

Problem 2

Nagaoka bags

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In this problem we face *Hall effect*-like resistivity density tensor. In the steady state there is no charge density $\nabla E = 0$, so we have Laplace equation $\nabla \phi = E$

$$\nabla^2 \phi = 0 \quad (1)$$

Potential difference is between $x = 0$ and $x = a$ sides of square. Also, let's shift the y axis so corresponding square side changes from $[0, a]$ to $[-a/2, a/2]$.

1 Boundary conditions

Boundary conditions are

$$\begin{aligned} \phi(x, a/2) &= \phi_0 \\ \phi(x, -a/2) &= -\phi_0 \end{aligned} \quad (2)$$

and we should satisfy the requirement that no current leave through $x = 0$ and $x = a$ edges

$$\begin{aligned} E_x(0, y) &= \lambda E_y(0, y) \\ E_x(a, y) &= \lambda E_y(a, y), \\ \lambda &= \frac{\rho_{xy}}{\rho_{xx}} \end{aligned} \quad (3)$$

Since $E = -\nabla \phi$

$$\begin{aligned} \frac{\partial \phi}{\partial x}(0, y) &= \lambda \frac{\partial \phi}{\partial y}(0, y) \\ \frac{\partial \phi}{\partial x}(a, y) &= \lambda \frac{\partial \phi}{\partial y}(a, y) \end{aligned} \quad (4)$$

2 Solution

One of possible solutions to (1) is linear function $\phi(x, y) = (ax + b)(cy + d)$. However, only

$$\phi(x, y) = \frac{2\phi_0}{a} [\lambda(x + b) + y] \quad (5)$$

satisfies boundary conditions (4).

Then we assume separable solutions $\phi(x, y) = X(x)Y(y)$. (1) splits into

$$\frac{d^2 X}{dx^2} = -k^2 X, \quad \frac{d^2 Y}{dy^2} = -k^2 Y \quad (6)$$

From boundaries (4) $X'_0 Y = \lambda Y' X_0$ (X_0 is for $x = 0$, $X' = \frac{dX}{dx}$). Then differentiate $X'_0 Y' = \lambda Y'' X_0$ and substitute into (6)

$$X'_0 Y' = \lambda k^2 Y X_0 \quad (7)$$

Eliminating Y' we have real k value

$$k^2 = \left(\frac{X'_0}{\lambda X_0} \right)^2 \quad (8)$$

That results in

$$\phi_k(x, y) = (A_k \cos kx + B_k \sin kx) (C_k e^{kY} + D_k e^{-kY}) \quad (9)$$

Applying the (4) boundary conditions

$$\frac{B_k}{A_k} = \lambda \frac{C_k e^{kY} - D_k e^{-kY}}{C_k e^{kY} + D_k e^{-kY}} \quad (10)$$

Either $C_k = 0, B_k = -\lambda A_k$ or $D_k = 0, B_k = \lambda A_k$.

Most general is linear combination

$$\begin{aligned} \phi_k(x, y) &= R_k (\cos kx + \lambda \sin kx) e^{ky} + S_k (\cos kx - \lambda \sin kx) e^{-ky}, \\ R_k &= A_k C_k, S_k = A_k D_k \end{aligned} \quad (11)$$

From boundaries (4) we have $\sin ka = 0 \rightarrow k_n = \frac{n\pi}{L}$ and

$$\begin{aligned} \phi(x, y) &= \frac{2\phi_0}{a} [\lambda(x + b) + y] + \\ &\sum_{n=1,2,3,..} \cos k_n x (R_{k_n} e^{k_n y} + S_{k_n} e^{-k_n y}) + \lambda \sin k_n x (R_{k_n} e^{k_n y} - S_{k_n} e^{-k_n y}) \end{aligned} \quad (12)$$

From symmetry of equations (1), (4) we find $S_n = (-1)^{n+1} R_n$ and

$$\begin{aligned} \phi(x, y) &= \frac{2\phi_0}{a} \left[\lambda \left(x - \frac{a}{2} \right) + y \right] + \\ &\sum_{m=1,3,..} T_m \left[\cos \left(\frac{m\pi}{a} x \right) \cosh \left(\frac{m\pi}{a} y \right) + \lambda \sin \left(\frac{m\pi}{a} x \right) \sinh \left(\frac{m\pi}{a} y \right) \right] + \\ &\sum_{n=2,4,..} U_n \left[\cos \left(\frac{n\pi}{a} x \right) \cosh \left(\frac{n\pi}{a} y \right) + \lambda \sin \left(\frac{n\pi}{a} x \right) \sinh \left(\frac{n\pi}{a} y \right) \right] \end{aligned} \quad (13)$$

where $T_m = 2R_m$ and $U_n = 2R_n$ (m odd, n even).

Applying boundary conditions at $y = \pm \frac{a}{2}$ one can achieve

$$T_m = \frac{8\phi_0 \lambda}{\pi^2 \cosh(m\pi/2)} - \frac{4\lambda}{\pi \cosh(m\pi/2)} \sum_{n=2,4,..} U_n \cosh \left(\frac{n\pi}{2} \right) \frac{n}{n^2 - m^2} \quad (14)$$

$$\frac{-4\lambda}{\pi \sinh(n\pi/2)} \sum_{m=1,3,..} T_m \sinh \left(\frac{m\pi}{2} \right) \frac{m}{m^2 - n^2} \quad (15)$$

To get solutions for I_y and R we should appropriate partial derivatives of ϕ and get $J_{x,y}$ from E

$$J_x = \frac{1}{\rho_{xx}} \sum_{m=1,3,..} T_m \frac{m\pi}{a} \sin \left(\frac{m\pi x}{a} \right) \cosh \left(\frac{m\pi y}{a} \right) - \frac{1}{\rho_{xx}} \sum_{n=2,4,..} U_n \frac{n\pi}{a} \sin \left(\frac{n\pi x}{a} \right) \sinh \left(\frac{n\pi y}{a} \right) \quad (16)$$

$$\begin{aligned} J_y &= -\frac{2\phi_0}{a\rho_{xx}} - \\ &\frac{1}{\rho_{xx}} \sum_{m=1,3,..} T_m \frac{m\pi}{a} \cos \left(\frac{m\pi x}{a} \right) \sinh \left(\frac{m\pi y}{a} \right) - \frac{1}{\rho_{xx}} \sum_{n=2,4,..} U_n \frac{n\pi}{a} \cos \left(\frac{n\pi x}{a} \right) \cosh \left(\frac{n\pi y}{a} \right) \end{aligned} \quad (17)$$

In order to get total I_y we should count integral $I_y = \int_0^a J_y dx$. All harmonics except the constant first part of J_y will give zero impact. $I_y = \frac{2\phi_0}{\rho_{xx}}$. Remember that potential difference is $\Phi_0 = \phi_0 - (-\phi_0) = 2\phi_0$. Finally,

$$I_y = \frac{\Phi_0}{\rho_{xx}} \quad (18)$$

$$R = \rho_{xx} \quad (19)$$

It is often considered in textbooks that magnetic force (in Hall effect magnetic force results in $\rho_{xy} \neq 0$) is balanced by induced Hall potential and the current flows only parallel to the y axis. It causes the same answer for I_y and R . However, in general J has both non-zero components everywhere except the boundary.