x_3 = 2 + 6 + \frac{1}{3} \times 3 = 9 \text{ cm}$$



$$y_1 = 2 \text{ cm}, \quad y_2 = 2 \text{ cm}, \quad y_3 = \frac{1}{3} \times 4 = 1.33 \text{cm}$$

$$\bar{x} = \frac{6.28 \times 1.15 + 24 \times 5 + 6 \times 9}{6.28 + 24 + 6}$$

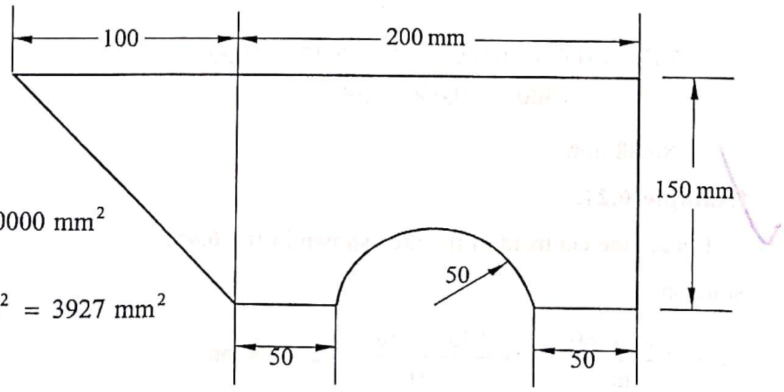
$$= 5 \text{ cm}$$

$$\bar{y} = \frac{6.28 \times 2 + 24 \times 2 + 6 \times 1.33}{6.28 + 24 + 6}$$

$$= 1.89 \text{ cm}$$

Q: Determine the centre of gravity of the thin homogeneous plane shown

Solution.



$$a_1 = \frac{1}{2} \times 100 \times 150$$

$$= 7500 \text{ mm}^2$$

$$a_2 = 200 \times 150 = 30000 \text{ mm}^2$$

$$a_3 = \frac{\pi r^2}{2} = \frac{\pi}{2} \times 50^2 = 3927 \text{ mm}^2$$

$$x_1 = (100 - \frac{1}{3} \times 100)$$

$$= 66.67 \text{ mm}$$

$$x_2 = 100 + \frac{200}{2}$$

$$= 200 \text{ mm}$$

$$x_3 = 100 + 50 + 50$$

$$= 200 \text{ mm}$$

$$y_1 = (150 - \frac{1}{3} \times 150) = 100 \text{ mm}$$

$$y_2 = \frac{150}{2} = 75 \text{mm}$$

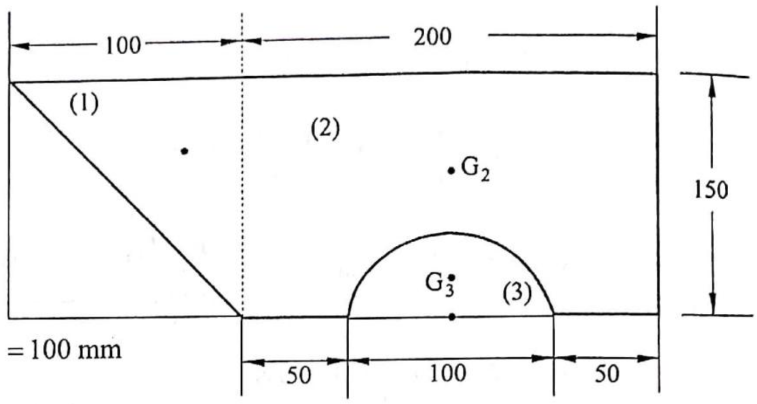


Fig.6.44

$$y_3 = \frac{4r}{3\pi} = \frac{4 \times 50}{3 \times \pi} = 21.22 \text{ mm}$$

$$\bar{x} = \frac{a_1 x_1 + a_2 x_2 - a_3 x_3}{a_1 + a_2 - a_3} = \frac{7500 \times 66.67 + 30{,}000 \times 200 - 3927 \times 200}{7500 + 30000 - 3927} = 170.21 \text{ mm.}$$

$$\bar{y} = \frac{a_1y_1 + a_2y_2 - a_3y_3}{a_1 + a_2 - a_3}$$

$$= \frac{7500 \times 100 + 30{,}000 \times 75 - 3927 \times 21.22}{7500 + 30{,}000 - 3927}$$

$$= 86.88 \text{ mm.}$$

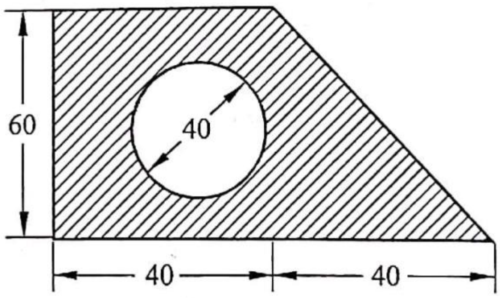
Q: Determine the coordinates of the centre of a 40 mm diameter circle to be cut in a thin plate so that this point will be the centroid of the remaining shaded area shown

Solution.

$$a_1 = 40 \times 60 = 2400 \text{ mm}^2$$

$$a_2 = \frac{1}{2} \times 40 \times 60$$

$$= 1200 \text{ mm}^2$$

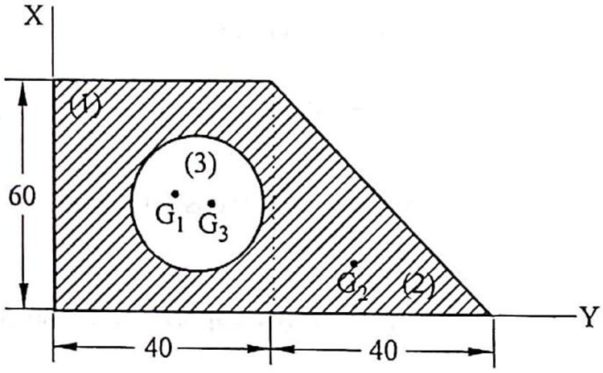


$$a_3 = \pi r^2 = \pi \times 20^2 =$$

$$= 1256.64 \text{ mm}^2$$

$$x_1 = \frac{40}{2} = 20 \text{ mm}$$

$$x_2 = 40 + \frac{1}{3} \times 40 = 53.33 \text{ mm}$$



$$x_3 = \bar{x}$$

$$y_1 = \frac{60}{2} = 30 \text{ mm}$$

$$y_2 = \frac{1}{3} \times 60 = 20 \text{ mm}$$

$$y_3 = \bar{y}$$

$$\bar{x} = \frac{a_1x_1 + a_2x_2 - a_3x_3}{a_1 + a_2 + a_3}$$

$$= \frac{2400 \times 20 + 1200 \times 53.33 - 1256.64 \times \bar{x}}{2400 + 1200 - 1256.64}$$

$$2343.36\,\bar{x} + 1256.64\,\bar{x} = 20 \times 2400 + 53.33 \times 1200$$

$$\bar{x} = 31.11 \text{ mm}$$

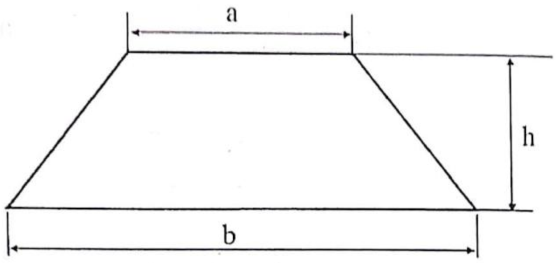
$$x_3 = \bar{x} = 31.11 \text{ mm}$$

$$\bar{y} = \frac{a_1y_1 + a_2y_2 - a_3y_3}{a_1 + a_2 - a_3}$$

$$= \frac{2400 \times 30 + 1200 \times 20 - 1256.64 \times \bar{y}}{2400 + 1200 - 1256.64}$$

$$\bar{y} = 26.67 \text{ mm}$$

Q: Locate the centroid of the trapezium with parallel side a and b and height h as shown

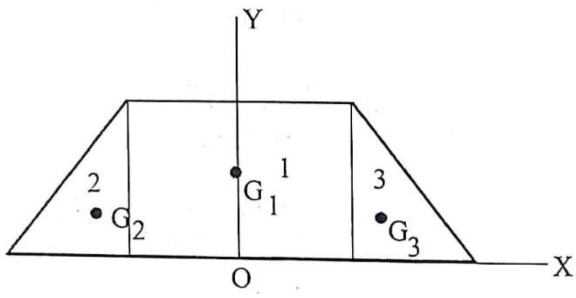


Solution

$$a_1 = a\,h$$

$$a_2 = a_3 = \frac{1}{2}(\frac{b-a}{2})\,h$$

$$a_1 + a_2 + a_3 = (\frac{a+b}{2})\,h$$



$$y_1 = \frac{h}{2}$$

$$y_2 = y_3 = \frac{1}{3}h$$

$$\bar{y} = \frac{a_1y_1 + a_2y_2 + a_3y_3}{a_1 + a_2 + a_3} = \frac{ah \times \frac{h}{2} + 2\left[\frac{1}{2}(\frac{b-a}{2})h\right]}{(\frac{a+b}{2})h} \times \frac{1}{3}h$$

$$= \frac{\frac{a\,h^2}{2} + (b-a) \times \frac{h^2}{6}}{(\frac{a+b}{2})\,h} = \frac{ah + (b-a)\,\frac{h}{3}}{(a+b)}$$

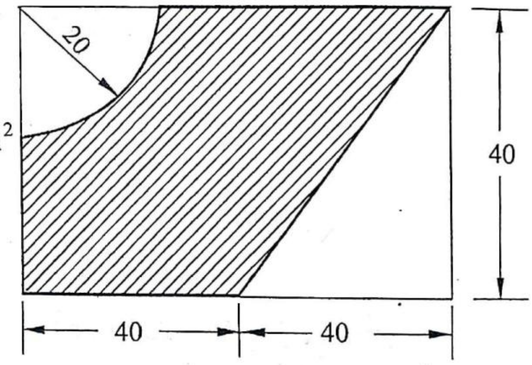
$$= \frac{3ah + bh - ah}{3(a+b)} = \frac{2ah + bh}{3(a+b)}$$

$$\boxed{\bar{y} = \frac{(2a+b)}{(a+b)} \times \frac{h}{3}}$$

Q: Locate the centroid of the shaded area shown

Solution.

$a_1 = 80 \times 40 = 3200\ mm^2$



$a_2 = \frac{1}{2} \times 40 \times 40 = 800\ mm^2$

$a_3 = \frac{\pi r^2}{4} = \frac{\pi \times 20^2}{4} = 314.16 mm^2$

$x_1 = \frac{80}{2} = 40\ mm$

$x_2 = (80 - \frac{1}{3} \times 40) = 66.67 mm$

$x_3 = \frac{4r}{3\pi} = \frac{4 \times 20}{3\pi} = 8.49 mm$

$y_1 = \frac{40}{2} = 20\ mm$

$y_2 = \frac{1}{3} \times 40 = 13.33\ mm$

$y_3 = (40 - \frac{4 \times 20}{3\pi}) = 31.51 mm$

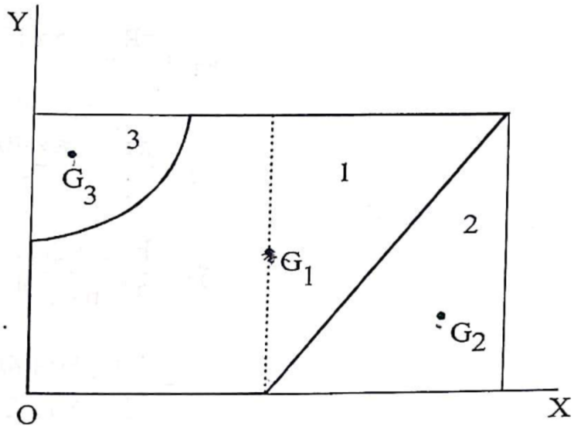


Fig.6.58

$$\bar{x} = \frac{a_1x_1 - a_2x_2 - a_3x_3}{a_1 - a_2 - a_3} = \frac{3200 \times 40 - 800 \times 66.67 - 314.16 \times 8.49}{3200 - 800 - 314.16}$$

$= 34.52\ mm$

$$\bar{y} = \frac{a_1y_1 - a_2y_2 - a_3y_3}{a_1 - a_2 - a_3} = \frac{3200 \times 20 - 800 \times 13.33 - 314.16 \times 31.51}{(3200 - 800 - 314.16)}$$

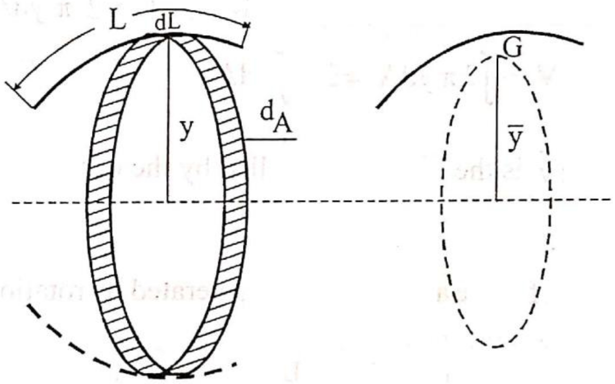
$= 20.82\ mm$

❖ THEOREM OF PAPPUS GULDINUS:

Theorem 1.

The area of surface generated by revolving a plane curve about a non-intersecting axis in the plane of the curve is equal to the product of length of curve and the distance travelled by the centroid of the curve while the surface is being generated.

Proof.



Consider an element of length dL of the curve of length L which is revolved about the X axis. The area generated by the element is equal to 2π y dL, where y is the distance of element from x axis. Therefore the entire area generated by the curve, $A = \int 2\pi y \, dL$

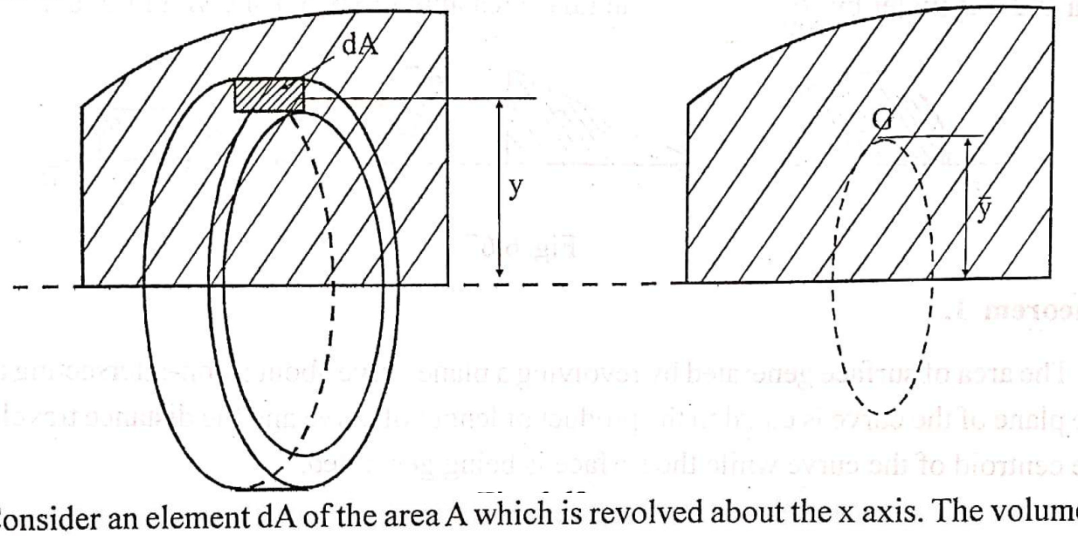
$= 2\pi \int (y \, dL) = 2\pi \bar{y} L$

$2\pi\bar{y}$ is the distance travelled by the centroid of curve of length L.

Theorem II

The volume of a body generated by revolving a plane area about a non-intersecting axis in the plane of the area is equal to the product of area and the distance travelled by the centroid of the plane area while the body is being generated.

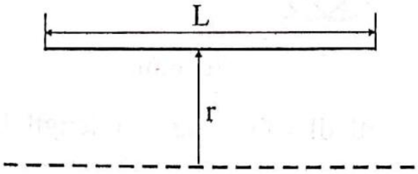
Proof.



Consider an element dA of the area A which is revolved about the x axis. The volume dv generated by the element dA in one revolution is equal to 2π ydA. Therefore the entire volume generated by A, $V = \int 2\pi y \, dA = 2\pi \int y \, dA$

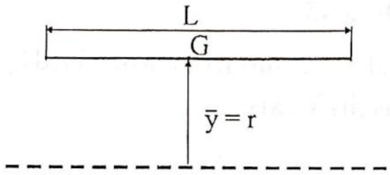
$= 2\pi\bar{y}A$. Where $2\pi\bar{y}$ is the distance travelled by the centroid of area A.

Q: Obtain the expression for the area of surface generated by rotation of a horizontal axis at a distance r from the line.



Solution.

Length of line = L. distance of centroid of line from the axis of rotation, r . Distance travelled by G in one revolution is $2\pi\bar{y} = 2\pi r$.



Surface area generated $= L \times 2\pi r$

$= 2\pi\ r \times L$

Q: Obtain an expression for the area of surface generated when a line of length L revolve about an axis. One end of the line is touching the axis and the other end is at a distance r from the axis.

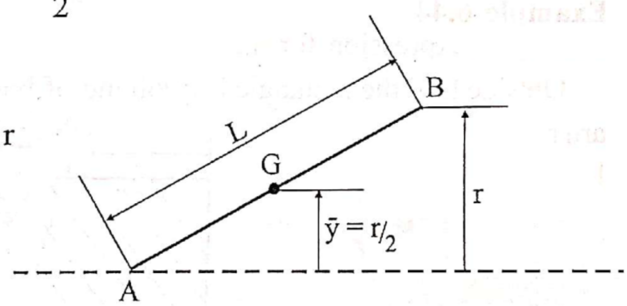
Solution.

Length of line = L

Distance of centroid G from the axis is $\frac{r}{2}$. Distance travelled by the centroid in one revolution is $2\pi\frac{r}{2} = \pi r$

$\therefore$ area of surface generated $= L \times \pi r$

$= \pi r L$

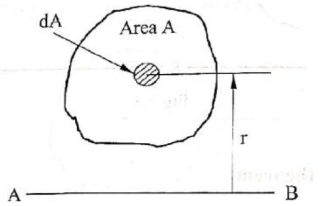


◆ Moment of inertia of area

Moment of Inertia of an area is a purely mathematical term which gives a quantitative estimate of the relative distribution of area with respect to some reference axis. If r is the distance of an elemental area, dA, from a reference axis AB, then the sum of the terms, $\Sigma r^2 dA$, to cover the entire area is called moment of inertia of the area about the reference axis AB and is denoted by I_{AB}.

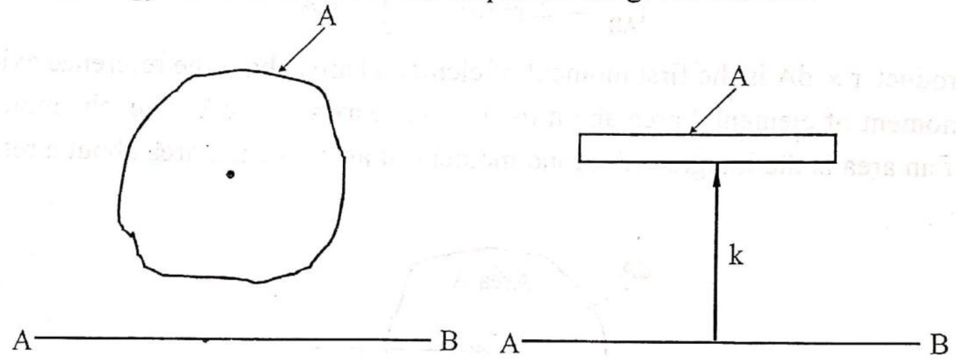
$$I_{AB} = \Sigma r^2\, dA = \int r^2\, dA$$

The product $r \times dA$ is the first moment of elemental area about the reference axis. The second moment of elemental area about the reference axis is $r^2 dA$ .Thus the moment of inertia of an area is the integral of second moment of an elemental area about a reference axis.



◆ Radius of gyration

Consider an area A which has a moment of inertia I with respect to a reference axisAB. Let us assume that this area is compressed to a thin strip parallel to the axis AB. For this strip to have the same moment of inertia I, with respect to the same reference axis AB, the strip should be placed at a distance k from the axis AB such that $I = A k^2$. $k = \sqrt{\frac{I}{A}}$ is called radius of gyration of the area with respect to the given axis AB.

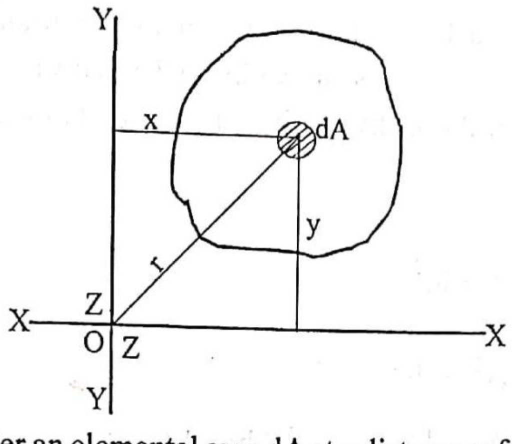


◆ Perpendicular axis theorem.

If I_{XX} and I_{YY} are the moment of inertia of an area A about mutually perpendicular axis XX and YY, in the plane of the area, then the moment of inertia of the area about the ZZ axis which is perpendicular to XX and YY axis and passing through the point of intersection of XX and YY axis is given by $I_{ZZ} = I_{XX} + I_{YY}$.

Proof:

Consider a plane area A. Let XX and YY be the two mutually perpendicular axes in the plane of the area, intersecting at O. Let ZZ be an axis through O and perpendicular to the



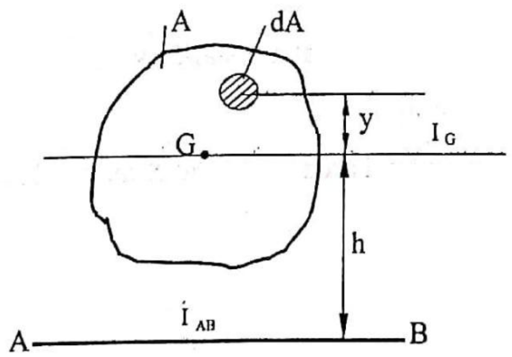
plane of area A. Consider an elemental area dA at a distance y from the XX axis, x from the YY axis and r from the ZZ axis, then $I_{ZZ} = \int r^2 dA = \int (x^2+y^2)\, dA$

$$= \int x^2 dA + \int y^2 dA = I_{YY} + I_{XX}$$

$$\boxed{I_{ZZ} = I_{XX} + I_{YY}}$$

◆ **Parallel axis theorem.**

It is a transfer theorem which is used to transfer moment of inertia from one axis to another axis . These two axes should be parallel to each other and one of these axes must be a centroidal axis.



It states that, if I_G is the moment of inertia of a plane lamina of area A, about its centroidal axis in the plane of the lamina, then the moment of inertia about any axis AB which is parallel to the centroidal axis and at a distance 'h' from the centroidal axis is given by

$$I_{AB} = I_G + A\,h^2$$

Proof.

Consider an elemental area dA at a distance y from the centroidal axis. The first moment of elemental area about the axis AB as shown in fig.7.5 is dA (y + h). Second moment of elemental area about the axis AB is $dA\,(y+h)^2$. The second moment of the area about the axis AB is $\int dA\,(y+h)^2$

$$I_{AB} = \int dA\,(y+h)^2 = \int dA\,(y^2 + h^2 + 2hy)$$

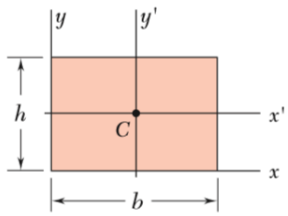
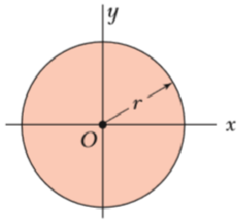
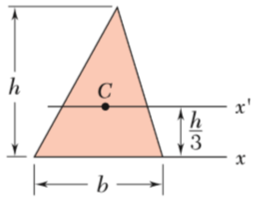
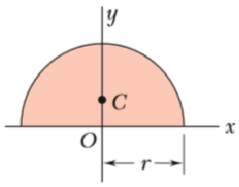
$$= \int y^2 dA + \int h^2 dA + \int 2\,h\,y\,d\,A = I_G + h^2 \int dA + 2h \int y\,dA$$

$$= I_G + h^2 A + 2\,h\,(A\bar{y})$$

$$I_{AB} = I_G + Ah^2$$

$\bar{y} = 0$, because it is the distance of centroid G from the axis from which y is measured. Here y is measured from the centroidal axis itself.

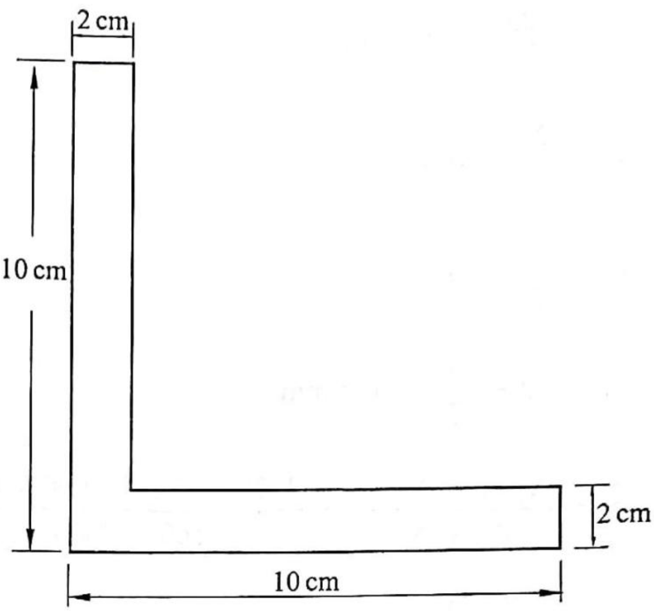
❖ MOMENT OF INERTIA OF STANDARD SECTIONS

Rectangle $\bar{I}_{x'} = \frac{1}{12}bh^3$ $\bar{I}_{y'} = \frac{1}{12}b^3h$ $I_x = \frac{1}{3}bh^3$ $I_y = \frac{1}{3}b^3h$		Circle $\bar{I}_x = \bar{I}_y = \frac{1}{4}\pi r^4$	
Triangle $\bar{I}_{x'} = \frac{1}{36}bh^3$ $I_x = \frac{1}{12}bh^3$		Semicircle $I_x = I_y = \frac{1}{8}\pi r^4$	

Q: Calculate the moment of inertia of the angle section having the dimensions shown about X and Y axis shown.

Solution.

$$A_1 = 10 \times 2 = 20 \text{ cm}^2$$



$$A_2 = 8 \times 2 = 16 \text{cm}^2$$

$$x_1 = \frac{10}{2} = 5 \text{ cm}$$

$$x_2 = \frac{2}{2} = 1 \text{ cm}$$

$$y_1 = \frac{2}{2} = 1 \text{ cm}$$

$$y_2 = 2 + \frac{8}{2} = 6 \text{ cm.}$$

$$I_X = (I_{G_{1XX}} + A_1 y_1^2)$$

$$+ (I_{G_{2XX}} + A_2 y_2^2)$$

Q: Calculate the moment of inertia of the area shown

Solution.

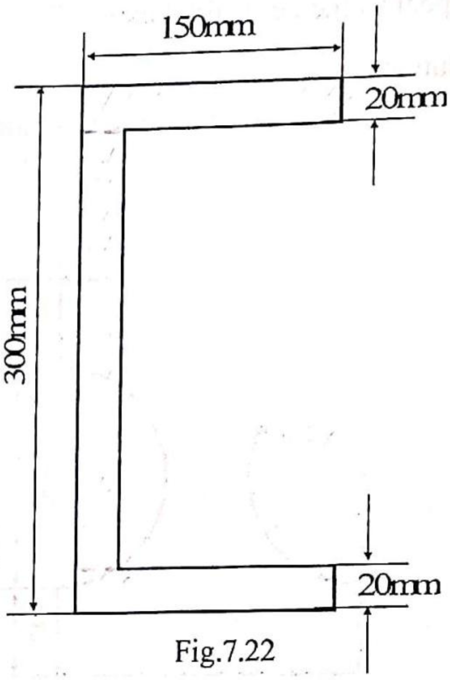


Fig.7.22

$a_1 = 130 \times 20 = 2600\ mm^2$

$a_2 = 300 \times 20 = 6000\ mm^2$

$a_3 = 130 \times 20 = 2600\ mm^2$

$x_1 = 85$ mm, $x_2 = 10$ mm, $x_3 = 85$ mm

$y_1 = 140$ mm, $y_2 = 0$, $y_3 = 140$ mm

$$\bar{x} = \frac{a_1x_1 + a_2x_2 + a_3x_3}{a_1 + a_2 + a_3}$$

$$= \frac{2600\times85+6000\times10+2600\times85}{2600+6000+2600}$$

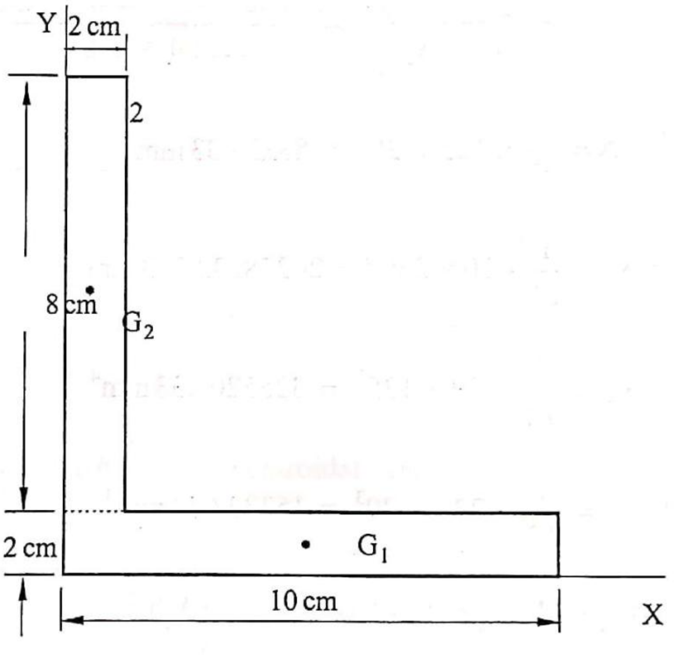
$= 44.821$ mm

$$IG_{YY} = (IG_{1YY} + A_1h_1^2) + (IG_{2YY} + A_2h_2^2) + (IG_{3YY} + A_3h_3^2)$$

$$= \frac{1}{12}\times20\times130^3+130\times20\times40.179^2+\frac{1}{12}\times300\times20^3+20\times300\times$$

$$(34.821)^2+\frac{1}{12}\times20\times130^3+130\times20\times40.179^2$$

$= 23192976.19\ mm^4$



$$= (\frac{1}{12}\times10\times2^3 + 20\times1^2)$$

$$+ (\frac{1}{12}\times2\times8^3 + 16\times6^2)$$

$= 688\ cm^4$

$$I_Y = (IG_{1YY} + A_1x_1^2) + (IG_{2YY} + A_2x_2^2)$$

$$= (\frac{1}{12}\times2\times10^3 + 20\times5^2) + (\frac{1}{12}\times8\times2^3 + 16\times1^2)$$

$= 688\ cm^4$

$$IG_{XX} = \frac{BD^3}{12} - \frac{bd^3}{12}$$

$$B = 130 + 20 = 150 \text{ mm}$$

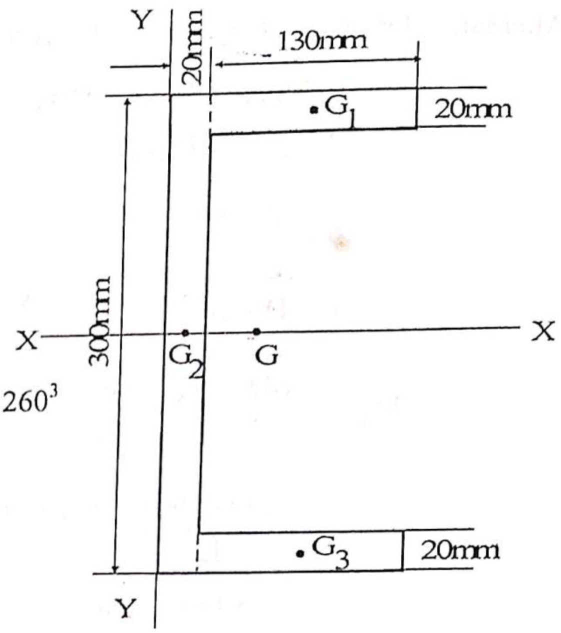
$$D = 300 \text{ mm}$$

$$b = 130 \text{ mm}$$

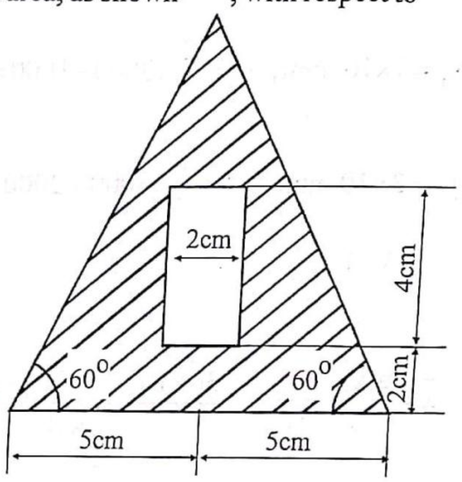
$$d = (300 - 40) = 260 \text{ mm}$$

$$IG_{XX} = \frac{1}{12} \times 150 \times 300^3 - \frac{1}{12} \times 130 \times 260^3$$

$$= 147093333.33 \text{ mm}^4$$



Q: Calculate the moment of inertia of the shaded area, as shown , with respect to the centroidal axes.



Solution.

$$a_1 = 2 \times \frac{1}{2} \times 5 \times h$$

$$\tan 60 = \frac{h}{5} \therefore \quad h = 5 \tan 60$$

$$h = 8.66 \text{ cm}$$

$$a_1 = 2 \times \frac{1}{2} \times 5 \times 8.66 = 43.3 \text{ cm}^2$$

$$a_2 = 2 \times 4 = 8 \text{ cm}^2$$

$$x_1 = 0, \ x_2 = 0$$

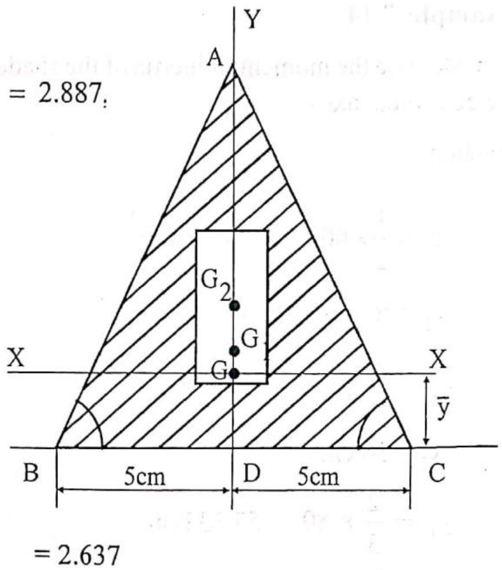
$$y_1 = \frac{1}{3} \times h = \frac{1}{3} \times 8.66 = 2.887.$$

$$y_2 = 2 + 2 = 4 \text{ cm}$$

$$\bar{x} = \frac{a_1 x_1 - a_2 x_2}{a_1 - a_2} = 0$$

$$\bar{y} = \frac{a_1 y_1 - a_2 y_2}{a_1 - a_2} =$$

$$= \frac{43.33 \times 2.887 - 8 \times 4}{43.3 - 8} \quad = 2.637$$



$$I_{G_{XX}} = (I_{G_{1XX}} + A_1 h_1^2) - (I_{G_{2XX}} + A_2 h_2^2)$$

$$h_1 = 2.887 - 2.637 = 0.25$$

$$h_2 = 4 - 2.637 = 1.363$$

$$I_{G_{XX}} = \frac{(10 \times (8.66)^3}{36} + 2 \times \frac{1}{2} \times 5 \times 8.66 \times (0.25)^2$$

$$- (\frac{1}{12} \times 2 \times 4^3 + 2 \times 4 \times (1.363)^2)$$

$$= (180.406 + 2.706) - (10.667 + 14.862)$$

$$I_{G_{XX}} = 157.583 \text{ cm}^4$$

$I_{G_{YY}}$ = M.I. of triangle ABD about its base AD + M.I. of triangle ADC about its base AD – M.I. of rectangle about its centroidal axis.

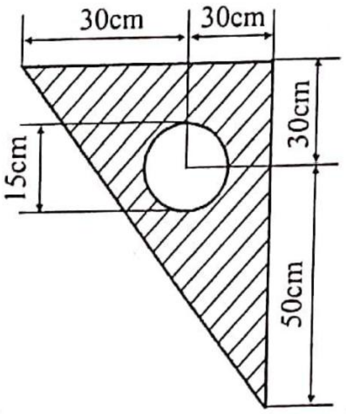
$$= \frac{1}{12} \times AD \times BD^3 + \frac{1}{12} \times AD \times CD^3 - \frac{1}{12} \times 4 \times 2^3$$

$$= \frac{1}{12} \times 8.66 \times 5^3 + \frac{1}{12} \times 8.66 \times 5^3 - \frac{1}{12} \times 4 \times 2^3$$

$$I_{G_{YY}} = 177.75 \text{ cm}^4$$

Q: Calculate the moment of inertia of the shaded area, as shown , with respect to the centroidal axes.

Solution.



$$a_1 = \frac{1}{2} \times 60 \times 80 = 2400 \text{ cm}^2$$

$$a_2 = \pi r^2 = 176.715 \text{ cm}^2$$

$$x_1 = \frac{2}{3} \times 60 = 40 \text{ cm}$$

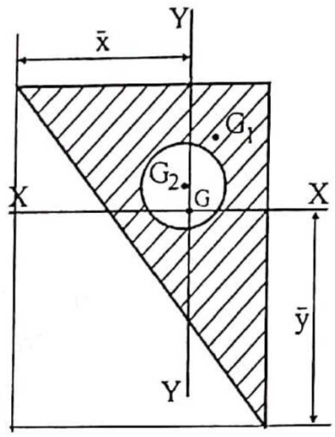
$$x_2 = 30 \text{ cm}$$

$$y_1 = \frac{2}{3} \times 80 = 53.333 \text{ cm}$$

$$y_2 = 50 \text{ cm}$$

$$\bar{x} = \frac{a_1 x_1 - a_2 x_2}{a_1 - a_2}$$

$$= \frac{2400 \times 40 - 176.715 \times 30}{2400 - 176.715} = 40.795 \text{ cm}$$



$$\bar{y} = \frac{a_1 y_1 - a_2 y_2}{a_1 - a_2}$$

$$= \frac{2400 \times 53.33 - 176.715 \times 50}{2400 - 176.715}$$

$$\bar{y} = 53.598 \text{ cm}$$

$$I_{G_{XX}} = (I_{G_{1XX}} + A_1 h_1^2) - (I_{G_{2XX}} + A_2 h_2^2)$$

$$= (\frac{60 \times 80^3}{36} + \frac{1}{2} \times 60 \times 80 \times (0.265)^2) - (\frac{\pi \times 15^4}{64} + \pi \times \left[\frac{15}{2}\right]^2 \times (3.598)^2)$$

$$= (853333.333 + 168.54) - (2485.049 + 2287.677)$$

$$= 853304.5 \text{ cm}^4$$

$$I_{G_{YY}} = (I_{G_{1YY}} + A_1 h_1^2) - (I_{G_{2YY}} + A_2 h_2^2)$$

$$= (\frac{80 \times 60^3}{36} + \frac{1}{2} \times 60 \times 80 \times (0.795)^2) - (\frac{\pi \times 15^4}{64} + \pi \times \left[\frac{15}{2}\right]^2 \times (10.795)^2)$$

$$= 480000 + 1516.86 - 2485.049 - 20592.957$$

$$I_{G_{YY}} = 458438.854 \text{ cm}^4$$

◆ **Mass moment of inertia of thin circular plate.**

Consider an elemental ring of radial thickness dr at a distance r from the centre.

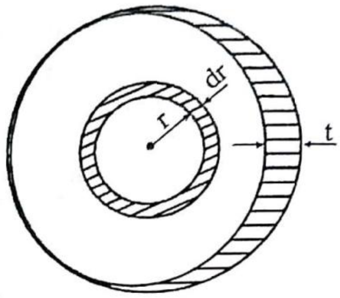
Volume of the ring = $2 \pi r \, dr \times t$

Mass of elemental ring = $2 \pi r \, dr \, t \times \rho$.

Moment of inertia of the plate about the axis through the centre and perpendicular to the plane of plate is $\int dm \, r^2$

$$I_{ZZ} = \int_0^R 2 \pi r \, dr \, t \, \rho r^2$$

$$= 2\pi t \rho \int_0^R r^3\, dr = 2\pi t \rho \left[\frac{r^4}{4}\right]_0^R$$



$$= \frac{2\pi t \rho}{4} R^4 = (\pi R^2 t \rho)\frac{R^2}{2}$$

$$= \frac{mR^2}{2}$$

Polar moment of inertia $I_{ZZ} = I_{XX} + I_{YY} = \frac{mR^2}{2}$

$$I_{XX} = I_{YY} = \frac{I_{ZZ}}{2} = \frac{mR^2}{4}$$

◆ **Mass moment of inertia of solid cylinder about its centroidal axes.**

Consider an elemental circular disc of thickness dy at a distance y from the centroidal XX axis of the cylinder.

Mass of the element, $dm = \pi R^2\, dy\, \rho$.

Moment of inertia thin circular disc about its centroidal XX axis, $dI = \frac{dm\, R^2}{4}$

$$dI_{XX} = dI + (dm)\, y^2$$

$$= \left[\frac{dmR^2}{4} + dm\, y^2\right]$$

$$= \pi R^2\, dy \frac{\rho R^2}{4} + \pi R^2\, dy \times \rho\, y^2$$

$$I_{XX} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \left(\frac{\pi R^4}{4}\rho\right) dy + \pi R^2 \rho \int_{-\frac{h}{2}}^{\frac{h}{2}} y^2\, dy$$

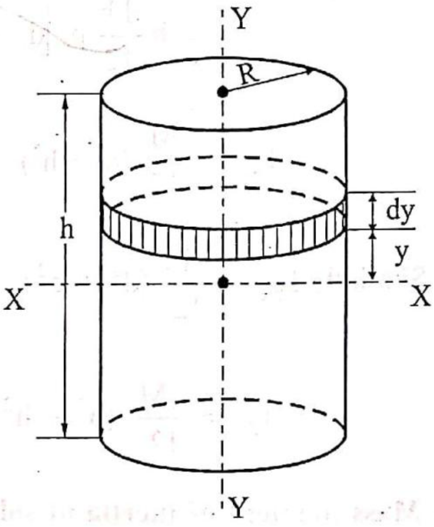
$$= 2\pi \frac{R^4}{4} \rho\, [y]_0^{\frac{h}{2}} + 2.\pi R^2 \rho \left[\frac{y^3}{3}\right]_0^{\frac{h}{2}}$$

$$= \frac{\pi R^4}{2}\frac{\rho h}{2} + \frac{2\pi R^2}{3}\rho\left[\frac{h}{2}\right]^3$$

$$= \frac{\pi R^2 h\rho}{4}\left(R^2 + \frac{h^2}{3}\right)$$

$$= \frac{M}{4}\left(\frac{3R^2 + h^2}{3}\right)$$

$$= \frac{M}{12}\left(3R^2 + h^2\right)$$



$$I_{zz} = I_{xx} = \frac{M}{12}\left(3R^2 + h^2\right)$$

◆**Force in space.**

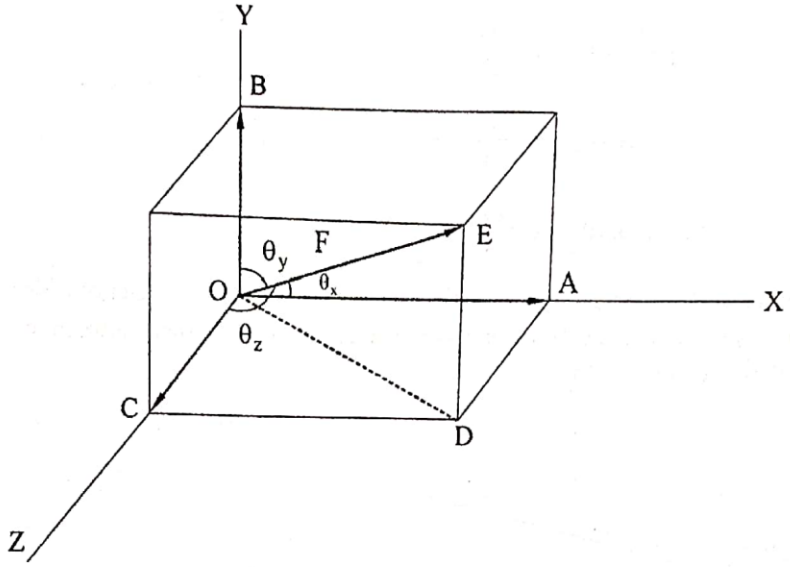
Consider a force F acting at the origin O of the system of rectangular co-ordinates X, Y and Z. The angles θ_x, θ_y and θ_z that the force F makes with the X, Y and Z axes define the direction of the force F. The components of the force F along the X, Y and Z directions are given by,

$$F_x = F\cos\theta_x$$

$$F_y = F\cos\theta_y \text{ and}$$

$$F_z = F\cos\theta_z$$

$$F = \sqrt{F_x^2 + F_y^2 + F_z^2}$$



The direction of force F acting at the origin of the system of co-ordinates can be specified by specifying a second point A on the line of action of force F which is at a distance d from the origin as shown. The co-ordinates of A are dx, dy and dz.

$$OA^2 = OC^2 + CA^2$$

$$= OB^2 + BC^2 + CA^2$$

$$d^2 = dx^2 + dy^2 + dz^2$$

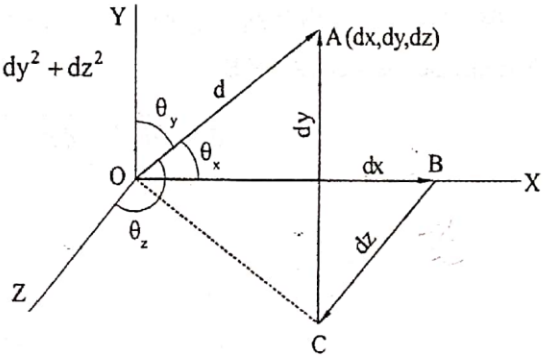
$$\therefore d = \sqrt{dx^2 + dy^2 + dz^2}$$

where, $dx = d\cos\theta_x$

$dy = d\cos\theta_y$ and

$dz = d\cos\theta_z$

$$\cos\theta_x = \frac{dx}{d}$$



$$\cos\theta_y = \frac{dy}{d}$$

$$\cos\theta_z = \frac{dz}{d}$$

$$F_x = F\cos\theta_x = F\times\frac{dx}{d}$$

$$F_y = F\cos\theta_y = F\times\frac{dy}{d}$$

$$F_z = F\cos\theta_z = F\times\frac{dz}{d}$$

When A and B are two points on the line of action of a force F, neither of which is at the origin, then the components dx, dy and dz are equal to the difference between the co-ordinates of B and A.

$$dx = x_B - x_A \qquad dy = y_B - y_A \text{ and } \qquad dz = z_B - z_A.$$

The distance between A and B,

$$d = \sqrt{dx^2 + dy^2 + dz^2}$$

◆ Resultant of concurrent forces in space.

Resultant of concurrent force in space can be obtained by summing their rectangular components. Summing the X components of the forces we will get the component R_x of the resultant force R.

$$R_x = \Sigma F_x \text{ Similarly } R_y = \Sigma F_y \text{ and } R_z = \Sigma F_z$$

$$\text{Resultant force } R = R_x i + R_y j + R_z k$$

The magnitude of the resultant and the angle θ_x, θ_y and θ_z it forms with the axes of coordinates are given by.

$$|R| = \sqrt{R_x^2 + R_y^2 + R_z^2} \text{ and}$$

$$\cos\theta_x = \frac{R_x}{R}; \quad \cos\theta_y = \frac{R_y}{R}; \quad \cos\theta_z = \frac{R_z}{R}$$

Q: Two cables AB and AC are attached at A as shown in Fig 2.48. Determine the resultant of the forces exerted at A by the two cables, if the tension is 2000 N in the cable AB and 1500 N in the cable AC.

Solution.

The coordinates of A, B and C are,

A (52,0,0),

B (0,50,40) and C (0, 62, –50)

$x_A = 52, y_A = 0, z_A = 0$

$x_B = 0, y_B = 50, z_B = 40$

$x_C = 0, y_C = 62, z_C = -50$

$$d_{AB} = \sqrt{(x_B - x_A)^2 + (y_B - y_A)^2 + (z_B - z_A)^2}$$

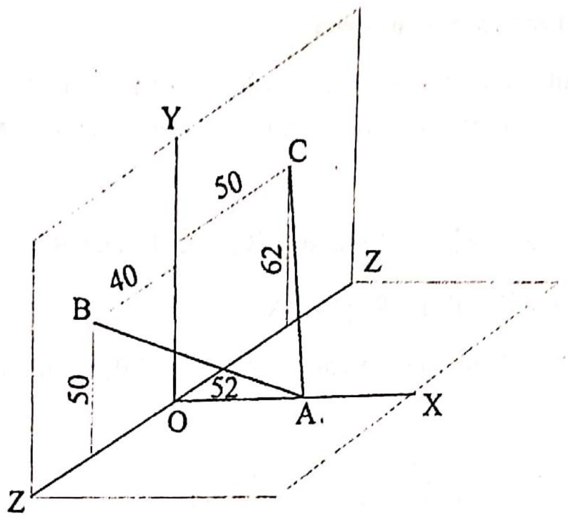
$$= \sqrt{(-52)^2 + 50^2 + 40^2}$$

$$= 82.5 \text{ m}$$

$$d_{AC} = \sqrt{(x_C - x_A)^2 + (y_C - y_A)^2 + (z_C - z_A)^2}$$

$$= \sqrt{(-52)^2 + 62^2 + (-50)^2}$$

$$= 95.12 \text{ m.}$$



Unit vector in the direction of AB,

$$= \frac{(0-52)i + (50-0)j + (40-0)k}{\sqrt{(-52)^2 + (50)^2 + (40)^2}}$$

$$= \frac{-52i + 50j + 40k}{82.49}$$

$$\text{Force vector along AB} = 2000\left(\frac{-52i + 50j + 40k}{82.49}\right)$$

$$= -1260.75\, i + 1212.27\, j + 969.81\, k$$

Unit vector in the direction of AC,

$$=\frac{(0-52)i+(62-0)j+(-50-0)k}{\sqrt{(-52)^2+(62)^2+(-50)^2}}$$

$$=\frac{-52i+62j-50k}{95.12}$$

Force vector along AC $= 1500\left(\frac{-52i+62j-50k}{95.12}\right)$

$=-820.02\ i+977.71\ j-788.48\ k$

Resultant force at A, $R = F_{AB} + F_{AC}$

$= (-1260.75\ i + 1212.27\ j + 969.81\ k) + (-820.02\ i + 977.71\ j - 788.48\ k)$

$=-2080.78\ i + 2189.98\ j + 181.33\ k$

The magnitude of resultant at A, $R = \sqrt{(-2080.78)^2+(2189.98)^2+(181.33)^2}$

$= 3026.31$ N

Direction of resultant is given by,

$$\cos\theta_x = \frac{R_x}{R} = \frac{-2080.78}{3026.31}$$

$$\theta_x = 133.44^0$$

$$\cos\theta_y = \frac{R_y}{R} = \frac{2189.98}{3026.31}$$

$$\theta_y = 43.64^0$$

$$\cos\theta_z = \frac{R_z}{R} = \frac{181.33}{3026.31}$$

$$\theta_z = 86.56^0$$

Q: A post is held in vertical position by three cable AB, AC and AD as shown . If the tension in the cable AB is 40 N, calculate the requird tension in AC and AD so that the resultant of the three forces applied at A is vertical.

Solution.

For the resultant force to be vertical, the components R_x and R_z must be zero.

Co-ordinates of A are (0, 48, 0). Co-ordinates of B are (16, 0, 12). Co-ordinates of C are (16, 0, – 24). Co-ordinates of D are (–14, 0, 0).

$$x_A = 0, y_A = 48, z_A = 0$$

$$x_B = 16, y_B = 0, z_B = 12$$

$$x_C = 16, y_C = 0, z_C = -24$$

$$x_D = -14, y_D = 0, z_D = 0$$

$$d_{AB} = \sqrt{(x_B - x_A)^2 + (y_B - y_A)^2 + (z_B - z_A)^2}$$

$$d_{AB} = \sqrt{(16)^2 + (-48)^2 + 12^2}$$

$$= 52 \text{ m}$$

$$d_{AC} = \sqrt{(x_C - x_A)^2 + (y_C - y_A)^2 + (z_C - z_A)^2}$$

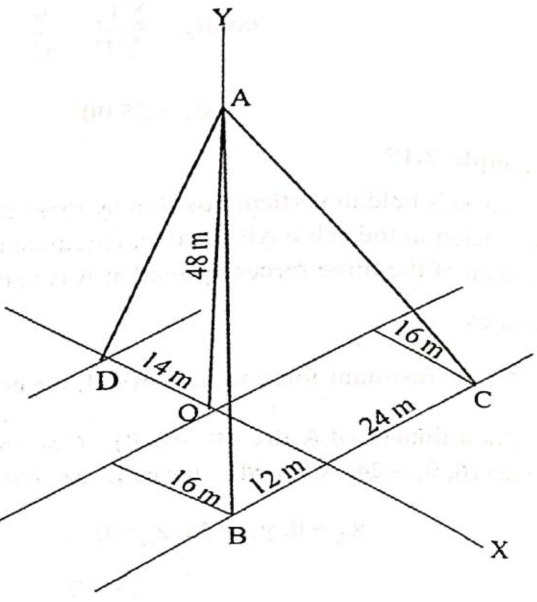
$$d_{AC} = \sqrt{16^2 + (-48)^2 + (-24)^2}$$

$$= 56 \text{ m}$$

$$d_{AD} = \sqrt{(x_D - x_A)^2 + (y_D - y_A)^2 + (z_D - z_A)^2}$$

$$d_{AD} = \sqrt{(-14)^2 + (-48)^2 + 0^2}$$

$$= 50 \text{ m}$$



Unit vector in the direction of AB,

$$= \frac{(16-0)i + (0-48)j + (12-0)k}{\sqrt{(16)^2 + (-48)^2 + (12)^2}}$$

$$= \frac{16i - 48j + 12k}{52}$$

$$\text{Force vector along AB} = 40\left(\frac{16i - 48j + 12k}{52}\right)$$

$$= 12.31\ i - 36.92\ j + 9.23\ k$$

Unit vector in the direction of AC,

$$= \frac{(16-0)i + (0-48)j + (0-24)k}{\sqrt{(16)^2 + (-48)^2 + (-24)^2}}$$

$$= \frac{16i - 48j - 24k}{56}$$

$$\text{Force vector along AC} = F_{AC}\left(\frac{16i - 48j - 24k}{56}\right)$$

$$= 0.29\ F_{AC}\, i - 0.86\ F_{AC}\, j - 0.43\ F_{AC}\, k$$

Unit vector in the direction of AD,

$$= \frac{(0-14)i + (0-48)j + (0-0)k}{\sqrt{(-14)^2 + (-48)^2 + (0)^2}}$$

$$= \frac{-14i - 48j + 0k}{50}$$

$$\text{Force vector along AC} = F_{AD}\left(\frac{-14i - 48j + 0k}{50}\right)$$

$$= -0.28\ F_{AD}\, i - 0.96\ F_{AD}\, j + 0\ k$$

Resultant force at A, $R = F_{AB} + F_{AC} + F_{AD}$

$$= (12.31\,i - 36.92\,j + 9.23\,k) + (0.29\,F_{AC}\,i - 0.86\,F_{AC}\,j - 0.43\,F_{AC}\,k) + (-0.28\,F_{AD}\,i - 0.96\,F_{AD}\,j + 0\,k)$$

$$= (12.31 + 0.29\,F_{AC} - 0.28\,F_{AD})\,i + (-36.96 - 0.86\,F_{AC} - 0.96\,F_{AD})\,j + (9.23 - 0.43\,F_{AC} + 0)\,k$$

For the resultant to the vertical, the X and Z components must be zero. i.e.,

$F_z = (9.23 - 0.43\, F_{AC} + 0) = 0$

$$0.43\, F_{AC} = 9.23$$

$$F_{AC} = \frac{9.23}{0.343}$$

$$= 21.47\ \text{N}$$

$$F_x = (12.31 + 0.29\, F_{AC} - 0.28\, F_{AD}) = 0$$

$$0.28\, F_{AD} = 12.31 + 0.29\, F_{AC}$$

$$= 1231 + 0.29 \times 21.47$$

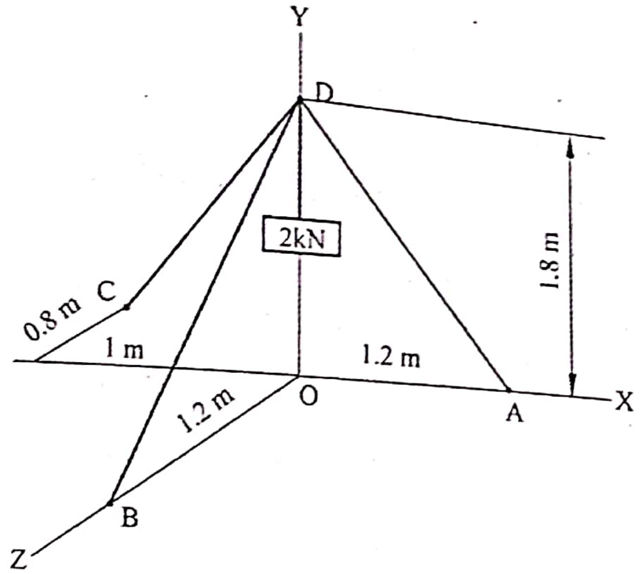
$$F_{AD} = 66.20\ \text{N}$$

◆ Equilibrium of concurrent forces in space.

A particle A will be in equilubrium if the resultant of all the forces acting on it is zero. For equilibrium the components R_x, R_y and R_z must be zero. ie.,

$$\Sigma F_x = R_x = 0, \quad \Sigma F_y = R_y = 0, \quad \Sigma F_z = R_z = 0.$$

Q: A tripod supports a load of 2 kN as shown. The ends A, B and C are in the X- Z plane. Find the force in the three legs of the tripod.



Solution:

Co-ordinates of A are (1.2, 0, 0)

Co-ordinates of B are (1, 1, 1.2)

Co-ordinates of C are (–1, 0, –0.8)

Co-ordinates of D are (0, 1.8, 0)

Position vector $AD = (0 - 1.2)\, i + (1.8 - 0)\, k + (0 - 0)\, k$

$$= -1.2\, i + 1.8\, k + 0\, k$$

Unit vector along AD = $\frac{-1.2\,i+1.8\,j+0k}{\sqrt{(-1.2)^2+(1.8)^2+(0)^2}}$

$= -0.56\,i + 0.83\,j + 0\,k$

Force vector along AD = F_{AD} $(-0.56\,i + 0.83\,j + 0k)$

Position vector BD = $(0-0)\,i + (1.8-0) + (0-1.2)\,k$

$= 0\,i + 1.8\,j - 1.2\,k$

Unit vector along AD = $\frac{0\,i+1.8j-1.2k}{\sqrt{0^2+1.8^2+(-1.2)^2}}$

$= 0\,i + 0.83\,j - 0.56\,k$

Force vector along BD = F_{BD} $(0\,i + 0.83\,j - 0.56\,k)$

Position vector CD = $[0-(-1)]\,i + (1.8-0)\,j + [0-(-0.8)]\,k$

$= i + 1.8\,j + 0.8\,k$

Unit vector along CD = $\frac{0\,i+1.8\,j+0.8k}{\sqrt{1^2+1.8^2+0.5^2}}$

$= 0.45\,i + 0.81\,j + 0.36\,k$

Force vector along CD = F_{CD} $(0.45\,i + 0.81\,j + 0.36\,k)$

Consider the equilibrium of point D, sum of force vector acts at A must be zero.

$F_{AD} = (-0.56\,i + 0.83\,j + 0\,k) + F_{BD}\,(0\,i + 0.83\,j - 0.56\,k) + F_{CD}\,(0.45\,i + 0.81\,j + 0.36\,k) - 2\,j = 0$

$(-0.56\,F_{AD} + 0 + 0.45\,F_{CD} + 0)\,i + (0.83\,F_{AD} + 0.83\,F_{BD} + 0.81\,F_{CD} - 2)\,j + (0 - 0.56\,F_{BD} + 0.36\,F_{CD} + 0)\,k = 0$

$-0.56\,F_{AD} + 0.45\,F_{CD} = 0$

$0.83\,F_{AD} + 0.83\,F_{BD} + 0.81\,F_{CD} = 2$

$-0.56\,F_{BD} + 0.36\,F_{CD} = 0$

Solving the above three equations

$F_{AD} = 802$ N

$F_{BD} = 642$ N and

$F_{CD} = 990$ N

Q: A rectangular concrete slab supports loads at its four corners as shown . Determine the resultant of these forces and the point of application of the resultant.

Solution.

$$\vec{F}_O = -125\,\vec{j},\quad \vec{F}_A = -250\,\vec{j},\quad \vec{F}_B = -150\,\vec{j} \text{ and } \vec{F}_C = -100\,\vec{j}$$

Resultant $\vec{R} = \Sigma\vec{F} = \vec{F}_O + \vec{F}_A + \vec{F}_B + \vec{F}_C$

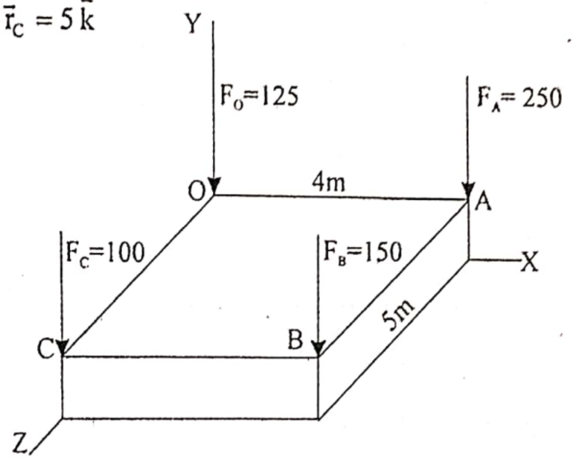
$$= -125\,\vec{j} + (-250\,\vec{j}) + (-150\,\vec{j}) + (-100\,\vec{j})$$
$$= -625\,\vec{j}.$$

Position vector of point O, $\vec{r}_O = 0$

Position vector of point A, $\vec{r}_A = 4\,\vec{i}$

Position vector of point B, $\vec{r}_B = 4\,\vec{i} + 5\,\vec{k}$

Position vector of point C, $\vec{r}_C = 5\,\vec{k}$



Moment of $\vec{F}_O$ about O is zero.

Moment of $\vec{F}_A$ about O is $\vec{r}_A \times \vec{F}_A$

$$= 4\,\vec{i} \times (-250\,\vec{j})$$
$$= -1000\,(\vec{i} \times \vec{j})$$
$$= -1000\,\vec{k}.$$

Moment of $\vec{F}_B$ about O is $\vec{r}_B \times \vec{F}_B$

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 4 & 0 & 5 \\ 0 & -150 & 0 \end{vmatrix} = 750\,\vec{i} - 600\,\vec{k}$$

Moment of $\vec{F}_C$ about O is $\vec{r}_C \times \vec{F}_C$

$$= 5\,\vec{k} \times (-100\,\vec{j}) = -500\,(\vec{k} \times j)$$
$$= 500\,\vec{i}.$$

Sum of moment of all the forces about O,

$$\Sigma M_O = 0 + (-1000\vec{k}) + (750\vec{i} - 600\vec{k}) + 500\vec{i}$$

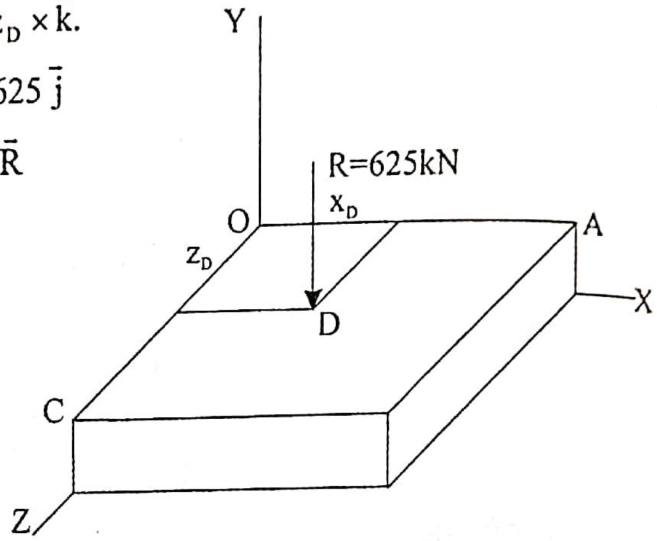
$$= 1250\vec{i} - 1600\vec{k}.$$

Let co-ordinates of point D, the point of application of resultant, be x_D and z_D.

Position vector of D is, $\vec{r}_D = x_D\vec{i} + z_D \times \vec{k}$.

$$\vec{F}_D = \vec{R} = -625\vec{j}$$

Moment of resultant about O is $\vec{r}_D \times \vec{R}$



$= (x_D\,\vec{i} + z_D\,\vec{k}) \times (-625\,\vec{j})$

$= -625\,x_D\,(\vec{i} \times \vec{j}) - 625\,z_D\,(\vec{k} \times \vec{j})$

$= -625\,x_D\,\vec{k} - 625\,z_D\,(-\vec{i})$

$= -625\,x_D\,\vec{k} + 625\,z_D\,\vec{i}.$

Equating this moment of resultant about O and the sum of moment of all the forces about O,

$$\vec{r}_D \times \vec{R} = \Sigma M_O.$$

$$625\,z_D\,\vec{i} - 625\,x_D\,\vec{k} = 1250\,\vec{i} - 1600\,\vec{k}.$$

$$625\,z_D = 1250$$

$$z_D = 2\text{ m}$$

$$-625\,x_D = -1600$$

$$x_D = \frac{1600}{625} = 2.56\text{ m}.$$