Computational Fluid Dynamics: The Finite-Volume Method

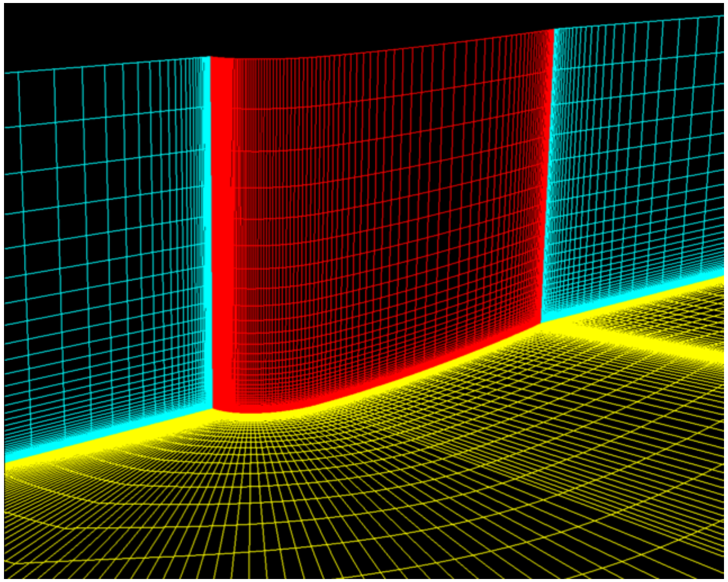
David Apsley

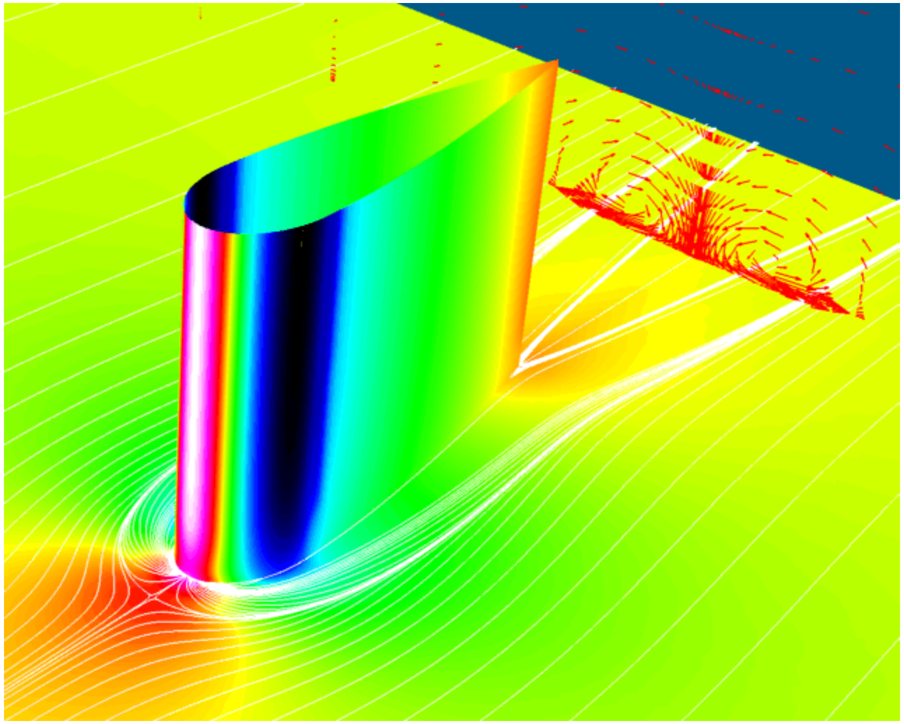
1. Introduction

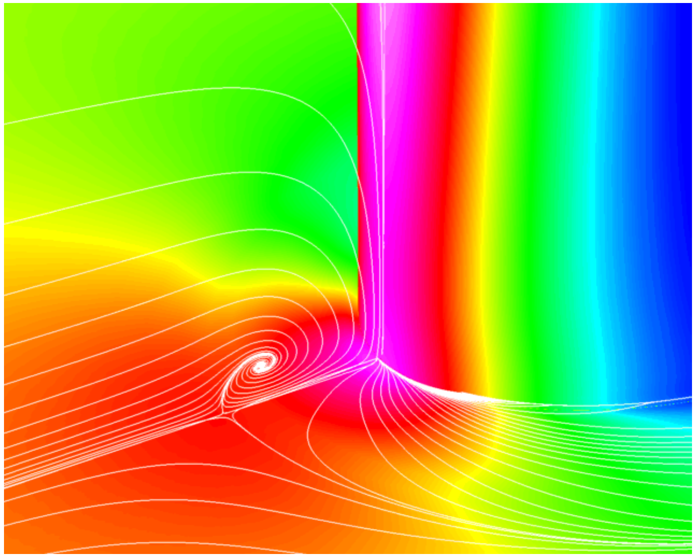
What is …
Computational Fluid Dynamics?

The use of computers and numerical methods to solve problems involving fluid flow

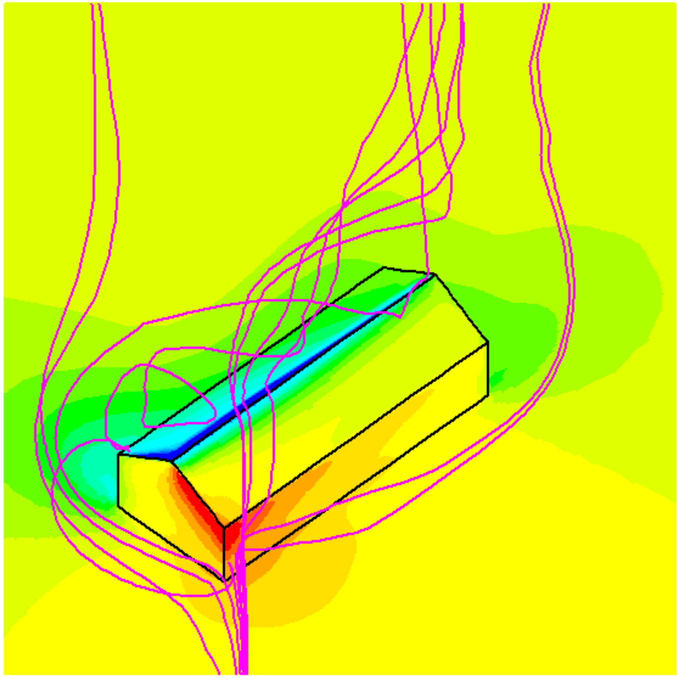
Aerodynamics

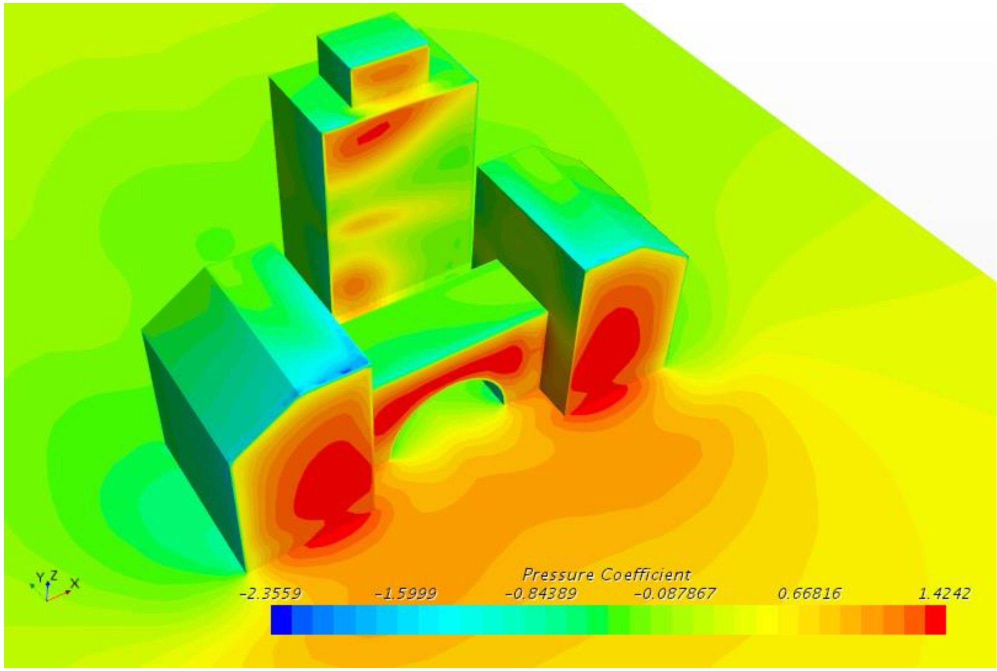




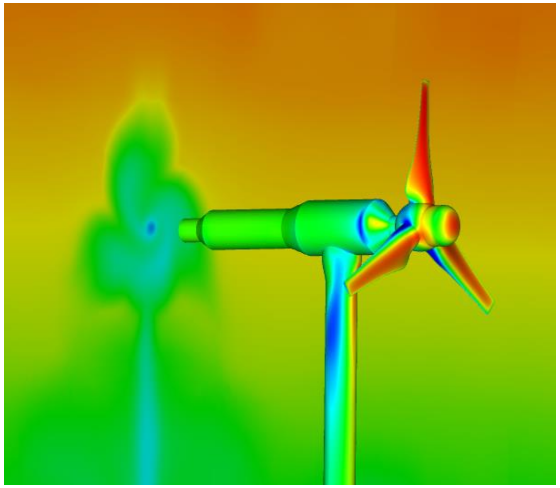


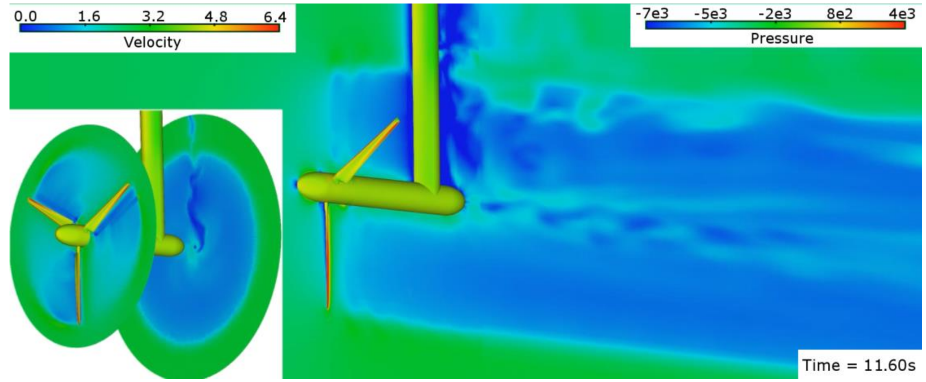
Wind Loading

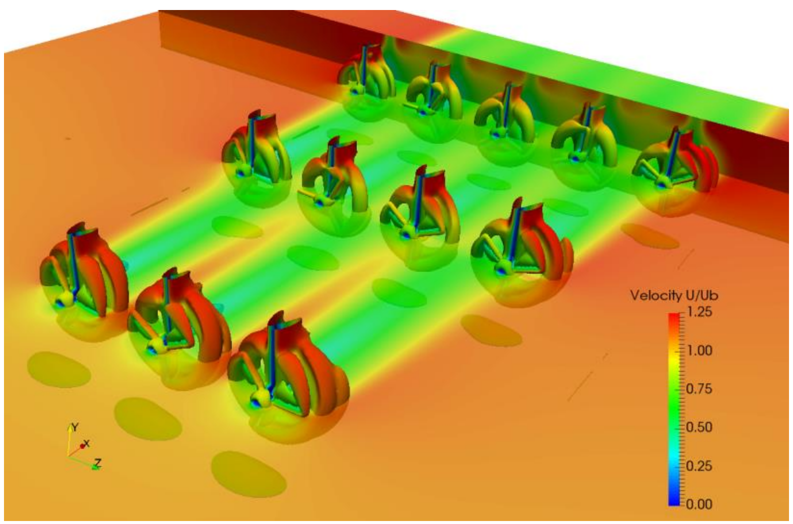


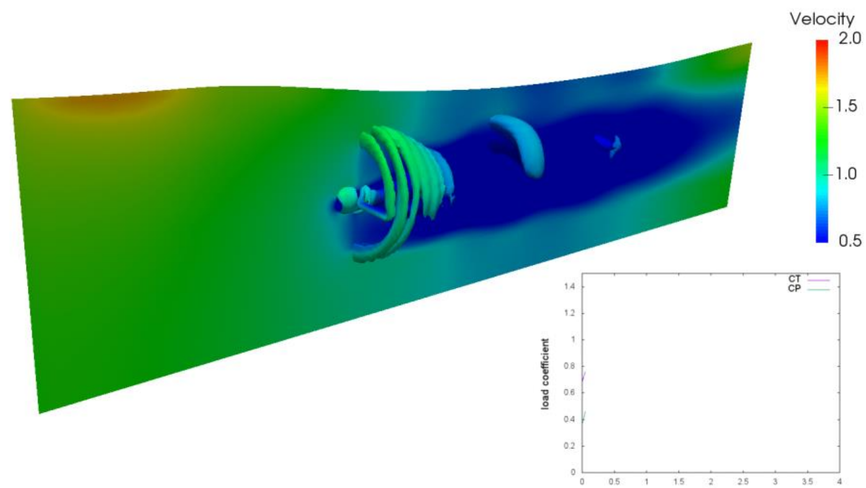


Turbine Technology

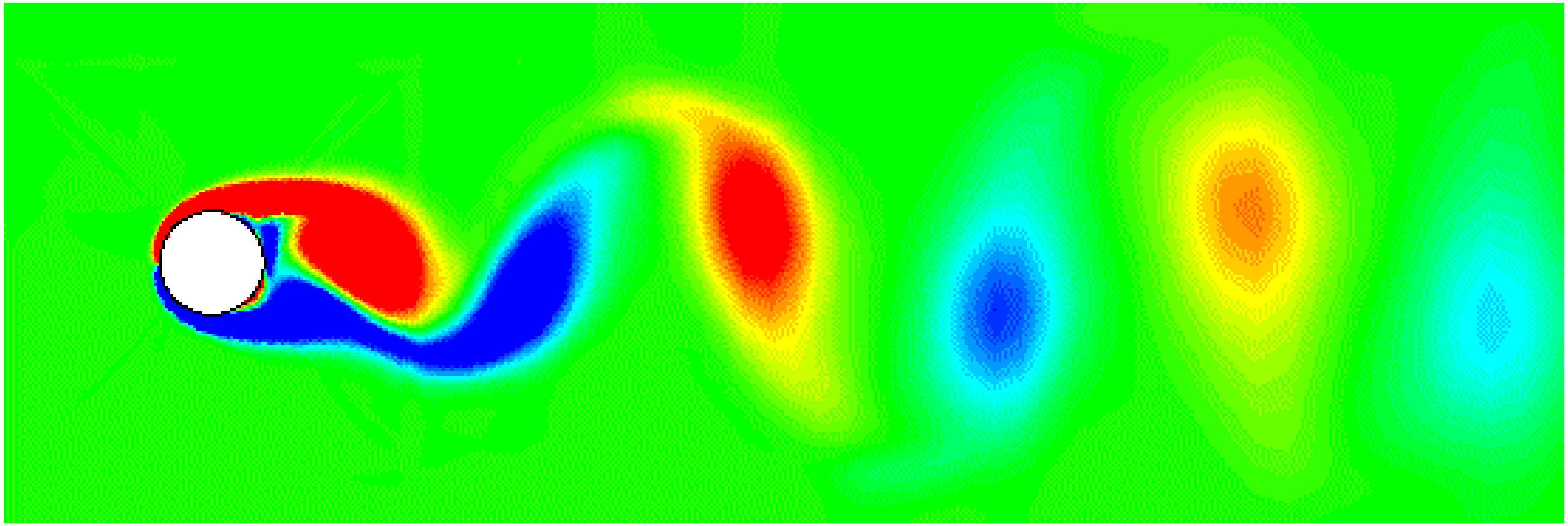




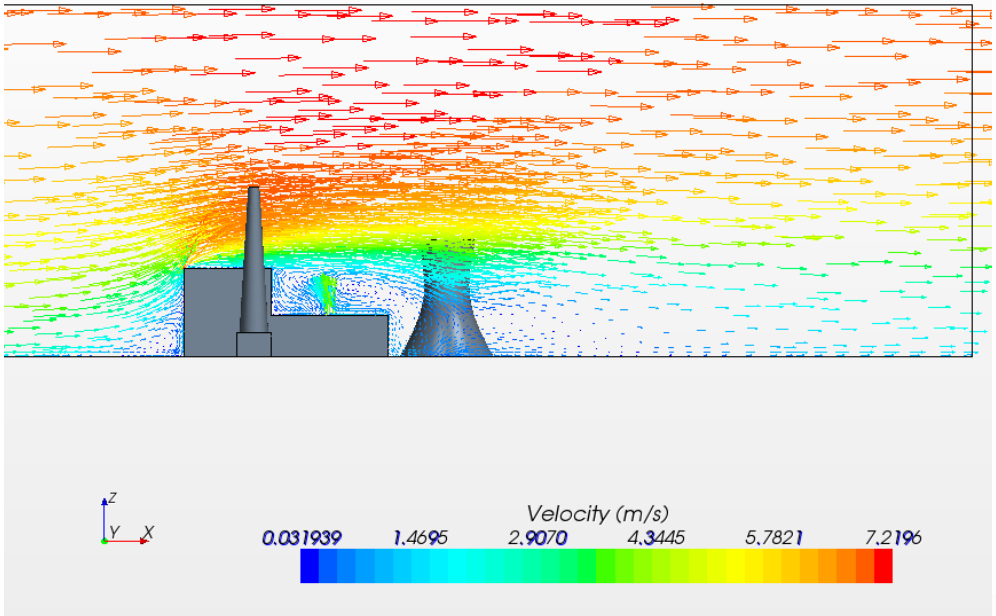


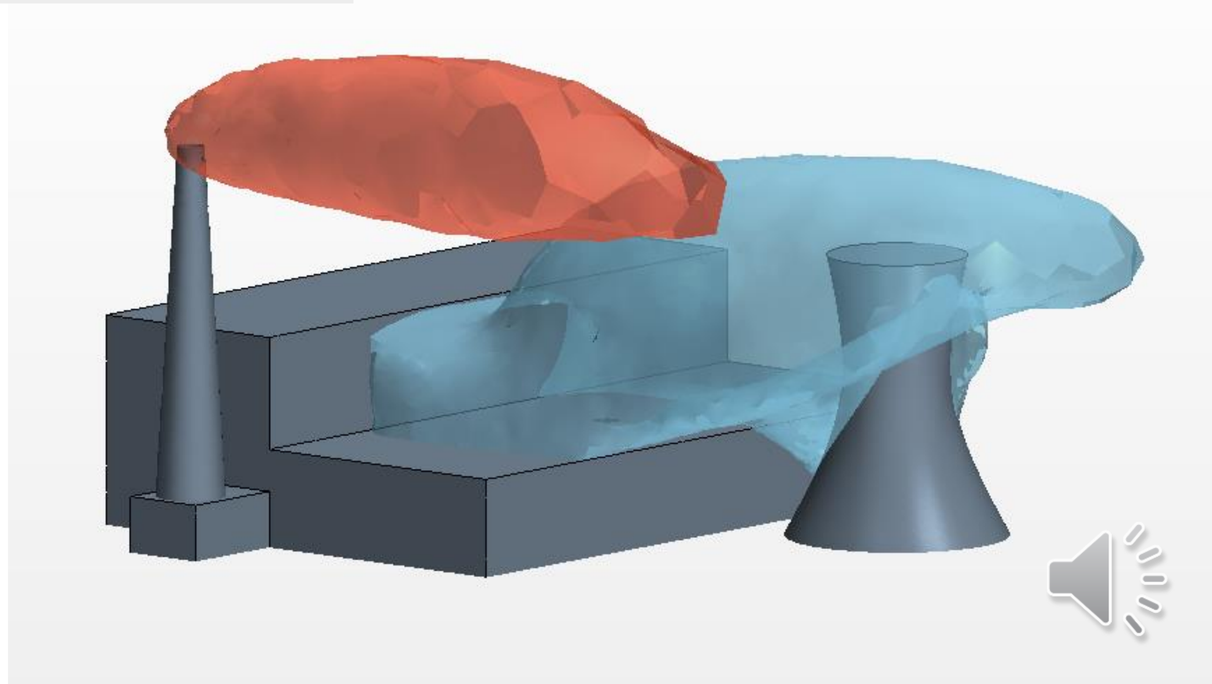


Vortex Shedding

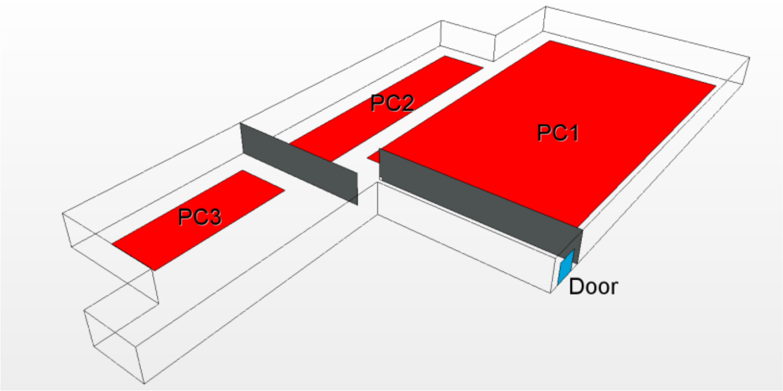


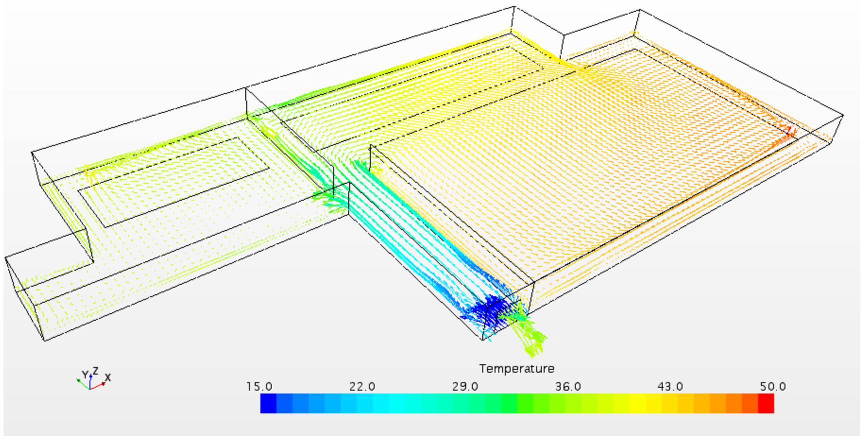
Dispersion of Pollution

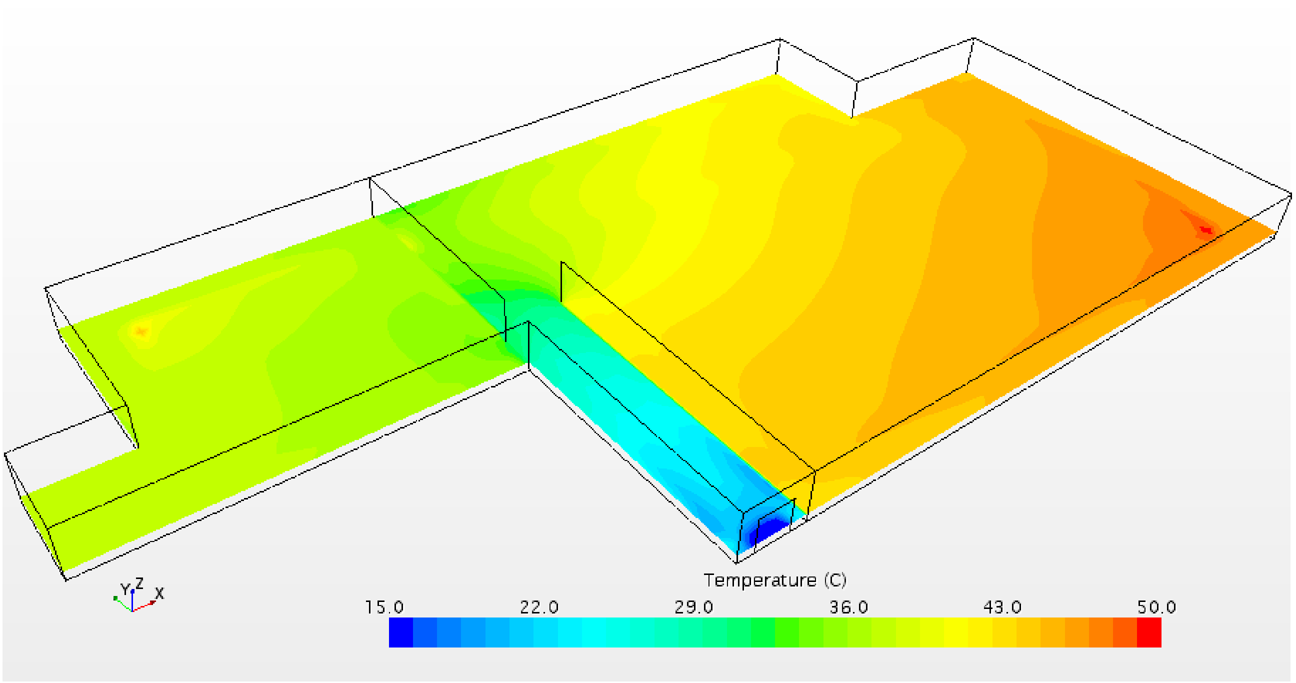




Ventilation

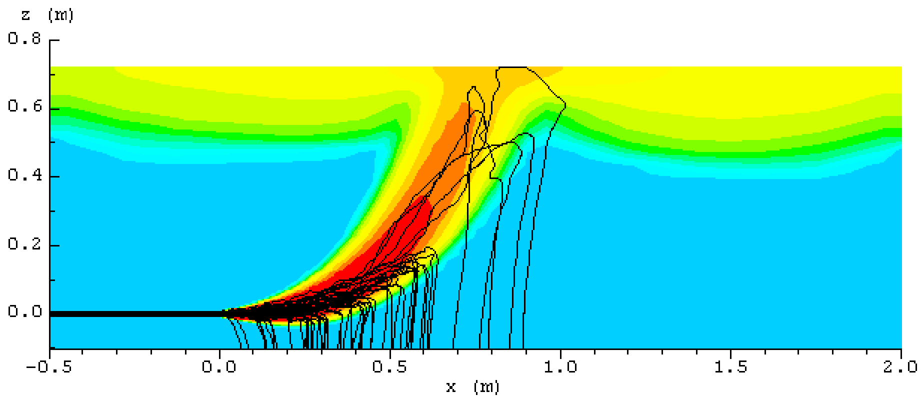


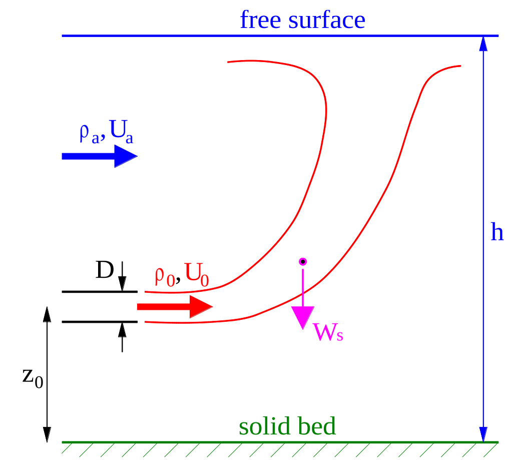




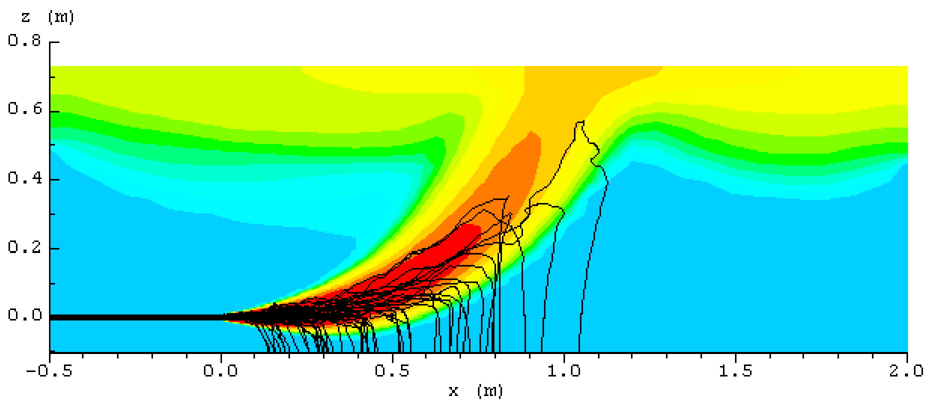
Particle-Laden Plumes (Sea Outfalls)

(ex302)
ρ_a=1040 kg/m^3

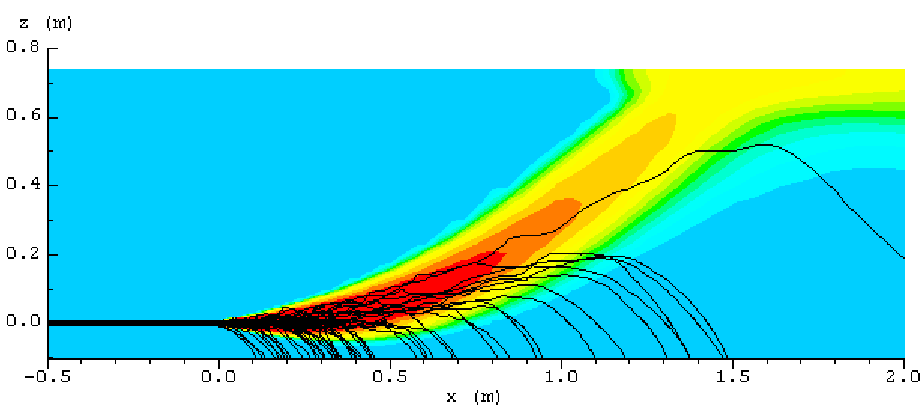




(ex322)
ρ_a=1020 kg/m^3

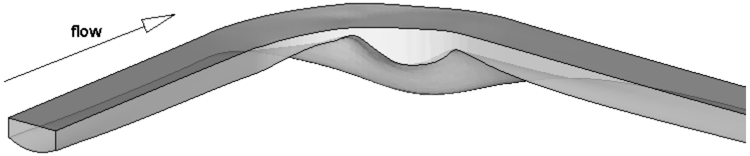


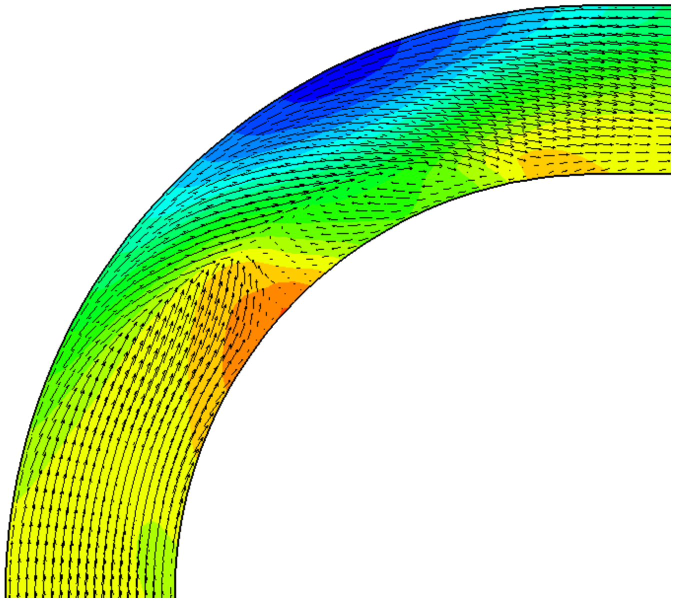
(ex336)
ρ_a=1020 kg/m^3
U_a=0.026 m/s



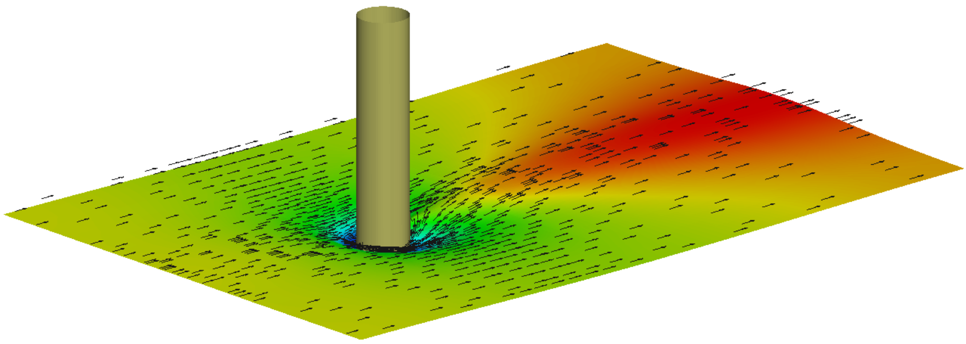
Sediment Scour

River bend



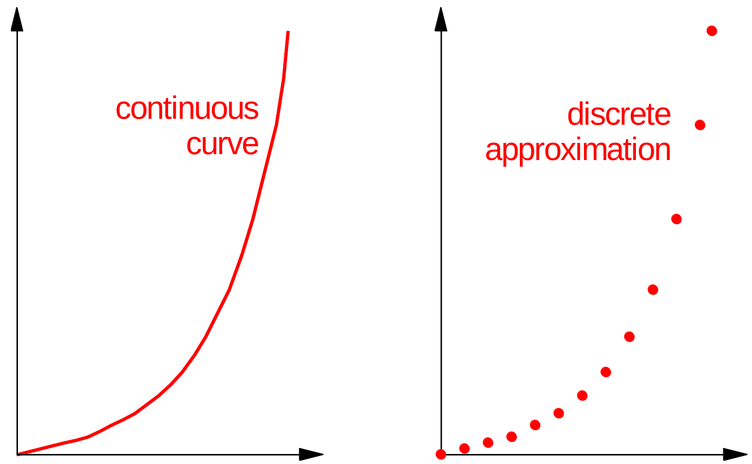


Bridge pier

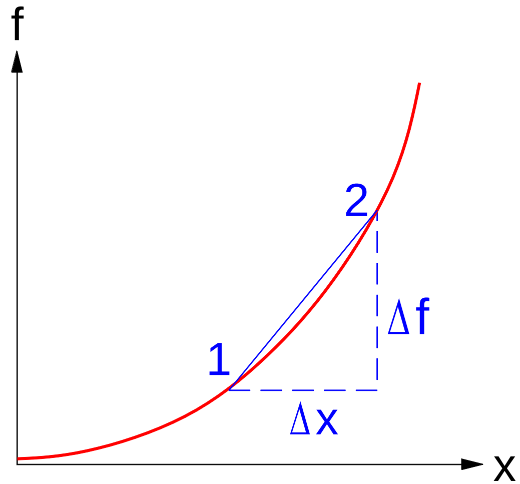


Discretisation

Field variables:



Equations:



$$\frac{\mathrm{d}f}{\mathrm{d}x} \approx \frac{\Delta f}{\Delta x} = \frac{f_2 - f_1}{x_2 - x_1}$$

Basic Principles of CFD

1. **Discretise space:**

   replace field variables $(\rho, u, v, w, p, \ldots)$ by values at a finite number of **nodes**

2. **Discretise equations:**

   continuum equations $\rightarrow$ **algebraic** equations

3. **Solve:**

   large system of **simultaneous equations**

Stages of a CFD Analysis

- **Pre-processing:**
  - formulate problem (geometry, equations, boundary conditions)
  - construct computational mesh
- **Solving:**
  - discretise
  - solve
- **Post-processing:**
  - analyse
  - visualise (graphs and plots)

Fluid-Flow Equations

- **Mass:** change of mass = 0
- **Momentum:** change of momentum = force × time
- **Energy:** change of energy = work + heat
- (**Other constituents**)

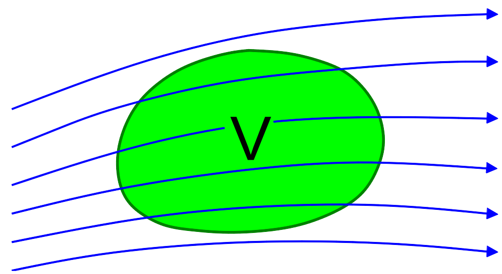
In fluid mechanics, these are normally expressed in **rate** form

Form of Equations

- **Integral** (control-volume)
- **Differential**

Integral (Control-Volume) Approach

Consider the budget of **any transported physical quantity** in **any control volume**



$$\textbf{CHANGE} = \textbf{IN} - \textbf{OUT} + \textbf{CREATED}$$

$$\frac{\textbf{CHANGE}}{\text{time}} + \frac{\textbf{OUT} - \textbf{IN}}{\text{time}} = \frac{\textbf{CREATED}}{\text{time}}$$

$$\begin{pmatrix}\textbf{TIME DERIVATIVE}\\ \text{of amount in } V\end{pmatrix} + \begin{pmatrix}\textbf{NET FLUX}\\ \text{through boundary of } V\end{pmatrix} = \begin{pmatrix}\textbf{SOURCE}\\ \text{inside } V\end{pmatrix}$$

$$\begin{pmatrix}\textbf{TIME DERIVATIVE}\\ \text{of amount in } V\end{pmatrix} + \begin{pmatrix}\textbf{ADVECTION} + \textbf{DIFFUSION}\\ \text{through boundary of } V\end{pmatrix} = \begin{pmatrix}\textbf{SOURCE}\\ \text{inside } V\end{pmatrix}$$

→ **Finite-volume** method for CFD

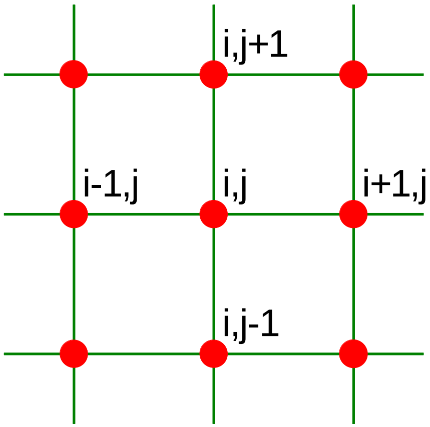
Differential Equations For Fluid Flow

- Derived by considering the rate of change **at a point**; i.e. using **infinitesimal control volumes**
- Discretisation gives a **finite-difference** method for CFD
- Several types:
  - fixed-point ("Eulerian"): **conservative**
  - moving with the flow ("Lagrangian"): **non-conservative**
  - **derived variables**; e.g. potential flow

Main Methods for CFD

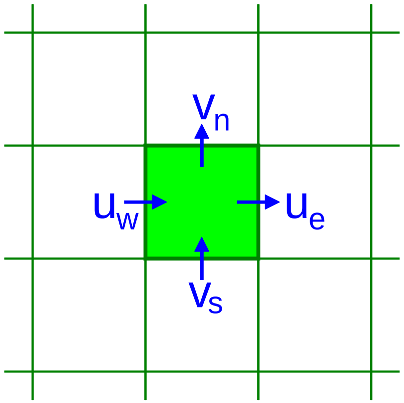
- **Finite-difference:**
  - discretise **differential** equations

$$0 = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \approx \frac{u_{i+1,j} - u_{i-1,j}}{2\Delta x} + \frac{v_{i,j+1} - v_{i,j-1}}{2\Delta y}$$



- **Finite-volume:**
  - discretise **control-volume** equations

$$0 = \text{net mass outflow} = (\rho u A)_e - (\rho u A)_w + (\rho v A)_n - (\rho v A)_s$$

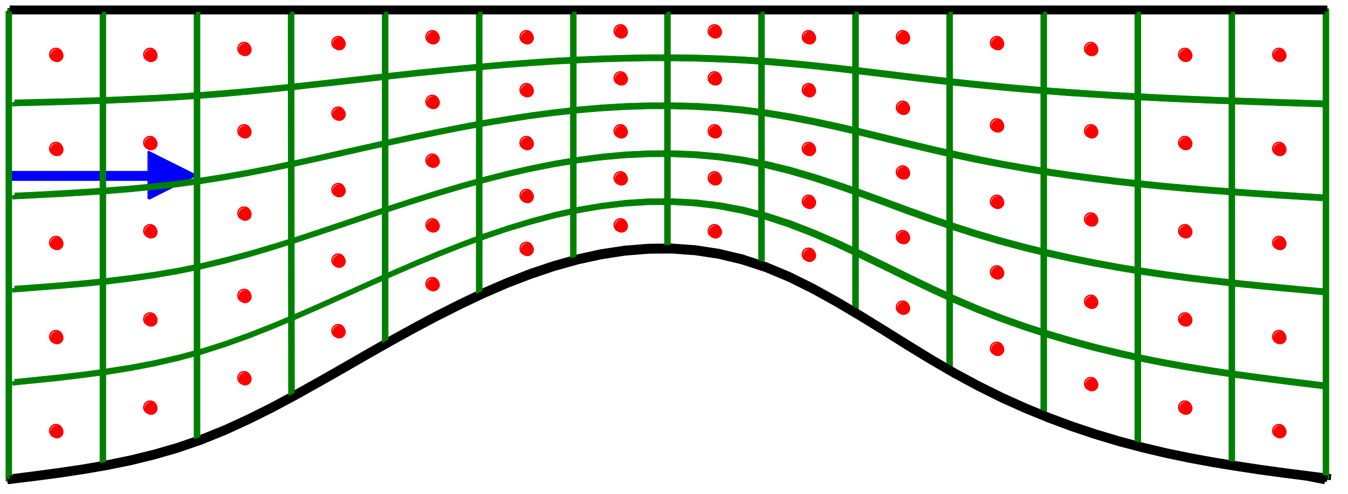


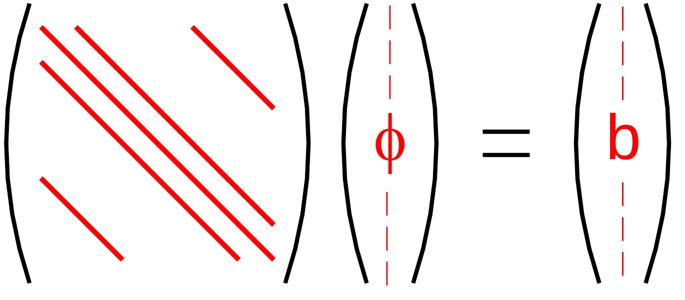
- **Finite-element:**
  - represent solution as a weighted sum of **basis functions**

$$u(\mathbf{x}) = \sum u_\alpha S_\alpha(\mathbf{x})$$

Advantages of the Finite-Volume Method in CFD

- Rigorously enforces **conservation**
- **Flexible** in terms of:
  - **geometry**
  - **fluid phenomena**
- Directly relatable to **physical quantities**

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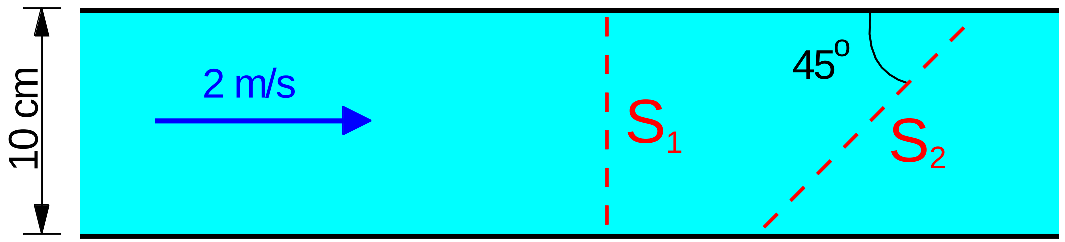
Examples

Example Q1

Water (density 1000 kg m^{-3}) flows at 2 m s^{-1} through a circular pipe of diameter 10 cm. What is the mass flux C across the surfaces S_1 and S_2?



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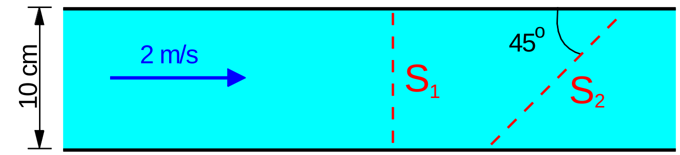


S_1: mass flux $(C) = \rho u A$

$$= 1000 \times 2 \times \frac{\pi \times 0.1^2}{4}$$

$$= 15.7 \text{ kg s}^{-1}$$

S_2: the same!

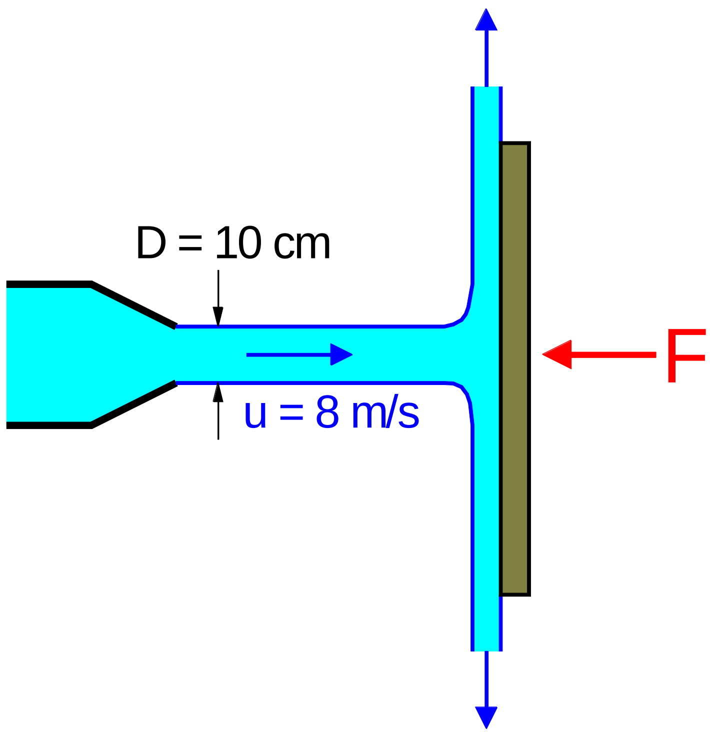
$$C = \rho(u\cos\theta)A$$

$$C = \rho u(A\cos\theta)$$

In general: $\boxed{C = \rho\mathbf{u} \bullet \mathbf{A}}$

Example Q2

A water jet strikes normal to a fixed plate as shown. Compute the force F required to hold the plate fixed.



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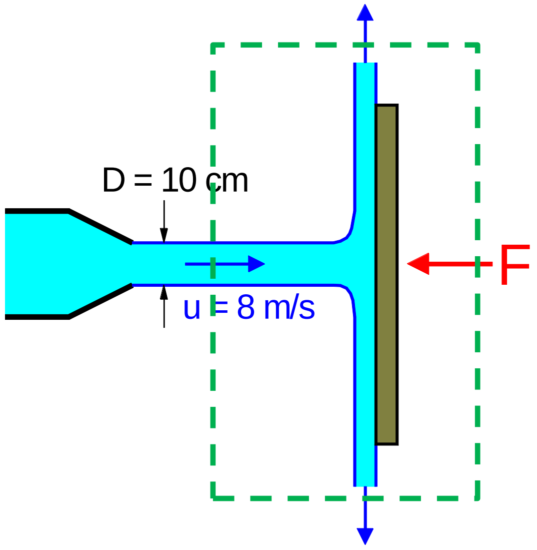

$$\text{force (on fluid)} = (\text{momentum flux})_{\text{out}} - (\text{momentum flux})_{\text{in}}$$

$$\text{momentum flux} = \text{mass flux} \times \text{velocity}$$

$$-F = 0 - (\rho U A)U$$

$$F = \rho U^2 A$$

$$= 1000 \times 8^2 \times \frac{\pi \times 0.1^2}{4}$$

$$= 503 \text{ N}$$

Example Q3

An explosion releases 2 kg of a toxic gas into a room of dimensions 30 m × 8 m × 5 m. Assuming the room air to be well-mixed and to be vented at a speed of 0.5 m s^{-1} through an aperture of area 6 m^2, calculate:

(a) the initial concentration of gas in ppm by mass;

(b) the time taken to reach a safe concentration of 1 ppm.

(Take the density of air as 1.2 kg m^{-3}.)

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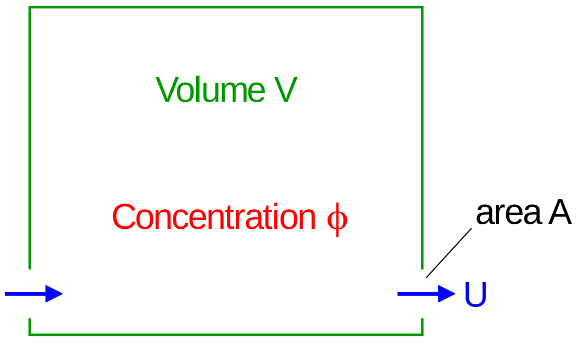

$$V = 30 \times 8 \times 5 \quad = 1200\ \mathrm{m}^3$$

$$\text{mass of fluid} \times \text{concentration} = \text{mass of toxin}$$

$$(\rho V)\phi_0 = 2\ \ \mathrm{kg}$$

$$\phi_0 = \frac{2}{1.2 \times 1200} \quad = 1.389 \times 10^{-3}$$

1389 ppm

An explosion releases 2 kg of a toxic gas into a room of dimensions 30 m × 8 m × 5 m. Assuming the room air to be well-mixed and to be vented at a speed of 0.5 m s^{-1} through an aperture of area 6 m^2, calculate:

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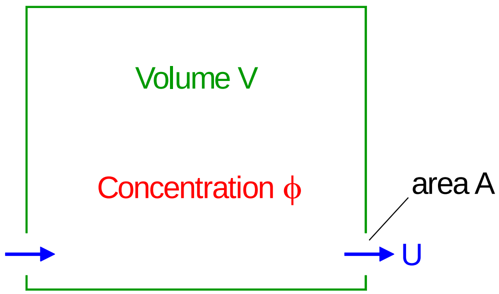

$$\phi_0 = 1389 \text{ ppm}$$

$$V = 1200 \text{ m}^3$$
$$A = 6 \text{ m}^2$$
$$U = 0.5 \text{ m s}^{-1}$$

$$\text{Change in amount of toxin} = \text{amount in} - \text{amount out}$$

$$\text{Rate of change of amount of toxin} = \text{rate of entering} - \text{rate of leaving}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(\rho V \phi) = 0 - (\rho u A)\phi$$

$$\frac{\mathrm{d}\phi}{\mathrm{d}t} = -\left(\frac{uA}{V}\right)\phi, \qquad \phi = \phi_0 \text{ at } t = 0$$

$$\frac{\mathrm{d}\phi}{\mathrm{d}t} = -\lambda\phi \qquad \lambda = \frac{uA}{V} = 0.0025 \text{ s}^{-1}$$

$$\phi = \phi_0 e^{-\lambda t}$$

$$e^{\lambda t} = \frac{\phi_0}{\phi}$$

$$t = \frac{1}{\lambda}\ln\frac{\phi_0}{\phi} = \frac{1}{0.0025}\ln(1389) = 2895 \text{ s}$$

$$\approx 48 \text{ min}$$