Claims Development Result in the Paid-Incurred Chain Reserving Method

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Abstract

We present the one-year claims development result (CDR) in the paid-incurred chain (PIC) reserving model. The PIC reserving model presented in Merz-Wüthrich [7] is a Bayesian stochastic claims reserving model that considers simultaneously claims payments and incurred losses information and allows for deriving the full predictive distribution of the outstanding loss liabilities. In this model we study the conditional mean square error of prediction (MSEP) for the one-year CDR uncertainty, which is the crucial uncertainty view under Solvency II.

Keywords: stochastic claims reserving, PIC method, outstanding loss liabilities, claims payments, incurred losses, prediction uncertainty, conditional mean square error, claims development result, solvency.

1 Introduction

A non-life insurance company needs to hold sufficient claims reserves (provisions) on its balance sheet in order to meet the outstanding loss liabilities. Therefore, a main task of the actuary in non-life insurance is to predict ultimate loss ratios and outstanding loss liabilities. For these predictions he often has different sources of information and the major difficulty is to combine these information channels appropriately.

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In the present paper we combine claims paid data and claims incurred data (case estimates for reported claims) to get a unified prediction for the outstanding loss liabilities. A well known method to combine claims paid data and claims incurred data for claims reserving is the Munich chain ladder (MCL) method introduced in Quarg-Mack [10]. However, to the best of our knowledge, there is no way to quantify the prediction uncertainty within the MCL method. Another approach was presented in Dahms [3]. Dahms [3] extended the complementary loss ratio (CLR) method for deriving unified predictions based on claims paid data and claims incurred data simultanously. Unlike the MCL method, the CLR method allows for the derivation of a mean square error of prediction (MSEP) estimate. A recent new approach is the paid-incurred (PIC) reserving method introduced in Posthuma et al. [9] and Merz-Wüthrich [7]. The PIC method was defined in a Bayesian framework and therefore allows for the derivation of the full predictive distribution for the outstanding loss liabilities. This means that within the Bayesian PIC model one is not only able to calculate the MSEP but one can also calculate any other risk measure, like Value-at-Risk or expected shortfall for the prediction uncertainty.

Under the new solvency regulations, such as Solvency II, the so-called one-year claims development result (CDR) is of central interest because it corresponds to a profit & loss statement position that directly influences the financial strength of an insurance company. The one-year CDR is defined as the difference between the prediction of the outstanding loss liabilities today and in one year's time (cf. Merz-Wüthrich [6]). This means that the one-year CDR measures the change in the expected outstanding loss liabilities over a one-year time horizon. Due to Solvency II, this one-year view has attracted a lot of attention in recent research. For references, we refer to Ohlsson-Lauzeningks [8], Merz-Wüthrich [6] and Bühlmann et al. [2]. Dahms et al. [4] analyze the one-year CDR in the framework of the CLR method, which is probably the first one-year CDR uncertainty analysis for combined claims paid and claims incurred data.

In the present paper we revisit the PIC method within this one-year solvency framework. This means that we consider the one-year CDR for the PIC reserving method. We are able to calculate the conditional MSEP for the one-year CDR and we can also derive the full predictive distribution of the one-year CDR via Monte-Carlo simulations.

Organization of the paper: In Section 2 we recapitulate the assumptions of the PIC model. The definition of the one-year CDR is given in Section 3. We then derive the best estimate of the ultimate claim, based on the paid and incurred data in one year, see Section 4. In Section 5.1 we split this best estimate in an appropiate way and derive the conditional MSEP of the one-year CDR for single accident years. In Section 5.2 we proceed with the conditional MSEP for aggregated accident years which provides the overall one-year CDR uncertainty. Finally, in Section 6 we present an example and compare it to the results derived in Dahms et al. [4] for the CLR method. Additionally, we provide the full predictive distribution of the one-year CDR via Monte-Carlo simulations. All proofs are provided in the Appendix.

2 Notation and Model Assumptions

The PIC reserving model combines two channels of information: i) claims payments, which correspond to the payments for reported claims; ii) incurred losses, which refer to the reported claim amounts. Claims payments and incurred losses data are usually aggregated in so-called claims development triangles:

In the following, we denote accident years by $i \in \{0, \ldots, J\}$ and development years by $j \in \{0, \ldots, J\}$. Cumulative payments in accident year i after j development years are denoted by $P_{i,j}$ and the corresponding incurred losses by $I_{i,j}$. We assume that all claims are settled and closed after development year J, i.e. $P_{i,J} = I_{i,J}$ holds with probability 1 for all $i \in \{0, \ldots, J\}$. After accounting year $t = J$ we have observations in the paid and incurred triangles given by (see Figure 1)

$$\mathcal{D}_J = \{P_{i,j}, I_{i,j};\ 0 \le i \le J,\ 0 \le j \le J,\ 0 \le i + j \le J\},$$

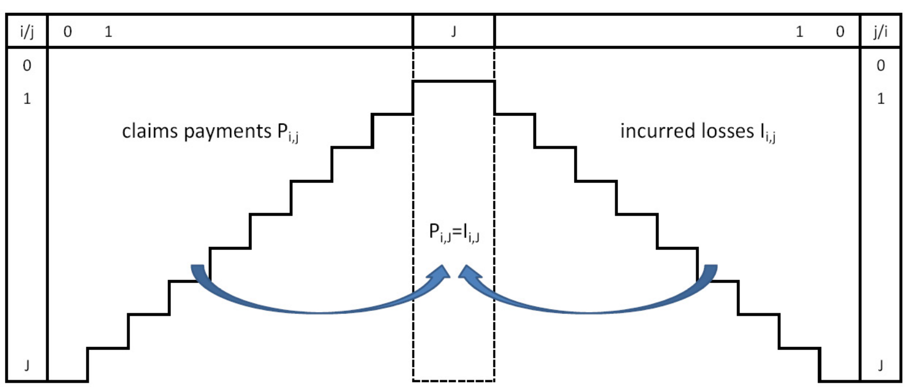

Figure 1: Cumulative claims payments $P_{i,j}$ and incurred losses $I_{i,j}$ observed after accounting year $t = J$ both leading to the ultimate loss $P_{i,J} = I_{i,J}$.

and after accounting year $t = J + 1$ we have observations in the paid and incurred trapezoids given by (see Figure 2)

$$\mathcal{D}_{J+1} = \{P_{i,j}, I_{i,j};\ 0 \le i \le J,\ 0 \le j \le J,\ 0 \le i + j \le J + 1\}.$$

This means the update of information $\mathcal{D}_J \mapsto \mathcal{D}_{J+1}$ adds a new diagonal to the observations. Our goal is to predict the ultimate losses $P_{i,J} = I_{i,J}$, $i = 1, \ldots, J$, based on the information $\mathcal{D}_J$ and $\mathcal{D}_{J+1}$, respectively.

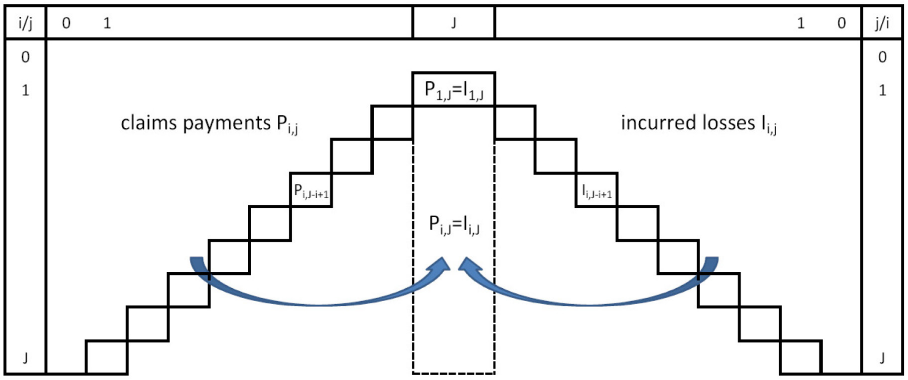

Figure 2: Cumulative claims payments $P_{i,j}$ and incurred losses $I_{i,j}$ observed after accounting year $t = J + 1$ both leading to the ultimate loss $P_{i,J} = I_{i,J}$.

We define the log-normal PIC model, which combines both cumulative payments and incurred losses information:

Model Assumptions 2.1 (log-normal PIC model)

- Conditionally, given the parameter $\Theta = (\Phi_0; \Phi_1, \Psi_1, \Phi_2, \Psi_2, \ldots, \Phi_J, \Psi_J)'$, we assume:
  - the random vectors $\Xi_i = (\xi_{i,0}; \xi_{i,1}, \zeta_{i,1}, \xi_{i,2}, \zeta_{i,2}, \ldots, \xi_{i,J}, \zeta_{i,J})'$ are i.i.d. with multivariate Gaussian distribution

$$\Xi_i \sim \mathcal{N}(\Theta, V) \qquad \text{for} \quad i \in \{0, 1, \ldots, J\},$$

    with positive definite covariance matrix V and individual development factors

$$\xi_{i,j} = \log \frac{P_{i,j}}{P_{i,j-1}} \qquad \text{and} \qquad \zeta_{i,l} = \log \frac{I_{i,l}}{I_{i,l-1}},$$

    for $j \in \{0, 1, \ldots, J\}$ and $l \in \{1, 2, \ldots, J\}$, where we have set $P_{i,-1} = 1$;
  - $P_{i,J} = I_{i,J}$, $\mathbb{P}$-a.s., for all $i = 0, 1, \ldots, J$.
- The components of Θ are independent with prior distributions

$$\Phi_j \sim \mathcal{N}(\phi_j, s_j^2) \quad \text{for } j \in \{0, \ldots, J\} \qquad \text{and} \qquad \Psi_l \sim \mathcal{N}(\psi_l, t_l^2) \quad \text{for } l \in \{1, \ldots, J\},$$

  with prior parameters ϕ_j, $\psi_l \in \mathbb{R}$ and $s_j^2 > 0$, $t_l^2 > 0$.

3 One-year Claims Development Result

In this paper we consider the short term (one-year) run-off risk described in Merz-Wüthrich [6]. This means, we study the uncertainty in the one-year CDR for accounting year $J+1$ given by

$$\mathrm{CDR}_i = \mathrm{CDR}_i(J+1) = \mathbb{E}\left[P_{i,J}\middle|\,\mathcal{D}_J\right] - \mathbb{E}\left[P_{i,J}\middle|\,\mathcal{D}_{J+1}\right], \quad i = 1, \ldots, J,$$

between the best estimates for the ultimate claim $P_{i,J}$ at times J and $J+1$.

The one-year CDR in accounting year $J+1$ measures the change in the prediction by updating the information from $\mathcal{D}_J$ to $\mathcal{D}_{J+1}$. With the tower property of the conditional expectation we obtain for the expected one-year CDR for accident year i, viewed from time J,

$$\mathbb{E}\left[\mathrm{CDR}_i\middle|\,\mathcal{D}_J\right] = 0,$$

which is the martingale property of successive predictions. This justifies the fact that, in the budget statement, the one-year CDR is usually predicted by 0 at time J. In the following we study the uncertainty in this prediction by means of the conditional MSEP, given the observations $\mathcal{D}_J$. In other words we calculate, see Wüthrich-Merz [11], Section 3.1,

$$\mathrm{msep}_{\mathrm{CDR}_i|\mathcal{D}_J}(0) = \mathbb{E}\left[\left(\mathrm{CDR}_i - 0\right)^2\middle|\mathcal{D}_J\right] = \mathrm{Var}\left(\mathrm{CDR}_i|\mathcal{D}_J\right) = \mathrm{Var}\left(\mathbb{E}\left[P_{i,J}|\mathcal{D}_{J+1}\right]|\mathcal{D}_J\right). \tag{1}$$

The conditional MSEP is probably the most popular uncertainty measure in claims reserving practice and has the advantage that it can be derived analytically in the PIC model. However, we also present the full predictive distribution below, which also allows to evaluate other uncertainty measures.

4 Expected Ultimate Claim at time $J+1$

In this section we derive the conditional expected ultimate claim $\mathbb{E}[P_{i,J}|\mathcal{D}_k]$ for $k \in \{J, J+1\}$ in two steps. In the first step we derive $\mathbb{E}[P_{i,J}|\Theta, \mathcal{D}_k]$, and in the second step we calculate $\mathbb{E}[P_{i,J}|\mathcal{D}_k]$, see Corollary 4.4.

In the following we can either work with the random vector $\Xi_i \in \mathbb{R}^{2J+1}$ (see Model Assumptions 2.1) or with the logarithmized observations of accident year i, namely,

$$X_i = (\log P_{i,0}, \log I_{i,0}, \log P_{i,1}, \ldots, \log P_{i,J-1}, \log I_{i,J-1}, \log P_{i,J})' \in \mathbb{R}^{2J+1}.$$

This is possible, since there exist an invertible matrix $B \in \mathbb{R}^{(2J+1)\times(2J+1)}$ such that $X_i = B\ \Xi_i$, i.e. there is a one-to-one correspondence between X_i and Ξ_i. This implies

$$X_i|_{\{\Theta\}} = B\ \Xi_i|_{\{\Theta\}} \sim \mathcal{N}(\mu = B\Theta, \Sigma = BVB').$$

Let $k \in \{J, J+1\}$ and define $n = 2J+1$ and $q = q_k(i) = 2(k-i+1)$. To simplify notation we define:

$$X_{i,k}^{(1)} = \begin{cases} (\log P_{i,0}, \log I_{i,0}, \log P_{i,1}, \log I_{i,1} \ldots, \log P_{i,k-i}, \log I_{i,k-i})' \in \mathbb{R}^q & \text{for } k-i<J, \\ X_i & \text{otherwise;} \end{cases}$$

$$X_{i,k}^{(2)} = \begin{cases} (\log P_{i,k-i+1}, \log I_{i,k-i+1}, \ldots, \log P_{i,J-1}, \log I_{i,J-1}, \log P_{i,J})' \in \mathbb{R}^{n-q} & \text{for } k-i<J, \\ \log(P_{i,J}) & \text{otherwise.} \end{cases}$$

$X_{i,k}^{(1)}$ describes the observations at time $k \in \{J, J+1\}$, i.e. it corresponds to the σ-field generated by $\mathcal{D}_k$, see Figures 1 and 2. $X_{i,k}^{(2)}$ is the part of claims development that needs to be predicted at time k for $i > k-J$.

We decompose the transformation matrix B in a similar way into

$$B = \begin{pmatrix} B_{i,k}^{(1)} \\ B_{i,k}^{(2)} \end{pmatrix} \quad \text{for} \quad k-i<J,$$

where $B_{i,k}^{(1)} \in \mathbb{R}^{q\times n}$. For $k-i \geq J$ we set $B_{i,k}^{(1)} = B$ and $B_{i,k}^{(2)} = B_{1,J}^{(2)}$. This provides, for $k-i<J$, a decomposition

$$\mu = B\Theta = \left(\mu_{i,k}^{(1)}, \mu_{i,k}^{(2)}\right)' \in \mathbb{R}^n$$

of the mean vector, where

$$\mu_{i,k}^{(1)} = \mathbb{E}[X_{i,k}^{(1)}|\Theta] = B_{i,k}^{(1)}\Theta \quad \text{and} \quad \mu_{i,k}^{(2)} = \mathbb{E}[X_{i,k}^{(2)}|\Theta] = B_{i,k}^{(2)}\Theta.$$

For $k-i<J$ the covariance matrix is decomposed in a similar way such that

$$\Sigma = BVB' = \begin{pmatrix} \Sigma_{11}^{(i,k)} & \Sigma_{12}^{(i,k)} \\ \Sigma_{21}^{(i,k)} & \Sigma_{22}^{(i,k)} \end{pmatrix} \in \mathbb{R}^{n\times n},$$

with $\Sigma_{11}^{(i,k)} \in \mathbb{R}^{q\times q}$. For $k-i \geq J$ we set $\Sigma_{11}^{(i,k)} = \Sigma$, $\Sigma_{12}^{(i,k)} = \Sigma_{12}^{(1,J)}$ and $\Sigma_{22}^{(i,k)} = \Sigma_{22}^{(1,J)}$.

Now having this notation we provide the following lemma:

Lemma 4.1 *Choose $k \in \{J, J+1\}$ and $i > k - J$. Under Model Assumptions 2.1 we obtain for the conditional distribution of $X_{i,k}^{(2)}$, given $\{\Theta, \mathcal{D}_k\}$,*

$$X_{i,k}^{(2)}|_{\{\Theta,\mathcal{D}_k\}} = X_{i,k}^{(2)}|_{\{\Theta,X_{i,k}^{(1)}\}} \sim \mathcal{N}\left(\tilde{\mu}_{i,k}^{(2)}, \tilde{\Sigma}_{22}^{(i,k)}\right),$$

where

$$\tilde{\mu}_{i,k}^{(2)} = \mu_{i,k}^{(2)} + \Sigma_{21}^{(i,k)}(\Sigma_{11}^{(i,k)})^{-1}\left(X_{i,k}^{(1)} - \mu_{i,k}^{(1)}\right) \in \mathbb{R}^{n-q},$$
$$\tilde{\Sigma}_{22}^{(i,k)} = \Sigma_{22}^{(i,k)} - \Sigma_{21}^{(i,k)}(\Sigma_{11}^{(i,k)})^{-1}\Sigma_{12}^{(i,k)}.$$

For $k = J$ we obtain

$$(\log P_{i,J-i+1}, \log I_{i,J-i+1})|_{\{\Theta,\mathcal{D}_J\}} \sim \mathcal{N}(\mu_i, \Sigma_i) \quad \textit{for} \quad i \in \{2, \ldots, J\},$$

with $(\mu_i)' = E_2\tilde{\mu}_{i,k}^{(2)}$ and $\Sigma_i = E_2\tilde{\Sigma}_{22}^{(i,k)}E_2'$, where $E_2 = E_2^{n-q}$ is the projection matrix on the first two coordinates of the vector of length $n - q$. Moreover, for $i = 1$ we have

$$\log P_{1,J}|_{\{\Theta,\mathcal{D}_J\}} \sim \mathcal{N}\left(\mu_1 = \tilde{\mu}_{1,J}^{(2)}, \Sigma_1 = \tilde{\Sigma}_{22}^{(1,J)}\right).$$

Remarks:

- The first claim in Lemma 4.1 describes the conditional distribution in a multivariate Gaussian model if some of the components are observed. This is a classical result in probability theory using the Schur complement.

- The second claim in Lemma 4.1 is used to derive the distribution of the elements in the next diagonal $\mathcal{D}_{J+1} \backslash \mathcal{D}_J$. This is needed for the calculation of the full predictive distribution of the CDR via Monte-Carlo methods. For details see Section 6.

As a direct consequence of the first claim in Lemma 4.1 we get for the ultimate claim, $i > k - J$,

$$\log I_{i,J}|_{\{\Theta,\mathcal{D}_k\}} = \log P_{i,J}|_{\{\Theta,\mathcal{D}_k\}} \sim \mathcal{N}\left(E_1\tilde{\mu}_{i,k}^{(2)}, E_1\tilde{\Sigma}_{22}^{(i,k)}E_1'\right),$$

where $E_1 = E_1^{n-q}$ is the projection on the last component of the vector of length $n - q$. This immediately implies the following corollary:

Corollary 4.2 *For the predictor of the ultimate claim $P_{i,J}$, given $\{\Theta, \mathcal{D}_J\}$, we obtain for $i > k - J$*

$$\mathbb{E}[P_{i,J}|\Theta, \mathcal{D}_k] = \exp\left\{E_1\tilde{\mu}_{i,k}^{(2)} + E_1\tilde{\Sigma}_{22}^{(i,k)}{E_1}'/2\right\}.$$

We see that the ultimate claim predictor in Corollary 4.2 still depends on Θ, namely through

$$\begin{aligned} E_1 \tilde{\mu}_{i,k}^{(2)} &= E_1 \left(\mu_{i,k}^{(2)} + \Sigma_{21}^{(i,k)} (\Sigma_{11}^{(i,k)})^{-1} \left(X_{i,k}^{(1)} - \mu_{i,k}^{(1)} \right) \right) \\ &= E_1 \left(B_{i,k}^{(2)} \Theta + \Sigma_{21}^{(i,k)} (\Sigma_{11}^{(i,k)})^{-1} \left(X_{i,k}^{(1)} - B_{i,k}^{(1)} \Theta \right) \right) \\ &= \Gamma_{i,k} \Theta + E_1 \Sigma_{21}^{(i,k)} (\Sigma_{11}^{(i,k)})^{-1} X_{i,k}^{(1)}, \end{aligned} \tag{2}$$

where $\Gamma_{i,k}$ is given by

$$\Gamma_{i,k} = E_1 \left(B_{i,k}^{(2)} - \Sigma_{21}^{(i,k)} (\Sigma_{11}^{(i,k)})^{-1} B_{i,k}^{(1)} \right).$$

Our aim now is to calculate the posterior distribution of Θ, conditionally given observations $\mathcal{D}_k$.

The likelihood of the logarithmized observations at time k, given Θ, is given by

$$L_{\mathcal{D}_k}(\Theta) \propto \prod_{i=0}^{J} \exp \left\{ -\frac{1}{2} \left(X_{i,k}^{(1)} - B_{i,k}^{(1)} \Theta \right)' (\Sigma_{11}^{(i,k)})^{-1} \left(X_{i,k}^{(1)} - B_{i,k}^{(1)} \Theta \right) \right\}. \tag{3}$$

With Model Assumptions 2.1 and Bayes' theorem follows that the posterior distribution $u(\Theta|\mathcal{D}_k)$ has the form

$$u(\Theta|\mathcal{D}_k) \propto L_{\mathcal{D}_k}(\Theta) \exp \left\{ -\frac{1}{2} (\Theta - \nu)' T^{-1} (\Theta - \nu) \right\}, \tag{4}$$

with prior mean

$$\nu = (\phi_0; \phi_1, \psi_1, \phi_2, \psi_2, \ldots, \phi_J, \psi_J)' \in \mathbb{R}^n,$$

and prior covariance matrix

$$T = \operatorname{diag}(s_0^2; s_1^2, t_1^2, s_2^2, t_2^2, \ldots, s_J^2, t_J^2) \in \mathbb{R}^{n \times n}.$$

Theorem 4.3 (posterior distribution of Θ)

Under Model Assumptions 2.1 the posterior distribution $u(\Theta|\mathcal{D}_k)$ is a multivariate Gaussian distribution with posterior mean $\nu(\mathcal{D}_k) \in \mathbb{R}^n$ and posterior covariance matrix $T(\mathcal{D}_k)$ with

$$T(\mathcal{D}_k) = \left(T^{-1} + \sum_{i=0}^{J} (B_{i,k}^{(1)})' (\Sigma_{11}^{(i,k)})^{-1} B_{i,k}^{(1)} \right)^{-1},$$

and posterior mean

$$\nu(\mathcal{D}_k) = T(\mathcal{D}_k) \left[T^{-1} \nu + \sum_{i=0}^{J} (B_{i,k}^{(1)})' (\Sigma_{11}^{(i,k)})^{-1} X_{i,k}^{(1)} \right].$$

From (2) we see that the exponent of the predictor given in Corollary 4.2 is a affin-linear function of Θ. Using Theorem 4.3 this implies the following corollary:

Corollary 4.4 (expected ultimate claim, given $\mathcal{D}_k$)
Choose $k \in \{J, J+1\}$. The conditional expected ultimate claim for accident year $i > k - J$, given $\mathcal{D}_k$, is given by

$$\mathbb{E}[P_{i,J}|\mathcal{D}_k] = \exp\left\{\Gamma_{i,k}\nu(\mathcal{D}_k) + \Gamma_{i,k}T(\mathcal{D}_k)(\Gamma_{i,k})'/2 + E_1\Sigma_{21}^{(i,k)}\left(\Sigma_{11}^{(i,k)}\right)^{-1}X_{i,k}^{(1)} + E_1\tilde{\Sigma}_{22}^{(i,k)}E_1'/2\right\}.$$

Remarks:

- For $k = J$ and diagonal covariance matrix V we obtain the same ultimate claim predictor as in Merz-Wüthrich [7].
- For $k = J + 1$ we get the ultimate claim predictor in the case that information $\mathcal{D}_{J+1}$ is available at time $J + 1$. It has a closed formula. This allows for the simulation of the full predictive distribution of the CDR. This is done in detail in Section 6.
- Markov-Chain-Monte-Carlo methods can be applied for other choices of prior distributions to calculate the posterior parameter distribution in Theorem 4.3. For details see Merz-Wüthrich [7].

5 Mean Square Error of Prediction of the CDR

5.1 Single Accident Years

In the last section we have calculated the expected ultimate claim in the PIC reserving model, given the observations $\mathcal{D}_k$ for $k \in \{J, J+1\}$. Our aim now is to calculate the prediction uncertainty in terms of the conditional MSEP. From (1) we see that the problem to derive the conditional MSEP for the one-year CDR is solved by calculating $\text{Var}\left(\mathbb{E}\left[P_{i,J}|\mathcal{D}_{J+1}\right]\middle|\, \mathcal{D}_J\right)$. Since $\mathbb{E}\left[P_{i,J}|\mathcal{D}_J\right]^2$ is given in Corollary 4.4 for $k = J$, this conditional variance can be derived by calculating $\mathbb{E}\left(\mathbb{E}\left[P_{i,J}|\mathcal{D}_{J+1}\right]^2\middle|\mathcal{D}_J\right)$.

We see that for $k = J + 1$ the exponential term from Corollary 4.4, namely,

$$\Gamma_{i,J+1}\nu(\mathcal{D}_{J+1}) + \Gamma_{i,J+1}T(\mathcal{D}_{J+1})(\Gamma_{i,J+1})'/2 + E_1\Sigma_{21}^{(i,J+1)}\left(\Sigma_{11}^{(i,J+1)}\right)^{-1}X_{i,J+1}^{(1)} + E_1\tilde{\Sigma}_{22}^{(i,J+1)}E_1'/2,$$

is affin-linear in the observations $\mathcal{D}_{J+1} \backslash \mathcal{D}_j$ given by

$$Y = (\log P_{1,J}, \log P_{2,J-1}, \log I_{2,J-1}, \ldots, \log P_{J,1}, \log I_{J,1})'.$$

That means that for all $i > 1$ there exist a matrix L_i and a $\mathcal{D}_J$-measurable random variable $g_i(\mathcal{D}_J)$ such that

$$\begin{aligned} &L_i Y + g_i(\mathcal{D}_J) \\ &= \Gamma_{i,J+1}\nu(\mathcal{D}_{J+1}) + \Gamma_{i,J+1} T(\mathcal{D}_{J+1})(\Gamma_{i,J+1})'/2 + E_1 \Sigma_{21}^{(i,J+1)} \left(\Sigma_{11}^{(i,J+1)}\right)^{-1} X_{i,J+1}^{(1)} + E_1 \tilde{\Sigma}_{22}^{(i,J+1)} E_1'/2. \end{aligned}$$

For $i = 1$ we set L_1 to be the projection on the first component, i.e. $L_1 Y = \log P_{1,J}$ and $g_1(\mathcal{D}_J) = 0$.

This implies for the ultimate claim predictor in Corollary 4.4

$$\mathbb{E}[P_{i,J} | \mathcal{D}_{J+1}] = \exp\{L_i Y + g_i(\mathcal{D}_J)\} \quad \text{for } i = 1, \ldots, J. \tag{5}$$

Different accident years are independent, given Θ. Thus, Lemma 4.1 leads to the joint distribution of Y, given $\{\mathcal{D}_J, \Theta\}$:

Lemma 5.1 *Under Model Assumptions 2.1 we have*

$$Y|_{\{\mathcal{D}_J,\Theta\}} = (\log P_{1,J}, \log P_{2,J-1}, \log I_{2,J-1}, \ldots, \log P_{J,1}, \log I_{J,1})'|_{\{\mathcal{D}_J,\Theta\}} \sim \mathcal{N}(\mu, \Sigma)$$

where $\mu = (\mu_1, \mu_2, \ldots, \mu_J)' \in \mathbb{R}^{2J-1}$ *and*

$$\Sigma = \begin{pmatrix} \Sigma_1 & 0 & 0 & \cdots & 0 \\ 0 & \Sigma_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \Sigma_J \end{pmatrix} \in \mathbb{R}^{(2J-1)\times(2J-1)},$$

with μ_i *and* Σ_i *defined in Lemma 4.1.*

In Lemma 5.1 the distribution of $Y|_{\{\mathcal{D}_J,\Theta\}}$ still depends on Θ via

$$\mu = (\mu_1, \mu_2, \ldots, \mu_J)' \in \mathbb{R}^{2J-1}$$

and recalling the definition of μ_i (see Lemma 4.1) we obtain, for $k = J$,

$$\mu_i = \begin{cases} \tilde{\Gamma}_{i,J}\Theta + \gamma_i & \text{for} \quad i \geq 2, \\ \tilde{\Gamma}_{1,J}\Theta + \gamma_1 & \text{for} \quad i = 1. \end{cases}$$

where

$$\tilde{\Gamma}_{i,J} = \begin{cases} E_2 \left(B^{(2)}_{i,J} - \Sigma^{(i,J)}_{21} \left(\Sigma^{(i,J)}_{11}\right)^{-1} B^{(1)}_{i} \right) & \text{for} \quad i \geq 2, \\ E_1 \left(B^{(2)}_{(1,J)} - \Sigma^{(1,J)}_{21} \left(\Sigma^{(1,J)}_{11}\right)^{-1} B^{(1)}_{(1,J)} \right) & \text{for} \quad i = 1. \end{cases}$$

and

$$\gamma_i = \begin{cases} E_2 \Sigma^{(i,J)}_{21} \left(\Sigma^{(i,J)}_{11}\right)^{-1} X^{(1)}_{i,J} & \text{for} \quad i \geq 1 \\ E_1 \Sigma^{(1,J)}_{21} \left(\Sigma^{(1,J)}_{11}\right)^{-1} X^{(1)}_{1,J} & \text{for} \quad i = 1, \end{cases}$$

Next, we define the matrix Γ with rows $\tilde{\Gamma}_{i,J}$, i.e.

$$\Gamma = \begin{pmatrix} \tilde{\Gamma}'_{1,J} & \tilde{\Gamma}'_{2,J} & \dots & \tilde{\Gamma}'_{J,J} \end{pmatrix}' \in \mathbb{R}^{(2J-1)\times n}.$$

and γ by $\gamma = (\gamma_1, \dots, \gamma_J)' \in \mathbb{R}^{(2J-1)}$. This shows that $\mu = \Gamma\Theta' + \gamma$ is a affin-linear function of Θ. This implies together with (5) the following theorem.

Theorem 5.2 *Under Model Assumptions 2.1 we obtain for* $i, l \in \{1, \dots, J\}$

$$\mathbb{E}\Big[\mathbb{E}[P_{i,J}|\mathcal{D}_{J+1}]\mathbb{E}[P_{l,J}|\mathcal{D}_{J+1}]\Big|\Theta,\mathcal{D}_J\Big] = \exp\left\{(L_i+L_l)\mu+(L_i+L_l)\Sigma(L_i+L_l)'/2+g_i(\mathcal{D}_J)+g_l(\mathcal{D}_J)\right\},$$

and

$$\mathbb{E}\Big[\mathbb{E}[P_{i,J}|\mathcal{D}_{J+1}]\mathbb{E}[P_{l,J}|\mathcal{D}_{J+1}]\Big|\mathcal{D}_J\Big] = \mathbb{E}[P_{i,J}|\mathcal{D}_J]\mathbb{E}[P_{l,J}|\mathcal{D}_J]\exp\{L_i\Gamma T(\mathcal{D}_J)\Gamma' L_l' + L_i\Sigma L_l'\}.$$

By means of this relationship between $\mathbb{E}\left[\mathbb{E}[P_{i,J}|\mathcal{D}_{J+1}]^2\Big|\mathcal{D}_J\right]$ and $\mathbb{E}[P_{i,J}|\mathcal{D}_J]^2$ it is straightforward to derive the MSEP for the one-year CDR of a single accident year, which is given in the next theorem:

Theorem 5.3 (conditional MSEP for single accident years)
Under Model Assumptions 2.1 the conditional MSEP, given $\mathcal{D}_J$*, for the one-year CDR of accident year* $i \in \{1, \dots, J\}$ *is given by*

$$\mathrm{msep}_{\mathrm{CDR}_i|\mathcal{D}_J}(0) = \left(\mathbb{E}[P_{i,J}|\mathcal{D}_J]\right)^2 \left(\exp\{L_i\Gamma T(\mathcal{D}_J)\Gamma' L_i' + L_i\Sigma L_i'\} - 1\right).$$

In the following section we consider the conditional MSEP for aggregated accident years.

5.2 Aggregated Accident Years

We study the conditional MSEP of the one-year CDR for aggregated accident years:

$$\begin{aligned}\mathrm{msep}_{\sum_{i=1}^J \mathrm{CDR}_i|\mathcal{D}_J}(0) &= \mathbb{E}\left[\left(\sum_{i=1}^J \mathrm{CDR}_i - 0\right)^2 \middle| \mathcal{D}_J\right] \\ &= \mathrm{Var}\left(\sum_{i=1}^J \mathrm{CDR}_i \middle| \mathcal{D}_J\right) = \mathrm{Var}\left(\sum_{i=1}^J \mathbb{E}\left[P_{i,J}|\mathcal{D}_{J+1}\right] \middle| \mathcal{D}_J\right).\end{aligned} \tag{6}$$

Using the tower property of conditional expectations and Theorem 5.2 we obtain for (6):

Theorem 5.4 (conditional MSEP for aggregated accident years)
Under Model Assumptions 2.1 the conditional MSEP, given $\mathcal{D}_J$, for the one-year CDR of aggregated accident years is given by

$$\begin{aligned}\mathrm{msep}_{\sum_{i=1}^J \mathrm{CDR}_i|\mathcal{D}_J}(0) = &\sum_{i=1}^J \mathrm{msep}_{\mathrm{CDR}_i|\mathcal{D}_J}(0) \\ &+ 2\sum_{l>i} \mathbb{E}[P_{i,J}|\mathcal{D}_J]\mathbb{E}[P_{l,J}|\mathcal{D}_J]\left(\exp\left\{L_i\Gamma T(\mathcal{D}_J)\Gamma' L_l' + L_i \Sigma L_l'\right\} - 1\right).\end{aligned}$$

6 Example

We revisit the data given in Dahms [3]. In Model Assumptions 2.1 we could choose any covariance matrices V as long as it is positive definite. This allows for modelling dependence between paid and incurred data. In this example we choose V as a diagonal matrix (i.e. paid and incurred ratios are independent) and estimate the variances on the diagonal with standard estimators. This is similar to Merz-Wüthrich [7]. Other choices of V are discussed in detail in Happ-Wüthrich [5]. Because we do not have any prior knowledge for prior distribution parameters ϕ_l and ψ_j we choose non-informative priors, i.e. we let $s_j^2 \to \infty$ and $t_l^2 \to \infty$. This implies that in Theorem 4.3 the term T^{-1} equals zero and no prior information is used in our calculations. In Table 1 we compare the prediction uncertainty measured by the square root of the conditional MSEP for the one-year CDR calculated by the PIC method and the complementary loss ratio (CLR) method (cf. Dahms et al. [4]). Under Model Assumptions 2.1, these values are calculated analytically with Theorem 5.3 for single accident years and with Theorem 5.4 for aggregated accident years. Note that in Dahms et al. [4] we obtain two values for the MSEP because we can estimate the variance in two ways. We observe in the PIC method for almost all accident years and totally a lower prediction uncertainty for the CDR than both prediction uncertainty estimators in the CLR method (see Table 1). This can partly be explained by fact that in the CLR method we have to estimate 44 parameters (cf. Dahms [3]) whereas in the Bayesian PIC model only 19 variance parameters have to be estimated leading to a lower standard error. We observe that in the PIC model it is not unlikely that the claims reserves increase about 3% in the one-year horizon. This is similar to the findings for the CLR method in Dahms et al. [4].

accident year i	claims reserves CLR	$\text{msep}^{1/2}_{\text{CDR}}$ CLR method Paid	$\text{msep}^{1/2}_{\text{CDR}}$ CLR method Incurred	claims reserves PIC	$\text{msep}^{1/2}_{\text{CDR}}$ PIC method	in % reserves PIC
1	314.902	194	14.639	337.799	2.637	0,78%
2	66.994	4.557	4.678	31.686	4.597	14,51%
3	359.384	5.597	6.628	331.890	7.656	2,31%
4	981.883	33.675	34.258	1.018.308	6.606	0,65%
5	1.115.768	30.574	30.997	1.104.816	31.594	2,86%
6	1.786.947	42.598	43.074	1.842.669	43.168	2,34%
7	1.942.518	166.154	166.255	1.953.767	139.352	7,13%
8	1.569.657	138.685	138.740	1.602.229	127.053	7,93%
9	2.590.718	210.899	210.979	2.402.946	173.721	7,23%
Total	10.728.771	346.576	350.534	10.626.108	292.879	2,76%

Table 1: Ultimate claim prediction and prediction uncertainty for the one-year CDR from the CLR method for claims payments and incurred losses (cf. Dahms et al. [4]) and from the PIC method.

Table 2 provides the ratios between the square root of the conditional MSEP for the one-year CDR and the square root of the conditional MSEP for the total run-off of the ultimate claim. We observe that for later accident years (i.e. $i \geq 7$) and aggregated accident years the values for the CLR method and for the PIC method only slightly differ. Moreover, we see that for aggregated accident years the one-year uncertainty is about 75% of the total run-off uncertainty. This result is in-line with the field study conducted by AISAM-ACME [1].

As mentioned in the introduction we can not only calculate the conditional MSEP for the one-year CDR. Based on Theorem 4.3 applied for the case $u(\Theta|\mathcal{D}_J)$, Corollary 4.4 and Lemma 4.1 we provide the full predictive distribution of the one-year CDR by means of Monte Carlo simulations. Firstly, we use Theorem 4.3 applied for the case $u(\Theta|\mathcal{D}_J)$ to generate Gaussian samples $\Theta^{(n)}$ with covariance matrix $T(\mathcal{D}_J)$ and mean $\nu(\mathcal{D}_J)$. Secondly, we generate independent two-dimensional Gaussians samples $(\log P_{i,J-i+1}, \log I_{i,J-i+1})_{\{\mathcal{D}_J,\Theta\}}$ and fill up the off-diagonal entries in the Paid and Incurred trapezoids (see Lemma 4.1). This way we obtain the data available at time $J+1$, i.e. $\mathcal{D}_{J+1}$, and can calculate $\mathbb{E}\left[P_{i,J}|\mathcal{D}_{J+1}\right]$ by means of Corollary 4.4. This provides Figure 3, where we compare the empirical density from 400.000 simulations to the

accident year	$\text{msep}^{1/2}_{\text{CDR}}/\text{msep}^{1/2}_{\text{Ultimate}}$ CLR method Incurred	CLR method Paid	$\text{msep}^{1/2}_{\text{CDR}}/\text{msep}^{1/2}_{\text{Ultimate}}$ PIC method Paid & Incurred
1	100.0%	100.0%	100.0%
2	100.0%	84.5%	87.6%
3	53.1%	52.7%	83.7%
4	91.5%	89.6%	62.4%
5	69.6%	69.1%	94.3%
6	65.5%	65.4%	80.8%
7	94.0%	93.9%	93.1%
8	70.1%	70.1%	70.3%
9	65.3%	65.3%	66.4%
Total	74.1%	74.3%	75.2%

Table 2: Ratio $\text{msep}^{1/2}_{\text{CDR}}/\text{msep}^{1/2}_{\text{Ultimate}}$ from the CLR method for claims payments and incurred losses (cf. Dahms [4]) and from the PIC method.

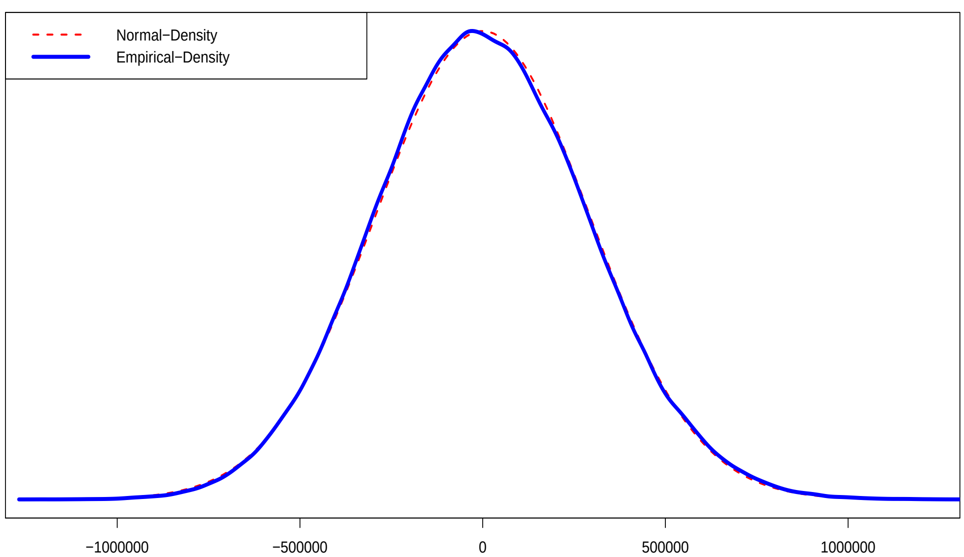

Figure 3: Empirical density for the one-year CDR from 100.000 simulations and fitted Gaussian density with mean 0 and standard deviation 292.879.

Gaussian density with the same mean ($\mu = 0$) and the same standard deviation ($\sigma = 292.879$) (cf. Table 1). We observe that these two densities look quite similar. To get a closer look on the left tail of the empirical density for the one-year CDR we show a QQ-plot for quantiles $q \in (0, 0.1)$. We observe that the tail behaviour of the empirical density of the one-year CDR and the fitted Gaussian density with mean 0 and standard deviation 292.879 only slightly differ (see

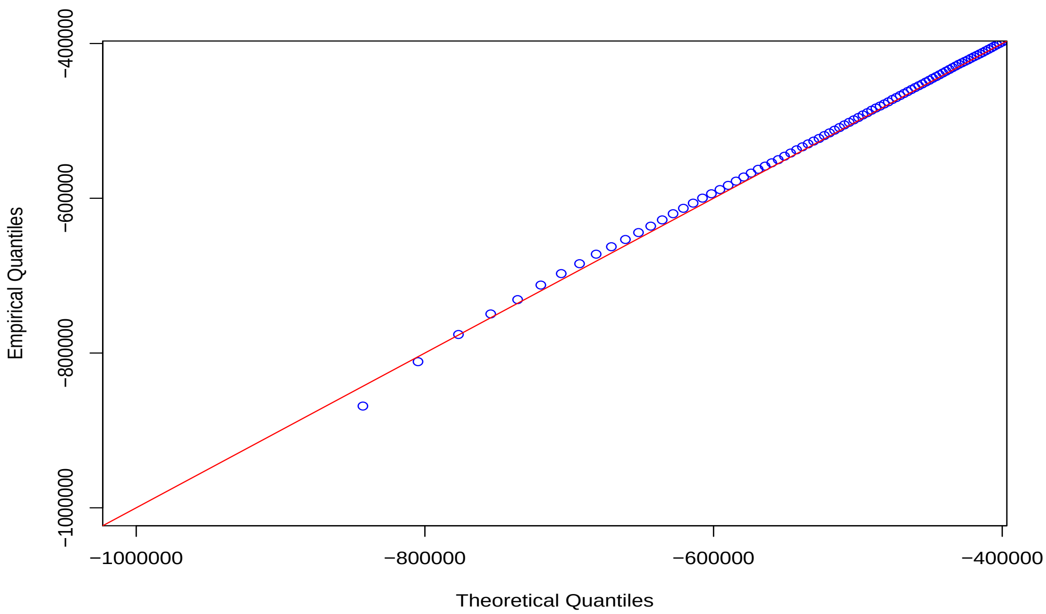

Figure 4: QQ-plot for lower quantiles $q \in (0, 0.1)$ to compare the left tail of the empirical density for the one-year CDR with the left tail of the fitted Gaussian density with mean 0 and standard deviation 292.879.

Figure 4). This is similar to the findings for the ultimate claim distribution in Merz-Wüthrich [7]. This means that using a Gaussian approximation for the density of the one-year CDR provides within our model and for our data set a good approximation of the shortfall risk in the one-year CDR.

7 Conclusions

The paid-incurred chain (PIC) claims reserving method provides a framework, where unified ultimate claim predictions can be calculated based on cumulative payments and incurred losses data simultaneously. Furthermore, it allows for the derivation of the mean square error of prediction for the ultimate claim in the long term run as well as for the claims development result in the one-year time horizon. Merz-Wüthrich [7] have derived the MSEP formula for the ultimate claim uncertainty. In the present paper we derive the MSEP formula for the one-year (CDR) uncertainty. However, in contrary to the complementary loss ratio method by Dahms [3], where also MSEP formulas for the ultimate claim and the claims development result exist, the PIC method allows for the calculation of the full predictive distribution of the ultimate claim and the claims development result via Monte-Carlo simulations. This implies that any other risk measure like Value-at-Risk or Expected Shortfall can be calculated for the ultimate claim risk (long term risk) as well as for the claims development result (one-year risk).

A Appendix

Proof of Lemma 4.1: Conditionally given the parameter Θ, X_i are independent for different accident years. Therefore, the conditional distribution of $X_{i,k}^{(2)}$ depends only on $\mathcal{D}_k$ through $X_{i,k}^{(1)}$. This shows the first equality in the first claim. The distributional claim is a well known result for multivariate normal distributions using the Schur complement for the calculation of the conditional covariance matrix. The second claim is a direct consequence of the first claim. This proves the lemma. $\square$

Proof of Theorem 4.3: From (3) immediately follows that the posterior distribution $u(\Theta|\mathcal{D}_k)$ is a multivariate Gaussian distribution. Therefore, it remains to calculate the first two moments of $u(\Theta|\mathcal{D}_k)$. This is done by squaring out all terms and analyzing quadratic and linear terms. $\square$

Proof of Theorem 5.2

Using standard properties of log-normal distribution, the first claim immediately follows by Lemma 5.1 and equation (5). The second claim follows with the identity $\mu = \Gamma\Theta' + \gamma$ and Theorem 4.3. $\square$

Proof of Theorem 5.4: With Theorem 5.2 we obtain

$$
\begin{aligned}
&\mathrm{msep}_{\sum_{i=1}^J \mathrm{CDR}_i|\mathcal{D}_J}(0) \\
&\quad = \sum_{i=1}^J \mathrm{Var}\left(\mathbb{E}\left[P_{i,J}|\mathcal{D}_{J+1}\right]\middle|\, \mathcal{D}_J\right) + 2\sum_{l>i}^J \mathbb{E}\left[\mathbb{E}[P_{i,J}|\mathcal{D}_{J+1}]\mathbb{E}[P_{l,J}|\mathcal{D}_{J+1}]\middle|\, \mathcal{D}_J\right] \\
&\qquad\qquad - 2\sum_{l>i}^J \mathbb{E}[P_{i,J}|\mathcal{D}_J]\,\mathbb{E}[P_{l,J}|\mathcal{D}_J] \\
&\quad = \sum_{i=1}^J \mathrm{msep}_{\mathrm{CDR}_i|\mathcal{D}_J}(0) + 2\sum_{l>i}^J \mathbb{E}[P_{i,J}|\mathcal{D}_J]\mathbb{E}[P_{l,J}|\mathcal{D}_J]\left(\exp\left\{L_i\Sigma L_l' + L_i\Gamma T(\mathcal{D}_J)\Gamma' L_l'\right\} - 1\right). \qquad \square
\end{aligned}
$$

Corollaries 4.2 and 4.4 and Lemma 5.1 are easy consequences of the previous statements. $\square$

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