Estimation of Tail Development Factors in the Paid-Incurred Chain Reserving Method

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Abstract

In many applied claims reserving problems in P&C insurance claims development goes beyond the latest observations available in the claims development triangles. This makes it necessary to estimate so-called tail development factors which account for the inexperienced part of the insurance claims. We estimate these tail development factors in a mathematically consistent way. Therefore, we modify the paid-incurred chain (PIC) reserving model from Merz-Wüthrich [5]. This modification then allows for the prediction of the outstanding loss liabilities and the corresponding prediction uncertainty under the inclusion of tail development factors.

1 Introduction and model assumptions

Often in P&C claims reserving problems the claims settlement process goes beyond the latest observations available in the claims development triangle. This means that there is still an inexperienced part of the insurance claims for which one needs to build claims reserves. In such situations claims reserving actuaries apply so-called tail development factors to the last column of the claims development triangle which accounts for the settlement that goes beyond the latest observations. Typically, one has only limited information for the estimation of such tail development factors. Therefore, various different techniques are applied to estimate this tail development factor. Most of these estimation methods are rather ad hoc methods that do not fit into any stochastic modeling framework. Popular estimation techniques, for example, fit parametric curves to the data using the right-hand corner of the claims development triangle

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(see Mack [6] and Boor [1]). By contrast, in practice, one often does a simultaneous study of claims payments and claims incurred data: incurred-paid ratios are used to determine tail development factors (see Section 3 in Boor [1]).

In this paper we review the paid-incurred chain (PIC) reserving method. The log-normal PIC reserving model introduced in Merz-Wüthrich [5] can easily be extended so that it allows for the inclusion of tail development factors in a natural and mathematically consistent way. Similar as in practice, the tail development factor estimates will then be based on incurred-paid ratios within our PIC reserving model.

In the following, we denote accident years by $i \in \{0, \ldots, J\}$ and development years by $j \in \{0, \ldots, J, J+1\}$. Development year J refers to the latest observed development year and the step from J to $J+1$ refers to the tail development factor. Cumulative payments in accident year i after j development periods are denoted by $P_{i,j}$ and the corresponding claims incurred by $I_{i,j}$. Moreover, for the ultimate claim we assume $P_{i,J+1} = I_{i,J+1}$ with probability 1 (see Figure 1). This means we assume that finally - possibly after several development periods beyond the

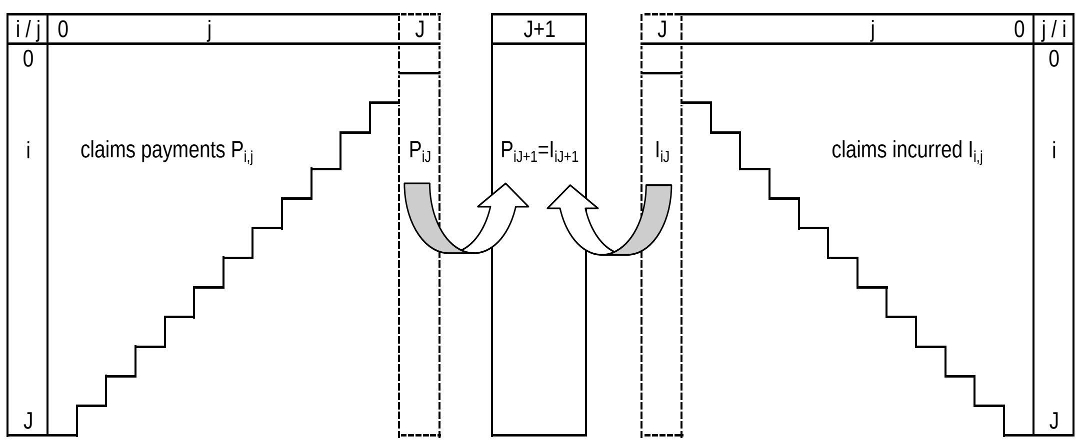

Figure 1: PIC reserving model. Left panel: cumulative payments $P_{i,j}$ development triangle; Right panel: claims incurred $I_{i,j}$ development triangle; both leading to the same ultimate claim $P_{i,J+1} = I_{i,J+1}$.

latest observed development year J - the cumulative payments and the claims incurred lead to the same ultimate claim. That is, ultimately when all claims are settled $I_{i,J+1}$ and $P_{i,J+1}$ must coincide.

Model Assumptions 1.1 (Log-normal PIC reserving model, Merz-Wüthrich [5])

- Conditionally, given the parameters $\mathbf{\Theta} = (\Phi_0, \ldots, \Phi_{J+1}, \Psi_0, \ldots, \Psi_J, \sigma_0, \ldots, \sigma_{J+1}, \tau_0, \ldots, \tau_J)$, we have
  - the random vector $(\xi_{0,0}, \ldots, \xi_{J,J+1}, \zeta_{0,0}, \ldots, \zeta_{J,J})$ has a multivariate Gaussian distribution with uncorrelated components given by

$$\begin{aligned}\xi_{i,j} &\sim \mathcal{N}\left(\Phi_j, \sigma_j^2\right) \qquad \text{for } i \in \{0, \ldots, J\} \text{ and } j \in \{0, \ldots, J+1\},\\ \zeta_{k,l} &\sim \mathcal{N}\left(\Psi_l, \tau_l^2\right) \qquad \text{for } k \in \{0, \ldots, J\} \text{ and } l \in \{0, \ldots, J\};\end{aligned}$$

  - cumulative payments $P_{i,j}$ are given by the recursion, $j = 0, \ldots, J+1$,

$$P_{i,j} = P_{i,j-1} \ \exp\{\xi_{i,j}\}, \qquad \text{with initial value } \ P_{i,-1} = 1;$$

  - claims incurred $I_{i,j}$ are given by the (backwards) recursion, $j = 0, \ldots, J$,

$$I_{i,j} = I_{i,j+1} \ \exp\{-\zeta_{i,j}\}, \qquad \text{with initial value } \ I_{i,J+1} = P_{i,J+1}.$$

- The components of $\mathbf{\Theta}$ are independent and $\sigma_j, \tau_j > 0$ for all j (with probability 1).

$\square$

For an extended model discussion we refer to Merz-Wüthrich [5]. Basically, the PIC Model Assumptions 1.1 are a combination of Hertig's log-normal model [4] (applied to cumulative payments) and Gogol's Bayesian claims reserving model [3] (applied to claims incurred). In contrast to the PIC reserving model in Merz-Wüthrich [5] we now add an extra development period from J to $J+1$. This is exactly the crucial step for allowing for tail development factors and it leads to the study of incurred-paid ratios for the inclusion of such tail development factors.

2 Estimation of tail development factors

At time J one has observed data given by

$$\mathcal{D}_J = \{P_{i,j}, I_{i,j} : \ i+j \le J, \ 0 \le i \le J, \ 0 \le j \le J\},$$

and one needs to predict the ultimate claims $P_{i,J+1} = I_{i,J+1}$, conditional on these observations $\mathcal{D}_J$. On the one hand this involves the calculation of the conditional expectations $E\left[P_{i,J+1} \middle| \mathcal{D}_J, \mathbf{\Theta}\right]$ and on the other hand it involves Bayesian inference on the parameters $\mathbf{\Theta}$, given $\mathcal{D}_J$ (see Theorems 2.4 and 3.4 in Merz-Wüthrich [5]). Here, we modify this outline which we are just going to discuss.

2.1 Ultimate claims prediction conditional on parameters

We would like to apply Model Assumptions 1.1 to the tail development factor estimation problem. Therefore we also need to specify the prior distribution of the parameter vector $\mathbf{\Theta}$.

Apart from the fact that claims incurred data $I_{i,j}$ can be very subjective and misleading it is often not considered for claims reserving. Reasons therefore are, for example, that different claims adjusters' may have rather different estimation methodologies that may, moreover, strongly vary over time. For example, the introduction of new claims adjusters' handling guidelines may imply that the whole claims incurred history may become almost worthless. Therefore, we use claims incurred data $I_{i,j}$ only for the estimation of tail development factors, i.e. we work under the assumption of having incomplete claims incurred triangles (see also Dahms [2] for another claims reserving method that copes with incomplete data). This is done as follows: Assume there exists $J^* \in \{0, \ldots, J\}$ such that with probability 1

$$\Psi_J \equiv \tau_J^2/2,$$

and if $J^* < J$

$$\tau_{J^*} = \tau_{J^*+1} = \ldots = \tau_{J-1} \equiv \tau,$$
$$\Psi_{J^*} = \Psi_{J^*+1} = \ldots = \Psi_{J-1} \equiv \tau^2/2.$$

Note that for $J^* = J$ we simply assume $\Psi_J \equiv \tau_J^2/2$. These assumptions imply that there is no substantial claims incurred development after claims development period J^*, i.e. for $j \in \{J^*, \ldots, J\}$

$$E\left[\exp\{-\zeta_{i,j}\}\right] = E\left[E\left[\exp\{-\zeta_{i,j}\}\middle|\, \mathbf{\Theta}\right]\right] = E\left[\exp\{-\Psi_j + \tau_j^2/2\}\right] = 1.$$

This implies that in the average claims incurred prediction is correct (and we have only pure random fluctuations around this prediction), i.e. for $j \in \{J^*, \ldots, J\}$

$$\begin{aligned} E\left[I_{i,j}\middle|\, I_{i,j+1}\right] &= I_{i,j+1}, \\ \mathrm{Vco}\left(I_{i,j}\middle|\, I_{i,j+1}\right) &= \left(\exp\{\tau_j^2\} - 1\right)^{1/2}, \end{aligned}$$

where $\mathrm{Vco}(\cdot)$ denotes the coefficient of variation. The fact that we allow τ_J to differ from τ corresponds to the difficulty that the tail development factor may cover several development periods beyond the last column in the claims development triangle.

We then define the restricted observations

$$\mathcal{D}_J^* = \{P_{i,j}, I_{k,l}:\ i+j \le J,\ k+l \le J,\ l \ge J^*\} = \mathcal{D}_J \cap \{P_{i,j}, I_{k,l}:\ l \ge J^*\}.$$

In this spirit, we consider all cumulative payments observations but only claims incurred observations from development year J^* on. That is, only the claims incurred $I_{i,j}$ from the latest $J - J^* + 1$ development periods $J^*, J^* + 1, \ldots, J$ are used to estimate tail development factors and the claims reserves. We define the following parameters

$$\eta_j = \sum_{m=0}^{j} \Phi_m \qquad \text{and} \qquad w_j^2 = \sum_{m=0}^{j} \sigma_m^2, \qquad \text{for } j = 0, \ldots, J+1,$$

$$\mu_l = \eta_{J+1} - \sum_{n=l}^{J} \Psi_n \qquad \text{and} \qquad v_l^2 = w_{J+1}^2 + \sum_{n=l}^{J} \tau_n^2, \qquad \text{for } l = J^*, \ldots, J.$$

Moreover, we define the parameters

$$\beta_j = \begin{cases} \frac{w_{J+1}^2 - w_j^2}{v_j^2 - w_j^2} > 0 & \text{for } j = J^*, \ldots, J, \\ 0 & \text{for } j = 0, \ldots, J^* - 1. \end{cases}$$

The following result shows that β_j can be interpreted as the credibility weight for the claims incurred observations:

Theorem 2.1 *Under Model Assumptions 1.1 we have, conditional on* $\mathbf{\Theta}$ *and* $\mathcal{D}_J^*$*,*

$$E\left[P_{i,J+1} \left| \mathcal{D}_J^*, \mathbf{\Theta}\right.\right] = P_{i,J-i}^{1-\beta_{J-i}} \; I_{i,J-i}^{\beta_{J-i}} \; \exp\left\{(1-\beta_{J-i}) \sum_{l=J-i+1}^{J+1} \left(\Phi_l + \sigma_l^2/2\right) + \beta_{J-i} \sum_{l=J-i}^{J} \Psi_l \right\}.$$

For the conditional variance we obtain

$$\mathrm{Var}\left(P_{i,J+1} \left| \mathcal{D}_J^*, \mathbf{\Theta}\right.\right) = E\left[P_{i,J+1} \left| \mathcal{D}_J^*, \mathbf{\Theta}\right.\right]^2 \left(\exp\left\{(1-\beta_{J-i}) \sum_{l=J-i+1}^{J+1} \sigma_l^2 \right\} - 1\right).$$

In the case $i > J - J^*$ it holds $\beta_{J-i} = 0$ and, therefore, we obtain a purely claims payment based prediction (see also Hertig's model [4] presented in Section 2.1 of Merz-Wüthrich [5])

$$P_{i,J-i} \; \exp\left\{ \sum_{l=J-i+1}^{J+1} \left(\Phi_l + \sigma_l^2/2\right) \right\}.$$

In the case $i \leq J - J^*$ it holds $\beta_{J-i} > 0$ and, therefore, we obtain a correction term to the purely claims payment based prediction which is based on the claims incurred-paid ratio $I_{i,J-i}/P_{i,J-i}$, i.e. for a large incurred-paid ratio we get a higher expected ultimate claim as can be seen from

$$\begin{aligned} P_{i,J-i}^{1-\beta_{J-i}} \; I_{i,J-i}^{\beta_{J-i}} &= \exp\left\{(1-\beta_{J-i}) \log P_{i,J-i} + \beta_{J-i} \log I_{i,J-i}\right\} \\ &= P_{i,J-i} \exp\left\{\beta_{J-i} \; \log \frac{I_{i,J-i}}{P_{i,J-i}}\right\}. \end{aligned}$$

2.2 Parameter estimation, the general case

The likelihood function of the restricted observations $\mathcal{D}_J^*$ is given by (see also (3.5) in Merz-Wüthrich [5])

$$\begin{aligned} l_{\mathcal{D}_J^*}(\mathbf{\Theta}) \;\propto\; & \prod_{j=0}^{J} \prod_{i=0}^{J-j} \frac{1}{\sigma_j} \exp\left\{ -\frac{1}{2\sigma_j^2} \left(\Phi_j - \log \frac{P_{i,j}}{P_{i,j-1}} \right)^2 \right\} \\ & \times \prod_{i=0}^{J-J^*} \frac{1}{\sqrt{v_{J-i}^2 - w_{J-i}^2}} \exp\left\{ -\frac{1}{2(v_{J-i}^2 - w_{J-i}^2)} \left(\mu_{J-i} - \eta_{J-i} - \log \frac{I_{i,J-i}}{P_{i,J-i}} \right)^2 \right\} \\ & \times \prod_{j=J^*}^{J-1} \prod_{i=0}^{J-j-1} \frac{1}{\tau_j} \exp\left\{ -\frac{1}{2\tau_j^2} \left(\Psi_j + \log \frac{I_{i,j}}{I_{i,j+1}} \right)^2 \right\}, \end{aligned}$$

where $\propto$ means that only relevant terms dependent on $\mathbf{\Theta}$ are considered. The first line describes the claims payments development, the last line describes the claims incurred development, and the middle line describes the gap between the diagonal claims incurred and the diagonal claims payment observations.

In order to do a Bayesian inference analysis on the parameters we need to specify prior distributions for $\mathbf{\Theta}$.

Model Assumptions 2.2 (PIC tail development factor model)

We assume Model Assumptions 1.1 hold true with positive constants $\sigma_0, \ldots, \sigma_{J+1}$ and $\tau_{J^*} = \ldots = \tau_{J-1} = \tau$ and $\Psi_{J^*} = \ldots = \Psi_{J-1} = \tau^2/2$ and $\Psi_J = \tau_J^2/2$. Moreover, it holds

$$\Phi_m \;\sim\; \mathcal{N}\left(\phi_m, s_m^2\right) \qquad \text{for } m \in \{0, \ldots, J+1\}.$$

$\square$

Under Model Assumptions 2.2 the posterior distribution $u\left(\mathbf{\Phi}\,|\mathcal{D}_J^*\right)$ of $\mathbf{\Phi} = (\Phi_0, \ldots, \Phi_{J+1})$, given $\mathcal{D}_J^*$, is given by

$$u\left(\mathbf{\Phi}\,|\mathcal{D}_J^*\right) \;\propto\; l_{\mathcal{D}_J^*}(\mathbf{\Theta}) \prod_{m=0}^{J+1} \exp\left\{ -\frac{1}{2s_m^2} \left(\Phi_m - \phi_m\right)^2 \right\}. \tag{2.1}$$

This immediately implies the following theorem:

Theorem 2.3 *Under Model Assumptions 2.2 the posterior* $u\left(\mathbf{\Phi}\,|\mathcal{D}_J^*\right)$ *of* $\mathbf{\Phi}$ *is a multivariate Gaussian distribution with posterior mean* $\left(\phi_0^{post}, \ldots, \phi_{J+1}^{post}\right)$ *and posterior covariance matrix* $\Sigma(\mathcal{D}_J^*)$. *Define the posterior standard deviation by*

$$s_j^{post} = \left(s_j^{-2} + (J-j+1)\sigma_j^{-2} \right)^{-1/2} \quad \textit{for } j = 0, \ldots J+1.$$

Then, the inverse covariance matrix $\Sigma(\mathcal{D}_J^*)^{-1} = (a_{n,m})_{0\le n,m\le J+1}$ *is given by*

$$a_{n,m} \;=\; (s_n^{post})^{-2}\, 1_{\{n=m\}} + \left[\sum_{i=J^*}^{(n-1)\wedge(m-1)} \left(v_i^2 - w_i^2\right)^{-1}\right] 1_{\{n,m\ge J^*+1\}}.$$

The posterior mean $\left(\phi_0^{post}, \ldots, \phi_{J+1}^{post}\right)$ *is obtained by*

$$\left(\phi_0^{post}, \ldots, \phi_{J+1}^{post}\right)' \;=\; \Sigma(\mathcal{D}_J^*)\; (c_0, \ldots, c_{J+1})',$$

with vector $(c_0, \ldots, c_{J+1})$ *given by*

$$c_j = \frac{\phi_j}{s_j^2} + \frac{1}{\sigma_j^2}\sum_{i=0}^{J-j}\log\frac{P_{i,j}}{P_{i,j-1}} + \left[\sum_{i=J-j+1}^{J-J^*}\frac{1}{v_{J-i}^2 - w_{J-i}^2}\left(\log\frac{I_{i,J-i}}{P_{i,J-i}} + \frac{i\;\tau^2+\tau_J^2}{2}\right)\right]1_{\{j\ge J^*+1\}}.$$

Note that the last term in the definition of $a_{n,m}$ and in the definition of c_j corresponds to the development years in $\mathcal{D}_J^*$ where we have both claims payments and claims incurred information. Theorem 2.3 immediately implies the following corollary:

Corollary 2.4 *Under Model Assumptions 2.2 the posterior* $u\left(\mathbf{\Phi}\,|\mathcal{D}_J^*\right)$ *of* $\mathbf{\Phi}$ *is a multivariate Gaussian distribution with* $\Phi_0, \ldots, \Phi_{J^*}, (\Phi_{J^*+1}, \ldots, \Phi_{J+1})$ *are independent with*

$$\Phi_j|_{\{\mathcal{D}_J^*\}} \sim \mathcal{N}\left(\phi_j^{post} = \gamma_j\overline{\phi_j} + (1-\gamma_j)\;\phi_j, (s_j^{post})^2\right) \tag{2.2}$$

for $j \le J^*$ *and credibility weight and empirical mean defined by*

$$\gamma_j = \frac{J-j+1}{J-j+1+\sigma_j^2/s_j^2} \quad \text{and} \quad \overline{\phi_j} = \frac{1}{J-j+1}\sum_{i=0}^{J-j}\log\frac{P_{i,j}}{P_{i,j-1}} \quad \text{for } j = 0,\ldots J^*.$$

Henceforth, Corollary 2.4 shows that for development years $j \le J^*$ we obtain the well-known credibility weighted average between the prior mean ϕ_j and the average observation $\overline{\phi_j}$. The case $j > J^*$ is more involved, one basically obtains a weighted average between the prior mean ϕ_j, the average observation $\overline{\phi_j}$ and the incurred-paid ratios $\log I_{i,J-i}/P_{i,J-i}$, $i \ge J-j+1$.

2.3 Parameter estimation, special case $J^* = J$

We consider the special case $J^* = J$, that is, only the claims incurred observation $I_{0,J}$ is considered in the tail development factor analysis. This immediately provides:

Corollary 2.5 *Choose* $J^* = J$. *Under Model Assumptions 2.2, the posterior distribution* $u\left(\mathbf{\Phi}\,|\mathcal{D}_J^*\right)$ *of* $\mathbf{\Phi}$ *is a multivariate Gaussian distribution with* $\Phi_0, \ldots, \Phi_{J+1}$ *are independent. For*

$m \le J^* = J$ the posterior distribution of Φ_m is given by (2.2). The posterior of Φ_{J+1} is given by

$$\Phi_{J+1}|_{\{\mathcal{D}_J^*\}} \sim \mathcal{N}\left(\phi_{J+1}^{post} = \gamma_{J+1}\ \left(\log \frac{I_{0,J}}{P_{0,J}} + \frac{\tau_J^2}{2}\right) + (1-\gamma_{J+1})\ \phi_{J+1}\ ,\ a_{J+1,J+1}^{-1}\right),$$

with inverse variance given by

$$a_{J+1,J+1} = s_{J+1}^{-2} + \left(\sigma_{J+1}^2 + \tau_J^2\right)^{-1},$$

and credibility weight given by

$$\gamma_{J+1} = \frac{1}{1 + (\sigma_{J+1}^2 + \tau_J^2)/s_{J+1}^2}.$$

This means that in the case $J^* = J$ we obtain a credibility weighted average between the prior tail development factor ϕ_{J+1} and the observation $\log \frac{I_{0,J}}{P_{0,J}}$. Henceforth, only the latest incurred-paid ratio is considered for the estimation of the tail development factor.

3 Posterior claims prediction and prediction uncertainty

3.1 General case

In view of Theorems 2.1 and 2.3 we can now predict the ultimate claim $P_{i,J+1}$, conditional on the restricted observations $\mathcal{D}_J^*$, under Model Assumptions 2.2.

Proposition 3.1 (Bayesian ultimate claims predictor) *Under Model Assumptions 2.2 we predict the ultimate claim $P_{i,J+1}$, given $\mathcal{D}_J^*$, by*

$$\begin{aligned} E\left[P_{i,J+1}\middle|\, \mathcal{D}_J^*\right] &= P_{i,J-i}^{1-\beta_{J-i}}\ I_{i,J-i}^{\beta_{J-i}}\ \exp\left\{(1-\beta_{J-i}) \sum_{l=J-i+1}^{J+1} \frac{\sigma_l^2}{2} + \beta_{J-i}\ \frac{i\ \tau^2 + \tau_J^2}{2}\right\} \\ &\quad \times\ \exp\left\{(1-\beta_{J-i}) \sum_{j=J-i+1}^{J+1} \phi_j^{post} + (1-\beta_{J-i})^2\ \frac{\mathbf{e}_{J-i+1}'\ \Sigma(\mathcal{D}_J^*)\ \mathbf{e}_{J-i+1}}{2}\right\}, \end{aligned}$$

where $\mathbf{e}_j = (0, \dots, 0, 1, \dots, 1)' \in \mathbb{R}^{J+2}$ with the first j components equal to 0.

Next we determine the prediction uncertainty. Model Assumptions 2.2 and Theorem 2.3 constitute a full distributional model which allows for the calculation of any risk measure (using Monte Carlo simulations) under the posterior distribution, given $\mathcal{D}_J^*$. Here, we use the most popular measure for the prediction uncertainty in claims reserving, the so-called conditional mean square error of prediction (MSEP). The conditional MSEP has the advantage that we can calculate it analytically. Analytical solutions have the advantage that they allow for more simple sensitivity analysis. The conditional MSEP is given by (see also Section 3.1 in Wüthrich-Merz [8])

$$\begin{aligned}
\mathrm{msep}_{\sum_{i=0}^J P_{i,J+1}|\mathcal{D}_J^*}\left(E\left[\left.\sum_{i=0}^J P_{i,J+1}\right|\mathcal{D}_J^*\right]\right) &= E\left[\left.\left(\sum_{i=0}^J P_{i,J+1} - E\left[\left.\sum_{i=0}^J P_{i,J+1}\right|\mathcal{D}_J^*\right]\right)^2\right|\mathcal{D}_J^*\right] \\
&= \mathrm{Var}\left(\left.\sum_{i=0}^J P_{i,J+1}\right|\mathcal{D}_J^*\right),
\end{aligned}$$

i.e. in this Bayesian setup the conditional MSEP is equal to the posterior variance. This posterior variance allows for the usual decoupling into average processes error and average parameter estimation error, see (A.3) below. The conditional MSEP satisfies

$$\mathrm{Var}\left(\left.\sum_{i=0}^J P_{i,J+1}\right|\mathcal{D}_J^*\right) = \sum_{i,k=0}^J \mathrm{Cov}\left(P_{i,J+1}, P_{k,J+1}\right|\mathcal{D}_J^*).$$

We obtain the following theorem:

Theorem 3.2 *Under Model Assumptions 2.2 the conditional MSEP of the Bayesian predictor* $E\left[\left.\sum_{i=0}^J P_{i,J+1}\right|\mathcal{D}_J^*\right]$ *for the aggregate ultimate claim* $\sum_{i=0}^J P_{i,J+1}$ *is given by*

$$\begin{aligned}
&\mathrm{msep}_{\sum_{i=0}^J P_{i,J+1}|\mathcal{D}_J^*}\left(E\left[\left.\sum_{i=0}^J P_{i,J+1}\right|\mathcal{D}_J^*\right]\right) \\
&= \sum_{0\le i,k\le J}\left(e^{(1-\beta_{J-i})(1-\beta_{J-k})\mathbf{e}'_{J-i+1}\Sigma(\mathcal{D}_J^*)\mathbf{e}_{J-k+1}+1_{\{i=k\}}(1-\beta_{J-i})\sum_{l=J-i+1}^{J+1}\sigma_l^2}-1\right) \\
&\qquad \times E\left[P_{i,J+1}\left|\mathcal{D}_J^*\right.\right] E\left[P_{k,J+1}\left|\mathcal{D}_J^*\right.\right].
\end{aligned}$$

3.2 Special case $J^* = J$ with non-informative priors

We revisit the special case $J^* = J$ and we also assume non-informative priors meaning that $s_j^2 \to \infty$. In that case we obtain that the posterior distributions of $\Phi_0, \ldots, \Phi_{J+1}$ are independent Gaussian distributions with

$$\Phi_j|_{\{\mathcal{D}_J^*\}} \sim \mathcal{N}\left(\phi_j^{post} = \overline{\phi_j} = \frac{1}{J-j+1}\sum_{i=0}^{J-j}\log\frac{P_{i,j}}{P_{i,j-1}}\ ,\ (s_j^{post})^2 = \frac{\sigma_j^2}{J-j+1}\right),$$

for $j \le J$, and

$$\Phi_{J+1}|_{\{\mathcal{D}_J^*\}} \sim \mathcal{N}\left(\phi_{J+1}^{post} = \log\frac{I_{0,J}}{P_{0,J}} + \frac{\tau_J^2}{2}\ ,\ (s_{J+1}^{post})^2 = a_{J+1,J+1}^{-1} = \sigma_{J+1}^2 + \tau_J^2\right).$$

This implies for the ultimate claim prediction for $i > 0$

$$E\left[P_{i,J+1}\right|\mathcal{D}_J^*] = P_{i,J-i}\ \exp\left\{\sum_{l=J-i+1}^{J+1}\phi_l^{post} + \frac{\sigma_l^2}{2} + \frac{(s_l^{post})^2}{2}\right\} = P_{i,J-i}\prod_{l=J-i+1}^{J}\widehat{f}_l\ \widehat{f}_{J+1}^{(ult)}, \quad (3.1)$$

with chain-ladder factors

$$\widehat{f}_l = \exp\left\{\phi_l^{post} + \left(1 + \frac{1}{J-l+1}\right)\frac{\sigma_l^2}{2}\right\}, \tag{3.2}$$

$$\widehat{f}_{J+1}^{(ult)} = \frac{I_{0,J}}{P_{0,J}} \exp\left\{\sigma_{J+1}^2 + \tau_J^2\right\}. \tag{3.3}$$

That is, the first terms in the product on the right-hand side of (3.1) are the classical chain-ladder factors for Hertig's log-normal model [4], see also (5.11)-(5.12) in Wüthrich-Merz [8]. The last term in (3.1), however, describes the tail development factor (adjusted for the variance). For $i = 0$ we have

$$E\left[P_{0,J+1}\middle|\, \mathcal{D}_J^*\right] = P_{0,J}\ \widehat{f}_{J+1}^{(ult)} = I_{0,J}\ \exp\left\{\sigma_{J+1}^2 + \tau_J^2\right\}. \tag{3.4}$$

4 Example

In this section we provide an example. We assume that $J = 9$ and that the claims payment data $P_{i,j}$ and the claims incurred data $I_{i,j}$ for $i+j \le J$ are given by Table 1 and Table 2, respectively.

	0	1	2	3	4	5	6	7	8	9
0	1,216,632	1,347,072	1,786,877	2,281,606	2,656,224	2,909,307	3,283,388	3,587,549	3,754,403	3,821,258
1	798,924	1,051,912	1,215,785	1,349,939	1,655,312	1,926,210	2,132,833	2,287,311	2,567,056	
2	1,115,636	1,387,387	1,930,867	2,177,002	2,513,171	2,931,930	3,047,368	3,182,511		
3	1,052,161	1,321,206	1,700,132	1,971,303	2,298,349	2,645,113	3,003,425			
4	808,864	1,029,523	1,229,626	1,590,338	1,842,662	2,150,351				
5	1,016,862	1,251,420	1,698,052	2,105,143	2,385,339					
6	948,312	1,108,791	1,315,524	1,487,577						
7	917,530	1,082,426	1,484,405							
8	1,001,238	1,376,124								
9	841,930									

Table 1: Observed claims payments data $P_{i,j}$, $i + j \le J$.

We first need to determine $J^* \le J$. We choose the value J^* such that there is no substantial claims incurred development after development period J^*. We therefore look at the individual chain ladder factors $I_{i,j+1}/I_{i,j}$, $j \ge 0$ and $i + j + 1 \le J$. These are provided in Table 3. In the upper right triangle in Table 3 (with the individual chain ladder factors for years 6, 7, 8) we see no further systematic development, therefore we do a case study for the choices $J^* \in \{6, \ldots, 9\}$. Choice of standard deviation parameters s_j, σ_j and τ_j: these standard deviation parameters should be determined with prior knowledge only. In our example we assume that we have

	0	1	2	3	4	5	6	7	8	9
0	3,362,115	5,217,243	4,754,900	4,381,677	4,136,883	4,094,140	4,018,736	4,001,591	4,001,391	4,001,258
1	2,640,443	4,643,860	3,869,954	3,248,558	3,102,002	3,019,980	2,976,064	2,966,941	2,959,955	
2	2,879,697	4,785,531	4,045,448	3,467,822	3,377,540	3,341,934	3,283,928	3,287,827		
3	2,933,345	5,299,146	4,451,963	3,700,809	3,553,391	3,469,505	3,413,921			
4	2,768,181	4,658,933	3,936,455	3,512,735	3,385,129	3,298,998				
5	3,228,439	5,271,304	4,484,946	3,798,384	3,702,427					
6	2,927,033	5,067,768	4,066,526	3,704,113						
7	3,083,429	4,790,944	4,408,097							
8	2,761,163	4,132,757								
9	3,045,376									

Table 2: Observed claims incurred data $I_{i,j}$, $i+j \leq J$.

	0	1	2	3	4	5	6	7	8	9
0	1.5518	0.9114	0.9215	0.9441	0.9897	0.9816	0.9957	1.0000	1.0000	
1	1.7587	0.8333	0.8394	0.9549	0.9736	0.9855	0.9969	0.9976		
2	1.6618	0.8453	0.8572	0.9740	0.9895	0.9826	1.0012			
3	1.8065	0.8401	0.8313	0.9602	0.9764	0.9840				
4	1.6830	0.8449	0.8924	0.9637	0.9746					
5	1.6328	0.8508	0.8469	0.9747						
6	1.7314	0.8024	0.9109							
7	1.5538	0.9201								
8	1.4967									
9										
average	1.6529	0.8561	0.8714	0.9619	0.9807	0.9834	0.9980	0.9988	1.0000	

Table 3: Individual chain ladder factors $I_{i,j+1}/I_{i,j}$ for $j \geq 0$ and $i+j+1 \leq J$.

non-informative priors, which means that we set $s_j = \infty$. For σ_j and τ_j we take an empirical Bayesian point of view and estimate them from the data: For $j = 0, \ldots, J-1$ we set

$$\widehat{\sigma}_j^2 = \frac{1}{J-j} \sum_{i=0}^{J-j} \left(\log \frac{P_{i,j}}{P_{i,j-1}} - \overline{\phi_j} \right)^2 .$$

Unfortunately, σ_J and σ_{J+1} cannot be estimated from the data, because we do not have sufficiently many observations. Therefore, we do the ad hoc choice

$$\widehat{\sigma}_{J+1} = \widehat{\sigma}_J = \min \left\{ \widehat{\sigma}_{J-1}, \widehat{\sigma}_{J-2}, \widehat{\sigma}_{J-1}^2 / \widehat{\sigma}_{J-2} \right\} .$$

We estimate the parameter $\tau = \tau_J^* = \ldots = \tau_{J-1}$ with the empirical standard deviation of $\log I_{i,j+1}/I_{i,j}$ for $i+j+1 \leq J$ and $j \geq 6$ (because we assume that there is no systematic claims incurred development after development period 6, see Table 3). Finally, for τ_J we do the ad hoc choice $\tau_J^2 = 3\ \tau_{J-1}^2$ which suggests that we have approximately another 3 development periods beyond $J = 9$ until the claims are finally settled. This provides the standard deviation parameters given in Table 4. Now, we are ready to calculate the claims reserves and the corresponding

	0	1	2	3	4	5	6	7	8	9	10
$\widehat{\sigma}_j$	0.1393	0.0650	0.0731	0.0640	0.0264	0.0271	0.0405	0.0227	0.0494	0.0227	0.0227
$\widehat{\tau}_j$							0.0021	0.0021	0.0021	0.0037	

Table 4: Estimated $\widehat{\sigma}_j$ for $j = 0, \ldots, J+1$, and $\widehat{\tau}_j$ for $j = 6, \ldots, J$.

prediction uncertainty in our model according to Proposition 3.1 and Theorem 3.2. We do this for $J^* = 6, \ldots, 9$. The results are provided in Table 5.

	reserves	msep$^{1/2}$	reserves	msep$^{1/2}$	reserves	msep$^{1/2}$	reserves	msep$^{1/2}$	reserves	msep$^{1/2}$
	Hertig's model [4]		$J^* = 9$		$J^* = 8$		$J^* = 7$		$J^* = 6$	
	no tail factor		PIC tail factor		PIC tail factor		PIC tail factor		PIC tail factor	
0	0	0	180,054	14,652	182,752	14,599	182,024	14,594	181,551	14,590
1	47,060	83,995	171,647	124,884	391,633	12,439	390,918	12,433	390,454	12,428
2	336,189	241,482	503,888	279,793	701,497	276,256	107,490	15,517	106,616	15,505
3	549,682	261,129	719,020	299,020	918,561	297,415	673,923	263,493	411,103	17,629
4	655,906	242,377	789,650	271,269	947,248	273,746	754,032	246,311	613,774	221,380
5	1,190,955	326,696	1,361,399	363,250	1,562,242	368,106	1,316,008	332,649	1,137,263	300,944
6	1,115,656	249,249	1,239,724	275,751	1,385,920	280,339	1,206,683	254,165	1,076,573	231,061
7	1,611,611	365,019	1,759,165	396,734	1,933,036	407,990	1,719,870	374,105	1,565,129	345,667
8	2,310,950	521,674	2,486,673	560,910	2,693,737	580,909	2,439,876	536,249	2,255,594	500,075
9	1,954,075	440,471	2,087,331	471,323	2,244,354	489,676	2,051,844	453,365	1,912,098	424,462
tot	9,772,084	1,519,464	11,298,552	1,747,672	12,960,980	1,624,873	10,842,668	1,292,329	9,650,155	1,022,505

Table 5: Estimated claims reserves and corresponding prediction standard deviation in the PIC tail development factor model (Model Assumptions 2.2) for $J^* = 6, \ldots, 9$, and the estimated claims reserves according to Hertig's model [4] (see Section 3.1 in Merz-Wüthrich [5]) without tail development factor.

Interpretations.

- Hertig's model [4] substantially underestimates the outstanding loss liabilities compared to the PIC tail development factor model for $J^* = 9, 8, 7$. Only the PIC tail development factor model for $J^* = 6$ gives similar reserves. This comes from the fact that the incurred development factors still give a downward trend to incurred losses in development period 6 and 7, see average in Table 3. This observation suggests to choose $J^* = 8$ or 9.

- Including tail development factors for $J^* = 8, 9$ also gives a higher prediction uncertainty msep$^{1/2}$ compared to Hertig's model [4] without tail development factors. Henceforth, this analysis suggests to use the PIC tail development factor model for $J^* = 8, 9$, otherwise both the claims reserves and the prediction uncertainty are underestimated.

- Note that for $J^* = 9$ we simultaneously consider claims payments and claims incurred information for accident year $i = 0$. For $J^* = 8$ we simultaneously consider claims payments and claims incurred information for accident years $i = 0, 1$. This results in these accident years in a much lower prediction uncertainty (above the horizontal line in the corresponding columns of Table 5). The reason therefore is that the claims incurred information has only little uncertainty (since we assume Ψ_j to be constant for $j \geq J^*$). This substantially reduces the prediction uncertainty.

5 Conclusion

We have modified the PIC reserving model from Merz-Wüthrich [5] so that it allows for the incorporation of tail development factors. These tail development factors are estimated considering claims incurred-paid ratios in an appropriate way. This extends the ad hoc methods used in practice and because we perform our analysis in a mathematically consistent way we also obtain formulas for the prediction uncertainty. These are obtained analytically for the conditional MSEP and these can be obtained numerically for other uncertainty measures using Monte Carlo simulations (because we work in a Bayesian setup). The case study highlights that one needs to incorporate tail development factors otherwise both the outstanding loss liabilities and the prediction uncertainty are underestimated.

A Appendix: Proofs

In this appendix we prove all the statements. We start with a well-known result for multivariate Gaussian distributions, see e.g. Appendix A in Posthuma et al. [7]:

Lemma A.1 *Assume $(X_1, \ldots, X_n)'$ is multivariate Gaussian distributed with mean $(m_1, \ldots, m_n)'$ and positive definite covariance matrix Σ. Then we have for the conditional distribution*

$$X_1 \mid_{\{X_2, \ldots, X_n\}} \sim \mathcal{N}\left(m_1 + \Sigma_{1,2}\Sigma_{2,2}^{-1}\left(X^{(2)} - m^{(2)}\right)\ ,\ \Sigma_{1,1} - \Sigma_{1,2}\Sigma_{2,2}^{-1}\Sigma_{2,1}\right),$$

where $X^{(2)} = (X_2, \ldots, X_n)'$ is multivariate Gaussian with mean $m^{(2)} = (m_2, \ldots, m_n)'$ and positive definite covariance matrix $\Sigma_{2,2}$, $\Sigma_{1,1}$ is the variance of X_1 and $\Sigma_{1,2} = \Sigma_{2,1}'$ is the covariance vector between X_1 and $X^{(2)}$.

Proof of Theorem 2.1. We first consider the case $i > J - J^*$, that is $I_{i,k} \notin \mathcal{D}_J^*$ for $k = 0, \ldots, J - i$, henceforth for accident years $i > J - J^*$ we do not consider claims incurred information. Using the conditional independence of accident years, given the parameters $\mathbf{\Theta}$, we obtain

$$E\left[P_{i,J+1} \left| \mathcal{D}_J^*, \mathbf{\Theta}\right.\right] = E\left[P_{i,J+1} \left| P_{i,0}, \ldots, P_{i,J-i}, \mathbf{\Theta}\right.\right].$$

Furthermore, $i > J - J^*$ implies $\beta_{J-i} = 0$. Therefore, the claim follows from Model Assumptions 1.1, see also (2.2) in Merz-Wüthrich [5], and because $\beta_j = 0$ for $j < J^*$. Similarly, we obtain for the conditional variance

$$\mathrm{Var}\left(P_{i,J+1} \left| \mathcal{D}_J^*, \mathbf{\Theta}\right.\right) = E\left[P_{i,J+1} \left| \mathcal{D}_J^*, \mathbf{\Theta}\right.\right]^2 \left(\exp\left\{ \sum_{l=J-i+1}^{J+1} \sigma_l^2 \right\} - 1 \right).$$

The case $i \leq J - J^*$ is more involved. Using again the independence of accident years conditional on $\mathbf{\Theta}$ we obtain

$$E\left[P_{i,J+1} \left| \mathcal{D}_J^*, \mathbf{\Theta}\right.\right] = E\left[P_{i,J+1} \left| P_{i,0}, \ldots, P_{i,J-i}, I_{i,J^*}, \ldots, I_{i,J-i}, \mathbf{\Theta}\right.\right],$$

henceforth, we now have both claims payments and claims incurred observations for accident year $i \leq J - J^*$. We set $j = J - i$, then using Lemma A.1 we obtain completely analogous to Theorem 2.4 and Corollary 2.5 in Merz-Wüthrich [5]

$$\begin{aligned} E&\left[P_{i,J+1}\right| \mathcal{D}_J^*, \mathbf{\Theta}] \\ &= \exp\left\{\eta_{J+1} + (1-\beta_j)(\log P_{i,j} - \eta_j) + \beta_j(\log I_{i,j} - \mu_j) + (1-\beta_j)(w_{J+1}^2 - w_j^2)/2\right\} \\ &= P_{i,j}^{1-\beta_j}\ I_{i,j}^{\beta_j}\ \exp\left\{(1-\beta_j) \sum_{l=j+1}^{J+1} \left(\Phi_l + \sigma_l^2/2\right) + \beta_j \sum_{l=j}^{J} \Psi_l \right\}. \end{aligned}$$

Analogously, Theorem 2.4 from Merz-Wüthrich [5] implies for the variance

$$\mathrm{Var}\left(P_{i,J+1} \left| \mathcal{D}_J^*, \mathbf{\Theta}\right.\right) = E\left[P_{i,J+1} \left| \mathcal{D}_J^*, \mathbf{\Theta}\right.\right]^2 \left(\exp\left\{ (1-\beta_j) \sum_{l=j+1}^{J+1} \sigma_l^2 \right\} - 1 \right).$$

This proves the theorem.

□

Proof of Theorem 2.3 and Corollary 2.4. We first write all the relevant terms of the likelihood of $\mathbf{\Phi}$, given $\mathcal{D}_J^*$. They are given by

$$
\begin{aligned}
u\left(\mathbf{\Phi}\left|\mathcal{D}_J^*\right.\right) \quad \propto \quad & \prod_{j=0}^{J^*} \exp\left\{-\frac{1}{2s_j^2}\left(\Phi_j-\phi_j\right)^2-\frac{1}{2\sigma_j^2}\sum_{i=0}^{J-j}\left(\Phi_j-\log\frac{P_{i,j}}{P_{i,j-1}}\right)^2\right\} \\
& \times \prod_{j=J^*+1}^{J} \exp\left\{-\frac{1}{2s_j^2}\left(\Phi_j-\phi_j\right)^2-\frac{1}{2\sigma_j^2}\sum_{i=0}^{J-j}\left(\Phi_j-\log\frac{P_{i,j}}{P_{i,j-1}}\right)^2\right\} \\
& \times \exp\left\{-\frac{1}{2s_{J+1}^2}\left(\Phi_{J+1}-\phi_{J+1}\right)^2\right\} \qquad\qquad \text{(A.1)} \\
& \times \prod_{i=0}^{J-J^*} \exp\left\{-\frac{1}{2(v_{J-i}^2-w_{J-i}^2)}\left(\sum_{m=J-i+1}^{J+1}\Phi_m-\frac{i\;\tau^2+\tau_J^2}{2}-\log\frac{I_{i,J-i}}{P_{i,J-i}}\right)^2\right\}.
\end{aligned}
$$

From this we easily see that the posterior distribution of $\mathbf{\Phi}$, given $\mathcal{D}_J^*$, is again multivariate Gaussian and there only remains to determine the posterior mean and covariance matrix. If we square out all terms in (A.1) for obtaining the Φ_j^2 and the $\Phi_j\Phi_n$ terms we find the covariance matrix $\Sigma(\mathcal{D}_J^*)$. First of all, we observe that the development periods with $j \le J^*$ are all on the first line of (A.1) which proves the independence statement on $\Phi_0,\ldots,\Phi_{J^*},(\Phi_{J^*+1},\ldots,\Phi_{J+1})$. Moreover, we see for $j \le J^*$ that the posterior variance of Φ_j, given $\mathcal{D}_J^*$, is given by

$$
s_j^{post} = \left(s_j^{-2}+(J-j+1)\sigma_j^{-2}\right)^{-1/2},
$$

this provides $a_{n,m}$ for $n,m=0,\ldots,J^*$. The posterior mean is given by

$$
\phi_j^{post} \quad = \quad (s_j^{post})^2\left(\frac{\phi_j}{s_j^2}+\frac{1}{\sigma_j^2}\sum_{i=0}^{J-j}\log\frac{P_{i,j}}{P_{i,j-1}}\right).
$$

Next, we square out all terms for $j > J^*$ to get the covariance matrix. We obtain

$$
\begin{aligned}
&\sum_{n=J^*+1}^{J+1}\left(\frac{1}{s_n^2}+\frac{J-n+1}{\sigma_n^2}\right)\Phi_n^2+\sum_{n,m=J^*+1}^{J+1}\Phi_n\Phi_m\sum_{i=(J-n+1)\vee(J-m+1)}^{J-J^*}\left(v_{J-i}^2-w_{J-i}^2\right)^{-1} \\
&= \sum_{n=J^*+1}^{J+1}\left(\frac{1}{s_n^2}+\frac{J-n+1}{\sigma_n^2}\right)\Phi_n^2+\sum_{n,m=J^*+1}^{J+1}\Phi_n\Phi_m\sum_{i=J^*}^{(n-1)\wedge(m-1)}\left(v_i^2-w_i^2\right)^{-1}.
\end{aligned}
$$

This provides $a_{n,m}$ for $n,m=J^*+1,\ldots,J+1$. The posterior mean is obtained by solving the posterior maximum likelihood functions for Φ_j, $j \ge J^*+1$. They are given by

$$
\begin{aligned}
\frac{\partial \log u\left(\mathbf{\Phi}\right|\mathcal{D}_J^*)}{\partial \Phi_j} \quad = \quad & \frac{\phi_j}{s_j^2}+\frac{1}{\sigma_j^2}\sum_{i=0}^{J-j}\log\frac{P_{i,j}}{P_{i,j-1}}+\sum_{i=J-j+1}^{J-J^*}\frac{\frac{i\;\tau^2+\tau_J^2}{2}+\log\frac{I_{i,J-i}}{P_{i,J-i}}}{v_{J-i}^2-w_{J-i}^2} \\
& -\sum_{m=J^*+1}^{J+1}\Phi_m\; a_{j,m} \;\overset{!}{=}\; 0. \qquad\qquad \text{(A.2)}
\end{aligned}
$$

Henceforth, this implies

$$(c_0, \dots, c_{J+1})' = \Sigma(\mathcal{D}_J^*)^{-1} (\Phi_0, \dots, \Phi_{J+1})',$$

from which the claim follows.

$\square$

Proof of Corollary 2.5. The corollary follows from Theorem 2.3 and Corollary 2.4.

$\square$

Proof of Proposition 3.1. From Theorem 2.1 we obtain

$$\begin{aligned}
E\left[P_{i,J+1} \left| \mathcal{D}_J^*\right.\right] &= E\left[E\left[P_{i,J+1} \left| \mathcal{D}_J^*, \mathbf{\Theta}\right.\right] \left| \mathcal{D}_J^*\right.\right] \\
&= P_{i,J-i}^{1-\beta_{J-i}} \; I_{i,J-i}^{\beta_{J-i}} \; E\left[\exp\left\{(1-\beta_{J-i}) \sum_{l=J-i+1}^{J+1} \left(\Phi_l + \sigma_l^2/2\right) + \beta_{J-i} \; \frac{i\;\tau^2 + \tau_J^2}{2}\right\}\right| \mathcal{D}_J^*\Bigg] \\
&= P_{i,J-i}^{1-\beta_{J-i}} \; I_{i,J-i}^{\beta_{J-i}} \; \exp\left\{(1-\beta_{J-i}) \sum_{l=J-i+1}^{J+1} \frac{\sigma_l^2}{2} + \beta_{J-i} \; \frac{i\;\tau^2 + \tau_J^2}{2}\right\} \\
&\qquad\qquad \times \; E\left[\exp\left\{(1-\beta_{J-i}) \sum_{l=J-i+1}^{J+1} \Phi_l\right\}\right| \mathcal{D}_J^*\Bigg].
\end{aligned}$$

From Theorem 2.3 we know that, given $\mathcal{D}_I^*$, $\mathbf{\Phi} = (\Phi_0, \dots, \Phi_{J+1})$ has a posterior multivariate Gaussian distribution with posterior mean $(\phi_0^{post}, \dots, \phi_{J+1}^{post})$ and posterior covariance matrix $\Sigma(\mathcal{D}_J^*)$. Henceforth, the posterior distribution of $\sum_{j=J-i+1}^{J+1} \Phi_j$ is Gaussian with mean $\sum_{j=J-i+1}^{J+1} \phi_j^{post}$ and variance $\mathbf{e}_{J-i+1}' \Sigma(\mathcal{D}_J^*) \mathbf{e}_{J-i+1}$. This proves the proposition.

$\square$

Proof of Theorem 3.2. We obtain with the tower property of conditional expectations

$$\begin{aligned}
&\text{Cov}\left(P_{i,J+1}, P_{k,J+1} \right| \mathcal{D}_J^*) \qquad\qquad (\text{A.3}) \\
&\quad = E\left[\text{Cov}\left(P_{i,J+1}, P_{k,J+1} \right| \mathcal{D}_J^*, \mathbf{\Theta}) \right| \mathcal{D}_J^*] + \text{Cov}\left(E\left[P_{i,J+1} \right| \mathcal{D}_J^*, \mathbf{\Theta}\right], E\left[P_{k,J+1} \right| \mathcal{D}_J^*, \mathbf{\Theta}\right] \right| \mathcal{D}_J^*).
\end{aligned}$$

This is the usual decomposition into average process (co-)variance and average parameter error. The first term in (A.3) is equal to 0 for $i \neq k$, because accident years i are independent, conditionally given $\mathbf{\Theta}$. Henceforth there remains the case $i = k$. Using Theorems 2.1 and 2.3

we obtain

$$
\begin{aligned}
&E\left[\operatorname{Var}\left(P_{i,J+1}\middle|\,\mathcal{D}_J^*,\mathbf{\Theta}\right)\middle|\,\mathcal{D}_J^*\right]\\
&\quad= E\left[E\left[P_{i,J+1}\left|\mathcal{D}_J^*,\mathbf{\Theta}\right]^2\right|\mathcal{D}_J^*\right]\left(\exp\left\{(1-\beta_{J-i})\sum_{l=J-i+1}^{J+1}\sigma_l^2\right\}-1\right)\\
&\quad= P_{i,J-i}^{2(1-\beta_{J-i})}\ I_{i,J-i}^{2\beta_{J-i}}\ \exp\left\{(1-\beta_{J-i})\sum_{l=J-i+1}^{J+1}\sigma_l^2+\beta_{J-i}\ \left(i\ \tau^2+\tau_J^2\right)\right\}\\
&\qquad\times\ E\left[\exp\left\{2(1-\beta_{J-i})\sum_{l=J-i+1}^{J+1}\Phi_l\right\}\middle|\,\mathcal{D}_J^*\right]\left(\exp\left\{(1-\beta_{J-i})\sum_{l=J-i+1}^{J+1}\sigma_l^2\right\}-1\right).
\end{aligned}
$$

From Theorem 2.3 we know that, given $\mathcal{D}_I^*$, $\mathbf{\Phi}=(\Phi_0,\ldots,\Phi_{J+1})$ has a posterior multivariate Gaussian distribution with posterior mean $(\phi_0^{post},\ldots,\phi_{J+1}^{post})$ and posterior covariance matrix $\Sigma(\mathcal{D}_J^*)$. Henceforth, the posterior distribution of $\sum_{j=J-i+1}^{J+1}\Phi_j$ is Gaussian with mean $\sum_{j=J-i+1}^{J+1}\phi_j^{post}$ and variance $\mathbf{e}_{J-i+1}'\Sigma(\mathcal{D}_J^*)\mathbf{e}_{J-i+1}$. This implies for the first term (A.3)

$$
\begin{aligned}
&E\left[\operatorname{Var}\left(P_{i,J+1}\middle|\,\mathcal{D}_J^*,\mathbf{\Theta}\right)\middle|\,\mathcal{D}_J^*\right]=E\left[P_{i,J+1}\left|\mathcal{D}_J^*\right]^2\right.\\
&\quad\times\exp\left\{(1-\beta_{J-i})^2\mathbf{e}_{J-i+1}'\Sigma(\mathcal{D}_J^*)\mathbf{e}_{J-i+1}\right\}\left(\exp\left\{(1-\beta_{J-i})\sum_{l=J-i+1}^{J+1}\sigma_l^2\right\}-1\right).
\end{aligned}
$$

Finally, we consider the last term in (A.3). Applying Theorems 2.1 and 2.3 we obtain

$$
\begin{aligned}
&\operatorname{Cov}\left(E\left[P_{i,J+1}\middle|\,\mathcal{D}_J^*,\mathbf{\Theta}\right],E\left[P_{k,J+1}\middle|\,\mathcal{D}_J^*,\mathbf{\Theta}\right]\middle|\,\mathcal{D}_J^*\right)\\
&\quad= P_{i,J-i}^{1-\beta_{J-i}}\ I_{i,J-i}^{\beta_{J-i}}\ \exp\left\{(1-\beta_{J-i})\sum_{l=J-i+1}^{J+1}\frac{\sigma_l^2}{2}+\beta_{J-i}\ \frac{i\ \tau^2+\tau_J^2}{2}\right\}\\
&\qquad\times P_{k,J-k}^{1-\beta_{J-k}}\ I_{k,J-k}^{\beta_{J-k}}\ \exp\left\{(1-\beta_{J-k})\sum_{l=J-k+1}^{J+1}\frac{\sigma_l^2}{2}+\beta_{J-k}\ \frac{k\ \tau^2+\tau_J^2}{2}\right\}\\
&\qquad\times\operatorname{Cov}\left(\exp\left\{(1-\beta_{J-i})\sum_{l=J-i+1}^{J+1}\Phi_l\right\},\exp\left\{(1-\beta_{J-k})\sum_{l=J-k+1}^{J+1}\Phi_l\right\}\middle|\,\mathcal{D}_J^*\right).
\end{aligned}
$$

Henceforth, we need to calculate this last covariance term. Due to Theorem 2.3 the joint distribution of the exponents is a multivariate Gaussian distribution with covariance $(1-\beta_{J-i})(1-\beta_{J-k})\ \mathbf{e}_{J-i+1}'\Sigma(\mathcal{D}_J^*)\mathbf{e}_{J-k+1}$. This implies

$$
\begin{aligned}
&\operatorname{Cov}\left(E\left[P_{i,J+1}\middle|\,\mathcal{D}_J^*,\mathbf{\Theta}\right],E\left[P_{k,J+1}\middle|\,\mathcal{D}_J^*,\mathbf{\Theta}\right]\middle|\,\mathcal{D}_J^*\right)\\
&\quad= E\left[P_{i,J+1}\middle|\,\mathcal{D}_J^*\right]\ E\left[P_{k,J+1}\middle|\,\mathcal{D}_J^*\right]\ \left(\exp\left\{(1-\beta_{J-i})(1-\beta_{J-k})\ \mathbf{e}_{J-i+1}'\Sigma(\mathcal{D}_J^*)\mathbf{e}_{J-k+1}\right\}-1\right),
\end{aligned}
$$

which is the well-known covariance formula for log-normal distributions. Collecting the terms for $i \neq k$ gives the off-diagonal terms. For $i = k$ we obtain the terms

$$
\begin{aligned}
& E\left[P_{i,J+1}\left|\mathcal{D}_J^*\right]^2 \exp\left\{(1-\beta_{J-i})^2 \mathbf{e}'_{J-i+1}\Sigma(\mathcal{D}_J^*)\mathbf{e}_{J-i+1}\right\} \left(\exp\left\{(1-\beta_{J-i})\sum_{l=J-i+1}^{J+1}\sigma_l^2\right\}-1\right) \\
&\quad + E\left[P_{i,J+1}\right|\mathcal{D}_J^*\right]^2 \left(\exp\left\{(1-\beta_{J-i})^2\, \mathbf{e}'_{J-i+1}\Sigma(\mathcal{D}_J^*)\mathbf{e}_{J-i+1}\right\}-1\right) \\
&\quad = E\left[P_{i,J+1}\left|\mathcal{D}_J^*\right]^2 \left(\exp\left\{(1-\beta_{J-i})^2 \mathbf{e}'_{J-i+1}\Sigma(\mathcal{D}_J^*)\mathbf{e}_{J-i+1} + (1-\beta_{J-i})\sum_{l=J-i+1}^{J+1}\sigma_l^2\right\}-1\right).
\end{aligned}
$$

This completes the proof.

$\square$

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