Nonequilibrium phase diagram of an atom-cavity system with three-level atoms

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- Quantum systems – Time crystals – Dark states
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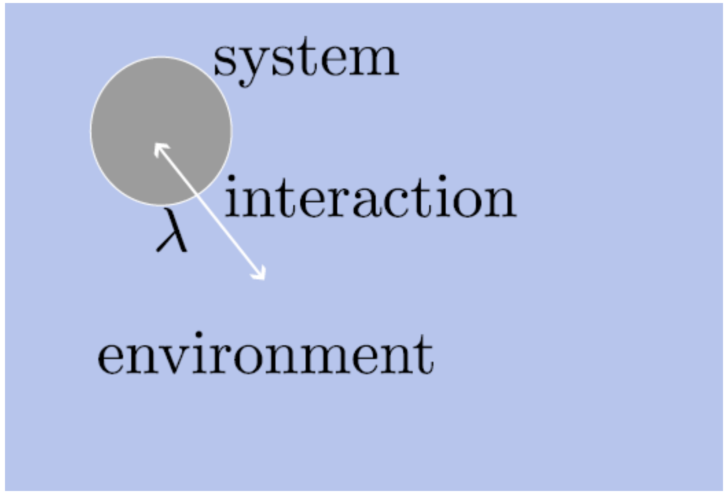
Theoretical backgrounds: Quantum systems

- Open quantum systems are those that cannot be considered entirely isolated, as they interact with their external environment.

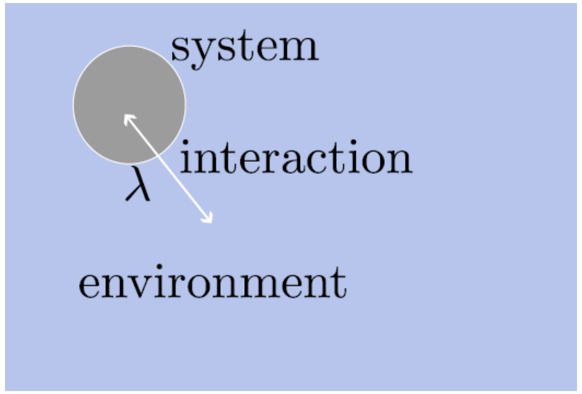


Unlike isolated systems, the evolution is not purely unitary in time.

These systems are influenced e.g. by - driving fields or dissipation rates.

We consisder a generic quantum systems whose state is encoded in the pure state $|\Psi\rangle$.

The compound systems environment dynamics can be described by



$$i\hbar \frac{\partial}{\partial t} |\Psi(t)\rangle = H|\Psi(t)\rangle$$

With $H = H_{sys} + H_{env} + \lambda H_{int}$.

H_{sys} - represents the „isolated“ system only
H_{env} - is the Hamiltonian of the environment alone
H_{int} - interaction between bath and system
λ - adimensional constant representing interaction strength

one is interested in the reduced dynamics of the system only. The latter reduced system state is

$$\rho_S(t) = \mathrm{Tr}_{env}(U\rho_{tot}(0)U^*)$$

U – time evolution operator
$\rho_{tot}(0)$ - initial state of the combined system and environment

Challenging or impossible to solve in most cases !

in the context of weak interaction strengths ($\lambda \ll 1$), it is common to assume no initiall correlations $\rho_{\text{tot}}(0) = \rho_S(0) \otimes \rho_{\text{env}}$.

This simplifies the analysis significantly!

We can further assume that both the system and its environment exhibit Markovian dynamics. Consequently, the time evolution of treduced state is well-approx. By

$$\frac{d}{dt}\rho(t) = \mathcal{L}[\rho_S(t)].$$

$\mathcal{L}[.]$ is the time independent generator, reflecting Markovian dynamics.

Its action on the density operator is given by

$$\mathcal{L}[\rho_S(t)] = -\frac{i}{\hbar}[H_{\text{sys}}, \rho] + \sum_k \gamma_k \left(L_k \rho L_k^* - \frac{1}{2}(\rho L_k^* L_k + L_k^* L_k \rho) \right).$$

L_k - „jump operators“

γ_k - rates for jump processes

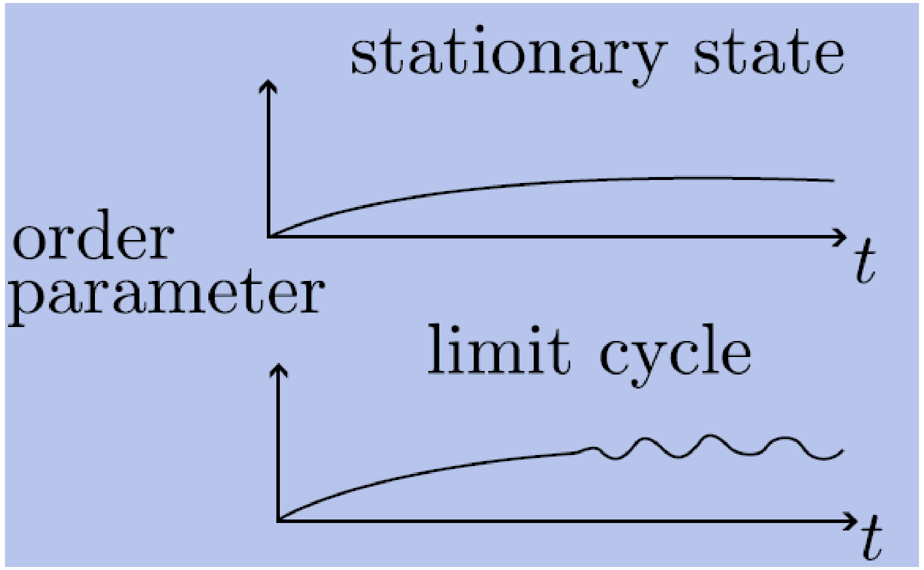
$\mathcal{O}$ - operator of the system

The evolution of system operators is given by the action of the adjoint generator

$$\mathcal{L}^*[\mathcal{O}] = \dot{\mathcal{O}} = \frac{i}{\hbar}[H, \mathcal{O}] + \sum_n \gamma_n \left(L_n^* \mathcal{O} L_n - \frac{1}{2}\{L_n^* L_n, \mathcal{O}\} \right).$$

Theoretical backgrounds: Time crystals

- Instead of relaxing to a stationary state, some systems exhibit periodic oscillations
- They brake temporal translation symmetry.
- A so-called limit cycle is reached.



In the context of Markovian open quantum systems, time crystals can be understood as follows.

The formal evolution of the state can be written as $\rho(t) = e^{\mathcal{L}t}[\rho(0)]$. $e^{\mathcal{L}t}$ - time translation operator

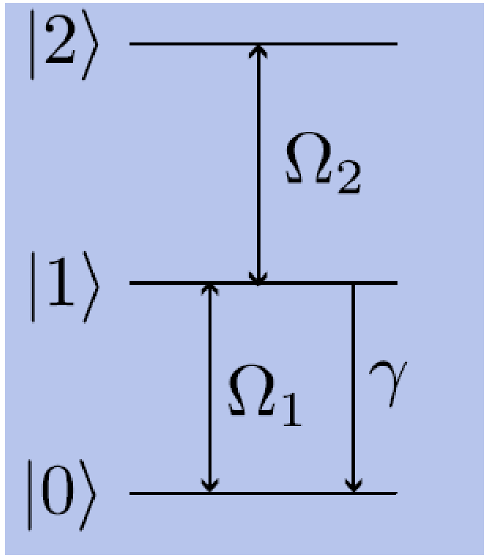
This evolution exhibits continuous time translation symmetry.

If $\rho_{\mathrm{LC}}(\mathrm{t}+\mathrm{T}) = \rho_{\mathrm{LC}}(\mathrm{t})$, a time crystal emerges. Therby spontaneously breaking the continous time translation symmetry. This can be seen as $e^{\mathcal{L}T}\left[\rho_{\mathrm{LC}}(t)\right] \neq \rho_{\mathrm{LC}}(t+T)$.

Theoretical backgrounds: Dark states

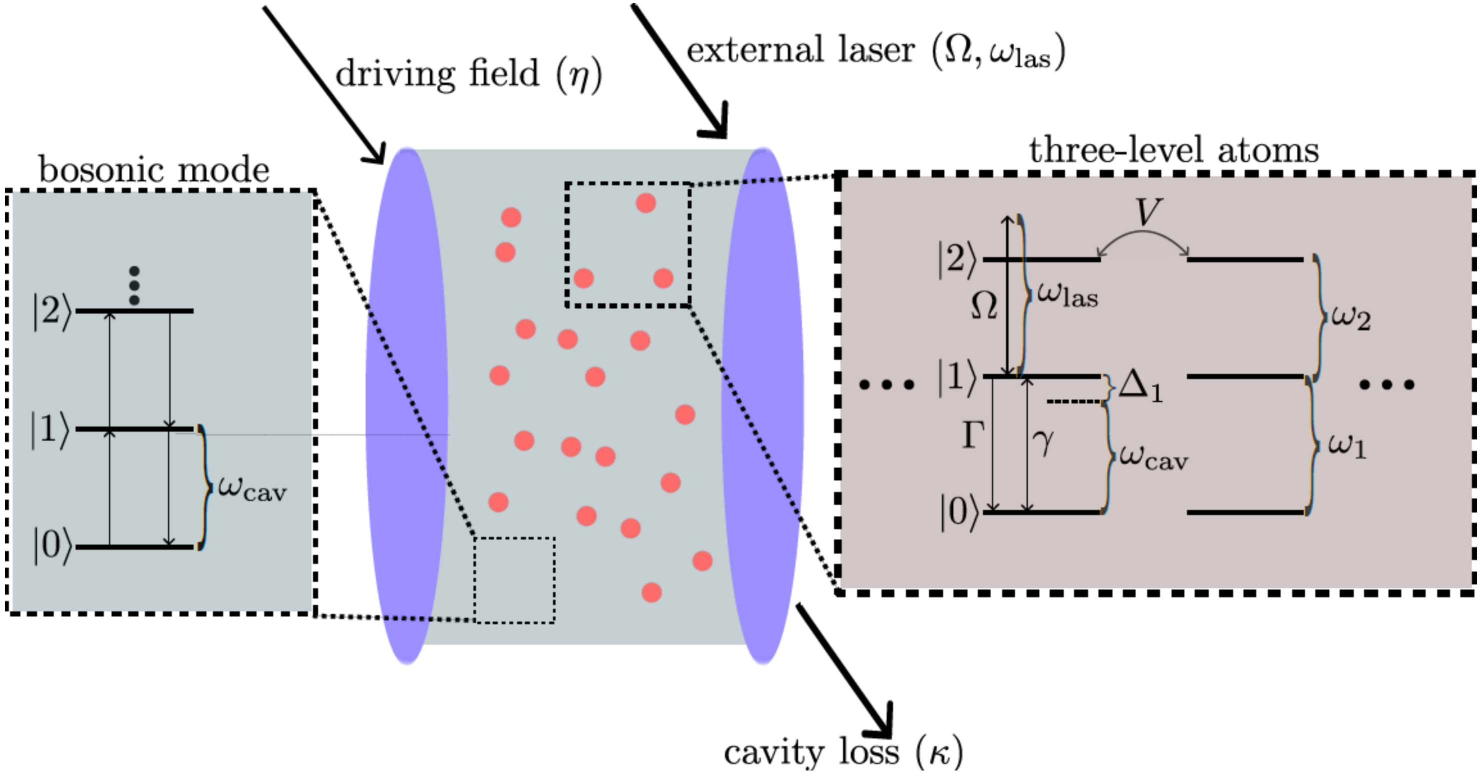
- A dark state is a pure stationary state $\rho_{SS} = |D\rangle\langle D|$.
- The state must fulfill the conditions:
- This implies, this state is an eigenstate of the Hamiltonian.
- Also not „interacting“ with the environment.
- A paradigm for such a system would be:
- With dark state:

$$|D\rangle = \frac{1}{\sqrt{\Omega_1^2 + \Omega_2^2}} \left(\Omega_2|0\rangle - \Omega_1|2\rangle\right)$$



Framework

- We look at a system composed of three-level atoms confined within an optical cavity.



ω_{cav} - cavity frequency

ω_{las} - laser frequency (resonant with cavity)

ω_1, ω_2 - cavity transition frequencys

η – driving field amplitude

Γ – atomic dissipation rate

κ – cavity dissipation rate

γ - how should i label this ?

Δ_1 - detuning of first excited state

Ω – Rabi frequency of external laser

V – all to all atom interaction potential

Framework - Formalism

We will use the creation and annihilation operators a, a^* to represent the degress of freedom of the cavity field and the operators $\sigma_{ab}^{(k)} = |a\rangle^{(k)} \langle b|^{(k)}$ for the atoms.

The Hamiltonian is given by $H = H_{\text{cav}} + H_{\text{at}} + H_{\text{int}} + H_{\text{las}} + H_{\text{at–at}} + H_{\text{pump}}$.
With

$$H_{\text{cav}} = \hbar\omega_{\text{cav}} a^* a,$$

$$H_{\text{pump}} = i\sqrt{N}\hbar\eta\left(a^* e^{i\omega_{\text{cav}} t} - a e^{-i\omega_{\text{cav}} t}\right),$$

$$H_{\text{at}} = \hbar \sum_k \left(\omega_1 \sigma_{11}^{(k)} + \omega_2 \sigma_{22}^{(k)}\right),$$

$$H_{\text{at–at}} = \frac{\hbar V}{N} \sum_{k,m} \sigma_{22}^{(k)} \sigma_{22}^{(m)},$$

$$H_{\text{las}} = \hbar\Omega \sum_k \left(\sigma_{12}^{(k)} + \sigma_{21}^{(k)}\right) \cos(\omega_{\text{las}} t),$$

$$H_{\text{int}} = \frac{\hbar\gamma g_0}{\sqrt{N}} \sum_k \left(\sigma_{01}^{(k)} + \sigma_{10}^{(k)}\right) (a + a^*).$$

To simplify the treatment, switch to rotating frame.

The Hamiltonian becomes time independen after this transformation.

We use $H_{\text{eff}} = ZHZ^* + i\frac{\partial Z}{\partial t}Z^*$, where Z describes the transformation, given by:

$$Z = e^{i\omega_{\text{cav}} a^* a \cdot t} \otimes \prod_k e^{i\omega_{\text{cav}}\sigma_{11}^{(k)} \cdot t} \cdot e^{i(\omega_{\text{cav}}+\omega_{\text{las}})\sigma_{22}^{(k)} \cdot t}.$$

While performing the calculation, we used a RWA.

The time independent Hamiltonian in the rotating frame reads

$$H_{\text{eff}} = \hbar \sum_k \left[\left(\Delta_1 \sigma_{11}^{(k)} + \Delta_2 \sigma_{22}^{(k)} \right) + \frac{\gamma g_0}{\sqrt{N}} \left(\sigma_{01}^{(k)} a^* + \sigma_{01}^{(k)} a \right) + \frac{\Omega}{2} \left(\sigma_{12}^{(k)} + \sigma_{21}^{(k)} \right) + i\sqrt{N}\eta(a^* - a) \right] + \hbar \frac{V}{N} \sum_{k,n} \sigma_{22}^{(k)} \sigma_{22}^{(n)}$$

$\Delta_1 = \omega_1 - \omega_{\text{cav}}$ $\Delta_2 = \omega_2 - \omega_{\text{las}} - \omega_{\text{cav}}$ carefull ! Δ_2 is not an actual detuning in the classical sense!

Framework - Equations of motion

- We will use the Heisenberg picture, where we evolve operaotrs.
- We do so by applying

$$\mathcal{L}^*[\mathcal{O}] = \dot{\mathcal{O}} = \frac{i}{\hbar}[H, \mathcal{O}] + \sum_n \gamma_n \left(L_n^* \mathcal{O} L_n - \frac{1}{2}\{L_n^* L_n, \mathcal{O}\} \right).$$

We have to sets of rates and jump operators:

- atomic dissipation with rate Γ and operator $L_k = \sigma_{01}^{(k)}$

- cavity dissipation with rate κ and operator $L_a = a$

- We are interested in the volution of collective, macroscopic behaviors.
- We introduce the rescaled bosonic operators α^N and average magnetization m_{ij}^N:

$$\alpha^N = \frac{a}{\sqrt{N}}, \qquad m_{ij}^N = \frac{\sum_{k=1}^{N} \sigma_{ij}^{(k)}}{N}.$$

With N being the particle number.

- These are our mean-field operators, they are experimentally accessible.
- They become classical variables in the thermodynamica limit.
- We assume no emergent correlations between these operators.

(mean-field assumption)

Our equations of motion read

$$
\begin{aligned}
\dot{\alpha}(t) \quad &= -\frac{\kappa\alpha(t)}{2} - i\gamma g_0 m_{01}(t) + \eta,\\
\dot{m}_{00}(t) &= \Gamma m_{11}(t) + i\gamma g_0\big(m_{10}(t)\alpha(t) - m_{01}(t)\alpha^*(t)\big),\\
\dot{m}_{11}(t) &= -\Gamma m_{11}(t) + i\gamma g_0\big(m_{01}(t)\alpha^*(t) - m_{10}(t)\alpha(t)\big) + \frac{i\Omega}{2}\big(m_{21}(t) - m_{12}(t)\big),\\
\dot{m}_{22}(t) &= i\frac{\Omega}{2}\big(m_{12}(t) - m_{21}(t)\big),\\
\dot{m}_{21}(t) &= -\frac{\Gamma}{2}m_{21}(t) + i\big(\Delta_2 m_{21}(t) - \Delta_1 m_{21}(t) - \gamma g_0 m_{20}(t)\alpha(t)\big)\, i\left(+\frac{\Omega}{2}\big(m_{11}(t) -\right.\\
&\left. m_{22}(t)\big) + 2V m_{21}(t) m_{22}(t)\right),\\
\dot{m}_{01}(t) &= -\frac{\Gamma}{2}m_{01}(t) + i\left(-\Delta_1 m_{01}(t) + \gamma g_0\big(m_{11}(t)\alpha(t) - m_{00}(t)\alpha(t)\big) - \frac{\Omega}{2}m_{02}(t)\right),\\
\dot{m}_{20}(t) &= i\left(\Delta_2 m_{20}(t) + \frac{\Omega}{2}m_{10}(t) + 2V m_{20}(t)m_{22}(t) - \gamma g_0 m_{21}(t)\alpha^*(t)\right).
\end{aligned}
$$

Note that, for the model at hand, the mean-field equations are exact in the thermodynamic limit.

Phase diagram

- We will numerically solve the mean-field equations.
- We use the time averaged order parameter $\bar{m}_{ij}$ and time variance ς_{ij}, defined by

$$\bar{m}_{ij} = \frac{1}{T}\int_{t_{\mathrm{fin}}-T}^{t_{\mathrm{fin}}} m_{ij}(t)dt\,, \qquad \varsigma_{ij} = \frac{1}{T}\int_{t_{\mathrm{fin}}-T}^{t_{\mathrm{fin}}} \left(m_{ij}(t) - \bar{m}_{ij}\right)^2 dt,$$

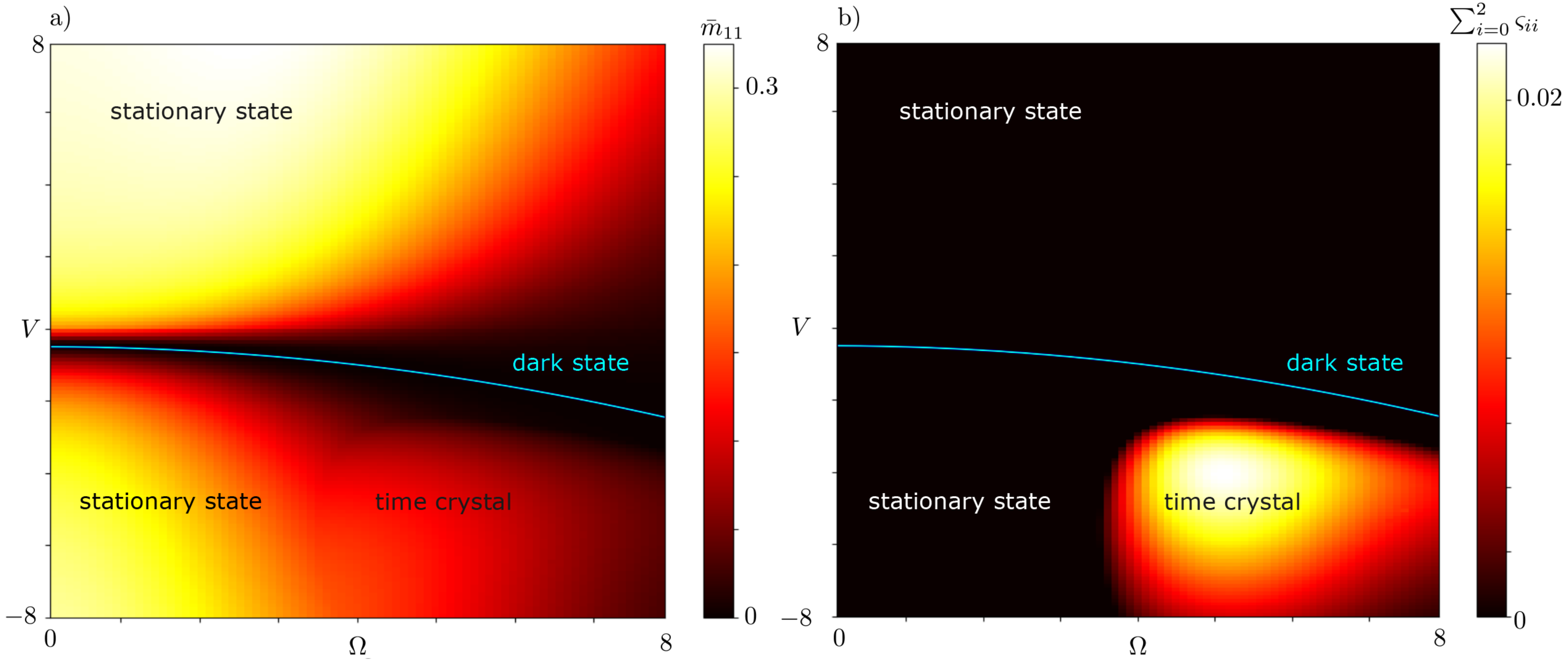
to classify phases.

t_{fin} - final simulated time step
T - window over which we average

- We have determind the condition for a stationary state being dark to be

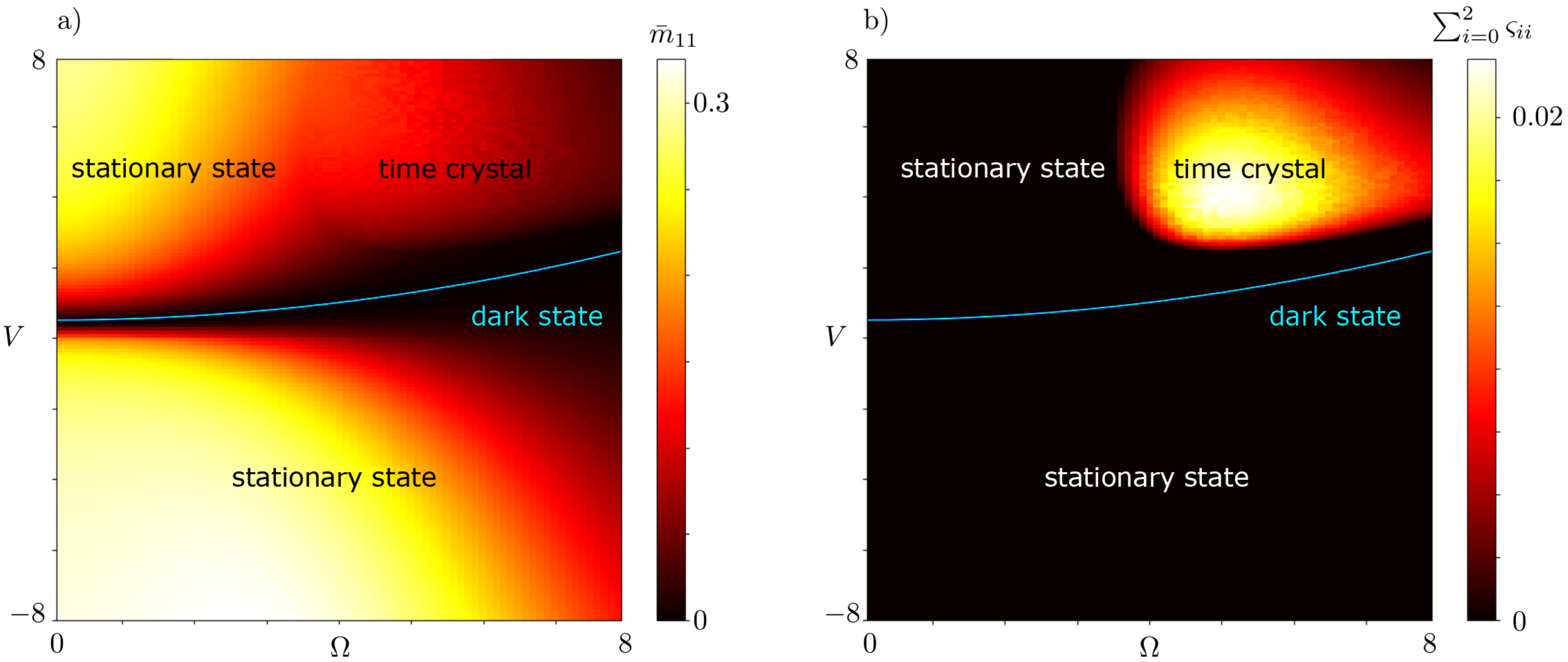
$$V = V_\mathrm{D} = -\frac{\Delta_2}{2}\left(\left(\frac{\Omega\kappa}{4\eta\gamma g_0}\right)^2 + 1\right).$$

- We use initial state $\mathrm{m}_{00}(0) = 1$, everything else zero.

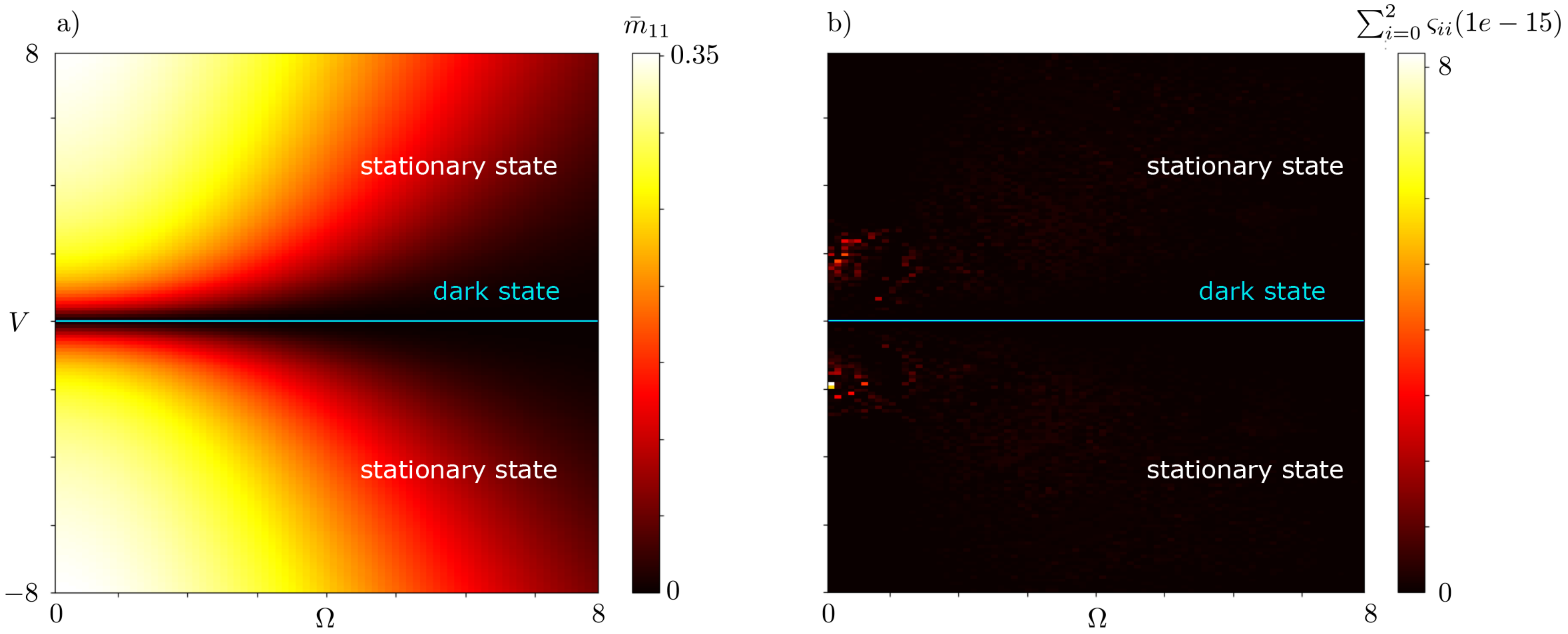


Γ=2 κ,
V,Ω in units of κ.
All other parameters are equal to κ

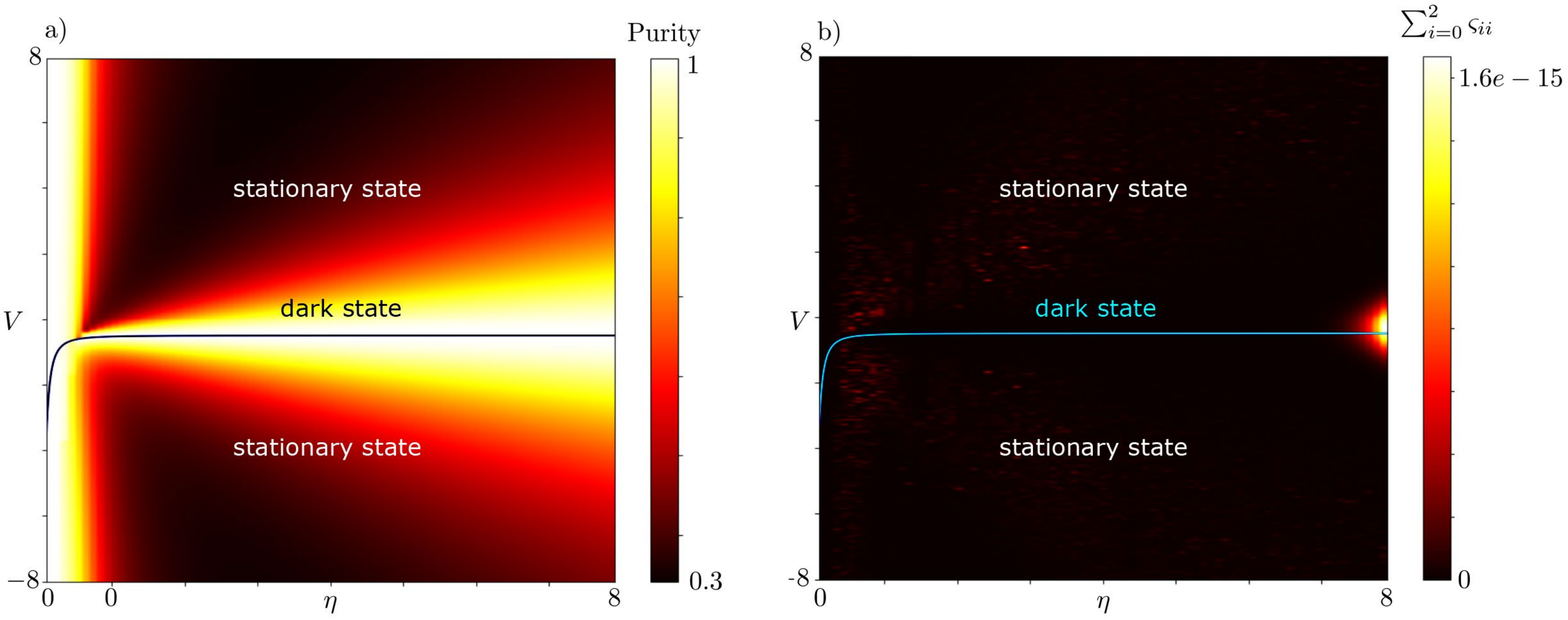
Here $\Delta_1 = \Delta_2 = -\kappa$



Here $\Delta_1 = \Delta_2 = 0$. No time crystal can be found!



$\Delta_1 = \Delta_2 = 0, \Omega = \kappa.$



Is it possible to link dark state to EIT ?

- Yes.
- Impose no interaction with first excited state

$$m_{11}(0) = m_{01}(0) = m_{21}(0) = m_{12}(0) = m_{10}(0) = 0.$$

$$\dot{\alpha} = \frac{-\kappa\alpha}{2} + \eta,$$

$$\dot{m}_{00} = 0,$$

$$\dot{m}_{11} = 0,$$

$$\dot{m}_{22} = 0,$$

$$\dot{m}_{20} = i(\Delta_2 m_{20} + 2V m_{20} m_{22}),$$

$$\dot{m}_{21} = i\left(-\gamma g_0 m_{20}\alpha + \frac{\Omega}{2} m_{11}\right).$$

Bosonic and atomic systems are decoupled!

This state is emergent -> only exact in thermodynamic limit.

Stability

- For EIT, small atomic perturbations could not be found propagating to α.
- We analysed the dark state and mixed stationary state stability. Small perturbations and linearistation were used. They were found to be mostly stable.
- The time crystal was analyzed using Floquet analysis and found to be stable.
- No bistability or coexistence could be found or indicated.
- A limit cycle was approached by the time crystal.

Outlook

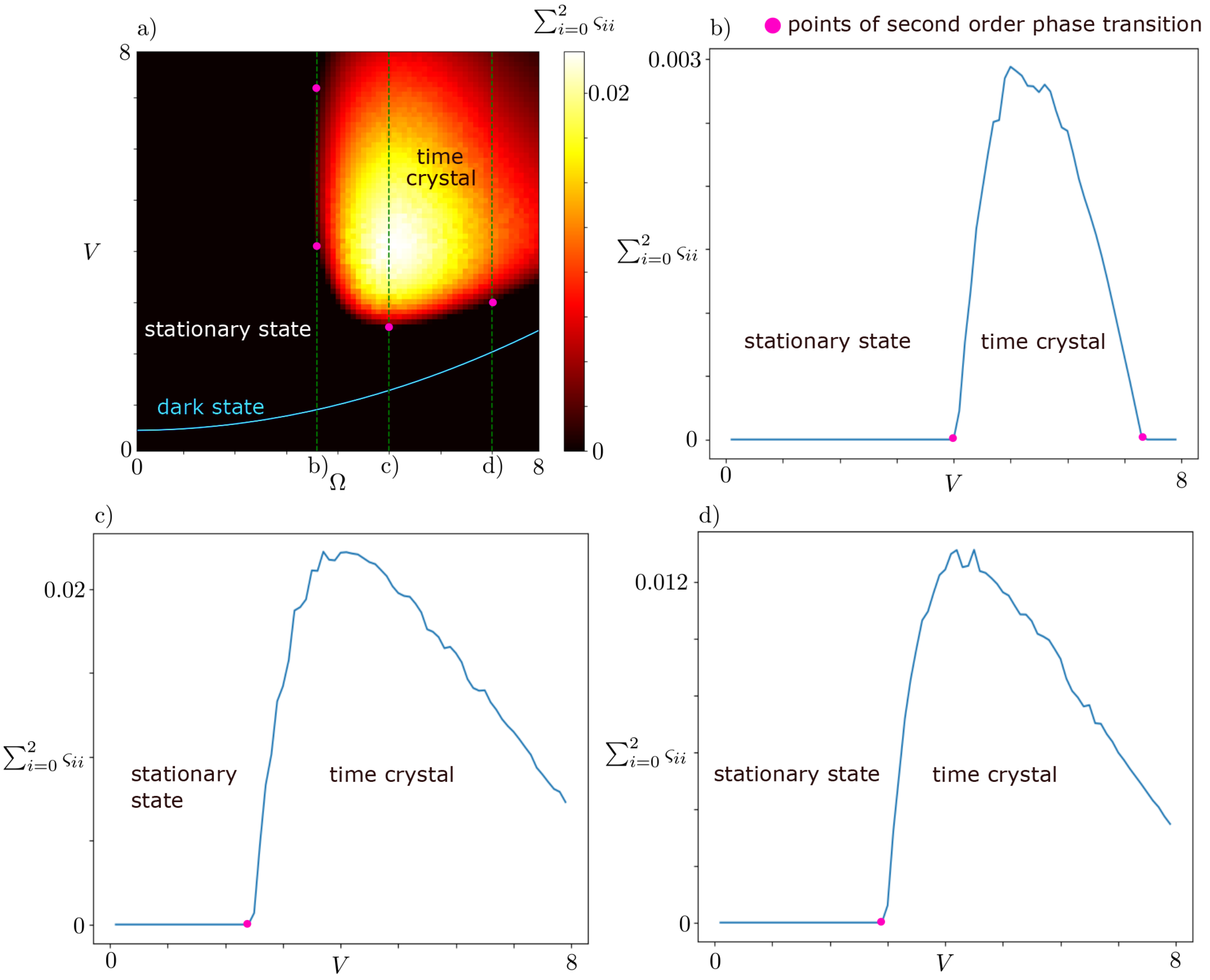
- Investigate quantum fluctuations.
- Compare with real experimental data of similar systems.
- Explore more parameter regimes.

Conclusion

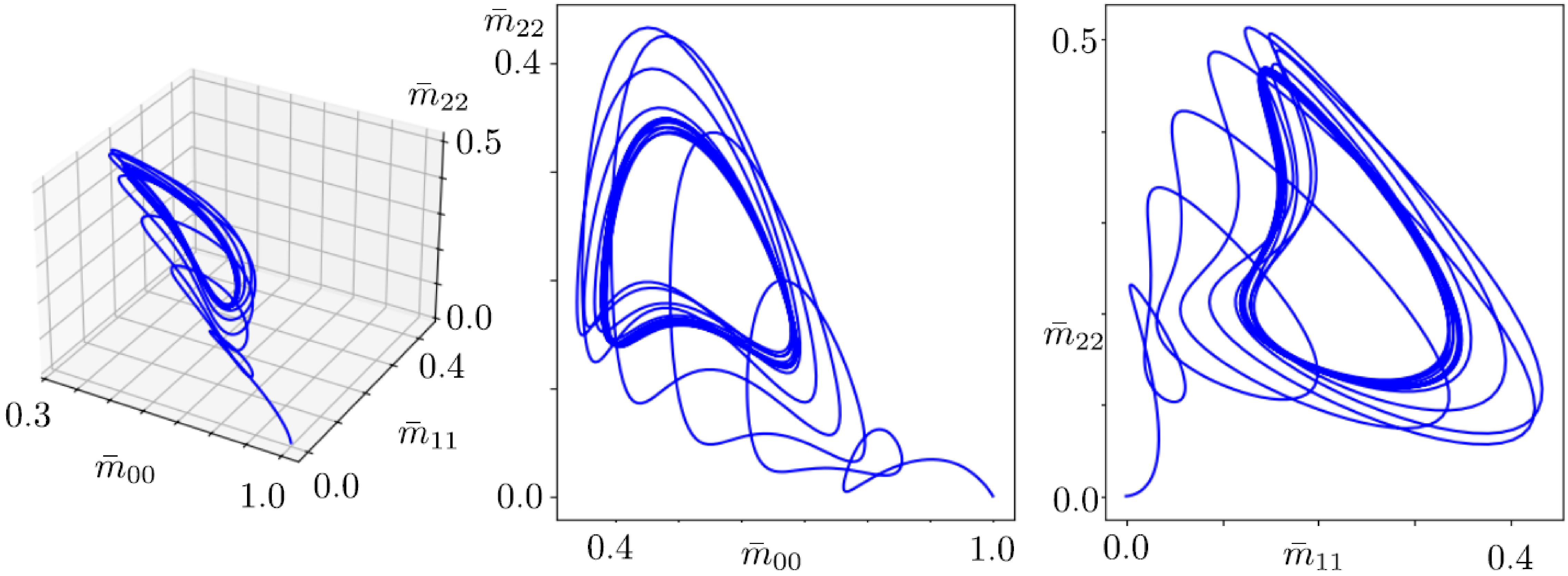
- We set up the equations of motion for our system
- We found a analytical classification for the dark state
- We investigated the stability of our phases

- Thanks for your attention!

Time crystal extra material



Limit cycle in magnetization space



Stationary- / Dark- state stability

- We define a fixed point and slightly perturb the system.
- We approximate by performing a Taylor expansion, only retaining linear terms.
- The full system equations read

$$A = \begin{bmatrix} -\frac{\kappa}{2} & 0 & 0 & 0 & 0 \\ 0 & -\frac{\kappa}{2} & 0 & 0 & 0 \\ i\gamma g_0 m_{10} & -i\gamma g_0 m_{01} & 0 & \Gamma & 0 \\ -i\gamma g_0 m_{10} & i\gamma g_0 m_{01} & 0 & -\Gamma & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\frac{d}{dt}\begin{pmatrix} \delta x \\ \delta y \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} \delta x \\ \delta y \end{pmatrix},$$

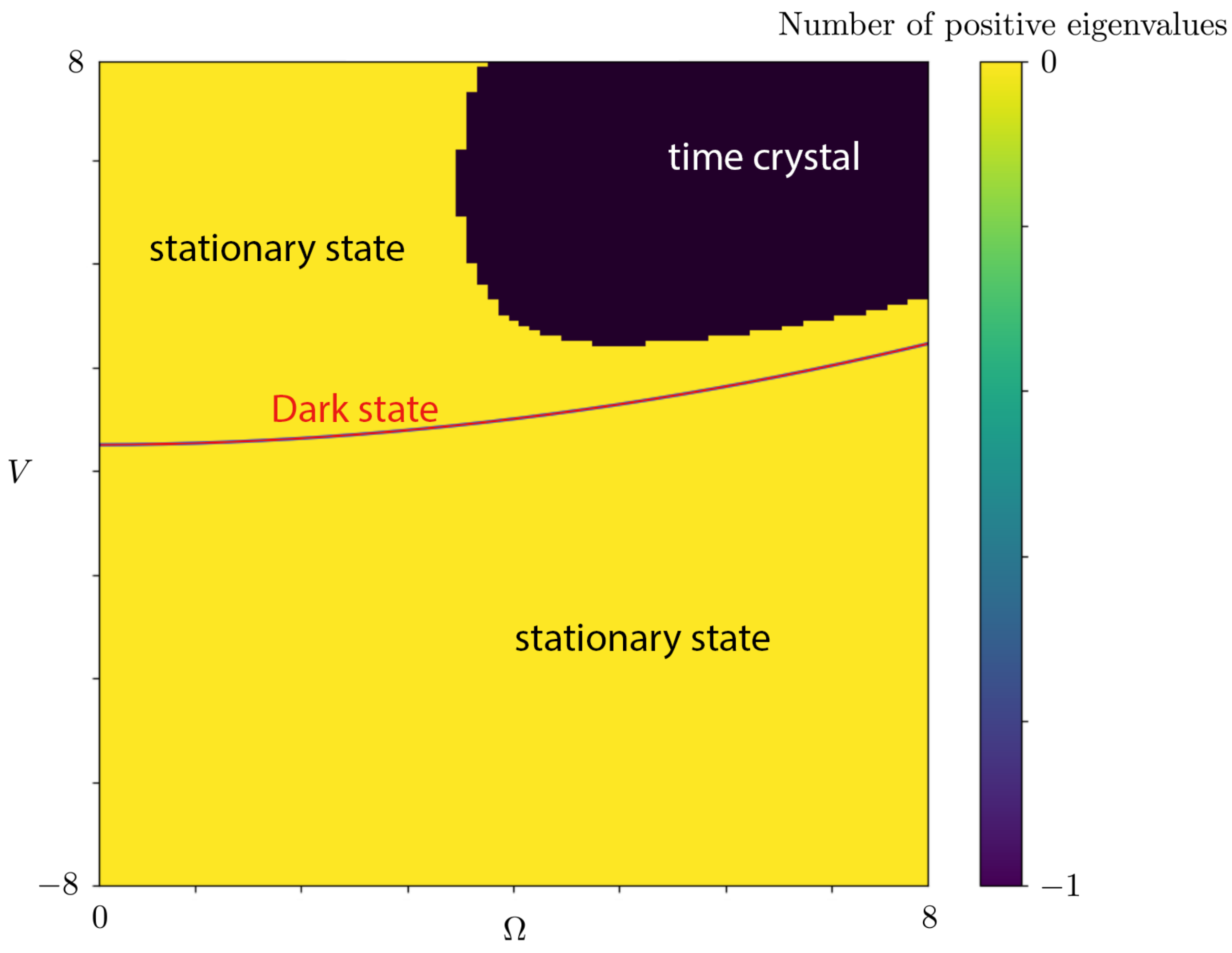
$$B = \begin{bmatrix} 0 & -i\gamma g_0 & 0 & 0 & 0 & 0 \\ i\gamma g_0 & 0 & 0 & 0 & 0 & 0 \\ i\gamma g_0\alpha & -i\gamma g_0\alpha^\dagger & 0 & 0 & 0 & 0 \\ -i\gamma g_0\alpha & i\gamma g_0\alpha^\dagger & \frac{i\Omega}{2} & -\frac{i\Omega}{2} & 0 & 0 \\ 0 & 0 & -\frac{i\Omega}{2} & \frac{i\Omega}{2} & 0 & 0 \end{bmatrix},$$

$$\delta x = \begin{bmatrix} \delta\alpha \\ \delta\alpha^\dagger \\ \delta m_{00} \\ \delta m_{11} \\ \delta m_{22} \end{bmatrix}, \qquad \delta y = \begin{bmatrix} \delta m_{10} \\ \delta m_{01} \\ \delta m_{21} \\ \delta m_{12} \\ \delta m_{20} \\ \delta m_{02} \end{bmatrix},$$

$$C = \begin{bmatrix} 0 & -i\gamma g_0(m_{11}-m_{00}) & i\gamma g_0\alpha^\dagger & -i\gamma g_0\alpha^\dagger & 0 \\ i\gamma g_0(m_{11}-m_{00}) & 0 & -i\gamma g_0\alpha & i\gamma g_0\alpha & 0 \\ -i\gamma g_0 m_{20} & 0 & 0 & \frac{i\Omega}{2} & 2iVm_{21} - \frac{i\Omega}{2} \\ 0 & i\gamma g_0 m_{02} & 0 & -\frac{i\Omega}{2} & -2iVm_{12} + \frac{i\Omega}{2} \\ 0 & -i\gamma g_0 m_{21} & 0 & 0 & 2iVm_{20} \\ i\gamma g_0 m_{12} & 0 & 0 & 0 & -2iVm_{02} \end{bmatrix},$$

$$\mathrm{D} = \begin{bmatrix} -\frac{\Gamma}{2} + i\Delta_1 & 0 & 0 & 0 & \frac{i\Omega}{2} & 0 \\ 0 & -\frac{\Gamma}{2} - i\Delta_1 & 0 & 0 & 0 & -\frac{i\Omega}{2} \\ 0 & 0 & -\frac{\Gamma}{2} + i(\Delta_2 - \Delta_1 + 2Vm_{22}) & 0 & -i\gamma g_0\alpha & 0 \\ 0 & 0 & 0 & -\frac{\Gamma}{2} + i(\Delta_1 - \Delta_2 - 2Vm_{22}) & 0 & i\gamma g_0\alpha^\dagger \\ \frac{i\Omega}{2} & 0 & -i\gamma g_0\alpha^\dagger & 0 & i(\Delta_2 + 2Vm_{22}) & 0 \\ 0 & -\frac{i\Omega}{2} & 0 & i\gamma g_0\alpha & 0 & -i(\Delta_2 + 2Vm_{22}) \end{bmatrix}.$$

for certain small Γ values, positive eigenvalues appear, indicating instability.



Γ=2 κ, $\Delta_1 = \Delta_2 = -\kappa$

V,Ω in units of κ.

All other parameters are equal to κ