Module 2

Error analysis

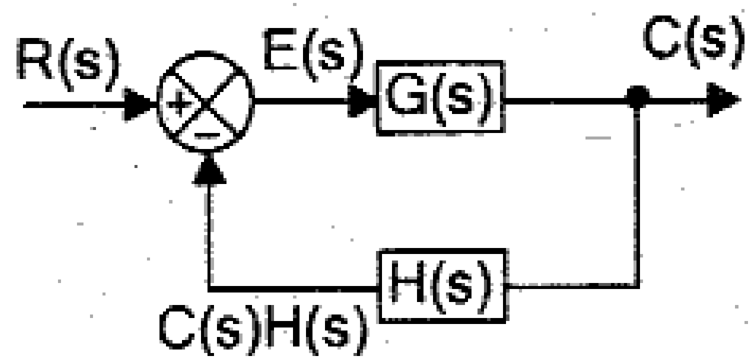
Steady state error

Let, $R(s)$ = Input signal

$E(s)$ = Error signal

$C(s)\,H(s)$ = Feedback signal

$C(s)$ = Output signal or response



$$E(s) = R(s) - C(s)\,H(s)$$

$$E(s) = R(s) - [E(s)\,G(s)]\,H(s)$$

$$E(s) + E(s)\,G(s)\,H(s) = R(s)$$

$$E(s)\,[1 + G(s)\,H(s)] = R(s)$$

$$\therefore E(s) = \frac{R(s)}{1 + G(s)\,H(s)}$$

$$\therefore e(t) = \mathcal{L}^{-1}\{E(s)\} = \mathcal{L}^{-1}\left\{\frac{R(s)}{1+G(s)\,H(s)}\right\}$$

Let, e_{ss} = steady state error.

The steady state error is defined as the value of e(t) when **t** tends to infinity.

$$\therefore e_{ss} = \underset{t\to\infty}{Lt}\ e(t)$$

The final value theorem of Laplace transform states that,

If, $\mathbf{F(s) = \mathcal{L}\{f(t)\}}$ **then,** $\mathbf{\underset{t\to\infty}{Lt}\ f(t) = \underset{s\to 0}{Lt}\ s\,F(s)}$

Using final value theorem,

The steady state error, $e_{ss} = \underset{t\to\infty}{Lt}\ e(t) = \underset{s\to 0}{Lt}\ sE(s) = \underset{s\to 0}{Lt}\ \dfrac{sR(s)}{1+G(s)\,H(s)}$

Static error constants

- Steady state error may be zero, constant or infinity depends on the type number and input signal
- **Type 0 and step input** – constant error
- **Type 1 and ramp input** (velocity signal) - constant error
- **Type 2 and parabolic input** (acceleration signal) - constant error

$$\underset{s \to 0}{\mathrm{Lt}} \frac{sR(s)}{1+G(s)\,H(s)}$$

Positional error constant, $K_p = \underset{s \to 0}{Lt}\ G(s)\,H(s)$

Velocity error constant, $K_v = \underset{s \to 0}{Lt}\ s\,G(s)\,H(s)$

Acceleration error constant, $K_a = \underset{s \to 0}{Lt}\ s^2 G(s)\,H(s)$

The K_p, K_v and K_a are in general called static error constants.

STEADY STATE ERROR WHEN THE INPUT IS UNIT STEP SIGNAL

Steady state error, $e_{ss} = \underset{s\to0}{\text{Lt}} \dfrac{sR(s)}{1+G(s)H(s)}$

When the input is unit step, $R(s) = 1/s$

$$\therefore e_{ss} = \underset{s\to0}{\text{Lt}} \frac{s\frac{1}{s}}{1+G(s)H(s)} = \underset{s\to0}{\text{Lt}} \frac{1}{1+G(s)H(s)} = \frac{1}{1+\underset{s\to0}{\text{Lt}}\, G(s)H(s)} = \frac{1}{1+K_p}$$

$$\text{where, } K_p = \underset{s\to0}{\text{Lt}}\, G(s)H(s)$$

The constant K_p is called ***positional error constant.***

Type-0 system

$$K_p = \underset{s\to 0}{Lt}\, G(s)\,H(s) = \underset{s\to 0}{Lt}\, K\frac{(s+z_1)\,(s+z_2)\,(s+z_3)......}{(s+p_1)\,(s+p_2)\,(s+p_3)......} \;=\; K\frac{z_1.z_2.z_3......}{p_1.p_2.p_3......} = \text{constant}$$

$$\therefore e_{ss} = \frac{1}{1+K_p} = \text{constant}$$

Hence in type-0 systems when the input is unit step there will be a constant steady state error.

Type-1 system

$$K_p = \underset{s\to 0}{Lt}\, G(s)\,H(s) = \underset{s\to 0}{Lt}\, K\frac{(s+z_1)\,(s+z_2)\,(s+z_3)......}{s\,(s+p_1)\,(s+p_2)\,(s+p_3)......} = \infty$$

$$\therefore e_{ss} = \frac{1}{1+K_p} = \frac{1}{1+\infty} = 0$$

In systems with type number 1 and above, for unit step input the value of K_p is infinity and so the steady state error is zero.

STEADY STATE ERROR WHEN THE INPUT IS UNIT RAMP SIGNAL

Steady state error, $e_{ss} = \underset{s \to 0}{\mathrm{Lt}} \dfrac{sR(s)}{1+G(s)\,H(s)}$

When the input is unit ramp, $R(s) = \dfrac{1}{s^2}$

$$\therefore e_{ss} = \underset{s \to 0}{\mathrm{Lt}} \frac{s\dfrac{1}{s^2}}{1+G(s)\,H(s)} = \underset{s \to 0}{\mathrm{Lt}} \frac{1}{s+sG(s)\,H(s)} = \frac{1}{\underset{s \to 0}{\mathrm{Lt}}\; sG(s)\,H(s)} = \frac{1}{K_v}$$

$$\text{where, } K_v = \underset{s \to 0}{\mathrm{Lt}}\; s\,G(s)\,H(s)$$

The constant K_v is called ***velocity error constant.***

Type-0 system

$$K_v = \underset{s\to 0}{Lt}\, sG(s)\,H(s) = \underset{s\to 0}{Lt}\; sK\frac{(s+z_1)\,(s+z_2)\,(s+z_3)\ldots\ldots}{(s+p_1)\,(s+p_2)\,(s+p_3)\ldots\ldots} = 0$$

$$\therefore e_{ss} = 1/K_v = 1/0 = \infty$$

Hence in type-0 systems when the input is unit ramp, the steady state error is infinity.

Type-1 system

$$K_v = \underset{s\to 0}{Lt}\, sG(s)\,H(s) = \underset{s\to 0}{Lt}\; sK\frac{(s+z_1)\,(s+z_2)\,(s+z_3)\ldots\ldots}{s\,(s+p_1)\,(s+p_2)\,(s+p_3)\ldots\ldots} = K\frac{z_1.z_2.z_3\ldots\ldots}{p_1.p_2.p_3\ldots\ldots} = \text{constant}$$

$$\therefore e_{ss} = 1/K_v = \text{constant}$$

- Hence in type-1 systems when the input is unit ramp there will be a constant steady state error.

Type-2 system

$$K_v = \underset{s\to 0}{Lt}\, sG(s)\,H(s) = \underset{s\to 0}{Lt}\; sK\frac{(s+z_1)\,(s+z_2)\,(s+z_3)\ldots\ldots}{s^2\,(s+p_1)\,(s+p_2)\,(s+p_3)\ldots\ldots} = \infty$$

$$\therefore e_{ss} = 1/K_v = 1/\infty = 0$$

In systems with type number 2 and above, for unit ramp input, the value of K_v is infinity so the steady state error is zero.

STEADY STATE ERROR WHEN THE INPUT IS UNIT PARABOLIC SIGNAL

Steady state error, $e_{ss} = \underset{s \to 0}{Lt} \frac{sR(s)}{1+G(s)H(s)}$

When the input is unit parabola, $R(s) = \frac{1}{s^3}$

$$\therefore e_{ss} = \underset{s \to 0}{Lt} \frac{s\frac{1}{s^3}}{1+G(s)H(s)} = \underset{s \to 0}{Lt} \frac{1}{s^2+s^2 G(s)H(s)} = \frac{1}{\underset{s \to 0}{Lt}\, s^2G(s)H(s)} = \frac{1}{K_a}$$

where, $K_a = \underset{s \to 0}{Lt}\, s^2G(s)H(s)$

The constant K_a is called ***acceleration error constant.***

Type-0 system

$$K_a = \underset{s\to 0}{Lt}\, s^2 G(s)\,H(s) = \underset{s\to 0}{Lt}\; s^2 K \frac{(s+z_1)\,(s+z_2)\,(s+z_3)......}{(s+p_1)\,(s+p_2)\,(s+p_3)......} = 0$$

$$\therefore e_{ss} = \frac{1}{K_a} = \frac{1}{0} = \infty$$

Hence in type-0 systems for unit parabolic input, the steady state error is infinity.

Type-1 system

$$K_a = \underset{s\to 0}{Lt}\, s^2 G(s)\,H(s) = \underset{s\to 0}{Lt}\; s^2 K \frac{(s+z_1)\,(s+z_2)\,(s+z_3)......}{s\,(s+p_1)\,(s+p_2)\,(s+p_3)......} = 0$$

$$\therefore e_{ss} = \frac{1}{K_a} = \frac{1}{0} = \infty$$

Hence in type-1 systems for unit parabolic input, the steady state error is infinity.

Type-2 system

$$K_a = \underset{s\to 0}{Lt}\, s^2 G(s)\,H(s) = \underset{s\to 0}{Lt}\; s^2 K \frac{(s+z_1)\,(s+z_2)\,(s+z_3)......}{s^2(s+p_1)\,(s+p_2)\,(s+p_3)......} = K\frac{z_1.z_2.z_3......}{p_1.p_2.p_3.....} = \text{constant}$$

$$\therefore e_{ss} = \frac{1}{K_a} = \text{constant}$$

Hence in type-2 system when the input is unit parabolic signal there will be a constant steady state error.

Type-3 system

$$K_a = \underset{s\to 0}{Lt}\, s^2 G(s)\, H(s) = \underset{s\to 0}{Lt}\; s^2 K \frac{(s+z_1)\,(s+z_2)\,(s+z_3)......}{s^3\,(s+p_1)\,(s+p_2)\,(s+p_3)......} = \infty$$

$$\therefore e_{ss} = \frac{1}{K_a} = \frac{1}{\infty} = 0$$

In systems with type number 3 and above for unit parabolic input the value of K_a is infinity and so the steady state error is zero.

TABLE-2.2 : Static Error Constant for Various Type Number of Systems

Error Constant	Type number of system			
	0	1	2	3
K_p	constant	∞	∞	∞
K_v	0	constant	∞	∞
K_a	0	0	constant	∞

TABLE-2.3 : Steady State Error for Various Types of Inputs

Input Signal	Type number of system			
	0	1	2	3
Unit Step	$\frac{1}{1+K_p}$	0	0	0
Unit Ramp	∞	$\frac{1}{K_v}$	0	0
Unit Parabolic	∞	∞	$\frac{1}{K_a}$	0

Generalized error coefficients

- Static error coefficients does not show variations of error with time and input should be standard input
- Generalised error coefficients give steady state error at any instants of time and for any type of input.

$$E(s) = \frac{R(s)}{1+G(s)\,H(s)} = \frac{1}{1+G(s)\,H(s)} R(s) = F(s)\,R(s)$$

$$\text{where,}\quad F(s) = \frac{1}{1+G(s)\,H(s)}$$

Let, $e(t) = \mathcal{L}^{-1}\{E(s)\}$ (error signal in time domain)

$f(t) = \mathcal{L}^{-1}\{F(s)\}$

$r(t) = \mathcal{L}^{-1}\{R(s)\}$ (input signal in time domain)

i.e., $\mathcal{L}\left\{f(t) * r(t)\right\} = F(s)\,R(s)$

where $*$ is the symbol for convolution operation

$$\therefore \mathcal{L}^{-1}\left\{F(s)\,R(s)\right\} = f(t) * r(t)$$

$$f(t) * r(t) = \int_{-\infty}^{+\infty} f(T)\, r(t-T)\, dT \quad ;$$

$$\therefore \quad e(t) = \int_{-\infty}^{+\infty} f(T)\, r(t-T)\, dT$$

Using Taylor's series expansion the signal r(t–T) can be expressed as,

$$r(t-T) = r(t) - T\,\dot{r}(t) + \frac{T^2}{2!}\,\ddot{r}(t) - \frac{T^3}{3!}\,\dddot{r}(t) + \ldots\ldots + (-1)^n\,\frac{T^n}{n!}\,\overset{n\bullet}{r}(t)\ldots$$

where, $\dot{r}(t) = 1^{st}$ derivative of r(t)

$\ddot{r}(t) = 2^{nd}$ derivative of r(t)

$\vdots$

$\overset{n\bullet}{r}(t) = n^{th}$ derivative of r(t)

On substituting the Taylor's series expansion of r(t – T), the error e(t) can be written as,

$$e(t) = \int_0^t f(T)\left[r(t) - T\,\dot{r}(t) + \frac{T^2}{2!}\,\ddot{r}(t) - \frac{T^3}{3!}\,\dddot{r}(t) + ... + (-1)^n \frac{T^n}{n!}\,\overset{n.}{r}(t)...\right] dT$$

$$e(t) = \int_0^t f(T)\,r(t)\,dT - \int_0^t f(T)\,T\,\dot{r}(t)\,dT + \int_0^t f(T)\frac{T^2}{2!}\,\ddot{r}(t)\,dT$$

$$- \int_0^t f(T)\frac{T^3}{3!}\,\dddot{r}(t) + ... + \int_0^t f(T)\,(-1)^n \frac{T^n}{n!}\,\overset{n.}{r}(t)\,dT....\infty$$

$$e(t) = r(t)\int_0^t f(T)\,dT - \dot{r}(t)\int_0^t Tf(T)\,dt + \frac{\ddot{r}(t)}{2!}\int_0^t T^2 f(T)\,dt$$

$$- \frac{\dddot{r}(t)}{3!}\int_0^t T^3 f(T)\,dt + \ldots + (-1)^n \frac{\overset{n.}{r}(t)}{n!}\int_0^t T^n f(T)\,dt \ldots$$

Let, $C_0 = +\int_0^t f(T)\,dT$

$C_1 = -\int_0^t Tf(T)\,dT$

$C_2 = +\int_0^t T^2 f(T)\,dT$

$C_3 = -\int_0^t T^3 f(T)\,dT$

$\vdots$

$C_n = (-1)^n \int_0^t T^n f(T)\,dT$

$$e(t) = r(t)\,C_0 + \dot{r}(t)\,C_1 + \ddot{r}(t)\,\frac{C_2}{2!} + \dddot{r}(t)\,\frac{C_3}{3!} + \dots + \overset{n.}{r}(t)\,\frac{C_n}{n!} + \dots$$

$$= C_0\,r(t) + C_1\,\dot{r}(t) + \frac{C_2}{2!}\,\ddot{r}(t) + \frac{C_3}{3!}\,\dddot{r}(t) + \dots + \frac{C_n}{n!}\,\overset{n.}{r}(t)\dots$$

The coefficients C_0, C_1, C_2 ,......C_n are called the generalized error coefficients or dynamic error coefficients.

The steady state error e_{ss} is obtained by taking limit $t \to \infty$ on e(t).

$$\therefore \text{ Steady state error, } e_{ss} = \underset{t\to\infty}{\mathrm{Lt}}\left[r(t)\,C_0 + \dot{r}(t)\,C_1 + \ddot{r}(t)\,\frac{C_2}{2!} + \dddot{r}(t)\,\frac{C_3}{3!} + \dots + \overset{n.}{r}(t)\,\frac{C_n}{n!} + \dots\right]$$

$$= C_0\,r(t) + C_1\,\dot{r}(t) + \frac{C_2}{2!}\,\ddot{r}(t) + \frac{C_3}{3!}\,\dddot{r}(t) + \dots + \frac{C_n}{n!}\,\overset{n.}{r}(t)\dots \qquad \dots(2.66)$$

The generalized error coefficient is given by,

$$C_n = (-1)^n \int_0^t T^n f(T)\, dT \,; \qquad \text{where } F(s) = \frac{1}{1+G(s)\,H(s)}$$

We know that $\mathcal{L}\{f(T)\} = F(s)$, hence by the definition of Laplace transform,

$$F(s) = \int_0^t f(T)\, e^{-sT} dT$$

On taking $\underset{s\to 0}{\mathrm{Lt}}$ on both sides of equation (2.67) we get,

$$\underset{s\to 0}{\mathrm{Lt}}\; F(s) = \underset{s\to 0}{\mathrm{Lt}} \int_0^t f(T)\, e^{-sT} dT$$

$$= \int_0^t f(T)\; \underset{s\to 0}{\mathrm{Lt}}\; e^{-sT} dT = \int_0^t f(T)\, dT = C_0$$

$$\therefore \boxed{\mathbf{C_0 = \underset{s\to 0}{Lt}\, F(s)}}$$

$$F(s) = \int_0^t f(T)\, e^{-sT} dT$$

On differentiating equation

$$\frac{d}{ds} F(s) = \frac{d}{ds}\int_0^t f(T)\, e^{-sT}\, dT$$

$$= \int_0^t f(T)\, \frac{d}{ds}\left(e^{-sT}\right) dT = \int_0^t f(T)\, (-T)\, e^{-sT} dT$$

$$= -\int_0^t Tf(T)\, e^{-sT} dT$$

On taking $\underset{s \to 0}{\mathrm{Lt}}$ on both sides of equation

$$\operatorname*{Lt}_{s\to 0} \frac{d}{ds}F(s) = \operatorname*{Lt}_{s\to 0} -\int_0^t Tf(T)\, e^{-sT} dT$$

$$= -\int_0^t Tf(T) \operatorname*{Lt}_{s\to 0} e^{-sT} dT = -\int_0^t Tf(T)\, dT = C_1$$

$$\therefore \boxed{\mathbf{C_1 = \operatorname*{Lt}_{s\to 0} \frac{d}{ds}F(s)}}$$

On differentiating equation (2.68) on both sides with respect to s we get,

$$\frac{d}{ds}\left[\frac{d}{ds}(F(s))\right] = \frac{d}{ds}\left[-\int_0^t Tf(T)\, e^{-sT}\, dT\right]$$

$$\frac{d^2}{ds^2} F(s) = \left[-\int_0^t Tf(T)\, \frac{d}{ds}\left(e^{-sT}\right) dT\right] = -\int_0^t Tf(T)\,(-T)\, e^{-sT} dT$$

$$\frac{d^2\,(F(s))}{ds^2} = \int_0^t T^2 f(T)\, e^{-sT} dT$$

Applying the limit $s \to 0$ on both sides of the equation (2.71) we get,

$$\underset{s\to 0}{\mathrm{Lt}} \frac{d^2}{ds^2} F(s) = \underset{s\to 0}{\mathrm{Lt}} \int_0^t T^2 \, f(T) \, e^{-st} \, dT$$

$$= \int_0^t T^2 \, f(T) \underset{s\to 0}{\mathrm{Lt}} \, e^{-st} \, dT = \int_0^t T^2 \, f(T) \, dT = C_2$$

$$\therefore \boxed{\mathbf{C_2 = \underset{s\to 0}{Lt} \frac{d^2}{ds^2} F(s)}}$$

Similarly it can be shown that,

$$\boxed{\mathbf{C_n = \underset{s\to 0}{Lt} \frac{d^n}{ds^n} F(s)}}$$

Relation between static and dynamic error coefficient

$$C_0 = \frac{1}{1+K_p}$$

$$C_1 = \frac{1}{K_v}$$

$$C_2 = \frac{1}{K_a}$$

Proof

$$C_0 = \underset{s\to 0}{\mathrm{Lt}}\, F(s) = \underset{s\to 0}{\mathrm{Lt}}\, \frac{1}{1+G(s)\,H(s)} = \frac{1`}{1+\underset{s\to 0}{\mathrm{Lt}}\, G(s)\,H(s)} = \frac{1}{1+K_p}$$

For a unity feedback control system the open loop transfer function, $G(s) = \frac{10(s+2)}{s^2(s+1)}$. Find

a) the position, velocity and acceleration error constants,

b) the steady state error when the input is R(s), where $R(s) = \frac{3}{s} - \frac{2}{s^2} + \frac{1}{3s^3}$

To find static error constants

For a unity feedback system , H(s)=1

Position error constant, $K_p = \underset{s\to 0}{Lt}\, G(s)H(s) = \underset{s\to 0}{Lt}\, G(s) = \underset{s\to 0}{Lt}\, \frac{10(s+2)}{s^2(s+1)} = \infty$

Velocity error constant, $K_v = \underset{s\to 0}{Lt}\, s\,G(s)H(s) = \underset{s\to 0}{Lt}\, s\,G(s) = \underset{s\to 0}{Lt}\, s\frac{10(s+2)}{s^2(s+1)} = \infty$

Acceleration error constant, $K_a = \underset{s\to 0}{Lt}\, s^2G(s)H(s) = \underset{s\to 0}{Lt}\, s^2G(s)$

$$= \underset{s\to 0}{Lt}\, s^2\,\frac{10(s+2)}{s^2(s+1)} = \frac{10\times 2}{1} = 20$$

b) To find steady state error

Method-I

Steady state error for non-standard input is obtained using generalized error series, given below.

The error signal, $e(t) = r(t)C_0 + \dot{r}(t)C_1 + \ddot{r}(t)\frac{C_2}{2!} + ... + \overset{n}{r}(t)\frac{C_n}{n!} + ...$

Given that, $R(s) = \frac{3}{s} - \frac{2}{s^2} + \frac{1}{3s^3}$

Input signal in time domain, $r(t) = \mathcal{L}^{-1}\{R(s)\} = \mathcal{L}^{-1}\left\{\frac{3}{s} - \frac{2}{s^2} + \frac{1}{3s^3}\right\}$

$$= 3 - 2t + \frac{1}{3}\frac{t^2}{2!} = 3 - 2t + \frac{t^2}{6}$$

$$\therefore \dot{r}(t) = \frac{d}{dt}r(t) = -2 + \frac{1}{6}2t = -2 + \frac{t}{3}$$

$$\ddot{r}(t) = \frac{d^2}{dt^2}r(t) = \frac{d}{dt}\dot{r}(t) = \frac{1}{3}$$

$$\dddot{r}(t) = \frac{d^3}{dt^3}r(t) = \frac{d}{dt}\ddot{r}(t) = 0$$

$$C_0 = \underset{s \to 0}{Lt} F(s) \, ; \qquad C_1 = \underset{s \to 0}{Lt} \frac{d}{ds} F(s) \, ; \qquad C_2 = \underset{s \to 0}{Lt} \frac{d^2}{ds^2} F(s)$$

$$C_0 = \underset{s \to 0}{Lt} F(s) \, ; \qquad C_1 = \underset{s \to 0}{Lt} \frac{d}{ds} F(s) \, ; \qquad C_2 = \underset{s \to 0}{Lt} \frac{d^2}{ds^2} F(s)$$

$$F(s) = \frac{1}{1 + G(s)H(s)} = \frac{1}{1 + G(s)} = \frac{1}{1 + \frac{10(s+2)}{s^2(s+1)}} = \frac{s^2(s+1)}{s^2(s+1) + 10(s+2)} = \frac{s^3 + s^2}{s^3 + s^2 + 10s + 20}$$

$$C_0 = \underset{s \to 0}{Lt} F(s) = \underset{s \to 0}{Lt} \left[\frac{s^3 + s^2}{s^3 + s^2 + 10s + 20} \right] = 0$$

$$C_1 = \underset{s\to 0}{Lt}\frac{d}{ds}F(s) = \underset{s\to 0}{Lt}\frac{d}{ds}\left[\frac{s^3+s^2}{s^3+s^2+10s+20}\right]$$

$$= \underset{s\to 0}{Lt}\left[\frac{\left(s^3+s^2+10s+20\right)\left(3s^2+2s\right)-\left(s^3+s^2\right)\left(3s^2+2s+10\right)}{\left(s^3+s^2+10s+20\right)^2}\right]$$

$$= \underset{s\to 0}{Lt}\left[\frac{3s^5+2s^4+3s^4+2s^3+30s^3+20s^2+60s^2+40s-3s^5-2s^4-10s^3-3s^4-2s^3-10s^2}{\left(s^3+s^2+10s+20\right)^2}\right]$$

$$= \underset{s\to 0}{Lt}\left[\frac{20s^3+70s^2+40s}{\left(s^3+s^2+10s+20\right)^2}\right] = 0$$

$$C_2 = \underset{s\to 0}{Lt}\frac{d^2}{ds^2}F(s) = \underset{s\to 0}{Lt}\frac{d}{ds}\left[\frac{d}{ds}F(s)\right] = \underset{s\to 0}{Lt}\frac{d}{ds}\left[\frac{20s^3+70s^2+40s}{\left(s^3+s^2+10s+20\right)^2}\right]$$

$$= \underset{s\to 0}{Lt}\left[\frac{\left(s^3+s^2+10s+20\right)^2\left(60s^2+140s+40\right) - \left(20s^3+70s^2+40s\right)2\times\left(s^3+s^2+10s+20\right)\left(3s^2+2s+10\right)}{\left(s^3+s^2+10s+20\right)^4}\right] = \frac{20^2\times 40}{20^4} = \frac{1}{10}$$

Error signal, $e(t) = r(t)C_0 + \dot{r}(t)C_1 + \ddot{r}(t)\frac{C_2}{2!} = \left(3 - 2t + \frac{t^2}{6}\right) \times 0 + \left(-2 + \frac{t}{3}\right) \times 0 + \frac{1}{3} \times \frac{1}{10} \times \frac{1}{2!} = \frac{1}{60}$

Steady state error, $e_{ss} = \underset{t \to \infty}{Lt}\, e(t) = \underset{t \to \infty}{Lt} \frac{1}{60} = \frac{1}{60}$

Method - II

The error signal in s-domain, $E(s) = \dfrac{R(s)}{1+G(s)H(s)}$

Given that, $R(s) = \dfrac{3}{s} - \dfrac{2}{s^2} + \dfrac{1}{3s^3}$; $G(s) = \dfrac{10(s+2)}{s^2(s+1)}$; $H(s) = 1$

$$\therefore E(s) = \frac{\dfrac{3}{s} - \dfrac{2}{s^2} + \dfrac{1}{3s^3}}{1 + \dfrac{10(s+2)}{s^2(s+1)}} = \frac{\dfrac{3}{s} - \dfrac{2}{s^2} + \dfrac{1}{3s^3}}{\dfrac{s^2(s+1)+10(s+2)}{s^2(s+1)}}$$

$$= \frac{3}{s}\left[\frac{s^2(s+1)}{s^2(s+1)+10(s+2)}\right] - \frac{2}{s^2}\left[\frac{s^2(s+1)}{s^2(s+1)+10(s+2)}\right] + \frac{1}{3s^3}\left[\frac{s^2(s+1)}{s^2(s+1)+10(s+2)}\right]$$

The steady state error e_{ss} can be obtained from final value theorem.

$$\text{Steady state error, } e_{ss} = \underset{t\to\infty}{\text{Lt}}\ e(t) = \underset{s\to 0}{\text{Lt}}\ s\,E(s)$$

$$\therefore e_{ss} = \underset{s\to 0}{\text{Lt}}\ s\left\{\frac{3}{s}\left[\frac{s^2(s+1)}{s^2(s+1)+10(s+2)}\right] - \frac{2}{s^2}\left[\frac{s^2(s+1)}{s^2(s+1)+10(s+2)}\right] + \frac{1}{3s^3}\left[\frac{s^2(s+1)}{s^2(s+1)+10(s+2)}\right]\right\}$$

$$= \underset{s\to 0}{\text{Lt}}\left\{\frac{3s^2(s+1)}{s^2(s+1)+10(s+2)} - \frac{2s(s+1)}{s^2(s+1)+10(s+2)} + \frac{(s+1)}{3s^2(s+1)+30\,(s+2)}\right\} = 0 - 0 + \frac{1}{60}$$

$$= \frac{1}{60}$$

For servomechanisms with open loop transfer function given below explain what type of input signal give rise to a constant steady state error and calculate their values.

a) $G(s) = \dfrac{20(s+2)}{s(s+1)(s+3)}$; b) $G(s) = \dfrac{10}{(s+2)(s+3)}$; c) $G(s) = \dfrac{10}{s^2(s+1)(s+2)}$

a) $G(s) = \dfrac{20(s+2)}{s(s+1)(s+3)}$

Let us assume unity feedback system, $\therefore H(s)=1$

The open loop system has a pole at origin. Hence it is a type-1 system. In systems with type number-1, the velocity (ramp) input will give a constant steady state error.

The steady state error with unit velocity input, $e_{ss} = \dfrac{1}{K_v}$

Velocity error constant, $K_v = \underset{s\to 0}{Lt}\ s\,G(s)\,H(s) = \underset{s\to 0}{Lt}\ s\,G(s)$

$$= \underset{s\to 0}{Lt}\ s\,\frac{20(s+2)}{s(s+1)(s+3)} = \frac{20\times 2}{1\times 3} = \frac{40}{3}$$

Steady state error, $e_{SS} = \dfrac{1}{K_v} = \dfrac{3}{40} = 0.075$

b) $G(s) = \dfrac{10}{(s+2)(s+3)}$

Let us assume unity feedback system, $\therefore$ H(s)=1.

The open loop system has no pole at origin. Hence it is a type-0 system. In systems with type number-0, the step input will give a constant steady state error.

The steady state error with unit step input, $e_{ss} = \dfrac{1}{1+K_p}$

Position error constant, $K_p = \underset{s\to 0}{\mathrm{Lt}}\, G(s)H(s) = \underset{s\to 0}{\mathrm{Lt}}\, G(s) = \underset{s\to 0}{\mathrm{Lt}}\, \dfrac{10}{(s+2)(s+3)} = \dfrac{10}{2\times 3} = \dfrac{5}{3}$

Steady state error, $e_{ss} = \dfrac{1}{1+K_p} = \dfrac{1}{1+\dfrac{5}{3}} = \dfrac{3}{3+5} = \dfrac{3}{8} = 0.375$

c) $G(s) = \dfrac{10}{s^2(s+1)(s+2)}$

Let us assume unity feedback system, $\therefore H(s)=1$.

The open loop system has two poles at origin. Hence it is a type-2 system. In systems with type number-2, the acceleration (parabolic) input will give a constant steady state error.

The steady state error with unit acceleration input, $e_{ss} = \dfrac{1}{K_a}$

Acceleration error constant, $K_a = \underset{s\to 0}{\mathrm{Lt}}\ s^2\,G(s)H(s) = \underset{s\to 0}{\mathrm{Lt}}\ s^2\,G(s) = \underset{s\to 0}{\mathrm{Lt}}\ s^2\,\dfrac{10}{s^2(s+1)(s+2)} = \dfrac{10}{1\times 2} = 5$

Steady state error, $e_{ss} = \dfrac{1}{K_a} = \dfrac{1}{5} = 0.2$

A unity feedback system has the forward transfer function $G(s) = \dfrac{K_1(2s+1)}{s\,(5s+1)(1+s)^2}$. When the input $r(t) = 1+6t$, determine the minimum value of K_1 so that the steady error is less than 0.1.

Given that, input $r(t) = 1 + 6t$

On taking laplace transform of r(t) we get R(s).

$$\therefore R(s) = \mathcal{L}\{r(t)\} = \mathcal{L}\{1+6t\} = \frac{1}{s} + \frac{6}{s^2}$$

The error signal in s-domain E(s) is given by,

$$\therefore E(s) = \frac{R(s)}{1+G(s)H(s)} = \frac{\frac{1}{s}+\frac{6}{s^2}}{1+\frac{K_1(2s+1)}{s(5s+1)(1+s)^2}} = \frac{\frac{1}{s}+\frac{6}{s^2}}{\frac{s\,(5s+1)\,(1+s)^2+K_1(2s+1)}{s\,(5s+1)\,(1+s)^2}}$$

$$= \frac{1}{s}\left[\frac{s\,(5s+1)\,(1+s)^2}{s\,(5s+1)\,(1+s)^2+K_1(2s+1)}\right] + \frac{6}{s^2}\left[\frac{s\,(5s+1)\,(1+s)^2}{s\,(5s+1)\,(1+s)^2+K_1(2s+1)}\right]$$

The steady state error e_{ss} can be obtained from final value theorem.

$$e_{ss} = \underset{t\to\infty}{Lt}\ e(t) = \underset{s\to 0}{Lt}\ s\,E(s)$$

$$= \underset{s\to 0}{Lt}\ s\left\{\frac{1}{s}\left[\frac{s(5s+1)(1+s)^2}{s(5s+1)(1+s)^2+K_1(2s+1)}\right]+\frac{6}{s^2}\left[\frac{s(5s+1)(1+s)^2}{s(5s+1)(1+s)^2+K_1(2s+1)}\right]\right\}$$

$$= \underset{s\to 0}{Lt}\left\{\frac{s(5s+1)(1+s)^2}{s(5s+1)(1+s)^2+K_1(2s+1)}+\frac{6(5s+1)(1+s)^2}{s(5s+1)(1+s)^2+K_1(2s+1)}\right\} = 0+\frac{6}{K_1}=\frac{6}{K_1}$$

Given that, $e_{ss} < 0.1$, $\quad \therefore\ 0.1=\dfrac{6}{K_1}$ $\quad$ or $\quad K_1=\dfrac{6}{0.1}=60$