Preface

Discontinuous Galerkin (dG) methods can be viewed as finite element methods allowing for discontinuities in the discrete trial and test spaces. Localizing test functions to single mesh elements and introducing numerical fluxes at interfaces, they can also be viewed as finite volume methods in which the approximate solution is represented on each mesh element by a polynomial function and not only by a constant function. From a practical viewpoint, working with discontinuous discrete spaces leads to compact discretization stencils and, at the same time, offers a substantial amount of flexibility, making the approach appealing for multi-domain and multi-physics simulations. Moreover, basic conservation principles can be incorporated into the method. Applications of dG methods cover a vast realm in engineering sciences. Examples can be found in the conference proceedings edited by Cockburn, Karniadakis, and Shu [106] and in recent special volumes of leading journals, e.g., those edited by Cockburn and Shu [114] and by Dawson [120]. There is also an increasing number of open source libraries implementing dG methods. A non-exhaustive list includes `deal.II` [27], `Dune` [36], `FEniCS` [251], `freeFEM` [118], `libmesh` [213], and `Life` [262].

A Brief Historical Perspective

Although dG methods have existed in various forms for more than 30 years, they have experienced a vigorous development only over the last decade, as illustrated in Fig. 1.

The first dG method to approximate first-order PDEs has been introduced by Reed and Hill in 1973 [268] in the context of steady neutron transport, while the first analysis for steady first-order PDEs was performed by Lesaint and Raviart in 1974 [227–229]. The error estimate was improved by Johnson and Pitkäranta in 1986 [204] who established an order of convergence in the L^2-norm of $(k + \frac{1}{2})$ if polynomials of degree k are used and the exact solution is smooth enough. Few years later, the method was further developed by Caussignac and Touzani [84, 85] to approximate the three-dimensional boundary-layer equations for incompressible steady fluid flows. At around the same time, dG methods were extended to time-dependent hyperbolic PDEs by Chavent and Cockburn [86] using the forward Euler scheme for time discretization together with limiters. The order of accuracy was improved by Cockburn and Shu [110, 111] using

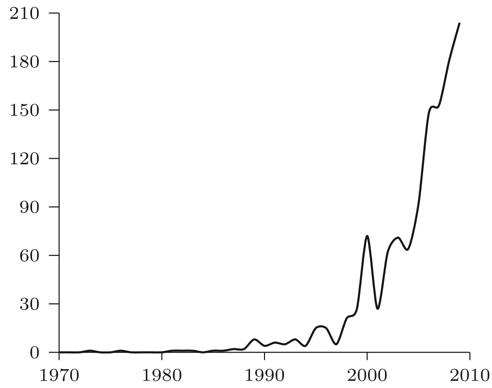

Fig. 1: Yearly number of entries with the keyword “discontinuous Galerkin” in the MathSciNet database

explicit Runge–Kutta schemes for time discretization, while a convergence proof to the entropy solution was obtained by Jaffré, Johnson, and Szepessy [200]. Extensions are discussed in a series of papers by Cockburn, Shu, and coworkers; see, e.g., [99, 108, 113].

For PDEs with diffusion, dG methods originated from the work of Nitsche on boundary-penalty methods in the early 1970s [248, 249] and the use of Interior Penalty (IP) techniques to weakly enforce continuity conditions imposed on the solution or its derivatives across interfaces, as in the work of Babuška [20], Babuška and Zlámal [24], Douglas and Dupont [134], Baker [25], Wheeler [306], and Arnold [14]. This latter work constitutes a milestone in the development of IP dG methods. In the late 1990s, following an approach more closely related to hyperbolic problems, dG methods were formulated using numerical fluxes by considering the mixed formulation of the diffusion term. Examples include the work of Bassi and Rebay [34] on the compressible Navier–Stokes equations and that of Cockburn and Shu [112] on convection-diffusion systems, leading to a new thrust in the development of dG methods. A unified analysis of dG methods for the Poisson problem can be found in the work of Arnold, Brezzi, Cockburn, and Marini [16], while a unified analysis encompassing both diffusive and first-order PDEs in the framework of Friedrichs' systems has been derived by Ern and Guermond [142–144].

Goals

The goal of this book is to provide the reader with the basic mathematical concepts to design and analyze dG methods for various model problems, starting at an introductory level and further elaborating on more advanced topics. Since the focus is on mathematical aspects of dG methods, linear model problems involving first-order PDEs and PDEs with diffusion play a central role.

Nonlinear problems, e.g., the incompressible Navier–Stokes equations and nonlinear conservation laws, are also treated, but in less detail for the latter, since very few mathematical results are yet available. This book also covers basic facts concerning the practical aspects of dG methods related to implementation; a more detailed treatment can be found in the recent textbook by Hesthaven and Warburton [192].

Some of the topics covered in this book stem from (very) recent work by the authors, e.g., on discrete functional analysis and convergence to minimal regularity solutions for PDEs with diffusion [131], the above-mentioned work with Guermond on Friedrichs' systems, and joint work with Vohralík on diffusive flux reconstruction and a posteriori error estimates for elliptic PDEs [147, 149, 151]. Another salient feature of this book is to bridge, as much as possible, the gap between dG methods and finite volume methods, where many important theoretical advances have been accomplished over the last decade (see, e.g., the work of Eymard, Gallouët, and Herbin [156, 157, 159]). In particular, we present so-called cell-centered Galerkin methods for elliptic PDEs with one degree of freedom per mesh cell, following recent work by Di Pietro and coworkers [5, 6, 128]. For the approximation of unsteady problems, we focus on the method of lines, whereby dG space semidiscretization is combined with various schemes to march in time. One important goal is to show how the stability of dG schemes interacts with the stability of the temporal scheme. This is reflected, for instance, in our covering the analysis based on energy estimates for explicit Runge–Kutta dG methods applied to first-order, linear PDEs, following the recent work of Burman, Ern, and Fernández [66]. For a description and analysis of space-time dG methods as an alternative to the method of lines, we refer the reader to the textbook by Thomée [294]. Finally, we consider a rather general setting for the underlying meshes so as to exploit, as much as possible, the flexibility offered by the dG setting.

Outline

Chapter 1 introduces basic concepts to formulate and analyze dG methods. In this chapter, we present:

(a) Two important results to assert the well posedness of linear model problems, namely the so-called Banach–Nečas–Babuška Theorem and the Lax–Milgram Lemma.

(b) The basic ingredients related to meshes and polynomials to build discrete functional spaces and, in particular, broken polynomial spaces.

(c) The three key properties for the convergence analysis of dG methods in the context of nonconforming finite elements, namely discrete stability, consistency, and boundedness.

(d) The basic analysis tools, in particular inverse and trace inequalities needed to assert discrete stability and boundedness, together with optimal polynomial approximation results, thereby leading to the concept of admissible mesh sequences.

In this book, we focus on mesh refinement as the main parameter to achieve convergence. Convergence analysis using, e.g., high-degree polynomials is possible by further tracking the dependency of the inverse and trace inequalities on the underlying polynomial degree and, in some cases, modifying accordingly the penalty strategy in the dG method to achieve discrete stability. Important tools in this direction can be found in the work of Babuška and Dorr [22], Babuška, Szabó, and Katz [23], Szabó and Babuška [288] and Schwab [275], and, in the context of dG methods, in the recent textbook of Hesthaven and Warburton [192].

The remaining chapters of this book are organized into three parts, each comprising two chapters. Part I, which comprises Chaps. 2 and 3, deals with scalar first-order PDEs. Chapter 2 focuses on the steady advection-reaction equation as a simple model problem. Therein, we present some mathematical tools to achieve a well-posed formulation at the continuous level and show how these ideas are built into the design of dG methods. Two methods are analyzed thoroughly, which correspond in the finite volume terminology to the use of centered and upwind fluxes.

Chapter 3 deals with unsteady first-order PDEs. Within the method of lines, we consider a dG method for space semidiscretization combined with an explicit scheme to march in time. We first present a detailed analysis in the linear case whereby the upwind dG method of Chap. 2 is used for space semidiscretization. Following the seminal work of Levy and Tadmor [231], the stability analysis hinges on energy estimates, and not on the more classical approach based on regions of absolute stability for explicit Runge–Kutta (RK) schemes. In particular, we show how the stability provided by the upwind dG scheme can compensate the anti-dissipative nature of the explicit time-marching scheme. We also show how this stability can be used to achieve quasi-optimal error estimates for smooth solutions. We present results for the forward Euler method combined with a finite volume scheme as an example of low-order approximation and then move on to explicit two- and three-stage RK schemes combined with a high-order dG approximation. The second part of Chap. 3 deals with nonlinear conservation laws. Taking inspiration from the linear case, we first design dG methods for space semidiscretization using the concept of numerical fluxes and provide various classical examples thereof, including Godunov, Rusanov, Lax–Friedrichs, and Roe fluxes. Then, we consider explicit RK schemes for time discretization and discuss Strong Stability-Preserving (SSP) RK schemes following the ideas of Gottlieb, Shu, and Tadmor [175, 176]. Finally, we discuss the use of limiters in the framework of one-dimensional total variation analysis and possible extensions to the multidimensional case.

Part II, which comprises Chaps. 4 and 5, addresses scalar, second-order PDEs. Chapter 4 covers various model problems with diffusion. We first present a heuristic derivation and a convergence analysis to smooth solutions for a purely diffusive problem approximated by the Symmetric Interior Penalty

(SIP) method of Arnold [14]. Then, we introduce the central concept of discrete gradients and present some important applications, including the link with the mixed dG approach and the local formulation of the discrete problem using numerical fluxes. Hybrid mixed dG methods are also briefly discussed. Focusing next on heterogeneous diffusion problems, we analyze a modification of the SIP method, the Symmetric Weighted Interior Penalty (SWIP) method, using weighted averages in the consistency term and harmonic means of the diffusion coefficient at interfaces in the penalty term. The SWIP method is then combined with the upwind dG method of Chap. 2 to approximate diffusion-advection-reaction problems. We examine singularly perturbed regimes due to dominant advection, and include in the analysis the case where the diffusion actually vanishes in some parts of the domain. Finally, as a simple example of time-dependent problem with diffusion, we consider the heat equation approximated by the SIP method in space and implicit time-marching schemes (backward Euler and BDF2).

Chapter 5 covers some additional topics related to diffusive PDEs, and, for the sake of simplicity, the scope is limited to purely diffusive problems. First, we present discrete functional analysis tools in broken polynomial spaces, namely discrete Sobolev embeddings, a discrete Rellich–Kondrachov compactness result, and a weak asymptotic consistency result for discrete gradients. As an example of application, we prove the convergence, as the meshsize goes to zero, of the sequence of discrete SIP solutions to minimal regularity solutions. These ideas are instrumental in Chap. 6 when analyzing the convergence of dG approximations for the steady incompressible Navier–Stokes equations. Then, we present some possible variations on symmetry and penalty for the SIP method. A further topic concerns the so-called cell-centered Galerkin methods which use a single degree of freedom per mesh cell and a suitable discrete gradient reconstruction, thereby providing a closer connection between dG and finite volume methods for elliptic PDEs. The last two topics covered in Chap. 5 are local postprocessing of the dG solution and a posteriori error estimates. One salient feature of local postprocessing is the possibility to reconstruct locally an accurate and conservative diffusive flux. An important application is contaminant transport where the diffusive flux resulting from a Darcy flow model is used as advection velocity. In turn, a posteriori error estimates provide fully computable error upper bounds that can be used to certify the accuracy of the simulation and to adapt the mesh. These last two topics can be nicely combined together since local postprocessing provides an efficient tool for deriving constant-free a posteriori error estimates.

Part III, which comprises Chaps. 6 and 7, deals with systems of PDEs. Chapter 6 is devoted to incompressible flows. Focusing first on the steady Stokes equations, we examine how the divergence-free constraint on the velocity field can be tackled using dG methods. We detail the analysis of equal-order approximations using both discontinuous velocities and pressures, whereby pressure jumps need to be penalized, and then briefly discuss alternative formulations avoiding the need for pressure jump penalty. The next step is the discretization of the nonlinear convection term in the momentum equation. To this purpose, we derive a discrete trilinear form that leads to the correct kinetic energy balance, using the so-called Temam's device to handle the fact that discrete velocities are only weakly divergence-free. Moreover, since the model problem is now nonlinear, the convergence analysis is performed under minimal regularity assumptions on the exact solution and without any smallness assumption on the data using the discrete functional analysis tools of Chap. 5. Finally, we discuss the approximation of the unsteady Navier–Stokes equations in the context of pressure-correction methods.

Chapter 7 presents a unified approach for the design and analysis of dG methods based on the class of symmetric positive systems of first-order PDEs introduced by Friedrichs [163]. Focusing first on the steady case, we review some examples of Friedrichs' systems and derive the main mathematical tools for asserting well posedness at the continuous level. Using these tools, we derive and analyze dG methods, and, in doing so, we follow a similar path of ideas to that undertaken in Chap. 2. Then, we consider more specifically the setting of two-field Friedrichs' systems and highlight the common ideas with the mixed dG approximation of elliptic PDEs. Finally, we consider unsteady Friedrichs' systems with explicit RK schemes in time and then specialize the setting to two-field Friedrichs' systems related to linear wave propagation, thereby addressing energy conservation and the possibility to accommodate local time stepping in the context of leap-frog schemes.

Appendix A covers practical implementation aspects of dG methods, focusing on matrix assembly and choice of local bases for which we discuss selection criteria and present various examples including both nodal and modal basis functions.

A bibliography comprising about 300 entries closes the manuscript. The amount of literature on dG methods is so vast that this bibliography is by no means exhaustive. We hope that the selected entries provide the reader with additional reading paths to examine more deeply the topics covered herein and to explore new ones. We mention, in particular, the recent textbooks by Hesthaven and Warburton [192], Kanschat [205], and Rivière [269].

Audience

This book is primarily geared to graduate students and researchers in applied mathematics and numerical analysis. It can be valuable also to graduate students and researchers in engineering sciences who are interested in further understanding the mathematical aspects that underlie the construction of dG methods, since these aspects are often important to formulate such methods when faced with new challenging applications. The reader is assumed to be familiar with conforming finite elements including weak formulations of model problems and error analysis (as presented, e.g., in the textbooks of Braess [49], Brenner and Scott [54], Ciarlet [92], or Ern and Guermond [141]) and to have some acquaintance with the basic PDEs in engineering and applied sciences. Special care has been devoted to making the material as much self-contained as possible. The general level of the book is best suited for a graduate-level course which can be built by drawing on some of the present chapters. The material is actually an elaboration on the lecture notes by the authors for a graduate course on dG methods at University Pierre et Marie Curie.

Acknowledgments

We are grateful to several colleagues for their constructive remarks that helped improve the manuscript. In particular, our warmest thanks go to Léo Agélas (IFP Énergies nouvelles), Erik Burman (University of Sussex), Miguel Fernández (INRIA), Jean-Luc Guermond (Texas A&M University), Rémi Joubaud (University Paris-Est), Olivier Le Maître (LIMSI), Serge Piperno (University Paris-Est), Quang Huy Tran (IFP Énergies nouvelles), and Martin Vohralík (University Pierre et Marie Curie). We are also thankful to Rémi Abgrall for offering us the opportunity to publish this book.