Rotation of rigid body.

15.1 Kinematics of rotation

When a rigid body rotates about a fixed axis, its position at any instant can be defined by angular displacement of a certain plane of the body passing through the axis of rotation. This angle of rotation is generally denoted by θ and is expressed in radians. When the body turns such that in equal intervals of time it describes equal angle of rotation, the motion is said to be uniform rotation. The rate of change of angular displacement with respect to time is called angular velocity and is denoted by ω.

$$\omega = \frac{d\theta}{dt} \text{ rad/s}$$

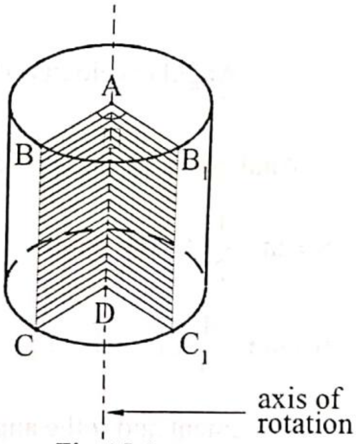

Fig.15.1

The rate of change of angular velocity with respect to time is called angular acceleration and is denoted by α.

$$\alpha = \frac{d\omega}{dt} = \frac{d}{dt}\left[\frac{d\theta}{dt}\right] = \frac{d^2\theta}{dt^2} \text{ rad/s}^2$$

Relationship between linear veocity and angular velocity.

Consider a particle moving along a circular arc of radius r. Let $d\theta$ be the angular rotation of the particle in dt seconds and AB be the displacement of the particle in dt seconds. The linear displacement , $AB = ds = r \times d\theta$.

The linear velocity $V = \frac{ds}{dt} = \frac{d}{dt}(rd\theta)$

$$= r \cdot \frac{d\theta}{dt} = r\omega$$

$$V = r\omega$$

Relationship between linear acceleration and angular acceleration.

The linear velocity, $V = r \times \omega$

Differentiating with respect to time,

$$\frac{dV}{dt} = \frac{d}{dt}(r\omega)$$

$$= r \cdot \frac{d\omega}{dt}$$

$$= r \times \alpha$$

$$a = r\alpha$$

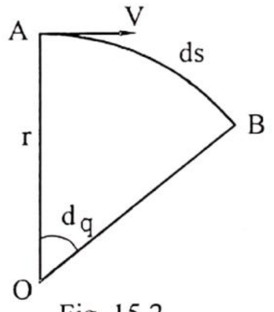

Fig. 15.2

When N is the number of revolutions of the body about the axis in one minitue, then number of rotations in one second is $\frac{N}{60}$. In one revolution, the angular displacement is 2π rad (360°). Therefore in $\frac{N}{60}$ revolution/s, the angular displcement/s is $2\pi \times \frac{N}{60}$ rad/s.

Angular velocity $\omega = \frac{2\pi N}{60}$ rad/s

Analogous to the expression in rectilinear motion, $V = u + at$, $V^2 = u^2 + 2\,aS$, and $S = ut + \frac{1}{2}at^2$, the expressions in rotation motion are, $\omega_2 = \omega_1 + \alpha\,t$, $\omega_2^2 = \omega_1^2 + 2\alpha\theta$ and $\theta = \omega_1 t + \frac{1}{2}\alpha\,t^2$, where ω_1 and ω_2 are the initial and final angular velocities, θ, angular displacement and α the angular acceleration.

Example 15.1

The armature of an electric motor, has angular speed of 1800 rpm at the instant when the power is cut off. If it comes to rest in 6 seconds, calculate the angular deceleration assuming it is constant. How many revolutions does the armature make during this period.

Solution:

$$N_1 = 1800\,\text{rpm}$$

$$\omega_1 = \frac{2\pi N_1}{60} = 2\pi \times 30\,\text{rad/s}$$

$$\omega_2 = \omega_1 + \alpha t$$

$$0 = 2\pi \times 30 - \alpha \times 6$$

$$\alpha = \frac{2\pi \times 30}{6} = 31.4\,\text{rad/s}^2$$

Angle turned, $\theta = \omega_1 t - \frac{1}{2}\alpha t^2$

$$= 2\pi \times 30 \times 6 - \frac{1}{2} \times 31.4 \times 6^2$$

$$= 565.77\,\text{rad}$$

$$= 90\,\text{revolutions}$$

Example 15.2

A flywheel rotates with a constant retardation due to brakeing. In the first 10 seconds, it made 300 revolutions. At t = 7.5 s, its angular veocity was 40π rad/s. Determine,

(i) the value of constant retardation

(ii) the total time taken to come to rest and

(iii) the total revolutions made till it comes to rest.

Solution:

Angular displacement $\theta = \omega_1 t + \frac{1}{2}\alpha t^2$

At t = 10, $\theta = 300 \times 2\pi$ rad

$$300 \times 2\pi = \omega_1 \times 10 + \frac{1}{2} \times 10^2 \times \alpha$$

$$60\,\pi = \omega_1 + 5\alpha \,..........(i)$$

$$\omega_2 = \omega_1 + \alpha t$$

at t = 7.5s, $\omega = 40\pi$ rad/s

$$40\pi = \omega_1 + 7.5\,\alpha \text{ ....(ii)}$$

From equations (i) and (ii)

$$\alpha = -\,8\pi \text{ rad/s}^2$$

From equation (i),

$$60\pi = \omega_1 + 5\times(-8\pi)$$

$$\omega_1 = 100\pi \text{ rad/s}$$

When the flywheel comes to rest, $\omega = 0$

$\omega_1 = 100\,\pi$, $\omega_2 = 0$, $\alpha = -8\pi$ rad/s²

$$\omega_2 = \omega_1 + \alpha t$$

$$0 = 100\pi + (-8\pi)\times t$$

Total time taken to come to rest $t = \dfrac{100\pi}{8\pi}$

$$= 12.5 \text{ s}$$

Total revolutions made by the flywheel before it comes to rest,

$$\omega_2^2 = \omega_1^2 + 2\alpha\theta$$

$$0 = (100\pi)^2 + 2\times(-8\pi)\theta$$

$$\theta = 19625 \text{ rad}$$

$$= 312.5 \text{ revolutions}$$

Example 15.3

A rotor of an electric motor is uniformly accelerated to a speed of 1800 rpm from rest in 5 seconds and then immediately power is switched off and the rotor decelerates uniformly to rest. If the total time elapsed is 12.5 s, determine the number of revolutions made in (i) acceleration and (ii) deceleration.

Solution:

The rotor is accelerated from rest

$$\therefore\ \omega_1 = 0$$

$$\omega_2 = \frac{2\pi\, N_2}{60} = \frac{2\pi \times 1800}{60}$$

$$= 60\ \pi \text{ rad/s}$$

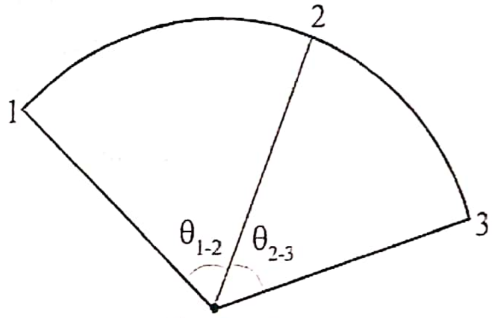

Fig. 15.3

$$\omega_2 = \omega_1 + \alpha_{1-2} \times t_{1-2}$$

$$60\ \pi = 0 + 5\ \alpha_{1-2}$$

$$\alpha_{1-2} = 12\ \pi \text{ rad/s}^2$$

$$\theta_{1-2} = \omega_1\, t_{1-2} + \frac{1}{2}\alpha_{1-2} \times t^2{}_{1-2}$$

$$= 0 + \frac{1}{2} \times 12\pi \times 5^2 \quad = 150\ \pi \text{ rad} \quad = 75 \text{ revolutions}$$

$$\omega_3 = 0$$

$$\omega_3 = \omega_2 + \alpha_{2-3} \times t_{2-3}$$

Total time of rotation is 12.5s

$$\therefore\ t_{2-3} = 12.5 - 5 = 7.5 \text{ s}$$

$$0 = 60\ \pi + \alpha_{2-3} \times 7.5$$

$$\alpha_{2-3} = -8\ \pi \text{ rad/s}^2$$

$$\theta_{2-3} = \omega_2 \times t_{2-3} + \frac{1}{2}\alpha_{2\text{-}3} \times t^2_{2\text{-}3} \quad = \ 60\ \pi \times 7.5 + \frac{1}{2}(-8\ \pi) \times 7.5^2$$

$$= 450\ \pi - 225\ \pi \quad = 225\ \pi \quad = 112.5 \text{ revolutions}$$

Example 15.6

A wheel rotates for 5 seconds with a constant angular acceleration and describes during that time 100 radian. It then rotates with constant angular velocity and during the next 5 second describes 80 radian. Find the initial angular velocity and the angular acceleration.

Solution:

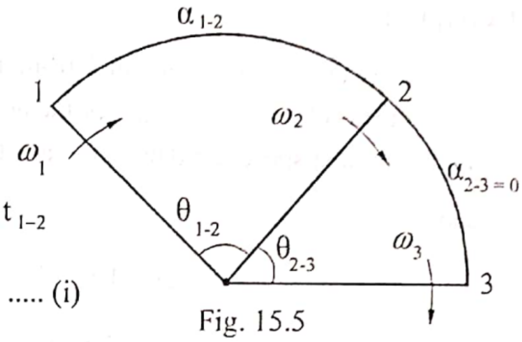

Fig. 15.5

Given: $\theta_{1-2} = 100$ rad $\quad \theta_{2-3} = 80$ rad

$t_{1-2} = 5$ s $\quad t_{2-3} = 5$ s $\quad \alpha_{2-3} = 0$

$$\omega_2 = \omega_1 + \alpha_{1-2} \times t_{1-2}$$

$$\omega_2 = \omega_1 + 5\,\alpha_{1-2} \quad \text{..... (i)}$$

$$\theta_{1-2} = \omega_1 t_{1-2} + \frac{1}{2} \times \alpha_{1-2} \times t_{1-2}^2$$

$$100 = 5\,\omega_1 + \frac{1}{2} \times \alpha_{1-2} \times 5^2$$

$$20 = \omega_1 + 2.5\,\alpha_{1-2} \quad \text{..... (ii)}$$

$$\theta_{2-3} = \omega_2 \times t_{2-3} + \frac{1}{2} \times \alpha_{2-3} \times t_{2-3}^2$$

$$80 = 5\,\omega_2 + 0$$

$$\omega_2 = 16 \text{ rad/s}$$

From equation (i)

$$16 = \omega_1 + 5\,\alpha_{1-2}$$

From equation (ii)

$$20 = \omega_1 + 2.5\,\alpha_{1-2}$$

$$-4 = 2.5\,\alpha_{1-2}$$

$$\alpha_{1-2} = -1.6 \text{ rad/s}^2$$

$$16 = \omega_1 + 5\,\alpha_{1-2}$$

$$16 = \omega_1 - 5 \times 1.6$$

$$\omega_1 = 24 \text{ rad/s}$$

Example 15.11

The angular acceleration of a wheel is given by $\alpha = 12 - t$ where α is in rad/s^2 and t is in seconds. If the angular velocity of the wheel is 60 rad/s at the end of 4 seconds, determine the angular velocity at the end of 6 seconds. Calculate the number of revolutions in these 6 seconds.

Solution:

$$\alpha = 12 - t$$

$$\frac{d\omega}{dt} = 12 - t$$

$$d\omega = (12 - t)\,dt$$

Integrating, $\omega = 12\,t - \frac{t^2}{2} + c_1$, c_1 is the constant of integration.

at t = 4s, ω = 60 rad/s

$$60 = 12 \times 4 - \frac{4^2}{2} + c_1$$

$$c_1 = 20$$

Angular velocity, $\omega = 12t - \frac{t^2}{2} + 20$

Angular velocity at the end of 6 seconds,

$$\omega = 12 \times 6 - \frac{6^2}{2} + 20$$

$$= 74 \text{ rad/s}$$

Angular velocity, $\omega = 12t - \frac{t^2}{2} + 20$

$$\frac{d\theta}{dt} = 12t - \frac{t^2}{2} + 20$$

$$d\theta = \left(12t - \frac{t^2}{2} + 20\right)dt$$

Integrating, $\theta = \frac{12t^2}{2} - \frac{t^3}{6} + 20t + c_2$

$$= 6t^2 - \frac{t^3}{6} + 20t + c_2$$

Let at t = 0, θ be θ_0

$$\theta_0 = 0 - 0 + 0 + c_2$$

$$c_2 = \theta_o$$

Angular displacement in 6 seconds,

$$\theta = 6t^2 - \frac{t^3}{6} + 20t + \theta_0$$

$$\theta - \theta_0 = 6t^2 - \frac{t^3}{6} + 20t$$

$$= 6 \times 6^2 - \frac{6^3}{6} + 20 \times 6$$

$$= 300 \text{ rad}$$

$$= \frac{300}{2\pi} \text{ revolutions}$$

$$= 47.77 \text{ revolutions}$$

15.2 Equation of motion of a rigid body rotating about a fixed axis.

When a rigid body rotates about a fixed axis, the resultant moment of all the exerted forces about the axis of rotation causes angular acceleration of the body. If I is the moment of inertia of the body with respect to the axis of rotation and α is the angular acceleration then, the resultant moment about the axis, torque, is given by, $T = I\alpha$.

This is the equation of motion for a rigid body rotating about a fixed axis.

Example 15.12

A right circular disc of weight 1500N and 750 mm diameter is free to rotate about its geometric axis and is constantly accelerated from rest to 300 rpm in 20s. Determine the constant torque required to produce this acceleration.

Solution:

$$\omega_2 = \omega_1 + \alpha t$$

$$\frac{2\pi \times 300}{60} = 0 + \alpha \times 20$$

$$\alpha = \frac{2\pi \times 300}{60 \times 20} = 1.57 \text{ rad/s}^2$$

Moment of inertia of disc, $I = mk^2$

$$= \frac{W}{g}\frac{r^2}{2}$$

$$= \frac{1500}{9.81} \times \frac{\left(\frac{0.75}{2}\right)^2}{2}$$

$$= 10.75 \text{ kg m}^2$$

Torque, $T = I\alpha$

$$= 10.75 \times 1.57$$

$$= 16.88 \text{ Nm}$$

15.3 Rotation under the action of a constant moment.

The equation of a rigid body rotating about a fixed axis is given by $T = I\alpha$.

$$T = I\frac{d^2\theta}{dt^2}$$

$$\frac{d^2\theta}{dt^2} = \frac{T}{I}$$

When the moment T is a constant,

$\int \frac{d^2\theta}{dt^2} = \frac{T}{I}t + c_1$, where c_1 is a constant of integration.

$$\frac{d\theta}{dt} = \frac{T}{I}t + c_1$$

Integrating, $\theta = \frac{T}{I}\frac{t^2}{2} + c_1 t + c_2$. c_2 is some other constant. The values of constants c_1 and c_2 can be obtained by applying the given conditions regarding the angular velocity $\frac{d\theta}{dt}$ and angular displacement θ.

Example 15.13

A shaft of radius r rotates with constant angular speed ω in bearings for which the coefficient of friction is μ. Through what angle θ will it rotate after the driving torque is removed.

Solution:

Frictional force = $\mu R_N = \mu W$.

Torque due to frictional force = μWr.

Initial angular velocity $\omega_1 = \omega$

Final angular velocity $\omega_2 = 0$

$$T = \mu Wr$$

$$I \times \alpha = \mu Wr$$

$$\alpha = \frac{\mu Wr}{I}$$

$$= \frac{\mu Wr}{m\frac{r^2}{2}}$$

$$\alpha = \frac{2\mu g}{r}$$

$$\omega_2^2 = \omega_1^2 - 2\alpha \times \theta$$

$$0 = \omega^2 - 2 \times \alpha\,\theta$$

$$\theta = \frac{\omega^2}{2\alpha}$$

Substituting for α,

$$\theta = \frac{\omega^2}{2 \times \left(\frac{2\mu g}{r}\right)}$$

$$\theta = \frac{\omega^2 r}{4\mu g}\ \text{rad}$$

Example 15.14

A solid right circular drum of radius 0.3 m and weight 150 N is free to rotate about its geometric axis as shown in fig 15.8. Wound around the circumference of the drum is a flexible cord carrying at its free end a weight 45N. If the weight is released from rest, find (i) the time 't' required for it to fall through the height 3m (ii) with what velocity will it strike the floor ?

Solution:

Let P be the tension in the string.

$$\text{Torque } T = P \times r$$

$$T = I\alpha = mk^2 \times \frac{a}{r}$$

$$= m\frac{r^2}{2} \times \frac{a}{r} = \frac{mar}{2}$$

$$Pr = \frac{mar}{2}$$

$$P = \frac{ma}{2}$$

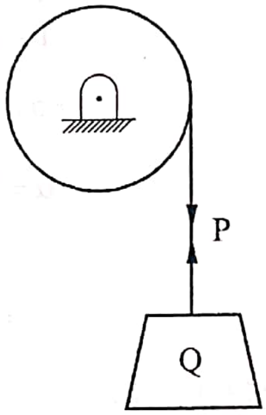

Fig.15.8

Consider the vertical motion of body Q.

$$Q - P = \frac{Q}{g} \times a$$

$$Q - \frac{ma}{2} = \frac{Q}{g}a$$

$$Q = a\left(\frac{Q}{g} + \frac{m}{2}\right)$$

$$45 = a\left(\frac{45}{9.81} + \frac{150}{2\times 9.81}\right) = 12.23$$

$$a = 3.68 \text{ m/s}^2$$

$$h = ut + \frac{1}{2}at^2$$

$$3 = 0 + \frac{1}{2}\times 3.68\times t^2$$

$$t = 1.28 \text{ s}$$

$$V = u + at$$

$$= 0 + 3.68\times 1.28$$

$$= 4.71 \text{ m/s}$$

Example 15.15

The rotor and the shaft weights 2500 N and the radius of gyration with respect to the axis of rotation is 250 mm. Calculate the acceleration of the falling weight 450 N if the shaft radius is 125 mm.

Solution:

Let P be the tension in the string

Torque due to the weight, $T = P\times r$

$$T = I\alpha$$

$$P\times r = mk^2\times\frac{a}{r}$$

$$P = mk^2\frac{a}{r^2} = \frac{2500}{9.81}\times\frac{0.25^2}{0.125^2}\times a$$

$$P = 1019.37 \text{ a}$$

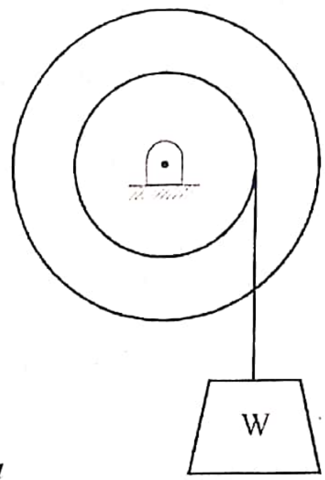

Fig. 15.9

Consider the downward motion of weight W

$$W - P = \frac{W}{g} \times a$$

$$W = P + \frac{W}{g} a$$

$$= 1019.37\,a + \frac{450}{9.81} \times a$$

$$= 1065.24\ a$$

$$a = \frac{450}{1065.24}$$

$$= 0.42\,\mathrm{m/s^2}$$

5.4 Concept of instantaneous centre

The motion of rotation and translation of a body, at a given instant, can be considered as that of pure rotation of the body about a point. This point about which the body can be assumed to be rotating at the given instant is called instantaneous centre of rotation. Since the velocity of this point at the given instant is zero, this point is also called instantaneous centre of zero velocity. This point is not a fixed point, and when the body changes its position, the position of instantaneous centre also changes. The locus of the instantaneous centre as the body goes on changing its position is called centrode.

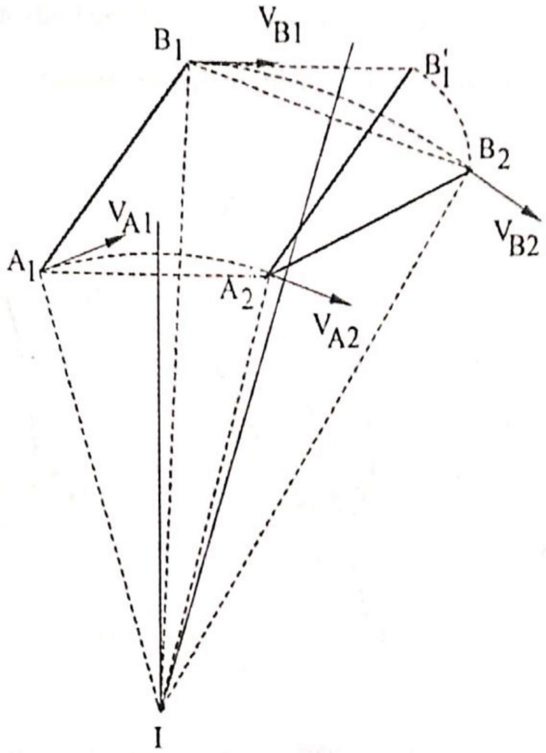

Fig. 5.24

Consider the plane motion of a body AB from A_1B_1 to A_2B_2. This plane motion can be considered as pure rotation of body AB about the point I, the instantaneous centre of rotation, which is the point of intersection of perpendicular bisectors of A_1A_2 and B_1B_2. Since I is on the perpendicular bisector of A_1A_2, $IA_1 = IA_2$. Similarly $IB_1 = IB_2$. Since $AB = A_1B_1 = A_2B_2$, the triangles IA_1B_1 and IA_2B_2 are congruent. Hence the motion of AB from A_1B_1 to A_2B_2 is pure rotation of triangle IA_1B_1 about I. Hence A_1B_1, at the given instant rotates about I. Since A_1B_1 rotates about I, the magnitude of velocity of A_1 is $\omega \times IA_1$ and that of B_1 is $\omega \times IB_1$.

The direction of velocity of points A_1 and B_1 are perpendicular to IA_1 and IB_1 respectively. Thus the properties of instantaneous centre are,

(i) the magnitude of velocity of any point on a body is proportional to its distance from the instantaneous centre and is equal to the angular velocity times the distance.

(ii) The direction of velocity of any point on a body is perpendicular to the line joining that point and the instantaneous centre.

The above properties are used to locate the instantaneous centre of a body.

Case (i)

When the direction of velocities of any two points on the body are known.

In this case the point of intersection of lines drawn perpendicular to the direction of velocities will be the instantaneous centre. Refer Fig. 5.25.

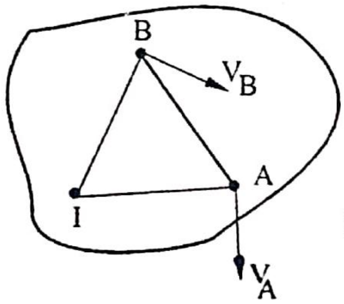

Fig. 5.25

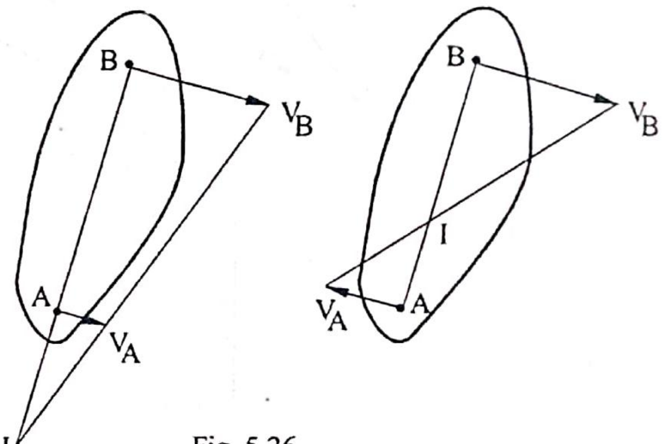

Fig. 5.26

Case (ii)

When the direction of velocities are parallel and magnitudes are unequal.

In this case the point of intersection of the line joining the tip of velocity vectors and either the line joining the points or extension of the line joining the points will be the instantaneous centre. Refer Fig. 5.26.

Vibrations

6.1 Mechanical vibration

A system is an assemblage of interacting elements constituted for a desired objective. The elements of a sytem may be a certain mechanical, electrical, electronics or some other devices which are characterised by their behaviour. A mechanical system comprises of mechanical elements. Vibration of mechanical system generally results when a system is displaced from its position of stable equilibrium. The system tends to return to its equilibrium position due to the action of restoring force. The restoring force may be an elastic force as in the case of mass attached to the end of a spring or gravitational force as in the case of simple pendulum.

The basic elements of a mechanical vibrating system are inertial element, compliance element and damping element. Inertial elements are mass m, in trans lation and mass moment of inertia I in rotation. The motion of inertial element is governed by the laws of mechanics, $F= m\,a$ in translational vibration and $T = I\,\alpha$ in rotational vibration. A compliance element complies with a steady force to produce a corresponding steady deformation. Compliance element such as springs are capable of conserving energy . A compliance element must return to its equilibrium position when the applied force is removed. The motion of compliance element is governed by the expression $F= k\,x$ in translational vibration and $M=k\theta$ in rotational vibration. F is the force, k is the stiffness, x is the linear displacement, M is the moment and θ is the angular displacement. Any resistance to vibratory motion is called damping. A damping element is associated with dissipative action. Mass and elasticity of the components of the dampers are assumed to be zero and a damping element stays where it is, when the applied force is removed. In translation motion the damping force, $F= cV$ and in rotational motion the damping moment $M= c\omega$ where c is the damping coefficient.

Classification of vibrations

Mechanical vibrations are classified as longitudinal, transverse and torsional vibrations. This classification is based on the direction of motion of the vibrating body with respect to the elastic mounting medium (spring, shaft etc) and based on the nature of stress induced in the mounting medium.

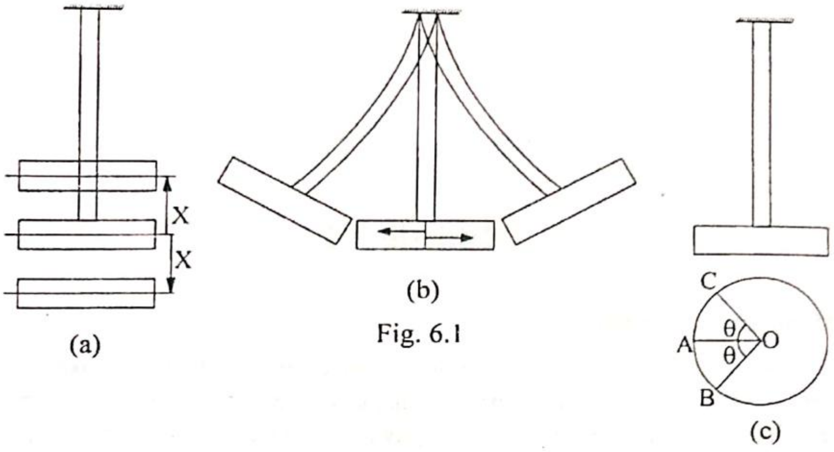

Fig. 6.1

Consider a shaft whose on end is fixed and the other end carrying a heavy disc. This system may execute one of the above mentioned type of vibrations.

1. Longitudinal vibration

The particles of the disc undergo to and fro motion, parallel to the axis of the shaft. The shaft is elongated and shortened alternately and is subjected to tensile and compressive stresses. Fig. 6.1 (a)

2. Transverse vibration

When the particles of the disc move approximately perpendicular to the axis of the shall, the vibration is called transverse vibration. The shaft is subjected to tensile and compressive stresses, induced due to the bending of the shaft. Fig. 6.1 (b)

3. Torsional vibration

When the particles of the disc move along circular arc about the axis of the shaft, the vibration is called torsional vibration. The shaft is subjected to twist and hence the stress induced is shear stress due to twisting moment. Fig. 6.1 (c)

6.2 Free vibration.

A mechanical element is said to have a free vibration if the periodic motion continuous after the cause of the original disturbance is removed. The free vibration of mechanical system will eventually cease because of loss of energy from the system.

6.3 Forced vibration.

A system is said to undergo forced vibration when a periodic disturbing force acts on the system. In forced vibration the system will vibrate at the frequency of the exciting force regardless of the initial conditions of the system.

6.4. Period of vibration.

It is the time interval after which the motion is repeated itself. The period of vibration or time period is usually expressed in seconds. The motion completed in one time period is called cycle.

6.5. Frequency.

The number of cycles per unit time is called frequency. Frequency of free vibration is called natural frequency. In SI unit the frequency is expressed in hertz (Hz). One Hz is one cycles per second (cps).

If t_p is the time period in seconds and f is the frequency in Hz, then $f = \frac{1}{t_p}$

Any resistance to the vibration of a system is called damping. In the case of free vibration the damping force decreases the amplitude of vibration and finally the system comes to rest in the equilibrium position. Amplitude of vibration is the distance between the mean position (equilibrium position) and extreme position of a vibrating body.

6.6. Stiffness of spring.

When a spring is compressed, the spring opposes the applied force which compress the spring. This opposing force is proportional to the displacement of the spring. That is the spring force is proportional to the displacement of spring. Spring force $F_S \ \alpha \ x$,

$F_S = \text{constant} \times x$. This constant of proportionality is called spring stiffness or constant of spring and is denoted by k.

$$\text{Thus } F_s = kx.$$

The stiffness, $k = \frac{F_S}{x}$, the unit of stiffness is N/m.

6.7 Natural frequency of longitudinal vibration

When the vibrating body of a system has to and fro motion along a straight line, the vibration is called linear vibration or longitudinal vibration. Consider a spring of stiffness k. When a body of mass m is attached to one end with the other end fixed, the spring elongates.

Let this elongation be δ . This δ is called static deflection of the spring. In the equilibrium position net force on the body is zero.

$mg - k\delta = 0$ where $k\ \delta$ is the spring force which is opposite to the direction of displacement.

$$m\,g = k\ \delta.$$

When the body is displaced by an amount X from the equilibrium position by an external force and if the external force is removed then the body will vibrate between two extreme positions with amplitude X. Consider the position of the body when it is at a distance x below the equilibrium position . The body is in motion and hence net force is equal to mass times acceleration.

$$\text{ie., } mg - k(x + \delta) = m \times a$$

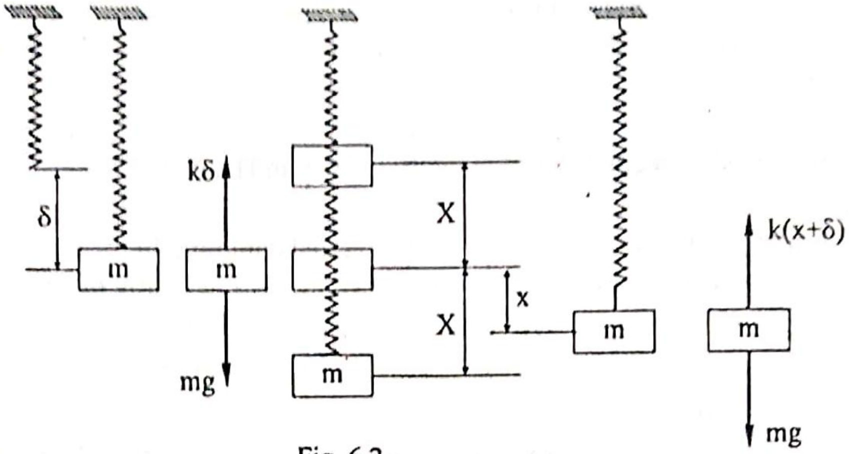

Fig. 6.2

$$mg - kx - k\delta = m\frac{d^2x}{dt^2}$$

Since $mg = k\delta$,

$$-kx = m\frac{d^2x}{dt^2}$$

$\therefore \frac{d^2x}{dt^2} + \frac{k}{m}x = 0$. This is the equation of motion of free vibration of a body.

$\frac{d^2x}{dt^2} = -\frac{k}{m}x$. Comparing this equation with the equation of simple harmonic motion, $\frac{d^2x}{dt^2} = -\omega^2 x$

$$\omega^2 = \frac{k}{m}$$

$\omega = \sqrt{\frac{k}{m}}$ Since the system vibrates freely, this frequency is called natural frequency and is denoted by ω_n

$$\omega_n = \sqrt{\frac{k}{m}} \quad \text{Since } \omega = 2\pi f,\ f = \frac{\omega}{2\pi}$$

$$f_n = \frac{1}{2\pi} \times \sqrt{\frac{k}{m}}.$$ This is the expression for the natural frequency of undamped free linear vibration.

6.8 Degree of freedom.

The number of degrees of freedom of a physical system is the number of independent spatial co ordinates required to define the configuration of the system. A rigid body in space has six degrees of freedom, three co ordinates (x, y and z) to define rectilinear position and three parameters to define the angular positions. The constraints to the motion reduce the degee of freedom of the system. The three systems shown in Fig. 6.3 are single degree of freedom systems. In Fig.6.3 (a), when the mass is constrained to move in the vertical

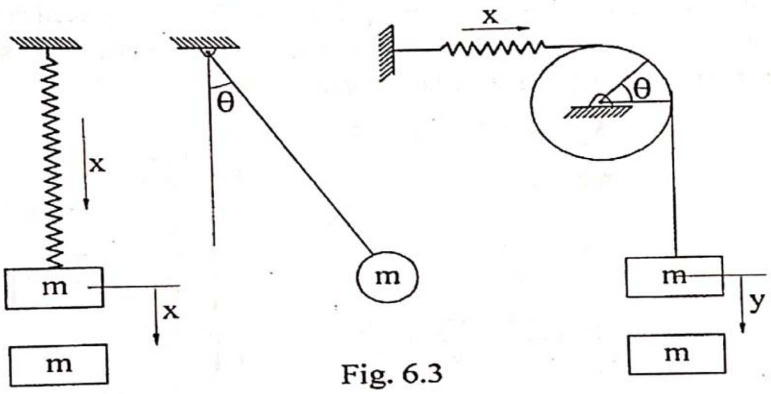

Fig. 6.3

direction only, the displacement of the mass will be same as that of the spring. In Fig. 6.3 (b) the only one co ordinate required to specify the position of the simple pendulum is θ. In Fig. 6.3 (c) the values of x and θ are functions of y and hence the independant co-ordinate required to specify the configuration of system is one.

Fig. 6.4 shows two systems having two degrees of freedom each.

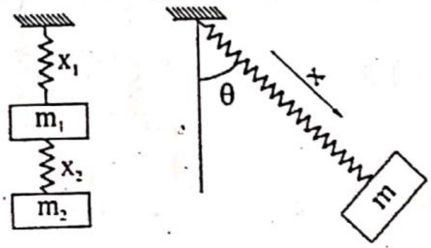

Fig. 6.4

When the spring-mass systems shown in Fig. 6.4 (a) and (b) are constrained to move in a vertical plane, only two co-ordinates are required to specify the configuration of the system. Hence the degrees of freedom of the systems are two.

6.9 Spring mass model

A mass subjected to a force along the axis of a spring which is attached to the mass is called spring mass system. If the mass is subjected to a damping force, it i called a damped spring mass system. In a spring mass model the springs can be attached to the mass either in series or in parallel.

A number of springs having stiffness k_1, k_2, k_3 etc can be replaced by a single spring of stiffness k_e. The stiffness of this single spring, k_e, is called equivalent stiffness. The expression for k_e depends on the arrangement of springs.

Fig 6.5 shows three springs of stiffness k_1, k_2, and k_3 arranged in series. Let δ_1, δ_2 and δ_3 be the elongation of each spring due to body of mass m.

Static deflection of mass, $\delta = \delta_1 + \delta_2 + \delta_3$. Let δ_e be the static deflection of the same mass when it is attached to the single spring of stiffness k_e. For this single spring to be equivalent to the three springs in series, the required condition is,

$$\delta_e = \delta = \delta_1 + \delta_2 + \delta_3.$$

$$m\,g = k\,\delta$$

$$\delta = \frac{mg}{k}$$

$$\therefore \frac{mg}{k_e} = \frac{mg}{k_1} + \frac{mg}{k_2} + \frac{mg}{k_3}$$

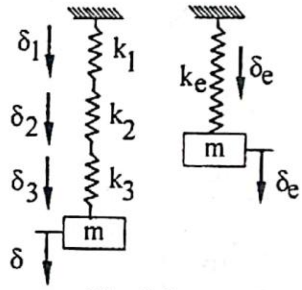

Fig. 6.5

$$\therefore \frac{1}{k_e} = \frac{1}{k_1} + \frac{1}{k_2} + \frac{1}{k_3}$$

Fig. 6.6 shows three springs arranged in parallel. The required condition for the single spring to be equivalent to the three springs in parallel is, $\delta_e = \delta$.

$$m\,g = k_1\,\delta + k_2\delta + k_3\delta$$

$$= (k_1 + k_2 + k_3)\,\delta$$

$$m\,g = k_e\,\delta_e$$

$$(k_1 + k_2 + k_3)\ \delta = k_e\ \delta_e\text{, since } \delta_e = \delta$$

$$k_e = k_1 + k_2 + k_3$$

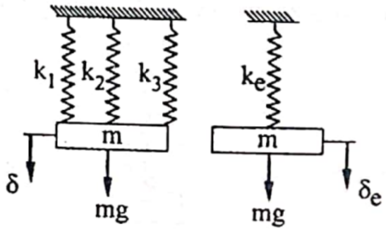

Fig. 6.6

Example 6.1

A 80 N weight is hung on the end of a helical spring and is set vibrating vertically. The weight makes 4 oscillations per second. Determine the stiffness of the spring

Solution.

$$m = \frac{W}{g} = \frac{80}{9.81}$$

$$f = 4 \text{ cps}$$

$$f_n = \frac{1}{2\pi} \times \sqrt{\frac{k}{m}}$$

$$f_n^2 = \frac{1}{4\pi^2} \times \frac{k}{m}$$

$$k = 4\pi^2 f_n^2 m = 4 \times \pi^2 \times 4^2 \times \frac{80}{9.81}$$

$$= 5151 \text{ N/m}$$

Example 6.2

If a helical spring having a stiffness of 90 N/cm is available, what weight should be hung on it so that it will oscillate with a periodic time of 1 sec.

Solution.

$$k = 90 \text{ N/cm} = 90 \times 100 \text{ N/m.}$$

$$t_p = 1 \text{ s}$$

$$f_n = \frac{1}{t_p} = 1 \text{ cps}$$

$$= \frac{1}{2\pi} \sqrt{\frac{k}{m}}$$

$$f_n^2 = \frac{1}{4\pi^2} \times \frac{k}{m}$$

$$m = \frac{k}{4\pi^2 f_n^2} = \frac{90 \times 100}{4 \times \pi^2 \times 1^2} = 227.97 \text{ kg}$$

Example 6.3

A weight of 50 N suspended from a spring vibrates vertically with an amplitude of 8 cm and a frequency of 1 oscillation per second. Find (a) the stiffness of the spring. (b) the maximum tension induced in the spring and (c) the maximum velocity of the weight.

Solution.

$$m = \frac{W}{g} = \frac{50}{9.81}$$

$$X = 8 \text{ cm} = 0.08 \text{ m}$$

$$f = 1 \text{ cps}$$

(a) $$f = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$

$$k = f^2 \times 4 \times \pi^2 \times m = 1 \times 4 \times \pi^2 \times \frac{50}{9.81}$$

$$= 201.22 \text{ N/m.}$$

(b) Maximum tension in the spring $= kX$

$$= 201.22 \times 0.08$$

$$= 16.1 \text{ N}$$

(c) Since the body vibrates with S. H. M,

the maximum velocity $= \omega X$

$$= (2\pi f) \times X$$

$$= 2\pi \times 1 \times 0.08$$

$$= 0.5 \text{ m/s}$$

Example 6.4

A body of mass 50 kg is suspended by two springs of stiffness 4 kN/m and 6 kN/m as shown in Fig. 6.7 (a), (b) and (c). The body is pulled 50 mm down from its equilibrium position and then released. Calculate

(i) the frequency of oscillation

(ii) maximum velocity and

(iii) maximum acceleration

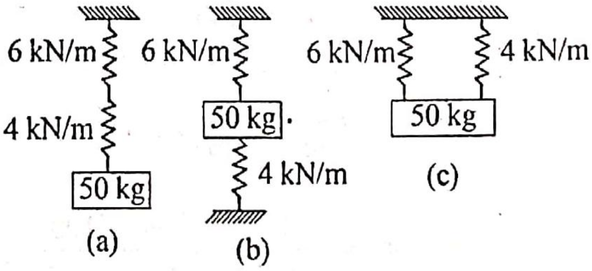

Fig. 6.7

Solution

When the springs are arranged as

shown in Fig 6.7 (a)

The equivalent stiffness is given by

$$\frac{1}{k_e} = \frac{1}{k_1} + \frac{1}{k_2} = \frac{1}{6} + \frac{1}{4} = \frac{10}{24}$$

$$k_e = \frac{24}{10} = 2.4 \quad \text{kN/m}$$

$$= 2.4 \times 1000 \text{ N/m}$$

(i)
$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m}} = \frac{1}{2\pi}\cdot\sqrt{\frac{2.4 \times 1000}{50}}$$

$$= 1.10 \text{ c p s}$$

$$\omega = 2\pi f = 6.93 \text{ rod/s.}$$

(ii) maximum velocity $= \omega X$

$= 6.93 \times 0.05$

$= 0.35$ m/s

(iii) acceleration $= \omega^2 X$

$= 6.93^2 \times .05$

$= 2.4 \text{ m/s}^2$

(b) When the springs are as shown in Fig 6.7 (b)

The springs are in parallel. Hence the equivalent stiffness;

$$k_e = k_1 + k_2$$

$$= 4 + 6 = 10 \text{ kN/m}$$

(i) Frequency
$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m}} = \frac{1}{2\pi}\sqrt{\frac{10 \times 1000}{50}}$$

$= 2.25$ c p s.

(ii) Maximum velocity $= \omega X$

$= (2\pi f) X$

$= 2\pi \times 2.25 \times 0.05$

$= 0.71$ m/s

$$\text{acceleration} = \omega^2 X = (2\pi f)^2 \times 0.05$$

$$= (2\pi \times 2.25)^2 \times 0.05$$

$$= 10 \text{ m/s}^2$$

When the springs are arranged as shown in Fig. 6.7 (c) the springs are in parallel.

Hence the frequency = 2.25 cps.

Maximum velocity = 0.71 m/s and

Acceleration = 10 m/s²

6.10 Simple harmonic motion.

Simple harmonic motion (SHM) is a periodic motion. Any motion which repeats after equal interval of time is called a periodic motion. For a periodic motion to be simple harmonic, it should satisfy two general conditions.

(i) The acceleration of the body performing periodic motion should be proportional to the distance of the body from a fixed point called centre of simple harmonic motion (mean position of the body)

(ii) The acceleration of the body should be directed towards the mean position.

Consider a particle moving along the circumference of a circle of radius r with a uniform angular velocity ω radians per second. Let P be the position of the particle after t seconds from the start of motion from the position A as shown in Fig. 6.8. M is the projection of particle on the horizontal diameter AB. When the particle moves along the arc ACB, the point M moves from A to B. Similarly when the particle moves along the are BDA, the point M moves from B to A. It can be proved that the acceleration of point M is proportional to its distance from O, and is directed towards O. Such a motion is called simple harmonic motion and hence the projection of particle on the horizontal diameter executes SHM. O is the mean position and A and B are the extreme positions. Similarly the projection of particle on the vertical diameter CD also executes SHM with C and D as extreme positions.

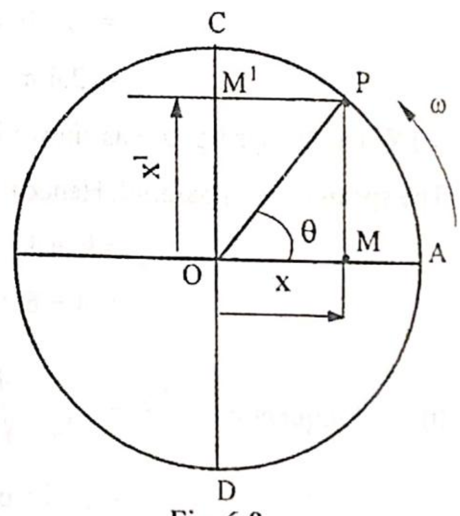

Fig. 6.8

Terms used with SHM

The following terms are generally used with SHM. -

1. **Amplitude.** It is the distance between extreme and mean position of the particle executing SHM. It is the maximum displacement of the particle from the mean position.

2. **Oscillation.** It is the motion of projection of particle along the diameter. Refering figure 16.2, one oscillation is said to be completed when the point M moves from A to B and then from B to A.

3. **Period.** It is the time for one oscillation. It is denoted by t_p.

From Fig. 6.8,

$$\theta = \omega t. \text{ For one oscillation, } t = t_p \text{ and } \theta = 2\pi$$

$$2\pi = \omega t_p$$

$$t_p = \frac{2\pi}{\omega}$$

The displacement of M from the mean position, OM $= x =$ OP $\cos\theta$

$$x = r \cos \omega t$$

Velocity of M, $v = \frac{dx}{dt} = -r\omega \sin \omega t$. The negative sign is because when the time t increases, the displacement x decreases.

The magnitude of velocity, $v = r\omega \sin \omega t$

$$= r\omega \sin\theta$$

$$= r\omega \frac{PM}{OP}$$

$$= r\omega \frac{\sqrt{r^2 - x^2}}{r}$$

$$= \omega\sqrt{r^2 - x^2}$$

Acceleration of M, $a = \frac{dv}{dt} = \frac{d}{dt}(r\omega \sin \omega t)$

$$= -r\omega^2 \cos \omega t$$

$$= -\omega^2 \ r \ \cos \ \omega \ t$$

$$a = -\omega^2 \ x.$$

Let M^1 be the projection of particle on the vertical diameter CD.

The displacement of M^1 from the mean position, $x = OP \sin \theta$

$$= r \sin \omega t$$

$$\text{Velocity of } M^1, \quad v = \frac{dx}{dt} = \frac{d}{dt} \ (r \sin \omega t)$$

$$= r\omega \ \cos \omega t$$

$$\text{Acceleration of } M^1, \quad a = \frac{dv}{dt}$$

$$= \frac{d}{dt} (r \ \omega \ cos \ \omega t)$$

$$= -\omega^2 r \sin \omega t$$

$$a = -\omega^2 x$$

$$\text{Again, } v = r \ \omega \ \cos \omega t$$

$$= r \ \omega \cos \theta$$

$$= r \ \omega \frac{OM}{OP}$$

$$= r \ \omega \frac{\sqrt{r^2 - x^2}}{r}$$

$$v = \omega \sqrt{r^2 - x^2}$$

It should be noted that the expression for displacement $x = r \sin \omega t$ is to be used when the time of motion is measured from the mean position. ie., in the case when at $t = 0$, the particle executing SHM is at the mean position. The expression $x = r \cos \omega t$ should be used when the time of motion is measured from the extreme position. ie., in the case when at $t = 0$, the particle executing SHM is at the extreme position. In both cases the expression for velocity is, $\omega \sqrt{r^2 - x^2}$ and acceleration, $a = -\omega^2 x$.

The maximum velocity is at $x = 0$. ie., at the mean position.

$$v_{max} = \omega\sqrt{r^2 - 0} = r\,\omega.$$

The maximum acceleration is at x = r. ie., at the extreme position.

$$a_{max} = -\,\omega^2\, r.$$

At the extreme position, the velocity is zero and at the mean position the acceleration is zero.

Example 6.5

A body, moving with simple harmonic motion, has an amptitude of 1 m and period of oscillation is 2 seconds. Find the velocity and acceleration of the body at t = 0.4 second, when time is measured from (i) the mean position and (ii) the extreme position.

Solution.

Case (i) when time is measured from the mean position.

$$r = 1 \text{ m}$$

$$t_p = 2 \text{ s}$$

$$t = 0.4 \text{ s}$$

$$\omega = \frac{2\pi}{t_p} = \frac{2\pi}{2} = \pi \text{ rad/s}$$

$$\omega t = \pi \times 0.4 \text{ rad}$$

$$= \left(\frac{180}{\pi} \times 0.4\,\pi\right)^0$$

$$= 72^0$$

$$x = r \sin \omega t$$

$$x = 1 \sin 72$$

$$= 0.95 \text{ m.}$$

$$\text{Velocity } v = \omega\sqrt{r^2 - x^2} = \pi \times \sqrt{1^2 - 0.95^2}$$

$$= 0.98 \text{ m/s}$$

$$\text{Acceleration } a = \omega^2 x$$

$$= \pi^2 \times 0.95$$

$$= 9.38 \text{ m/s}^2$$

Case (ii) when time is measured from the extreme position.

$$x = r \cos \omega t$$

$$= 1 \times \cos 72$$

$$= 0.31 \text{ m}$$

$$v = \omega \sqrt{r^2 - x^2}$$

$$= \pi \sqrt{1^2 - 0.31^2}$$

$$= 2.99 \text{ m/s}$$

$$\text{Acceleration } a = \omega^2 x$$

$$= \pi^2 \times 0.31$$

$$= 3.06 \text{ m/s}^2$$

Example 6.6

A body moving with simple harmonic motion has velocities of 10m/s and 4 m/s at 2 and 4 m distance from the mean position. Find the amplitude and time period of the body.

Solution.

At $x = 2$ m, $v = 10$ m/s

At $x = 4$ m, $v = 4$ m/s.

$$\text{Velocity, } v = \omega \sqrt{r^2 - x^2}$$

$$10 = \omega\sqrt{r^2 - 2^2}$$

$$4 = \omega\sqrt{r^2 - 4^2}$$

$$\frac{10}{4} = \frac{\sqrt{r^2 - 4}}{\sqrt{r^2 - 16}}$$

$$6.25 = \frac{r^2 - 4}{r^2 - 16}$$

$$6.25\, r^2 - 100 = r^2 - 4$$

$$5.25\, r^2 = 96$$

$$r = 4.28 \text{ m}$$

$$10 = \omega\sqrt{r^2 - 2^2}$$

$$10 = \omega\sqrt{4.28^2 - 4}$$

$$= 3.78\,\omega$$

$$\omega = 2.64 \text{ rad/s}$$

$$\text{Time period } t_p = \frac{2\pi}{\omega} = \frac{2\pi}{2.64} = 2.38\text{s}$$

Example 6.7

The piston of an IC engine moves with simple harmonic motion. The crank rotates at 420 rpm and the stroke length is 40 cm. Find the velocity and acceleration of the piston when it is at a distance of 10 cm from the mean position.

Solution.

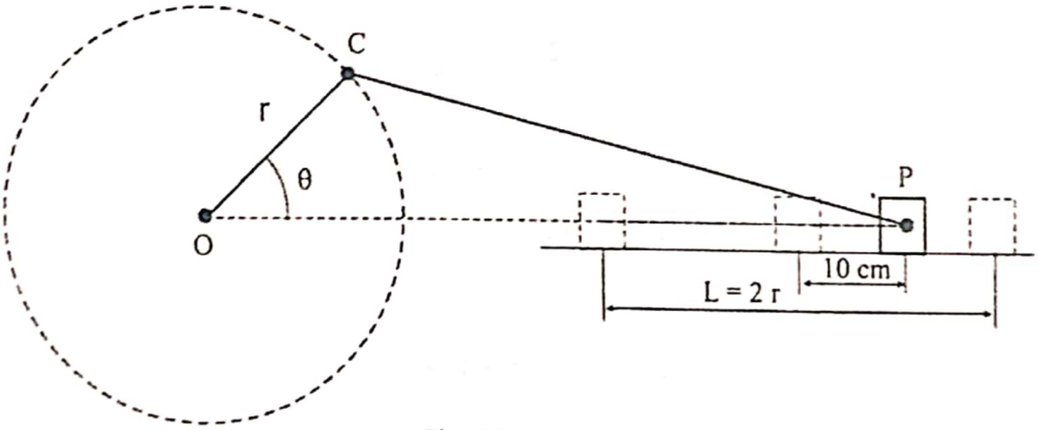

Fig. 6.9

Speed of crank = 420 rpm.

$$\omega = \frac{2\pi N}{60} = \frac{2\pi \times 420}{60} = 43.98 \text{ rad/s}$$

Stroke length L = 2 × crank radius

$$\text{crank radius, } r = \frac{L}{2} = \frac{40}{2} = 20 \text{ cm}$$

$$= 0.2 \text{ m}$$

$$x = 10\text{ cm} = 0.1\text{ m}$$

$$\text{Velocity } v = \omega\sqrt{r^2 - x^2}$$

$$= 43.98\sqrt{0.2^2 - 0.1^2}$$

$$= 7.62\text{ m/s.}$$

$$\text{Acceleration of the piston } a = \omega^2\ r$$

$$= 43.98^2 \times 0.1$$

$$= 193.42\text{ m/s}^2$$

Example 6.8

A particle moving with SHM has an amplitude of 4.5 m and period of oscillation is 3.5 second. Find the time required by the particle to pass two points which are at a distance of 3.5 m and 1.5 m from the centre and on the same side of mean position.

Solution.

$$\text{Amplitude, } r = 4.5\text{ m}$$

$$\text{Time period } t_p = 3.5\text{ s.}$$

$$\omega = \frac{2\pi}{t_p} = \frac{2\pi}{3.5} = 1.8\text{ rad / s}$$

Let x_1 and x_2 be the distance of the first and second points from the mean position.

$$x = r\cos\omega t$$

$$x_1 = r\cos\omega t_1$$

$$3.5 = 4.5\cos\left(1.8 \times t_1 \times \frac{180}{\pi}\right)^0$$

$$t_1 = 0.38\text{ s}$$

$$x_2 = r\cos\omega t_2$$

$$(1.5 = 4.5\cos\left(1.8 \times t_2 \times \frac{180}{\pi}\right)$$

$$t_2 = 0.68\text{ s}$$

Time required to pass the two points,

$$t = t_2 - t_1$$

$$= 0.68 - 0.38$$

$$= 0.3\text{ s}$$