Bootstrapping: Lessons Learnt in the Last 10 Years

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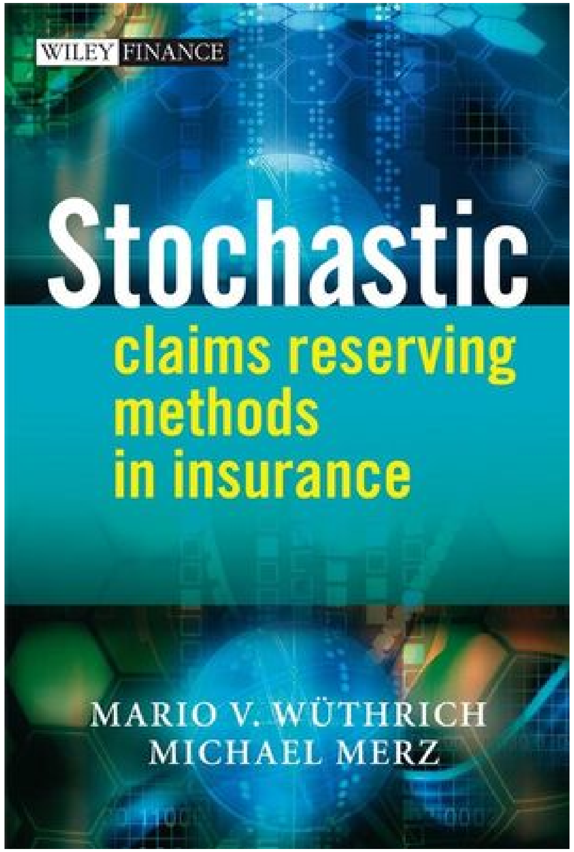
"We can do this the easy way, or we can do it the hard way"

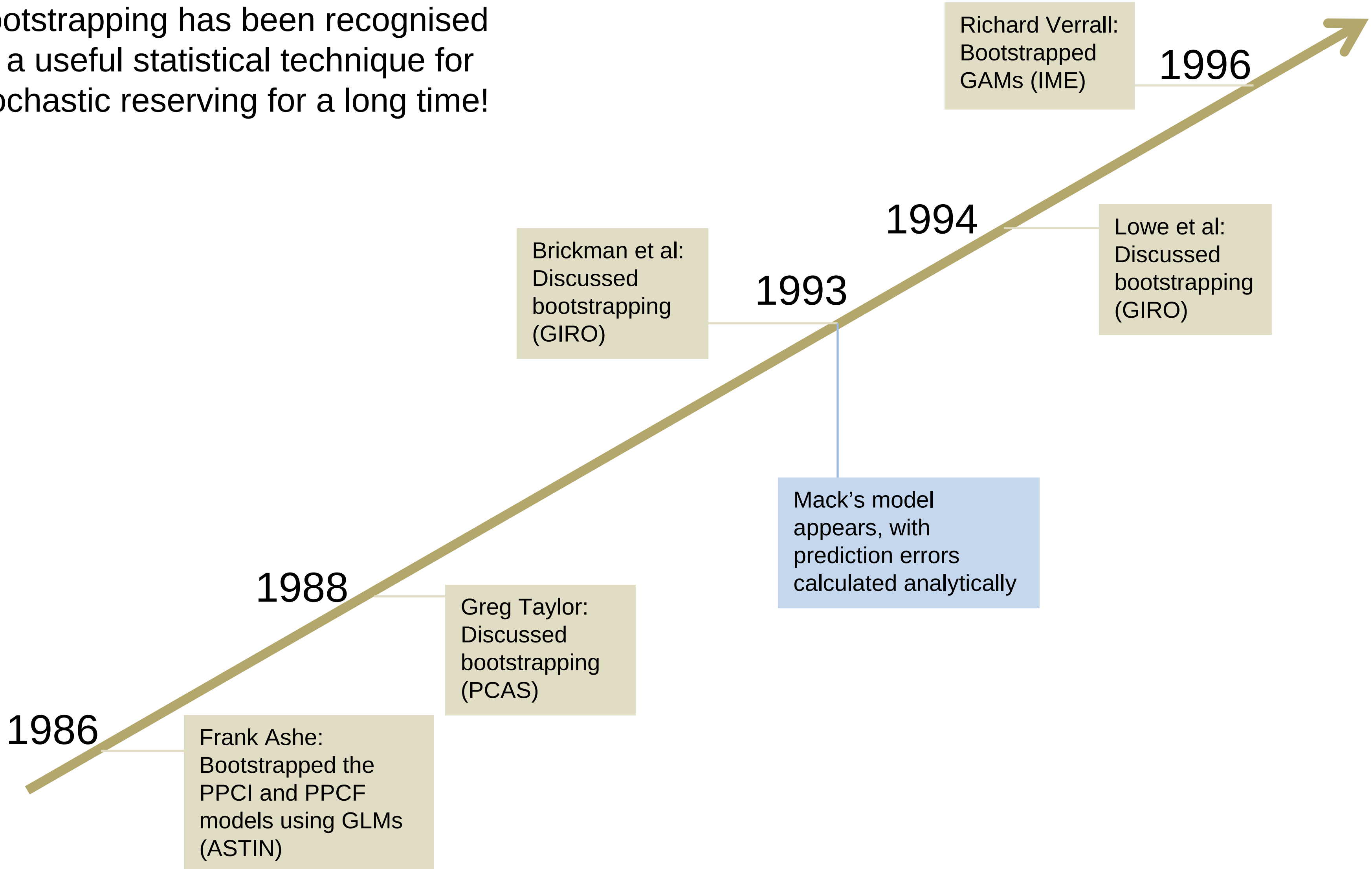
Stochastic claims reserving

- This has become a new academic discipline
- It has spawned several PhDs
- Numerous papers appearing in academic journals
- Presentations at every actuarial conference
- A book has appeared
- There is a Wikipedia page

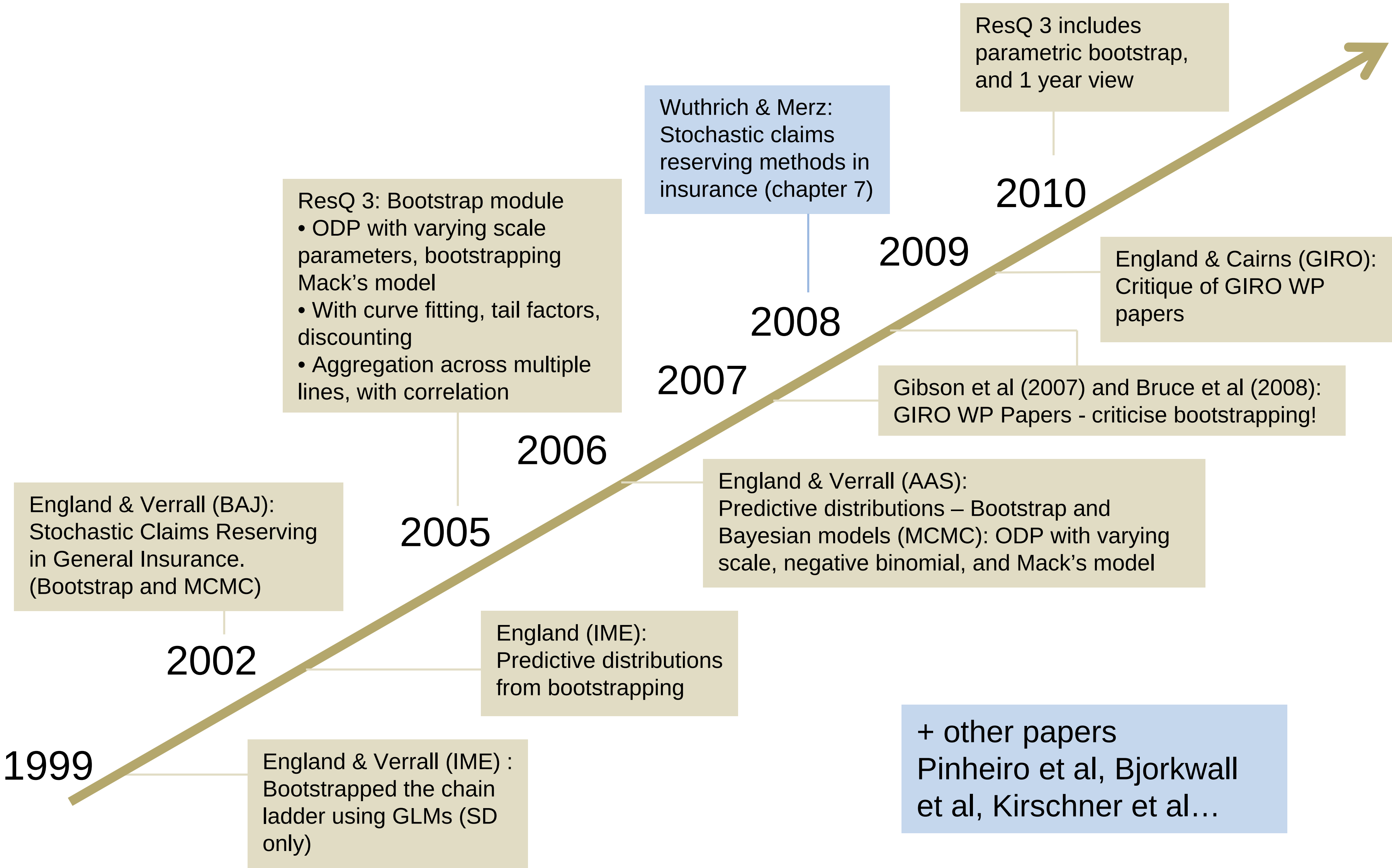


Let’s go back a bit further…

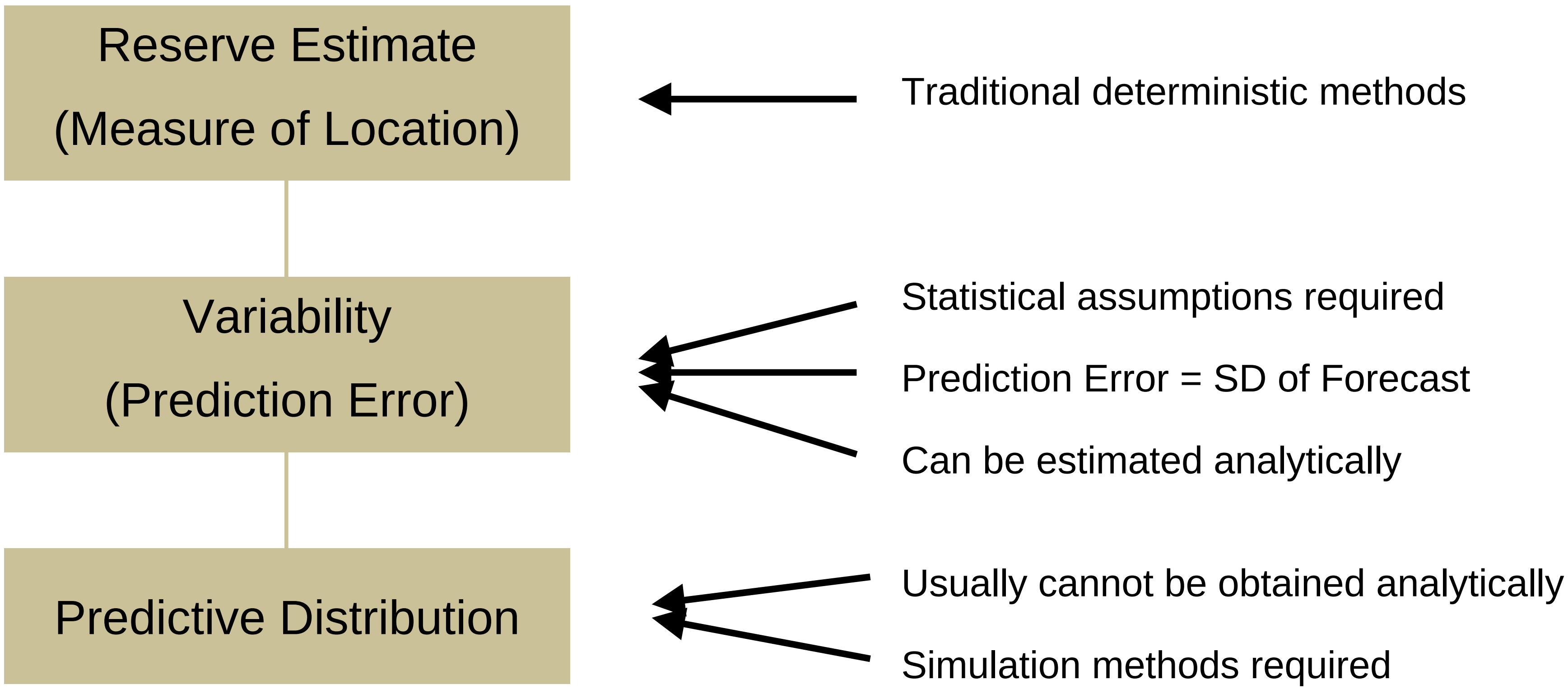
Bootstrapping has been recognised as a useful statistical technique for stochastic reserving for a long time!



Bootstrapping: The last 10 years (or so)…



Conceptual Framework



England & Verrall (1999) and England (2002)

- E&V (1999) compared analytic and bootstrap estimates of the prediction error for an over-dispersed Poisson GLM with constant scale parameter that gives identical forecasts to the chain ladder model.
  - The aim was to devise an easy way to approximate the prediction error in a spreadsheet, without requiring statistical software
  - An appropriate residual definition and bootstrap procedure was used to ensure consistency between the analytic and bootstrap results
  - We fell into the trap of only considering the SD of the predictive distribution
- England (2002) extended E&V (1999) by adopting a two-stage simulation process to provide a complete predictive distribution of the reserves (and all cash-flows)
  - It was later discovered that Ashe (1986) and Taylor (1988) had already advocated this approach.

England & Verrall (2002)

- E&V (2002) reviewed several models and frameworks, and explored the connections between them
- Presented Mack's model within a GLM framework
  - Provided alternative formulae for the analytic prediction error under Mack's model (this is still the HARD way)
- Discussed curve fitting using GLMs, and smoothers using GAMs
- Discussed the BF method as a stochastic model
- Described bootstrapping the ODP model
- Introduced MCMC methods as an alternative to bootstrapping for obtaining predictive distributions
- Illustrated all models with examples
- Discussed the results in the context of simulation based capital models

Bootstrapping begins to take off…

- E&V (2002) became surprisingly popular, especially the bootstrapping parts
- The bootstrapping methods were applied without deviation
  - For example, ODP chain ladder model only, with constant scale parameter
- “Bootstrapping” and “Mack” were seen as mutually exclusive, which fostered a misconception
  - Bootstrapping is just a statistical procedure, which can be applied to any well-defined model
  - So bootstrapping can be applied to Mack’s model, and the log-normal models of Zehnwirth et al
- Furthermore, bootstrapping is not the only technique that will give predictive distributions. E&V (2002) also discussed MCMC methods.
- England & Verrall (2006) was written to tackle the misconceptions, and compare bootstrapping with MCMC methods

England & Verrall (2006)

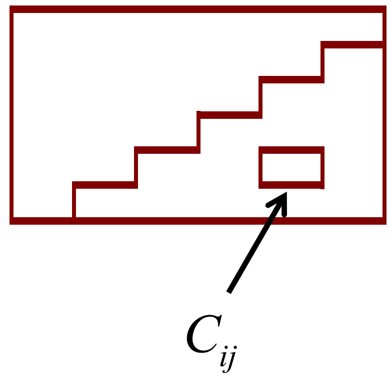
- E&V (2006) presented a general framework for bootstrapping GLMs, and for Bayesian GLMs
- Provided detailed results (with examples) for the over-dispersed Poisson model, over-dispersed Negative Binomial model, and Mack's model.
  - Note: the bootstrap and Bayesian versions of Mack's model give predictive distributions where the SDs are consistent with Mack's formulae
- Discussed the estimation of non-constant dispersion parameters (see Appendix), and illustrated this in the examples
- Furthermore, a description of how to bootstrap log-normal regression models was also provided (see Discussion)

Stochastic Reserving

Over-dispersed Poisson Model

Over-Dispersed Poisson Model

C_{ij} = Incremental claims in origin year i and development year j



$$C_{ij} \sim ODP(\mu_{ij}, \phi_j)$$

$$E\left[C_{ij}\right] = \mu_{ij}$$

$$Var\left[C_{ij}\right] = \phi_j \mu_{ij}$$

Variance proportional to expected value

With constant scale parameters, $\phi_j = \phi \ \forall \ j$

Writing $\log(\mu_{ij}) = \eta_{ij}$ with $\eta_{ij} = c + a_i + b_j$

gives the same forecasts as the chain ladder model

Variability in Claims Reserves

- Variability of a forecast
- Includes estimation variance and process variance

$$\text{prediction error} = (\text{process variance} + \text{estimation variance})^{\frac{1}{2}}$$

- Calculated analytically, the problem reduces to estimating the two components. See, for example, E & V (1999, 2002)
  - This is doing it the HARD way
- Note: “prediction error” is also known as “root mean square error of prediction” (RMSEP), and is just the SD of the forecast

England & Verrall (1999)
ODP Chain Ladder Model with constant scale parameter

- E&V (1999) used bootstrapping to give an estimate of the estimation variance only. This was adjusted to take account of the 'degrees of freedom'
- The process variance was calculated analytically and added

$$\text{prediction error}_{bs} = \sqrt{\phi R + \frac{N}{N\text{-}p}\left(SE_{bs}(R)\right)^2}$$

- The results were compared to their analytic counterpart, and shown to be very close, demonstrating the usefulness of the approach
- The aim was to find an EASY way to calculate the prediction error in a spreadsheet, without the need for statistical software
  - Obviously we had to do it the HARD way as well, to make the comparison

England (2001/2002) – Addendum to 1999 paper

- England (2001/2002) introduced two developments
  - Adjusting the residuals *before* bootstrapping to automatically take account of the "degrees of freedom". This increases the variance and eliminates the need to adjust the estimation variance *after* bootstrapping.

$$r'_P = r_P \sqrt{\frac{N}{N\text{-}p}}$$

  - Using simulation at the forecasting stage for the "process variance"
- This gives a full predictive distribution of all cash-flows
  - The prediction error is the standard deviation of the predictive distribution, but just focusing on the standard deviation is missing the point.

Stochastic Reserving

Over-dispersed Poisson Model

Doing it the EASY way

Reserving and Bootstrapping

Define and fit statistical model

Obtain residuals and pseudo data

Re-fit statistical model to pseudo data

Obtain forecast, including process error

Any model that can be clearly defined can be bootstrapped

Bootstrapping the Chain Ladder
Over-dispersed Poisson model

1. Fit chain ladder model and obtain fitted incremental values
2. Obtain (scaled) Pearson residuals → $r_{ij} = \frac{C_{ij} - \mu_{ij}}{\sqrt{\phi_j \mu_{ij}}}$
3. Resample residuals[1]
4. Obtain pseudo data, given $r_{ij}^{*}, \mu_{ij}, \phi_j$

$$C_{ij}^{*} = r_{ij}^{*} \sqrt{\phi_j \mu_{ij}} + \mu_{ij}$$

5. Use chain ladder to re-fit model, and estimate future incremental payments

[1] At this stage, we use the adjusted scaled residuals

Bootstrapping the Chain Ladder

6. Simulate observation from process distribution assuming mean is incremental value obtained at Step 5
7. Repeat many times, storing the reserve estimates (this gives the predictive distribution)
8. Prediction error is then standard deviation of results

Note: Where curve fitting has been used for smoothing and extrapolation (for tail estimation), replace the chain ladder model in steps 1 and 5 by the actual model used

Example

Taylor & Ashe Data
Observed incremental values

	1	2	3	4	5	6	7	8	9	10
1	357,848	766,940	610,542	482,940	527,326	574,398	146,342	139,950	227,229	67,948
2	352,118	884,021	933,894	1,183,289	445,745	320,996	527,804	266,172	425,046	
3	290,507	1,001,799	926,219	1,016,654	750,816	146,923	495,992	280,405		
4	310,608	1,108,250	776,189	1,562,400	272,482	352,053	206,286			
5	443,160	693,190	991,983	769,488	504,851	470,639				
6	396,132	937,085	847,498	805,037	705,960					
7	440,832	847,631	1,131,398	1,063,269						
8	359,480	1,061,648	1,443,370							
9	376,686	986,608								
10	344,014									
Dev Factors	**3.49061**	**1.74733**	**1.45741**	**1.17385**	**1.10382**	**1.08627**	**1.05387**	**1.07656**	**1.01772**	**1.00000**

Taylor & Ashe Data

Fitted incremental values (chain ladder model)

	1	2	3	4	5	6	7	8	9	10	Reserve
1	270,061	672,617	704,494	753,438	417,350	292,571	268,344	182,035	272,606	67,948	0
2	376,125	936,779	981,176	1,049,342	581,260	407,474	373,732	253,527	379,669	94,634	94,634
3	372,325	927,316	971,264	1,038,741	575,388	403,358	369,957	250,966	375,833	93,678	469,511
4	366,724	913,365	956,652	1,023,114	566,731	397,290	364,391	247,190	370,179	92,268	709,638
5	336,287	837,559	877,254	938,200	519,695	364,316	334,148	226,674	339,456	84,611	984,889
6	353,798	881,172	922,933	987,053	546,756	383,287	351,548	238,477	357,132	89,016	1,419,459
7	391,842	975,923	1,022,175	1,093,189	605,548	424,501	389,349	264,121	395,534	98,588	2,177,641
8	469,648	1,169,707	1,225,143	1,310,258	725,788	508,792	466,660	316,566	474,073	118,164	3,920,301
9	390,561	972,733	1,018,834	1,089,616	603,569	423,113	388,076	263,257	394,241	98,266	4,278,972
10	344,014	856,804	897,410	959,756	531,636	372,687	341,826	231,882	347,255	86,555	4,625,811

Total	18,680,856

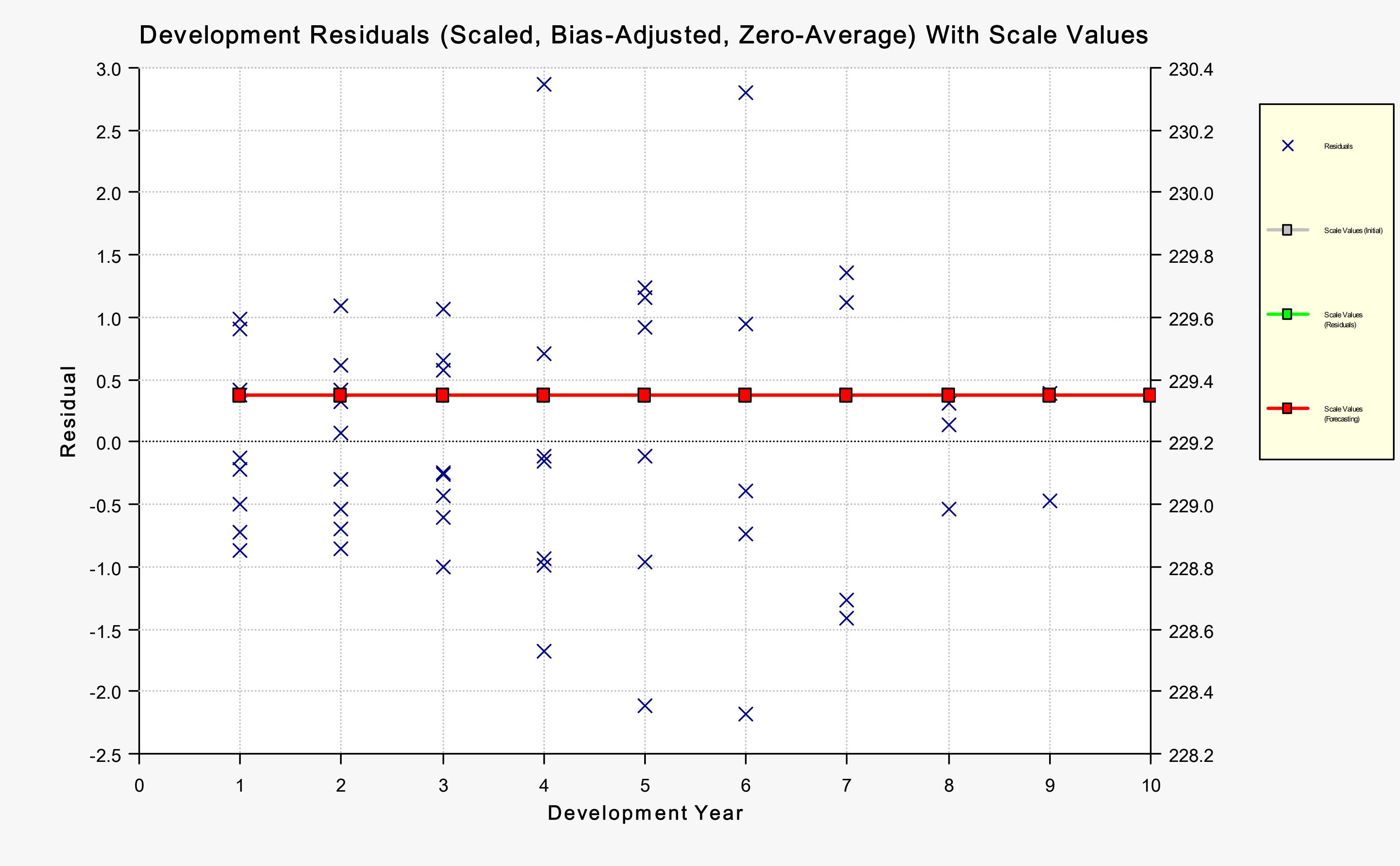
Taylor & Ashe Data
Scaled residuals : ODP with constant scale parameter

	1	2	3	4	5	6	7	8	9	10
1	0.737	0.501	-0.488	-1.359	0.742	2.272	-1.027	-0.430	-0.379	0.000
2	-0.171	-0.238	-0.208	0.570	-0.775	-0.591	1.099	0.110	0.321	
3	-0.585	0.337	-0.199	-0.094	1.008	-1.760	0.903	0.256		
4	-0.404	0.889	-0.804	2.325	-1.704	-0.313	-1.142			
5	0.804	-0.688	0.534	-0.759	-0.090	0.768				
6	0.310	0.260	-0.342	-0.799	0.939					
7	0.341	-0.566	0.471	-0.125						
8	-0.701	-0.436	0.860							
9	-0.097	0.061								
10	0.000									

Scale^0.5	**229.3**	**229.3**	**229.3**	**229.3**	**229.3**	**229.3**	**229.3**	**229.3**	**229.3**	**229.3**

Taylor & Ashe Data

Scaled residuals : ODP with constant scale parameter



Note that the volatility is lower at the earlier and later development periods

ODP model: Non-constant scale parameters

- With constant scale parameters, the scale parameter is estimated as the sum of (unscaled) residuals squared, divided by the degrees of freedom
- Alternatively, we could view this as the average of the sum of adjusted residuals squared

adjusted unscaled residual

$$\phi = \frac{\sum r_u^{\ 2}}{N-p} \qquad = \frac{N}{N-p} \times \frac{\sum r_u^{\ 2}}{N} \qquad = \frac{\sum \left(\left(\frac{N}{N-p} \right)^{1/2} r_u \right)^2}{N}$$

- For non-constant scale parameters, with a scale parameter at each development period, we just take the average of the adjusted residuals squared at each development period

$$\phi_j = \frac{\sum_{i=1}^{n_j} \left(\left(\frac{N}{N-p} \right)^{1/2} (r_u)_{ij} \right)^2}{n_j}$$

- This gives residuals that are standardised better

residuals at development period *j*

Taylor & Ashe Data

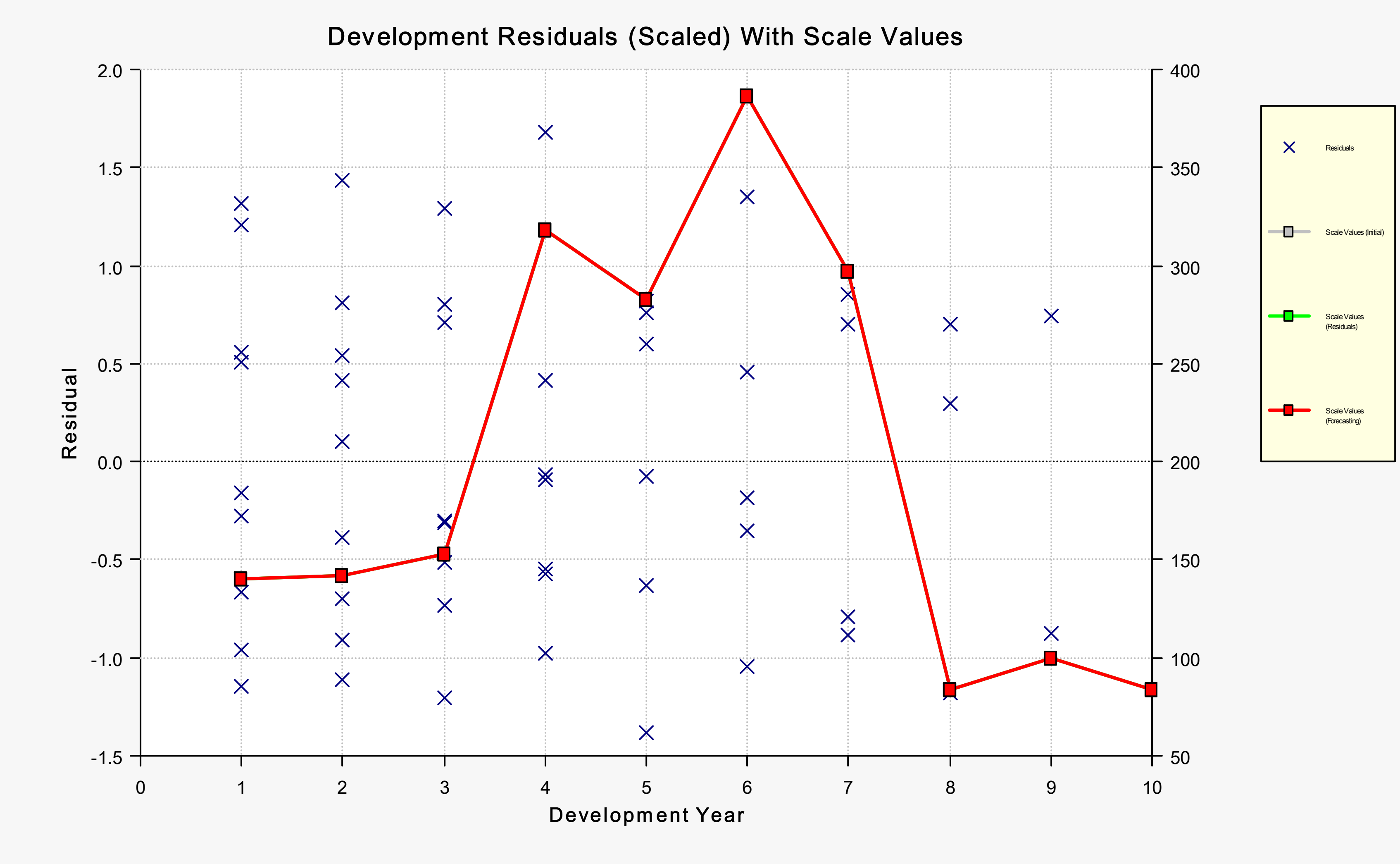
Scaled residuals : ODP with non-constant scale parameter

	1	2	3	4	5	6	7	8	9	10
1	1.207	0.808	-0.731	-0.980	0.602	1.348	-0.794	-1.176	-0.873	0.000
2	-0.280	-0.383	-0.312	0.411	-0.629	-0.350	0.849	0.299	0.740	
3	-0.958	0.544	-0.299	-0.068	0.818	-1.045	0.698	0.701		
4	-0.662	1.433	-1.206	1.676	-1.383	-0.186	-0.883			
5	1.317	-1.109	0.800	-0.548	-0.073	0.456				
6	0.509	0.419	-0.513	-0.576	0.762					
7	0.559	-0.913	0.706	-0.090						
8	-1.149	-0.702	1.288							
9	-0.159	0.099								
10	0.000									

Scale^0.5	**139.9**	**142.3**	**153.0**	**318.1**	**282.6**	**386.6**	**296.7**	**83.9**	**99.6**	**83.9**

Taylor & Ashe Data

Scaled residuals: ODP with non-constant scale parameter



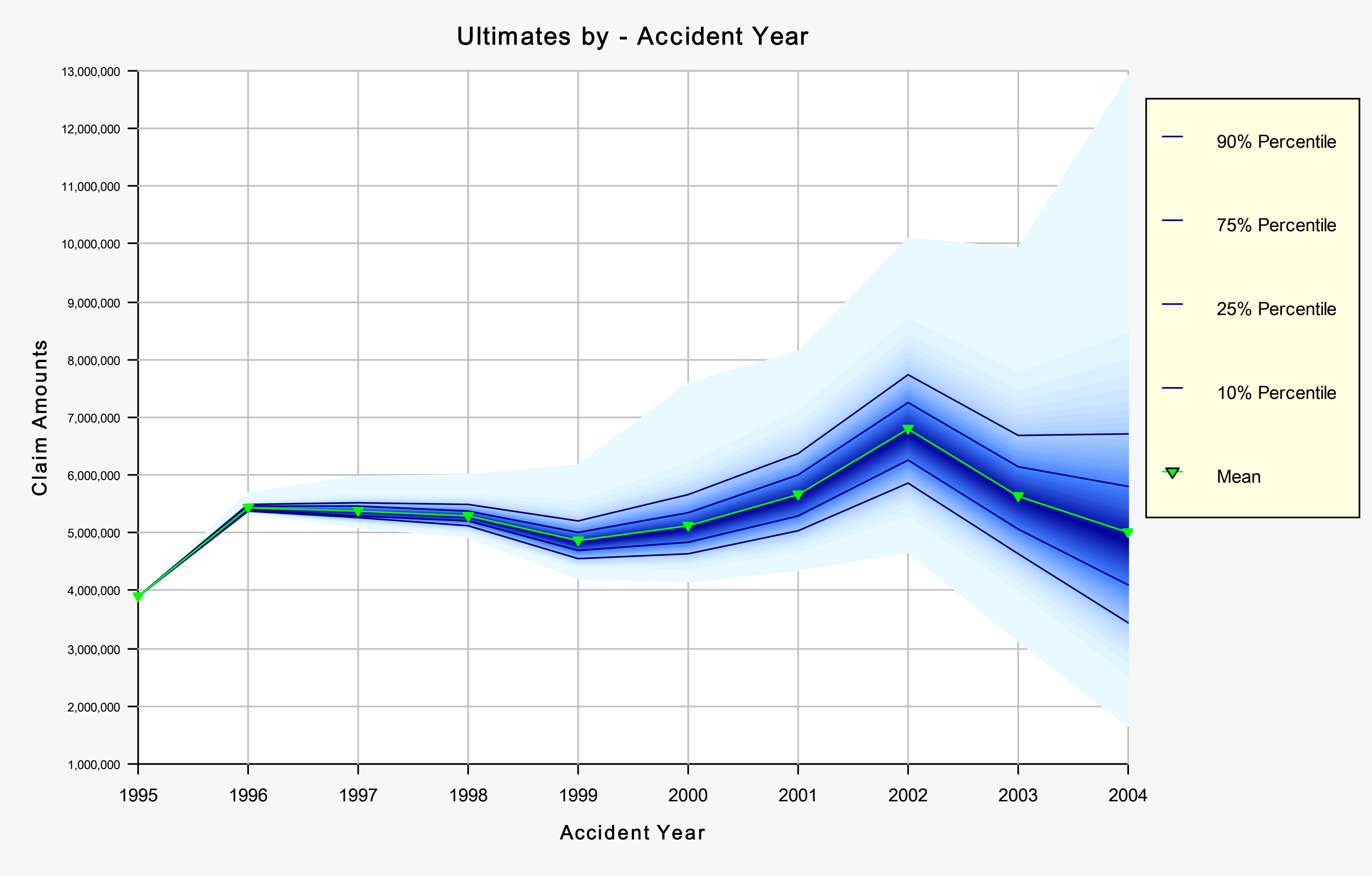
Note that the residuals are standardised better when using non-constant scale parameters

Taylor & Ashe Data

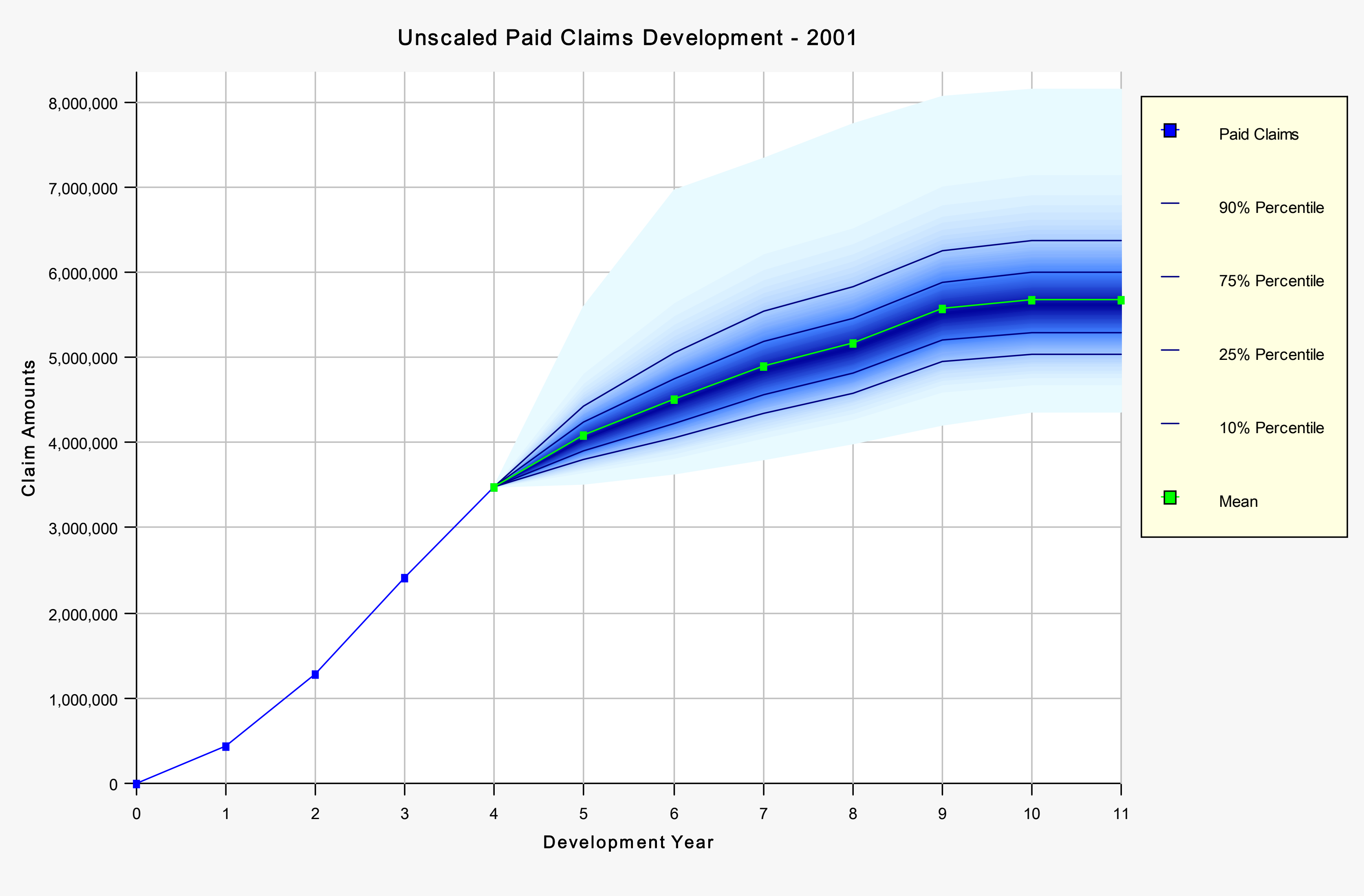
ODP: Constant vs Non-Constant Scale Parameters

	Simulated		Simulated	
	Constant Scale		Non-Constant Scale	
Accident Year	Prediction Error	Prediction Error %	Prediction Error	Prediction Error %
1	0	0.0%	0	0.0%
2	112,552	119.0%	43,882	45.3%
3	217,547	46.2%	109,449	23.0%
4	262,934	36.9%	141,509	19.8%
5	306,595	31.0%	256,031	25.7%
6	375,745	26.4%	398,377	27.8%
7	500,332	22.9%	529,898	24.2%
8	791,481	20.1%	735,245	18.7%
9	1,060,473	24.7%	809,457	18.9%
10	2,025,898	43.3%	1,285,560	27.6%
Total	2,992,296	15.9%	2,228,677	11.9%

Distribution of Ultimates by Origin Period



Claims Development Percentile Fan Chart
Single Origin Year



Over-dispersed Poisson model
Recommendations 1

- For the ODP model, we recommend using non-constant scale parameters
  - This is analogous to using varying *alpha* parameters in Mack's model
  - Some people have tried to avoid the heteroscedasticity issues by sampling residuals from different sections of the residuals triangle, but this is not ideal – a fundamental principle of bootstrapping is that the "observations" being re-sampled are *iid,* and standardising using non-constant scale parameters is a way of achieving this.
  - England & Verrall have tried to highlight this:

"This paper only considers the special case of a constant dispersion parameter ϕ. Evidence of heteroskedasticity would require extensions to the approach … The dispersion parameters would need to be included in the residual definition …" - England (2002)

"The restriction that the scale parameter is constant for all observations can be relaxed. It is common to allow the scale parameters to depend on development period …"

"We recommend using non-constant scale parameters (or at least checking the assumption that using a constant scale parameter is appropriate)…" - England & Verrall (2006)

Over-dispersed Poisson model

- There is the possibility of obtaining negative pseudo incremental values when using non-parametric bootstrapping (resampling residuals), which could in turn lead to negative pseudo cumulative values.
- This is a known issue with non-parametric bootstrapping. For example:

  "Although the [non-parametric] bootstrap/ simulation procedure provides prediction errors that are consistent with their analytic counterparts, the predictive distribution produced in this way might have some undesirable properties. For example, for some origin year reserves, the minimum values of the predictive distribution could be negative…" – England (2002)

- *"It [non-parametric bootstrapping] is not without its difficulties, for example: a small number of sets of pseudo data may be incompatible with the underlying model…" – England & Verrall (2006)*

If $r_{ij} = \dfrac{C_{ij} - \mu_{ij}}{\sqrt{\phi_j \mu_{ij}}}$ then $C_{ij}^* = r_{ij}^* \sqrt{\phi_j \mu_{ij}} + \mu_{ij}$

$$C_{ij}^* < 0 \text{ if } r_{ij}^* < -\sqrt{\frac{\mu_{ij}}{\phi_j}}$$

$C =$ incremental amounts

$\mu =$ expected incremental amounts

$\phi =$ scale parameter

$r =$ scaled Pearson residual

This issue disappears with parametric bootstrapping

Over-dispersed Poisson model

- The (non-parametric) procedure of re-sampling residuals and inverting is simply used to obtain pseudo-data that has the same characteristics as the observed data:

$$E\left[C_{ij}^{*}\right] = \mu_{ij}$$
$$Var\left[C_{ij}^{*}\right] = \phi_{j}\mu_{ij}$$

- Once the expected values and scale parameters have been estimated, we could use a parametric distribution to generate the pseudo data instead, calibrated to give the same mean and variance
  - For example, using a gamma or log-normal (or some other distribution) as a proxy to an over-dispersed Poisson will force all pseudo values to be positive

Over-dispersed Poisson model
Recommendations 2

- We also recommend investigating parametric bootstrapping
  - In some ways, the common approach of using a non-parametric approach for bootstrapping (parameter uncertainty), then a parametric distribution at the forecasting stage (to include process uncertainty) is a little inconsistent.
  - Using non-parametric bootstrapping of residuals, the pseudo-data extremes are naturally limited by the extremes of the residuals (although the effect of this will reduce as triangle size increases)
  - Using an appropriate parametric approach, this issue is ameliorated
  - [For a given triangle however, there is no guarantee that a parametric approach will be more extreme]
  - For further information, see Björkwall *et al* 2008.

ODP Model – Characteristics

- It is a model of *incremental* amounts
- It is not suitable when development factors are less than 1
- When forecasting, by using a distribution that only allows positive values (eg Gamma or Lognormal), forecast incremental values will be positive
  - That is, simulated cumulative amounts will be strictly increasing
  - Simulated reserves can never be negative
  - The ultimate claims will be at least as big as the observed cumulative paid for each origin period
  - (Although note comments on parametric vs non-parametric bootstrapping above)

Stochastic Reserving

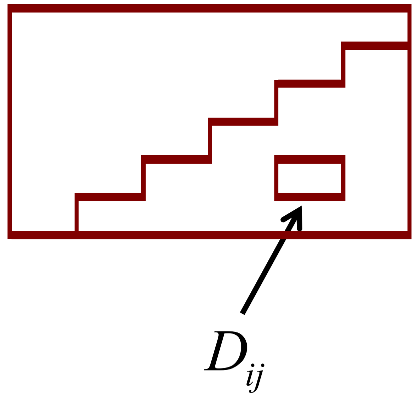
Mack's Model

Doing it the HARD way

Mack's Model

Mack, T (1993), *Distribution-free calculation of the standard error of chain-ladder reserve estimates*. ASTIN Bulletin, 22, 93-109

D_{ij} = Cumulative claims in origin year i and development year j



Specified mean and variance only:

$$E(D_{ij}) = \lambda_j D_{i,j-1}$$

← Expected value proportional to previous cumulative

$$V(D_{ij}) = \sigma_j^2 D_{i,j-1}$$

← Variance proportional to previous cumulative

Mack's Model

$$\hat{\lambda}_j = \frac{\sum_{i=1}^{n-j+1} w_{ij} f_{ij}}{\sum_{i=1}^{n-j+1} w_{ij}}$$

Estimator for lambda

$$\hat{\sigma}_j^2 = \frac{1}{n-j} \sum_{i=1}^{n-j+1} w_{ij} \left(f_{ij} - \hat{\lambda}_j \right)^2$$

Estimator for sigma squared

$$w_{ij} = D_{i,j-1} \text{ and } f_{ij} = \frac{D_{ij}}{D_{i,j-1}}$$

Variability in Claims Reserves

- Variability of a forecast
- Includes estimation variance and process variance

$$\text{prediction error} = (\text{process variance} + \text{estimation variance})^{1/2}$$

- Problem reduces to estimating the two components. For example, for the reserves in origin year *i*:

$$RMSEP\left[\hat{R}_i\right] \approx \sqrt{\hat{D}_{in}^2 \sum_{k=n-i+1}^{n-1} \frac{\hat{\sigma}_{k+1}^2}{\hat{\lambda}_{k+1}^2} \left(\frac{1}{\hat{D}_{ik}} + \frac{1}{\sum_{q=1}^{n-k} D_{qk}} \right)}$$

Mack's Model as a GLM

Step 1: Reformulate Mack's model as a model of the ratios

Step 2: Recognise that Mack's "Scale" parameters are derived from the squared residuals of a weighted normal regression model

$$E\left(\frac{D_{ij}}{D_{i,j-1}}\right) = E\left(f_{ij}\right) = \lambda_j$$

$$V\left(\frac{D_{ij}}{D_{i,j-1}}\right) = V\left(f_{ij}\right) = \frac{\sigma_j^2}{w_{ij}}$$

$$w_{ij} = D_{i,j-1} \text{ and } f_{ij} = \frac{D_{ij}}{D_{i,j-1}}$$

$$D_{ij} = \text{Cumulative claims in origin year } i \text{ and development year } j$$

Mack's Model as a GLM

$$f_{ij} \sim N\left(\lambda_j, \frac{\sigma_j^2}{w_{ij}}\right)$$

Weighted normal errors GLM with weights *w*

$$r_{ij} = \sqrt{w_{ij}}\left(f_{ij} - \hat{\lambda}_{ij}\right)$$

Pearson residual (unscaled)

$$\hat{\sigma}_j^2 = \frac{1}{n-j}\sum_{i=1}^{n-j+1} w_{ij}\left(f_{ij} - \hat{\lambda}_j\right)^2$$

Note: Mack's model was not derived as a weighted normal GLM, but a weighted normal GLM gives the same estimator of sigma

Stochastic Reserving

Mack's Model

Doing it the EASY way

Reserving and Bootstrapping

Define and fit statistical model

Obtain residuals and pseudo data

Re-fit statistical model to pseudo data

Obtain forecast, including process error

Any model that can be clearly defined can be bootstrapped

Bootstrapping the Chain Ladder

1. Fit chain ladder model to the observed link ratios
2. Obtain (scaled) Pearson residuals → $r_{ij} = \dfrac{\sqrt{w_{ij}}\left(f_{ij} - \lambda_j\right)}{\sigma_j}$
3. Resample residuals
4. Obtain pseudo data, given r_{ij}^*, λ_j

$$f_{ij}^* = \frac{r_{ij}^* \sigma_j}{\sqrt{w_{ij}}} + \lambda_j$$

5. Use chain ladder model to re-estimate the development factors (as a weighted average of the pseudo-link ratios, using the original weights w)

Bootstrapping the Chain Ladder
Mack's model

6. Given the observed cumulative payments to date, move 1 period ahead by multiplying the previous cumulative claims by the appropriate simulated development factor obtained at Step 5
   - Include the process error by sampling a single observation from the underlying process distribution
7. Move to the next period, where the forecast cumulative amounts are now conditional on the simulated 1 period ahead forecast obtained at Step 6 (including the process error)
8. Repeat many times, storing the reserve estimates (this gives the predictive distribution)
9. Prediction error is then standard deviation of results

Note: Where curve fitting has been used for smoothing and extrapolation (for tail estimation), replace the chain ladder model in steps 1 and 5 by the actual model used

Example

Taylor & Ashe Data

Scaled residuals: Mack's model

	1	2	3	4	5	6	7	8	9
1	-0.519	-1.117	-1.152	0.772	1.490	-0.850	-1.189	-0.759	0.000
2	0.030	0.047	0.632	-0.608	-0.322	0.939	0.346	0.651	
3	1.290	-0.179	0.006	0.850	-1.141	0.758	0.682		
4	1.500	-1.228	1.840	-1.594	-0.282	-0.907			
5	-1.540	0.689	-0.683	0.005	0.543				
6	-0.197	-0.664	-0.636	0.878					
7	-0.942	0.764	-0.137						
8	0.693	1.647							
9	0.197								

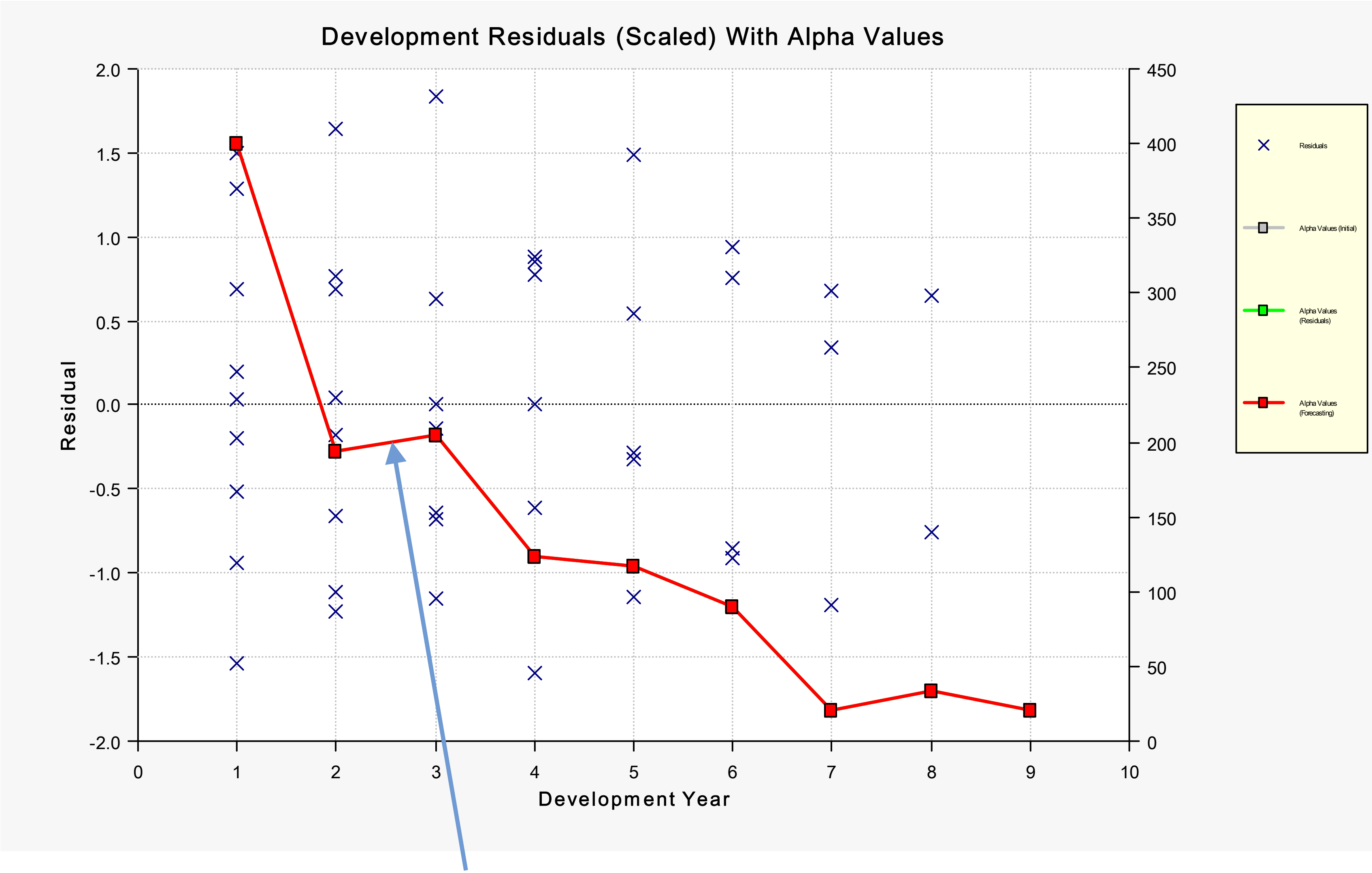
Note that the σ parameters decrease rapidly

Mack's sigma	**400.4**	**194.3**	**204.9**	**123.2**	**117.2**	**90.5**	**21.1**	**33.9**	**21.1**

This parameter is highly influential on the overall variability

Taylor & Ashe Data

Scaled residuals: Mack's model



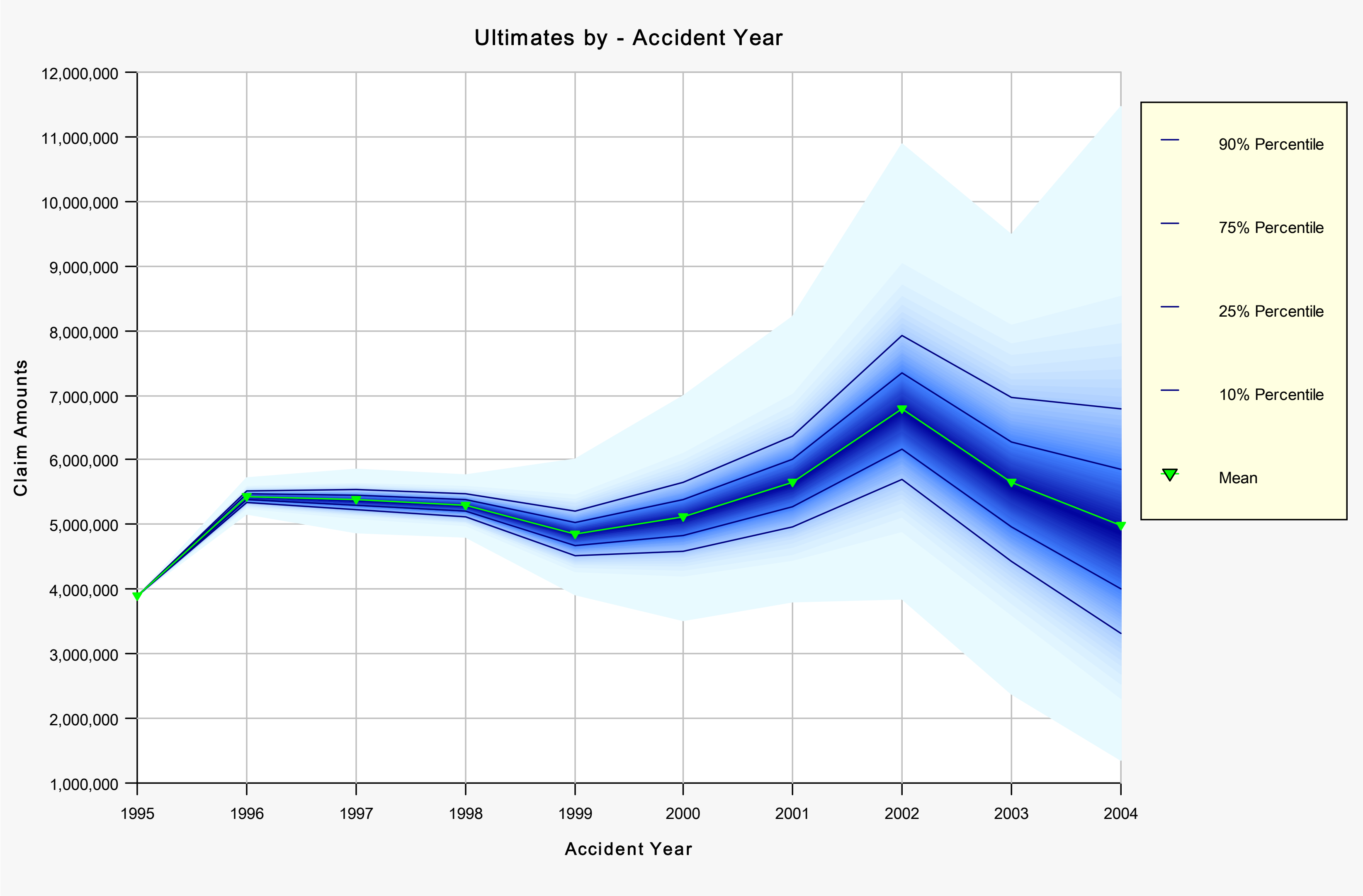
Note that the σ parameters decrease rapidly

Taylor & Ashe Data

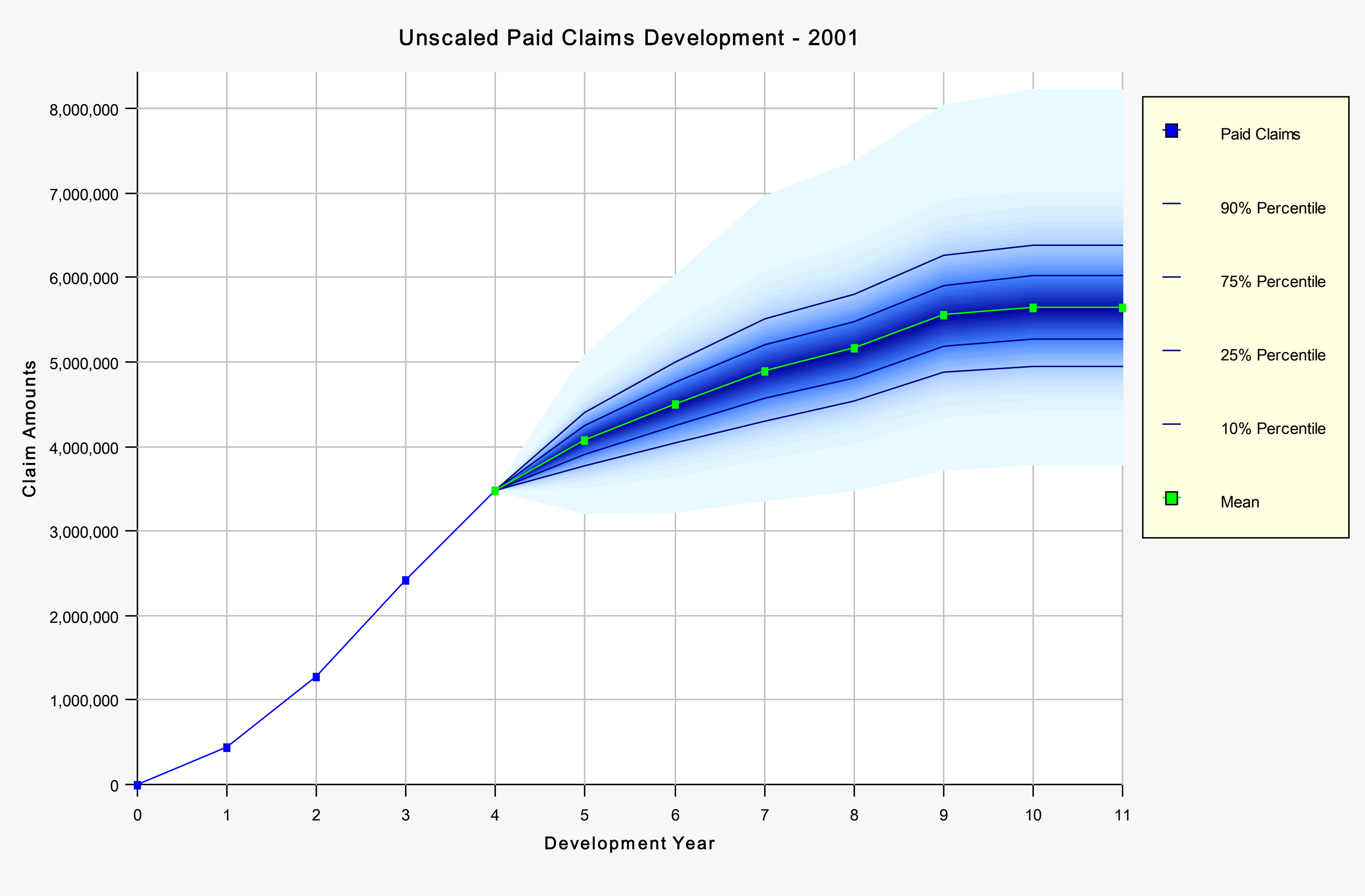
Bootstrapping Mack's Model

	Analytic		Simulated	
Accident Year	Prediction Error	Prediction Error %	Prediction Error	Prediction Error %
1	0	0.0%	0	0.0%
2	75,535	79.8%	75,001	78.4%
3	121,699	25.9%	121,578	26.0%
4	133,549	18.8%	132,939	18.7%
5	261,406	26.5%	261,911	26.5%
6	411,010	29.0%	414,910	29.1%
7	558,317	25.6%	558,639	25.7%
8	875,328	22.3%	880,184	22.4%
9	971,258	22.7%	979,052	22.8%
10	1,363,155	29.5%	1,368,720	29.4%
Total	2,447,095	13.1%	2,454,616	13.1%

Distribution of Ultimates by Origin Period



Claims Development Percentile Fan Chart
Single Origin Year



Practical Issues

- Choice of process distribution?
  - Normal: theoretically correct, but allows negative cumulative amounts
  - Gamma: pragmatic alternative
- When used with incurred data:
  - Provides distribution of ultimate claims
  - Provides distribution of IBNR+IBNER (not outstanding claims)
  - Also provides a distribution of Ultimates
    - By subtracting the observed paid amounts gives a distribution of outstanding amounts
  - Requires (simulated) paid to incurred ratios if paid cash-flows are required

Bootstrapping Mack's Model
Characteristics

- It is a model of the *cumulative* amounts
- It will work with negative incremental observed claims where development factors are less than 1
- Although it is possible to force simulated cumulative amounts to be positive, there is nothing to stop a simulated cumulative amount being less than the previous amount. That is, negative incremental amounts are always possible.
  - This may be beneficial with incurred data, but possibly a disadvantage with paid data
- Note: bootstrapping provides predictive distributions for Mack's model (including cash-flows)

Stochastic Reserving

Lognormal Models

Lognormal Models

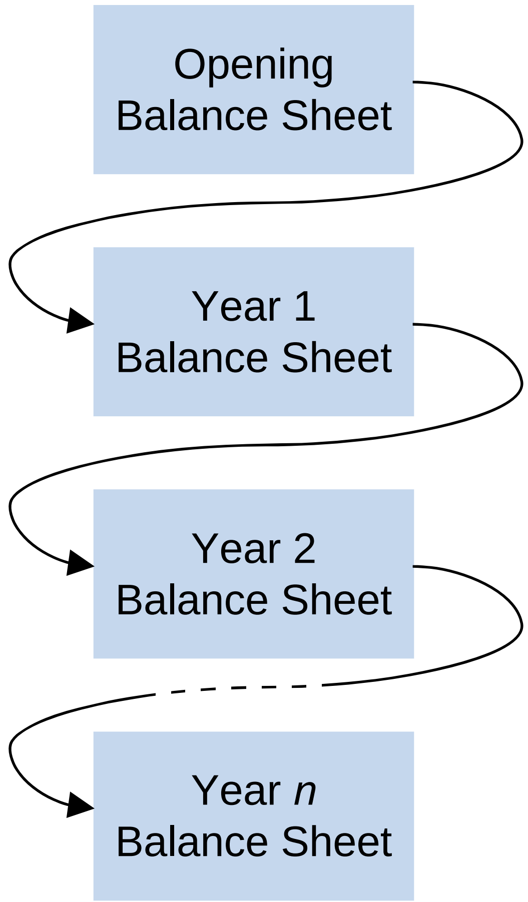
- It is also possible to fit linear regression models to the log of the incremental claims (log-normal models)
- Again, the prediction error can be calculated the HARD way (analytically) or the EASY way (using bootstrap or MCMC methods)
- To bootstrap the lognormal models, simply follow the steps outlined above for reserving and bootstrapping (see E&V 2006)

Stochastic Reserving

The one-yr view of risk

A Projected Balance Sheet View

- When projecting Balance Sheets for solvency, we have an opening balance sheet with expected outstanding liabilities
- We then project one year forwards, simulating the payments that emerge in the year
- We then require a closing balance sheet, with (simulated) expected outstanding liabilities conditional on the payments in the year
- In a multi-year model, the closing balance sheet after one year becomes the opening balance sheet in the second year, and so on



Solvency II

- For Solvency II, a 1 year perspective is taken, requiring a distribution of the *expected* value of the liabilities after 1 year, for the 1 year ahead balance sheet in internal capital models
- If the standard formula is used, a 1 year-ahead "reserve risk" standard deviation % is required. This could be:
  - The standard parameter for the line-of-business
  - An undertaking specific parameter
- The 1 year-ahead "reserve risk" standard deviation is the SD of the distribution of profit/loss on reserves after 1 year

The one-year run-off result (undiscounted)

(the view of profit or loss on reserves after one year)

For a particular origin year, let:

The opening reserve estimate be R_0

The reserve estimate after one year be R_1

The payments in the year be C_1

The run-off result (claims development result) be CDR_1

Then

$$CDR_1 = R_0 - C_1 - R_1 = U_0 - U_1$$

Where the opening estimate of ultimate claims and the estimate of the ultimate after one year are U_0, U_1

The one-year run-off result
(the view of profit or loss on reserves after one year)

Merz & Wuthrich (2008) derived analytic formulae for the standard deviation of the claims development result after one year assuming:

- The opening reserves were set using the pure chain ladder model (no tail)
- Claims develop in the year according to the assumptions underlying Mack's model
- Reserves are set after one year using the pure chain ladder model (no tail)
- The mathematics is quite challenging. This is the HARD way

The M&W method is gaining popularity, but has limitations. What if:

- We need a tail factor to extrapolate into the future?
- Mack's model is not used – other assumptions are used instead?
- We want another risk measure (say, VaR @ 99.5%)?
- We want a distribution of the CDR?

Merz & Wuthrich (2008)

Data Triangle

Accident Year	12m	24m	36m	48m	60m	72m	84m	96m	108m
0	2,202,584	3,210,449	3,468,122	3,545,070	3,621,627	3,644,636	3,669,012	3,674,511	3,678,633
1	2,350,650	3,553,023	3,783,846	3,840,067	3,865,187	3,878,744	3,898,281	3,902,425	
2	2,321,885	3,424,190	3,700,876	3,798,198	3,854,755	3,878,993	3,898,825		
3	2,171,487	3,165,274	3,395,841	3,466,453	3,515,703	3,548,422			
4	2,140,328	3,157,079	3,399,262	3,500,520	3,585,812				
5	2,290,664	3,338,197	3,550,332	3,641,036					
6	2,148,216	3,219,775	3,428,335						
7	2,143,728	3,158,581							
8	2,144,738								

Merz & Wuthrich (2008)
Prediction errors

	Analytic Prediction Errors	
Accident Year	1 Year Ahead CDR	Mack Ultimate
0	0	0
1	567	567
2	1,488	1,566
3	3,923	4,157
4	9,723	10,536
5	28,443	30,319
6	20,954	35,967
7	28,119	45,090
8	53,320	69,552
Total	$\boxed{81,080}$	108,401

Expressed as a percentage of the opening reserves, this forms a basis of the reserve risk parameter under Solvency II (QIS 5 Technical Specification)

The one-year run-off result in a simulation model
The EASY way

For a particular origin year, let:

The opening reserve estimate be R_0

The expected reserve estimate after one year be $R_1^{(i)}$

The payments in the year be $C_1^{(i)}$

The run-off result (claims development result) be $CDR_1^{(i)}$

Then

$$CDR_1^{(i)} = R_0 - C_1^{(i)} - R_1^{(i)} = U_0 - U_1^{(i)}$$

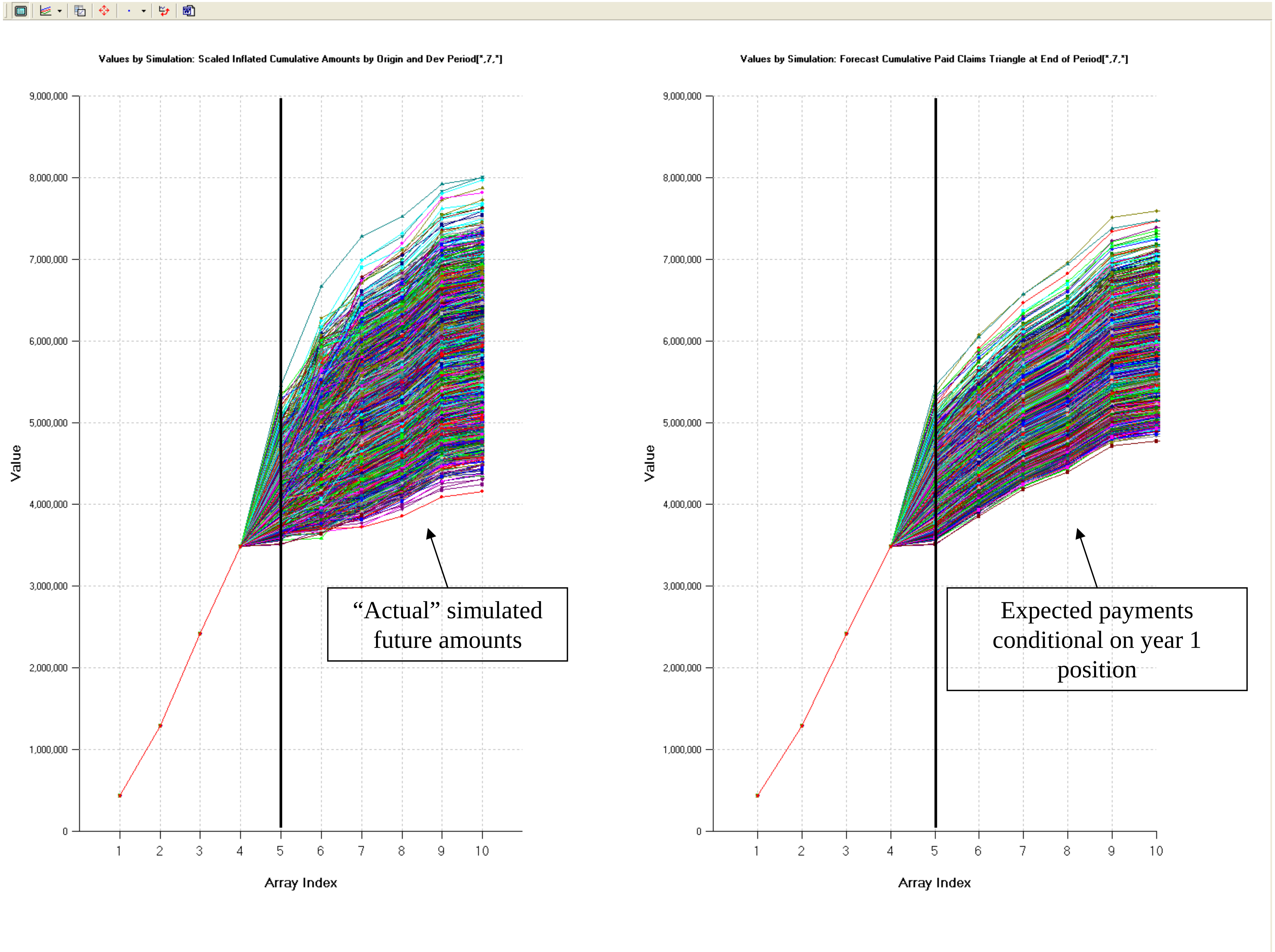
Where the opening estimate of ultimate claims and the expected ultimate after one year are $U_0, U_1^{(i)}$

for each simulation *i*

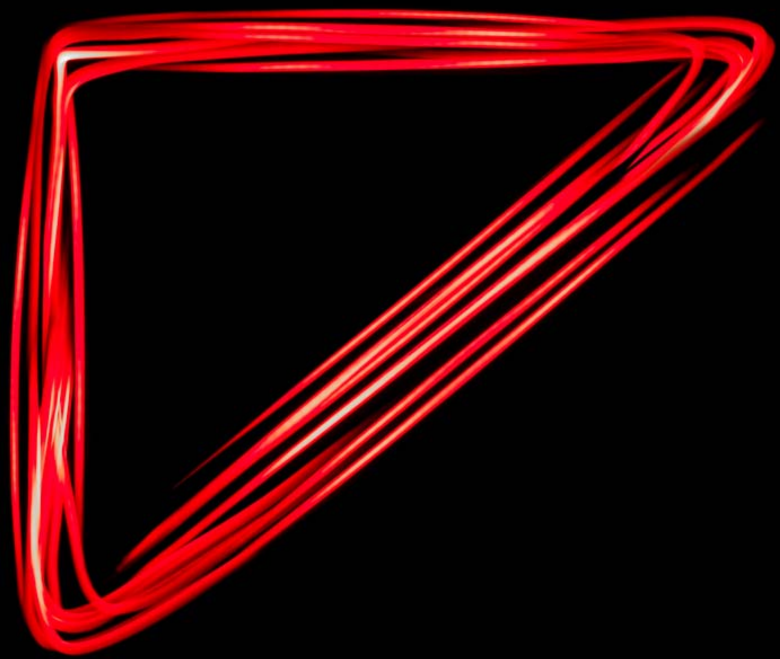
The one-year run-off result in a simulation model
The EASY way

1. Given the opening reserve triangle, simulate all future claim payments to ultimate using bootstrap (or Bayesian MCMC) techniques.
2. Now forget that we have already simulated what the future holds.
3. Move one year ahead. Augment the opening reserve triangle by one diagonal, that is, by the simulated payments from step 1 in the next calendar year only. An actuary only sees what emerges in the year.
4. For each simulation, estimate the outstanding liabilities, conditional only on what has emerged to date. (The future is still "unknown").
5. A reserving methodology is required for each simulation – an "actuary-in-the-box" is required*. We call this re-reserving.
6. For a one-year model, this will underestimate the true volatility at the end of that year (even if the mean across all simulations is correct).

** The term "actuary-in-the-box" was coined by Esbjörn Ohlsson*


Values by Simulation: Scaled Inflated Cumulative Amounts by Origin and Dev Period[*,7,*]
9,000,000
8,000,000
7,000,000
6,000,000
5,000,000
4,000,000
3,000,000
2,000,000
1,000,000
0
Value
1 2 3 4 5 6 7 8 9 10
Array Index
“Actual” simulated future amounts
Values by Simulation: Forecast Cumulative Paid Claims Triangle at End of Period[*,7,*]
9,000,000
8,000,000
7,000,000
6,000,000
5,000,000
4,000,000
3,000,000
2,000,000
1,000,000
0
Value
1 2 3 4 5 6 7 8 9 10
Array Index
Expected payments conditional on year 1 position

EMB ResQ Example



ResQ Example

Bootstrap Results Summary – “Ultimo” perspective

EMB ResQ Enterprise - W&M Astin - [Edit Bootstrap Method: "W&M\DFM Paid Claims Ultimate - Bootstrap"]

File Edit Administration Windows Help

Project | Details | Residuals | Simulation | Results | Output | Notes | Audit Log

Unscaled Results | Targets | Scaled Results | Discounting | Discounted Results | Diagnostics | Consolidation

Summary | Detail | Aggregates | Reserve Correlations | Cumulative Probability | Probability Density | Ultimates Graph

Show Mack prediction errors

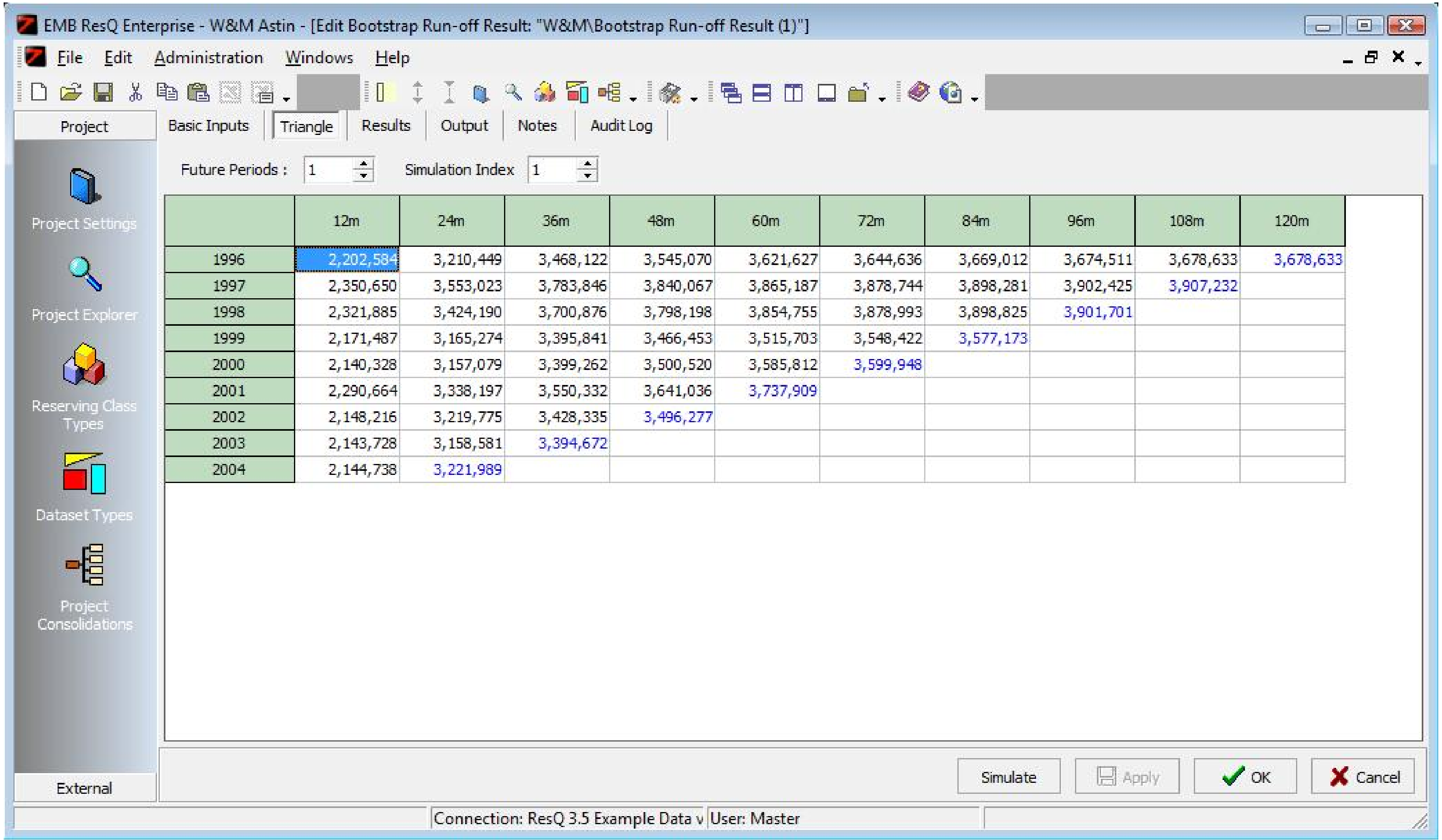
Accident Year	Latest	Bootstrap Expected Reserve	Bootstrap Prediction Error	Bootstrap Prediction Error %	Expected Ultimate	DFM Reserve	Reserve Difference
1996	3,678,633	0	0	0.00%	3,678,633	0	0
1997	3,902,425	4,379	568	12.98%	3,906,804	4,378	1
1998	3,898,825	9,345	1,564	16.73%	3,908,170	9,347	-3
1999	3,548,422	28,389	4,147	14.61%	3,576,811	28,392	-3
2000	3,585,812	51,472	10,569	20.53%	3,637,284	51,444	28
2001	3,641,036	111,961	30,296	27.06%	3,752,997	111,811	150
2002	3,428,335	187,170	35,951	19.21%	3,615,505	187,084	86
2003	3,158,581	411,687	44,996	10.93%	3,570,268	411,864	-177
2004	2,144,738	1,433,443	69,713	4.86%	3,578,181	1,433,505	-62
Total	**30,986,807**	**2,237,846**	**108,992**	**4.87%**	**33,224,653**	**2,237,826**	**20**

Project Settings
Project Explorer
Reserving Class Types
Dataset Types
Project Consolidations
External

Simulate | Apply | OK | Cancel

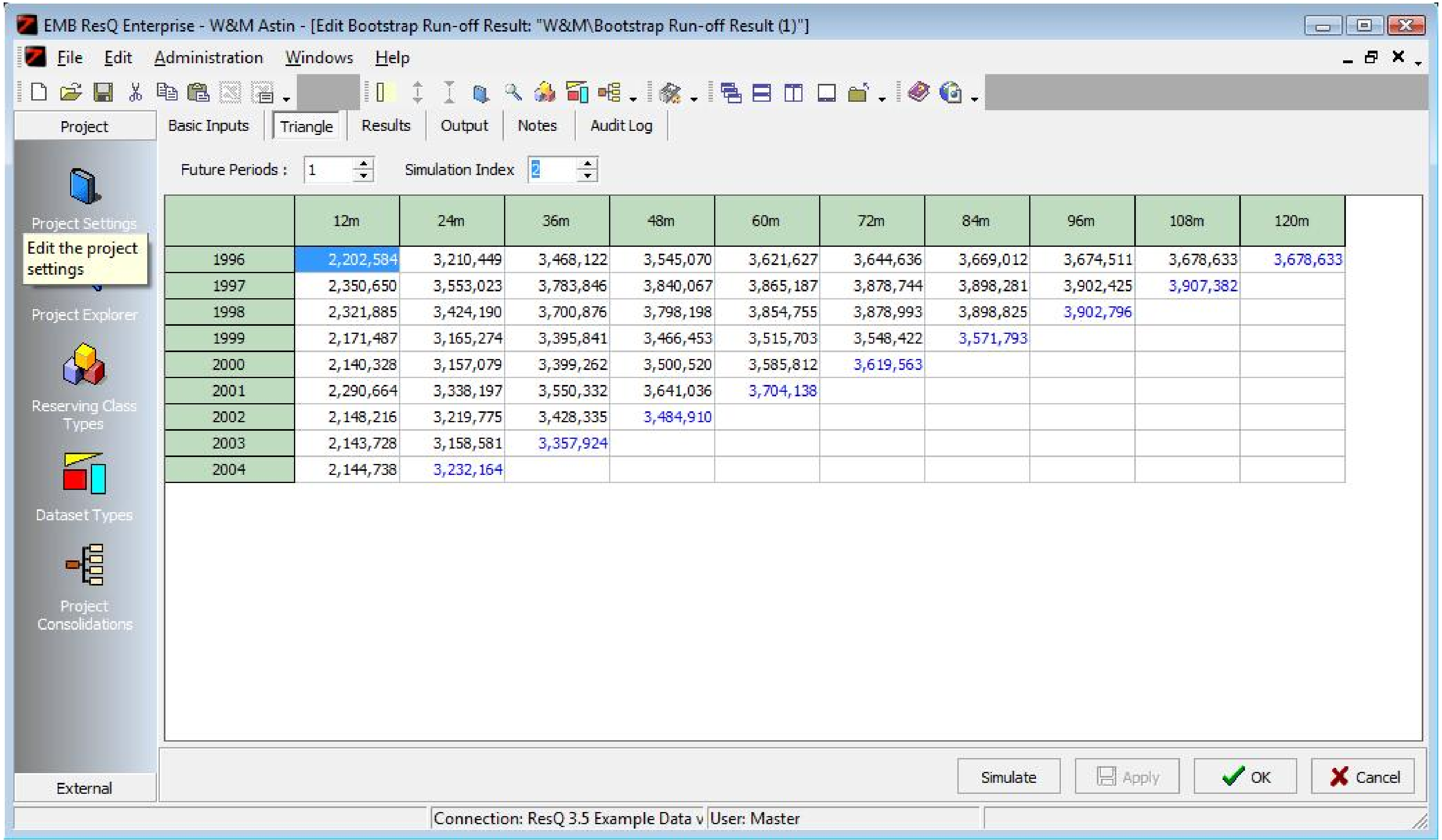
Connection: ResQ 3.5 Example Data v | User: Master

ResQ Example
1 Year ahead – Simulation 1



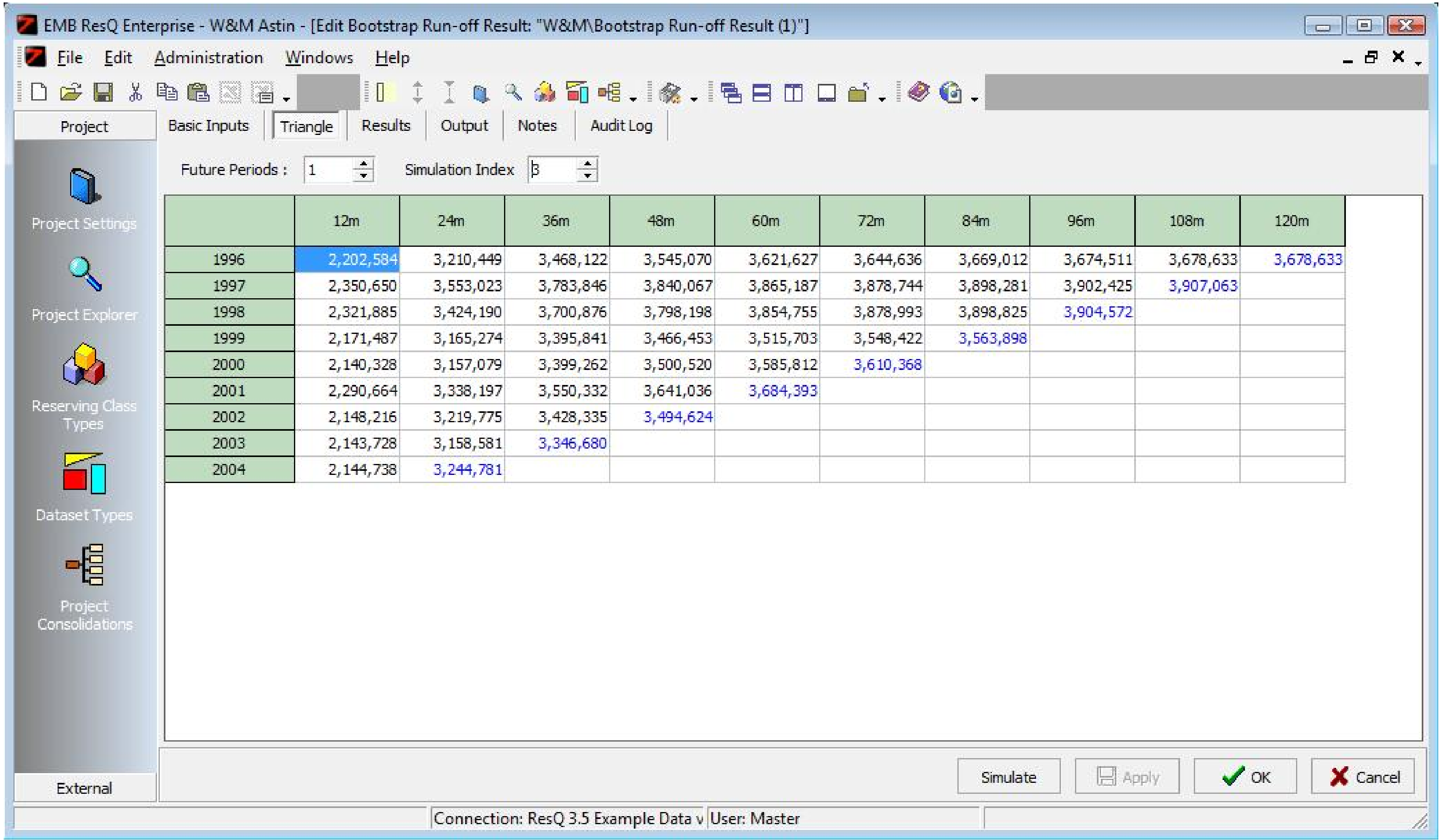
	12m	24m	36m	48m	60m	72m	84m	96m	108m	120m
1996	2,202,584	3,210,449	3,468,122	3,545,070	3,621,627	3,644,636	3,669,012	3,674,511	3,678,633	3,678,633
1997	2,350,650	3,553,023	3,783,846	3,840,067	3,865,187	3,878,744	3,898,281	3,902,425	3,907,232	
1998	2,321,885	3,424,190	3,700,876	3,798,198	3,854,755	3,878,993	3,898,825	3,901,701		
1999	2,171,487	3,165,274	3,395,841	3,466,453	3,515,703	3,548,422	3,577,173			
2000	2,140,328	3,157,079	3,399,262	3,500,520	3,585,812	3,599,948				
2001	2,290,664	3,338,197	3,550,332	3,641,036	3,737,909					
2002	2,148,216	3,219,775	3,428,335	3,496,277						
2003	2,143,728	3,158,581	3,394,672							
2004	2,144,738	3,221,989								

ResQ Example
1 Year ahead – Simulation 2



	12m	24m	36m	48m	60m	72m	84m	96m	108m	120m
1996	2,202,584	3,210,449	3,468,122	3,545,070	3,621,627	3,644,636	3,669,012	3,674,511	3,678,633	3,678,633
1997	2,350,650	3,553,023	3,783,846	3,840,067	3,865,187	3,878,744	3,898,281	3,902,425	3,907,382	
1998	2,321,885	3,424,190	3,700,876	3,798,198	3,854,755	3,878,993	3,898,825	3,902,796		
1999	2,171,487	3,165,274	3,395,841	3,466,453	3,515,703	3,548,422	3,571,793			
2000	2,140,328	3,157,079	3,399,262	3,500,520	3,585,812	3,619,563				
2001	2,290,664	3,338,197	3,550,332	3,641,036	3,704,138					
2002	2,148,216	3,219,775	3,428,335	3,484,910						
2003	2,143,728	3,158,581	3,357,924							
2004	2,144,738	3,232,164								

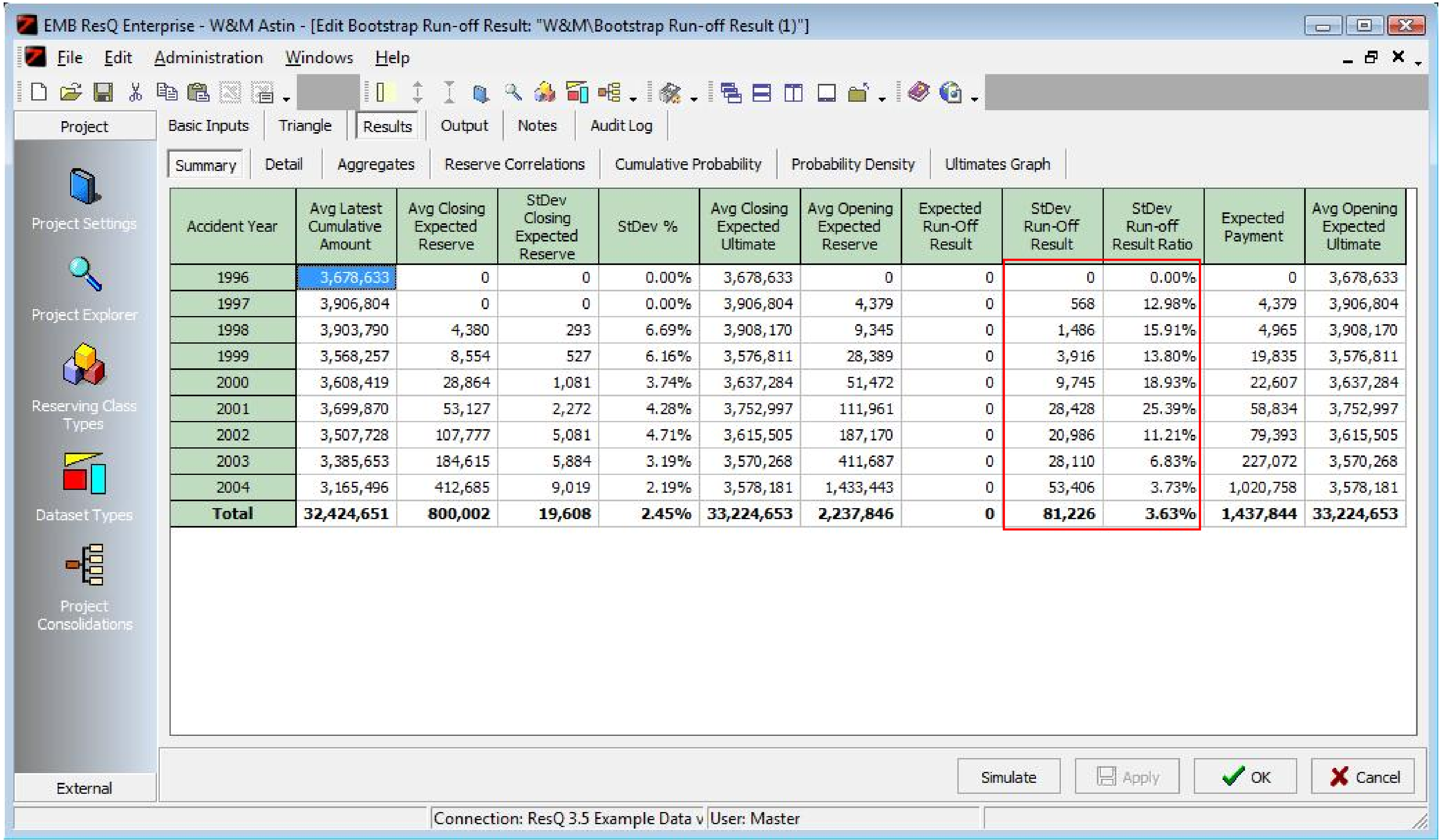
ResQ Example
1 Year ahead – Simulation 3



	12m	24m	36m	48m	60m	72m	84m	96m	108m	120m
1996	2,202,584	3,210,449	3,468,122	3,545,070	3,621,627	3,644,636	3,669,012	3,674,511	3,678,633	3,678,633
1997	2,350,650	3,553,023	3,783,846	3,840,067	3,865,187	3,878,744	3,898,281	3,902,425	3,907,063	
1998	2,321,885	3,424,190	3,700,876	3,798,198	3,854,755	3,878,993	3,898,825	3,904,572		
1999	2,171,487	3,165,274	3,395,841	3,466,453	3,515,703	3,548,422	3,563,898			
2000	2,140,328	3,157,079	3,399,262	3,500,520	3,585,812	3,610,368				
2001	2,290,664	3,338,197	3,550,332	3,641,036	3,684,393					
2002	2,148,216	3,219,775	3,428,335	3,494,624						
2003	2,143,728	3,158,581	3,346,680							
2004	2,144,738	3,244,781								

ResQ Example

Bootstrap Run-off Results Summary – 1 year perspective



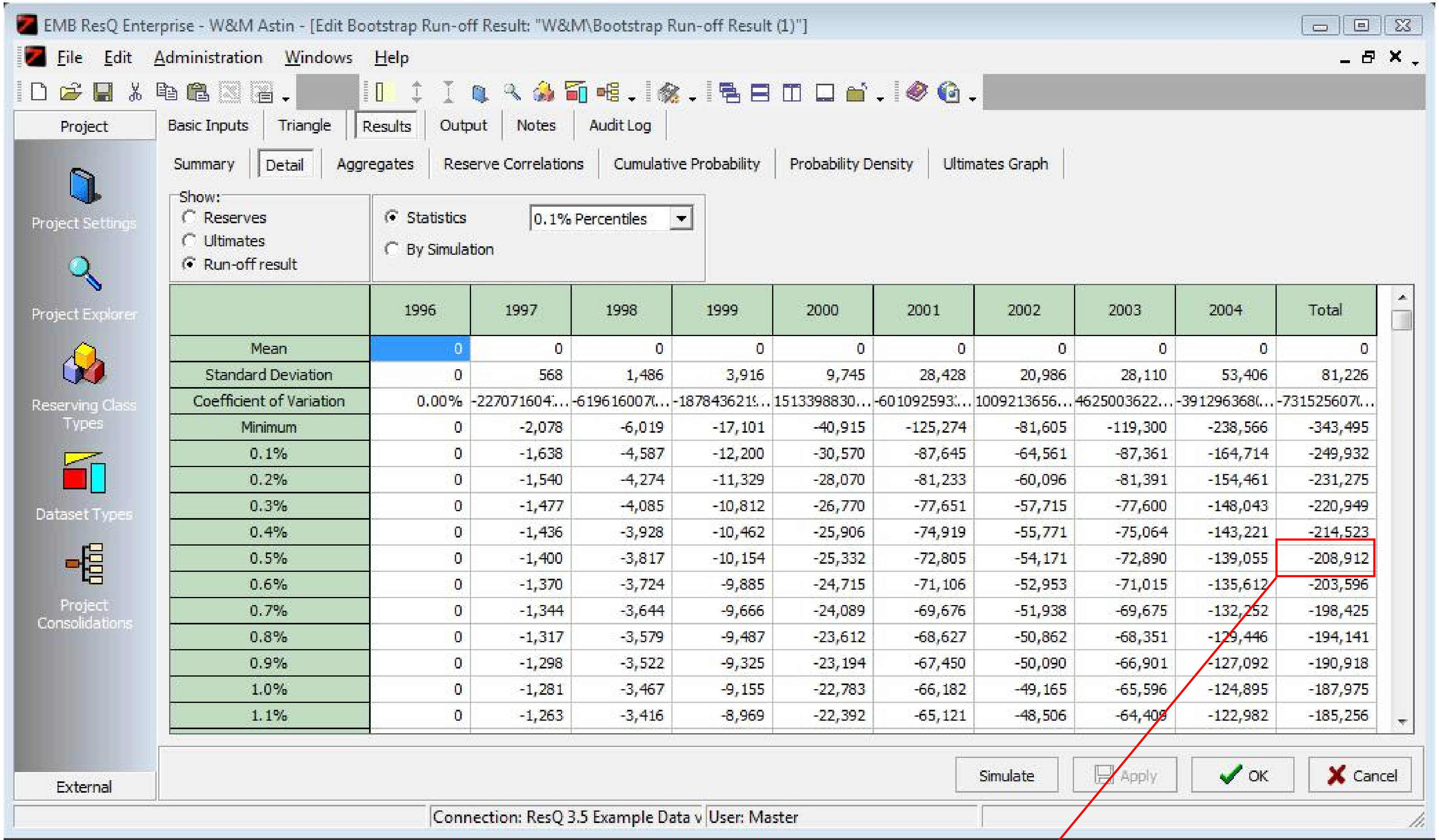
Accident Year	Avg Latest Cumulative Amount	Avg Closing Expected Reserve	StDev Closing Expected Reserve	StDev %	Avg Closing Expected Ultimate	Avg Opening Expected Reserve	Expected Run-Off Result	StDev Run-Off Result	StDev Run-off Result Ratio	Expected Payment	Avg Opening Expected Ultimate
1996	3,678,633	0	0	0.00%	3,678,633	0	0	0	0.00%	0	3,678,633
1997	3,906,804	0	0	0.00%	3,906,804	4,379	0	568	12.98%	4,379	3,906,804
1998	3,903,790	4,380	293	6.69%	3,908,170	9,345	0	1,486	15.91%	4,965	3,908,170
1999	3,568,257	8,554	527	6.16%	3,576,811	28,389	0	3,916	13.80%	19,835	3,576,811
2000	3,608,419	28,864	1,081	3.74%	3,637,284	51,472	0	9,745	18.93%	22,607	3,637,284
2001	3,699,870	53,127	2,272	4.28%	3,752,997	111,961	0	28,428	25.39%	58,834	3,752,997
2002	3,507,728	107,777	5,081	4.71%	3,615,505	187,170	0	20,986	11.21%	79,393	3,615,505
2003	3,385,653	184,615	5,884	3.19%	3,570,268	411,687	0	28,110	6.83%	227,072	3,570,268
2004	3,165,496	412,685	9,019	2.19%	3,578,181	1,433,443	0	53,406	3.73%	1,020,758	3,578,181
Total	**32,424,651**	**800,002**	**19,608**	**2.45%**	**33,224,653**	**2,237,846**	**0**	**81,226**	**3.63%**	**1,437,844**	**33,224,653**

Merz & Wuthrich (2008)
Analytic vs Simulated: Summary

Accident Year	Analytic Prediction Errors 1 Year Ahead CDR	Analytic Prediction Errors Mack Ultimate	Simulated Prediction Errors 1 Year Ahead CDR	Simulated Prediction Errors Mack Ultimate
0	0	0	0	0
1	567	567	568	568
2	1,488	1,566	1,486	1,564
3	3,923	4,157	3,916	4,147
4	9,723	10,536	9,745	10,569
5	28,443	30,319	28,428	30,296
6	20,954	35,967	20,986	35,951
7	28,119	45,090	28,110	44,996
8	53,320	69,552	53,406	69,713
Total	81,080	108,401	81,226	108,992

ResQ Example

99.5th percentile of the Bootstrap Run-off Result

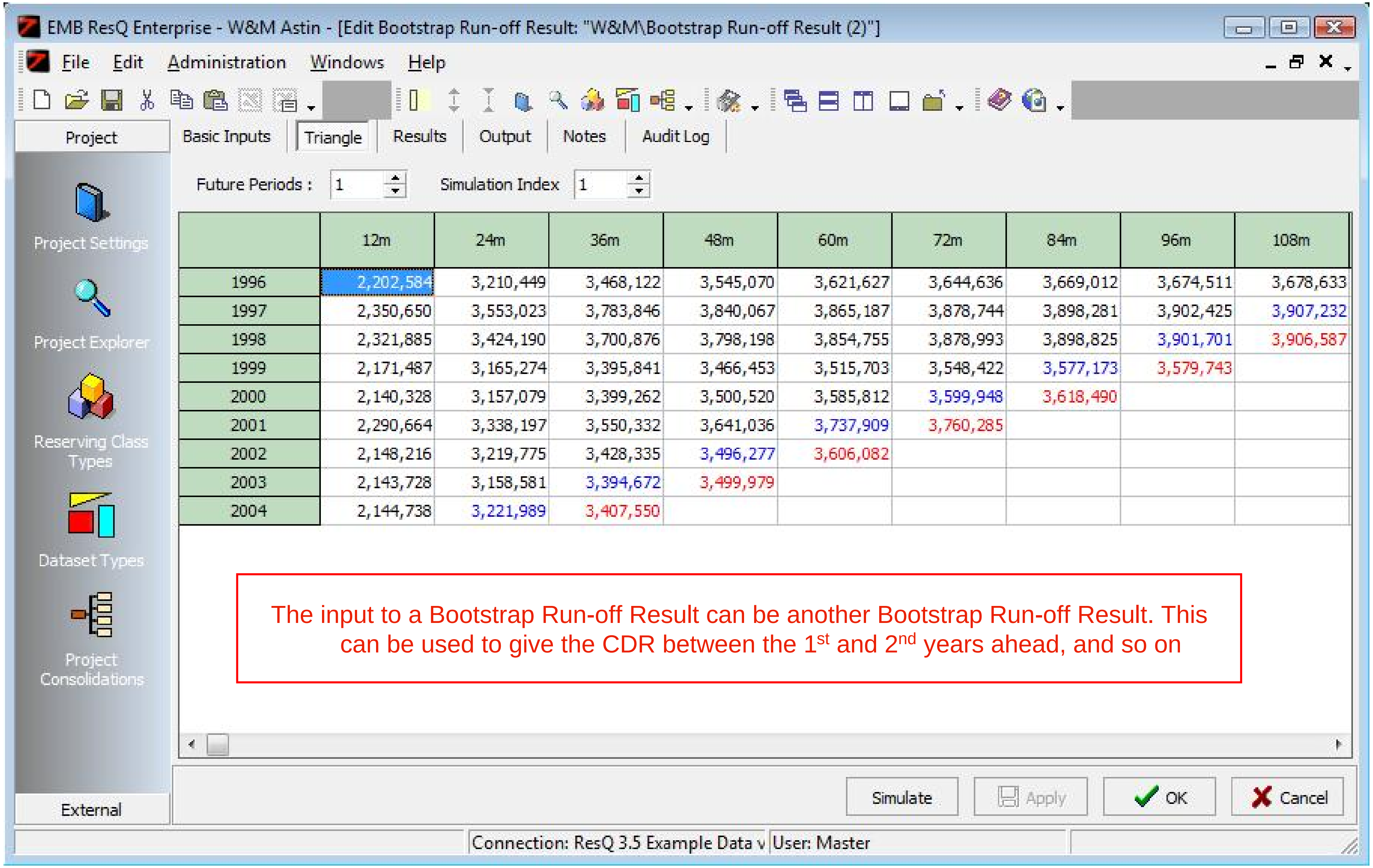


	1996	1997	1998	1999	2000	2001	2002	2003	2004	Total
Mean	0	0	0	0	0	0	0	0	0	0
Standard Deviation	0	568	1,486	3,916	9,745	28,428	20,986	28,110	53,406	81,226
Coefficient of Variation	0.00%	-227071604…	-619616007…	-187843621…	1513398830…	-601092593…	1009213656…	4625003622…	-391296368…	-731525607…
Minimum	0	-2,078	-6,019	-17,101	-40,915	-125,274	-81,605	-119,300	-238,566	-343,495
0.1%	0	-1,638	-4,587	-12,200	-30,570	-87,645	-64,561	-87,361	-164,714	-249,932
0.2%	0	-1,540	-4,274	-11,329	-28,070	-81,233	-60,096	-81,391	-154,461	-231,275
0.3%	0	-1,477	-4,085	-10,812	-26,770	-77,651	-57,715	-77,600	-148,043	-220,949
0.4%	0	-1,436	-3,928	-10,462	-25,906	-74,919	-55,771	-75,064	-143,221	-214,523
0.5%	0	-1,400	-3,817	-10,154	-25,332	-72,805	-54,171	-72,890	-139,055	-208,912
0.6%	0	-1,370	-3,724	-9,885	-24,715	-71,106	-52,953	-71,015	-135,612	-203,596
0.7%	0	-1,344	-3,644	-9,666	-24,089	-69,676	-51,938	-69,675	-132,252	-198,425
0.8%	0	-1,317	-3,579	-9,487	-23,612	-68,627	-50,862	-68,351	-129,446	-194,141
0.9%	0	-1,298	-3,522	-9,325	-23,194	-67,450	-50,090	-66,901	-127,092	-190,918
1.0%	0	-1,281	-3,467	-9,155	-22,783	-66,182	-49,165	-65,596	-124,895	-187,975
1.1%	0	-1,263	-3,416	-8,969	-22,392	-65,121	-48,506	-64,409	-122,982	-185,256

Var @ 99.5% = - (0.5th percentile) = 208,912

ResQ Example

Cascading Bootstrap Run-off Results



	12m	24m	36m	48m	60m	72m	84m	96m	108m
1996	2,202,584	3,210,449	3,468,122	3,545,070	3,621,627	3,644,636	3,669,012	3,674,511	3,678,633
1997	2,350,650	3,553,023	3,783,846	3,840,067	3,865,187	3,878,744	3,898,281	3,902,425	3,907,232
1998	2,321,885	3,424,190	3,700,876	3,798,198	3,854,755	3,878,993	3,898,825	3,901,701	3,906,587
1999	2,171,487	3,165,274	3,395,841	3,466,453	3,515,703	3,548,422	3,577,173	3,579,743	
2000	2,140,328	3,157,079	3,399,262	3,500,520	3,585,812	3,599,948	3,618,490		
2001	2,290,664	3,338,197	3,550,332	3,641,036	3,737,909	3,760,285			
2002	2,148,216	3,219,775	3,428,335	3,496,277	3,606,082				
2003	2,143,728	3,158,581	3,394,672	3,499,979					
2004	2,144,738	3,221,989	3,407,550						

The input to a Bootstrap Run-off Result can be another Bootstrap Run-off Result. This can be used to give the CDR between the 1st and 2nd years ahead, and so on

Multiple 1 yr ahead CDRs
An interesting result

- Creating cascading CDRs over all years gives the following results:

Accident Year	Number of years ahead 1 Yr	2 Yrs	3 Yrs	4 Yrs	5 Yrs	6 Yrs	7 Yrs	8 Yrs	Sqrt(Sum of Squares)	Mack Ultimate
1	0	0	0	0	0	0	0	0	-	0
2	568	0	0	0	0	0	0	0	568	568
3	1,486	487	0	0	0	0	0	0	1,564	1,564
4	3,916	1,306	431	0	0	0	0	0	4,151	4,147
5	9,745	3,837	1,277	425	0	0	0	0	10,560	10,569
6	28,428	9,679	3,824	1,272	425	0	0	0	30,303	30,296
7	20,986	27,438	9,343	3,693	1,226	409	0	0	35,998	35,951
8	28,110	20,404	26,922	9,162	3,613	1,208	402	0	45,055	44,996
9	53,406	27,798	20,236	26,687	9,111	3,600	1,203	402	69,600	69,713
Total	81,226	52,344	38,513	29,010	10,120	3,879	1,285	402	108,543	108,992

- The sum of the variances of the repeated 1 yr ahead CDRs (over all years) equals the variance over the lifetime of the liabilities
  - Under Mack's assumptions/chain ladder, this can be proved
  - (The simulation based results are approximate due to numerical error)
- This means that we can partition the "ultimo" variance into a sequence of year-on-year variances, and furthermore we expect the risk under the Solvency II 1 year view to be lower than the standard "ultimo" perspective

Re-reserving in Simulation-based Capital Models

The advantage of investigating the claims development result (using re-reserving) in a simulation environment is that the procedure can be generalised:

- Not just the chain ladder model
- Not just Mack's assumptions
- Can include curve fitting and extrapolation for tail estimation
- Can incorporate a Bornhuetter-Ferguson step
- Can be extended beyond the 1 year horizon to look at multi-year forecasts
- Provides a *distribution* of the CDR, not just a standard deviation
- Can be used to help calibrate Solvency II internal models

The Future

Looking ahead into the future

- Bootstrapping has had its critics
  - For example, GIRO ROC WP Reports 2007/8
  - Most of the criticisms are unfounded (see for example, England & Cairns GIRO 2009)
- As a statistical procedure, bootstrapping is difficult to criticise
  - It is remarkably robust when applied correctly
- However, it is inevitable that the techniques will be scrutinised more closely, given the focus of Solvency II
- The obvious area of attack is the underlying model structure
  - This is a valid area of attack: “Model specification” error
- It is likely that over time, there will be a move towards MCMC methods
- It is also likely that triangle based methods will come under attack, with a move towards “granular” reserving and “timeline” reserving for some lines of business

Conclusions

- There is plenty more we could talk about, for example
  - Other practical adjustments
  - Inflation adjusting, and links to an Economic Scenario Generator
  - Dependencies between lines-of business (use of copulas, and “synchronous” bootstrapping)
  - Bootstrapping other models (separation technique, Schnieper’s model, Munich C-L, etc)
  - MCMC as an alternative to bootstrapping

> *“Modern computer simulation techniques open up a wide field of practical applications for risk theory concepts, without the restrictive assumptions, and sophisticated mathematics, of many traditional aspects of risk theory”.*
>
> - Daykin, Pentikainen and Pesonen, 1996

