Available online at www.sciencedirect.com
ScienceDirect
Applied Mathematics and Computation 182 (2006) 1623–1628
APPLIED MATHEMATICS AND COMPUTATION
www.elsevier.com/locate/amc

Stabilized discontinuous Galerkin methods for scalar linear hyperbolic equations

Mi-Young Kim

Department of Mathematics, Inha University, Inchon 402-751, Republic of Korea

Abstract

Brezzi et al. [F. Brezzi, L.D. Marini, E. Süli, Discontinuous Galerkin methods for first-order hyperbolic problems, M^3AS, 14(12) (2004) 1893–1903] have proposed "stabilized" discontinuous Galerkin finite element methods to approximate the solution to scalar linear hyperbolic problems under a coercivity condition. In this paper, we relax such a condition. Stabilized DG methods are then applicable to more general problems arising in biology, for example. Strong stability of the approximate solution is shown and optimal order a priori error estimates are obtained.

Keywords: Stabilized discontinuous Galerkin method; Hyperbolic equations; Upwind; Jump stabilizations; Jump penalty

1. Introduction

For the last several decades, discontinuous Galerkin methods (DG methods) have been successfully applied to many problems. We refer the readers to [1,3] and the references cited therein, for more details. Recently, Brezzi et al. [2] have proposed "stabilized" discontinuous Galerkin methods to approximate the solution to scalar linear hyperbolic problems under a coercivity condition. In this paper, we relax such a condition. Stabilized DG methods, which are obtained by introducing a jump penalty to a weak form of the differential equation, include as a special case, the classical DG method [7,10]. We prove the strong stability of the approximate solution and show the optimal order of convergent rate. The advantages of the stabilized DG methods are stated in [2].

We now consider the following scalar linear hyperbolic boundary value problem:

$$\begin{aligned}&\boldsymbol{\beta}\cdot\nabla u+\gamma u=f\quad\text{in }\Omega,\\&u=g\quad\text{on }\Gamma_-.\end{aligned}\tag{1.1}$$

We assume that Ω is a bounded polygonal domain in $\mathbf{R}^2$ and the advective velocity field $\boldsymbol{\beta}$ is a constant vector. Γ_- denotes the inflow part of the boundary $\Gamma=\partial\Omega$: $\Gamma_-=\{x\in\partial\Omega:\boldsymbol{\beta}\cdot\mathbf{n}(x)<0\}$, where $\mathbf{n}(x)$ is the outward unit normal to $\partial\Omega$ at x. Further, we let $\Gamma_+=\{x\in\partial\Omega:\boldsymbol{\beta}\cdot\mathbf{n}(x)>0\}$ denote the outflow part of the boundary.

E-mail address: mikim@inha.ac.kr

0096-3003/$ - see front matter © 2006 Elsevier Inc. All rights reserved.
doi:10.1016/j.amc.2006.05.047

We also assume that $f \in L^2(\Omega)$, $g \in L^2(\Gamma_-)$. Brezzi et al. [2] assumed the existence of a positive constant c_0 such that

$$\gamma(x) \geqslant c_0 \quad \text{for all } x \in \overline{\Omega}. \tag{1.2}$$

Assumption (1.2) was in fact made in many situations. (See, for example [4] and the references cited therein.) However, for problems in biology such an assumption is sometimes too severe [8]. In this paper we relax (1.2) and we assume that γ is bounded, measurable, and continuous on $\overline{\Omega}$. Following the same line of the proofs given here, one can then check that the methods is also applicable to the case that γ is nonnegative, $\gamma \in L^1_{\text{loc}}(\Omega)$, and thus that the solution is not regular, which case often arises in modelling of the biological population [5,6,8,9] (see also the references cited therein).

The organization of the remainder of this article is as follows. In the next section, we present the stabilized DG methods to approximate the solution to (1.1). In Section 3, we show the consistency and the strong stability of the approximate solution. Finally, in Section 4, we prove the optimal order of convergent rate of the approximate solution to the problem.

2. Numerical method

In what follows we use the space $L^p(\Omega)$ and the Sobolev spaces $W^{m,p}(\Omega)$ and $H^m(\Omega) = W^{m,2}(\Omega)$, $m \geqslant 1$, $1 \leqslant p \leqslant \infty$ in their usual meaning for Ω a domain in R^2. The norm in $L^p(\Omega)$ is denoted by $\|\cdot\|_{p,\Omega}$, if $p \neq 2$ and by $\|\cdot\|_{0,\Omega}$, if $p = 2$. Similarly, if Λ is a piecewise smooth curve or a union of such curves, $\|\cdot\|_{p,\Lambda}$ denotes the norm in $L^p(\Lambda)$, with the subindex 0 if $p = 2$.

Let $\mathscr{T}$ be a triangulations of Ω. We assume that $\mathscr{T}$ is a regular family of triangulations in the sense that

$$\text{there exists a } \kappa > 0 \text{ such that } \frac{h}{h_{\min}} < \kappa, \quad h_{\min} = \min_{K \in \mathscr{T}} h_K, \tag{2.1}$$

where h_K is the diameter of $K \in \mathscr{T}$ and $h = \max_{K \in \mathscr{T}} h_K$. In order to define a discontinuous Galerkin method, we first introduce the *jumps* and *averages* of scalar- and vector-valued functions across the edges of $\mathscr{T}$. Following [2], let e be an interior edge shared by elements K_1 and K_2. Define the unit normal vectors $\mathbf{n}^1$ and $\mathbf{n}^2$ on e pointing exterior to K_1 and K_2, respectively. For a function φ, piecewise smooth on $\mathscr{T}$, with $\varphi^i := \varphi|_{K_i}$, we define

$$\{\varphi\} = \frac{1}{2}(\varphi^1 + \varphi^2), \qquad [|\varphi|] = \varphi^1\mathbf{n}^1 + \varphi^2\mathbf{n}^2 \quad \text{on } e \in \mathscr{E}_0,$$

where $\mathscr{E}_0$ is the set of interior edges e. For a vector-valued function $\boldsymbol{\tau}$, piecewise smooth on $\mathscr{T}$, with analogous meaning for $\boldsymbol{\tau}^1$ and $\boldsymbol{\tau}^2$, we define

$$\{\boldsymbol{\tau}\} = \frac{1}{2}(\boldsymbol{\tau}^1 + \boldsymbol{\tau}^2), \qquad [|\boldsymbol{\tau}|] = \boldsymbol{\tau}^1 \cdot \mathbf{n}^1 + \boldsymbol{\tau}^2 \cdot \mathbf{n}^2 \quad \text{on } e \in \mathscr{E}_0.$$

Notice that the jump $[|\varphi|]$ of the scalar function φ across $e \in \mathscr{E}_0$ is a vector parallel to the normal to e, and the jump $[|\boldsymbol{\tau}|]$ of the vector function $\boldsymbol{\tau}$ ia a scalar quantity. For $e \in \mathscr{E}_\partial$, the set of boundary edges, we let

$$[|\varphi|] = \varphi\mathbf{n}, \quad \{\boldsymbol{\tau}\} = \boldsymbol{\tau}.$$

We dot not require either of the quantities $\{\varphi\}$ or $[|\boldsymbol{\tau}|]$ on boundary edges, and leave them undefined there.

Next, let $P_r(K)$ denote the space of all polynomials of degree $\leqslant r$ on K. Let W_h be the family of subspaces of $L^2(\Omega)$ given by

$$W_h = \{v \in L^2(\Omega) : v_{|K} \in P_r(K)\}. \tag{2.2}$$

Then, on multiplying (1.1) by $v_h \in W_h$ and integrating by parts, we have that

$$\sum_{K \in \mathscr{T}} \left(\int_K (-u\boldsymbol{\beta} \cdot \nabla v_h + \gamma u v_h)\,\mathrm{d}x + \int_{\partial K} \boldsymbol{\beta} \cdot \mathbf{n} u v_h \mathrm{d}s \right) = \int_\Omega f v_h \,\mathrm{d}x. \tag{2.3}$$

Note that, for all vectors $\boldsymbol{\tau}$ and scalars φ piecewise smooth on $\mathscr{T}$,

$$\sum_{K\in\mathscr{T}}\int_{\partial K}\boldsymbol{\tau}\cdot\mathbf{n}\varphi\,\mathrm{d}s=\sum_{e\in\mathscr{E}}\int_e\{\boldsymbol{\tau}\}\cdot[|\varphi|]\,\mathrm{d}s+\sum_{e\in\mathscr{E}_0}\int_e[|\boldsymbol{\tau}|]\{\varphi\}\,\mathrm{d}s. \tag{2.4}$$

Since $[|\boldsymbol{\beta}u|]=0$ on internal edges, we then have, from (2.4) with $\boldsymbol{\tau}=\boldsymbol{\beta}u$, $\varphi=v_h$, that

$$\sum_{K\in\mathscr{T}}\int_{\partial K}\boldsymbol{\beta}\cdot\mathbf{n}uv_h\,\mathrm{d}s=\sum_{e\in\mathscr{E}}\int_e\{\boldsymbol{\beta}u\}\cdot[|v_h|]\,\mathrm{d}s=\sum_{e\nsubseteq\Gamma_-}\int_e\{\boldsymbol{\beta}u\}\cdot[|v_h|]\,\mathrm{d}s+\sum_{e\nsubseteq\Gamma_-}\int_e\boldsymbol{\beta}\cdot\mathbf{n}gv_h\,\mathrm{d}s. \tag{2.5}$$

On the other hand, since $\boldsymbol{\beta}$ is continuous, we have the following:

$$\{\boldsymbol{\beta}v_h\}\cdot[|v_h|]=\frac{1}{2}\{\boldsymbol{\beta}\}\cdot[|v_h^2|]\text{ on }\mathscr{E}_0,\quad\{\boldsymbol{\beta}v_h\}\cdot[|v_h|]=\{\boldsymbol{\beta}\}\cdot[|v_h^2|]\text{ on }\mathscr{E}_\partial. \tag{2.6}$$

Substituting (2.5) into (2.3) we now obtain a weak form of (1.1) as follows:

$$a_h(u,v_h)=(f,v_h)+\langle g,v_h\rangle,\quad v_h\in W_h, \tag{2.7}$$

where

$$a_h(u,v_h)=\sum_{K\in\mathscr{T}}\int_K(-u\boldsymbol{\beta}\cdot\nabla v_h+\gamma uv_h)\,\mathrm{d}x+\sum_{e\nsubseteq\Gamma_-}\int_e\{\boldsymbol{\beta}u\}\cdot[|v_h|]\,\mathrm{d}s, \tag{2.8}$$

$$(f,v_h)=\int_\Omega fv_h\,\mathrm{d}x,\ \langle g,v_h\rangle=-\sum_{e\subseteq\Gamma_-}\int_e\boldsymbol{\beta}\cdot\mathbf{n}gv_h\,\mathrm{d}s.$$

The stabilized form of discontinuous Galerkin methods we shall analyze is then found by introducing a jump penalty. That is, given r and $\mathscr{T}$, find $u_h\in W_h$ such that for any $v_h\in W_h$,

$$a_h(u_h,v_h)+b_h(u_h,v_h)=(f,v_h)+\langle g,v_h\rangle, \tag{2.9}$$

where

$$b_h(u_h,v_h)=\sum_{e\in\mathscr{E}_0}\int_e c_e[|u_h|]\cdot[|v_h|]\,\mathrm{d}s, \tag{2.10}$$

and

$$c_e\geqslant v_0|\boldsymbol{\beta}\cdot\mathbf{n}| \tag{2.11}$$

with v_0 a positive constant on $\mathscr{E}_0$. We define

$$c_e=\begin{cases}\boldsymbol{\beta}\cdot\mathbf{n}_\Omega/2 & \text{on } e\subseteq\Gamma_+,\\ -\boldsymbol{\beta}\cdot\mathbf{n}_\Omega/2 & \text{on } e\subseteq\Gamma_-,\end{cases} \tag{2.12}$$

where $\mathbf{n}_\Omega$ is the unit outward normal vector to $\partial\Omega$. We then see from (2.11) and (2.12), that

$$c_e\geqslant 0\quad\text{for all } e\in\mathscr{E}. \tag{2.13}$$

We here note that (2.9) includes as a special case, the classical DG which has the numerical flux function taken as the upwind flux. That is, the DG method (2.9) coincides with the one of [7] when $c_e=|\boldsymbol{\beta}\cdot\mathbf{n}|/2$ on $e\in\mathscr{E}$. Below, we denote by C a positive constant which may take different values on different occurrences. The constant may depend on the above parameter κ and r but not on other parameters, unless indicated explicitly.

3. Consistency and stability of the method

Consistency follows immediately from (2.7) and (2.9) on observing that, since $[|u|]=0$ on internal edges, $b_h(u,v_h)=0$. In particular, the following Galerkin orthogonality holds:

$$a_h(u-u_h,v_h)+b_h(u-u_h,v_h)=0,\quad v_h\in W_h. \tag{3.1}$$

Now, let $\widetilde{\chi}$ denote the locally defined L^2-projection of χ into W_h, for each $\chi \in L^2(\Omega)$. That is, $\widetilde{\chi}$ is defined by

$$\int_K (\chi - \widetilde{\chi}) v_h \, dx = 0 \qquad \forall v_h \in P_r(K), \quad K \in \mathcal{T}.$$

Then we have the following standard approximation properties:

$$\begin{aligned} \|\chi - \widetilde{\chi}\|_{0,K} &\leqslant C h_K^{r+1} |\chi|_{H^{r+1}(K)}, \\ \|\chi - \widetilde{\chi}\|_{0,\partial K} &\leqslant C h_K^{r+1/2} |\chi|_{H^{r+1}(K)}. \end{aligned} \tag{3.2}$$

Define

$$|||\cdot||| = \left(\|\cdot\|_{0,\Omega}^2 + \sum_{e \in \mathcal{E}} \|c_e^{1/2} [|\cdot|]\|_{0,e}^2 \right)^{1/2}. \tag{3.3}$$

We then prove that u_h is (strongly) stable with the norm $|||\cdot|||$.

Theorem 3.1. *There is a constant $C > 0$ depending on $\|\gamma\|_{\infty,\Omega}$ and* diam(Ω), *such that*

$$|||u_h||| \leqslant C \left\{ \|f\|_{0,\Omega} + \|g\|_{0,\Gamma_-} \right\}.$$

Proof. Consider $\psi(x) = \exp(-\delta \boldsymbol{\beta} \cdot (x - x_0)/|\boldsymbol{\beta}|^2)$, where $x \in \Omega$, $x_0 \in \partial\Omega$, and $\delta = 2\|\gamma\|_{\infty,\Omega} + 1$. We then see that $\|\psi\|_{\infty,\Omega} = 1$, $\psi \geqslant \exp(-\mathrm{diam}(\Omega)\delta/|\boldsymbol{\beta}|)$. We also observe, by integration by parts, that

$$\int_K (\boldsymbol{\beta} \cdot \nabla u_h) \psi u_h \, dx = \frac{1}{2} \left\{ \int_{\partial K} \psi u_h^2 \boldsymbol{\beta} \cdot \mathbf{n} \, ds + \delta \int_K \psi u_h^2 \, dx \right\}.$$

Thus, we have that

$$a_h(u_h, \psi u_h) = \sum_{K \in \mathcal{T}} \int_K \left(\frac{1}{2} \delta \psi u_h^2 + \psi \gamma u_h^2 \right) dx - \frac{1}{2} \sum_{K \in \mathcal{T}} \int_{\partial K} \boldsymbol{\beta} \cdot \mathbf{n} \psi u_h^2 \, ds + \sum_{e \not\subseteq \Gamma_-} \{\boldsymbol{\beta} u_h\} \cdot [|\psi u_h|] \, ds. \tag{3.4}$$

Now, by taking $\boldsymbol{\tau} = \boldsymbol{\beta}$, $\varphi = \psi u_h^2$, since $[|\boldsymbol{\beta}|] = 0$, we see from (2.4) that

$$\sum_{K \in \mathcal{T}} \int_{\partial K} (\boldsymbol{\beta} \cdot \mathbf{n}) \psi u_h^2 \, ds = \sum_{e \in \mathcal{E}} \int_e \{\boldsymbol{\beta}\} \cdot [|\psi u_h^2|] \, ds = \sum_{e \in \mathcal{E}_0} \int_e \{\boldsymbol{\beta}\} \cdot [|\psi u_h^2|] \, ds + \sum_{e \in \mathcal{E}_\partial} \int_e \{\boldsymbol{\beta}\} \cdot [|\psi u_h^2|] \, ds. \tag{3.5}$$

Combining (3.4) and (3.5) and using (2.6) we obtain that

$$\begin{aligned} a_h(u_h, \psi u_h) &= \sum_{K \in \mathcal{T}} \int_K \left(\frac{1}{2} \delta \psi u_h^2 + \psi \gamma u_h^2 \right) dx \quad + \sum_{e \subseteq \Gamma_+} \int_e \{\boldsymbol{\beta} u_h\} \cdot [|\psi u_h|] \, ds - \frac{1}{2} \sum_{e \subseteq \Gamma_-} \int_e \{\boldsymbol{\beta}\} \cdot [|\psi u_h^2|] \, ds \\ &= \sum_{K \in \mathcal{T}} \int_K \left(\frac{1}{2} \delta \psi u_h^2 + \psi \gamma u_h^2 \right) dx + \frac{1}{2} \sum_{e \subseteq \Gamma_+} \int_e \{\boldsymbol{\beta}\} \cdot [|\psi u_h^2|] \, ds - \frac{1}{2} \sum_{e \subseteq \Gamma_-} \int_e \{\boldsymbol{\beta}\} \cdot [|\psi u_h^2|] \, ds \\ &= \sum_{K \in \mathcal{T}} \int_K \left(\frac{1}{2} \delta \psi u_h^2 + \psi \gamma u_h^2 \right) dx + \sum_{e \subseteq \Gamma_+} \int_e c_e \psi |[|u_h|]|^2 \, ds + \sum_{e \subseteq \Gamma_-} \int_e c_e \psi |[|u_h|]|^2 \, ds. \end{aligned}$$

Therefore, we have that

$$\begin{aligned} a_h(u_h, \psi u_h) + b_h(u_h, \psi u_h) &= \sum_{K \in \mathcal{T}} \int_K \left(\frac{1}{2} \delta \psi u_h^2 + \psi \gamma u_h^2 \right) dx + \sum_{e \in \mathcal{E}} \int_e c_e \psi |[|u_h|]|^2 \, ds \\ &\geqslant \frac{1}{2} \exp(-\mathrm{diam}(\Omega)\delta/|\boldsymbol{\beta}|) |||u_h|||^2 \geqslant C |||u_h|||^2. \end{aligned} \tag{3.6}$$

In order to estimate $a_h(u_h, \psi u_h - \widetilde{\psi u_h}) + b_h(u_h, \psi u_h - \widetilde{\psi u_h})$, we first note that $\|\psi\|_{W^{l,\infty}(K)} \leqslant C\delta^l$, $\|u_h\|_{H^l(K)} \leqslant C h_K^{-l} \|u_h\|_{0,K}$, and $|u_h|_{H^{r+1}(K)} = 0$. We then have, by Leibniz's rule and (3.2), the following properties of the local projection: If $\delta h \leqslant 1$, then

$$\|\psi u_h - \widetilde{\psi u_h}\|_{0,K} + h_K^{1/2}\|\psi u_h - \widetilde{\psi u_h}\|_{0,\partial K} \leqslant Ch_K^{r+1}|\psi u_h|_{H^{r+1}(K)} \leqslant C(r+1)\delta h_K\|u_h\|_{0,K}. \tag{3.7}$$

We also have, by the inverse estimate and (3.2), that

$$\|\psi u_h - \widetilde{\psi u_h}\|_{\infty,K} + h_K^{1/2}\|\psi u_h - \widetilde{\psi u_h}\|_{\infty,\partial K} \leqslant Ch_K^{r+1}|\psi u_h|_{W^{r+1,\infty}(K)} \leqslant C(r+1)\delta\|u_h\|_{0,K}.$$

On the other hand, since $\int_K(\psi u_h - \widetilde{\psi u_h})w\,\mathrm{d}x = 0 \quad \forall w \in P_r(K)$, we see that

$$\int_K(\psi u_h - \widetilde{\psi u_h})\boldsymbol{\beta}\cdot\nabla u_h\,\mathrm{d}x = 0.$$

We thus have, by Hölder's inequality and (3.7), that

$$\begin{aligned} a_h(u_h, \psi u_h - \widetilde{\psi u_h}) &= \sum_{K\in\mathcal{T}}\int_K \gamma u_h(\psi u_h - \widetilde{\psi u_h})\,\mathrm{d}x + \sum_{e\not\subseteq\Gamma_-}\int_e\{\boldsymbol{\beta}u_h\}\cdot[|\psi u_h - \widetilde{\psi u_h}|]\,\mathrm{d}s \\ &\leqslant \frac{1}{4}\sum_{K\in\mathcal{T}}\|u_h\|_{0,K}^2 + C\sum_{K\in\mathcal{T}}h_K^2\|\psi u_h\|_{0,K}^2 + \sum_{e\not\subseteq\Gamma_-}\|\boldsymbol{\beta}u_h\|_{0,e}\|[|\psi u_h - \widetilde{\psi u_h}|]\|_{0,e} \\ &\leqslant C\|u_h\|_{0,\Omega}^2. \end{aligned} \tag{3.8}$$

Similarly, we see that

$$b_h(u_h, \psi u_h - \widetilde{\psi u_h}) \leqslant C\|c_e^{1/2}[|u_h|]\|^2. \tag{3.9}$$

Combining (3.6), (3.8) and (3.9), we then have that

$$a_h(u_h, \widetilde{\psi u_h}) + b_h(u_h, \widetilde{\psi u_h}) \geqslant C\||u_h\||^2. \tag{3.10}$$

On the other hand, we see that

$$a_h(u_h, \widetilde{\psi u_h}) + b_h(u_h, \widetilde{\psi u_h}) = (f, \widetilde{\psi u_h}) + \langle g, \widetilde{\psi u_h}\rangle \leqslant C\Big\{\|f\|_{0,\Omega} + \|g\|_{0,\Gamma_-}\Big\}\|u_h\|_{0,\Omega}. \tag{3.11}$$

Thus, from (3.10) and (3.11), we obtain that

$$\||u_h\|| \leqslant C\{\|f\|_{0,\Omega} + \|g\|_{\Gamma_-}\}.$$

This completes the proof. □

4. A priori error estimates

To derive an error bound, we adopt the following notations: $E = u_h - u$, $\theta = u_h - \tilde{u}$, $-\rho = \tilde{u} - u$.

Theorem 4.1. *Assume that u belongs to $H^{r+1}(K)$. Then, there is a constant $C > 0$ depending on $\|\gamma\|_{\infty,\Omega}$ and* diam(Ω), *such that for $h > 0$ sufficiently small,*

$$\||u - u_h\|| \leqslant Ch^{r+1/2}\|u\|_{r+1,\Omega}.$$

Proof. Since (3.1) holds with $v_h = \widetilde{\psi\theta}$, we have that

$$a_h(\theta, \widetilde{\psi\theta}) + b_h(\theta, \widetilde{\psi\theta}) = 0. \tag{4.1}$$

We now note that (3.10) holds for u_h replaced by θ. We thus have that

$$a_h(\theta, \widetilde{\psi\theta}) + b_h(\theta, \widetilde{\psi\theta}) \geqslant C\||\theta\||^2. \tag{4.2}$$

On the other hand, we note that

$$|\{\boldsymbol{\beta}\rho\}\cdot\mathbf{n}| = |\boldsymbol{\beta}\cdot\mathbf{n}||\{\rho\}| \leqslant \frac{c_e}{v_0}|\{\rho\}|. \tag{4.3}$$

Using the fact that $[|\theta|]$ is also normal to e, and using (2.13) once again, we then have that

$$\int_e \{\boldsymbol{\beta}\rho\} \cdot [|\theta|]\,\mathrm{d}s \leqslant \frac{1}{v_0}\|c_e^{1/2}\{\rho\}\|_{0,e}\|c_e^{1/2}[|\theta|]\|_{0,e},$$
$$\int_e c_e[|\rho|] \cdot [|\theta|]\,\mathrm{d}s \leqslant \frac{1}{v_0}\|c_e^{1/2}[|\rho|]\|_{0,e}\|c_e^{1/2}[|\theta|]\|_{0,e}. \tag{4.4}$$

Since $\int_K \rho\boldsymbol{\beta} \cdot \nabla(\widetilde{\psi\theta})\,\mathrm{d}x \equiv 0$, we see, from (2.8) and (4.4) and by integration by parts, that

$$a_h(\rho, \widetilde{\psi\theta}) = \sum_{K\in\mathscr{T}} \int_K \gamma\rho\widetilde{\psi\theta}\,\mathrm{d}x + \sum_{e\not\subseteq\Gamma_-} \int_e \{\boldsymbol{\beta}\rho\} \cdot [|\widetilde{\psi\theta}|]\,\mathrm{d}s$$
$$\leqslant C\left\{\|\rho\|_{0,\Omega}\|\theta\|_{0,\Omega} + \sum_{e\not\subseteq\Gamma_-} \|c_e^{1/2}\{\rho\}\|_{0,e}\|c_e^{1/2}[|\theta|]\|_{0,e}\right\}, \tag{4.5}$$
$$b_h(\rho, \widetilde{\psi\theta}) \leqslant C\sum_{e\in\mathscr{E}_0} \|c_e^{1/2}[|\rho|]\|_{0,e}\|c_e^{1/2}[|\theta|]\|_{0,e}.$$

We thus have, from (3.2) and (4.5), that

$$a_h(\rho, \widetilde{\psi\theta}) + b_h(\rho, \widetilde{\psi\theta}) \leqslant Ch^{k+1/2}\|u\|_{k+1,\Omega}|||\theta|||. \tag{4.6}$$

The result then directly follows from (4.2) and (4.6). □

Acknowledgement

This research was supported by KRF(2003-070-C00007).

References

[1] D.N. Arnold, F. Brezzi, B. Cockburn, L.D. Marini, Unified analysis of discontinuous Galerkin methods for elliptic problems, SIAM J. Numer. Anal. 39 (2002) 1749–1779.
[2] F. Brezzi, L.D. Marini, E. Süli, Discontinuous Galerkin methods for first-order hyperbolic problems, M^3AS 14 (12) (2004) 1893–1903.
[3] B. Cockburn, G.E. Karniadakis, C.-W. Shu (Eds.), Discontinuous Galerkin Methods, Springer, Berlin, 2000.
[4] P. Houston, C. Schwab, E. Süli, Stabilized hp-finite element methods for first order hyperbolic problems, SIAM J. Numer. Anal. 37 (2000) 1618–1643.
[5] M. Iannelli, Mathematical Theory of Age-Structured Population Dynamics, Applied Mathematics Monographs, Comitato Nazionale per le Scienze Matematiche, Consiglio Nazionale delle Ricerche (C.N.R.), Giardini-Pisa, 1995.
[6] M. Iannelli, M.-Y. Kim, E.-J. Park, A. Pugliese, Global boundedness of the solutions to a Gurtin–MacCamy system, Nonlinear Differ. Equ. Appl. 9 (2002) 197–216.
[7] C. Johnson, J. Pitkaranta, An analysis of the discontinuous Galerkin method for a scalar hyperbolic equation, Math. Comput. 46 (1986) 1–26.
[8] M.-Y. Kim, Discontinuous Galerkin methods for a model of population dynamics with unbounded mortality, SIAM J. Sci. Comput. 27 (4) (2006) 1371–1393.
[9] M.-Y. Kim, Y. Kwon, A collocation method for the Gurtin–MacCamy equation with finite life-span, SIAM J. Numer. Anal. 39 (6) (2002) 1914–1937.
[10] P. Lesaint, P.-A. Raviart, On a finite element method for solving the neutron transport equation, in: C.A. deBoor (Ed.), Mathematical Aspects of Finite Elements in Partial Differential Equations, Academic Press, New York, 1974, pp. 89–123.