\documentclass[12pt,titlepage]{article}

\usepackage[utf8]{inputenc}

\usepackage[T1]{fontenc}

\usepackage[ngerman]{babel}

\usepackage{amsmath}

\usepackage{amsfonts}

\usepackage{amssymb}

\usepackage{lmodern}

\usepackage{graphicx}

\usepackage{enumerate}

\usepackage{stmaryrd}

\usepackage[dvipsnames,svgnames,x11names]{xcolor}

\usepackage[left=2cm,right=3.5cm,top=0cm,bottom=2cm,includeheadfoot]{geometry}

\usepackage[markup=underlined]{changes}

\usepackage{microtype}

\usepackage{braket}

\usepackage{makeidx}

\usepackage{hyperref}

\usepackage{float}

\usepackage{subcaption}

\usepackage{tikz}

\usepackage{amsmath}

\usepackage{graphicx}

\usepackage{geometry}

\usetikzlibrary{arrows.meta, decorations.pathmorphing, positioning}

\usepackage{array}

\setcounter{MaxMatrixCols}{11}

\setlength {\marginparwidth }{2cm}

\newcommand{\ketbramode}[2]{\sigma\_{#1}^{(#2)}}

\newcommand{\dotketbra}[1]{\Dot{\sigma}\_{#1}^{(k)}}

\newcommand{\expect}[1]{\left\langle #1\right\rangle}

\newcommand{\ketbra}[1]{\sigma\_{#1}}

\geometry{a4paper, total={170mm,257mm}, left=10mm, right=10mm, top=20mm, bottom=20mm}

\begin{document}

\begin{titlepage}

\begin{center}

\LARGE{EBERHARD KARLS UNIVERSITÄT TÜBINGEN}

\vspace{5mm}

\\ \large{Physikalisches Institut}

\vspace{5mm}

\\ \LARGE{Test}

\end{center}

\vspace{15mm}

\begin{center}

{Some calcs}

\end{center}

\vspace{30mm}

\begin{minipage}[t]{0.47\textwidth}

Haffner, Paul

\end{minipage}

\hfill

\end{titlepage}

\newpage

\tableofcontents

\newpage

\section{Model}

First off, I want to talk about the model which I will be using throughout this Thesis. It consists of a cavity with a frequency $\omega\_{cav}$, the cavity is pumped by an external laser field with rate $\eta$ and frequency $\omega\_p$. The cavity has a loss rate $\kappa$. Inside the Cavity is an atom gas consisting of 3 level atoms. All Atoms are supposed to be equal. The atom has the usual transition frequencies, $\omega\_{1,2}$.

The Ground state of the Atom couples to the first excited state via the cavity Field with rate $\gamma$. The First excited state of the Atoms has a spontaneous decay with rate $\Gamma$. The second excited state is coupled to an external laser which has a frequency $\Omega$. I neglect the position dependence of the atoms in terms of coupling and assume all atoms are placed in a way that maximum coupling strength comes up. The Atom-Atom interaction is described by a potential $V$.\\

\newline

\begin{tikzpicture}[>=Stealth, thick]

% Cavity

\draw[blue, thick] (0,-3) -- (0,3);

\draw[blue, thick] (8,-3) -- (8,3);

\draw[black, thick] (0.5,0) -- (3.5,0);

\draw[black, thick] (4.5,0) -- (7.5,0);

\draw[black, thick] (0.5,2) -- (3.5,2);

\draw[black, thick] (4.5,2) -- (7.5,2);

\draw[black, thick] (0.5,-2) -- (3.5,-2);

\draw[black, thick] (4.5,-2) -- (7.5,-2);

\draw[black, thick] (4.5,-2) -- (7.5,-2);

%pump field

\draw[red,decorate, decoration={snake, amplitude=0.3mm, segment length=3mm}, ->] (-4,0) -- (0,0) node[midway, above] {$\eta$};

%cavity loss

\draw[green,decorate, decoration={snake, amplitude=0.3mm, segment length=3mm}, ->] (8,-1) -- (10,-3) node[midway, right] {$\kappa$};

%Cavity - Atom Interaction

\draw[red, <->] (1.5,-2) -- (1.5,0) node[midway, left] {$\gamma$};

%Loss rate atom

\draw[green,decorate, decoration={snake, amplitude=0.3mm, segment length=3mm}, ->] (2.5,0) -- (2.5,-2) node[midway, right] {$\Gamma$};

%Laser - Atom Interaction

\draw[red, <->] (1.5,0) -- (1.5,2) node[midway, left] {$\Omega$};

%Laser - Atom Interaction

\draw[red, <->] (2.5,2) to[out=45, in=135] (5.5,2) ;

\node[red] at (4,3) {$V$};

%draw in atomic levels

\node at (4,-2){$\ket{0}$};

\node at (4,0) {$\ket{1}$};

\node at (4,2) {$\ket{2}$};

\end{tikzpicture}

\section{Constants}

\textbf{TODO} Re-check for physicality of my system, currently a bit "not thought thru".\\

A number of N atomic level system resonantly coupled to a cavity with coupling constant $g\_0$ is here. \textbf{Problem} Usually 2-Level.\\

Cavity mode volume $V\_{mode}$,$\mu\_m$ dipole matrix element of the corresponding transition where $\mu\_m=\mu\_{red}c\_m$ where $\mu\_{red}$ is the reduced dipole matrix element and $c\_m$ the clebsch-Gordon coefficient.\cite{Rubidium87D}.

\begin{align\*}

g\_m=g\_0 = \mu\_0 \sqrt{\frac{\omega\_a}{\hbar \epsilon\_0 V\_{mode}}}

\end{align\*}

\section{Deriving the Differential equation}

\subsection{Setting up the Hamiltonian}

Now that I have described the model I will be working with, I will start by setting up a Hamiltonian which represents my system. For this purpose, I'm going to introduce the Detuning $\Delta\_{1,2,c}=\omega\_{1,2,c}-\omega\_{p}$. Where again $\omega\_{1,2}$ describes the Atomic frequencies and $\omega\_p$ the pump frequency and $\omega\_c$ the cavity frequency. Also, I will use the usual creation and annihilation operators $a,a^\dagger$ for the cavity Field and $\ketbramode{ab}{k}$ for the atomic transition from $a$ to $b$ of the $k-$th atom.\\

Now my system can be Described by the Hamiltonian:

\begin{align\*}

H=&H\_{cav}+H\_{atom}+H\_{int}+H\_{Laser}+H\_{atom-atom}+H\_{pump}

\end{align\*}

with \cite{Gripp1997}

\begin{align\*}

H\_{cav}=&\hbar [\Delta\_c a^\dagger a ] \\

H\_{atom}=& \hbar \sum\_k\left( \Delta\_1 \ketbramode{11}{k}+\Delta\_2 \ketbramode{22}{k}\right) \\

H\_{int}=&\frac{\hbar \gamma g\_0}{\sqrt{N}} \sum\_k\left(\ketbramode{01}{k}+\ketbramode{10}{k}\right)(a+a^\dagger)& \\

H\_{Laser}=& \hbar \Omega\sum\_k\left(\ketbramode{12}{k}+\ketbramode{21}{k}\right)cos(\omega\_Lt) \\

H\_{atom-atom}=& \hbar V\sum\_{k,n}(\ketbramode{22}{k}\otimes\ketbramode{22}{m}) \\

H\_{pump}=& i \hbar\eta(a^\dagger e^{i\Delta\_c t}-a e^{-i\Delta\_c t}) \\

\end{align\*}

I now want to go ahead and Transform my hamiltonian in a way that it becomes the time independent effective Hamiltonian, therefore I use \cite{OlmosSanchez2023},\cite{breuer2002theory}

\begin{align\*}

H\_e=UHU^\dagger+i\frac{\partial U}{\partial t}U^\dagger

\end{align\*}

Where $U$ describes the Transformation in another frame. In my case A frame Rotating with the Laser.\\

For $U$ I chose this Transformation because it seemed fitting

\begin{align\*}

U=&e^{i\Delta\_c a^\dagger a\cdot t}\otimes\prod\_k e^{i\frac{\Delta\_c}{\sqrt{N}} \ketbramode{11}{k}\cdot t}\cdot e^{i\frac{\Delta\_c+\omega\_L}{\sqrt{N}}\ketbramode{22}{k}\cdot t}\\

\frac{\partial U}{\partial t}=&\left(

i\Delta\_c a^\dagger a e^{i\Delta\_c t a^\dagger a } \otimes \prod\_k e^{i\frac{\Delta\_c}{\sqrt{N}} \ketbramode{11}{k}\cdot t}\cdot e^{i\frac{\Delta\_c+\omega\_L}{\sqrt{N}}\ketbramode{22}{k}\cdot t}\cdot\sum\_j\left[ i\frac{\Delta\_c}{\sqrt{N}} \ketbramode{11}{j}+

i\frac{\Delta\_c+\omega\_L}{\sqrt{N}}\ketbramode{22}{j}\right]

\right)\\

\rightarrow \frac{\partial U}{\partial t}U^\dagger =& i\frac{\Delta\_c}{\sqrt{N}} a^\dagger a \otimes\left(\sum\_k i\Delta\_c \ketbramode{11}{k}+\sum\_k i\frac{\Delta\_c+\omega\_L}{\sqrt{N}}\ketbramode{22}{k}\right)

\end{align\*}

Now to the Hamiltonian, every part of the Hamiltonian needs to be transformed, so I do it step by step. Starting with the Cavity Hamiltonian:

\begin{align\*}

UH\_{cav}U^\dagger = H\_{cav}

\end{align\*}

This can be easily seen through the commutation of all exponential factors with the Hamiltonian.\\

For the Atomic Hamiltonian, the same thought applies, so

\begin{align\*}

UH\_{atom}U^\dagger= H\_{atom}

\end{align\*}

Now for the Interaction Hamiltonian it gets a bit more interesting.

\begin{align\*}

UH\_{int}U^\dagger=&e^{i\Delta\_c a^\dagger a\cdot t}\otimes\prod\_k e^{i\frac{\Delta\_c}{\sqrt{N}} \ketbramode{11}{k}\cdot t}\cdot e^{i\frac{\Delta\_c+\omega\_L}{\sqrt{N}}\ketbramode{22}{k}\cdot t}

\frac{\hbar \gamma g\_0}{\sqrt{N}} \sum\_k\left(\ketbramode{01}{k}+\ketbramode{10}{k}\right)(a+a^\dagger)\\&e^{-i\Delta\_c a^\dagger a\cdot t}\otimes\prod\_k e^{-i\frac{\Delta\_c}{\sqrt{N}} \ketbramode{11}{k}\cdot t}\cdot e^{-i\frac{\Delta\_c+\omega\_L}{\sqrt{N}}\ketbramode{22}{k}\cdot t}

\end{align\*}

The Commutation of the creation and Annihilation operators as well as the atomic transition can be taken into account separately \cite{OlmosSanchez2023} so

\begin{align}

U(a+a^\dagger)U^\dagger =& ae^{-i\frac{\Delta\_c}{\sqrt{N}}t } + a^\dagger e^{i\frac{\Delta\_c}{\sqrt{N}}t}\\

U\sum\_k(\ketbramode{01}{k} + \ketbramode{10}{k})U^\dagger =&

\sum\_k \ketbramode{01}{k} e^{-i\frac{\Delta\_c}{\sqrt{N}}t } + \ketbramode{10}{k} e^{i\frac{\Delta\_c}{\sqrt{N}}t}

\end{align}

So in total I get

\begin{align\*}

UH\_{int}U^\dagger=& \frac{\hbar \gamma g\_0}{\sqrt{N}}\left(\sum\_k \ketbramode{01}{k} e^{-i\Delta\_ct}+\ketbramode{10}{k}e^{+i\Delta\_ct}\right) \left[ae^{-i\Delta\_ct } +a^\dagger e^{i\Delta\_ct}\right]\\

=&\frac{\hbar \gamma g\_0}{\sqrt{N}} \sum\_k \left(\ketbramode{01}{k}a e^{-2i\Delta\_ct}

+\ketbramode{01}{k} a^\dagger

+\ketbramode{10}{k} e^{2i\Delta\_c t }a^\dagger

+\ketbramode{10}{k} a

\right)\\

\approx&\frac{\hbar \gamma g\_0}{\sqrt{N}} \sum\_k \left(

\ketbramode{01}{k} a^\dagger+

\ketbramode{10}{k} a

\right)

\end{align\*}

Where in the last step I have applied a rotating wave approximation (RWA).\\

Next I have the Laser Hamiltonian

\begin{align\*}

UH\_{Laser}U^\dagger=&U\hbar\frac{\Omega}{\sqrt{N}}\sum\_k\left(\ketbramode{12}{k} + \ketbramode{21}{k}\cos(\omega\_Lt)\right)U^\dagger\\

=&\hbar \frac{\Omega}{\sqrt{N}} \sum\_k \left( e^{+i\Delta\_ct}\ketbramode{12}{k} e^{-i(\Delta\_c+\omega\_L)t} + e^{-i\Delta\_ct}\ketbramode{21}{k} e^{+i(\Delta\_c+\omega\_L)t} \right) \cos(\omega\_Lt)\\

=&\hbar\frac{\Omega}{\sqrt{N}} \sum\_k \left( e^{+i\Delta\_ct}\ketbramode{12}{k} e^{-i(\Delta\_c+\omega\_L)t} + e^{-i\Delta\_ct}\ketbramode{21}{k} e^{+i(\Delta\_c+\omega\_L)t} \right) \frac{e^{i\omega\_Lt} + e^{-i\omega\_Lt}}{2}\\

=&\frac{\hbar\Omega}{2\sqrt{N}} \sum\_k \left(\ketbramode{12}{k}(e^{2i\omega\_Lt} + 1) + \ketbramode{21}{k}(e^{-2i\omega\_Lt} + 1) \right)\\

=&\frac{\hbar\Omega}{2\sqrt{N}} \sum\_k \left(\ketbramode{12}{k} + \ketbramode{21}{k} \right)\\

\end{align\*}

Where again in the last step I have performed a RWA. One can immediately see that the term is no longer time-dependent.\\

Now to the Atom-Atom Interaction Hamiltonian and the Pump Hamiltonian.

For the Atom-Atom Hamiltonian, it is easy to see that again everything commutes so

\begin{align\*}

UH\_{atom-atom}U^\dagger =&H\_{atom-atom} \\

\end{align\*}

For the Pump we get

\begin{align\*}

UH\_{pump}U^\dagger=& U i \hbar\eta(a^\dagger e^{i\Delta\_c t}-a e^{-i\Delta\_c t})U^\dagger \\

=& i\hbar \cdot \eta (a^\dagger-a)

\end{align\*}

Where I again have applied (1).\\

Now that I got everyting together I can proceed to calculate the total Effective Hamiltonian

\begin{align\*}

H\_{eff}=&UHU^\dagger+i\frac{\partial U}{\partial t}U^\dagger\\

=&\hbar \sum\_k \left[\left( \frac{\Delta\_1}{\sqrt{N}} \ketbramode{11}{k}+\frac{\Delta\_2}{\sqrt{N}} \ketbramode{22}{k}\right) +\Delta\_c a^\dagger a +\frac{ \gamma g\_0}{\sqrt{N}} \left(

\ketbramode{01}{k} a^\dagger+

\ketbramode{10}{k} a

\right)+\frac{\Omega}{2\sqrt{N}} \left(\ketbramode{12}{k} + \ketbramode{21}{k} \right)+i \cdot \eta (a^\dagger-a)\right]\\

&+\hbar \frac{V}{N}\sum\_{k,n}(\ketbramode{22}{k}\otimes\ketbramode{22}{m}) -\Delta\_c a^\dagger a \otimes\left(-\sum\_k \frac{\Delta\_c}{\sqrt{N}} \ketbramode{11}{k}-\sum\_k \frac{\Delta\_c+\omega\_L}{\sqrt{N}}\ketbramode{22}{k}\right) \\

=&\hbar \sum\_k \left[\left( \frac{\textcolor{red}{\Delta\_1}}{\sqrt{N}} \ketbramode{11}{k}+\frac{\textcolor{red}{\Delta\_2}}{\sqrt{N}} \ketbramode{22}{k}\right) +\frac{ \gamma g\_0}{\sqrt{N}} \left(

\ketbramode{01}{k} a^\dagger+

\ketbramode{10}{k} a

\right)+\frac{\Omega}{2\sqrt{N}} \left(\ketbramode{12}{k} + \ketbramode{21}{k} \right)+i \cdot \eta (a^\dagger-a)\right]\\&+\hbar \frac{V}{N}\sum\_{k,n}(\ketbramode{22}{k}\otimes\ketbramode{22}{m})

\end{align\*}

In the last line $\Delta\_1=\omega\_1-\omega\_p$ switches to $\Delta\_1=\omega\_1-\omega\_c$ and $\Delta\_2=\omega\_2-\omega\_p$ switches to $\Delta\_2=\omega\_2+\omega\_L-\omega\_c$

\textbf{TODO} Theoretically speaking this is sloppy notation because every term in theory needs an $\otimes1$ if It's not already given, but it makes everything a bit confusing.\\

\subsection{Deriving the EOM from the Hamiltonian}

Now that I got a time-independent Hamiltonian. I can now proceed to derive the Heisenberg equations of motion. To simplify the process, I use the so-called Adjoint Lindblad master equation \cite{breuer2002theory}\cite{lindblad}:

\begin{align\*}

\dot{\mathcal{O}} =& \frac{i}{\hbar} [H, \mathcal{O}] + \sum\_{n} \gamma\_n \left( L\_n^\dagger \mathcal{O} L\_n - \frac{1}{2} \{ L\_n^\dagger L\_n, \mathcal{O} \} \right)

\end{align\*}

Where $\rho$ denotes the Density matrix of the system. $L$ denotes the so-called Jump operators, $\gamma\_n$ the rate, and $\mathcal{O}$ describes an Operator.\\

The Equation can from the Standard Lindblad master equation by using the cyclic property of the trace $Tr\_S(A \mathcal{L}\rho)=Tr\_S(\mathcal{L}^\*A\rho)$.\\

It can also be adviced to not use this exact form to calculate the results, one should rather use

\begin{align\*}

\dot{\mathcal{O}} =& \frac{i}{\hbar} [H, \mathcal{O}] +\frac{1}{2}\sum\_n\gamma\_n(L\_n^\dagger [\mathcal{O},L\_n]+[L\_n^\dagger,\mathcal{O}]L\_n)

\end{align\*}

(Here $[a,b]$ denotes the commutator.)

Which is just another representation, a bit more fit for the following calculations.\\

Because I have two dissipative Processes I have two Jump Operators. First the atomic Dissipation with rate $\Gamma$ and $L=\ketbramode{01}{k}$ and the cavity Dissipation with $L=a$ and rate $\kappa$.\\

I will not go into depth and calculate all EOM, but I will use the example of the operator $a$ to show how the computation works.

\begin{align\*}

\Dot{a}=&\frac{i}{\hbar}\left[\hbar \sum\_k \frac{ \gamma g\_0}{\sqrt{N}} \left(

\ketbramode{01}{k} a^\dagger+

\ketbramode{10}{k} a

\right)+i \hbar \eta(a^\dagger-a) ,a\right]+\frac{\kappa}{2}(a^\dagger [a,a]+[a^\dagger,a]a)\\

=& [a^\dagger,a](\sum\_ki\frac{\gamma g\_0}{\sqrt{N}}-\eta)+\frac{\kappa}{2}[a^\dagger,a]a\\

=&-\frac{\kappa a}{2}-\sum\_k \frac{i\gamma g\_0}{\sqrt{N}}\ketbramode{01}{k}+\eta

\end{align\*}

Now for all EOM:

\begin{align\*}

\dot{a} =& -\frac{\kappa a}{2} - \frac{i}{\sqrt{N}} \left[\sum\_k \gamma g\_0 \ketbramode{01}{k}\right] + \eta \\

\dotketbra{00} =& +\Gamma \sigma\_{11}^{(j)} + \frac{i}{\sqrt{N}}\left(\gamma g\_0 \sum\_k (\ketbramode{10}{k} a - \ketbramode{01}{k} a^\dagger) \right) \\

\dotketbra{00} =& -\Gamma \ketbramode{11}{j} + \frac{i}{\sqrt{N}} \left( \gamma g\_0 \sum\_k (\ketbramode{01}{k} a^\dagger - \ketbramode{10}{k} a) + \sum\_k \frac{\Omega}{2} (\ketbramode{21}{k} - \ketbramode{12}{k}) \right) \\

\dotketbra{22} =& i \frac{\Omega}{2\sqrt{N}} \sum\_k (\ketbramode{12}{k} - \ketbramode{21}{k}) \\

\dotketbra{21} =& -\frac{\Gamma}{2} \ketbramode{21}{j} + \frac{i}{\sqrt{N}} \left( \sum\_k \left[\Delta\_2 \ketbramode{21}{k} - \Delta\_1 \ketbramode{21}{k} - \gamma g\_0 \ketbramode{20}{k} a + \frac{\Omega}{2} (\ketbramode{11}{k} - \ketbramode{22}{k}) \right] \right. \\

& \left. + \frac{2V}{\sqrt{N}} \sum\_{k,m} (\ketbramode{21}{k} \ketbramode{22}{m}) \right) \\

\dotketbra{01} =& -\frac{\Gamma}{2} \ketbramode{01}{j} + \frac{i}{\sqrt{N}} \left( \sum\_k \left[-\Delta\_1 \ketbramode{01}{k} + \gamma g\_0 (\ketbramode{11}{k} a - \ketbramode{00}{k} a) + \frac{\Omega}{2} (-\ketbramode{02}{k}) \right] \right) \\

\dotketbra{20} =& \frac{i}{\sqrt{N}} \left( \sum\_k \left(\Delta\_2 \ketbramode{20}{k} + \frac{\Omega}{2} \ketbramode{10}{k} \right) + \frac{2V}{\sqrt{N}} \sum\_{k,m} (\ketbramode{20}{k} \ketbramode{22}{m}) - \gamma g\_0 \sum\_k \ketbramode{21}{k} a^\dagger \right) \\

\text{h.c.}

\end{align\*}

Obviously this set of Equations isn't Solvable for an Infinite or Large number of Atoms, there will always be a next order non-linear term which one would have to calculate. So what I will do to make the System actually solvable is perform a so-called Mean-field approximation \cite{Carollo2021ExactnessMF} \cite{Mean-Field}. This will lead to all Operators becoming the Expectation Values of themselves and combined Operators like $\ketbramode{10}{k}a$ will become $\expect{\ketbramode{10}{k}}\expect{a}$. It is to be noted that while performing this Approximation the Sums will vanish, I will once again perform one example why this happens and afterward just give results.

\begin{align\*}

\left\langle \frac{2V}{N} \sum\_{k,m} \ketbramode{20}{k} \ketbramode{22}{m} \right\rangle = 2V \expect{\ketbra{20}} \expect{\ketbra{22}}

\end{align\*}

It's easy to see that each summation yields a factor of $\sqrt{N}$ witch cancels the existing once. So now we can write the mean-field equations of motion:

\begin{align\*}

\expect{\dot{a}} =& -\frac{\kappa \expect{a}}{2} - i \gamma g\_0 \expect{\ketbra{01}} + \eta \\

\expect{\dot{\ketbra{00}}} =& \Gamma \expect{\ketbra{11}} + i \gamma g\_0 \left( \expect{\ketbra{10}} \expect{a} - \expect{\ketbra{01}} \expect{a^\dagger} \right) \\

\expect{\dot{\ketbra{00}}} =& -\Gamma \expect{\ketbra{11}} + i \gamma g\_0 \left( \expect{\ketbra{01}} \expect{a^\dagger} - \expect{\ketbra{10}} \expect{a} \right) + \frac{\Omega}{2} \left( \expect{\ketbra{21}} - \expect{\ketbra{12}} \right) \\

\expect{\dot{\ketbra{22}}} =& i \frac{\Omega}{2} \left( \expect{\ketbra{12}} - \expect{\ketbra{21}} \right) \\

\expect{\dot{\ketbra{21}}} =& -\frac{\Gamma}{2} \expect{\ketbra{21}} + i \left( \Delta\_2 \expect{\ketbra{21}} - \Delta\_1 \expect{\ketbra{21}} - \gamma g\_0 \expect{\ketbra{20}} \expect{a} + \frac{\Omega}{2} \left( \expect{\ketbra{11}} - \expect{\ketbra{22}} \right) \right. \\

& \left. + 2V \expect{\ketbra{21}} \expect{\ketbra{22}} \right) \\

\expect{\dot{\ketbra{01}}} =& -\frac{\Gamma}{2} \expect{\ketbra{01}} + i \left( -\Delta\_1 \expect{\ketbra{01}} + \gamma g\_0 \left( \expect{\ketbra{11}} \expect{a} - \expect{\ketbra{00}} \expect{a} \right) + \frac{\Omega}{2} \left(-\expect{\ketbra{02}} \right) \right) \\

\expect{\dot{\ketbra{20}}} =& i \left( \Delta\_2 \expect{\ketbra{20}} + \frac{\Omega}{2} \expect{\ketbra{10}} + 2V \expect{\ketbra{20}} \expect{\ketbra{22}} - \gamma g\_0 \expect{\ketbra{21}} \expect{a^\dagger} \right) \\

\text{h.c.}

\end{align\*}

Now we got a Solvable System whitch we can analyse.

\section{Stationary State}

\subsection{Non one one State}

We now Interested in solving the EOM Above, to take a closer look at the Possibilitys I will start by exploring the Stationary state. Not the most general case but the case that the Initial state has no Population in the first excited state.

So we make a Ansatz of the form

\begin{align\*}

\ket{\psi}=c\_1\ket{0}+c\_2\ket{2}

\end{align\*}

So we Instantly know $\expect{\ketbra{11}},\expect{\ketbra{01}},\expect{\ketbra{21}}$ are zero. If we know take a Look at the Equations in the stationary state Given in 3 we get :

\begin{align\*}

0 =& -\frac{\kappa \expect{a}}{2} + \eta \\

0 =& i \gamma g\_0 \left( \expect{\ketbra{10}} \expect{a} \right) \\

0 =& i \gamma g\_0 \left( - \expect{\ketbra{10}} \expect{a} \right) + \frac{\Omega}{2} \left( - \expect{\ketbra{12}} \right) \\

0 =& i \frac{\Omega}{2} \expect{\ketbra{12}} \\

0 =& - i \gamma g\_0 \expect{\ketbra{20}} \expect{a} - \frac{i \Omega}{2} \expect{\ketbra{22}} \\

0 =& i \left( \Delta\_2 \expect{\ketbra{20}} + \frac{\Omega}{2} \expect{\ketbra{10}} + 2V \expect{\ketbra{20}} \expect{\ketbra{22}} \right) \\

\text{h.c.}

\end{align\*}

Thus we also already know

\begin{align}

\expect{\ketbra{22}}=&-\frac{\Delta\_2}{2V}\rightarrow\expect{\ketbra{00}}=\frac{\Delta\_2}{2V}

-1\\

\expect{\ketbra{11}}=&\expect{\ketbra{01}}=\expect{\ketbra{21}}=\expect{\ketbra{12}}=\expect{\ketbra{10}}=0\\

\expect{\ketbra{20}}=&-\frac{\Omega\kappa\Delta\_2}{(8\eta\gamma g\_0 V)}=\expect{\ketbra{02}}\\

\expect{\ketbra{02}}=& \frac{4\gamma g\_0 \eta }{\Omega\kappa}\left(\frac{\Delta\_2}{2V}+1\right)\\

\expect{a}=&\frac{2\eta}{\kappa}\\

\expect{a^\dagger}=&\frac{2\eta}{\kappa}\\

\end{align}

So we have found an analytical Solution to our system. Also we have found a Condition for $V$ which reads

\begin{align\*}

V = -\frac{\Delta\_2}{2}\left(\left(\frac{\Omega \kappa}{4\eta \gamma}\right)^2 + 1\right)

\end{align\*}

We will later find out that this condition gives us a complete pure state with constant Purity$=1$.

\subsubsection{Stability}

To take a look at the Stability of this Ansatz we made we will take the aproach of linearasing the system of equations and looking at the jacobi matrix and its eigenvalues\cite{ETHZurich2023} .

$\dot{x}=f(x),\Bar{x}$ is a fixed point if $f(\Bar{x})=0$. For $x$ near $\Bar{x}$, $x=\Bar{x}+\Delta x$, $\Delta x=x-\Bar{x}$ is small

\begin{align\*}

\frac{d}{dt}\Vec{x}=f(x)=f(\Bar{x}+\Delta x)=\underbrace{f(\Bar{x})}\_{\text{$0$ via definition}}+\underbrace{\frac{Df}{Dx}(\Bar{x})\Delta x}\_{\text{only surviving term}}+\underbrace{\frac{D^2f}{Dx^2}(\Bar{x})\Delta x^2+....+\text{h.o.t}}\_{\text{Very small if $\Delta x$ very small}}

\end{align\*}

Thus $\dot{x}=\frac{d}{dt}(\Bar{x}+\Delta x)=\underbrace{\frac{d}{dt}\Bar{x}=\frac{Df}{Dx}(\Bar{x}\Delta x)}\_{\text{Linear ODE in } \Delta x}$

We thus now write $x=\expect{x}\_{ss}+\delta x$ with SS subscripted meaning stationary state. \\

We will keep in general and thus not impose, e.g. %$\expect{\ketbra{11}=0$

I Will write one example line how it is done, afterward I will provide the rest of the results.

\begin{align\*}

\expect{\dot{a}} =& \frac{-\kappa}{2} \expect{a} - i (\gamma g\_0 \expect{\ketbra{01}}) + \eta = 0 \\

\delta \expect{\dot{a}} =& \frac{-\kappa}{2} \left( \expect{a} + \delta \expect{a} \right) - i \left( \gamma g\_0 (\expect{\ketbra{01}} + \delta \expect{\ketbra{01}}) \right) + \eta \\

=& \frac{-\kappa}{2} \delta \expect{a} - i \gamma g\_0 (\delta \expect{\ketbra{01}}) \\

\rightarrow \expect{\dot{a^\dagger}} =& -\frac{\kappa}{2} \delta \expect{a^\dagger} + i \gamma g\_0 \delta \expect{\ketbra{10}}

\end{align\*}

So the result matrix looks like

\begin{equation\*}

\resizebox{1.0\textwidth}{!}{$

\frac{d}{dt}

\begin{array}{c}

\begin{pmatrix}

\delta \expect{a} \\

\delta \expect{a^\dagger} \\

\delta \expect{\ketbra{00}} \\

\delta \expect{\ketbra{11}} \\

\delta \expect{\ketbra{22}} \\

\delta \expect{\ketbra{10}}\\

\delta \expect{\ketbra{01}} \\

\delta \expect{\ketbra{21}} \\

\delta \expect{\ketbra{12}} \\

\delta \expect{\ketbra{20}} \\

\delta \expect{\ketbra{02}} \\

\end{pmatrix}

=

\underbrace{

\begin{pmatrix}

-\frac{\kappa}{2} & 0 & 0 & 0 & 0 & 0 & -i\gamma g\_0 & 0 & 0 & 0 & 0 \\

0 & -\frac{\kappa}{2} & 0 & 0 & 0 & i\gamma g\_0 & 0 & 0 & 0 & 0 & 0 \\

i\gamma g\_0 \langle \ketbra{10} \rangle & -i\gamma g\_0 \langle \ketbra{01} \rangle & 0 & \Gamma & 0 & ig\_0\gamma \langle a \rangle & -ig\_0\gamma \langle a^\dagger \rangle & 0 & 0 & 0 & 0 \\

-i\gamma g\_0 \langle \ketbra{10} \rangle & i\gamma g\_0 \langle \ketbra{01} \rangle & 0 & -\Gamma & 0 & -i\gamma g\_0 \langle a \rangle & i\gamma g\_0 \langle a^\dagger \rangle & \frac{i\Omega}{2} & -\frac{i\Omega}{2} & 0 & 0 \\

0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{i\Omega}{2} & \frac{i\Omega}{2} & 0 & 0 \\

0 & -i\gamma g\_0 ( \langle \ketbra{11} \rangle - \langle \ketbra{00} \rangle) & i\gamma g\_0 \langle a^\dagger \rangle & -i\gamma g\_0 \langle a^\dagger \rangle & 0 & -\frac{\Gamma}{2} + i\Delta\_1 & 0 & 0 & 0 & \frac{i\Omega}{2} & 0 \\

i\gamma g\_0 (\langle \ketbra{11} \rangle - \langle \ketbra{00} \rangle) & 0 & -i\gamma g\_0 \langle a \rangle & i\gamma g\_0 \langle a \rangle & 0 & 0 & -\frac{\Gamma}{2} - i\Delta\_1 & 0 & 0 & 0 & \frac{-i\Omega}{2} \\

-i\gamma g\_0 \langle \ketbra{20} \rangle & 0 & 0 & \frac{i\Omega}{2} & 2iV\langle \ketbra{21} \rangle -\frac{i\Omega}{2} & 0 & 0 & -\frac{\Gamma}{2} + i(\Delta\_2 - \Delta\_1 + 2V \langle \ketbra{22} \rangle) & 0 & -i\gamma g\_0 \langle a \rangle & 0 \\

0 & i\gamma g\_0 \langle \ketbra{02} \rangle & 0 & -\frac{i\Omega}{2} & -2iV \langle \ketbra{12} \rangle + \frac{i\Omega}{2} & 0 & 0 & 0 & -\frac{\Gamma}{2}+i(\Delta\_1-\Delta\_2-2V\langle \ketbra{22} \rangle) & 0 & +i\gamma g\_0 \langle a^\dagger \rangle \\

0 & -i\gamma g\_0 \langle \ketbra{21} \rangle & 0 & 0 & 2iV \langle \ketbra{20} \rangle & \frac{i\Omega}{2} & 0 & -i\gamma g\_0 \langle a^\dagger \rangle & 0 & i(\Delta\_2 + 2V \langle \ketbra{22} \rangle) & 0 \\

ig\_0\gamma \langle \ketbra{12} \rangle & 0 & 0 & 0 & -2iV \langle \ketbra{02} \rangle & 0 & -\frac{i\Omega}{2} & 0 & i\gamma g\_0 \langle a \rangle & 0 & -i(\Delta\_2 + 2V \langle \ketbra{22} \rangle)

\end{pmatrix}

}\_{M}

\begin{pmatrix}

\delta \expect{a} \\

\delta \expect{a^\dagger} \\

\delta \expect{\ketbra{00}} \\

\delta \expect{\ketbra{11}} \\

\delta \expect{\ketbra{22}} \\

\delta \expect{\ketbra{10}}\\

\delta \expect{\ketbra{01}} \\

\delta \expect{\ketbra{21}} \\

\delta \expect{\ketbra{12}} \\

\delta \expect{\ketbra{20}} \\

\delta \expect{\ketbra{02}} \\

\end{pmatrix}

\end{array}

$}

\end{equation\*}

If we now go ahead and numerically simulate this, we can go ahead and now imply our conditions which we found before. So now we apply (3)-(9) to our system. Which yields

\begin{equation\*}

\resizebox{1.0\textwidth}{!}{$

\frac{d}{dt}

\begin{array}{c}

\begin{pmatrix}

\delta \expect{a} \\

\delta \expect{a^\dagger} \\

\delta \expect{\ketbra{00}} \\

\delta \expect{\ketbra{11}} \\

\delta \expect{\ketbra{22}} \\

\delta \expect{\ketbra{10}}\\

\delta \expect{\ketbra{01}} \\

\delta \expect{\ketbra{21}} \\

\delta \expect{\ketbra{12}} \\

\delta \expect{\ketbra{20}} \\

\delta \expect{\ketbra{02}} \\

\end{pmatrix}

=

\begin{pmatrix}

-\frac{\kappa}{2} & 0 & 0 & 0 & 0 & 0 & -i\gamma g\_0 & 0 & 0 & 0 & 0 \\

0 & -\frac{\kappa}{2} & 0 & 0 & 0 & i\gamma g\_0 & 0 & 0 & 0 & 0 & 0 \\

i\gamma g\_0 \frac{2\eta}{\kappa} & -i\gamma g\_0 \frac{2\eta}{\kappa} & 0 & \Gamma & 0 & ig\_0\gamma \frac{2\eta}{\kappa} & -ig\_0\gamma \frac{2\eta}{\kappa} & 0 & 0 & 0 & 0 \\

-i\gamma g\_0 \frac{2\eta}{\kappa} & i\gamma g\_0 \frac{2\eta}{\kappa} & 0 & -\Gamma & 0 & -i\gamma g\_0 \frac{2\eta}{\kappa} & i\gamma g\_0 \frac{2\eta}{\kappa} & \frac{i\Omega}{2} & -\frac{i\Omega}{2} & 0 & 0 \\

0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{i\Omega}{2} & \frac{i\Omega}{2} & 0 & 0 \\

0 & -i\gamma g\_0 (\frac{\Delta\_2}{2V}-1) & i\gamma g\_0 \frac{2\eta}{\kappa} & -i\gamma g\_0 \frac{2\eta}{\kappa} & 0 & -\frac{\Gamma}{2} + i\Delta\_1 & 0 & 0 & 0 & \frac{i\Omega}{2} & 0 \\

i\gamma g\_0 (\frac{\Delta\_2}{2V}-1) & 0 & -i\gamma g\_0 \frac{2\eta}{\kappa} & i\gamma g\_0 \frac{2\eta}{\kappa} & 0 & 0 & -\frac{\Gamma}{2} - i\Delta\_1 & 0 & 0 & 0 & \frac{-i\Omega}{2} \\

-i\gamma g\_0 (-\frac{\Omega\kappa\Delta\_2}{8\eta\gamma g\_0 V}) & 0 & 0 & \frac{i\Omega}{2} & 2iV(\frac{4\gamma g\_0 \eta }{\Omega\kappa}(\frac{\Delta\_2}{2V}+1)) -\frac{i\Omega}{2} & 0 & 0 & -\frac{\Gamma}{2} + i(\Delta\_2 - \Delta\_1 + 2V (-\frac{\Delta\_2}{2V})) & 0 & -i\gamma g\_0 \frac{2\eta}{\kappa} & 0 \\

0 & i\gamma g\_0 (\frac{4\gamma g\_0 \eta }{\Omega\kappa}(\frac{\Delta\_2}{2V}+1)) & 0 & -\frac{i\Omega}{2} & -2iV (-\frac{\Omega\kappa\Delta\_2}{8\eta\gamma g\_0 V}) + \frac{i\Omega}{2} & 0 & 0 & 0 & -\frac{\Gamma}{2}+i(\Delta\_1-\Delta\_2-2V(-\frac{\Delta\_2}{2V})) & 0 & +i\gamma g\_0 \frac{2\eta}{\kappa} \\

0 & -i\gamma g\_0 (\frac{4\gamma g\_0 \eta }{\Omega\kappa}(\frac{\Delta\_2}{2V}+1)) & 0 & 0 & 2iV (-\frac{\Omega\kappa\Delta\_2}{8\eta\gamma g\_0 V}) & \frac{i\Omega}{2} & 0 & -i\gamma g\_0 \frac{2\eta}{\kappa} & 0 & i(\Delta\_2 + 2V (-\frac{\Delta\_2}{2V})) & 0 \\

ig\_0\gamma (\frac{4\gamma g\_0 \eta }{\Omega\kappa}(\frac{\Delta\_2}{2V}+1)) & 0 & 0 & 0 & -2iV (-\frac{\Omega\kappa\Delta\_2}{8\eta\gamma g\_0 V}) & 0 & -\frac{i\Omega}{2} & 0 & i\gamma g\_0 \frac{2\eta}{\kappa} & 0 & -i(\Delta\_2 + 2V (-\frac{\Delta\_2}{2V}))

\end{pmatrix}

\begin{pmatrix}

\delta \expect{a} \\

\delta \expect{a^\dagger} \\

\delta \expect{\ketbra{00}} \\

\delta \expect{\ketbra{11}} \\

\delta \expect{\ketbra{22}} \\

\delta \expect{\ketbra{10}}\\

\delta \expect{\ketbra{01}} \\

\delta \expect{\ketbra{21}} \\

\delta \expect{\ketbra{12}} \\

\delta \expect{\ketbra{20}} \\

\delta \expect{\ketbra{02}} \\

\end{pmatrix}

\end{array}

$}

\end{equation\*}

If we now also imply the conditions which we found for $V$ and run a couple of numeric simulations we actually find, that all eigenvalues of this matrix are negative with respect to a small numerical error $\sim 10^{-16} $. But I can't possible test all Numerical values, so \textbf{TODO}: IS THIS ENOUGH?

\subsubsection{Results}

Now that we have spend quite some time on Just preparing our system, we want to go Ahead and actually look at it. But before we actually start looking at the Results we need two more things, we need to be sure our system is physically meaningful and actually makes sense. To do so, we look at the purity of the system. As well as Real positive eigenvalues of the Density matrix (with respect to some numerical error).

We know that the Purity $\frac{1}{d}\leq Tr(\rho^2)\leq 1$\cite{jaeger2006} Where $d$ is the Dimension of our Hilbert space, so for our Stationary state we have $d=2$ and else $d=3$. We will now express my density matrix,

\begin{align\*}

\rho =&\bigotimes\_{i=1}^N\rho\_i\\

\rho\_1=&\sum\_{n,m=1}^{2 \text{ or } 3} r\_{m,n} \ket{m}\bra{n}

\end{align\*}

The Density matrix of the Stationary state can be written as

\begin{align\*}

\rho=\begin{pmatrix}

\ketbra{22} &\ketbra{02} \\

\ketbra{20} &\ketbra{00} \\

\end{pmatrix}

\end{align\*}

So now each time that we will compute something we will a) check what the Purity of the System is and if it's in a physical regime. b) See if the eigenvalues of the Density matrix are real and positive. From now on we assume this condition to be fulfilled because we won't take data into account which doesn't reach these conditions. Also, if any of the Expectation values take negative values or the condition $\expect{\ketbra{00}}+\expect{\ketbra{11}}+\expect{\ketbra{22}}=0$ isn't satisfied anymore, we won't consider the data.

\\

\section{Non Stationary}

I will now take a look at the Non-stationary system,

\begin{align\*}

\rho=\begin{pmatrix}

\rho\_{2,2} &\rho\_{1,2} & \rho\_{0,2} \\

\rho\_{2,1} &\rho\_{1,1} & \rho\_{0,1} \\

\rho\_{2,0} &\rho\_{1,0} & \rho\_{0,0}\\

\end{pmatrix}=

\begin{pmatrix}

\ketbra\_{22} &\ketbra\_{12} & \ketbra\_{02} \\

\ketbra\_{21} &\ketbra\_{11} & \ketbra\_{01} \\

\ketbra\_{20} &\ketbra\_{10} & \ketbra\_{00}\\

\end{pmatrix}

\end{align\*}

I can now start to look at the heatmaps of the parameters with the variances. I always perform a sanity check to see if the eigenvalues are positive and also look at the Purity of the state.\\

\bibliographystyle{plain}

\bibliography{Clean\_Writing/lib}

\end{document}