Solution by Gauss Elimination

In this method, the unknowns are eliminated successively & the system is reduced to upper triangular system (upper triangular matrix) from which the unknowns can be found by back ~~substitution~~ substitution.

Problems:

1) Solve the linear system by Gauss Elimination.

$$x_1 - x_2 + x_3 = 0$$
$$-x_1 + x_2 - x_3 = 0$$
$$10x_2 + 25x_3 = 90$$
$$20x_1 + 10x_2 = 80$$

Solution:-

Consider the matrix Equation $Ax = b$ as

$$\begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 0 & 10 & 25 \\ 20 & 10 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 90 \\ 80 \end{bmatrix}$$

Consider the augmented matrix

$$\tilde{A} = \begin{bmatrix} 1 & -1 & 1 & 0 \\ -1 & 1 & -1 & 0 \\ 0 & 10 & 25 & 90 \\ 20 & 10 & 0 & 80 \end{bmatrix}.$$

Reduce the system to an upper triangular ~~system~~ by a sequence of Elementary row operations.

$$\sim \begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 10 & 25 & 90 \\ 0 & 30 & -20 & 80 \end{bmatrix} \begin{matrix} \\ R_2 \to R_2 + R_1 \\ R_4 \to R_4 - 20R_1 \\ \end{matrix}$$

$$\sim \begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 30 & -20 & 80 \\ 0 & 10 & 25 & 90 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad R_2 \leftrightarrow R_4$$

$$\sim \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 30 & -20 & 80 \\ 0 & 0 & 95/3 & 190/3 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad R_3 \to R_3 - \frac{1}{3} R_2$$

which is in upper triangular form.

<u>Back Substitution:-</u>

Consider the Equivalent System of Equations

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 30 & -20 \\ 0 & 0 & 95/3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} m_1 \\ m_2 \\ m_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 80 \\ 190/3 \\ 0 \end{bmatrix}$$

$$\Rightarrow m_1 - m_2 + m_3 = 0 \longrightarrow (1)$$

$$30m_2 - 20m_3 = 80 \longrightarrow (2)$$

$$\frac{95}{3} m_3 = \frac{190}{3} \longrightarrow (3)$$

From (3), $m_3 = \frac{\cancel{190}^{2}}{\cancel{3}} \times \frac{\cancel{3}}{\cancel{95}} = 2$

$\therefore \boxed{m_3 = 2}$

From (2), $30m_2 = 80 + 20m_3$

$$\Rightarrow m_2 = \frac{1}{30}\left[80 + 20m_3\right]$$

$$= \frac{1}{30}\left[80 + 20 \times 2\right]$$

$$= \frac{1}{30}\left[80 + 40\right] = \frac{120}{30} = \underline{\underline{4}}$$

$\therefore \boxed{x_2 = 4}$

From (1), $x_1 = x_2 - x_3$

$= 4 - 2$

$= 2$

$\therefore \boxed{x_1 = 2}$

Thus, by Back substitution, we found the solutions of the given systems of

$x_1 = 2, \; x_2 = 4, \; x_3 = 2.$

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2) Solve the linear system by Gauss elimination:

$$3.0x_1 + 2.0x_2 + 2.0x_3 - 5.0x_4 = 8.0$$
$$0.6x_1 + 1.5x_2 + 1.5x_3 - 5.4x_4 = 2.7$$
$$1.2x_1 - 0.3x_2 - 0.3x_3 + 2.4x_4 = 2.1$$

Solution:-

Consider the matrix equation $Ax = b$ of

$$\begin{bmatrix} 3.0 & 2.0 & 2.0 & -5.0 \\ 0.6 & 1.5 & 1.5 & -5.4 \\ 1.2 & -0.3 & -0.3 & 2.4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 8.0 \\ 2.7 \\ 2.1 \end{bmatrix}$$

Consider the Augmented matrix $\tilde{A}$ & reduce the system to an upper triangular form.

$$\tilde{A} = \left[\begin{array}{cccc|c} 3.0 & 2.0 & 2.0 & -5.0 & 8.0 \\ 0.6 & 1.5 & 1.5 & -5.4 & 2.7 \\ 1.2 & -0.3 & -0.3 & 2.4 & 2.1 \end{array}\right]$$

$$\cancel{\sim \begin{bmatrix} 3.0 & 2.0 & 2.0 & -5.0 \\ 0 & 1.1 & 1.1 & \\ & & & \end{bmatrix} \begin{matrix} R_2 \to R_2 - 0.2R_1 \\ R_3 \to R_3 - 0.4R_1 \end{matrix}}$$

$$\sim \left[\begin{array}{cccc|c} 3.0 & 2.0 & 2.0 & -5.0 & 8.0 \\ 0 & 1.1 & 1.1 & -4.4 & 1.1 \\ 0 & -1.1 & -1.1 & 4.4 & -1.1 \end{array}\right] \begin{matrix} R_2 \to R_2 - 0.2R_1 \\ R_3 \to R_3 - 0.4R_1 \end{matrix}$$

$$\sim \left[\begin{array}{cccc|c} 3.0 & 2.0 & 2.0 & -5.0 & 8.0 \\ 0 & 1.1 & 1.1 & -4.4 & 1.1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \quad R_3 \to R_3 + R_2$$

which is in upper triangular form.

Now, consider the equivalent system of equations as follows:

$$\begin{bmatrix} 3.0 & 2.0 & 2.0 & -5.0 \\ 0 & 1.1 & 1.1 & -4.4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 8.0 \\ 1.1 \\ 0 \end{bmatrix}$$

$$\Rightarrow 3.0x_1 + 2.0x_2 + 2.0x_3 - 5.0x_4 = 8.0 \to (1)$$

$$1.1x_2 + 1.1x_3 - 4.4x_4 = 1.1 \to (2)$$

Now, x_1 & x_2 are leading variables & x_3 & x_4 are free variables.

Choose $x_3 = a$, $x_4 = b$ where a & b are arbitrary real numbers.

From (2), $1.1x_2 = 1.1 - 1.1x_3 + 4.4x_4$

$$\Rightarrow x_2 = 1 - x_3 + 4x_4$$

$\Rightarrow m_2 = 1 - a + 4b$

$\therefore \boxed{m_2 = 1 - a + 4b}$

from (i), $3.0 m_1 = 8.0 - 2.0 m_2 - 2.0 m_3 + 5.0 m_4$

$$\Rightarrow m_1 = \frac{1}{3.0}\left[8.0 - 2.0(1-a+4b) - 2.0a + 5.0b\right]$$

$$= \frac{1}{3}\left[8 - 2 + \cancel{2a} - 8b - \cancel{2a} + 5b\right]$$

$$= \frac{1}{3}\left[6 - 3b\right]$$

$$= 2 - b$$

$\therefore \boxed{m_1 = 2 - b}$

Thus, by Back substitution, we found the solution of the given system of

$m_1 = 2-b$, $m_2 = 1-a+4b$, $m_3 = a$, $m_4 = b$,

where a & b are arbitrary real numbers.

Since a & b are arbitrary, the given system has infinitely many solutions.

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3) Solve the linear system by Gauss-Elimination,

$$3m_1 + 2m_2 + m_3 = 3$$
$$2m_1 + m_2 + m_3 = 0$$
$$6m_1 + 2m_2 + 4m_3 = 6.$$

Solution:-

Consider the matrix equation $Ax=b$ as

$$\begin{bmatrix} 3 & 2 & 1 \\ 2 & 1 & 1 \\ 6 & 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 6 \end{bmatrix}$$

Consider the Augmented matrix $\tilde{A}$ & reduce the system into an upper triangular form.

$$\tilde{A} = \begin{bmatrix} 3 & 2 & 1 & 3 \\ 2 & 1 & 1 & 0 \\ 6 & 2 & 4 & 6 \end{bmatrix}$$

$$\sim \begin{bmatrix} 3 & 2 & 1 & 3 \\ 0 & -1/3 & 1/3 & -2 \\ 0 & -2 & 2 & 0 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 - \frac{2}{3} R_1 \\ R_2 \to R_2 - 2R_1 \end{matrix}$$

$$\sim \left[\begin{array}{ccc|c} 3 & 2 & 1 & 3 \\ 0 & -1/3 & 1/3 & -2 \\ 0 & 0 & 0 & 12 \end{array}\right] \quad R_3 \to R_3 - 6R_2$$

which is in upper triangular form.

Now, consider the equivalent systems of equations as follows:

$$\begin{bmatrix} 3 & 2 & 1 \\ 0 & -1/3 & 1/3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ 12 \end{bmatrix}$$

$$\Rightarrow \quad 3x_1 + 2x_2 + x_3 = 3 \longrightarrow (1)$$

$$-\tfrac{1}{3}x_2 + \tfrac{1}{3}x_3 = -2 \longrightarrow (2)$$

$$0x_1 + 0x_2 + 0x_3 = 12 \longrightarrow (3)$$

(3) is a false statement (0 = 12), which shows that the given system has no solution.

Practice Problems

1) Using Gauss elimination method, solve the linear system of equations:

$$x + 2y + z = 3$$
$$2x + 3y + 2z = 5$$
$$3x - 5y + 5z = 2$$
$$3x + 9y - z = 4.$$

2) Using Gauss elimination method, find the solution of the system of equations

$$4y + 3z = 8$$
$$2x - z = 2$$
$$3x + 2y = 5.$$

3) Using Gauss elimination method, solve the linear system:

$$2x - 3y + 7z = 5$$
$$3x + y - 3z = 13$$
$$2x + 19y - 47z = 32.$$

4) Using Gauss elimination method, solve the linear system:

$$10x + 4y - 2z = 14$$
$$-3w - 15x + y + 2z = 0$$
$$w + x + y = 6$$
$$8w - 5x + 5y - 10z = 26$$