Problem 2

Nagaoka bags

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In this problem we face *Hall effect*-like resistivity density tensor. In the steady state there is no charge density $\nabla E = 0$, so we have Laplace equation $\nabla \phi = E$

$$\nabla^2 \phi = 0 \tag{1}$$

Potential difference is between $x = 0$ and $x = a$ sides of square. Also, let's shift the y axis so corresponding square side changes from $[0, a]$ to $[-a/2, a/2]$.

1 Boundary conditions

Boundary conditions are

$$\begin{aligned} \phi(x, a/2) &= \phi_0 \\ \phi(x, -a/2) &= -\phi_0 \end{aligned} \tag{2}$$

and we should satisfy the requirment that no current leave through $x = 0$ and $x = a$ edges

$$\begin{aligned} E_x(0, y) &= \lambda E_y(0, y) \\ E_x(a, y) &= \lambda E_y(a, y), \\ \lambda &= \frac{\rho_{xy}}{\rho_{xx}} \end{aligned} \tag{3}$$

Since $E = -\nabla \phi$

$$\begin{aligned} \frac{\partial \phi}{\partial x}(0, y) &= \lambda \frac{\partial \phi}{\partial y}(0, y) \\ \frac{\partial \phi}{\partial x}(a, y) &= \lambda \frac{\partial \phi}{\partial y}(a, y) \end{aligned} \tag{4}$$

2 Solution

One of possible solutions to (1) is linear function $\phi(x, y) = (ax + b)(cy + d)$. However, only

$$\phi(x, y) = \frac{2\phi_0}{a} \left[\lambda \left(x + b\right) + y\right] \tag{5}$$

satisfies boundary conditions (4).
Then we assume separable solutions $\phi(x, y) = X(x)Y(y)$. (1) splits into

$$\frac{d^2 X}{dx^2} = -k^2 X, \frac{d^2 Y}{dy^2} = -k^2 Y \tag{6}$$

From boundaries (4) $X_0' Y = \lambda Y' X_0$ (subscript X_0 is for $x = 0$, $X' = \frac{dX}{dx}$). Then differentiate $X_0' Y' = \lambda Y'' X_0$ and substitute into (6)

$$X_0' Y' = \lambda k^2 Y X_0 \tag{7}$$

Eliminating Y' we have real k value

$$k^2 = \left(\frac{X_0'}{\lambda X_0}\right)^2 \tag{8}$$

That results in

$$\phi_k(x,y) = (A_k \cos kx + B_k sinkx)\left(C_k e^{kY} + D_k e^{-kY}\right) \tag{9}$$

Applying the (4) boundary conditions

$$\frac{B_k}{A_k} = \lambda \frac{C_k e^{kY} - D_k e^{-kY}}{C_k e^{kY} + D_k e^{-kY}} \tag{10}$$

Either $C_k = 0, B_k = -\lambda A_k$ or $D_k = 0, B_k = \lambda A_k$.
Most general is linear combination

$$\begin{aligned}\phi_k(x,y) = R_k\left(coskx + \lambda \sin kx\right)e^{ky} + S_k\left(coskx - \lambda \sin kx\right)e^{-ky},\\ R_k = A_k C_k, S_k = A_k D_k\end{aligned} \tag{11}$$

From boundaries (4) we have $\sin ka = 0 \to k_n = \frac{n\pi}{L}$ and

$$\begin{aligned}\phi(x,y) = \frac{2\phi_0}{a}\left[\lambda\left(x+b\right)+y\right] + \\ \sum_{n=1,2,3,..} cosk_n x\left(R_{kn}e^{k_n y} + S_{kn}e^{-k_n y}\right) + \lambda \sin k_n x\left(R_{kn}e^{k_n y} - S_{kn}e^{-k_n y}\right)\end{aligned} \tag{12}$$

From symmetry of equations (1), (4) we find $S_n = (-1)^{n+1}R_n$ and

$$\begin{aligned}\phi(x,y) = \frac{2\phi_0}{a}\left[\lambda\left(x-\frac{a}{2}\right)+y\right] + \\ \sum_{m=1,3..} T_m\left[\cos\left(\frac{m\pi}{a}x\right)\cosh\left(\frac{m\pi}{a}y\right) + \lambda\sin\left(\frac{m\pi}{a}x\right)\sinh\left(\frac{m\pi}{a}y\right)\right] + \\ \sum_{n=2,4..} U_n\left[\cos\left(\frac{n\pi}{a}x\right)\cosh\left(\frac{n\pi}{a}y\right) + \lambda\sin\left(\frac{n\pi}{a}x\right)\sinh\left(\frac{n\pi}{a}y\right)\right]\end{aligned} \tag{13}$$

where $T_m = 2R_m$ and $U_n = 2R_n$ (m odd, n even).
Applying boundary conditions at $y = \pm\frac{a}{2}$ one can achieve

$$T_m = \frac{8\phi_0\lambda}{\pi^2\cosh(m\pi/2)} - \frac{4\lambda}{\pi\cosh(m\pi/2)}\sum_{n=2,4,..} U_n \cosh\left(\frac{n\pi}{2}\right)\frac{n}{n^2-m^2} \tag{14}$$

$$\frac{-4\lambda}{\pi\sinh(n\pi/2)}\sum_{m=1,3,..} T_m \sinh\left(\frac{m\pi}{2}\right)\frac{m}{m^2-n^2} \tag{15}$$

To get solutions for I_y and R we should appropriate partial derivatives of ϕ and get $J_{x,y}$ from E

$$J_x = \frac{1}{\rho_{xx}}\sum_{m=1,3,..} T_m\frac{m\pi}{a}\sin\left(\frac{m\pi x}{a}\right)\cosh\left(\frac{m\pi y}{a}\right) - \frac{1}{\rho_{xx}}\sum_{n=2,4,..} U_n\frac{n\pi}{a}\sin\left(\frac{n\pi x}{a}\right)\sinh\left(\frac{n\pi y}{a}\right) \tag{16}$$

$$\begin{aligned}J_y = -\frac{2\phi_0}{a\rho_{xx}} - \\ \frac{1}{\rho_{xx}}\sum_{m=1,3,..} T_m\frac{m\pi}{a}\cos\left(\frac{m\pi x}{a}\right)\sinh\left(\frac{m\pi y}{a}\right) - \frac{1}{\rho_{xx}}\sum_{n=2,4,..} U_n\frac{n\pi}{a}\cos\left(\frac{n\pi x}{a}\right)\cosh\left(\frac{n\pi y}{a}\right)\end{aligned} \tag{17}$$

In order to get total I_y we should count integral $I_y = \int_0^a J_y dx$. All harmonics except the constant first part of J_y will give zero impact. $I_y = \frac{2\phi_0}{\rho_{xx}}$. Remember that potential difference is $\Phi_0 = \phi_0 - (-\phi_0) = 2\phi_0$. Finally,

$$I_y = \frac{\Phi_0}{\rho_{xx}} \tag{18}$$

$$R = \rho_{xx} \tag{19}$$

It is often considered in textbooks that magnetic force (in Hall effect magnetic force results in $\rho_{xy} \neq 0$) is balanced by induced Hall potential and the current flows only parallel to the y axis. It causes the same answer for I_y and R. However, in general J has both non-zero components everywhere except the boundary.