6.042J/18.062J, Spring '15: Mathematics for Computer Science

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1 Introduction

2 In-Class Problems & Solutions

2.1 Week 1, Fri

2.1.1 Problem 1.

$P :=$ **Prove that if $a \cdot b = n$, then either a or b must be $\leq \sqrt{n}$, where a, b and n are nonnegative real numbers. *Hint:* by contradiction Section 1.8 in the course textbook.**

Solution: We use proof by contradiction. Suppose P is false. Therefore, there exists a and b are $> \sqrt{n}$. Whilst $a \cdot b = n$.

$$a \cdot b > \sqrt{n} \cdot \sqrt{n} = n$$

This is a contradiction. Therefore P must be true. ■

2.1.2 Problem 2.

Generalise the proof of Theorem 1.8.1 repeated below that $\sqrt{2}$ is irrational in the course textbook. For example, how about $\sqrt{3}$?.

Solution: We use proof by contradiction. Suppose $\sqrt{3}$ is rational.

$$\Rightarrow \sqrt{3} = \frac{n}{d}$$

Where n and d are the lowest terms i.e., without common prime factors.

$$\Rightarrow 3 = \frac{n^2}{d^2}$$

$$\Rightarrow 3d^2 = n^2$$

n^2 is a factor of 3 which is only possible if n is also a factor of 3:

$$\Rightarrow n = 3k$$

Where $k \in \mathbb{N}$.

$$\Rightarrow n^2 = (3k)^2 = 9k^2$$

$$\Rightarrow 3d^2 = 9k^2$$

$$\Rightarrow d^2 = 3k^2$$

Therefore, 3 is a factor of d^2 which is only possible if 3 is factor of d as well.

Above we prove n and d have a common factor of 3, therefore n and d are not the lowest terms. This a contradiction. Therefore $\sqrt{3}$ is an irrational number. ■

My Extra(s):
1. Prove that the square root of all prime numbers are irrational.
2. Since 3 is a factor n^2, prove that 3 must also be a factor of n.

2.1.3 Problem 3.

If we raise an irrational number to an irrational power, can the result be rational? Show that it can by considering $\sqrt{2}^{\sqrt{2}}$ and arguing by cases.

Solution: We argue by cases. Where $a = \sqrt{2}^{\sqrt{2}}, b = \sqrt{2}$.

Case 1: *We assume a is irrational.* b is known to be irrational - **Theorem.** *$\sqrt{2}$ is irrational.*

$$\Rightarrow a^b = \left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}}$$

$$\Rightarrow \left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}} = \sqrt{2}^{\sqrt{2}\times\sqrt{2}} = \sqrt{2}^{2} = 2.$$

2 can be written in the form $\frac{n}{d}$ where its lowest terms are $n = 2$ and $d = 1$. Therefore by definition, 2 is a rational number. *Case 1* thus shows a irrational number raised to the power of irrational number can produce a rational number.

Case 2: *We assume a is rational.* We introduce a third term c where $c = \sqrt{2}$ - an irrational number.

$$\Rightarrow b^c = \sqrt{2}^{\sqrt{2}}$$

Therefore b, an irrational number, raised to the power of c also an irrational number produces a rational number, a.

Extra: Chris Reineke - He came up with a constructive proof i.e., a specific pair of irrational numbers with this property.

Let $a = \sqrt{10}$, $b = \log_{10} 4$.

$$\Rightarrow a^b = \sqrt{10}^{\log_{10} 4}$$

$$\Rightarrow \sqrt{10}^{\log_{10} 4} = 10^{\log_{10} 4^{1/2}} = 10^{\log_{10} 2} = 2$$

■

2.1.4 Problem 4.

The fact that that there are irrational numbers a, b such that a^b is rational was proved earlier by cases. Unfortunately, that proof was nonconstructive: it didn't reveal a specific pair, a, b with this property. But in fact, it's easy to do this: let $a := \sqrt{2}$ and $b := 2\log_2 3$. We know that $a = \sqrt{2}$ is irrational and $a^b = 3$ by definition. Finish the proof that these values for a, b work, by showing that $2\log_2 3$ is irrational.

We use proof by contradiction. Suppose $2\log_2 3$ is a rational number.

$$\Rightarrow 2\log_2 3 = \frac{n}{d}$$

$$\Rightarrow \log_2 3^2 = \frac{n}{d}$$

$$\Rightarrow 2^{\frac{n}{d}} = 3^2$$

$$\Rightarrow 2^n = 3^{2d}$$

This is a contradiction as 2 (an even number) can never be a prime factor of an odd number and vice versa, provided $n, d \neq 0$. If $n, d = 0$:

$$\Rightarrow 2^0 = 3^{2\times 0}$$

$$\Rightarrow 2^0 = 3^0$$

$$\Rightarrow 1 = 1$$

As any number to the power of zero is 1, is case where these two numbers are equal. Otherwise proof that $2\log_2 3$ is an irrational number. ■

2.1.5 My Extra(s)

Since 3 is a factor n^2, prove that 3 must also be a factor of n.

We consider the initial statement, 3 is a factor of n^2, then 3 is also a factor of n and its contra-positive i.e., 3 is not a factor of n, then 3 is also not a factor of n^2.

Case 1. 3 is a factor of n.

Let $n = 3k$.

$$\Rightarrow n^2 = (3k)(3k) = 9k^2$$

$$\Rightarrow n^2 = 3(3k^2)$$

3 is also a prime factor of n^2. Therefore, 3 is not a factor n, implies 3 is also a factor of n^2.

Case 2. 3 is not a factor of n.

Let $n = 3k + 1$.

$$\Rightarrow n^2 = (3k + 1)(3k + 1)$$

$$\Rightarrow n^2 = 9k^2 + 6k + 1$$

In this case, 3 is not a prime factor of n^2. Thus, 3 is not a factor of n implies 3 is also not a factor of n^2. Since both cases are true, the initial statement, if 3 is a factor of n^2 then 3 is a factor of n is true.