SUPPLEMENTAL MATERIAL

Dynamical phases and quantum correlations in an emitter-waveguide system with feedback

G. Buonaiuto,[1] F. Carollo,[1] B. Olmos,[1,2] and I. Lesanovsky[1,2]

[1]*Institut für Theoretische Physik, Universität Tübingen, Auf der Morgenstelle 14, 72076 Tübingen, Germany*
[2]*School of Physics and Astronomy and Centre for the Mathematics and Theoretical Physics of Quantum Non-Equilibrium Systems, The University of Nottingham, Nottingham, NG7 2RD, United Kingdom*

RECASTING THE DYNAMICAL GENERATOR

In order to find the contribution to the dynamical equations of the different terms in the generator, we rewrite the latter in a convenient form. In addition, we work in the Heisenberg picture, where only operators are evolved. In particular, the Heisenberg representation of the generator appearing in equation (2) in the main text can be written as the sum of four different terms

$$\mathcal{L}[X] = \mathcal{H}_{\rm L}[X] + \mathcal{H}_{\rm C}[X] + \mathcal{A}[X] + \mathcal{B}[X]\,. \tag{S1}$$

The first contribution contains the local terms of the Hamiltonian and is given by

$$\mathcal{H}_{\rm L}[X] = {\rm i}[H_{\rm L}, X]\,, \qquad \text{with} \qquad H_{\rm L} = \sum_{\mu=x,y,z} \omega_\mu J_\mu\,, \qquad \text{and} \qquad \omega = (2\Omega, 0, 0)\,. \tag{S2}$$

The second contribution is still a coherent Hamiltonian one, and accounts for the coherent all-to-all interaction between emitters generated by the feedback. It can be written as

$$\mathcal{H}_{\rm C}[X] = {\rm i}[H_{\rm C}, X]\,, \qquad \text{with} \qquad H_{\rm C} = \frac{1}{N}\sum_{\mu,\nu} h_{\mu\nu} J_\mu J_\nu\,,$$

where h is a 3×3 Hermitian matrix. In our specific setup, such a matrix assumes the form

$$h = -\frac{g\Gamma}{2}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\,.$$

The last two contributions are due to the dissipative part of the generator, which is split into two terms:

$$\mathcal{A}[X] = \frac{1}{N}\sum_{\mu,\nu} \frac{A_{\mu\nu}}{2}\left[[J_\mu, X]\,, J_\nu\right]\,, \quad \text{and} \quad \mathcal{B}[X] = \frac{\rm i}{N}\sum_{\mu,\nu} \frac{B_{\mu\nu}}{2}\left\{[J_\mu, X]\,, J_\nu\right\}\,, \tag{S3}$$

where the matrices A and B are given by

$$A = \frac{\Gamma}{2}\begin{pmatrix} 1+\kappa^2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \text{and} \qquad B = \frac{\Gamma}{2}\begin{pmatrix} 0 & -(\kappa+1) & 0 \\ (\kappa+1) & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and $\kappa = 2g+1$.

MEAN-FIELD DYNAMICS

In this section we give some details on the dynamics of the magnetization operators for the emitters. Given the collective operators J_α, these are defined as

$$m_\alpha = \frac{J_\alpha}{N}\,, \tag{S4}$$

and, due to the scaling $\frac{1}{N}$, such operators form a commutative algebra in the limit $N \to \infty$. This can be seen from the fact that $\|[m_\alpha, m_\beta]\| \propto \frac{1}{N} \to 0$, for $N \gg 1$.

In addition, the dynamics considered in the main text, as well as the initial state we focus on, are such that only very weak collective correlations between pairs of emitters [79, 80] are present. These correlations are not inherited by the operators m_α. As such, under the expectation $\langle\cdot\rangle$ over the state, these operators m_α obey a sort of law of large numbers and converge, for $N \gg 1$, to their expectation value

$$m_\alpha \longrightarrow \langle m_\alpha\rangle\,. \tag{S5}$$

This also means that the expectation of any product of magnetization operators converges to the product of the limiting operators, i.e. the expectation value factorizes as

$$\langle m_\alpha m_\beta\rangle \longrightarrow \langle m_\alpha\rangle\langle m_\beta\rangle\,. \tag{S6}$$

Our aim is thus to obtain the equations governing the dynamics of the quantities $\langle m_\alpha\rangle$ in the thermodynamic limit. These are the equations appearing in the main text in Eq. (3). To this end, we first compute the action of the generator on the collective operators J_α (using collective spin commutation relations):

$$\mathcal{L}\left[J_\alpha\right] = -\sum_\delta\sum_\mu \omega_\mu\epsilon_{\mu\alpha\delta}J_\delta - \frac{1}{N}\sum_{\mu\nu}h_{\mu\nu}\left(\sum_\delta \epsilon_{\nu\alpha\delta}J_\mu J_\delta + \sum_\delta \epsilon_{\mu\alpha\delta}J_\delta J_\nu\right) - \frac{1}{N}\sum_{\mu\nu}\frac{B_{\mu\nu}}{2}\sum_\delta \epsilon_{\mu\alpha\delta}\left\{J_\delta, J_\nu\right\} + \mathcal{A}[J_\alpha]\,.$$

We do not explicitly compute $\mathcal{A}[J_\alpha]$ since this is not relevant for the dynamics of m_α. Using the above relation, we straightforwardly find the action of the generator on the m_α:

$$\mathcal{L}\left[m_\alpha\right] \approx \sum_\delta D^{\mathrm{L}}_{\alpha\delta}\, m_\delta - \sum_{\mu\nu\delta}\left(h_{\mu\nu} + h^T_{\mu\nu}\right)\epsilon_{\nu\alpha\delta}\, m_\mu m_\delta - \sum_{\mu\nu\delta} B_{\mu\nu}\epsilon_{\mu\alpha\delta} m_\delta m_\nu\,.$$

The approximate symbol appears because we have set to zero the commutator of two average operators and because we have dropped the term $\mathcal{A}[J_\alpha]/N$ since both these terms converge to zero in norm, for large N. We have further defined

$$D^{\mathrm{L}}_{\alpha\delta} = -\sum_\mu \omega_\mu\epsilon_{\mu\alpha\delta}\,.$$

In order to obtain the semi-classical equations of motion Eq. (3) in the main text, which are exact in the thermodynamic limit, we consider the expectation of the above equation with respect to the quantum expectation $\langle\cdot\rangle$ recalling that, for such operators, expectation values factorize [see Eq. (S6)]. In this way, we find

$$\langle\dot m_\alpha\rangle = \sum_\delta D^{\mathrm{L}}_{\alpha\delta}\langle m_\delta\rangle - \sum_{\mu\nu\delta}\left(h_{\mu\nu} + h^T_{\mu\nu}\right)\epsilon_{\nu\alpha\delta}\,\langle m_\mu\rangle\langle m_\delta\rangle - \sum_{\mu\nu\delta} B_{\mu\nu}\epsilon_{\mu\alpha\delta}\,\langle m_\delta\rangle\langle m_\nu\rangle\,; \tag{S7}$$

specializing the above equation for $\alpha = x, y, z$, we find the equations reported in Eq. (3) in the main text.

PERSISTENT OSCILLATIONS AND DYNAMICAL SYMMETRIES

In the dissipative continuous time crystal phase, the state of the quantum system approaches a limit cycle as witnessed by the oscillations displayed by the magnetization operators. This implies that the stationary-state manifold, for this region of the parameter space, is degenerate and, for long times, the quantum state undergoes a non-trivial motion within this manifold. The emergence of a highly dimensional degenerate stationary manifold is also witnessed by the behavior of the real part of the spectrum of the Lindblad generator $\mathcal{L}$.

The emergence of such a many-body non-stationary phase can also be understood in terms of dynamical symmetries [26]. To discuss this, we make use of the dynamical generator $\mathcal{L}$ for the dynamics of operators introduced in Eq. (S1). This generator can be brought into a diagonal form by finding eigenmatrices ℓ_k and eigenvalues μ_k satisfying

$$\mathcal{L}[\ell_k] = \mu_k\ell_k\,.$$

The eigenvalues μ_k are in general complex numbers. Their real part encodes a decay rate while their imaginary part encodes coherent oscillations. Importantly for us, any operator O can be decomposed into the (basis) elements ℓ_k, which also provides a convenient formulation of the time evolved operator

$$O = \sum_k c_k\ell_k \quad \mapsto \quad O(t) := e^{t\mathcal{L}}[O] = \sum_k c_k e^{\mu_k t}\ell_k\,.$$

The presence of strong dynamical symmetries implies that there exist operators $A_m, A_m^\dagger$ such that, for $N \gg 1$,

$$\mathcal{L}[A_m] = -\mathrm{i}m\omega A_m\,, \qquad \mathcal{L}[A_m^\dagger] = \mathrm{i}m\omega A_m^\dagger\,.$$

The above relations are eigenvalue relations for the operators $A_m, A_m^\dagger$, with eigenvalues which are purely imaginary. This means that the operators $A_m, A_m^\dagger$ are not affected by decoherence and dissipation, but just evolve displaying coherent oscillations. (In particular, notice that the operators $A_m, A_m^\dagger$ can be seen as powers of fundamental "ladder" operators $A, A^\dagger$ such that $A_m = A^m$ and $A_m^\dagger = (A^\dagger)^m$.) In the next section of this Supplemental Material, we show indeed how the Lindblad generator $\mathcal{L}$ possesses several eigenvalues which converge to purely imaginary number, in the large N limit, which are all multiples of a fundamental frequency ω.

Since the operators $A_m, A_m^\dagger$ are the only ones which are not damped, we have that in the long time limit the expectation value of any operator O can be written as

$$\langle O\rangle_t \approx \sum_m \tilde{c}_m^- e^{-im\omega t}\langle A_m\rangle_0 + \sum_m \tilde{c}_m^+ e^{+im\omega t}\langle A_m^\dagger\rangle_0\,,$$

where $\tilde{c}_m^\pm$ are the coeffients of the decomposition of O for the operators $A_m, A_m^\dagger$. The approximation symbol is due to the fact that we are neglecting the part of the decomposition of O on the elements of the eigenbasis of the dynamical generator which are damped in the large time limit.

We now consider the magnetization operator m_z. From the mean-field equations reported in the main text, we see that for the special point $g = -1/2$ we have that with $A = m_y + \mathrm{i}m_z$, the dynamical symmetry relations are satified since

$$\mathcal{L}[A] \approx i[\tilde{H}, A] = -\mathrm{i}2\Omega A\,.$$

Since $m_z = (A + A^\dagger)/2$ and $\langle A\rangle_0 = \langle A^\dagger\rangle_0 = -1/2$ for our choice of the initial state, we have

$$\langle m_z\rangle_t = -\frac{1}{4}e^{-\mathrm{i}2\Omega t} - \frac{1}{4}e^{\mathrm{i}2\Omega t} = -\frac{1}{2}\cos(2\Omega t)\,.$$

On the other hand, for $g \neq -1/2$, it is not possible to find an easy decomposition of m_z into the eigenmodes A_m and $A_m^\dagger$. This suggests that in this case m_z must be decomposed onto all the eigenmodes A_m and $A_m^\dagger$, so that its expectation value (for large t) assumes the form

$$\langle m_z\rangle_t \approx \sum_m \tilde{c}_m^- e^{-im\omega t}\langle A_m\rangle_0 + \sum_m \tilde{c}_m^+ e^{-im\omega t}\langle A_m^\dagger\rangle_0\,.$$

The above relation shows that the expectation value $\langle m_z\rangle_t$ is in principle an infinite Fourier series that could thus give rise to any periodic function, including the tilted oscillations reported in Fig. 2(c).

SPECTRUM OF THE LINDBLAD OPERATOR IN THE (CONTINUOUS) TIME-CRYSTAL PHASE

In this section we report numerical results on the behavior of the spectrum of the Lindblad operator in the dissipative (continuous) time-crystal phase. We consider the Lindblad operator restricted to the fully symmetric (N-spin) subspace, which is enough to understand the dynamics ensuing from the initial state considered. We look at two exemplary parameter choices. In one case, we consider the special case with $g = -1/2$ [see Fig. S1(a-b)], while in the other we consider the case $g = +1/2$ [see Fig. S1(c-d)]. As shown in Fig. S1, the qualitative behavior of the spectrum is the same in the two cases. In particular, we observe how several eigenvalues show a trend compatible with a vanishing real part in the large N limit, and how these converge to purely imaginary numbers which are multiple of a fundamental frequency ω. These results support the interpretation in terms of strong dynamical symmetries given in the previous section.

QUANTUM FLUCTUATION OPERATORS

In order to quantify quantum correlations in the emitter ensemble, as well as to derive their dynamical behavior, we need to be able to control correlations between different collective spin operators, J_α, with $\alpha = x, y, z$. We are interested in correlations of the form

$$\Sigma_{\alpha\beta} = \frac{1}{N}\left(\frac{1}{2}\langle\{J_\alpha, J_\beta\}\rangle - \langle J_\alpha\rangle\langle J_\beta\rangle\right)\,,$$

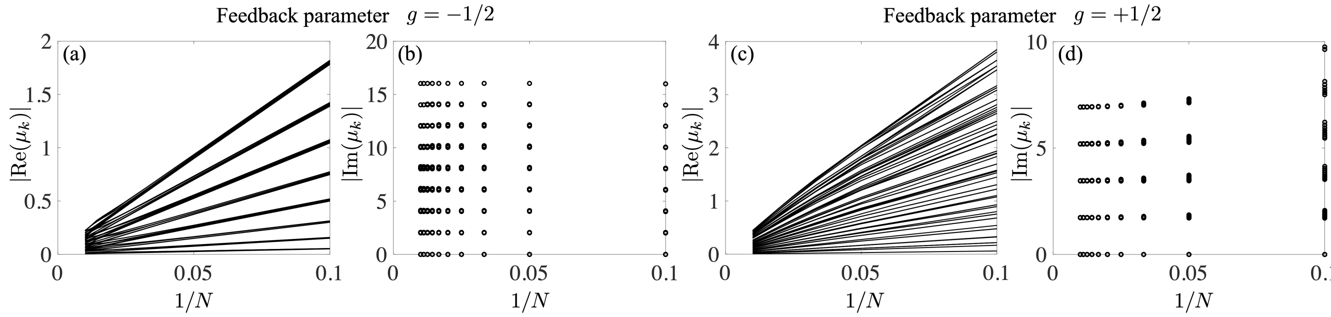

FIG. S1. **Spectrum of the Lindblad generator in the (continuous) time-crystal phase.** We present here the behavior of the first 80 non-zero eigenvalues of the Lindblad operator with smallest real part in absolute value. We consider $\Omega/\Gamma = 1$ and $g = -1/2$ [panels (a)-(b)] and $g = +1/2$ [panels (c)-(d)]. We consider system sizes up to $N = 100$. (a) Feedback parameter $g = -1/2$. Real part of the 80 eigenvalues with smallest real part in absolute value as a function of the inverse system size. Consistently with the behavior of the gap in the time-crystal phase, all these eigenvalues show a $1/N$ behavior. Small oscillations can be observed. (b) Feedback parameter $g = -1/2$. Behavior of the absolute value of the imaginary part of the eigenvalues considered in (a), as a functon of the inverse system size. It is shown how these eigenvalues tend to accumulate at multiple values of a fundamental frequency ω, which, in the thermodynamic limit, is expected to be $\omega = 2\,\Omega/\Gamma$ [cf. Eq. (S7) in the main text]. (c) Same as (a) but for a feedback parameter $g = 1/2$. The real part of the eigenvalues show a decaying behavior with $1/N$. (d) Same as (b) but for a feedback parameter $g = 1/2$. The absolute values of the imaginary part of the considered eigenvalues accumulate at values which are multiples of a fundamental frequency ω, which in this case is $\omega \approx 1.73\,\Omega/\Gamma$.

where Σ is nothing but a covariance matrix for collective spin operators. We want to look at the behavior of these correlations in the thermodynamic limit. To this end, it is convenient to exploit the formalism of quantum fluctuation operators [59–62].

For a given quantum expectation $\langle\cdot\rangle$, we can define the fluctuation operator F_α ($\alpha = x, y, z$) as

$$F_\alpha = \frac{1}{\sqrt{N}}\left(J_\alpha - \langle J_\alpha\rangle\right) .$$

This is essentially the collective operator J_α renormalized with respect to its expectation value in the state, and rescaled by a factor of $\frac{1}{\sqrt{N}}$. This operator is a quantum version of the fluctuation variable studied in central limit theorems. Furthermore, note that

$$\langle F_\alpha\rangle = 0\,, \qquad \text{and} \qquad \Sigma_{\alpha\beta} = \frac{1}{2}\langle\{F_\alpha, F_\beta\}\rangle\,. \tag{S8}$$

As shown in Refs. [59–62], under appropriate conditions on the quantum state defining the expectation $\langle\cdot\rangle$, which are always satisfied in our analysis, these fluctuation operators converge, for large N, to bosonic field operators. To show this, it is convenient to look at the commutation relations between fluctuation operators. These can be derived from the finite-N commutation

$$[F_\alpha, F_\beta] = \frac{\mathrm{i}}{N}\sum_\gamma \epsilon_{\alpha\beta\gamma} J_\gamma = \mathrm{i} s_{\alpha\beta}\,.$$

In the above relations, the definition of $s_{\alpha\beta}$ makes apparent that such quantity has the scaling of a magnetization operator and thus converges to a scalar multiple of the identity [see Eq. (S6)] under any expectation taken with the state $\langle\cdot\rangle$. In particular, the actual value to which this operator converges is

$$s_{\alpha\beta} \to \sum_\gamma \epsilon_{\alpha\beta\gamma}\langle m_\gamma\rangle\,.$$

This shows that the commutator between two fluctuation operators, in the large N limit, is actually a scalar multiple of the identity. Thus, fluctuation operators obey canonical commutation relations and can naturally be identified with bosonic operators. In our specific setting, we have three bosonic field operators whose commutation relations are encoded in the 3×3 antisymmetric matrix

$$s = \begin{pmatrix} 0 & \langle m_z\rangle & -\langle m_y\rangle \\ -\langle m_z\rangle & 0 & \langle m_x\rangle \\ \langle m_y\rangle & -\langle m_x\rangle & 0 \end{pmatrix}\,. \tag{S9}$$

In particular, for the setup considered here, we can find a mapping from these three generalized bosonic field operators to the standard bosonic position-like and momentum-like operators. Indeed, one can always find a rotation R which brings s into its canonical form

$$\tilde{s} = RsR^T = \begin{pmatrix} 0 & j & 0 \\ -j & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} ,$$

with R being a real unitary matrix, and $j = \sqrt{\langle m_x \rangle^2 + \langle m_y \rangle^2 + \langle m_z \rangle^2}$. Basically, the rotation simply consists in aligning the z-axis of the reference frame with the principal direction of the magnetization, $\vec{n} = j^{-1} \left(\langle m_x \rangle , \langle m_y \rangle , \langle m_z \rangle \right)$. This rotation R thus identifies three new fluctuation operators $\tilde{F}_\alpha$ whose commutation relations are

$$\left[\tilde{F}_\alpha, \tilde{F}_\beta \right] = i \tilde{s}_{\alpha\beta} ,$$

and whose correlations are encoded in the appropriately rotated covariance matrix

$$\tilde{\Sigma} = R \Sigma R^T .$$

We can then construct effective position and momentum operators

$$\hat{x} := \frac{\tilde{F}_x}{\sqrt{j}} , \qquad \hat{p} := \frac{\tilde{F}_y}{\sqrt{j}} ,$$

which are such that $[\hat{x}, \hat{p}] = \mathrm{i}$. We also note that the operator $\tilde{F}_z$ plays the role of a scalar (classical) random variable since it commutes with the rest of the fluctuation algebra. For the relevant quantum degrees of freedom, $\hat{x}$ and $\hat{p}$, we can also compute the covariance matrix

$$\hat{\Sigma} = \begin{pmatrix} \langle \hat{x}^2 \rangle & \frac{1}{2} \langle \{ \hat{x}, \hat{p} \} \rangle \\ \frac{1}{2} \langle \{ \hat{x}, \hat{p} \} \rangle & \langle \hat{p}^2 \rangle \end{pmatrix} ,$$

as the 2×2 principal minor of the matrix $\tilde{\Sigma}$, rescaled by the factor j. This covariance matrix contains the correlations between collective operators in the directions perpendicular to the principal magnetization vector $\vec{n}$. The smallest eigenvalue of $\hat{\Sigma}$ thus corresponds to the minimal variance that a collective operator, perpendicular to the direction $\vec{n}$, can have, further rescaled by j^{-1}. When multiplied by 2, the smallest eigenvalue of $\hat{\Sigma}$ is thus nothing but the spin squeezing parameter ξ, evaluated in the thermodynamic limit of a large number of emitters.

TIME-EVOLUTION OF THE COVARIANCE MATRIX

Now that we have understood the structure of fluctuation operators for a fixed state of the quantum system, we need to be able to recover the covariance matrix Σ, at each time t. Essentially, we have to find how this matrix propagates in time according to the generator of the dynamics from a given initial condition. Before, though, note that, from Eq. (S9), it is clear that the dynamics of the anti-symmetric matrix s is completely specified by the dynamics of the magnetization operators.

To simplify the discussion, we introduce the matrix

$$C = \langle F_\alpha F_\beta \rangle ,$$

whose connection with the covariance matrix is explicitly

$$\Sigma = \frac{1}{2} \left(C + C^T \right) .$$

We start by taking the derivative of C. The time derivative of F_α is a scalar quantity since the derivative can only act on the state. This gives

$$\frac{d}{dt} F_\alpha = - \frac{1}{\sqrt{N}} \frac{d}{dt} \langle J_\alpha \rangle ;$$

recalling also that F_α has a vanishing expectation on the state $\langle \cdot \rangle$, the time-derivative of the matrix C is simply determined by the generator $\mathcal{L}$ as

$$\frac{d}{dt} C_{\alpha\beta} = \langle \mathcal{L} \left[F_\alpha F_\beta \right] \rangle .$$

We now study the action of the different terms of the generator, appearing in Eq. (S1), on the product of two fluctuation operators.

First of all, we notice that

$$\mathcal{H}_{\rm L}\left[F_\alpha F_\beta\right]=\mathcal{H}_{\rm L}\left[F_\alpha\right]F_\beta+F_\alpha\mathcal{H}_{\rm L}\left[F_\beta\right]=\sum_\gamma D^{\rm L}_{\alpha\gamma}\frac{J_\gamma}{\sqrt{N}}F_\beta+F_\alpha\sum_\gamma D^{\rm L}_{\beta\gamma}\frac{J_\gamma}{\sqrt{N}}\,. \tag{S10}$$

Then, considering that the expectation value of a single fluctuation operator is zero under expectation over the state, we can freely subtract terms like $F_\alpha\sum_\gamma D^{\rm L}_{\beta\gamma}\frac{\langle J_\gamma\rangle}{\sqrt{N}}$ to the ones above, to reconstruct

$$\langle\mathcal{H}_{\rm L}\left[F_\alpha F_\beta\right]\rangle=\sum_\gamma D^{\rm L}_{\alpha\gamma}C_{\gamma\beta}+\sum_\gamma D^{\rm L}_{\beta\gamma}C_{\alpha\gamma}\,.$$

For the second term of the generator, the one with all-to-all interacting Hamiltonian, we also have

$$\mathcal{H}_{\rm C}\left[F_\alpha F_\beta\right]=\mathcal{H}_{\rm C}\left[F_\alpha\right]F_\beta+F_\alpha\mathcal{H}_{\rm C}\left[F_\beta\right]\,. \tag{S11}$$

We can thus focus on the action of $\mathcal{H}_{\rm C}$ on a single fluctuation operator. This gives

$$\begin{aligned}\mathcal{H}_{\rm C}\left[F_\alpha\right]=&\frac{\rm i}{N}\sum_{\mu\nu}h_{\mu\nu}J_\mu\left[J_\nu,F_\alpha\right]+\frac{\rm i}{N}\sum_{\mu\nu}h_{\mu\nu}\left[J_\mu,F_\alpha\right]J_\nu=\\ =&{\rm i}\sum_{\mu,\nu}h_{\mu\nu}F_\mu\left[F_\nu,F_\alpha\right]+{\rm i}\sum_{\mu,\nu}h_{\mu\nu}\left[F_\mu,F_\alpha\right]F_\nu\\ &+{\rm i}\sum_{\mu,\nu}h_{\mu\nu}\frac{\langle J_\mu\rangle}{N}\left[J_\nu,F_\alpha\right]+{\rm i}\sum_{\mu,\nu}h_{\mu\nu}\left[J_\mu,F_\alpha\right]\frac{\langle J_\nu\rangle}{N}\,.\end{aligned} \tag{S12}$$

To obtain the second equality, we have simply added and subtracted the proper expectation values, appearing in the last line of the above equation, in order to reconstruct fluctuation operators. We now divide the terms on the right hand side of the second equality into two contributions. We call $\mathcal{H}'_{\rm C}$ the ones in the second line of the above equation and $\mathcal{H}''_{\rm C}$ those in the third line. The first term can be understood by looking at commutation relation between fluctuation operators. Indeed, recalling the fact that $s_{\alpha\beta}=-s_{\beta\alpha}$ we can write

$$\begin{aligned}\mathcal{H}'_{\rm C}\left[F_\alpha\right]=&-\sum_{\mu,\nu}h_{\mu\nu}F_\mu s_{\nu\alpha}-\sum_{\mu,\nu}h_{\mu\nu}s_{\mu\alpha}F_\nu=\\ \approx&\sum_\gamma Q^{\rm C}_{\alpha\gamma}F_\gamma\,,\end{aligned} \tag{S13}$$

where we have

$$Q^{\rm C}=s\left(h^T+h\right)\,.$$

We now come to the second term; this can be rewritten as

$$\mathcal{H}''_{\rm C}\left[F_\alpha\right]\approx\sum_\gamma D^{\rm C}_{\alpha\gamma}J_\gamma\,, \tag{S14}$$

where we have

$$D^{\rm C}_{\alpha\gamma}=-\sum_{\mu,\nu}\left[h_{\mu\nu}+h^T_{\mu\nu}\right]\epsilon_{\nu\alpha\gamma}\langle m_\mu\rangle\,.$$

The approximate symbol in Eq. (S13) is due to the fact that, in $D^{\rm C}$, we have considered $\langle m_\mu\rangle$ instead of m_μ: this is only valid in the limit $N\to\infty$ [see also discussion of Eq. (S5)]. In addition, when considering the expectation over the state of the term in Eq. (S11), we have that we can safely add or substract a scalar quantity to the term $\mathcal{H}''_{\rm C}\left[F_\alpha\right]$ using the fact that, in any case, F_β has zero expectation. Overall, we have found that

$$\langle\mathcal{H}_{\rm C}\left[F_\alpha F_\beta\right]\rangle\approx\sum_\gamma\left(\left[Q^{\rm C}\right]_{\alpha\gamma}+\left[D^{\rm C}\right]_{\alpha\gamma}\right)C_{\gamma\beta}+\sum_\gamma\left(\left[Q^{\rm C}\right]_{\beta\gamma}+\left[D^{\rm C}\right]_{\beta\gamma}\right)C_{\alpha\gamma}\,.$$

This concludes the contribution which comes from the Hamiltonian term of the generator.

We now turn to the dissipative contributions and start with the one encoded in $\mathcal{A}$. We have

$$\langle \mathcal{A}\left[F_\alpha F_\beta\right]\rangle = \left\langle \frac{1}{N}\sum_{\mu,\nu}\frac{A_{\mu\nu}}{2}\left[\left[J_\mu, F_\alpha F_\beta\right], J_\nu\right]\right\rangle . \tag{S15}$$

To understand this contribution, we look at a single term in the above summation. This can be reduced to

$$\begin{aligned}\left[\left[J_\mu, F_\alpha F_\beta\right], J_\nu\right] = &\left[\left[F_\mu, F_\alpha\right], F_\nu\right] F_\beta + F_\alpha\left[\left[F_\mu, F_\beta\right], F_\nu\right] \\ &+ \left[F_\alpha, F_\nu\right]\left[F_\mu, F_\beta\right] + \left[F_\mu, F_\alpha\right]\left[F_\beta, F_\nu\right] .\end{aligned} \tag{S16}$$

The first two terms on the right hand side of the above equality are zero when taking the expectation over the state and in the thermodynamic limit. This can be understood as follows. All terms in the above equation actually consist of the product of two operators which have the scaling of magnetization operators. For the first two terms, one of the two is given by a fluctuation operator having a further scaling $\frac{1}{\sqrt{N}}$, which indeed transforms it into an operator with a scaling $\frac{1}{N}$. However, fluctuation operators are rescaled with respect to their average over the state so that, under any expectation over the state, the operator $F_\alpha/\sqrt{N} \to 0$.

Concerning the remaining two terms [the ones in the second line of Eq. (S16)] we can use the fact that they are magnetization operators to argue that they converge to the product of their expectation. This leads to

$$\langle \mathcal{A}\left[F_\alpha F_\beta\right]\rangle \approx -\left(sA\,s\right)_{\alpha\beta} .$$

We are thus left with the second dissipative contribution, the one related to the matrix B, which acts on fluctuation operators as

$$\begin{aligned}\mathcal{B}\left[F_\alpha F_\beta\right] &= \frac{\mathrm{i}}{N}\sum_{\mu\nu}\frac{B_{\mu\nu}}{2}\left\{\left[J_\mu, F_\alpha F_\beta\right], J_\nu\right\} = \\ &= \mathrm{i}\sum_{\mu\nu}\frac{B_{\mu\nu}}{2}\left\{\left[F_\mu, F_\alpha F_\beta\right], F_\nu\right\} + \mathrm{i}\sum_{\mu\nu}B_{\mu\nu}\frac{\langle J_\nu\rangle}{N}\left[J_\mu, F_\alpha F_\beta\right] .\end{aligned} \tag{S17}$$

We divide the above term into two pieces. The first on the right hand side of the second equality we call it $\mathcal{B}'$ while the second $\mathcal{B}''$. We focus on the first and, exploiting commutation relations of fluctuation operators, obtain

$$\mathcal{B}'\left[F_\alpha F_\beta\right] = -\sum_{\mu\nu}\frac{B_{\mu\nu}}{2}\left\{s_{\mu\alpha}F_\beta, F_\nu\right\} - \sum_{\mu\nu}\frac{B_{\mu\nu}}{2}\left\{F_\alpha s_{\mu\beta}, F_\nu\right\} ; \tag{S18}$$

under expectation over the state this contributes with

$$\langle \mathcal{B}'\left[F_\alpha F_\beta\right]\rangle \approx -\sum_{\mu\nu}B_{\mu\nu}s_{\mu\alpha}\Sigma_{\nu\beta} - \sum_{\mu\nu}B_{\mu\nu}s_{\mu\beta}\Sigma_{\alpha\nu} . \tag{S19}$$

We are then left with the second part of this term. This gives

$$\begin{aligned}\mathcal{B}''\left[F_\alpha F_\beta\right] &= \mathrm{i}F_\alpha\sum_{\mu\nu}B_{\mu\nu}\frac{\langle J_\nu\rangle}{N}\left[J_\mu, F_\beta\right] + \mathrm{i}\sum_{\mu\nu}B_{\mu\nu}\frac{\langle J_\nu\rangle}{N}\left[J_\mu, F_\alpha\right]F_\beta \\ &= -\sum_\gamma\left[\sum_{\mu\nu}B_{\mu\nu}\frac{\langle J_\nu\rangle}{N}\epsilon_{\mu\beta\gamma}\right]F_\alpha\frac{J_\gamma}{\sqrt{N}} - \sum_\gamma\left[\sum_{\mu\nu}B_{\mu\nu}\frac{\langle J_\nu\rangle}{N}\epsilon_{\mu\alpha\gamma}\right]\frac{J_\gamma}{\sqrt{N}}F_\beta .\end{aligned} \tag{S20}$$

Under expectation we thus have

$$\langle \mathcal{B}''\left[F_\alpha F_\beta\right]\rangle \approx \sum_\gamma D^{\mathrm{B}}_{\alpha\gamma}C_{\gamma\beta} + \sum_\gamma D^{\mathrm{B}}_{\beta\gamma}C_{\alpha\gamma} \tag{S21}$$

with

$$D^{\mathrm{B}}_{\alpha\gamma} = -\sum_{\mu\nu}B_{\mu\nu}\langle m_\nu\rangle\epsilon_{\mu\alpha\gamma} .$$

Putting these results together, we obtain the exact differential equation for the evolution of C which, in thermodynamic limit $N \to \infty$, becomes

$$\frac{d}{dt}C = -sAs + \left(D + Q^{\mathrm{C}}\right)C + C\left(D + Q^{\mathrm{C}}\right)^T + Q^{\mathrm{B}}\Sigma + \Sigma\left(Q^{\mathrm{B}}\right)^T$$

where

$$D := D^{\mathrm{L}} + D^{\mathrm{C}} + D^{\mathrm{B}}\,, \qquad \text{with} \qquad Q^{\mathrm{B}} = sB\,.$$

Finally, using the fact that $\Sigma = [C + C^T]/2$ we find [79, 80]

$$\frac{d}{dt}\Sigma = -sAs + G\Sigma + \Sigma G^T\,, \tag{S22}$$

with

$$G = D + Q\,, \qquad \text{and} \qquad Q = Q^{\mathrm{C}} + Q^{\mathrm{B}}$$

In our specific setting, the relevant matrices have the following form

$$D = \begin{pmatrix} 0 & 0 & \Gamma\langle m_x\rangle \\ 0 & 0 & \kappa\Gamma\langle m_y\rangle - 2\Omega \\ -\Gamma\langle m_x\rangle & 2\Omega - \kappa\Gamma\langle m_y\rangle & 0 \end{pmatrix} \qquad Q = \begin{pmatrix} \Gamma\langle m_z\rangle & 0 & 0 \\ 0 & \kappa\Gamma\langle m_z\rangle & 0 \\ -\Gamma\langle m_x\rangle & -\kappa\Gamma\langle m_y\rangle & 0 \end{pmatrix}\,. \tag{S23}$$

We recall here that all quantities $\langle m_\alpha\rangle$ are actually time-dependent and obey the system of differential equations (S7). Whenever they appear in Eq. (S22) they must be considered at the running time t. As such the actual solution of Eq. (S22) is given by

$$\Sigma(t) = X_{t,0}\Sigma(0)X_{t,0}^T - \int_0^t du X_{t,u}s(u)As(u)X_{t,u}^T\,, \tag{S24}$$

where the matrix $s(t)$ is the matrix s in Eq. (S9) evaluated at time t and $X_{t,u}$ is the time-order exponential of the matrix G, such that

$$\frac{d}{dt}X_{t,u} = G(t)X_{t,u}\,, \qquad \text{and} \qquad \frac{d}{du}X_{t,u} = -X_{t,u}G(u)\,.$$