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Mobile Robotics

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Introduction to Mobile Robotics - Lecture 4 - Probability Revision

Lecture Content

- Probability Revision
- Axioms
- Law of Total Probability
- Distributions
- Causal vs Diagnostic Reasoning
- Robotics Examples
- Markov Assumption
- Bayes Formula

Probability

Ω is the measure of all possible outcomes of a mathematical experiment, the sample space. The result of a single execution of an experiment is called an **Event**. The Probability of an event A is written as

$$P(A), \qquad p(A), \qquad or \qquad Pr(A)$$

The probability of an event is a number between 0 and 1. 0 indicates an impossible event, and 1 indicates a definite event.

Probability Revision

Example: Probability Experiment

1. Flipping a coin Ω = {Heads, Tails} oder Ω = {0,1}
2. Rolling a dice: Ω = {1,2,3,4,5,6}
3. The Braking Distance of a car as a function of its speed $\Omega = \{(x)|x \subset \mathcal{R}\}$

A possible subset of the sample space Ω will be defined with capital letters. It can include the impossible event $\emptyset = \{\}$ and the definite event (the complete Ω itself).

Example of subsets

From a pack of 52 Cards, the subset A:= " All Aces"

1. $A = \{\heartsuit\text{Ace}, \clubsuit\text{Ace}, \diamondsuit\text{Ace}, \spadesuit\text{Ace}\}$

From the set of possible results with N individual experiments, there are a possible 2^N results, inclusive the impossible and definite events.
For Example: The possible events from the Set $\Omega = \{1, 2, 3\}$ are:
$\{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$. i.e. $2^3 = 8$ Results.

Probability

Axioms of Probability

For the examples column in the table, $\Omega = \{a, b, c, d, e\}$, A:=$\{a, b, c\}$ und B:=$\{c, d\}$

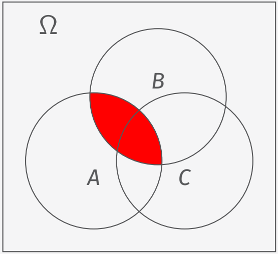
Axiom	Syntax	Description	Example
Joint probability	$A \cap B$. A and B	All events from Ω, that belong to both A **and** B	$A \cap B = \{c\}$
Alternate Probability	$A \cup B$. A or B	All events from Ω, that belong to A **or** B	$A \cup B = \{a, b, c, d\}$
Differences of Probabilities	$A - B$	All events from Ω, that belong to A **but** not B	$A - B = \{a, b\}$
Complementary Probability	$\bar{A}$	All events from Ω, that **don't** belong to A	$\bar{A} = \{d, e\}$

Commutative Law	$A \cap B = B \cap A$ and $A \cup B = B \cup A$
Associative Law	$A \cap (B \cap C) = (A \cap B) \cap C$ and $A \cup (B \cup C) = (A \cup B) \cup C$
Distributive Law	$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ and $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
De Morgan's laws	$\overline{A \cup B} = \bar{A} \cap \bar{B}$ und $\overline{A \cap B} = \bar{A} \cup \bar{B}$

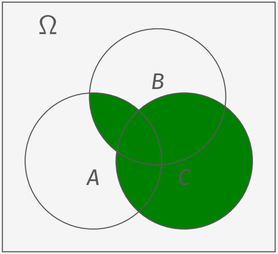
Probability

Venn-Diagrams

The previously defined Axioms, can be visually shown and analysed in Venn-Diagrams.



$A \cap B$



$(A \cap B) \cup C$

Probability

Probability

The Probability of an event, p is the number, that is approached by repeated experiments. $p \in [0;1]$ For every possible event,

$$0 \leq P(A) \leq 1 \qquad (1)$$

Calculations / Syntax
$P(\emptyset) = 0$
$P(\Omega) = 1$
$P(\bar{A}) = 1 - P(A)$
$P(A \cap B) = P(A) + P(B) - P(A \cup B)$

The De Moivre–Laplace theorem

Exists a sample space Ω with a limited number of possible events, that all have the same probability, then for every event $A \subseteq \Omega$ the probability is:

$$P(A) = \frac{|A|}{|\Omega|} \qquad (2)$$

$$= \frac{\text{Number of elements in subset A}}{\text{Number of elements in sample space}}$$

Probability

Disjunctive, Independant Events and conditional probabilities

Two events A and B are Disjunctive when:

$$P(A \cap B) = \emptyset \tag{3}$$

Two Events A and B are independant when:

$$P(A \cap B) = P(A)P(B) \tag{4}$$

The conditional probability is defined as the probability of A occuring, given that B occurs. It is written as $P(A|B)$ and assuming $P(B) > 0$

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \tag{5}$$

Probability

The Law of Total Probabilities

Discrete

$$P(x) = \sum_{y} P(x|y)P(y) \tag{6}$$

Continuous

$$p(x) = \int p(x|y)p(y)dy \tag{7}$$

Probability

Likelihood Evidence Priors, and Bayes Formula

Based on the Law of Total Probability:

$$P(x, y) = P(x|y)P(y) = P(y|x)P(x) \tag{8}$$

implies:

$$P(x|y) = \frac{P(y|x)P(x)}{P(y)} = \frac{\text{likelihood} \cdot \text{prior}}{\text{evidence}} \tag{9}$$

this is known as Bayes Formula.

Normalisation

$$P(x|y) = \frac{P(y|x)P(x)}{P(y)} = \eta P(y|x)P(x)$$

$$\eta = P(y)^{-1} = \frac{1}{\sum_x P(y|x)P(x)}$$

Probability

Distribution Function

Given X as a discrete random variable with the values $x_1, x_2, \cdots$. The event $X = x_j$ belonging to each value x_j has the probability

$$p_j := P(X = x_j), j = 1, 2, \cdots \tag{10}$$

The Function $F : \mathcal{R} \mapsto [0; 1]$ with

$$F(x) := P(X \leq x) = \sum_{x_j \leq x} p_j \tag{11}$$

is termed the Distribution Function for X.

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Probability

Example: discrete Distribution Function

A Coin is flipped three times. The random variable X stands for the number of times a head is flipped. We use the terms H and T for Heads and Tails.

$$P(X=0)=P(TTT)=\left(\frac{1}{2}\right)^3=0.125$$

$$P(X=1)=P(HTT)+P(THT)+P(TTH)$$
$$=3\cdot\left(\frac{1}{2}\right)^3=0.375$$

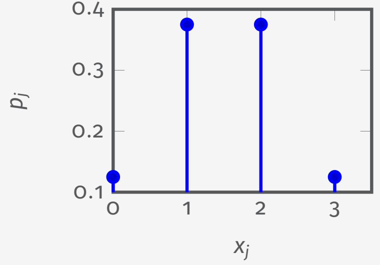
$$P(X=2)=P(HHT)+P(THH)+P(HTH)$$
$$=3\cdot\left(\frac{1}{2}\right)^3=0.375$$

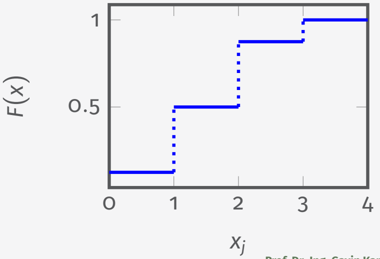
$$P(X=3)=P(HHH)=\left(\frac{1}{2}\right)^3=0.125$$

Distribution

The Distribution is thus:

x_j	0	1	2	3
$p_j = P(X=x_j)$	0.125	0.375	0.375	0.125





Probability

Continuous random Variables and their Probability and Density Functions

For a continuous sample space, all values from $x = -\infty$ to $x = \infty$ are possible. Because the Probability of the complete sample space is still equal to 1, the function $f : \mathcal{R} \mapsto \mathcal{R}$ is given the name the Probablity Density Function.

$$\int_{-\infty}^{\infty} f(x)dx = 1 \tag{12}$$

The Probability $P(X \leq x)$ is defined through the Cumulative distribution function $F(x)$ as a definite Integral:

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(u)du \tag{13}$$

To calculate the probability of a desired range:

$$P(a \leq X \leq b) = \int_{b}^{a} f(x)dx = F(b) - F(a) \tag{14}$$

Probability

Mean and Standard Deviation

The occurance of a random variable can be described with two real values: the Mean Value, that describes the most likely location of the event X, and the Standard Deviation, that describes how wide the values of X are distributed from the Mean Value.

Mean Value

The Mean Value of X is defined as:

$$\mu = E[X] := \begin{cases} \sum_j x_j p_j, & \text{Discrete} \\ \int_{-\infty}^{\infty} x f(x) dx, & \text{Continuous} \end{cases} \tag{15}$$

Standard Deviation

The Standard Deviation of X is defined as:

$$\sigma^2 = E[(X-\mu)^2] := \begin{cases} \sum_j (x_j - \mu)^2 p_j, \\ \int_{-\infty}^{\infty} (x-\mu)^2 f(x) dx \end{cases} \tag{16}$$

For practical calculations the following can be useful:

$$E[(X-\mu)^2] = E[X^2 - 2X\mu + \mu^2] = E[X^2] - 2\mu \underbrace{E[X]}_{\mu} + \mu^2 = E[X^2] - \mu^2 \tag{17}$$

Probability

Normal Distribution

A normal distribution is one of the most important distributions in Probability theorem. The definition the distribution is:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \tag{18}$$

In the following picture, three different probability density functions are shown:

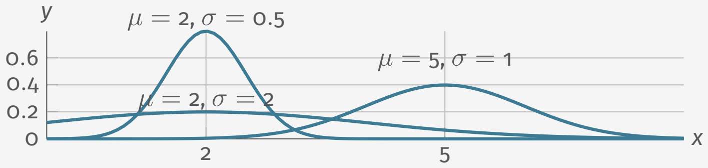
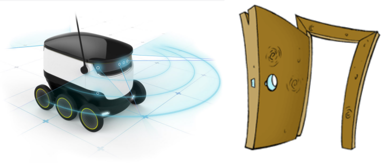


Figure: Example of Normal Distribution Density Function

Probability

Whats Probability got to do with Robotics?

- Suppose a robot obtains measurement z.
- What is P(open|z)?



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Probability

Causal vs Diagnostic Reasoning

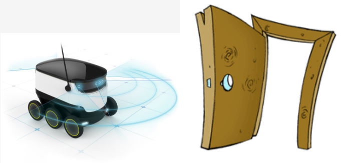
- $P(open|z)$ is diagnostic
- $P(z|open)$ is causal

Often causal knowledge is easier to obtain. For example, it can be obtained through counting measurements up-front.
Bayes rule allows us to use causal knowledge:

$$P(open|z) = \frac{P(z|open)P(open)}{P(z)} \tag{19}$$

Probability

Example



$$P(z|open) = 0.6 \qquad P(z|\overline{open}) = 0.3$$

$$P(open) = P(\overline{open}) = 0.5$$

Applying Bayes Rule together with Marginalistion

$$P(open|z) = \frac{P(z|open)P(open)}{P(z|open)P(open) + P(z|\overline{open})P(\overline{open})}$$

$$P(open|z) = \frac{0.6 \cdot 0.5}{0.6 \cdot 0.5 + 0.3 \cdot 0.5} = 0.67$$

After the measurement, the probability the door is open, has increased from 0.5 to 0.67

Probability

Multiple Measurements

How can we improve our certainty with increased measurements? i.e. Mathematically spoken:

$$P(x|z_1, z_2, \cdots, z_n) = ?$$

Recursive Bayesian Updating

$$P(x|z_1, z_2, \cdots, z_n) = \frac{P(z_n|x, z_1, z_2, \cdots, z_{n-1})P(x|z_1, z_2, \cdots, z_{n-1})}{P(z_n|z_1, z_2, \cdots, z_{n-1})}$$

Using the Markov assumption, that all measurements are independant:

$$P(x|z_1, \cdots, z_n) = \frac{P(z_n|x)P(x|z_1, \cdots, z_{n-1})}{P(z_n|z_1, \cdots, z_{n-1})}$$

$$P(x|z_1, \cdots, z_n) = \eta P(z_n|x)P(x|z_1, \cdots, z_{n-1})$$

$$P(x|z_1, \cdots, z_n) = \eta \prod_{i=1..n} P(z_n|x)P(x)$$

Probability

Example - continued, second measurement

A different measurement is made, with different probabilities.

$$P(z_2|open) = 0.25 \qquad P(z_2|\overline{open}) = 0.3$$

$$P(open|z_1) = 0.67 \qquad P(\overline{open}|z_1) = 0.33$$

$$P(open|z_1, z_2) = \frac{P(z_2|open)P(open|z_1)}{P(z_2|open)P(open|z_1) + P(z_2|\overline{open})P(\overline{open}|z_1)}$$

$$P(open|z_1, z_2) = \frac{0.25 \cdot 0.67}{0.25 \cdot 0.67 + 0.3 \cdot 0.33} = 0.625$$

In this case, the measurement had not increased our certainty on the state of the door. More measurements needed?

Probability

Including Actions

Actions performed by the robot (and other dynamic events) can change the world. Our model of the world must be adapted accordingly.

- How certain are the actions performed?
- How do we include these actions in our model?

Using the Conditional Probability Density Function:

$$P(x|u, x') \tag{20}$$

where u is the action performed, and we calculate the probability of a state x, given a previous state x'.

Integrating the outcome of multiple Actions

Continuous Case:

$$P(x|u) = \int P(x|u, x')P(x')dx'$$

Discrete Case:

$$P(x|u) = \sum P(x|u, x')P(x')$$

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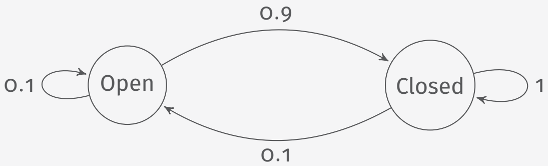
Probability

Example - Closing a Door

P(x|u,x') for u = "close door":



$$P(closed|closedoor, open) = 0.9$$

$$P(closed|closedoor, closed) = 1$$

$$P(open|closedoor, open) = 0.1$$

$$P(open|closedoor, closed) = 0.1$$

When the door is open, and we attempt to close it, we have a 90% chance of success.

Probability

Example - resulting certainty

$$
\begin{aligned}
P(closed|u) &= \sum P(closed|u, x')P(x') \\
&= P(closed|u, open)P(open) \\
&+ P(closed|u, closed)P(closed) \\
&= 0.9 \cdot 0.625 + 1 \cdot 0.375 = 0.9375
\end{aligned} \tag{21}
$$

$$
\begin{aligned}
P(open|u) &= \sum P(open|u, x')P(x') \\
&= P(open|u, open)P(open) \\
&+ P(open|u, closed)P(closed) \\
&= 0.1 \cdot 0.625 + 0 \cdot 0.375 = 0.0625 \\
&= 1 - P(closed|u)
\end{aligned} \tag{22}
$$

Probability

Generalised Concept - Actions and Sensors

Given a stream of sensor data together with known actions, estimate the system state.
Using:

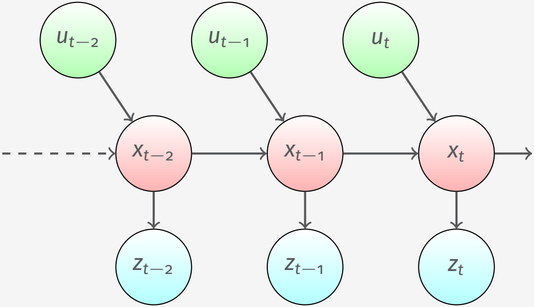
- Previous State Estimate x
- The probability of the previous estimate P(x),
- The Sensor Model $P(x|z)$, and
- The Action Model $P(x|u, x')$

Estimate the new state of the System x.
The posterior of the state is also called the Belief:

$$Bel(x_t) = P(x_t|u_1, z_1, \cdots, u_t, z_t) \tag{23}$$

Probability

Markov Assumption



$$p(z_t | x_{0:t}, z_{1:t-1}, u_{1:t}) = p(z_t | x_t) \tag{24}$$

$$p(x_t | x_{1:t-1}, z_{1:t-1}, u_{1:t}) = p(x_t | x_{t-1}, u_t) \tag{25}$$

Calculating the Belief

$$
\begin{aligned}
Bel(x_t) &= P(x_t|u_1, z_1, \cdots, u_t, z_t) \\
\text{(Bayes)} &= \eta P(z_t|x_t, u_1, z_1, \cdots, u_t) P(x_t|u_1, z_1, \cdots, u_t) \\
\text{(Markov)} &= \eta P(z_t|x_t) P(x_t|u_1, z_1, \cdots, u_t) \\
\text{(Total prob.)} &= \eta P(z_t|x_t) \int P(x_t|u_1, z_1, \cdots, u_t, x_{t-1}) P(x_{t-1}|u_1, z_1, \cdots, u_t) dx_{t-1} \\
\text{(Markov)} &= \eta P(z_t|x_t) \int P(x_t|u_t, x_{t-1}) P(x_{t-1}|u_1, z_1, \cdots, u_t) dx_{t-1} \\
\text{(Markov)} &= \eta P(z_t|x_t) \int P(x_t|u_t, x_{t-1}) P(x_{t-1}|u_1, z_1, \cdots, u_t, z_t) dx_{t-1} \\
&= \eta P(z_t|x_t) \int P(x_t|u_t, x_{t-1}) Bel(x_{t-1}) dx_{t-1}
\end{aligned}
\tag{26}
$$

Probability

Summary

- Probability Rules are the fundamentals for all decision making in robotics
- Uncertanty, Noise and Time, all add to a Dynamic Environemt that can never be perfectly known
- A Robots Sensors allow information to be gained about this environment. Improving the likelyhood of its world model.
- A Robots own Actions allow an estimation of the future state of the environment, which can be updated in its world model.