Fundamental theorem for linear systems ①

Case: I (Non-homogeneous System)

(a) Existence of solutions

A linear system of 'm' equations in 'n' unknowns $x_1, x_2, \ldots, x_n$

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \longrightarrow (1)$$
$$\cdots\cdots\cdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m$$

is consistent (ie, has solutions) if and only if $\rho(\tilde{A}) = \rho(A)$, where $\tilde{A}$ is the augmented matrix,

$$\tilde{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{bmatrix}$$

and A is the coefficient matrix,

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

(b) Uniqueness of solution

The system (1) has precisely one solution if and only if $\rho(\tilde{A}) = \rho(A) = n$ (n is the number of unknowns)

(c) Infinitely many solutions.

If $\rho(\tilde{A}) = \rho(A) < n$, the system (1) has infinitely many solutions.

Note:- (Summary)

The Non-homogeneous system $Ax=b$ is consistent $\Leftrightarrow$ $\rho(\tilde{A})=\rho(A)$

The system $Ax=b$ has unique solution $\Leftrightarrow$ $\rho(\tilde{A})=\rho(A)=n$ (the no: of unknowns)

If $\rho(\tilde{A})=\rho(A)<n$ (the no: of unknowns), the system $Ax=b$ has infinitely many solutions.

Case: II (Homogeneous System)

A homogeneous linear system of 'm' equations in 'n' unknowns $x_1, x_2, \dots, x_n$

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = 0$$
$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = 0$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = 0$$

always has the trivial solution $x_1=0, x_2=0, \dots, x_n=0$.

Non-trivial solutions (or non-zero solution) exist if and only if $\rho(A)<n$ (the no. of unknowns)

Note:- (Summary)

The homogeneous system $Ax=0$ is always consistent (due to the existence of trivial solution).

The system $Ax=0$ has non-trivial solution $\Leftrightarrow$ $\rho(A)<n$ (the no: of unknowns).

Note:-

A homogeneous linear system with fewer equations than unknowns always has non-trivial solutions.

Problems:-

1) Check if the following system of equations is consistent or inconsistent:

$x+y+z=1, \; x+2y+4z=3, \; x+4y+10z=9.$

Ans:-

Given system of equations are

$x+y+z=1$

$x+2y+4z=3$

$x+4y+10z=9.$

Consider the matrix equation $AX=b$ of

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 4 & 10 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 1 \\ 3 \\ 9 \end{bmatrix}.$$

To check the consistency we consider the Augmented matrix

$$\tilde{A}=\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 3 \\ 1 & 4 & 10 & 9 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 2 \\ 0 & 3 & 9 & 8 \end{bmatrix} \begin{matrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{matrix}$$

$$\sim \left[\begin{array}{ccc:c} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 2 \end{array}\right] R_3 \to R_3 - 3R_2$$

which is in echelon form.

Now, $\rho(\tilde{A})=3$

$\rho(A)=2$

$\therefore \rho(\tilde{A}) \neq \rho(A)$, the given system of equations is inconsistent so that it has no solutions.

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2) Test for consistency and solve the following system of equations:

$$2x-y+3z=8$$
$$-x+2y+z=4$$
$$3x+y-4z=0.$$

Ans:-

Consider the matrix equation $AX=b$ as

$$\begin{bmatrix} 2 & -1 & 3 \\ -1 & 2 & 1 \\ 3 & 1 & -4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ 4 \\ 0 \end{bmatrix}$$

To test the consistency, consider the Augmented matrix

$$\tilde{A} = \begin{bmatrix} 2 & -1 & 3 & 8 \\ -1 & 2 & 1 & 4 \\ 3 & 1 & -4 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} -1 & 2 & 1 & 4 \\ 2 & -1 & 3 & 8 \\ 3 & 1 & -4 & 0 \end{bmatrix} \quad R_1 \leftrightarrow R_2$$

$$\sim \begin{bmatrix} -1 & 2 & 1 & 4 \\ 0 & 3 & 5 & 16 \\ 0 & 7 & -1 & 12 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2+2R_1 \\ R_3 \to R_3+3R_1 \end{matrix}$$

$$\sim \left[\begin{array}{ccc|c} -1 & 2 & 1 & 4 \\ 0 & 3 & 5 & 16 \\ 0 & 0 & -\frac{38}{3} & -\frac{76}{3} \end{array}\right] \quad R_3 \to R_3 - \frac{7}{3}R_2$$

which is in Echelon form.

Now, $\rho(\tilde{A}) = 3$

$\rho(A) = 3$

$\because \rho(\tilde{A}) = \rho(A)$, the given system is consistent.

Now, $n = 3$, i.e., $\rho(\tilde{A}) = \rho(A) = 3$ (no: of unknowns)

Hence the given system has a unique solution.

Now, the solution can be obtained by considering the equivalent system

$$\begin{bmatrix} -1 & 2 & 1 \\ 0 & 3 & 5 \\ 0 & 0 & -38/3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 16 \\ -76/3 \end{bmatrix}$$

$$\Rightarrow -x + 2y + z = 4 \longrightarrow (1)$$

$$3y + 5z = 16 \longrightarrow (2)$$

$$-\frac{38}{3} z = \frac{-76}{3} \longrightarrow (3)$$

From (3), $z = -\frac{\cancel{76}^{2}}{\cancel{3}} \times \frac{-\cancel{3}}{\cancel{38}} = 2$

ie, $\boxed{z = 2}$

From (2), $3y = 16 - 5z$

$$\Rightarrow y = \frac{16}{3} - \frac{5}{3} z$$

$$= \frac{16}{3} - \frac{5}{3} \times 2$$

$$= \frac{16}{3} - \frac{10}{3}$$

$$= \frac{6}{3}$$

$$= 2$$

i'e, $\boxed{y = 2}$

NOTE:-

Equivalent systems have exactly the same solutions.

NOTE:-

In order to solve the system $Ax = b$, it is enough to solve the system arised from the Echelon form.

From (1), $-x = 4 - 2y - z$

$= 4 - 2(2) - 2$

$= \cancel{4} - \cancel{4} - 2$

$\Rightarrow \boxed{x = 2}$

Thus, we have $x = 2$, $y = 2$, $z = 2$, the required unique solution.

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3) Check for consistency of the system

$x + y + z = 1$, $x + 2y + 4z = 2$, $x + 4y + 10z = 4$.

If it is consistent, solve it.

Solution:-

Consider the matrix equation $AX = b$ i.e.

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 4 & 10 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$$

To check the consistency, consider the Augmented matrix

$$\tilde{A} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 2 \\ 1 & 4 & 10 & 4 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 1 \\ 0 & 3 & 9 & 3 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{matrix}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad R_3 \to R_3 - 3R_2$$

which is in ~~a~~ echelon form.

Now, $\rho(\tilde{A}) = 2$, $\rho(A) = 2$.

$\therefore$ $\rho(\tilde{A}) = \rho(A) = 2$, hence the given system of equations is consistent.

Now, $n = 3$ (the no. of unknowns).

$\therefore$ $\rho(\tilde{A}) = \rho(A) < 3$ (the no. of unknowns), the system has infinitely many solutions.

Now, to solve the system, we consider the equivalent system

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$\Rightarrow x + y + z = 1 \longrightarrow (1)$$

$$y + 3z = 1 \longrightarrow (2)$$

Now, x & y are leading variables & z is a free variable.

Let $z = k$ (k is an arbitrary ~~constant~~ real number)

From (2), $y = 1 - 3z$
$= 1 - 3k$

$\therefore$ $\boxed{y = 1 - 3k}$

From (1), $x = 1 - y - z$
$= 1 - (1 - 3k) - k$
$= \cancel{1} - \cancel{1} + 3k - k$
$= 2k$

$\therefore$ $\boxed{x = 2k}$

Thus, the required ~~[illegible]~~ infinitely many solutions are $x=2k$, $y=1-3k$, $z=k$.

Note:-

Leading Variables are those variables corresponding to the leading entries in the Echelon form. The variables which are not leading is taken to be free.

Note:-

The arbitrariness of k guarantees the infinitely many solutions of the system.

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4) For what values of λ and μ the given system of equations

$$x+y+z=6,\ x+2y+3z=10,\ x+2y+\lambda z=\mu$$

has

(a) no solution

(b) a unique solution.

(c) infinite number of solutions.

Answer:-

Consider the matrix equation $AX=b$ as

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & \lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 10 \\ \mu \end{bmatrix}$$

Now, Consider the Augmented matrix

$$\tilde{A} = \begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 10 \\ 1 & 2 & \lambda & \mu \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 1 & \lambda-1 & \mu-6 \end{bmatrix} \begin{matrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & \lambda-3 & \mu-10 \end{bmatrix} \quad R_3 \to R_3 - R_2$$

which is in echelon form.

(a) We know that the system has no solution if $\rho(\tilde{A}) \neq \rho(A)$. This is possible only if $\lambda = 3$ & $\mu \neq 10$, so that $\rho(A) = 2$ and $\rho(\tilde{A}) = 3$ and hence the inconsistency of the system.

(b) We know that the system possesses a unique solution only if $\rho(\tilde{A}) = \rho(A) =$ the number of unknowns. Since the number of unknowns is found to be 3, then to obtain a unique solution, we must have $\rho(\tilde{A}) = \rho(A) = 3$. This is possible only if $\lambda \neq 3$ & μ has any value.

(c) We know the system possesses infinite number of solutions only if

$\rho(\tilde{A}) = \rho(A) < 3$ (the no. of unknowns).

Now, this happens only if $\rho(\tilde{A}) = \rho(A) = 2$.

$\therefore$ We must have $\lambda = 3$, $\mu = 10$.

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5) Test for consistency & if possible find solutions :

$$x+y+z=1$$
$$x+2y+3z=4$$
$$x+3y+5z=7$$
$$x+4y+7z=10.$$

(ii) Answer:-

Consider the matrix equation $AX=b$ as

$$\begin{bmatrix}1&1&1\\1&2&3\\1&3&5\\1&4&7\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}1\\4\\7\\10\end{bmatrix}$$

To test the consistency, we have to consider the Augmented matrix

$$\tilde{A}=\begin{bmatrix}1&1&1&1\\1&2&3&4\\1&3&5&7\\1&4&7&10\end{bmatrix}$$

$$\sim\begin{bmatrix}1&1&1&1\\0&1&2&3\\0&2&4&6\\0&3&6&9\end{bmatrix}\quad\begin{matrix}R_2\to R_2-R_1\\R_3\to R_3-R_1\\R_4\to R_4-R_1\end{matrix}$$

$$\sim\left[\begin{array}{ccc:c}1&1&1&1\\0&1&2&3\\0&0&0&0\\0&0&0&0\end{array}\right]\quad\begin{matrix}R_3\to R_3-2R_2\\R_4\to R_4-3R_2\end{matrix}$$

which is in echelon form.

Now, $\rho(\tilde{A})=2$, $\rho(A)=2$ $\therefore$ $AX=b$ is consistent.

Since $n=3$ (the no. of unknowns), we have $\rho(\tilde{A})=\rho(A)=2<3$ (the no. of unknowns).

$\therefore$ The system possesses infinitely many solutions.

To solve the system, we consider the Equivalent system $\begin{bmatrix}1&1&1\\0&1&2\\0&0&0\\0&0&0\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}1\\3\\0\\0\end{bmatrix}$

$\Rightarrow x+y+z=1 \longrightarrow (1)$

$y+2z=3 \longrightarrow (2)$

Thus, we identified that x & y are the leading variables & z is a free variable.

So let $z=a$, where a is an arbitrary real number.

Now, from (2), $y=3-2z$
$=3-2a$

$\therefore \boxed{y=3-2a}$

From (1), we have $x=1-y-z$
$=1-(3-2a)-a$
$=1-3+2a-a$
$=-2+a$
$=a-2$

$\therefore \boxed{x=a-2}$

Hence the infinitely many solutions of the given system is

$x=a-2,\ y=3-2a,\ z=a$, where a is an arbitrary real number.

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6)(a) Find the value of λ for which the system of equations $x+y+z=1$, $x+2y+4z=\lambda$, $x+4y+10z=\lambda^2$ will be consistent.

(b) S.T for each value of λ obtained in part (a) the system has a one-parameter family of solutions, and find these solutions.

∴ Solution:-

Consider the matrix equation $AX = b$ —(1)

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 4 & 10 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ \lambda \\ \lambda^2 \end{bmatrix}$$

Now, Consider the Augmented matrix

$$\tilde{A} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & \lambda \\ 1 & 4 & 10 & \lambda^2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & \lambda-1 \\ 0 & 3 & 9 & \lambda^2-1 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{matrix}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & \lambda-1 \\ 0 & 0 & 0 & \lambda^2-3\lambda+2 \end{array}\right] \quad R_3 \to R_3 - 3R_2$$

which is in Echelon form.

Now, the System of equations will be Consistent only if $\rho(\tilde{A}) = \rho(A)$.

∵ $\rho(A) = 2$, then we must have $\rho(\tilde{A}) = 2$ & this is possible only if $\lambda^2 - 3\lambda + 2 = 0$

$\Rightarrow (\lambda-1)(\lambda-2) = 0$

$\Rightarrow \lambda = 1, 2$

(b) Case (I) $(\lambda = 1)$

When $\lambda = 1$, we then have $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow x+y+z = 1 \longrightarrow (1)$$
$$y+3z = 0 \longrightarrow (2)$$

Thus x & y are leading variables & z is a free variable.

Let $z = t$, where t is an arbitrary real number.

From (2), $y = -3z = -3t$

$\therefore \boxed{y = -3t}$

From (1), $x = 1-y-z$

$= 1-(-3t)-t$

$= 1+3t-t$

$= 1+2t$

$\therefore \boxed{x = 1+2t}$

Thus, when $\lambda = 1$, the system has one-parameter (here t is the one parameter) family of solutions, $\boxed{x = 1-2t,\ y = -3t,\ z = t}$

Case (ii) ($\lambda = 2$)

When $\lambda = 2$, we have $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$\Rightarrow x+y+z=1 \longrightarrow (3)$$
$$y+3z=1 \longrightarrow (4)$$

Then x & y are leading variables & z is a free variable.

Choose $z=k$, where k is an arbitrary real number.

From (4), $y = 1-3z$
$= 1-3k$

$\therefore \boxed{y = 1-3k}$

From (3), $x = 1-y-z$
$= 1-(1-3k)-k$
$= \cancel{1}-\cancel{1}+3k-k$
$= 2k$

$\therefore \boxed{x = 2k}$

Thus, when $\lambda=2$, the system has one-parameter (here k is the one parameter) family of solutions, $\boxed{x=2k, y=1-3k, z=k}$

Practice Problems:

1) Test the consistency & hence solve

$$x+2y+z=3$$
$$2x+3y+2z=5$$
$$3x-5y+5z=2$$
$$3x+9y-z=4.$$

2) Test the consistency & solve

$$-x+2y+3z=-2$$
$$2x-5y+z=2$$
$$3x-8y+5z=2$$
$$5x-12y-z=6$$

3) Test the consistency & solve

$$2x-2z=6$$
$$y+z=1$$
$$2x+y-z=7$$
$$3y+3z=0.$$

4) Find the values of λ and μ for which the system of equations

$$2x+3y+5z=9$$
$$7x+3y-2z=8$$
$$2x+3y+\lambda z=\mu$$

has (i) no solution (ii) a unique solution (iii) a one-parameter family of solutions.

5) Show that the equations $x+y+z=a$, $3x+4y+5z=b$, $2x+3y+4z=c$
(i) have no solutions if $a=b=c=1$

(ii) have many solutions if $a = \frac{b}{2} = c = 1$.