# Section E
# PRÉCIS OF OTHER ACTUARIAL PAPERS

This section provides a series of précis of several other papers published since the first edition of the Claims Reserving Manual was produced. The intention is to give the reader an overview of the paper, together with a description of the reserving model on which the paper is based. A few observations are also made about the applicability of the model, what data are required, what level of statistical and computational ability is needed, plus some thoughts on the strengths and weaknesses of the model.

The first three of these papers provide variations on the theme of regression models based on log-incremental payments. The paper by R J Verrall and Z Li gives a suggestion to overcome the problem of negative incremental payments. The paper by R J Verrall uses the log-incremental regression model as a basis for allowing the practitioner to enter prior information, or to estimate the parameters dynamically. A Bayesian method is used and the data are analysed recursively, using the Kalman filter. Finally, the paper by B Zehnwirth sets out a framework based on the log-incremental payments. His models include systematic components by development year, accident year and calendar year, as well as a random component.

The next two papers make use of more detailed claims information than just aggregate claims payments. The paper by T S Wright sets out a comprehensive approach using Generalised Linear Models to fit Operational Time models. These allow estimates of reserves and different components of reserve variability to be produced. The paper by D H Reid extends the basic Operational Time concept to allow for sudden changes in the nature and mix of business.

The paper by D M Murphy examines the standard link-ratio methods from the point of view of classical regression theory, and considers the circumstances under which the standard link-ratio methods may be considered optimal.

The final, brief, paper by D Gogol describes an approach to estimating loss reserves, using recent loss experience and two probability distributions. The distributions are of the ultimate losses, based on prior experience and rate adequacy changes, and the ratio of the estimator based on recent experience to the true ultimate loss.

[E1]
# NEGATIVE INCREMENTAL CLAIMS: CHAIN LADDER AND LINEAR MODELS

By R J Verrall and Z Li

(13 pages)
*Journal of the Institute of Actuaries*, Vol. 120, p. 171 (1993)

**Summary**

One of the problems of many models based on log-incremental payments is the inability to deal with negative incremental payments. One approach to this problem is to add a suitably large arbitrary constant to all the payments, and then subtract the constant after the forecasts are made.

The paper shows that the addition of such a constant (a threshold parameter) is equivalent to modelling the incremental payments by a three parameter log-normal distribution, for which the choice of constant can be performed by maximum likelihood estimation rather than arbitrarily.

**Description of the model**

The basic model, based on log-incremental payments (see for example the Christofides paper in Section D5 of Volume 2), is adjusted as follows:

$$\text{Log}(P_{ij}+c) = Y_{ij} = a_i + b_j + e_{ij}$$ ($e_{ij}$ are independent identically distributed normal random errors)

where $P_{ij}$ are the incremental payments for accident year i, development period j, and c is the threshold parameter.

Standard procedures for producing maximum likelihood estimates yield a set of equations that can be solved iteratively for $a_i$, $b_j$ and c. The technique could also be applied to other models for $Y_{ij}$. The implications for the standard errors of the estimated future payments are not discussed.

**General comments**

As for most models based on log-incremental payments, the technique is not restricted to any particular class of business, and the only data required are incremental payments.

The technique gives a theoretically sound solution to the problem of negative incremental payments, rather than relying on arbitrary adjustments to the data. The user should, however, examine the sensitivity of the results to the level of threshold parameter used.

The paper requires a basic level of statistical knowledge. The authors include a worked example, but the steps in the iterative process to calculate the parameters are not spelt out. Familiarity with matrix manipulation and regression in a spreadsheet is therefore essential.

[E2]

# A STATE SPACE REPRESENTATION OF THE CHAIN LADDER MODEL

**By R J Verrall**

**(21 pages)**

***Journal of the Institute of Actuaries*, Vol. 116, p. 589 (1989)**

## Summary

The model treats the development triangle as a dynamic system, with development taking place over time, t, in the direction of the diagonal (calendar year). A recursive relationship between the parameters at time t and t+1 is developed, with the ability to enter prior information. The recursive estimation of the parameters is based on a process known as the Kalman Filter.

## Description of the model

The basic model begins with the familiar:

$\text{Log}(X_{ij}) = Y_{ij} = \mu + a_i + b_j + e_{ij}$ ($e_{ij}$ are independent identically distributed Normal random errors)

where $X_{ij}$ are the incremental payments for accident year i, development period j.

The model then becomes quite unfamiliar as it defines:

| | |
|---|---|
| The Observation equation, | $\mathbf{Y}_t = F_t \, . \, \theta_t + \mathbf{e}_t$ |
| The System equation, | $\theta_{t+1} = G_t \, . \, \theta_t + H_t \, . \, _t + \mathbf{w}_t$ |

The bold symbols denote vectors. For example, $\mathbf{Y}_t$ is a vector of the $Y_{ij}$ at time t, and $\mathbf{e}_t$ is a vector of the $e_{ij}$ at time t. $\theta_t$ is known as the State vector, which is a vector of the parameter estimates (that is estimates of $a_i$ and $b_j$) at time t. $\mathbf{u}_t$ is a stochastic input vector assumed to be independent of $\theta_t$, and $\mathbf{w}_t$ is a disturbance vector. $F_t$, $G_t$ and $H_t$ are matrices. The Observation and System equations together comprise the "State Space representation" of this particular chain ladder model.

When $\mathbf{u}_t = \mathbf{w}_t = \mathbf{0}$, the System equation reduces to $\theta_{t+1} = G_t.\theta_t$, and $G_t$ can be defined such that the parameters at times t and t+1 are equal. This equates to least squares estimation when the parameters are identical for each row and each column.

When $\mathbf{w}_t = \mathbf{0}$, and $\mathbf{u}_t$ has the prior distribution of the new parameters, Bayesian estimates are obtained with distinct parameters.

When $\mathbf{w}_t \neq \mathbf{0}$, the parameters at times t and t+1 are related but not necessarily the same. This is known as dynamic parameter estimation, which in a sense lies in between the two previous cases of identical and distinct parameter estimation.

The paper considers a specific case of the State Space system, where $\mathbf{e}_t$, $\mathbf{u}_t$, $\mathbf{w}_t$ and $\theta_{t|t-1}$ are independent and normally distributed with defined means and variances, for which the State vector, $\theta_t$, can be calculated recursively by a series of matrix manipulations.

**General comments**

The paper illustrates by way of examples how prior information and dynamic estimation of parameters can enhance traditional chain ladder methods, and squeeze the maximum amount of information from the available data. The use of a Bayesian approach should lead to greater parameter and predictor stability than ordinary chain ladder models.

The standard of mathematics and computational ability is very high and may be beyond the scope of most people. Whilst numerical examples are given, the intermediate steps in arriving at the results are not, so it may be tricky to replicate the examples. Realistically, anyone wanting to use these methods may be best advised to do so using commercially available software packages, although it is important to understand the theory underlying the model when doing so.

One possible problem when using this type of model is that the assumptions and inputs can become somewhat divorced from reality, including as they do estimates of the variances of parameters of a model of the logs of incremental payments. These are not concepts that are readily translatable to ones knowledge of the payment of claims, and it is not always easy to understand the implications for the future payments of changes in these inputs to the model.

[E3]
# PROBABILISTIC DEVELOPMENT FACTOR MODELS WITH APPLICATION TO LOSS RESERVE VARIABILITY, PREDICTION INTERVALS AND RISK BASED CAPITAL

**By B Zehnwirth**

**(159 pages)**

***Casualty Actuarial Society Spring Forum,* Vol. 2, p. 447 (1994)**


## Summary

The paper describes a statistical modelling framework. Each model contained in the framework has four components. The first three components are the trends in the development year, accident year and calendar year, and the fourth component is random fluctuation (or distribution of the deviations) about the trends. The emphasis of the paper is to focus on the calendar year direction.

The modelling framework is relatively simple, allowing the testing of assumptions (for example, looking at the stability of models) and the quantification of reserve variability.


## Description of the model

The family of models includes:

$$\mathrm{Log}(P_{ij}) = Y_{ij} = a_i + b_j + c_k + e_{ij} \quad (e_{ij} \text{ are independent identically distributed Normal random errors})$$

where $P_{ij}$ are the incremental payments for accident year i, development period j, and $k = i + j$.

Models are fitted by using weighted least squares regression. As a result of multi-collinearity, principally due to the non-orthogonality of the calendar year direction with the other two directions, varying parameter models are necessary and are also included in the framework. Other Bayesian approaches are included, which are of particular use if estimates of certain parameters in a parsimonious model are subject to large uncertainties.

## General comments

This family of models is an extension of the type of model described by S Christofides in Section D5 of Volume 2, and can be used for a variety of types of business or types of incremental data. Whilst the basic model can be easily programmed in a spreadsheet, the more complex variations are probably beyond the means of most programmers.

As for other models based on the logs of incremental payments, the models do not work for negative incremental payments, and there is a limit in a spreadsheet to the number of future payments that can be predicted.

Most of the paper requires a basic level of statistical knowledge, whilst some of the variations on the basic model require a more advanced level. Familiarity with matrix manipulation and regression in a spreadsheet is required to implement the models.

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[E4]
# STOCHASTIC CLAIMS RESERVING WHEN PAST CLAIM NUMBERS ARE KNOWN

**By T S Wright**

**(93 pages)**

***Proceedings of the Casualty Actuarial Society,* Vol. 79, p. 255 (1992)**

**Summary**

The model attempts to represent the underlying claims settlement process.

The starting premise is that the cost of settling claims and the order in which they are settled are related — that is, typically, the longer the period to settlement, the greater the final settlement cost is likely to be. The method therefore develops a model of the claim settlement cost as a function of the relative proportion of claims settled (this time-frame is known as Operational Time).

The model is fitted using the theory of Generalised Linear Modelling ("GLMs"). Because it is a statistical model, standard errors (as a measure of the variability of the estimate) for the future incremental payments can be calculated, and statistical techniques used to test the fit of the model.

**Description of the model**

Operational Time ($\tau$) is the number of claims closed to date, expressed as a proportion of the ultimate number of claims. The mean claim size, $m(\tau)$, can be modelled by a wide variety of different types of function of $\tau$. For example:

$$m(\tau) = \exp(\beta_0 + \beta_1\tau + \ldots..\beta_n\tau^n)$$

Alternatives include polynomial functions of $\tau$, or functions such as $\beta_n\tau^{-n}$, or some combination of these functions. The parameters of the model are fitted using GLMs, for example using the software package GLIM.

The modelling technique involves fitting a basic model that adheres closely to the data, then examining alternative models. The nature of the model means that familiar measures of goodness of fit, such as "sums of squares", are not appropriate. An alternative measure, deviance, is therefore considered, as well as other indications as to the goodness of fit of the model.

Certain restrictive assumptions are made at the initial fitting stage, which are then examined and may subsequently be relaxed.

## General comments

The method is likely to be of most use where the greatest cause of uncertainty in predicting ultimate claims is due to individual claim costs — for example, classes involving bodily injury claims. It should also be of particular use when it is believed that settlement rates are changing, as the model may be able to capture these changes more effectively than traditional techniques.

It requires data on both the amount and number of claims settled.

The model is sensitive to the estimated future number of settled claims, and these estimates need careful scrutiny. Inflation is a parameter that may be modelled, and this is also an area where close scrutiny is required. The approach to comparing different functions for $m(\tau)$ is open to some criticism, as the comparison of non-nested models using deviances is not strictly valid — though the author recognises that this is a pragmatic approach.

A high degree of statistical knowledge is required to implement and understand the model, as well as considerable computer literacy. Knowledge of a GLM software package such as GLIM is essential.

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[E5]

# OPERATIONAL TIME AND A FUNDAMENTAL PROBLEM OF INSURANCE IN A DATA-RICH ENVIRONMENT

**By D H Reid**

**(13 pages)**

***Applied Stochastic Models and Data Analysis*, Vol. 11, No. 3, Wiley (1995)**

## Summary

This paper is the latest in a series developing a particular approach to claims reserving where relatively complete information on individual claims is available, and where past years' claims patterns are relevant — albeit with modifications — to the development of more recent years' experience.

Specifically, this paper addresses a problem which has arisen in recent years, where relatively rapid changes in size and factor mix of the claims portfolio are taking place. Most, if not all, previous claim reserving methodologies have implicitly assumed that factors change relatively slowly, to such an extent that the effect of this trend on claim development is not significant.

The present paper, by contrast, models the effect of factor trends explicitly, both on the level of claim cost itself and on the development patterns. By doing so, it proposes an approach which may then be applied directly to the development of premium rates, as well as reserves.

## Description of the model

The model proposed is based upon that described in Section D4 of Volume 2, but develops that model to allow for the incorporation of a rating factor or set of classificatory factors into the analysis. This is done by first elaborating the structure of claim cost development for recent years as represented by the original model, and then introducing an approach which makes the resulting complex picture more readily comprehensible and, at the same time, statistically estimable.

## General comments

Given that this methodology is intended for situations where significant resources are available for claims modelling, and where it is important to achieve as close an understanding of the claims development process as possible, the proposed methodology is relatively flexible and can be adapted to a wide range of situations.

[E6]
# UNBIASED LOSS DEVELOPMENT FACTORS
**By D M Murphy**
**(60 pages)**
*Casualty Actuarial Society Spring Forum,* **Vol. 1, p. 183 (1994)**


## Summary

Standard link ratio methods are examined from the viewpoint of classical regression theory. The circumstances under which the standard link ratio methods could be considered optimal are discussed. Formulae for variances of, and confidence intervals around, point estimates of ultimate loss and loss reserves are derived. A triangle of incurred losses is used to demonstrate the techniques.

A summary of a simulation study is presented which suggests that the performance of the link ratio method, using least squares linear estimates, may approach that of the Bornhuetter-Ferguson and Stanard-Bühlmann techniques in some situations.


## Description of the model

The estimates of ultimate loss for n accident years are derived using recursion:

$$\hat{M}_1 = \hat{a}_1 + \hat{b}_1 x_{0,0}$$

$$\hat{M}_n = n\hat{a}_n + \hat{b}_n(\hat{M}_{n-1} + x_{n-1,n-1})$$

$x_{i,j}$ denotes the cumulative incurred loss from accident year i, development year j, and

$$\hat{M}_n = E\left(\sum_{i=0}^{n-1} x_{i,n} | x_{i,i}\right)$$

The variance is given by the sum of the parameter risk and the process risk. Each are defined for n=1, and then recursively for n>1, as follows:

*parameter risk*

$$\mathrm{Var}(\hat{M}_1) = \frac{\sigma_1^2}{I_1} + \left(x_{0,0} - \bar{x}_0\right)^2 \mathrm{Var}(\hat{b}_1)$$

$$\mathrm{Var}(\hat{M}_n) = n^2 \frac{\sigma_n^2}{I_n} + \left(\hat{M}_{n-1} + x_{n-1,n-1} - n\bar{x}_{n-1}\right)^2 \mathrm{Var}(\hat{b}_n) + b_n^2 \mathrm{Var}(\hat{M}_{n-1}) + \mathrm{Var}(\hat{b}_n)\mathrm{Var}(\hat{M}_{n-1})$$

where:

$$\bar{x}_{n-1} = \frac{1}{I_n}\sum_{i=n}^{N} x_{i,n-1}$$

is the average "x value" and $I_n = N - n + 1$ (assuming a full column in the triangle) is the number of data points in the regression estimate of the $n^{th}$ link ratio.

*process risk*

$$\mathrm{Var}(E_1) = \sigma_1^2$$

$$\mathrm{Var}(E_n) = n\sigma_n^2 + b_n^2 \mathrm{Var}(E_{n-1})$$

where $E_i$ is an error term.

**General comments**

A modest level of mathematics is required to follow the paper. Proofs of the theory are relegated to a bulky appendix. The example provided helps the reader to follow the theory by showing practical application of the formulae. The calculation of the least squares estimates and their variances can readily be done in most spreadsheet packages.

The use of the confidence intervals depends on whether the assumptions made regarding the probability distribution of the error terms are appropriate. The paper does not address how these should be tested.

The Benjamin-Eagles paper in Section D3 of the Manual describes a method which is the same as the least squares linear method described in this paper, but without the mathematical rigour.

The Stanard-Bühlmann technique (also known as the "Cape Cod Method") is not explained. Reference would need to be made to the paper by J Stanard in the 1985 Proceedings of the CAS (Casualty Actuarial Society) for explanantion.

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## [E7]
## USING EXPECTED LOSS RATIOS IN RESERVING
**By D F Gogol**
**(3 pages)**
***Casualty Actuarial Society Fall Forum*, p. 241 (1995)**


### Summary

The paper describes an approach to estimating loss reserves using the recent loss experience and two probability distributions. The first distribution is that of the ultimate losses for the recent period, based on prior experience and rate adequacy changes. The second distribution is that of the ratio of the estimator based on recent experience to the true ultimate loss.


### Description of the model

The model is:

$$h(x|y) = g(y|x)f(x) \,/\, \int_0^\infty g(y|x)f(x)dx$$

where, for losses in respect of an exposure period E:

> f(x) is the probability density function of the distribution of ultimate losses for exposure period E, prior to considering the losses for exposure period E.
>
> g(y|x) is the probability density function of the distribution of y, the developed losses at the point of time under consideration, for exposure period E, given that the ultimate losses are x.
>
> h(x|y) is the probability density function of the distribution of the ultimate losses, given that the developed losses are y.

The functions f(x) and g(y|x) are estimated, and the mean of the distribution given by h(x|y) is the estimate of ultimate losses. For certain choices of f(x) and g(y|x), an explicit formula for the mean of h(x|y) is known, for example when f(x) and g(y|x) are both log-normal.

The paper compares the Bayesian estimate of the ultimate loss ratio with the actual developed loss ratio and the Bornhuetter-Ferguson estimate of the ultimate loss ratio.

### General comments

The model is particularly useful for recent accident years and for lines of business with slow development. The model should be capable of fairly easy implementation in most spreadsheet packages.

A modest level of statistical knowledge is required. One approach to estimating the distributions given by f(x) and g(y|x) is to assume f(x) and g(y|x) are of a known type, such as log-normal, and estimate their means and variances to obtain the parameters of the distributions. To do this, a certain amount of judgment may be needed, as the estimates will usually have to be based on somewhat limited information. Thus, although the model provides a rigorous way of incorporating prior information, some of the information used in applying the model may be rather unreliable.