NLS and Sharp Interpolation Estimates

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Introduction: Prelude to the NLS

Finding the "best constant" of various inequalities usually has some geometric or analytic significance. As an example:

Theorem (Isoperimetric Inequality)

Consider a domain $D \in \mathbb{R}^n$ with boundary ∂D. Then,

$$V^{n-1} \leqslant n^{-n}\omega_n^{-1}A^n$$

- *V is the volume (Lebesgue Measure) of D*
- *ω_n is the volume of the n-ball of radius 1*
- *A is the "surface area" of D, generalized by by $M_{n-1}(\partial D)$, the (n-1)-dimensional Minkowski Content of ∂D.*

Equality is obtained only when D is the n-ball!

Introduction: Prelude to the NLS

How does this geometric fact concern inequalities? Consider the following Sobolev-type Embedding theorem:

Theorem

Consider $u \in C_0^\infty(\mathbb{R}^N)$. Then

$$\|u\|_{L^{\frac{n}{n-1}}} \leqslant C(n)\|\nabla u\|_{L^1}$$

and the best constant is the same as in the Isoperimetric Inequality, ie. $C(n)_{sharp} = n^{-n}\omega_n^{-1}$.

Introduction

Weinstein's 1983 paper presents a relationship between the sharp constant for a classical interpolation inequality by Nirenberg and Gagliardo and a criterion for global existence for the Nonlinear Schrödinger Equation:

$$i\frac{\partial \phi}{\partial t} + \Delta\phi + |\phi|^{2\sigma}\phi = 0 \tag{NLS}$$

where $x \in \mathbb{R}^N$, $t \in \mathbb{R}^+$ and initial conditions $\phi(x,0) = \phi_0(x)$ in the L^2-critical case $N\sigma = 2$.

Introduction

This presentation will tackle three things:

I. Sharpest Constant for the (classical) Gagliardo-Nirenberg Interpolation Inequality and the ground state solution of the NLS: Solution of a Variational Problem

II. A sufficient condition for global existence of H^1 solutions of the NLS: Global Existence for the IVP in the L^2 critical case

III. (time-permitting) comments on stability and finite blow-up

I. Sharpest Constant and the Variational Problem

Consider a particular Gagliardo-Nirenberg Interpolation Inequality:

$$\|f\|_{2\sigma+2}^{2\sigma+2} \leqslant C_{\sigma,N}^{2\sigma+2}\|\nabla f\|_2^{\sigma N}\|f\|_2^{2+\sigma(2-N)}$$

for $0 < \sigma < \frac{2}{N-2}$ and $N \geqslant 2$. This is a particular case of the general inequality:

$$\|D^j f\|_p \leqslant C\|D^m f\|_r^{\alpha}\|f\|_q^{1-\alpha}$$

for some function $f : \mathbb{R}^N \to \mathbb{R}$ with $p = 2\sigma + 2,\ m = 1,\ q = r = 2,$ which fixes $\alpha = n(\frac{1}{2} - \frac{1}{2\sigma+2})$

I. Sharpest Constant and the Variational Problem

In order to find the sharpest constant $C_{\sigma,N}$, we define the functional:

$$J^{\sigma,N}(f) = \frac{\|\nabla f\|_2^{\sigma N} \|f\|_2^{2+\sigma(2-N)}}{\|f\|_{2\sigma+2}^{2\sigma+2}}$$

Now this turns into a minimization problem!
We will show that the minimum is attained at some H^1 function ψ^*. We obtain explicit forms for $C_{\sigma,N}$ and the equation it solves by applying the Euler-Lagrange equations to $J^{\sigma,N}$.
Note that by the original estimate $J^{\sigma,N}$ is defined on $H^1(\mathbb{R}^N)$ for $0 < \sigma < \frac{2}{N-2}$.

I. Sharpest Constant and the Variational Problem

Theorem

For $0 < \sigma < \frac{2}{N-2}$,

$$\alpha \equiv \inf_{u \in H^1(\mathbb{R}^N)} J^{\sigma,N}(u)$$

is attained at a function ψ *with the following properties*

1) ψ *is positive and a function of* $|x|$ *alone*
2) $\psi \in H^1(\mathbb{R}^N) \cap C^\infty(\mathbb{R}^N)$
3) ψ *is a solution of:*

$$\frac{\sigma N}{2}\Delta\psi - (1 + \frac{\sigma}{2}(2 - N))\psi + \psi^{2\sigma+1} = 0$$

of minimal L^2 *norm (the ground state), and in addition*

$$\alpha = \frac{\|\psi\|_2^{2\sigma}}{\sigma + 1}$$

I. Sharpest Constant and the Variational Problem

Ideas behind the proof:

(1) show that positive minimizers satisfy the same minimization problem, and a "symmetrization" argument to show the same for radial minimizers (Strauss 1977)

(2) Compactness lemma gives compact embedding of radial functions, obtain a convergent minimizing sequence!

(3) Apply the Euler-Lagrange equations to $J^{\sigma,N}$

I. Sharpest Constant and the Variational Problem

Lemma (Compactness Lemma)

For $0 < \sigma < \frac{2}{N-2}$, the embedding

$$H^1_{radial}(\mathbb{R}^N) \to L^{2\sigma+2}(\mathbb{R}^N)$$

is compact.

Proof.

Proof follows from the interpolation estimate:

$$\|u\|_{2\sigma+2}^{2\sigma+2} \leqslant C\|u\|_{H^1}^{\sigma N}\|u\|_2^{2+\sigma(2-N)}$$

for $0 < \sigma < \frac{2}{N-2}$ and $u \in H^1(\Omega)$ where Ω is a bounded domain.

I. Sharpest Constant and Variational Problem

We extend to $u \in H(\mathbb{R}^N)$ if we can show that a bounded sequence in $H(\mathbb{R}^N)_{radial}$ is uniformly small at infinity. This follows by Strauss' Radial Lemma. □

Lemma (Radial Lemma 1)

Let $N \geqslant 2$. Every radial function $u \in H^1$ is almost everywhere equal to a function $U(x)$, continuous for $x \neq 0$, such that

$$|U(x)| \leqslant c|x|^{\frac{1-N}{2}} \|u\|_{H^1}$$

for $|x| \geqslant 1$, and where c depends only on n.

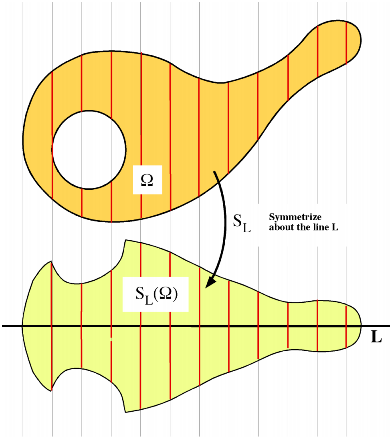
I. Sharpest Constant and the Variational Problem

Sketch of 1).

Argue that if the minimization problem has a solution, then it also has a solution that is non-negative and radial.
Let u be a solution. Argue that $u^+(x) := max(u(x), 0)$ solves the same minimization problem (observe that $\nabla(u^+) = (\nabla u)^+$ and that $u^+ \in H^1$).
To assume that u is radial we will argue using Steiner Symmetrization, imitating Strauss 1977 who in turn extended the classical version of this technique (Polya and Szegö 1951).

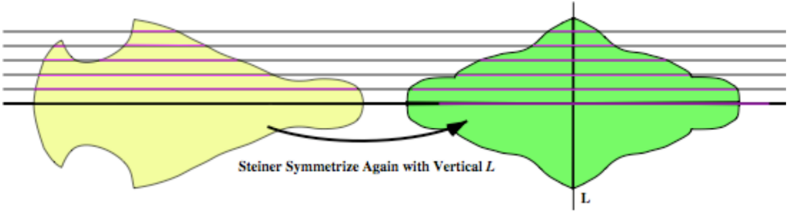
I. Sharpest Constant and the Variational Problem

"Symmetrize" a given bounded domain with "nice" boundary about a hyperplane that passes through the origin.



I. Sharpest Constant and the Variational Problem

Applying the symmetrization multiple times...



I. Sharpest Constant and the Variational Problem

Theorem (Ljusternik-Gross Sphericalization Theorem)

Let Ω be a non-empty compact set and $\mathbb{G}$ the family of all multiple symmetrizations (finite composition of symmetrizations) of Ω. Then there is a subsequence $\Omega_n \subset \mathbb{G}$ and a closed ball $\overline{B}$ with the same volume as Ω and

$$\Omega_n \to \overline{B} \ \text{ as } \ n \to \infty$$

I. Sharpest Constant and Variational Problem

Define

$$D = \{(x, t) \in \mathbb{R}^{N+1} : 0 \leqslant t \leqslant u(x)\}$$

Let D^* be the symmetrization of D around the hyperplane $x_1 = 0$. Then, D^* is of the form

$$D = \{(x, t) \in \mathbb{R}^{N+1} : 0 \leqslant t \leqslant u^*(x)\}$$

and we argue that u^* is also a solution of the same problem. So by successive choice of hyperplanes, we get a (non-negative) radial solution. $\square$

I. Sharpest Constant and Variational Problem

Sketch of 2).

Since $J^{\sigma,N}(u) \geqslant 0$ there exists a minimizing sequence $u_\nu \in H^1(\mathbb{R}^N) \cap L^{2\sigma+2}(\mathbb{R}^N)$. We assume this u_ν to be positive and radial.

Define the scaling

$$u^{\lambda,\mu}(x) \equiv \mu u(\lambda x)$$

and fix values of λ and μ as follows.

$$\lambda_\nu = \frac{\|u_\nu\|_2}{\|\nabla u_2\|_2}$$

and

$$\mu_\nu == \frac{\|u_\nu\|_2^{\frac{N}{2}-1}}{\|\nabla u_2\|_2^{\frac{N}{2}}}$$

I. Sharpest Constant and Variational Problem

Then the sequence $\psi_\nu(x) = u^{\lambda_\nu,\mu_\nu}(x)$ with the following properties:

(a) $\psi_\nu \geqslant 0$, $\psi_\nu = \psi_\nu(|x|)$

(b) $\psi_\nu \in H^1(\mathbb{R}^N)$

(c) $\|\psi_\nu\|_2 = 1$ and $\|\nabla\psi_\nu\|_2 = 1$

(d) $J^{\sigma,N}(\psi_\nu) \downarrow \alpha$ as $\nu \to \infty$

Since the sequence ψ_ν is bounded in $H^1(\mathbb{R}^N)$ some subsequence has a weak H^1 limit ψ^* (Recall the Banach-Alaoglu Theorem). We can use the Compactness Lemma to take ψ_ν strongly convergent to ψ^* in $L^{2\sigma+2}(\mathbb{R}^N)$ for $0 < \sigma < \frac{2}{N-2}$.

I. Sharpest Constant and Variational Problem

By weak convergence we have $\|\psi^*\|_2 \leqslant 1$ and $\|\nabla\psi^*\|_2 \leqslant 1$. So using the definition of the operator $J^{\sigma,N}$:

$$\begin{aligned}\alpha &\leqslant J^{\sigma,N}(\psi^*) \\ &\leqslant \frac{1}{\int |\psi^*|^{2\sigma+2} dx} \\ &= \lim_{\nu\uparrow\infty} J(\psi_n u) = \alpha\end{aligned}$$

It follows that

$$\|\psi^*\|_2^{\sigma N} \|\psi^*\|_2^{2+\sigma(2-N)} = 1$$

so then $\|\psi^*\| = \|\nabla\psi^*\|_2 = 1$ and we have strong convergence of $\psi_\nu \to \psi^*$ in H^1. $\square$

I. Sharpest Constant and Variational Problem

Sketch of 3).

Applying the Euler-Lagrange equation to $J^{\sigma,N}$:

$$\frac{d}{d\epsilon}\bigg|_{\epsilon=0} J^{\sigma,N}(\psi^* + \epsilon\eta) = 0$$

for all $\eta \in C_0^\infty(\mathbb{R}^N)$. Using $\|\psi^*\| = \|\nabla\psi^*\|_2 = 1$ we obtain from the E-L equation:

$$\frac{\sigma N}{2}\Delta\psi^* - (1 + \frac{\sigma}{2}(2-N))\psi^* + \alpha(\sigma+1)(\psi^*)^{2\sigma+1} = 0$$

Rescaling we let $\psi^* = [\alpha(\sigma+1)]^{-\frac{1}{2}\sigma}\psi$ and we obtain a solution the equation in Theorem B with

$$\alpha = \frac{\|\psi\|_2^{2\sigma}}{(\sigma+1)}$$

I. Sharpest Constant and Variational Problem

Theorem

For $0 < \sigma < \frac{2}{N-2}$,

$$\alpha \equiv \inf_{u \in H^1(\mathbb{R}^N)} J^{\sigma,N}(u)$$

is attained at a function ψ *with the following properties*

1) ψ *is positive and a function of* $|x|$ *alone*
2) $\psi \in H^1(\mathbb{R}^N) \cap C^\infty(\mathbb{R}^N)$
3) ψ *is a solution of:*

$$\frac{\sigma N}{2}\Delta\psi - (1 + \frac{\sigma}{2}(2 - N))\psi + \psi^{2\sigma+1} = 0$$

of minimal L^2 *norm (the ground state), and in addition*

$$\alpha = \frac{\|\psi\|_2^{2\sigma}}{\sigma + 1}$$

I. Sharpest Constant and Variational Problem

Also obtain two corollaries:

Corollary (Corollary B.1)

The best (smallest) constant for which the classical interpolation estimate holds is given by

$$C_{\sigma,N} = \left(\frac{\sigma+1}{\|\psi\|_2^{2\sigma}}\right)^{\frac{1}{2\sigma+2}}$$

where ψ is the ground state of the NLS.

Corollary (Corollary B.2)

Let $0 < \sigma < \frac{2}{N-2}$. Then, the equation

$$\Delta u - u + u^{2\sigma+1} = 0$$

has a positive, radial solution of class $H^1(\mathbb{R}^N)$.

II. Global Existence of IVP

Building on the results of Ginibre and Velo for global existence:

Theorem (Theorem 3.1)

Let $\phi_0 \in H^1(\mathbb{R}^N)$. Then:

(i) If $0 < \sigma < \frac{2}{N}$, then there exists a unique solution $\phi \in C([0, \infty]; H^1(\mathbb{R}^N))$ of the IVP (NLS) in the sense of the equivalent integral equation:

$$\phi = U(t - t_0)\phi - i \int_{t_0}^{t} U(t - s)(|\phi|^{2\sigma}\phi)(s)d2s$$

where $U(t)$ is the Schrödinger Kernel.

II. Global Existence of IVP

Theorem (Theorem 3.1)

(ii) If $\sigma = \frac{2}{N}$, then for $\|\phi_0\|_2$ sufficiently small, the conclusion of (i) holds.

(iii) As long as $\phi(x,t)$ remains in $H^1(\mathbb{R}^N)$, the quantities

$$\mathcal{N}(\phi) \equiv \int |\phi(x,t)|^2 dx$$

and

$$\mathcal{H}(\phi) \equiv \int (|\nabla\phi(x,t)|^2 - \frac{1}{\sigma+1}|\phi(x,t)|^{2\sigma+2})$$

are constants in time.

Remark If $\sigma \geqslant \frac{2}{N}$, solutions may develop singularities in finite time.

II. Global Existence of IVP

Theorem (Theorem A)

Let $\phi_0 \in H^1(\mathbb{R}^N)$. For $\sigma = \frac{2}{N}$, a sufficient condition for global existence in the IVP is:

$$\|\phi_0\|_2 \leq \|\psi\|_2$$

where ψ is a positive solution of the equation

$$\Delta u - u + u^{\frac{4}{N}+1} = 0$$

of minimal L^2 norm (the ground state), and $\psi e^{\frac{it}{2}}$ is an exact solution of the IVP.

II. Global Existence of IVP

The idea behind the proof of Theorem A:
In the theorem of Ginibre and Velo ($0 < \sigma < \frac{2}{N-2}$) they show that the length T of the interval of existence $[t_0, t_0 + T]$ can be taken to depend only on $\|\phi(t_0)\|_{H^1}$.
Then if $\phi(x,t)$ is a maximally defined solution on $[t_0, t_{max}]$ we have two possibilities:

(i) $t_{max} = +\infty$

(ii) $\lim_{t \uparrow t_{max}} \|\phi(t)\|_{H^1} = +\infty$

In Ginibre and Velo's proof they use the invariants $\mathcal{N}(\phi)$ and $\mathcal{H}(\phi)$ to obtain an *a priori* bound of the type:

$$\|\phi(t)\|_{H^1(\mathbb{R}^N)} \leqslant C(\mathcal{N}, \mathcal{H})$$

II. Global Existence of IVP

We seek to imitate their proof by showing a particular version of the bound.

Sketch of Proof of Theorem A: Using the constants of motion and the interpolation estimate:

$$\|\nabla\phi(t)\|_2^2 \leqslant \mathcal{H} + \frac{C_{\sigma,N}^{2\sigma+2}}{\sigma+1}\|\phi_0\|_2^{2+\sigma(2-N)}\|\nabla\phi(t)\|_2^{\sigma N}$$

If the case $\sigma = \frac{2}{N}$, we re-arrange:

$$(1 - \frac{C_N^{\frac{4}{N}}}{\frac{2}{N}+1}\|\phi_0\|_2^{\frac{4}{N}})\|\nabla\phi(t)\|_2^2 \leqslant \mathcal{H}$$

II. Global Existence of IVP

Using Corollary 1.1 (explicit expression for $C_{\sigma,N}^{2\sigma+2}$), we obtain the estimate:

$$(1-(\frac{\|\phi_0\|_2}{\|\psi\|_2})^{\frac{4}{N}})\|\nabla\phi(t)\|_2^2 \leqslant \mathcal{H}$$

Taking $\|\phi_0\|_2 \leq \|\psi\|_2$, we obtain a time-independent bound on $\|\nabla\phi(t)\|_2^2$.
We use the fact that the scaling $f(x) \to \lambda^{\frac{1}{\sigma}} f(\lambda x)$ leaves the L^2 norm of f unchanged when $\sigma = \frac{2}{N}$. Since ψ solves the E-L equation for $J^{\sigma,N}$ in the critical case, then re-scaling ψ by $\frac{1}{\sigma}$ yields the equation

$$\Delta\psi - \psi + \psi^{\frac{4}{N}+1} = 0$$

$\square$

II. Global Existence of IVP

Theorem (Theorem A)

Let $\phi_0 \in H^1(\mathbb{R}^N)$. For $\sigma = \frac{2}{N}$, a sufficient condition for global existence in the IVP is:

$$\|\phi_0\|_2 \leq \|\psi\|_2$$

where ψ is a positive solution of the equation

$$\Delta u - u + u^{\frac{4}{N}+1} = 0$$

of minimal L^2 norm (the ground state), and $\psi e^{\frac{it}{2}}$ is an exact solution of the IVP.

III. Comments on Stability and Finite Blow-up

First, recall some conservation laws:

Theorem

Let $|x|\phi_0(x) \in L^2$, and let $\phi(x,t)$ be an H^1 solution of the NLS for $0 \leqslant t \leqslant T$. Then, for $0 \leqslant t \leqslant T$:

(i) $\frac{d}{dt}\int\{|x\phi - it\nabla\phi|^2 - \frac{t^2}{\sigma+1}|\phi|^{2\sigma+1}\}dx = t\frac{\sigma N-2}{\sigma+1}\int|\phi|^{2\sigma+2}dx$

(ii) $\frac{d^2}{dt^2}\int|\phi|^2|x|^2dx = 2\mathcal{H}(\phi_0) + \frac{N}{\sigma+1}(\frac{2}{N} - \sigma)\int|\phi|^{2\sigma+2}dx$

Remark Identity (i) is referred to as "pseudoconformal conservation law". (i) proved by Ginibre and Velo 1979, (ii) proved by Vlasov et al 1971.

III. Comments on Stability and Finite Blow-up

Glassey 1977 proved a result on finite time blow up of solutions to the NLS in the case $\sigma \geqslant \frac{2}{N}$. Tsutsumi strengthened this:

Theorem (Tsutsumi)

Let either

(i) $\mathcal{H}(\phi_0) < 0$

(ii) $\mathcal{H}(\phi_0) = 0$ *and* $Im \int x \cdot \overline{\phi_0} \nabla \phi_0 dx < 0$

(iii) $\mathcal{H}(\phi_0) > 0$ *and* $Im \int x \cdot \overline{\phi_0} \nabla \phi_0 dx \leqslant -2\sqrt{\mathcal{H}(\phi_0)} \|x\phi_0\|$

Then, there exists a time $0 < T < \infty$ *such that*

$$\lim_{t \uparrow T} \|\nabla \phi(t)\|_2 = +\infty$$

III. Comments on Stability and Finite Blow-up

In the critical case $\sigma = \frac{N}{2}$ the identity due to Vlasov reduces to:

$$\frac{d^2}{dt^2}\int |\phi|^2 |x|^2 dx = 2\mathcal{H}(\phi_0)$$

If we consider particular solutions $\Phi(x,t) = e^{it/2}R(x)$ where $R(x)$ is an H^1 function satisfying the equation:

$$\Delta u - u + u^{\frac{4}{N}+1} = 0$$

then we have that $\mathcal{H}(R) = 0$. As a consequence Weinstein obtains the following instability result when $\sigma = 2/N$, which expresses the "sharpness" of the condition in his global existence theorem.

III. Comments on Stability and Finite Blow-up

Theorem (Instability Theorem)

Let $\sigma = 2/N$. The nontrivial H^1 solutions of

$$\Delta u - u + u^{\frac{4}{N}+1} = 0$$

are unstable for the nonlinear Schrödinger equation in the following sense:
Let $R \in H^1 (R \neq 0)$ solve the above equation. Then for any $\delta > 0$, there is a function η, with $\|\eta - R\|_2 < \delta$ such that for $\phi(x,t)$ the solution of the IVP with $\phi_0 = \eta$ and:

$$\lim_{t \to T^-} \|\nabla \phi(t)\|_2 = \infty$$

for some $0 < T < \infty$

III. Comments on Stability and Finite Blow-up

The following picture emerges in the critical case $\sigma = 2/N$:

(1) If $\phi_0 \in H^1(\mathbb{R}^N)$ and $\|\phi_0\|_2 < \|\psi\|_2$, where ψ is the ground state of

$$\Delta u - u + u^{\frac{4}{N}+1} = 0$$

(ie. positive radial and H^1 solution of minimal L^2 norm), then the IVP has a unique global solution $\phi(x,t)$ of class $C([0,\infty) : H^1(\mathbb{R}^N))$.

(2) If $\mathcal{H}(\phi_0) < 0$ then the solution $\phi(x,t)$ of the NLS blows up in finite time in $H^1(\mathbb{R}^N)$.

(3) By Tsutsumi's Theorem $\mathcal{H}(\phi_0) \geqslant 0$ is not sufficient for global existence.

(4) If $\|\phi_0\|_2 < \|\psi\|_2$, then $\mathcal{H}(\phi_0) \geqslant 0$

III. Comments on Stability and Finite Blow-up

(5) If R is a nontrivial H^1 solution of the above equation then $Re^{it/2}$ is an exact solution of the NLS and $\mathcal{H}(Re^{it/2}) = 0$. These solutions are unstable in the sense of the Instability Theorem.

Conclusion

Weinstein obtains a sharp sufficient condition for global existence for the NLS

$$\frac{\partial \phi}{\partial t} + \Delta \phi + |\phi|^{2\sigma} \phi = 0$$

in the L^2 critical case $\sigma = \frac{2}{N}$, in terms of an exact stationary solution of the NLS.
This condition is derived by investigating the the sharpest constant for a classic interpolation inequality of the type Gagliardo-Nirenberg.