DISCONTINUOUS GALERKIN METHOD FOR TIME DEPENDENT PROBLEMS: SURVEY AND RECENT DEVELOPMENTS

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Abstract. In these lectures we give a general survey on discontinuous Galerkin methods for solving time dependent partial differential equations. We also present a few recent developments on the design, analysis and application of these discontinuous Galerkin methods.

Key words. Discontinuous Galerkin method, time dependent partial differential equations, superconvergence, positivity-preserving, δ-functions.

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1. Introduction. Discontinuous Galerkin (DG) methods belong to the class of finite element methods. The finite element function space corresponding to DG methods consists of piecewise polynomials (or other simple functions) which are allowed to be completely discontinuous across element interfaces. Therefore, using finite element terminologies, DG methods are the most extreme case of nonconforming finite element methods.

The first DG method was introduced in 1973 by Reed and Hill in a Los Alamos technical report [72]. It solves the equations for neutron transport, which are time independent linear hyperbolic equations. A major development of the DG method is carried out by Cockburn et al. in a series of papers [28, 27, 25, 23, 29], in which the authors have established a framework to easily solve nonlinear time dependent hyperbolic equations, such as the Euler equations of compressible gas dynamics. The DG method of Cockburn et al. belongs to the class of method-of-lines, namely the DG discretization is used only for the spatial variables, and explicit, nonlinearly stable high order Runge-Kutta methods [81] are used to discretize the time variable. Other important features of the DG method of Cockburn et al. include the usage of exact or approximate Riemann solvers as interface fluxes and total variation bounded (TVB) nonlinear limiters [79] to achieve non-oscillatory properties for strong shocks, both of which are borrowed from the methodology of high resolution finite volume schemes.

The DG method has found rapid applications in such diverse areas as aeroacoustics, electro-magnetism, gas dynamics, granular flows, magnetohydrodynamics, meteorology, modeling of shallow water, oceanography, oil recovery simulation, semiconductor device simulation, transport of contaminant in porous media, turbomachinery, turbulent flows, viscoelastic flows and weather forecasting, among many others. For earlier work on

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DG methods, we refer to the survey paper [24], and other papers in that Springer volume, which contains the conference proceedings of the First International Symposium on Discontinuous Galerkin Methods held at Newport, Rhode Island in 1999. The lecture notes [21] is a good reference for many details, as well as the extensive review paper [31]. The review paper [99] covers the local DG method for partial differential equations (PDEs) containing higher order spatial derivatives. More recently, there are three special journal issues devoted to the DG method [32, 35, 33], which contain many interesting papers on DG method in all aspects including algorithm design, analysis, implementation and applications. There are also a few recent books and lecture notes [38, 54, 59, 75, 80] on DG methods.

2. DG methods for hyperbolic conservation laws. As we mentioned in the previous section, the first DG method [72] was designed to solve linear hyperbolic equations in neutron transport. Let us use the following simple example to demonstrate the idea of this method. We consider a one-dimensional linear steady state hyperbolic equation

$$u_x = f, \quad x \in [0,1]; \qquad u(0) = g. \tag{2.1}$$

First, we divide [0,1] into N cells

$$0 = x_{\frac{1}{2}} < x_{\frac{3}{2}} < \cdots < x_{N+\frac{1}{2}} = 1,$$

and denote

$$I_j = \left(x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}\right), \quad x_j = \frac{1}{2}\left(x_{j-\frac{1}{2}} + x_{j+\frac{1}{2}}\right), \quad h_j = x_{j+\frac{1}{2}} - x_{j-\frac{1}{2}}$$

as the cells, cell centers and cell lengths respectively. We also define $h = h_{\max} = \max_j h_j$ and $h_{\min} = \min_j h_j$, and we consider only regular meshes, that is $h_{\max} \le \lambda h_{\min}$ where $\lambda \ge 1$ is a constant during mesh refinement. If $\lambda = 1$, then the mesh is uniformly distributed. Define the discontinuous Galerkin finite element space as

$$V_h^k = \{v : v|_{I_j} \in \mathcal{P}^k(I_j), j = 1, \cdots, N\}, \tag{2.2}$$

where $\mathcal{P}^k(I_j)$ denotes the space of polynomials in I_j of degree at most k. This polynomial degree k can actually change from cell to cell, but we assume it is a constant in these lectures for simplicity. The DG scheme for solving (2.1) is: find $u_h \in V_h^k$, such that for any $v_h \in V_h^k$ and all $1 \le j \le N$,

$$\begin{aligned} &-\int_{I_j} u_h \, (v_h)_x dx + (u_h)^-_{j+\frac{1}{2}} (v_h)^-_{j+\frac{1}{2}} - (u_h)^-_{j-\frac{1}{2}} (v_h)^+_{j-\frac{1}{2}} \\ &= \int_{I_j} f v_h dx. \end{aligned} \tag{2.3}$$

Here we define $(u_h)^-_{\frac{1}{2}} = g$ using the given boundary condition in (2.1). If a local basis of $P^k(I_j)$ is chosen and denoted as $\varphi^\ell_j(x)$ for $\ell = 0, 1, \cdots, k$, we can express the numerical solution as

$$u_h(x) = \sum_{\ell=0}^{k} u^\ell_j \varphi^\ell_j(x), \qquad x \in I_j,$$

and we should solve for the coefficients

$$u_j = \begin{pmatrix} u^0_j \\ \vdots \\ u^k_j \end{pmatrix}, \tag{2.4}$$

which, according to the scheme (2.3), satisfies the linear equation

$$A_j u_j = b_j \tag{2.5}$$

where A_j is a $(k+1) \times (k+1)$ matrix whose (ℓ, m)-th entry is given by

$$a^{\ell,m}_j = -\int_{I_j} \varphi^m_j(x)(\varphi^\ell_j(x))_x dx + \varphi^m_j(x_{j+\frac{1}{2}})\varphi^\ell_j(x_{j+\frac{1}{2}}) \tag{2.6}$$

and the ℓ-th entry of the right-hand-side vector b_j is given by

$$b^\ell_j = u_h(x^-_{j-\frac{1}{2}})\varphi^\ell_j(x_{j-\frac{1}{2}}) + \int_{I_j} f(x)\varphi^\ell_j(x)dx,$$

which depends on the information of u_h in the left cell I_{j-1}, if it is in the computational domain, or on the boundary condition, if it is outside the computational domain (i.e. when $j = 1$). It is easy to verify that the matrix A_j in (2.5) with entries given by (2.6) is invertible, hence the numerical solution u_h in the cell I_j can be easily obtained by solving the small linear system (2.5), once the solution at the left cell I_{j-1} is already known, or if the left cell is outside the computational domain. Therefore, we can obtain the numerical solution u_h in the following ordering: first we obtain it in the cell I_1, since its left boundary is equipped with the prescribed boundary condition in (2.1). We then obtain the solution in the cell I_2, as the numerical solution u_h in its left cell I_1 is already available. This process can be repeated sequentially to obtain solutions in I_j with $j = 3, 4, \cdots$ until we obtain the solution u_h for all cells in the computational domain.

Notice that this method does not involve any large linear system solvers and is very easy to implement. The first order version ($k = 0$) is a well-known upwind finite difference scheme, however it is more difficult to generalize the same scheme for higher order finite difference schemes which involve a wide stencil. On the other hand, this DG scheme can be designed for any polynomial degree k, and it is easy to be generalized to

two and higher spatial dimensions. In [56], Lesaint and Raviart proved that this DG method is convergent with the optimal order of accuracy, namely $O(h^{k+1})$, in the L^2 norm, when piecewise tensor product polynomials of degree k are used as basis functions in multi-dimensions. Numerical experiments indicate that the convergence rate is also optimal when the usual piecewise polynomials of degree k are used instead in multi-dimensions.

Notice that, even though the method (2.3) is designed for the steady state problem (2.1), it can be easily used on initial-boundary value problems of linear time dependent hyperbolic equations: we just need to identify the time variable t as one of the spatial variables. Also, this DG method can be easily designed and efficiently implemented on arbitrary triangulations. L^2 error estimates of $O(h^{k+1/2})$ where k is again the polynomial degree and h is the mesh size can be obtained when the solution is sufficiently smooth, for arbitrary meshes, see, e.g. [53]. This estimate is actually sharp for the most general situation [67], however in many cases the optimal $O(h^{k+1})$ error bound can be proved [74, 22]. In actual numerical computations, one almost always observe the optimal $O(h^{k+1})$ accuracy.

Unfortunately, even though the method (2.3) is easy to implement, accurate, and efficient, it cannot be easily generalized to linear systems, where the characteristic information comes from different directions, or to nonlinear problems, where the characteristic wind direction depends on the solution itself. This difficulty can be overcome when the DG discretization is only used for the spatial variables, and the time discretization is achieved by the explicit Runge-Kutta methods. This is the approach of the so-called Runge-Kutta discontinuous Galerkin (RKDG) method [28, 27, 25, 23, 29]. We demonstrate the RKDG method with the one dimensional conservation law

$$u_t + f(u)_x = 0. \tag{2.7}$$

The semi-discrete DG method for solving (2.7) is defined as follows: find the unique function $u_h = u_h(t) \in V_h^k$ such that, for all test functions $v_h \in V_h^k$ and all $1 \le j \le N$, we have

$$\begin{aligned} \int_{I_j} (u_h)_t \, v_h dx - \int_{I_j} f(u_h)(v_h)_x dx \\ + \hat{f}_{j+\frac{1}{2}}(v_h)^-_{j+\frac{1}{2}} - \hat{f}_{j-\frac{1}{2}}(v_h)^+_{j-\frac{1}{2}} = 0. \end{aligned} \tag{2.8}$$

Here, $\hat{f}_{i+\frac{1}{2}}$ is the numerical flux, which is a single-valued function defined at the cell interfaces and in general depends on the values of the numerical solution u_h from both sides of the interface

$$\hat{f}_{i+\frac{1}{2}} = \hat{f}(u_h(x^-_{i+\frac{1}{2}}, t), u_h(x^+_{i+\frac{1}{2}}, t)).$$

We use the so-called monotone fluxes from finite difference and finite volume schemes for solving conservation laws, which satisfy the following conditions:

- Consistency: $\hat{f}(u,u) = f(u)$;
- Continuity: $\hat{f}(u^-, u^+)$ is at least Lipschitz continuous with respect to both arguments u^- and u^+.
- Monotonicity: $\hat{f}(u^-, u^+)$ is a non-decreasing function of its first argument u^- and a non-increasing function of its second argument u^+. Symbolically $\hat{f}(\uparrow, \downarrow)$.

We refer to, e.g., [57] for more details about monotone fluxes.

The semi-discrete version of this DG scheme can again be written in the compact form

$$\frac{d}{dt}u_j = A(u_{j-1}) + B(u_j) + C(u_{j+1})$$

where the vectors u_j are defined in (2.4), and A, B, C are vector functions. If the conservation law is linear, then A, B, C are linear operators, namely

$$A(u_{j-1}) = Au_{j-1}; \qquad B(u_j) = Bu_j; \qquad C(u_{j+1}) = Cu_{j+1}$$

with constant matrices A, B and C (scaled by the local mesh sizes). This makes the implementation of the RKDG method local and highly efficient. It also makes the method easy for parallel implementation. The method can achieves almost 100% parallel efficiency for static meshes and over 80% parallel efficiency for dynamic load balancing with adaptive meshes [9, 73].

As a finite element method, the DG method can be designed in multi-dimensions on arbitrary triangulations (even those with hanging nodes) in the same fashion as in the one-dimensional case. It is easy to handle complicated geometry and boundary conditions.

2.1. Stability. It is well known that weak solutions of (2.7) may not be unique and the unique, physically relevant weak solution (the so-called entropy solution) satisfies the following entropy inequality

$$U(u)_t + F(u)_x \le 0 \tag{2.9}$$

in distribution sense, for any convex entropy $U(u)$ satisfying $U''(u) \ge 0$ and the corresponding entropy flux $F(u) = \int^u U'(u)f'(u)du$. It will be nice if a numerical approximation to (2.7) also shares a similar entropy inequality as (2.9). It is usually quite difficult to prove a discrete entropy inequality for finite difference or finite volume schemes, especially for high order schemes and when the flux function $f(u)$ in (2.7) is not convex or concave. However, it turns out that it is easy to prove that the solution u_h to the semi-discrete DG scheme (2.8) satisfies a cell entropy inequality [51]:

$$\frac{d}{dt}\int_{I_j} U(u_h)\,dx + \hat{F}_{j+\frac{1}{2}} - \hat{F}_{j-\frac{1}{2}} \le 0 \tag{2.10}$$

for the square entropy $U(u) = \frac{u^2}{2}$, with a consistent entropy flux

$$\hat{F}_{i+\frac{1}{2}} = \hat{F}(u_h(x^-_{i+\frac{1}{2}}, t), u_h(x^+_{i+\frac{1}{2}}, t))$$

satisfying $\hat{F}(u, u) = F(u)$.

An easy corollary of the cell entropy inequality is the following L^2 stability. For periodic or compactly supported boundary conditions for the computational domain $[a, b]$, the solution u_h to the semi-discrete DG scheme (2.8) satisfies the following L^2 stability

$$\frac{d}{dt}\int_a^b (u_h)^2 dx \leq 0, \tag{2.11}$$

or

$$\|u_h(\cdot, t)\| \leq \|u_h(\cdot, 0)\|. \tag{2.12}$$

Here and below, an unmarked norm is the usual L^2 norm.

Notice that both the cell entropy inequality (2.10) and the L^2 stability (2.11) are valid even when the exact solution of the conservation law (2.7) is discontinuous. Both conclusions are valid in multi-dimensions on arbitrary triangulations [51], and for both scalar equations and symmetric hyperbolic systems [39]. They also hold true for fully discrete RKDG methods for linear conservation laws [110].

2.2. Error estimates and superconvergence. If we assume the exact solution of (2.7) is smooth, we can obtain optimal L^2 error estimates. Namely, the solution u_h of the DG scheme (2.8) for the PDE (2.7) with a smooth solution u, using the space of k-th degree piecewise polynomials (2.2), satisfies the following error estimate

$$\|u - u_h\| \leq Ch^{k+1} \tag{2.13}$$

where C depends on u and its derivatives but is independent of h. Such error estimates can be obtained for the general nonlinear scalar conservation law (2.7) and hyperbolic systems, and for both semi-discrete DG methods and fully discretized RKDG methods, see [108, 109, 110]. The results also hold in multi-dimensions in tensor-product meshes and basis functions.

In recent years, there are a lot of efforts in the literature to obtain superconvergence results for DG methods solving hyperbolic conservation laws. These results consist of two categories.

The first category is to explore the superconvergence of the DG solution to the exact smooth solution in negative norms for linear hyperbolic equations:

$$\|u - u_h\|_{-k} \leq Ch^{2k+1} \tag{2.14}$$

where C depends on u and its derivatives but is independent of h [26]. Here $\|\cdot\|_{-k}$ is the negative Sobolev norm defined by

$$\|v\|_{-k} = \max_{\varphi \in H^k,\, \varphi \neq 0} \frac{(v,\varphi)}{\|\varphi\|_{H^k}}$$

where $(\cdot,\cdot)$ is the standard L^2 inner product and H^k is standard Sobolev space of order k. This result (and similar results for divided differences on uniform meshes), together with a local post-processing technique [12], allows us to obtain a post-processed solution $w_h = P(u_h)$ (where P is a local post-processing operator) on uniform meshes which is superconvergent in the strong L^2 norm:

$$\|u - w_h\| \leq Ch^{2k+1} \tag{2.15}$$

where C depends on u and its derivatives but is independent of h [26]. These results have been generalized to one-sided post-processing near the boundaries [76], structured triangular meshes [65], non-uniform meshes [34], and nonlinear problems [50]. It has also been applied to aeroacoustics [77] and computer graphics [82].

The second category is to explore the superconvergence of the DG solution to a special projection of the exact smooth solution, or superconvergence of the DG solution to the exact smooth solution at certain Gauss-Radau quadrature points.

The superconvergence of the DG solution to a special projection of the exact smooth solution takes the form

$$\|Pu - u_h\| \leq Ch^{k+1+\alpha} \tag{2.16}$$

where Pu is a projection (of the Gauss-Radau type) of the exact solution u into the finite element space V_h^k, and $\alpha > 0$ is the rate of superconvergence. In [18], Cheng and Shu started this line of study by obtaining (2.16) with $\alpha = 1/2$ for linear, time-dependent hyperbolic equations in one-dimension, with uniform meshes and periodic boundary conditions. The proof is based on Fourier analysis and is carried out only for the piecewise linear $k = 1$ case, however numerical results confirm the validity for higher k's. Another important consequence of this superconvergence result is that the constant C in (2.16) only grows linearly with time t, therefore the standard error $\|u - u_h\|$ does not grow for a very long time $t \sim 1/\sqrt{h}$. This analysis verifies an observation by practitioners, that the error of the DG solution for wave propagation does not seem to grow much with time. The result in [18] is improved in [20] to general polynomial degree k, on non-uniform regular meshes, and without periodic boundary conditions. The technique used in [20] is a finite element type, not a Fourier analysis. In [105], the result in [20] is further improved to $\alpha = 1$. This half-order increase in the analysis is highly non-trivial and involves subtle handling of cancellation of errors

during time evolution. The result in [105] is optimal. In [64], (2.16) with $\alpha = 1/2$ is proved for scalar nonlinear conservation laws with a fixed wind direction in one space dimension.

The superconvergence of the DG solution to the exact smooth solution at certain Gauss-Radau quadrature points has been explored in the literature. In [2, 3], Adjerid et al. proved the $(k+2)$-th order superconvergence of the DG solutions at the downwind-biased Radau points for ordinary differential equations. Later, Adjerid and Weihart [4, 5] investigated the local DG error for multi-dimensional first-order linear symmetric and symmetrizable hyperbolic systems of partial differential equations. The authors showed the projection of the local DG error is also $(k+2)$-th order superconvergent at the downwind-biased Radau points by performing a local error analysis on Cartesian meshes. The global superconvergence is given by numerical experiments. In [4, 5], only initial-boundary value problems are considered, and the local DG error estimate is valid for t sufficiently large. Subsequently, Adjerid and Baccouch [1] investigated the global convergence of the implicit residual-based a posteriori error estimates, and proved that these estimates at a fixed time t converge to the true spatial error in the L^2 norm under mesh refinement. In [117], using Fourier analysis, Zhong and Shu showed that the error between the DG numerical solution and the exact solution is $(k+2)$-th order superconvergent at the downwind-biased Radau points and $(2k+1)$-th order superconvergent at the downwind point in each cell on uniform meshes with periodic boundary conditions for $k = 1$, 2 and 3, for linear time-dependent hyperbolic equations in one-dimension, with uniform meshes and periodic boundary conditions.

One of the applications of these superconvergence results is the design of asymptotically exact *a posteriori* error indicators, which are useful in adaptive computations.

2.3. Nonlinear limiters. For computing solutions with strong discontinuities, the cell entropy inequality (2.10) and the L^2 stability (2.11), although helpful, are often not enough to control spurious numerical oscillations. In practice, especially for nonlinear problems containing strong discontinuities, we often need to apply nonlinear limiters to control these oscillations. Most of the limiters studied in the literature come from the methodologies of finite volume and finite difference high resolution schemes.

A limiter can be considered as a post-processor of the computed DG solution. In any cell which is deemed to contain a possible discontinuity (the so-called *troubled cells*), the DG polynomial is replaced by a new polynomial of the same degree, while maintaining the original cell average for conservation. Different limiters compute this new polynomial in different fashions. The main idea is to require that the new polynomial is less oscillatory than the old one, and, if the solution in this cell happens to be smooth, then the new polynomial should have the same high order accuracy as the old one. Some of the limiters are applied to all cells, while they

should take effect (change the polynomial in the cell) only in the cells near the discontinuities. The total variation diminishing (TVD) limiters [37] belong to this class. Unfortunately, such limiters tend to take effect also in some cells in which the solution is smooth, for example in cells near smooth extrema of the exact solution. Accuracy is therefore lost in such cells. The total variation bounded (TVB) limiters [79], applied to RKDG schemes in [27, 25, 23, 29], attempt to remove this difficulty and to ensure that the limiter takes effect only in cells near the discontinuities. The TVB limiters are widely used in applications, because of their simplicity in implementation. However, the TVB limiters involve a parameter M, related to the value of the second derivative of the exact solution near smooth extrema, which must be chosen by the user for different test cases. The moment-based limiter [9] and the improved moment limiter [13] also belong to this class, and they are specifically designed for DG methods and limit the moments of the polynomial sequentially, from the highest order moment downwards. Unfortunately, the moment-based limiters may also take effect in certain smooth cells, thereby destroying accuracy in these cells.

The limiters based on the weighted essentially non-oscillatory (WENO) methodology are designed with the objective of maintaining the high order accuracy even if they take effect in smooth cells. These limiters are based on the WENO methodology for finite volume and finite difference schemes [62, 52], and involve nonlinear reconstructions of the polynomials in troubled cells using the information of neighboring cells. The WENO reconstructed polynomials have the same high order of accuracy as the original polynomials when the solution is smooth, and they are (essentially) non-oscillatory near discontinuities. Qiu and Shu [70] and Zhu et al. [119] designed WENO limiters using the usual WENO reconstruction based on cell averages of neighboring cells as in [52, 40, 78], to reconstruct the values of the solutions at certain Gaussian quadrature points in the target cells, and then rebuild the solution polynomials from the original cell average and the reconstructed values at the Gaussian quadrature points through a numerical integration for the moments. This limiter needs to use the information from not only the immediate neighboring cells but also neighbors' neighbors, making it complicated to implement in multi-dimensions, especially for unstructured meshes [119, 40, 116]. The effort in [68, 71] attempts to construct Hermite type WENO approximations, which use the information of not only the cell averages but also the lower order moments such as slopes, to reduce the spread of reconstruction stencils. However for higher order methods the information of neighbors' neighbors is still needed. More recently, Zhong and Shu [118] developed a new WENO limiting procedure for RKDG methods on structured meshes. The idea is to reconstruct the entire polynomial, instead of reconstructing point values or moments in the classical WENO reconstructions. That is, the entire reconstruction polynomial on the target cell is a convex combination of polynomials on this cell and its immediate neighboring cells, with suitable

adjustments for conservation and with the nonlinear weights of the convex combination following the classical WENO procedure. The main advantage of this limiter is its simplicity in implementation, as it uses only the information from immediate neighbors and the linear weights are always positive. This simplicity is more prominent for multi-dimensional unstructured meshes, which is studied in [120] for two-dimensional unstructured triangular meshes.

The WENO limiters are typically applied only in designated "troubled cells", in order to save computational cost and to minimize the influence of accuracy in smooth regions. Therefore, a troubled cell indicator is needed, to correctly identify cells near discontinuities in which the limiters should be applied. Qiu and Shu in [69] have compared several troubled cell indicators. In practice, the TVB indicator [79] and the KXRCF indicator [55] are often the best choices.

Finally, let us mention the recently developed positivity-preserving limiters for DG schemes [114]. These limiters involve only simple scaling of the polynomials and is very inexpensive to implement. They can guarantee maximum principle in the scalar case [111, 115] and positivity-preserving for certain systems, for example positivity-preserving for density and pressure for Euler equations of compressible gas dynamics [112, 113, 115] and positivity-preserving for water height for shallow water equations [88]. They are also proved to maintain the original high order accuracy of the DG scheme. It is worth mentioning that, in [83], the RKDG method with only the positivity-preserving limiter is used to compute the very demanding gaseous detonations in two-dimensional structured and unstructured meshes, with stable and high resolution results.

2.4. Hyperbolic equations involving δ-functions. In a hyperbolic conservation law

$$\begin{aligned} &u_t + f(u)_x = g(x,t), && (x,t) \in R \times (0,T], \\ &u(x,0) = u_0(x), && x \in R, \end{aligned} \tag{2.17}$$

the initial condition u_0, or the source term $g(x,t)$, or the solution $u(x,t)$ may contain δ-singularities. Such problems appear often in applications and are difficult to approximate numerically. Many numerical techniques rely on modifications with smooth kernels and hence may severely smear such singularities, leading to large errors in the approximation. On the other hand, the DG methods are based on weak formulations and can be designed directly to solve such problems without modifications, leading to very accurate results.

In [106], DG methods to solve hyperbolic equations (2.17) involving δ-singularities are explored. Negative-order norm error estimates for the accuracy of DG approximations to δ-singularities are investigated. First, linear hyperbolic conservation laws in one space dimension with singular initial data are investigated. It is proved that, by using piecewise k-th degree polynomials, at time t, the error in the $H^{-(k+2)}$ norm over the whole

domain is $(k+1/2)$-th order, and the error in the $H^{-(k+1)}(\mathbb{R}\backslash\mathcal{R}_t)$ norm is $(2k+1)$-th order, where $\mathcal{R}_t$ is the pollution region due to the initial singularity with the width of order $\mathcal{O}(h^{1/2}\log(1/h))$ and h is the maximum cell length. As an application of the negative-order norm error estimates, the numerical solution can be convolved with a suitable kernel which is a linear combination of B-splines, to obtain L^2 error estimate of $(2k+1)$-th order for the post-processed solution. Second, high order superconvergence error estimates for linear hyperbolic conservation laws with singular source terms are obtained in [106] by applying Duhamel's principle. Numerical examples including an acoustic equation and the nonlinear rendez-vous algorithms are given to demonstrate the good performance of DG methods for solving hyperbolic equations involving δ-singularities. The results in [106] give us evidence that the DG method is a good algorithm for problems involving δ-singularities in their solutions. In future work we will apply the DG method to more nonlinear hyperbolic equations involving δ-singularities.

2.5. Generalization to Hamilton-Jacobi equations. Time dependent Hamilton-Jacobi equations take the form

$$\varphi_t + H(\varphi_{x_1}, ..., \varphi_{x_d}) = 0, \qquad \varphi(x,0) = \varphi^0(x), \tag{2.18}$$

where H is a Lipschitz continuous function. H could also depend on φ, x and t in some applications. Hamilton-Jacobi equations appear often in many applications. Examples include front propagation, level set methods, image processing and computer vision, control and differential games.

At least in the one dimensional case, there is a strong relationship between the Hamilton-Jacobi equation

$$\varphi_t + H(\varphi_x) = 0, \qquad \varphi(x,0) = \varphi^0(x) \tag{2.19}$$

and the hyperbolic conservation law

$$u_t + H(u)_x = 0, \qquad u(x,0) = u^0(x). \tag{2.20}$$

In fact, if we identify $u = \varphi_x$, the two equations (2.19) and (2.20) are equivalent. This equivalency provides motivations for designing algorithms for one equation based on the success for another equation. Therefore, it is very natural to attempt an adaptation of the DG methods designed for the conservation laws (2.20) to solve the Hamilton-Jacobi equation (2.19).

The first attempt to design a DG method was based exactly on this observation: at least in one dimension, the viscosity solution of the Hamilton-Jacobi equation (2.19) is equivalent to the entropy solution of the conservation law (2.20), when we identify $\varphi_x = u$. Therefore, a DG scheme for solving the conservation law (2.20), as given by (2.8) (with f there replaced by H), can be directly used to approximate the derivative of the viscosity solution of the Hamilton-Jacobi equation (2.19). This leads to the following DG algorithm of Hu and Shu [41]: Find $\varphi_h \in V_h^{k+1}$, such

that $u_h = (\varphi_h)_x \in V_h^k$ is determined by the DG scheme (2.8) (with f there replaced by H), and the missing degree of freedom is determined by

$$\int_{I_j} ((\varphi_h)_t + H(u_h))\, dx = 0.$$

This algorithm is well defined for one dimension. Additional complications exist for multi-dimensional cases. We take two space dimensions as an example. The Hamilton-Jacobi equation

$$\varphi_t + H(\varphi_x, \varphi_y) = 0 \tag{2.21}$$

is (in some sense) equivalent to the following system of conservation laws

$$u_t + H(u, v)_x = 0, \qquad v_t + H(u, v)_y = 0 \tag{2.22}$$

when we identify $u = \varphi_x$ and $v = \varphi_y$. We would like to still use a piecewise polynomial φ_h as our solution variable and take its derivatives to approximate u and v. The DG algorithm of Hu and Shu [41], as re-interpreted by Li and Shu [43], can be formulated as follows: Find $\varphi_h \in V_h^{k+1}$, such that $(u_h, v_h) = ((\varphi_h)_x, (\varphi_h)_y) \in W_h^k$ is determined by the standard DG scheme solving the conservation laws (2.22), and the missing degree of freedom is determined by

$$\int_{I_j} ((\varphi_h)_t + H(u_h, v_h))\, dxdy = 0.$$

Here, I_j denotes two-dimensional elements (triangles or rectangles), and W_h^k is the locally curl-free subspace of $V_h^k \times V_h^k$:

$$W_h^k = \{(u, v) \in V_h^k \times V_h^k : u_y - v_x = 0 \quad \forall (x, y) \in I_j\}.$$

Some analysis for this DG method (including L^2 stability for a specific class of the Hamiltonian H) is given in [42]. A priori L^2 error estimates for smooth solutions are given in [89].

Even though the DG schemes in [41, 43] are successful in approximating the Hamilton-Jacobi equation (2.18), it involves rewriting it as a conservation law satisfied by the derivatives of the solution φ. It is desirable to design a DG method which solves directly the solution φ to the Hamilton-Jacobi equation (2.18). The scheme of Cheng and Shu [16] serves this purpose. The scheme is defined as: find $\varphi_h \in V_h^k$, such that

$$\begin{aligned} &\int_{I_j} ((\varphi_h)_t + H((\varphi_h)_x)\, v_h(x) dx \\ &+ \left(\min_{x \in I_{j+1/2}} H'((\tilde{\varphi}_h)_x) \right)_- [\varphi_h]_{j+\frac{1}{2}} (v_h)^-_{j+\frac{1}{2}} \\ &+ \left(\max_{x \in I_{j-1/2}} H'((\tilde{\varphi}_h)_x) \right)_+ [\varphi_h]_{j-\frac{1}{2}} (v_h)^+_{j-\frac{1}{2}} = 0 \end{aligned} \tag{2.23}$$

holds for any $v_h \in V_h^k$ and all $1 \le j \le N$. Here $a_- = \min(a, 0)$, $a_+ = \max(a, 0)$, $[w] = w^+ - w^-$ denotes the jump of w, and $H'(u)$ denotes the derivative of $H(u)$ with respect to u. The interval $I_{j+1/2} = [x_j, x_{j+1}]$, and the function $\tilde{\varphi}_h$ is the L^2 projection of φ_h (which is discontinuous at the interface point $x_{j+\frac{1}{2}}$) into $\mathcal{P}^{2k+1}(I_j \cup I_{j+1})$. It can be easily verified that, when the Hamiltonian $H(u) = au$ is linear and the Hamilton-Jacobi equation is also a conservation law with a possible source term (when a depends on x), the scheme (2.23) becomes the standard DG scheme for this conservation law. Extension of this method to multi-dimensions is straightforward. Further development and application of this method to problems in optimal control are given in [10, 11]. A priori L^2 error estimates for smooth solutions are given in [89].

Another DG method which solves directly the Hamilton-Jacobi equations (2.18) is that of Yan and Osher [102]. This method is motivated by the local discontinuous Galerkin (LDG) method for solving second order partial differential equations [30], to be described in next section. We refer the readers to [102] for more details of this method. A priori L^2 error estimates for smooth solutions are given in [89].

3. DG methods for convection-diffusion equations. DG methods are most suitable for convection equations, however they are also good methods for solving convection dominated convection diffusion equations, such as Navier-Stokes equations with high Reynolds numbers. In this section we discuss the DG methods for time dependent convection-diffusion equations

$$u_t + \sum_{i=1}^{d} f_i(u)_{x_i} - \sum_{i=1}^{d}\sum_{j=1}^{d} (a_{ij}(u)u_{x_j})_{x_i} = 0, \tag{3.1}$$

where $(a_{ij}(u))$ is a symmetric, semi-positive definite matrix.

For equations containing higher order spatial derivatives, such as the convection-diffusion equation (3.1), discontinuous Galerkin methods designed for hyperbolic conservation laws cannot be directly applied. Let us look at the heat equation as an example

$$u_t = u_{xx}. \tag{3.2}$$

Comparing with the hyperbolic conservation law (2.7), we can treat the heat equation (3.2) also as a "conservation law" by identifying $f(u)$ in (2.7) with $-u_x$ in (3.2). Therefore, it would appear that we could change the DG scheme (2.8), designed for solving the conservation law (2.7), to the following scheme for solving the heat equation (3.2). Find $u_h \in V_h^k$ such that, for all test functions $v_h \in V_h^k$ and all $1 \le j \le N$, we have

$$\begin{aligned} &\int_{I_j} (u_h)_t \, v_h dx + \int_{I_j} (u_h)_x (v_h)_x dx \\ &\qquad -\widehat{u_x}_{j+\frac{1}{2}} (v_h)^-_{j+\frac{1}{2}} + \widehat{u_x}_{j-\frac{1}{2}} (v_h)^+_{j-\frac{1}{2}} = 0, \end{aligned} \tag{3.3}$$

where $f(u)$ in (2.8) is replaced by $-u_x$ in (3.3). Of course, we still need to define the numerical fluxes $\widehat{u_x}_{j+\frac{1}{2}}$. Upwinding and monotone fluxes are no longer relevant, as we are dealing with the heat equation (3.2) for which there is no preferred wind direction. It would appear that the average flux

$$\widehat{u_x}_{j+\frac{1}{2}} = \frac{1}{2}\left(((u_h)_x)^-_{j+\frac{1}{2}} + ((u_h)_x)^+_{j+\frac{1}{2}}\right) \tag{3.4}$$

is a reasonable choice. If we take this flux in the scheme (3.3), we would observe numerically a very strange phenomenon. The numerical solution appears to be convergent when the mesh is refined, however it does not seem to converge to the correct solution of the PDE. One would then suspect that the scheme is stable but inconsistent. However, the scheme can be written as a standard finite difference scheme and standard linear analysis for finite difference schemes can be performed. It turns out that the scheme (3.3) with the flux (3.4) is consistent but (very weakly) unstable [31, 107]. Therefore, we should be very careful when generalizing DG schemes from first order hyperbolic equations to PDEs with higher order spatial derivatives.

3.1. Local discontinuous Galerkin methods. One possible way to design a stable and convergent DG method for solving convection-diffusion equations is to rewrite the equation into a first order system, then apply the discontinuous Galerkin method on the system. A key ingredient for the success of such methods is the correct design of interface numerical fluxes. These fluxes must be designed to guarantee stability and local solvability of all the auxiliary variables introduced to approximate the derivatives of the solution. The local solvability of all the auxiliary variables is why the method is called a "local" discontinuous Galerkin (LDG) method in [30].

The first local discontinuous Galerkin method was developed by Cockburn and Shu [30], for the convection-diffusion equation (3.1) containing second derivatives. Their work was motivated by the successful numerical experiments of Bassi and Rebay [7] for the compressible Navier-Stokes equations.

We will use the heat equation (3.2) to demonstrate the idea of LDG schemes. We rewrite equation (3.2) as the following system

$$u_t - q_x = 0, \qquad q - u_x = 0, \tag{3.5}$$

which "looks like" a system of conservation laws, except that the second equation does not have a time derivative in q. We can then formally write down the DG scheme (2.8) for each equation in (3.5), resulting in the following scheme. Find $u_h, q_h \in V_h^k$ such that, for all test functions $v_h, p_h \in V_h^k$ and all $1 \le j \le N$, we have

$$\int_{I_j} (u_h)_t \, v_h dx + \int_{I_j} q_h \, (v_h)_x dx - \hat{q}_{j+\frac{1}{2}} (v_h)^-_{j+\frac{1}{2}} + \hat{q}_{j-\frac{1}{2}} (v_h)^+_{j-\frac{1}{2}} = 0;$$

$$\int_{I_j} q_h p_h dx + \int_{I_j} u_h (p_h)_x dx - \hat{u}_{j+\frac{1}{2}} (p_h)^-_{j+\frac{1}{2}} + \hat{u}_{j-\frac{1}{2}} (p_h)^+_{j-\frac{1}{2}} = 0. \tag{3.6}$$

Of course, we would still need to define the numerical fluxes $\hat{u}_{j+\frac{1}{2}}$ and $\hat{q}_{j+\frac{1}{2}}$. Again, based on the fact that upwinding and monotone fluxes are no longer relevant, as we are dealing with the heat equation (3.2) for which there is no preferred wind direction, we could still try the average flux

$$\hat{u}_{j+\frac{1}{2}} = \frac{1}{2}\left((u_h)^-_{j+\frac{1}{2}} + (u_h)^+_{j+\frac{1}{2}}\right), \ \hat{q}_{j+\frac{1}{2}} = \frac{1}{2}\left((q_h)^-_{j+\frac{1}{2}} + (q_h)^+_{j+\frac{1}{2}}\right). \tag{3.7}$$

Notice that, from the second equation in the scheme (3.6), we can solve q_h explicitly and locally (in the cell I_j) in terms of u_h, by inverting the small mass matrix inside the cell I_j. This is why the method is referred to as the "local" discontinuous Galerkin method. It turns out that the LDG scheme (3.6) with the central fluxes (3.7) is stable and convergent, but it loses one order of accuracy, to $O(h^k)$ only in the L^2 norm, for odd k. A better choice of the numerical fluxes is the so-called "alternating fluxes", defined as

$$\hat{u}_{j+\frac{1}{2}} = (u_h)^-_{j+\frac{1}{2}}, \qquad \hat{q}_{j+\frac{1}{2}} = (q_h)^+_{j+\frac{1}{2}}. \tag{3.8}$$

The important point is that $\hat{q}$ and $\hat{u}$ should be chosen from different directions. Thus, the choice

$$\hat{u}_{j+\frac{1}{2}} = (u_h)^+_{j+\frac{1}{2}}, \qquad \hat{q}_{j+\frac{1}{2}} = (q_h)^-_{j+\frac{1}{2}}$$

is also fine. It can be proved that the LDG scheme (3.6) with the alternating fluxes (3.8) is stable and convergent, with optimal $O(h^{k+1})$ error order in the L^2 norm, see, e.g. [80].

The beauty of the DG method is that, once it is designed for the linear equation (3.2) and proved to be stable and accurate, it can be easily generalized to fully nonlinear convection-diffusion equations

$$u_t + f(u)_x = (a(u)u_x)_x \tag{3.9}$$

with $a(u) \geq 0$. We again rewrite this equation as the following system

$$u_t + f(u)_x - (b(u)q)_x = 0, \qquad q - B(u)_x = 0, \tag{3.10}$$

where

$$b(u) = \sqrt{a(u)}, \qquad B(u) = \int^u b(u)du. \tag{3.11}$$

The semi-discrete LDG scheme is defined as follows. Find $u_h, q_h \in V_h^k$ such that, for all test functions $v_h, p_h \in V_h^k$ and all $1 \leq i \leq N$, we have

$$\begin{aligned} &\int_{I_j} (u_h)_t \, v_h dx - \int_{I_j} (f(u_h) - b(u_h)q_h)(v_h)_x dx \\ &+ (\hat{f} - \hat{b}\hat{q})_{j+\frac{1}{2}} (v_h)^-_{j+\frac{1}{2}} - (\hat{f} - \hat{b}\hat{q})_{j-\frac{1}{2}} (v_h)^+_{j-\frac{1}{2}} = 0, \\ &\int_{I_j} q_h p_h dx + \int_{I_j} B(u_h)(p_h)_x dx - \hat{B}_{j+\frac{1}{2}} (p_h)^-_{j+\frac{1}{2}} + \hat{B}_{j-\frac{1}{2}} (p_h)^+_{j-\frac{1}{2}} = 0. \end{aligned} \tag{3.12}$$

In [30], sufficient conditions for the choices of the numerical fluxes to guarantee the stability of the scheme (3.12) are given. Here, we only discuss a particularly attractive choice, i.e. the so-called "alternating fluxes" discussed before for the linear heat equation, now defined as

$$\hat{b} = \frac{B(u_h^+) - B(u_h^-)}{u_h^+ - u_h^-}, \qquad \hat{q} = q_h^+, \qquad \hat{B} = B(u_h^-). \tag{3.13}$$

The important point is that $\hat{q}$ and $\hat{B}$ should be chosen from different directions. Thus, the choice

$$\hat{b} = \frac{B(u_h^+) - B(u_h^-)}{u_h^+ - u_h^-}, \qquad \hat{q} = q_h^-, \qquad \hat{B} = B(u_h^+)$$

is also fine.

Notice that, even for this fully nonlinear case, from the second equation in the scheme (3.12), we can still solve q_h explicitly and locally (in cell I_j) in terms of u_h, by inverting the small mass matrix inside the cell I_j, thus justifying the terminology "local" discontinuous Galerkin methods.

In [30], it is proved that, for the solution u_h, q_h to the semi-discrete LDG scheme (3.12), we still have the following "cell entropy inequality"

$$\frac{1}{2}\frac{d}{dt}\int_{I_j} (u_h)^2\, dx + \int_{I_j} (q_h)^2 dx + \hat{F}_{j+\frac{1}{2}} - \hat{F}_{j-\frac{1}{2}} \le 0 \tag{3.14}$$

for a consistent entropy flux

$$\hat{F}_{j+\frac{1}{2}} = \hat{F}(u_h(x^-_{j+\frac{1}{2}}, t), q_h(x^-_{j+\frac{1}{2}}, t); u_h(x^+_{j+\frac{1}{2}}, t), q_h(x^+_{j+\frac{1}{2}}))$$

satisfying $\hat{F}(u, u) = F(u) - ub(u)q$ where, as before, $F(u) = \int^u uf'(u)du$. This, together with periodic or compactly supported boundary conditions, implies the following L^2 stability

$$\frac{d}{dt}\int_a^b (u_h)^2 dx + 2\int_a^b (q_h)^2 dx \le 0, \tag{3.15}$$

or

$$\|u_h(\cdot, t)\| + 2\int_0^t \|q_h(\cdot, \tau)\| d\tau \le \|u_h(\cdot, 0)\|. \tag{3.16}$$

A priori L^2 error estimates for smooth solutions are provided in [94].

3.2. Internal penalty discontinuous Galerkin methods. Another important class of DG methods for solving diffusion equations is the class of internal penalty discontinuous Galerkin methods. We will use the simple heat equation (3.2) to demonstrate the idea. If we multiply

both sides of (3.2) by a test function v and integrate over the cell I_j, and integrate by parts for the right-hand-side, we obtain the equality

$$\int_{I_j} u_t v dx = -\int_{I_j} u_x v_x dx + (u_x)_{j+\frac{1}{2}} v^-_{j+\frac{1}{2}} - (u_x)_{j-\frac{1}{2}} v^+_{j-\frac{1}{2}} \tag{3.17}$$

where we have used superscripts $\pm$ on v at cell boundaries to prepare for numerical schemes involving functions which are discontinuous at those cell boundaries. Summing over j, we obtain, with periodic boundary conditions for simplicity, the following equality

$$\int_a^b u_t v dx = -\sum_{j=1}^N \int_{I_j} u_x v_x dx - \sum_{j=1}^N (u_x)_{j+\frac{1}{2}} [v]_{j+\frac{1}{2}} \tag{3.18}$$

where $[w] \equiv w^+ - w^-$ denotes the jump of w at the cell interface. If we attempt to convert the equality (3.18) into a numerical scheme, we could try the following. Find $u_h \in V_h^k$ such that, for all test functions $v_h \in V_h^k$, we have

$$\int_a^b (u_h)_t (v_h) dx = -\sum_{j=1}^N \int_{I_j} (u_h)_x (v_h)_x dx - \sum_{j=1}^N \{(u_h)_x\}_{j+\frac{1}{2}} [v_h]_{j+\frac{1}{2}} \tag{3.19}$$

where $\{w\} \equiv \frac{1}{2}(w^+ + w^-)$ denotes the average of w at the cell interface. This scheme is actually exactly the same as the scheme (3.3) with the numerical flux (3.4), which is known to be unstable as mentioned above [31, 107]. Notice that the right-hand-side of (3.19) is not symmetric with respect to u_h and v_h. We can therefore add another term to symmetrize it, obtaining the following scheme. Find $u_h \in V_h^k$ such that, for all test functions $v_h \in V_h^k$, we have

$$\begin{aligned} \int_a^b (u_h)_t (v_h) dx = &-\sum_{j=1}^N \int_{I_j} (u_h)_x (v_h)_x dx \\ &-\sum_{j=1}^N \{(u_h)_x\}_{j+\frac{1}{2}} [v_h]_{j+\frac{1}{2}} - \sum_{j=1}^N \{(v_h)_x\}_{j+\frac{1}{2}} [u_h]_{j+\frac{1}{2}}. \end{aligned} \tag{3.20}$$

Notice that, since the exact solution is continuous, the additional term $-\sum_{j=1}^N \{(v_h)_x\}_{j+\frac{1}{2}} [u_h]_{j+\frac{1}{2}}$ is zero if the numerical solution u_h is replaced by the exact solution u, hence the scheme is consistent. Scheme (3.20) is symmetric, unfortunately it is still unconditionally unstable. In order to stabilize the scheme, a further penalty term must be added, resulting in the following symmetric internal penalty discontinuous Galerkin (SIPG) method [84, 6]

$$\int_a^b (u_h)_t (v_h) dx = -\sum_{j=1}^N \int_{I_j} (u_h)_x (v_h)_x dx - \sum_{j=1}^N \{(u_h)_x\}_{j+\frac{1}{2}} [v_h]_{j+\frac{1}{2}}$$

$$-\sum_{j=1}^{N}\{(v_h)_x\}_{j+\frac{1}{2}}[u_h]_{j+\frac{1}{2}}-\sum_{j=1}^{N}\frac{\alpha}{h}[u_h]_{j+\frac{1}{2}}[v_h]_{j+\frac{1}{2}}. \tag{3.21}$$

Clearly, the scheme (3.21) is still symmetric, and it can be proved [84, 6] that, for sufficiently large α, it is stable and has optimal $O(h^{k+1})$ order convergence in L^2. The disadvantage of this scheme is that it involves a parameter α which has to be chosen adequately to ensure stability. Another possible way to obtain a stable scheme is to change the sign of the last term in the unstable scheme (3.20), resulting in the following non-symmetric internal penalty discontinuous Galerkin (NIPG) method [8, 66] of Baumann and Oden

$$\begin{aligned}\int_a^b (u_h)_t(v_h)dx = &-\sum_{j=1}^{N}\int_{I_j}(u_h)_x(v_h)_x dx\\ &-\sum_{j=1}^{N}\{(u_h)_x\}_{j+\frac{1}{2}}[v_h]_{j+\frac{1}{2}}+\sum_{j=1}^{N}\{(v_h)_x\}_{j+\frac{1}{2}}[u_h]_{j+\frac{1}{2}}.\end{aligned} \tag{3.22}$$

This scheme is not symmetric, however it is L^2 stable and convergent, although it has a sub-optimal $O(h^k)$ order of L^2 errors for even k [8, 66, 107].

There are other types of DG methods involving the internal penalty methodology, for example the direct discontinuous Galerkin (DDG) methods [60, 61].

3.3. Ultra weak discontinuous Galerkin methods. Ultra weak discontinuous Galerkin methods are designed in [17]. Let us again use the simple heat equation (3.2) to demonstrate the idea. If we multiply both sides of (3.2) by a test function v and integrate over the cell I_j, and integrate by parts twice for the right-hand-side, we obtain the equality

$$\begin{aligned}\int_{I_j} u_t v dx = &\int_{I_j} u v_{xx} dx + (u_x)_{j+\frac{1}{2}}v_{j+\frac{1}{2}} - (u_x)_{j-\frac{1}{2}}v_{j-\frac{1}{2}}\\ &-u_{j+\frac{1}{2}}(v_x)_{j+\frac{1}{2}}+u_{j-\frac{1}{2}}(v_x)_{j-\frac{1}{2}}.\end{aligned} \tag{3.23}$$

We can then follow the general principle of designing DG schemes, namely converting the solution u and its derivatives at the cell boundary into numerical fluxes, and taking values of the test function v and its derivatives at the cell boundary by values inside the cell I_j, to obtain the following scheme. Find $u_h \in V_h^k$ such that, for all test functions $v_h \in V_h^k$ and all $1 \le j \le N$, we have

$$\begin{aligned}\int_{I_j}(u_h)_t\, v_h dx = &\int_{I_j} u_h\,(v_h)_{xx}dx + \widehat{u_x}_{j+\frac{1}{2}}(v_h)^-_{j+\frac{1}{2}} - \widehat{u_x}_{j-\frac{1}{2}}(v_h)^+_{j-\frac{1}{2}}\\ &-\hat{u}_{j+\frac{1}{2}}((v_h)_x)^-_{j+\frac{1}{2}}+\hat{u}_{j-\frac{1}{2}}((v_h)_x)^+_{j-\frac{1}{2}}.\end{aligned} \tag{3.24}$$

The crucial ingredient for the stability of the scheme (3.24) is still the choice of numerical fluxes. It is proved in [17] that the following choice of numerical fluxes

$$\hat{u}_{j+\frac{1}{2}} = (u_h)^-_{j+\frac{1}{2}}, \qquad \widehat{u_x}_{j+\frac{1}{2}} = ((u_h)_x)^+_{j+\frac{1}{2}} + \frac{\alpha}{h}[u_h]_{j+\frac{1}{2}} \tag{3.25}$$

would yield a stable DG scheme if the constant $\alpha > 0$ is sufficiently large. Notice that the choice in (3.25) is a combination of alternating fluxes and internal penalty. The following choice of alternating fluxes would also work

$$\hat{u}_{j+\frac{1}{2}} = (u_h)^+_{j+\frac{1}{2}}, \qquad \widehat{u_x}_{j+\frac{1}{2}} = ((u_h)_x)^-_{j+\frac{1}{2}} + \frac{\alpha}{h}[u_h]_{j+\frac{1}{2}}.$$

Suboptimal L^2 error estimates are given in [17] for the scheme (3.24) with the fluxes (3.25) for $k \geq 1$. In numerical experiments, optimal L^2 convergence rate of $O(h^{k+1})$ is observed for all $k \geq 1$. The scheme can be easily generalized to the general nonlinear convection-diffusion equation (3.9) with the same stability property [17].

3.4. Superconvergence. Results for superconvergence of DG methods, similar to those for hyperbolic equations discussed in Section 2.2, have been obtained for convection-diffusion equations in the literature. Superconvergence of the DG solution to the exact smooth solution in negative norms for convection-diffusion equations is studied in [49]. The superconvergence of the DG solution to a special projection of the exact smooth solution is addressed in [19, 20].

4. DG methods for third order convection-dispersion equations. In this section we study convection-dispersion equations which are wave equations involving third spatial derivatives. We study the following general KdV type equations

$$u_t + \sum_{i=1}^{d} f_i(u)_{x_i} + \sum_{i=1}^{d} \left(r_i'(u) \sum_{j=1}^{d} g_{ij}(r_i(u)_{x_i})_{x_j} \right)_{x_i} = 0, \tag{4.1}$$

where $f_i(u)$, $r_i(u)$ and $g_{ij}(q)$ are arbitrary (smooth) nonlinear functions. The one-dimensional KdV equation

$$u_t + (\alpha u + \beta u^2)_x + \sigma u_{xxx} = 0, \tag{4.2}$$

where α, β and σ are constants, is a special case of the general class (4.1).

It is important to realize that third order dispersive equations are wave equations, sharing similarity with first order hyperbolic conservation laws and being quite different from diffusion equations. For example, the third order linear equation

$$u_t + u_{xxx} = 0 \tag{4.3}$$

admits the following simple wave solution

$$u(x,t) = \sin(x+t),$$

that is, information propagates from right to left. Therefore, upwinding is a relevant concept for the design of DG schemes for convection-dispersion equations.

We will discuss two classes of DG schemes for third order dispersive PDEs.

4.1. Local discontinuous Galerkin methods. We can again design local discontinuous Galerkin (LDG) methods for third order dispersive PDEs, by rewriting such a PDE into a first order system and then applying the discontinuous Galerkin method on the system. Of course, a key ingredient for the success of such methods is still the correct design of interface numerical fluxes. These fluxes must be designed to guarantee stability and local solvability of all the auxiliary variables introduced to approximate the derivatives of the solution, thus justifying the terminology "local" DG. As mentioned above, upwinding should participate in the guiding principles for the design of numerical fluxes for dispersive PDEs.

We will use the simple linear equation (4.3) to demonstrate the idea of LDG schemes. We rewrite equation (4.3) as the following system

$$u_t + p_x = 0, \qquad p - q_x = 0, \qquad q - u_x = 0, \tag{4.4}$$

which "looks like" a system of conservation laws, except that the second and third equations do not have time derivatives. We can then formally write down the DG scheme (2.8) for each equation in (4.4), resulting in the following scheme. Find $u_h, p_h, q_h \in V_h^k$ such that, for all test functions $v_h, w_h, z_h \in V_h^k$ and all $1 \le j \le N$, we have

$$\begin{aligned}
&\int_{I_j} (u_h)_t \, v_h dx - \int_{I_j} p_h \, (v_h)_x dx + \hat{p}_{j+\frac{1}{2}} (v_h)^-_{j+\frac{1}{2}} - \hat{p}_{j-\frac{1}{2}} (v_h)^+_{j-\frac{1}{2}} = 0; \\
&\int_{I_j} p_h w_h dx + \int_{I_j} q_h \, (w_h)_x dx - \hat{q}_{j+\frac{1}{2}} (w_h)^-_{j+\frac{1}{2}} + \hat{q}_{j-\frac{1}{2}} (w_h)^+_{j-\frac{1}{2}} = 0; \\
&\int_{I_j} q_h z_h dx + \int_{I_j} u_h \, (z_h)_x dx - \hat{u}_{j+\frac{1}{2}} (z_h)^-_{j+\frac{1}{2}} + \hat{u}_{j-\frac{1}{2}} (z_h)^+_{j-\frac{1}{2}} = 0.
\end{aligned} \tag{4.5}$$

Of course, we would still need to define the numerical fluxes $\hat{u}_{j+\frac{1}{2}}$, $\hat{q}_{j+\frac{1}{2}}$ and $\hat{p}_{j+\frac{1}{2}}$. Since the wind blows from right to left, intuitively, we should choose more information from the right. If we take all three fluxes from the right

$$\hat{u} = (u_h)^+, \qquad \hat{q} = (q_h)^+, \qquad \hat{p} = (p_h)^+,$$

we obtain an unstable scheme. It is often the case that we should not attempt to be completely upwind, only upwind-biased. The next logical

thing to try is to take two numerical fluxes from the right and one from the left. Analysis helps us to pinpoint the following upwind-biased alternating fluxes

$$\hat{u} = (u_h)^-, \qquad \hat{q} = (q_h)^+, \qquad \hat{p} = (p_h)^+. \tag{4.6}$$

In fact, it turns out that the most important thing is to take $\hat{q}$, which approximates u_x, from the upwind side $(q_h)^+$. The fluxes $\hat{u}$ and $\hat{p}$, approximating u and u_{xx} respectively, can be taken in alternating sides. Therefore, the following choice of upwind-biased alternating fluxes

$$\hat{u} = (u_h)^+, \qquad \hat{q} = (q_h)^+, \qquad \hat{p} = (p_h)^-$$

is also fine.

It is proved in [103] that the LDG scheme (4.5) with the upwind-biased alternating fluxes (4.6) is L^2 stable

$$\frac{d}{dt}\int_a^b (u_h)^2 dx \le 0, \tag{4.7}$$

or

$$\|u_h(\cdot,t)\| \le \|u_h(\cdot,0)\|. \tag{4.8}$$

A sub-optimal L^2 error estimate of order $O(h^{k+1/2})$ is also proved in [103]. In a more recent work [101], Xu and Shu proved optimal L^2 error estimate of order $O(h^{k+1})$ for this scheme. This extra half order turns out to be difficult to obtain, mainly because of the wave nature of the equation (4.3) and hence a lack of control of the derivatives. The approach in [101] is to establish stability not only for u_h as in (4.7), but also for q_h and p_h approximating u_x and u_{xx}.

The LDG scheme can be designed for the general nonlinear convection-dispersion equation (4.1). Let us use the one-dimensional case to describe the scheme

$$u_t + f(u)_x + (r'(u)g(r(u)_x)_x)_x = 0, \tag{4.9}$$

where $f(u)$, $r(u)$ and $g(q)$ are arbitrary (smooth) nonlinear functions. The LDG method is based on rewriting it as the following system

$$u_t + (f(u) + r'(u)p)_x = 0, \quad p - g(q)_x = 0, \quad q - r(u)_x = 0. \tag{4.10}$$

The semi-discrete LDG scheme is defined as follows. Find $u_h, p_h, q_h \in V_h^k$ such that, for all test functions $v_h, w_h, z_h \in V_h^k$ and all $1 \le i \le N$, we have

$$\begin{aligned}&\int_{I_j} (u_h)_t\, v_h dx - \int_{I_j} (f(u_h) + r'(u_h)p_h)(v_h)_x dx \\ &+(\hat{f} + \widehat{r'}\hat{p})_{j+\frac{1}{2}}(v_h)^-_{j+\frac{1}{2}} - (\hat{f} + \widehat{r'}\hat{p})_{j-\frac{1}{2}}(v_h)^+_{j-\frac{1}{2}} = 0;\end{aligned} \tag{4.11}$$

$$\int_{I_j} p_h w_h dx + \int_{I_j} g(q_h)\,(w_h)_x dx - \hat{g}_{j+\frac{1}{2}}(w_h)^-_{j+\frac{1}{2}} + \hat{g}_{j-\frac{1}{2}}(w_h)^+_{j-\frac{1}{2}} = 0;$$
$$\int_{I_j} q_h z_h dx + \int_{I_j} r(u_h)\,(z_h)_x dx - \hat{r}_{j+\frac{1}{2}}(z_h)^-_{j+\frac{1}{2}} + \hat{r}_{j-\frac{1}{2}}(z_h)^+_{j-\frac{1}{2}} = 0.$$

By our experience with linear equations discussed above, we would like to use the following upwind-biased alternating fluxes

$$\widehat{r'} = \frac{r((u_h)^+) - r((u_h)^-)}{(u_h)^+ - (u_h)^-},\ \hat{r} = r((u_h)^-),\ \hat{g} = \hat{g}((q_h)^-, (q_h)^+),\ \hat{p} = (p_h)^+. \tag{4.12}$$

Here, $-\hat{g}((q_h)^-, (q_h)^+)$ is a monotone flux for $-g(q)$, namely $\hat{g}$ is a non-increasing function in the first argument and a non-decreasing function in the second argument. The important point is again the "alternating fluxes", namely $\hat{r}$ and $\hat{p}$ should come from opposite sides. Thus

$$\widehat{r'} = \frac{r((u_h)^+) - r((u_h)^-)}{(u_h)^+ - (u_h)^-},\ \hat{r} = r((u_h)^+),\ \hat{g} = \hat{g}((q_h)^-, (q_h)^+),\ \hat{p} = (p_h)^-$$

would also work.

It is quite interesting to observe that monotone fluxes, which are originally designed for hyperbolic conservation laws, can be used also for nonlinear dispersive equations to obtain stability. Also notice that, from the third equation in the scheme (4.11), we can solve q_h explicitly and locally (in cell I_j) in terms of u_h, by inverting the small mass matrix inside the cell I_j. Then, from the second equation in the scheme (4.11), we can solve p_h explicitly and locally (in cell I_j) in terms of q_h. Thus only u_h is the global unknown and the auxiliary variables q_h and p_h can be solved in terms of u_h locally. This justifies again the terminology of "local" discontinuous Galerkin method.

It is proved in [103] that the LDG scheme (4.11) with the upwind-biased alternating fluxes (4.12) is L^2 stable, i.e. (4.7) or (4.8) holds. This is also true for the multi-dimensional case (4.1). A sub-optimal L^2 error estimate of order $O(h^{k+1/2})$ is also proved in [94].

4.2. Ultra weak discontinuous Galerkin methods. Ultra weak discontinuous Galerkin methods are designed in [17]. Let us again use the simple linear equation (4.3) to demonstrate the idea. If we multiply both sides of (4.3) by a test function v and integrate over the cell I_j, and integrate by parts three times for the right-hand-side, we obtain the equality

$$\begin{aligned} &\int_{I_j} u_t v dx - \int_{I_j} u v_{xxx} dx + (u_{xx})_{j+\frac{1}{2}} v_{j+\frac{1}{2}} - (u_{xx})_{j-\frac{1}{2}} v_{j-\frac{1}{2}} \\ &-(u_x)_{j+\frac{1}{2}} (v_x)_{j+\frac{1}{2}} + (u_x)_{j-\frac{1}{2}} (v_x)_{j-\frac{1}{2}} \\ &+u_{j+\frac{1}{2}} (v_{xx})_{j+\frac{1}{2}} - u_{j-\frac{1}{2}} (v_{xx})_{j-\frac{1}{2}} = 0. \end{aligned} \tag{4.13}$$

We can then follow the general principle of designing DG schemes, namely converting the solution u and its derivatives at the cell boundary into numerical fluxes, and taking values of the test function v and its derivatives at the cell boundary by values inside the cell I_j, to obtain the following scheme. Find $u_h \in V_h^k$ such that, for all test functions $v_h \in V_h^k$ and all $1 \le j \le N$, we have

$$\begin{aligned} &\int_{I_j} (u_h)_t \, v_h dx - \int_{I_j} u_h \, (v_h)_{xxx} dx + \widehat{u_{xx}}_{j+\frac{1}{2}} (v_h)^-_{j+\frac{1}{2}} \\ &-\widehat{u_{xx}}_{j-\frac{1}{2}} (v_h)^+_{j-\frac{1}{2}} - \widehat{u_x}_{j+\frac{1}{2}} ((v_h)_x)^-_{j+\frac{1}{2}} + \widehat{u_x}_{j-\frac{1}{2}} ((v_h)_x)^+_{j-\frac{1}{2}} \\ &+\hat{u}_{j+\frac{1}{2}} ((v_h)_{xx})^-_{j+\frac{1}{2}} - \hat{u}_{j-\frac{1}{2}} ((v_h)_{xx})^+_{j-\frac{1}{2}} = 0. \end{aligned} \tag{4.14}$$

The crucial ingredient for the stability of the scheme (4.14) is still the choice of numerical fluxes. It is proved in [17] that the following choice of upwind-biased alternating fluxes

$$\hat{u} = (u_h)^-, \qquad \widehat{u_x} = ((u_h)_x)^+, \qquad \widehat{u_{xx}} = ((u_h)_{xx})^+, \tag{4.15}$$

would yield a stable DG scheme. Notice that the choice of numerical fluxes (4.15) is exactly the same as that for the stable LDG scheme (4.6). The most important thing is to take $\widehat{u_x}$ from the upwind side $((u_h)_x)^+$. The fluxes $\hat{u}$ and $\widehat{u_{xx}}$ can be taken in alternating sides. Therefore, the following choice of upwind-biased alternating fluxes

$$\hat{u} = (u_h)^+, \qquad \widehat{u_x} = ((u_h)_x)^+, \qquad \widehat{u_{xx}} = ((u_h)_{xx})^-$$

is also fine.

It is proved in [17] that the ultra weak DG scheme (4.14) with the upwind-biased alternating fluxes (4.15) is L^2 stable, namely (4.7) or (4.8) holds. Suboptimal L^2 error estimates are also given in [17] for this scheme with $k \ge 2$. In numerical experiments, optimal L^2 convergence rate of $O(h^{k+1})$ is observed for all $k \ge 2$. The scheme can be easily generalized to general nonlinear convection-dispersion equations with the same stability property [17].

5. DG methods for other dispersive wave equations. DG methods have also been designed for other dispersive wave equations containing higher order (usually odd order) derivatives. We will describe briefly some of these schemes in this section.

5.1. Equations with fifth order spatial derivatives. An LDG scheme for solving the following fifth order convection-dispersion equation

$$u_t + \sum_{i=1}^{d} f_i(u)_{x_i} + \sum_{i=1}^{d} g_i(u_{x_i x_i})_{x_i x_i x_i} = 0, \tag{5.1}$$

where $f_i(u)$ and $g_i(q)$ are arbitrary functions, was designed in [104]. The numerical fluxes are chosen following the same upwind-biased alternating fluxes principle similar to the third order KdV type equations (4.1), namely the flux corresponding to u_{xx} should be chosen according to upwinding, and the flux pairs corresponding to u and u_{xxxx}, and the flux pairs corresponding to u_x and u_{xxx}, should be chosen in an alternating fashion within each pair. A cell entropy inequality and the L^2 stability of the LDG scheme for the nonlinear equation (5.1) can be proved [104], which again do not depend on the smoothness of the solution of (5.1), the order of accuracy of the scheme, or the triangulation. For the linear fifth order equation

$$u_t + u_{xxxxx} = 0, \tag{5.2}$$

optimal $O(h^{k+1})$ order L^2 error estimate is obtained in [101]. Similar results can be obtained for PDEs of higher odd order spatial derivatives.

Ultra weak DG methods for solving (5.2) are designed in [17]. The choice of numerical fluxes are identical to that for the LDG schemes described above. The resulting scheme can be proved to be L^2 stable, and a sub-optimal L^2 error estimate for $k \geq 4$ is proved in [17]. Numerical experiments indicate optimal convergence in L^2 for all $k \geq 4$. The scheme as well as the stability analysis can be generalized to certain nonlinear fifth order PDEs [17]. Similar results can also be obtained for PDEs of higher odd order spatial derivatives.

5.2. The $K(m,n)$ equation. The so-called $K(m,n)$ equation

$$u_t + (u^m)_x + (u^n)_{xxx} = 0 \tag{5.3}$$

arises from mathematical physics and has the *compactons* solutions. In [58], an LDG scheme is designed for (5.3), which is proved to be L^{n+1} stable for the $K(n,n)$ equation with odd n. For all other cases, the LDG scheme is proved to be linearly stable. Computational results including those for compactons indicate excellent performance of these schemes.

This example indicates that we do not always seek to prove L^2 stability for DG schemes. In fact, most time-dependent PDEs arising from physics and applications have certain "energy", which is a positive functional of the solution, and the energy usually does not increase with time. An ideal DG scheme would produce numerical solutions for which the same energy also does not increase with time. In this particular example ($K(n,n)$ equation with odd n), this "energy" is the square of the L^{n+1} norm.

5.3. The KdV-Burgers type (KdVB) equations. LDG methods for solving the KdV-Burgers type (KdVB) equations

$$u_t + f(u)_x - (a(u)u_x)_x + (r'(u)g(r(u)_x)_x)_x = 0, \tag{5.4}$$

where $f(u)$, $a(u) \geq 0$, $r(u)$ and $g(q)$ are arbitrary functions, are designed in [90]. The design of numerical fluxes follows the same lines as that

for the convection-diffusion equation (3.9) for the second derivative term $(a(u)u_x)_x$ and for the KdV type equation (4.9) for the third derivative term $(r'(u)g(r(u)_x)_x)_x$. A cell entropy inequality and the L^2 stability of the LDG scheme for the nonlinear equation (5.4) are proved [90], which again do not depend on the smoothness of the solution of (5.4) and the order of accuracy of the scheme. For smooth solutions, a sub-optimal $O(h^{k+1/2})$ order L^2 error estimate for the linearized version is proved [90]. The LDG scheme is used in [90] to study different regimes when one of the dissipation and the dispersion mechanisms dominates, and when they have comparable influence on the solution. An advantage of the LDG scheme designed in [90] is that it is stable regardless of which mechanism (convection, diffusion, dispersion) actually dominates.

5.4. The fifth-order KdV type equations. An LDG scheme is designed in [90] for the fifth-order KdV type equations

$$u_t + f(u)_x + (r'(u)g(r(u)_x)_x)_x + (s'(u)h(s(u)_{xx})_{xx})_x = 0, \tag{5.5}$$

where $f(u)$, $r(u)$, $g(q)$, $s(u)$ and $h(p)$ are arbitrary functions, and a cell entropy inequality and L^2 stability are proved. A special case is the Kawahara equation

$$u_t + uu_x + u_{xxx} - \delta u_{xxxxx} = 0$$

which has very interesting close-form exact solutions that can be used to test the accuracy of the scheme [90]. Other special cases of (5.5) include the generalized Kawahara equation, Ito's fifth-order KdV equation, and a fifth-order KdV type equations with high nonlinearity, which are also explored in [90].

5.5. The fully nonlinear $K(n, n, n)$ equations. LDG methods for solving the fifth-order fully nonlinear $K(n, n, n)$ equations

$$u_t + (u^n)_x + (u^n)_{xxx} + (u^n)_{xxxxx} = 0, \tag{5.6}$$

where n is a positive integer, have been designed in [90]. The design of numerical fluxes follows the same lines as that for the $K(m, n)$ equations (5.3). For odd n, stability in the L^{n+1} norm of the resulting LDG scheme can be proved for the nonlinear equation (5.6) [90]. This scheme is used to simulate compacton propagation in [90].

5.6. The nonlinear Schrödinger (NLS) equations. The nonlinear Schrödinger (NLS) equation

$$i\,u_t + u_{xx} + i\,(g(|u|^2)u)_x + f(|u|^2)u = 0, \tag{5.7}$$

the two-dimensional version

$$i\,u_t + \Delta u + f(|u|^2)u = 0, \tag{5.8}$$

and the coupled nonlinear Schrödinger equation

$$\begin{cases} i\,u_t + i\,\alpha u_x + u_{xx} + \beta\,u + \kappa v + f(|u|^2,|v|^2)u = 0 \\ i\,v_t - i\,\alpha v_x + v_{xx} - \beta\,u + \kappa v + g(|u|^2,|v|^2)v = 0, \end{cases} \tag{5.9}$$

where $f(q)$ and $g(q)$ are arbitrary functions and α, β and κ are constants, are also dispersive wave equations, even though they involve second order spatial derivatives with $i = \sqrt{-1}$ as the coefficient. In [91], LDG methods are designed for these equations. The cell entropy inequality and L^2 stability are proved for these schemes in [91]. For smooth solutions, an L^2 error estimate of $O(h^{k+1/2})$ for the linearized version is also obtained in [91]. The LDG scheme is used in [91] to simulate the soliton propagation and interaction, and the appearance of singularities. The easiness of h-p adaptivity of the LDG scheme and rigorous stability for the fully nonlinear case make it an ideal choice for the simulation of Schrödinger equations, for which the solutions often have quite localized structures.

5.7. The Ito-type coupled KdV equations. An LDG method is developed in [93] to solve the Ito-type coupled KdV equations

$$u_t + \alpha u u_x + \beta v v_x + \gamma u_{xxx} = 0, v_t + \beta(uv)_x = 0,$$

where α, β and γ are constants. An L^2 stability is proved for the LDG method. For the Ito's equation

$$\begin{aligned} u_t - (3u^2 + v^2)_x - u_{xxx} &= 0, \\ v_t - 2(uv)_x &= 0, \end{aligned}$$

the result for u behaves like dispersive wave solutions and the result for v behaves like shock wave solutions. Simulation for such solutions is performed in [93] using the LDG scheme.

5.8. The Kadomtsev-Petviashvili (KP) equations. The two dimensional Kadomtsev-Petviashvili (KP) equations

$$(u_t + 6uu_x + u_{xxx})_x + 3\sigma^2 u_{yy} = 0, \tag{5.10}$$

where $\sigma^2 = -1$ (referred to as KP-I) or $\sigma^2 = 1$ (referred to as KP-II), are generalizations of the one-dimensional KdV equations and are important models for water waves.

This equation is equivalent to

$$u_t + 6(uu_x) + u_{xxx} + 3\sigma^2 \partial_x^{-1} u_{yy} = 0 \tag{5.11}$$

where the non-local operator ∂_x^{-1} makes the equation well-posed only in the restricted space

$$\mathcal{V}(R^2) = \left\{ f : \int_{R^2} (1 + \xi^2 + \frac{\eta^2}{\xi^2}) |\hat{f}(\xi,\eta)|^2 d\xi d\eta < \infty \right\}.$$

It is therefore complicated to design an efficient LDG scheme which relies on local operations. In [92], an LDG scheme for (5.10) is designed by carefully choosing locally supported bases which satisfy the global constraint needed by the solution of (5.10). The LDG scheme satisfies a cell entropy inequality and is L^2 stable for the fully nonlinear equation (5.10). Numerical simulations are performed in [92] for both the KP-I equations and the KP-II equations. Line solitons and lump-type pulse solutions have been simulated.

5.9. The Zakharov-Kuznetsov (ZK) equation. The two dimensional Zakharov-Kuznetsov (ZK) equation

$$u_t + (3u^2)_x + u_{xxx} + u_{xyy} = 0 \tag{5.12}$$

is another generalization of the one-dimensional KdV equations.

An LDG scheme is designed for (5.12) in [92]. A cell entropy inequality and the L^2 stability are proved. A sub-optimal L^2 error estimate is given in [94]. Various nonlinear waves have been simulated by this scheme in [92].

5.10. The Camassa-Holm (CH) equation. The Camassa-Holm (CH) equation is given as

$$u_t - u_{xxt} + 2\kappa u_x + 3uu_x = 2u_x u_{xx} + uu_{xxx},$$

where κ is a constant. An LDG scheme is designed in [95]. L^2 stability for general solutions and a sub-optimal L^2 error estimate for smooth solutions are provided in [95].

5.11. The Hunter-Saxton (HS) equation. The Hunter-Saxton (HS) equation is given as

$$u_{xxt} + 2u_x u_{xx} + uu_{xxx} = 0.$$

A regularization with viscosity is given as

$$u_{xxt} + 2u_x u_{xx} + uu_{xxx} - \varepsilon_1 u_{xxxx} = 0,$$

and a regularization with dispersion is given as

$$u_{xxt} + 2u_x u_{xx} + uu_{xxx} - \varepsilon_2 u_{xxxxx} = 0,$$

where $\varepsilon_1 \geq 0$ and ε_2 are small constants.

In [96, 98], we design LDG schemes for these equations and prove their energy stability.

5.12. The generalized Zakharov system. The following system

$$iE_t + \Delta E - Nf(|E|^2)E + g(|E|^2)E = 0,$$
$$\epsilon^2 N_{tt} - \Delta(N + F(|E|^2)) = 0$$

is referred to as the generalized Zakharov system and is originally introduced to describe the Langmuir turbulence in a plasma. In [87], we design an LDG scheme for this system and prove two energy conservations for this scheme. Numerical experiments for the Zakharov system are presented to illustrate the accuracy and capability of the methods, including accuracy tests, plane waves, soliton-soliton collisions of the standard and generalized Zakharov system and a two-dimensional problem.

5.13. The Degasperis-Procesi (DP) equation. The Degasperis-Procesi (DP) equation is given as

$$u_t - u_{txx} + 4f(u)_x = f(u)_{xxx},$$

where $f(u) = \frac{1}{2}u^2$. The solution may be discontinuous regardless of the smoothness of the initial conditions.

In [100], we develop LDG methods and prove L^2 stability for the general polynomial spaces and total variation stability for P^0 elements. The numerical simulation results for different types of solutions of the nonlinear Degasperis-Procesi equation are provided to illustrate the accuracy and capability of the LDG method in [100].

6. DG methods for other dissipative equations. DG methods have also been designed for other dissipative equations containing higher even order derivatives. We will describe briefly some of these schemes in this section.

6.1. The bi-harmonic type equations. An LDG scheme for solving the time dependent convection bi-harmonic equation

$$u_t + \sum_{i=1}^{d} f_i(u)_{x_i} + \sum_{i=1}^{d} (a_i(u_{x_i})u_{x_i x_i})_{x_i x_i} = 0, \tag{6.1}$$

where $f_i(u)$ and $a_i(q) \geq 0$ are arbitrary functions, was designed in [104]. The numerical fluxes are chosen following the same "alternating fluxes" principle similar to the second order convection-diffusion equation (3.1), namely the flux pairs corresponding to u and u_{xxx}, and the flux pairs corresponding to u_x and u_{xx}, should be chosen in an alternating fashion within each pair. A cell entropy inequality and the L^2 stability of the LDG scheme for the nonlinear equation (6.1) can be proved [104], which do not depend on the smoothness of the solution of (6.1), the order of accuracy of the scheme, or the triangulation. Optimal L^2 error estimates can be proved for the linear biharmonic equation

$$u_t + \triangle^2 u = 0, \tag{6.2}$$

for both structured and unstructured meshes, see [36]. In [63], superconvergence of the LDG method for linear fourth order equations is studied.

Ultra weak DG methods for solving (6.2) are designed in [17]. The choice of numerical fluxes are similar to that for the LDG schemes described above, with additional internal penalty term on the flux corresponding to u_{xxx}. The resulting scheme can be proved to be L^2 stable, and a sub-optimal L^2 error estimate for $k \geq 3$ is proved in [17]. Numerical experiments indicate optimal convergence in L^2 for all $k \geq 3$. The scheme as well as the stability analysis can be generalized to certain nonlinear fourth order PDEs [17].

Both the LDG schemes and the ultra weak DG methods can be generalized to PDEs of higher even order spatial derivatives. For example, [36] contains optimal L^2 error estimates for linear diffusion PDEs with higher even orders.

6.2. The Kuramoto-Sivashinsky type equations. LDG methods are developed in [93] to solve the Kuramoto-Sivashinsky type equations

$$u_t + f(u)_x - (a(u)u_x)_x + (r'(u)g(r(u)_x)_x)_x + (s(u_x)u_{xx})_{xx} = 0, \tag{6.3}$$

where $f(u)$, $a(u)$, $r(u)$, $g(q)$ and $s(p) \geq 0$ are arbitrary functions. The Kuramoto-Sivashinsky equation

$$u_t + uu_x + \alpha u_{xx} + \beta u_{xxxx} = 0, \tag{6.4}$$

where α and $\beta \geq 0$ are constants, which is a special case of (6.3), is a canonical evolution equation which has attracted considerable attention over the last decades. When the coefficients α and β are both positive, its linear terms describe a balance between long-wave instability and short-wave stability, with the nonlinear term providing a mechanism for energy transfer between wave modes. The LDG method developed in [93] can be proved to satisfy a cell entropy inequality and is therefore L^2 stable, for the general nonlinear equation (6.3). The LDG scheme is used in [93] to simulate chaotic solutions of (6.4).

6.3. Semi-conductor device simulations. Device simulation models in semi-conductor device simulations include drift-diffusion, hydrodynamic, energy transport, high field, kinetic and Boltzmann-Poisson models. DG or LDG methods have been designed for these models, many of them with stability analysis and error estimates.

In [44, 45], an LDG method is designed to solve time dependent and steady state moment models including the hydrodynamic (HD) models and the energy transport (ET) models, for semiconductor device simulations, in which both the first derivative convection terms and second derivative diffusion (heat conduction) terms exist and are discretized by the DG method and the LDG method respectively. The potential equation for the electric field is also discretized by the LDG method, thus the numerical tool is

based on a unified discontinuous Galerkin methodology for different components and is hence potentially viable for efficient h-p adaptivity and parallel implementation. One dimensional n^+-n-n^+ diode and two dimensional MESFET device are simulated by the DG methods using the HD and ET models and comparison is made with earlier finite difference essentially non-oscillatory (ENO) simulation results. In [46], we obtain L^2 error estimates for smooth solutions of the drift-diffusion (DD) and high-field (HF) models using the LDG method.

In [14, 15], a discontinuous Galerkin scheme applied to deterministic computations of the transients for the Boltzmann-Poisson system describing electron transport in semiconductor devices is developed and applied to simulate hot electron transport in bulk silicon, in a silicon n^+-n-n^+ diode and in a double gated 12nm MOSFET. Additionally, the obtained results are compared to those of a high order WENO scheme simulation.

6.4. Cahn-Hilliard equations. An important class of high order nonlinear diffusion equations is the class of the Cahn-Hilliard equation

$$u_t = \nabla \cdot \Big(b(u) \nabla \big(-\gamma \Delta u + \Psi'(u) \big) \Big), \tag{6.5}$$

and the Cahn-Hilliard system

$$\mathbf{u}_t = \nabla \cdot (\mathbf{B}(\mathbf{u}) \nabla \omega), \qquad \omega = -\gamma \Delta \mathbf{u} + D\Psi(\mathbf{u}), \tag{6.6}$$

where $\{D\Psi(\mathbf{u})\}_l = \frac{\partial \Psi(\mathbf{u})}{\partial u_l}$ and γ is a positive constant. Here $b(u)$ is the nonnegative diffusion mobility and $\Psi(u)$ is the homogeneous free energy density for the scalar case (6.5). For the system case (6.6), $\mathbf{B}(\mathbf{u})$ is the symmetric positive semi-definite mobility matrix and $\Psi(\mathbf{u})$ is the homogeneous free energy density.

In [85, 86], LDG methods are designed for the Cahn-Hilliard equation (6.5) and the Cahn-Hilliard system (6.6) respectively. The proof of the energy stability for the LDG schemes is given for the general nonlinear solutions. Many simulation results are given. In [36], optimal L^2 error estimate is obtained for the LDG method for solving the linearized Cahn-Hilliard equation.

6.5. The surface diffusion equations. The surface diffusion equation is given as

$$u_t + \nabla \cdot \left(Q \left(\mathbf{I} - \frac{\nabla u \otimes \nabla u}{Q^2} \right) \nabla H \right) = 0 \tag{6.7}$$

where Q is the area element

$$Q = \sqrt{1 + |\nabla u|^2}$$

and H is the mean curvature of the domain boundary Γ

$$H = \nabla \cdot \left(\frac{\nabla u}{Q} \right).$$

The Willmore flow equation is given as

$$u_t + Q\nabla\cdot\left(\frac{1}{Q}\left(\mathbf{I}-\frac{\nabla u\otimes\nabla u}{Q^2}\right)\nabla(QH)\right) -\frac{1}{2}Q\nabla\cdot\left(\frac{H^2}{Q}\nabla u\right)=0. \tag{6.8}$$

In [97], LDG methods are designed for both the surface diffusion equation (6.7) and the Willmore flow equation (6.8). Energy stability is proved. In [48, 47], L^2 error estimates are given for these LDG methods.

7. Concluding remarks. In these lectures we have given a brief summary of discontinuous Galerkin (DG) methods for time-dependent partial differential equations (PDEs). Clearly, DG schemes can be designed for a wide class of PDEs, both of the dispersive type and of the dissipative type, and often energy stability similar to those for the exact solution of the PDEs can be obtained. Among the current and future research topics for discontinuous Galerkin method, we would like to point out the following.

First, it is worthwhile to study efficient time discretization techniques. While the explicit TVD Runge-Kutta time discretization might be suitable for hyperbolic equations or strongly convection dominated problems, for other equations, the time step restriction is too severe to use explicit Runge-Kutta time discretization for the semi-discrete DG schemes. Suitable time discretization techniques, such as exponential time stepping, preconditioning and multigrid techniques, are being investigated. It is particularly challenging to design efficient time discretization techniques for PDEs with high and odd leading order of spatial derivatives (dispersive type PDEs), especially when the leading term is nonlinear.

Second, it is worthwhile to study effective and efficient error indicators and *a posteriori* error estimates, to guide the design of both h and p adaptivity. DG methods have the flexibility in h-p adaptivity, however this potential can only be fully realized if we have reliable error indicators to tell us where to refine or coarsen the mesh and where to increase or decrease the polynomial degree. Again, it is particularly challenging to design reliable error indicators for PDEs with high and odd leading order of spatial derivatives (dispersive type PDEs).

Finally, it is worthwhile to study the design and stability analysis of DG schemes for more demanding nonlinear PDEs from applications.

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