[D2]
EXPONENTIAL RUN-OFF
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1. Introduction

This method has been used to assess reserves for personal injury liabilities within direct motor third party insurance. It is a relatively straightforward method which simply models claims run-off by an exponential distribution. It is based on the observation that the claims in each development year for a particular year of business often show an exponentially decreasing shape apart perhaps from the first two years of development.

Thus, if the first few development years (often the first two years) are ignored, an exponential model can be applied. Care must obviously be taken that the model fits the data accurately and an examination of the residuals would perhaps be useful. The method has the advantage that prediction is possible for later development years than any in the triangle, unlike the chain ladder method.

A separate model is applied to each year of business written, but the results are inspected for trends and possible pooling of years of incurrence for which there is insufficient data for estimation.

The method is described first without taking account of inflation; inflation is dealt with in section 6.

2. The general case

In this section the model is discussed in detail, without inflation adjustments.

Define C_{ij} = amount paid in development year j in respect of claims incurred in year i.

It is assumed that all payments are made before year A.

i.e. $C_{ij} = 0$ for $j \geq A$ (2.1)

Also, after year α ($\alpha < A$) the claim payments are modelled by

$$C_{i\,j+1} = q_i\, C_{ij} \qquad (2.2)$$

for some fixed q_i.

This is equivalent to an exponential tail, since, under the exponential model,

$$C_{ij} = \lambda_i \, e^{\lambda_i j} \qquad \text{for some } \lambda \tag{2.3}$$

then $$C_{i\,j+1} = \lambda_i \, e^{\lambda_i (j+1)} = e^{\lambda_i} C_{ij} \tag{2.4}$$

and (2.4) is equivalent to (2.2) with a parameter transformation.

The following is a non-rigorous motivation of the likelihood which is used to estimate q_i. Considering one particular incurrence year i, the suffix on q_i is dropped, and it is assumed that there have been T years of run-off ($T > \alpha$).

Let X = development year by the end of which a claim is paid ($X > \alpha$).

Then set $P(X = j) = \beta q^{j-\alpha} \qquad j = \alpha, \ldots, T$ (2.5)

Summing over j gives

$$\beta = \frac{1}{\left(\dfrac{1 - q^{T-\alpha+1}}{1 - q}\right)} = \frac{1 - q}{1 - q^{T-\alpha+1}} \tag{2.6}$$

The likelihood function is

$$L(q) = \prod_{j=\alpha}^{T} (\beta q^{j-\alpha})^{N_{ij}} \tag{2.7}$$

where N_{ij} = number of claims in development year j in respect of claims incurred in year i

It can be shown that, if each pound of claim is independent of the rest, (2.7) can be replaced by

$$L(q) = \prod_{j=\alpha}^{T} (\beta q^{j-\alpha})^{C_{ij}} \tag{2.8}$$

and (2.8) is used in all cases, even when the above assumption does not hold.

Taking logs of (2.8),

$$\log L(q) = \sum_{j=\alpha}^{T} C_{ij} (\log\beta + (j - \alpha) \log q) \tag{2.9}$$

(2.9) can be maximised and the maximum likelihood estimate of q found. This is done for each row and thus a set of q_i estimates found.

The mathematical maximisation is contained in the appendix: this shows the uniqueness and existence of the maximum. However, it is more straightforward to maximise (2.9) numerically using a simple search algorithm such as interval bisection. This is illustrated by the example in section 5.

3. Reserves

Let R_{ij} be the claims reserve at the end of the development year j (where $j > \alpha$) assuming no inflation.

So

$$R_{ij} = \sum_{k=\alpha} C_{ik} \tag{3.1}$$

Now

$$R_{ij} = \sum_{k=\alpha} C_{ik} \left[\frac{\sum_{k=j+1}^{\infty} C_{ik}}{\sum_{k=\alpha}^{j} C_{ik}} \right] \tag{3.2}$$

and, according to the model in section 2,

$$\frac{\sum_{k=j+1}^{\infty} C_{ik}}{\sum_{k=\alpha}^{j} C_{ik}} = \frac{\sum_{k=j+1}^{A} \gamma_i \, q_i^{k-\alpha}}{\sum_{k=\alpha}^{j} \gamma_i \, q_i^{k-\alpha}} \quad \text{where } \gamma_i = C_{i\alpha}$$

$$= \frac{\sum_{k=j+1}^{A} q_i^{k-\alpha}}{\sum_{k=\alpha}^{j} q_i^{k-\alpha}}$$

$$= \frac{q_i^{j-\alpha+1} \, (1 - q_i^{A-j}) \, (1 - q_i)^{-1}}{(1 - q_i^{j-\alpha+1})(1 - q_i)^{-1}}$$

$$= \frac{q_i^{j-\alpha} - q_i^{A-\alpha}}{q_i^{-1} - q_i^{j-\alpha}}$$

$$= S_j(q_i), \text{ say} \tag{3.3}$$

Now, since we are using maximum likelihood estimation, the maximum likelihood estimate of the reserve is

$$\hat{R}_{ij} = \sum_{k=\alpha}^{j} C_{ik} S_j (\hat{q}_i) \tag{3.4}$$

The reserves in the example (section 5) have been calculated using (3.4) and the estimate of q_i from section 2.

4. The model in practice

The model which has been used in practice can be summarised in the following table.

For year of incurrence i,

development year	data	model
0	C_{i0}	C_{i0}
1	C_{i1}	C_{i1}
2	C_{i2}	γ_i
3	C_{i3}	$\gamma_i q_i$
4	C_{i4}	$\gamma_i q_i^2$
.	.	.
.	.	.
.	.	.
A-1	$C_{i\,A-1}$	$\gamma_i q_i^{A-3}$
A	C_{iA}	$\gamma_i q_i^{A-2}$
A+1	$C_{i\,A+1}$	0
.	.	.
.	.	.
.	.	.

where $\gamma_i = C_{i2}$.

It can be seen that this case has $\alpha = 2$. From the data, q_i is usually estimated to be around 0.9 for a succession of origin years i, and A is about 19.

Thus, in practice, the first two years are not modelled: the forecasting is applied only to run-off years of delay 3 or more. This means that the two most recent accident years have no forecast values of ultimate claims.

5. Example

The method is illustrated in this section by applying it to some actual data. In the example, the theoretical derivation of q_i is used (as set out in the appendix). As stated earlier, it is much easier to use a simple search method, but the theoretical approach is used in order to illustrate the method.

Year of origin 1974

$$T = 10, \ \frac{(T-2)}{2} = 4 \qquad \sum(j-2)\,C_{ij} = 30{,}483 \qquad \sum C_{ij} = 10{,}335$$

From equation (A.9)

$$\bar{X} = 2.9495 < \frac{T-2}{2}$$

Hence a solution of f(q) = 0 is needed where

$$f(q) = \frac{9}{1-q^9} - \frac{1}{1-q} - (8-\bar{X}) \qquad \text{(equation (A10))}$$

As a first approximation, using equation (A11)

$$q = 1 - \left(\frac{T-2}{2} - \bar{X}\right)\frac{12}{T(T-2)} = 0.842$$

q	f(q)
0.842	$5.102 - 5.05 > 0$
0.860	$4.976 - 5.05 < 0$
0.850	$5.046 - 5.05 \approx 0$

$\therefore \hat{q} \approx 0.85$

Year of origin 1975

$$T = 9, \ \frac{(T-2)}{2} = 3.5 \qquad \sum(j-2)\,C_{ij} = 24{,}090 \qquad \sum C_{ij} = 8{,}354$$

$$\bar{X} = 2.8836 < \frac{T-2}{2}$$

$$f(q) = \frac{8}{1-q^8} - \frac{1}{1-q} - (7-\bar{X})$$

As a first approximation, $q = 0.883$

q	$f(q)$
0.883	$4.143 - 4.1164 > 0$
0.890	$4.103 - 4.1164 < 0$
0.885	$4.131 - 4.1164 > 0$

$\therefore\ \hat{q} \approx 0.89$

and so on.

Continuing the process for years of origin 1976 to 1981 gives the following table:

Year of origin	$\hat{q}$
1974	0.85
1975	0.89
1976	0.84
1977	0.78
1978	0.74
1979	0.69
1980	0.78
1981	0.79

These values of $\hat{q}$ can be substituted into the formula in section 3 to calculate the reserves.

The simpler search method can be illustrated by considering, for example, year of origin 1974. The values of q and $I = \log L(q)$ (which has to be maximised) in the relevant range are

q	I
0.82	-21875.7
0.83	-21854.2
0.84	-21841.7
0.85	-21837.9
0.86	-21842.7
0.87	-21855.8
0.88	-21877.2

Thus the maximum likelihood estimate of q is

$$\hat{q} \approx 0.85 \qquad \text{(as before)}$$

For the most recent years of origin there is very little data to use in the estimation procedure, and an IBNR computation is needed. For these years a "smoothed" common q value may be chosen which is a conservative estimate (e.g. 0.85 or 0.90) in the sense that it over-reserves: it is preferable that the predicted claims should be greater than the actual claims.

6. Adjustment for future inflation

Future inflation can be taken into account by modifying the claims reserve at the end of year j, R_{ij}.

Suppose future inflation with inflation factor r per year is to be taken into account.

This implies that R_{ij} has to be increased by a factor

$$\frac{\sum_{k=j+1}^{A} \gamma_i \, q_i^{k-\alpha} \, r^{k-j-½}}{\sum_{k=j+1}^{A} \gamma_i \, q_i^{k-\alpha}}$$

$$= \frac{q_i^{j-\alpha+1} \, r^{½}(1 - (q_i \, r)^{A-j}) \, (1 - q_i \, r)^{-1}}{q_i^{j-\alpha+1} \, (1 - q_i^{A-j}) \, (1 - q_i)^{-1}}$$

$$= \frac{r^{½}[1 - (q_i \, r)^{A-j}](1 - q_i)}{(1 - q_i^{A-j})(1 - q_i \, r)} \qquad \text{if } q_i \, r \neq 1$$

or

$$= \frac{r^{½}(A - j)(1 - q_i)}{1 - q_i^{A-j}} \qquad \text{if } q_i \, r = 1 \qquad (6.1)$$

This factor is called $I_j(q_i, r)$.

If future inflation is to be taken into account, but its influence limited to n years ahead, then the factor by which R_{ij} has to be increased is instead (for $j \le A - n$)

$$\frac{\sum_{k=j+1}^{j+n-1} \gamma_i\, q_i^{k-\alpha}\, r^{k-j-½} + \sum_{k=j+n}^{A} \gamma_i\, q_i^{k-\alpha}\, r^{n-½}}{\sum_{k=j+1}^{A} \gamma_i\, q_i^{k-\alpha}}$$

$$= \frac{r^{½}\, q_i^{j-\alpha+1}\,(1-(q_i r)^{n-1})\,(1-q_i r)^{-1} + r^{n-½}\, q_i^{j+n-\alpha}\,(1-q_i^{A-j-n+1})\,(1-q_i)^{-1}}{q_i^{j-\alpha+1}\,(1-q_i^{A-j})\,(1-q_i)^{-1}}$$

$$= \frac{r^{½}(1-(q_i\, r)^{n-1})(1-q_i) + r^{n-½}\, q_i^{n-1}\,(1-q_i^{A-j-n+1})(1-q_i\, r)}{(1-q_i^{A-j})\,(1-q_i\, r)} \qquad \text{if } q_i\, r \ne 1$$

or

$$= \frac{r^{½}(n-1)\,(1-q_i) + r^{½}\,(1-q_i^{A-j-n+1})}{(1-q_i^{A-j})} \qquad \text{if } q_i\, r = 1$$

(6.2)

This factor is called $I_j^{(n)}(q_i, r)$.

Summarising, it can be seen that if future inflation is taken into account then the reserve must be

$$R_{ij}\, I_j(\hat{q}_i, r) \qquad \text{or} \qquad R_{ij}\, I_j^{(n)}(\hat{q}_i, r)$$

depending on how many years' inflation are taken into account.

Appendix

In section 2, the log likelihood is derived in equation (2.9) as

$$\log L(q) = \sum_{j=\alpha}^{T} C_{ij}\,(\log\beta + (j-\alpha)\log q)$$

This expression has to be differentiated with respect to q. First of all, note that

$$\frac{d\beta}{dq} = \frac{-1}{(1-q^{T-\alpha+1})} + \frac{(1-q)\,q^{T-\alpha}\,(T-\alpha+1)}{(1-q^{T-\alpha+1})^2} \qquad \text{(A.1)}$$

and so

$$\frac{d}{dq}\log L = \sum_{j=\alpha}^{T} C_{ij}\left[\frac{(1-q)(T-\alpha+1)q^{T-\alpha}}{(1-q^{T-\alpha+1})^2}\cdot\frac{(1-q^{T-\alpha+1})}{(1-q)}\right.$$

$$\left.-\frac{1}{(1-q^{T-\alpha+1})}\cdot\frac{(1-q^{T-\alpha+1})}{(1-q)}+\frac{(j-\alpha)}{q}\right]$$

$$= \sum_{j=\alpha}^{T} C_{ij}\left[\frac{(T-\alpha+1)q^{T-\alpha}}{(1-q^{T-\alpha+1})}-\frac{1}{(1-q)}+\frac{(j-\alpha)}{q}\right] \tag{A.2}$$

Put $\bar{X} = \dfrac{1}{\sum_{j=\alpha}^{T} C_{ij}} \sum_{j=\alpha}^{T} (j-\alpha)\, C_{ij}$ and note that

$$\frac{q^{T-\alpha}}{(1-q^{T-\alpha+1})} = \frac{1}{q}\left(-1+\frac{1}{(1-q^{T-\alpha+1})}\right). \tag{A.3}$$

Substituting into (A.2) it can be seen that

$$\frac{d}{dq}\log L = \sum_{j=\alpha}^{T} C_{ij}\left\{\frac{(T-\alpha+1)}{q}\left(\frac{1}{(1-q^{T-\alpha+1})}-1\right)-\frac{1}{1-q}+\frac{\bar{X}}{q}\right\}$$

$$= \left(\sum_{j=\alpha}^{T} C_{ij}\right)\left(\frac{1}{q}\left[\bar{X}-\frac{q}{1-q}-(T-\alpha+1)+\frac{T-\alpha+1}{1-q^{T-\alpha+1}}\right]\right)$$

$$= \left(\sum_{j=\alpha}^{T} C_{ij}\right)\frac{1}{q}\left(\bar{X}-\left[\frac{1}{1-q}+(T-\alpha)-\frac{T-\alpha+1}{1-q^{T-\alpha+1}}\right]\right) \tag{A.4}$$

A solution of $\frac{d}{dq}\log L = 0$ is needed, so consider

$$f(q) = \bar{X} - \left[\frac{1}{1-q}+(T-\alpha)-\frac{T-\alpha+1}{1-q^{T-\alpha+1}}\right] \tag{A.5}$$

and note that

$$f(q) = \frac{T - \alpha + 1}{1 - q^{T - \alpha + 1}} - \frac{1}{1 - q} - (T - \alpha - \bar{X}) \tag{A.6}$$

Hence $f(0_+) = \bar{X} > 0$.

It is easy to show that f is a decreasing function of q for $0 < q < 1$ (just differentiate and show that the derivative is always < 0). So if it can be shown that $f(1_-) < 0$ it will have been proved that there is an unique solution of $f(q) = 0$ in $0 < q < 1$ and that this is the maximum likelihood estimator of q.

To calculate $f(1_-)$, first consider $q = 1 - \epsilon$. Then

$$f(q) = \frac{T - \alpha + 1}{1 - (1 - \epsilon)^{T-\alpha+1}} - \frac{1}{\epsilon} - (T - \alpha - \bar{X})$$

$$= \frac{T-\alpha+1}{1-[1-(T-\alpha+1)\epsilon+½(T-\alpha+1)(T-\alpha)\epsilon^2-\frac{1}{6}(T-\alpha+1)(T-\alpha)(T-\alpha-1)\epsilon^3+O(\epsilon^4)]}$$

$$- \frac{1}{\epsilon} - (T - \alpha - \bar{X})$$

$$= \frac{1}{\epsilon[(1 - ½(T - \alpha) + \frac{1}{6}(t - \alpha)(T - \alpha - 1)\epsilon^2 + O(\epsilon^3)]} - \frac{1}{\epsilon} - \left(T - \alpha - \bar{X}\right)$$

$$= \frac{1}{\epsilon} - \left(\frac{T - \alpha}{2}\right) - \frac{(T - \alpha)(T - \alpha - 1)\epsilon}{6} + \left(\frac{T - \alpha}{2}\right)^2 \epsilon + O(\epsilon^2) - \frac{1}{\epsilon} - (T - \alpha - \bar{X})$$

$$= \frac{T - \alpha}{2} - \left(\frac{T - \alpha}{2}\right)\left(\frac{T - \alpha - 1}{3} - \frac{T - \alpha}{2}\right)\epsilon + O(\epsilon^2) - (T - \alpha - \bar{X}) \tag{A.7}$$

$$\rightarrow \bar{X} - \frac{T - \alpha}{2} \quad \text{as } \epsilon \rightarrow 0$$

So $f(1_-) < 0$ if $\bar{X} < \frac{T - \alpha}{2}$.

A value of q is needed such that f(q) = 0. If $\bar{X} < \frac{T - \alpha}{2}$, and if $q = 1 - \epsilon$ then from the above equation (A.7) it can be seen that a first order approximation for ϵ is given by

$$\epsilon = \left(\frac{T - \alpha}{2} - \bar{X} \right) \cdot \frac{12}{(T - \alpha)(T - \alpha + 2)} \qquad \text{(A.8)}$$

For the model in practice (section 4),

$$\bar{X} = \frac{1}{\sum_{j=2}^{T} C_{ij}} \sum_{j=2}^{T} (j - 2)\, C_{ij} \qquad \text{(A.9)}$$

Then if $\bar{X} < \frac{T - \alpha}{2}$, q is a solution of

$$f(q) = \frac{T - 1}{1 - q^{T-1}} - \frac{1}{1 - q} - (T - 2 - \bar{X}) \qquad \text{(A.10)}$$

and a first order approximation for ϵ is given by

$$\epsilon = \left(\frac{T - 2}{2} - \bar{X} \right) \cdot \frac{12}{T(T - 2)} \qquad \text{(A.11)}$$

A more accurate maximum likelihood estimate of q can be found by a numerical search method around this first order approximation. This is illustrated in the example in section 5.

The approximate maximum likelihood estimate of q can be found from equation (A.8) using

$$\hat{q} = 1 - \epsilon.$$

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