Theoretical physics olympiad rehearsal

January 25, 2017

1 A rare bird

A bird of mass m sits at rest on a bank of a river of width L. Let's model its flight capacity by assuming that the bird can propel itself with force of magnitude at most $F(t)$, where $t = 0$ corresponds to the start of the flight, and $F(t)$ decreases with t as the bird gets tired. Additionally, it is affected by the drag force $F_d = -kv$, and of course gravity. What is the largest width of the river the bird can cross?

Give an explicit answer for $F = F_0(1 + t/t_0)^{-2}$ and a high-stamina species with $t_0 \gg m/k$.

2 Spin cluster, again

Hamiltonian of a Heisenberg spin cluster formed by N particles of spin 1/2 is given by

$$H = -\frac{J}{N}\sum_{i<j} \mathbf{s}_i \mathbf{s}_j, \qquad 1 \leqslant i, j \leqslant N.$$

We are interested in the dynamic correlation function $K(t) = \langle s_i^+(t) s_i^-(0)\rangle$ for some arbitrary spin s_i. Find the *exact* value of $K(t)$ for any N in the case of infinite temperature, when equidistribution amounts to $K(t) = \frac{1}{2^N}\operatorname{Tr}(s_i^+(t)s_i^-(0))$. Give an exact integral representation for $K(t)$ at any temperature T. As usual, closed forms should not involve sums of unbounded length.

3 Kronig Penney Klein Gordon

The space is filled with an array of small metallic spheres of radius ε each, arranged in a cubic lattice with lattice constant $a \gg \varepsilon$. Consider the Klein–Gordon (massive electrodynamics) field

$$-\Delta\Psi + \kappa^2\Psi = 0$$

in the situation when the sphere at $\mathbf{r} = 0$ carries unit charge and others are grounded. Find the asymptotic behavior of the solution at large r. How does it differ from the case of no grounded spheres?

What happens in the limit $\kappa a \ll 1$ and for $\kappa = 0$ in particular (the latter case being equivalent to usual electrodynamics)? Does the solution depend on the system size essentially, or there is a correct limit of an infinite system?

For all questions, consider dimensions $D = 2$ and $D = 3$.

4 Almost a phase transition

A long polymer chain put into a suitable solvent releases negatively charged ions and remains positively charged with linear density ρ. Due to Coulomb self-repulsion it straightens, and may be considered as an infinite rigid rod. Negative ions of charge $-q$ move freely in the solvent, which we otherwise assume to be electrically inert. If we view the system as a cylinder of radius R coaxial with the chain, local density $n(r)$ of negative ions as a function of temperature has singular behavior in the limit of large R. Find the critical temperature T_c in the mean field model. What is the exact value of total negative charge localized in the vicinity of the chain? How should one properly define vicinity?

Hint: although the relevant equations may be solved in closed form, it may be useful to consider negative ions in a solvent between two walls charged with surface densities σ_1 and σ_2 first.

5 Not a Dyson diffusion

A large number of particles in one dimension participate in diffusion processes $x_k(t)$ with diffusion coefficient D each, $k \in \mathbb{Z}$; the particles are impenetrable for each other, but otherwise do not interact. Initially the closest distances $d_k = x_{k+1}(0) - x_k(0)$ were random, independent and identically distributed with average $\langle d \rangle = a$ and variance σ^2.

Assume we follow the particle x_0, initially at $x_0(0) = 0$. Find A and B in the asymptotic behavior $\langle x_0^2(t) \rangle \sim At^B$ at large t. For partial credit, consider the lattice case $\sigma^2 = 0$.

6 Everything cancels

In a problem following §51 of "Statistical physics" by Landau and Lifshitz, rotational partition function of methane CH_4 is given as the first few terms in its low-temperature expansion,

$$Z = \frac{5}{16} + \frac{9}{16} e^{-\frac{\hbar^2}{IT}} + \frac{25}{16} e^{-3\frac{\hbar^2}{IT}} + \frac{77}{16} e^{-6\frac{\hbar^2}{IT}} + \frac{117}{16} e^{-10\frac{\hbar^2}{IT}} + \dots,$$

where I stands for the moment of inertia of the molecule. What are the next two terms?

Find the leading quantum correction to the classical rotational *heat capacity* of CH_4 in the high temperature limit $\hbar^2/IT \ll 1$.

7 No child left behind

Two coaxial solenoids at distance L from each other carry steady codirectional currents I each. The solenoids are identical with radius $R \ll L$, length much greater than L, and ρ turns per unit length. Additionally, a weak uniform electric field E is applied parallel to the common axis.

A particle of charge e and mass m starts its motion at the midpoint between solenoids, with velocity v forming angle $\frac{\pi}{2}(1-\varepsilon)$ with the axis, $\varepsilon \ll 1$. Describe the motion of the particle assuming that its cyclotron radius is much smaller than L.