S9 Minimal surface solver in the rhinoceros and grasshopper software package

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Contents

Introduction

In the module *S9 - Computational optimization* we take a deeper look at the theory behind minimal surfaces and will develop a simple *polygon mesh based* solver integrated into the *Rhinoceros* and *Grasshopper* software package. This solver will be able to process arbitrary three dimensional polygon meshed, as long as they are triangular and have only **unique** vertices. The solver also gives users the opportunity to define certain *boundary conditions* for any given vertex of the input mesh. Those conditions will modify how the optimization is run and thus yield different results.

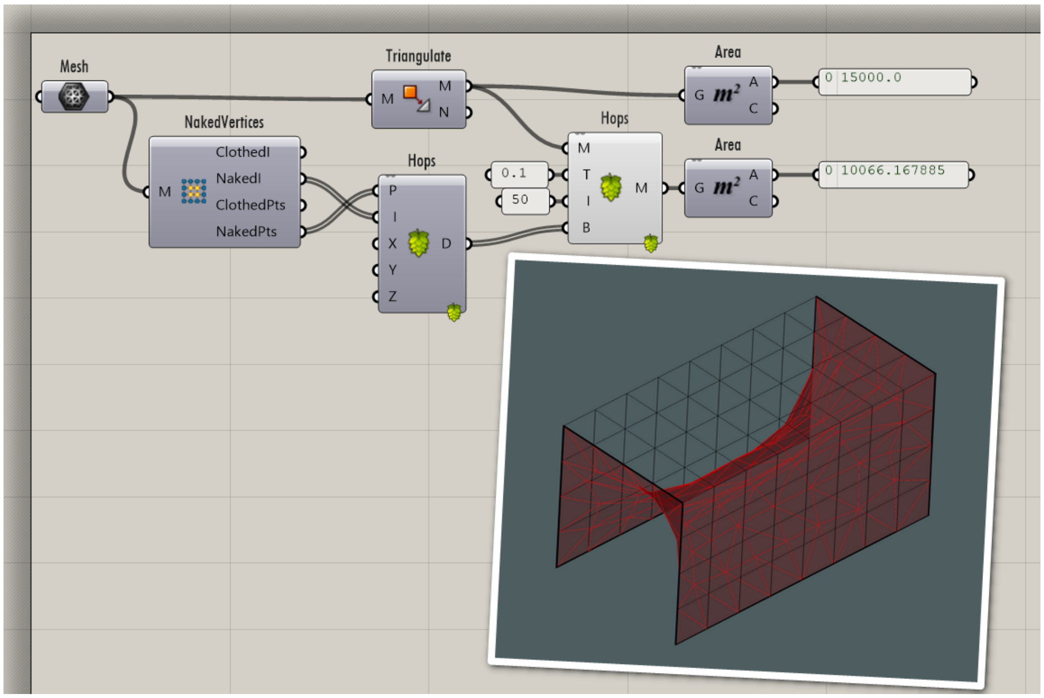


Figure 1: A simple example of running the solver in grasshopper

Minimal surfaces

A surface is considered *minimal*, if it's *mean curvature* c is 0 at any given point on the surface. This property also ensures that if we compare a minimal surface to another one spanning the same space, the area A_{min} of the minimal surface will be smaller as the area of the other surface. If we consider a surface to be parametrized over it's u and v domain D, this gives:

$$c_{u,v} = 0, \forall u, v \in D$$

Minimal polygon meshes

For the solver, we simplify the representation of a surface to that of a *triangulated* polygon mesh M, where all it's vertices lie on the target, minimal surface. The area of a triangle mesh can be simply calculated by calculating the sum of areas of all it's triangular faces, regardless of it's shape. Let M be a polygon mesh, which has the following elementary parts:

- $I = 1, 2, ..., N$ is the set of vertex indices of M, where N is the number of total *unique* vertices
- $P : I \to R^3$ is a mapping from a given vertex index to it's location in 3d space
- T is the set of all triangle faces of M and each triangle $t \in T$ is defined as a triplet of vertex indices $t =< i, j, k >$, which maps to the vertices in 3d space P_i, P_j and P_k

To calculate the area A of a given triangle mesh M, we calculate the sum of areas of all $t \in T$:

$$A = \sum_{t=<i,j,k>\in T} \frac{1}{2} |P_j P_k \times P_j P_i|$$

Solver architecture

Our main goal with the software developed for this module, was to give an intuitive and accessible interface to running the solver. Users should be able to iterate on solver runs quickly and visually. To achieve this we integrate the library into the *rhinocers* and *grasshopper* software package. With the use of the *Hops* plugin, we can run any *CPython* code on a server and call into endpoints on that server from grasshopper.

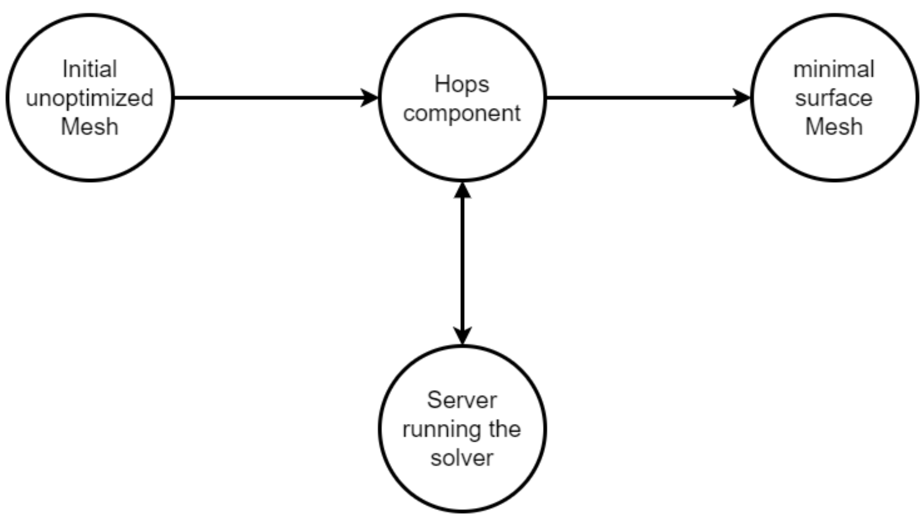


Figure 2: Data flow to minimize an input mesh

The solver itself is hosted at GitHub, together with instructions on how to properly install it, as well as some example files.

Modules

The solver is written in python and the code is distributed into multiple files, which handle different functionalities of the solver:

- `algorithms.py` contains the actual solver implementation
- `boundary_conditions.py` contains all boundary conditions that users can define
- `geometry.py` defines geometry types used inside of the solver
- `conversion.py` defines conversion routines from *opennurbs* types to our custom geometry types used in the solver
- `components.py` defines all *'components'* as endpoints reachable from Hops

algorithms.py

In this file we define all private functions needed to run the solver, as well as the main public function to call, `minimize_mesh`

```
def minimize_mesh(
    vertices: MeshVertexCollection,
    faces: MeshFaceCollection,
    connectivity: MeshVertexConnectivity,
    tolerance: float,
    max_iterations: typing.Union[float, None],
    boundary_conditions: VertexBoundaryConditionsCollection
    ) -> MeshVertexCollection:
    """
    Minimize the vertex positions of the given mesh,
    so that the total area of the mesh is minimal.
    """

    #code omitted
```

boundary_conditions.py

We define boundary conditions in *object oriented programming* fashion as objects which inherit from a common base class, `VertexBoundaryCondition`. These objects have some shared functionality:

- Store a reference to the bound vertex coordinates
- Store a reference to the bound vertex index inside of the mesh vertex buffer
- Define how to serialize to *json* and deserialize back to a class instance from a *json* string.

We need to implement json serialization because the different boundary conditions are defined as different *component* endpoints and *hops* does not support sending custom objects. The workaround is the aforementioned serialization to json strings, which can in turn be deserialized inside of the *component* endpoint.

The Boundary conditions defined for users to use are:

- `VertexAnchorCondition`
- `OnCircleBoundaryCondition`

components.py

In here we define the endpoints for hops components to call into. This is done by annotating a regular python function with the `@hops.component` attribute defined in the *gh-hops-server* python library. Additionaly we need to run a *flask* server that runs *rhinoinside* and serves the endpoints we defined.

As a reference, the endoint for the minimization solving is defined as follows:

```
@hops.component(
    "/minimize",
    name="Minimize",
    nickname="MNMZ",
    description="Minimize mesh area",
    category="Mesh",
    subcategory="CPython",
    inputs=[
```

```
        hs.HopsMesh("Mesh", "M", "The mesh to minimize"),
        hs.HopsNumber("Tolerance", "T", "The tolerance for
↪   minimization"),
        hs.HopsInteger("Max Iterations", "I", "Optional upper limit
↪   on iteration count", optional=True, default=-1),
        hs.HopsString("Boundary conditions", "B", "List of vertex
↪   boundary conditions to enforce",
↪   access=hs.HopsParamAccess.LIST, optional=True, default=None),
    ],
    outputs=[
        hs.HopsMesh("Mesh", "M", "The minimized mesh"),
        hs.HopsNumber("Areas", "A", "The areas at every iteration
↪   step", hs.HopsParamAccess.LIST)],
)
def minimize(mesh, tol, iterations, boundary_conditions):

    start = time.time()

    if iterations == -1:
        iterations = None

    # Convert from opennurbs mesh to internal types
    vertices, faces, connectivity, boundary_indices =
↪   conversion.convert_from_mesh(mesh)

    # if there are no boundary conditions defined, explicitly
     ↪   convert the boundary indices
    # to anchor points
    if boundary_conditions is None:
        boundary_conditions = dict([(i,
↪   VertexAnchorCondition(vertices[i], i)) for i in
↪   boundary_indices])
    else:
        # Deserialize boundary conditions from json
        conditions = []
        for condition in boundary_conditions:
            if "center" in condition:
                condition =
↪   OnCircleBoundaryCondition.from_json(condition)
```

```
            else:
                condition =
↪   VertexAnchorCondition.from_json(condition)

            conditions.append(condition)

        # boundary_conditions is a flat list, we need to convert it
         ↪  to a dict
        boundary_conditions = build_boundary_collection(conditions)
↪

    # Call into the minimize function
    optimized, areas = algorithms.minimize_mesh(vertices, faces,
↪   connectivity, tol, iterations, boundary_conditions)

    # Convert output back to a opennurbs mesh instance
    converted = conversion.convert_to_mesh(optimized, faces)

    # Time runtime and give print feedback
    end = time.time()
    print(f"Minimization took {end - start}ms")

    # Return back result
    return (converted, areas)
```

Minimization algorithm

Area minimization

Let $NT(i) \in T$ be the set of all triangles that contain the vertex P_i and let

$$\frac{\partial A}{\partial P_h} = (\frac{\partial A}{\partial (P_h)_x}, \frac{\partial A}{\partial (P_h)_y}, \frac{\partial A}{\partial (P_h)_z})^T$$

We can generalize this over all vertices in the mesh, as defined by their face-triplets t

$$\frac{1}{2} \sum_{t=<i,j,k>\in T} \frac{\partial A}{\partial P_h} \sqrt{(P_jP_k \times P_jP_i)^2}$$

$$= \frac{1}{2} \sum_{t=<h,j,k>\in NT(h)} \frac{1}{2} \frac{\frac{\partial A}{\partial P_h}(P_jP_k \times P_jP_h)^2}{\sqrt{(P_jP_k \times P_jP_i)^2}}$$

$$= \frac{1}{2} \sum_{t=<h,j,k>\in NT(h)} \frac{(P_jP_k)^2 P_jP_h - (P_jP_k * P_jP_h)P_jP_k}{(P_jP_k \times P_jP_i)^2}$$

We introduce a value C to simplify the calculation with

$$C = \sum_{t=<h,j,k>\in NT(h)} \frac{(P_jP_k)^2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - (P_jP_k)(P_jP_k)^T}{\sqrt{P_jP_k \times P_jP_h}^2}$$

And by setting $\frac{\partial A}{\partial P_h} = 0$ we can now solve for P_h

$$P_h' = -C^{-1} \sum_{t=<h,j,k>\in NT(h)} \frac{(P_jP_k \cdot P_j)P_jP_k - (P_jP_k)^2 P_j}{\sqrt{P_jP_k \times P_jP_h}^2}$$

As we can see, this is not an *explicit* solution for P_h', but we can iteratively calculate new values of lower overall area A for all P_h'.

Boundary conditions

Vertex anchor

On circle

Tests

Triangle Prism

For this test, we have a simple setup of a triangle boundary, similar to experiments done by *Frei Otto*.

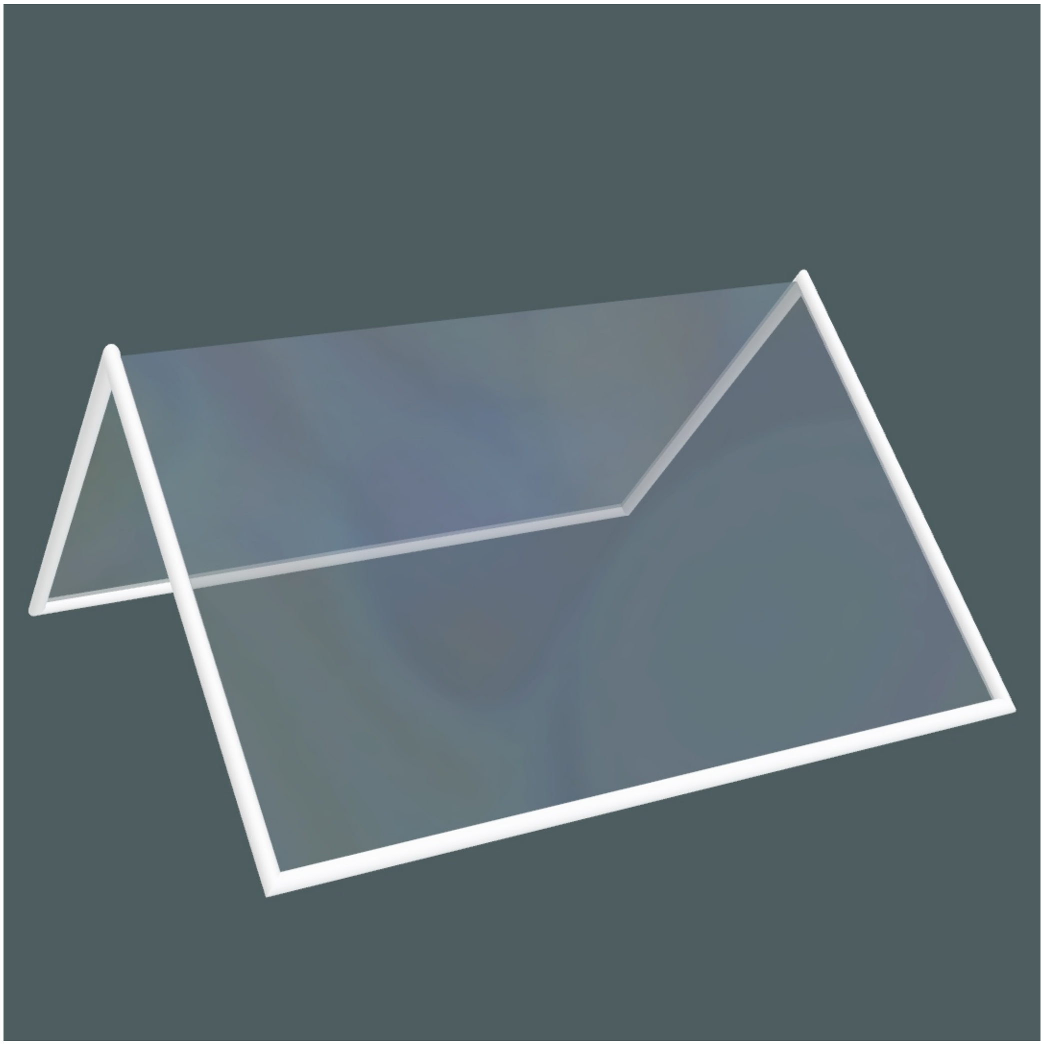
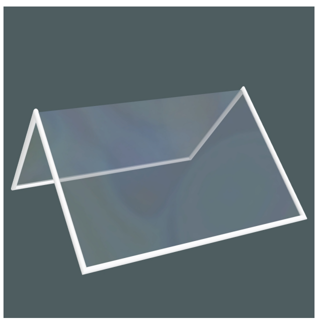
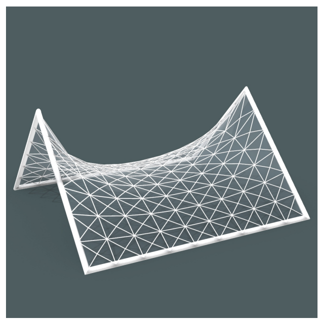


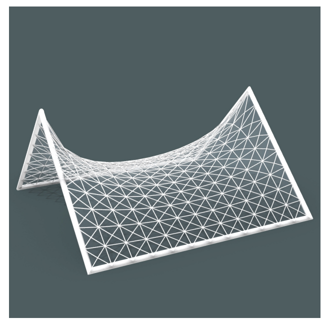
Figure 3: Base shape to minimize, fixed boundary in white

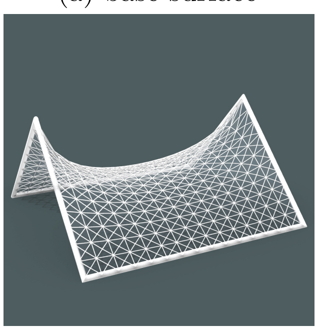
We ran the solver on the different subdivision levels and got some unexpected results in the first run. Since the solver takes a tolerance value, the runs where stopped once area changes where less than the given tolerance value. This leads to meshes with less vertices to finish their run in less iterations. As we see from our values, meshes with more vertices actually have a **bigger** area after our test run.

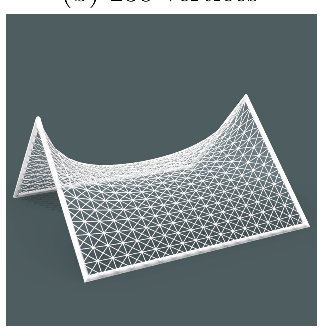
Vertex count	Iterations	Runtime [s]	Area	Min curvature	Max curvature	Average curvature
0	0	0	1.200,00	-2,07E-01	0,00E+00	1,17E-02
238	127	16,579	1.024,89	-1,17E-01	8,97E-03	8,65E-03
413	188	45	1.025,73	-1,41E-01	8,15E-03	9,66E-03
636	250	95,588	1.028,23	-1,64E-01	7,04E-03	1,08E-02
858	300	159,787	1.031,74	-1,90E-01	5,86E-03	1,21E-02
1034	329	207,141	1.034,21	-2,00E-01	5,11E-03	1,27E-02


(a) base surface


(b) 238 vertices


(c) 413 vertices


(d) 636 vertices


(e) 858 vertices

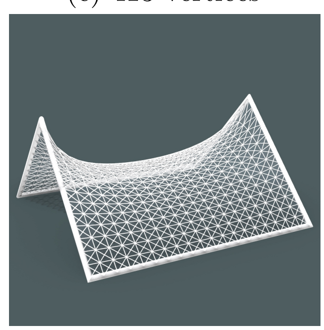

(f) 1034 vertices

Figure 4: 6 Test cases for the triangle prism test

After some more tests it became clear, that meshes with more vertices take more iterations to get to the same area as meshes with less vertices. So we ran the test again, with roughly 1000 iterations for all subdivisions, which equals roughly 3 times the iteration count of the hightest subdivision level of the last run.

Vertex count	Iterations	Runtime [s]	Area	Min curvature	Max curvature	Average curvature
0	0	0	1.200,00	-2,07E-01	0,00E+00	1,17E-02
238	1000	150	1.019,03	-1,07E-01	1,18E-02	6,28E-03
413	1000	251	1.019,01	-1,24E-01	1,27E-02	5,51E-03
636	1000	401,285	1.018,00	-1,39E-01	1,33E-02	5,04E-03
858	1000	543,977	1.017,74	-1,57E-01	1,32E-02	5,45E-03
1034	1000	693,474	1.017,64	-1,64E-01	1,30E-02	5,62E-03

As we can see, the values now correspond more closely to what we would expect, more vertices equate to a better area value, at the cost of a higher runtime. When plotting the area values for all subdivision levels per iteration, this becomes visually apparent

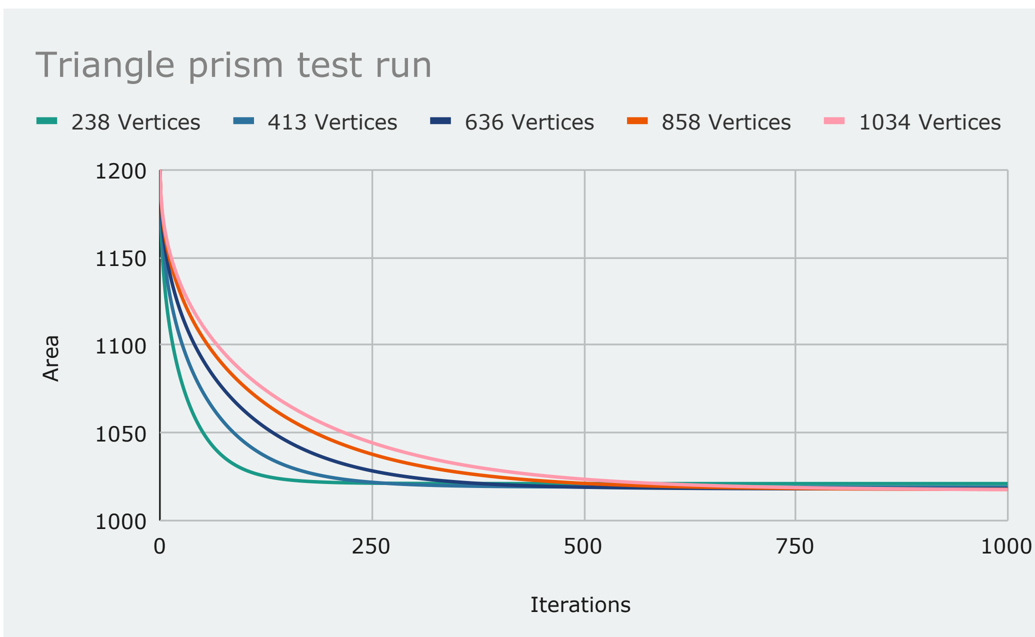
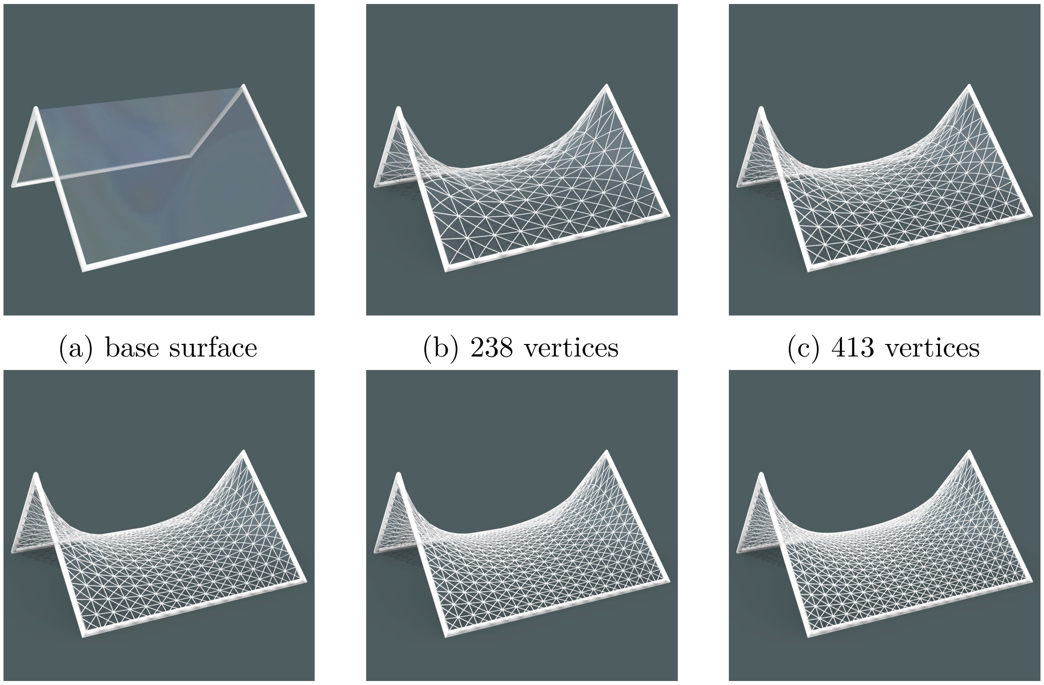


Figure 5: Triangle prism test run area values

Additionally, the minimization seems to yield more relaxed meshed, as the result is visually more lowered than the results after the first run



(a) base surface (b) 238 vertices (c) 413 vertices

(d) 636 vertices (e) 858 vertices (f) 1034 vertices

Figure 6: 6 Test cases for the triangle prism test