$$\left(\frac{m}{m+1}\right)^m = \left(1+\frac{m}{m+1}-1\right)^m = \left(1+\frac{(-1)}{m+1}\right)^m$$

$$= \left(\underbrace{\left(1+\frac{1}{\underbrace{-(m+1)}_{\to\infty}}\right)^{-(m+1)}}_{\to e}\right)^{\frac{m}{-(m+1)}} \longrightarrow e^{\lim\limits_{m\to\infty}\frac{m}{-m-1}} = e^{-1} = \frac{1}{e}$$

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exemplu serie $\quad \sum_{m=1}^{\infty} = \underbrace{\frac{1}{m^3-m+1}}_{=x_m} = ?$

Considerăm $\quad \sum_{m=1}^{\infty} \underbrace{\frac{1}{m^3}}_{=y_m} =$ convergentă

$$\sum_{m=1}^{\infty} = \frac{1}{m^p} = \begin{cases} \text{conv} & \text{pt } p>1 \\ \text{div} & \text{pt } p\le 1 \end{cases}$$

Seria armonică generalizată

$\sum_{m=1}^{\infty} \frac{1}{m}$, $\quad \sum_{m=1}^{\infty} \frac{1}{m^2}$

DIVERGENTĂ $\qquad$ CONVERGENTĂ

Criteriul de divergență: Dacă $x_m \not\xrightarrow{m\to\infty} 0$ nu tinde

atunci $\sum_{m=0}^{\infty} x_m =$ DIV

$$l = \lim_{n\to\infty} \frac{x_n}{y_n} = \lim_{n\to\infty} \frac{n^3}{n^3-n+1} = 1 \in (0,\infty)$$

$$\xrightarrow[\text{forma la limită}]{\text{CRIT. COMP}} \underbrace{\sum_{n=1}^{\infty} \frac{1}{n^3-n+1}}_{\text{conv}} \sim \sum_{n=1}^{\infty} \frac{1}{n^3} = \text{CONV}$$

<u>not</u> x_n

<u>exemplul 2</u>

$$\sum_{n=1}^{\infty} \frac{\sqrt{n^3}+1}{n^5-2n+2}$$

$$\sum_{n=1}^{\infty} y_n = \sum_{n=1}^{\infty} x_n = \frac{n^{\frac{3}{2}}+1}{n^5-2n+2} = \sum_{n=1}^{\infty} \frac{1}{n^{\frac{7}{2}}}$$

$$5-\frac{3}{2} = \boxed{\frac{7}{2}}$$

$$l = \lim_{x\to\infty} \frac{x_n}{y_n} = \lim_{n\to\infty} \frac{n^{\frac{3}{2}}+1}{n^5-2n+2} \cdot n^{\frac{7}{2}} =$$

$$= \lim_{n\to\infty} \frac{n^5+n^{\frac{7}{2}}}{n^5-2n+2} = 1$$

$$l = 1 \in (0,\infty) \xrightarrow[\text{la limită}]{\text{CRIT COMP}} / \sum \frac{\sqrt{n^3}+1}{n^5-2n+2} = \text{CONV}$$

<u>3 Criteriul Raportului (D'Alambert)</u>

Fie $\sum_{n=0}^{\infty} x_n$, cu $x_n > 0$ și $l = \lim_{n\to\infty} \frac{x_{n+1}}{x_n}$

(i) Dacă $l < 1 \Rightarrow \sum_{n=1}^{\infty} x_n = \text{CONV}$

(ii) Dacă $l > 1 \Rightarrow \sum_{n=1}^{\infty} x_n = \text{DIV}$

ex $\sum_{m=1}^{\infty} \frac{2^m}{m!} = ?$

$\frac{2^m}{m!} = x_m$

$$l = \lim_{m\to\infty} \frac{x_{m+1}}{x_m} = \lim_{m\to\infty} \frac{2^{\cancel{m+1}}}{\cancel{(m+1)!}} \cdot \frac{\cancel{m!}}{\cancel{2^m}} =$$

$$= \lim_{m\to\infty} \frac{2}{m+1} = 0 < 1 \; !$$

$\xRightarrow[(i)]{\text{CRIT RAP}} \sum \frac{2^m}{m!} =$ CONVERGENTĂ

4. Criteriul lui RAABE - DUHAMEL

Fie $\sum_{m=0}^{\infty} x_m, \quad x_m > 0$

Calculați: $L = \lim_{m\to\infty} m \cdot \left(\frac{x_m}{x_{m+1}} - 1\right)$

(i) Dacă $L < 1 \Rightarrow \sum_{m=0}^{\infty} x_m =$ DIV

(ii) Dacă $L > 1 \Rightarrow \sum_{m=0}^{\infty} x_m =$ CONV

ex 4 $\sum_{m=1}^{\infty} \frac{(2m-1)!!}{(2m)!!} = \sum_{m=1}^{\infty} \frac{1\cdot 3\cdot 5 \cdot \_\_\_ \cdot (2m-1)}{2\cdot 4\cdot 6 \cdot \_\_\_ \cdot (2m)}$

$$l = \lim_{m\to\infty} \frac{x_{m+1}}{x_m} = \lim_{m\to\infty} \frac{\cancel{1\cdot 3\cdot 5 \cdot \_\_\_ \cdot (2m-1)}(2m+1)}{\cancel{2\cdot 4\cdot 6\cdot \_\_\_ \cdot (2m)}(2m+2)} \cdot$$

$$\cdot \frac{\cancel{2\cdot 4\cdot 6 \cdot \_\_\_ \cdot (2m)}}{\cancel{1\cdot 3\cdot 5\cdot \_\_\_ \cdot (2m-1)}} = \lim_{m\to\infty} \frac{2m+1}{2m+2} = 1$$

$\Rightarrow$ CRIT. RAP. nu decide natura seriei

$$L = \lim_{n\to\infty} n\cdot\left(\frac{x_n}{x_{n+1}} - 1\right) = \lim_{n\to\infty} n\cdot\left(\frac{2n+2}{2n+1} - \frac{2n+1}{1}\right)$$

$$= \lim_{n\to\infty} n\cdot\frac{1}{2n+1} = \frac{1}{2} < 1 \xRightarrow{R-D}$$

$$\sum_{n=1}^{\infty} \frac{(2n-1)!!}{(2n)!!} = \text{DIV}$$

5. Criteriul rădăcinii (radicalului)

Fie $\sum_{n=0}^{\infty} x_n$, $x_n > 0$. Calculăm $l = \lim_{n\to\infty} \sqrt[n]{x_n}$.

(i) Dacă $l < 1 \Rightarrow \sum_{n=1}^{\infty} x_n = \text{CONV}$

$\boxed{(ii)}$ Dacă $l > 1 \Rightarrow \sum_{n=1}^{\infty} x_n = \text{DIV}$

ex 5 $\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^{n^2}$ $= x_n$

$$l = \lim_{n\to\infty} \sqrt[n]{x_n} = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e > 1$$

$$\xRightarrow[\text{răd (ii)}]{\text{crit}} \sum \left(1+\frac{1}{n}\right)^{n^2} = \text{DIV}$$

6. Criteriul de CONDENSARE a lui CAUCHY

Fie $(x_n)_{n\geq 0}$ un șir de nr. reale pozitive care este descrescător.

Seriile $\sum_{n=0}^{\infty} x_n$ și $\sum_{n=0}^{\infty} 2^n \cdot x_{2^n}$ au aceeași NATURĂ!

ex.6 $\sum_{n=1}^{\infty} \frac{\ln n}{n} = x_n \xrightarrow{n\to\infty} 0$

Se obs că x_n

$$\sum_{n=1}^{\infty} 2^n \cdot x_{2^n} = \sum_{n=1}^{\infty} \cancel{2^n} \cdot \frac{\ln 2^n}{\cancel{2^n}} =$$

$$= \sum_{n=1}^{\infty} n \cdot \ln 2 = \ln 2 \cdot \sum_{n=1}^{\infty} n = \text{DIV}$$

ex7. $\sum_{n=1}^{\infty} \frac{1}{n^p} = \begin{cases} \text{conv pt } p>1 \\ \text{div pt } p \le 1 \end{cases}$

(i) pt $p \le 0 \Rightarrow x_n \not\xrightarrow{n\to\infty} 0 \Rightarrow \sum_{n=1}^{\infty} x_n = \text{DIV}$

(ii) pt $p > 0$ avem că $x_n = \frac{1}{n^p}$ este șir descrescător $(x_n \ge x_{n+1})$

$$\cdot \sum_{n=1}^{\infty} 2^n \cdot x_{2^n} = \sum_{n=1}^{\infty} 2^n \cdot \frac{1}{(2^n)^p} = \sum_{n=1}^{\infty} \frac{1}{(2^n)^{p-1}} =$$

$$= \sum_{n=1}^{\infty} \underbrace{\left(\frac{1}{2^{p-1}}\right)}_{=q}{}^n = \text{CONV} \Leftrightarrow$$

$$q = \frac{1}{2^{p-1}} \in (-1,1) \Rightarrow \frac{1}{2^{p-1}} < 1$$

$$= \frac{1}{2^0} \iff p-1>0 \iff p>1$$

$$\xrightarrow{\text{CRIT CONV}} \sum_{n=1}^{\infty} \frac{1}{n^p} = \text{CONV pt. } p>1$$

Seria geometrică

$$\sum_{n=0}^{\infty} q^n = \text{CONV pt } q \in (-1,1)$$

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§ Serii cu termeni oarecare

① Criteriul lui Dirichlet

Fie $\sum x_n = \sum_{n=0}^{\infty} a_n \cdot u_n$; $(a_n)_{n\geq 0}$, $(u_n)_{n\geq 0}$ = șiruri numerice

Dacă: (i) $(a_n)_{n\geq 0}$ șir descrescător de nr. reale pozitive.

(ii) $(u_n)_{n\geq 0}$ are proprietatea că șirul sumelor parțiale

$$\underbrace{t_n := u_0 + u_1 + \dots + u_n}$$ este

mărginit ($\exists M>0$ a.î. $|t_n| < M, \forall n$),

atunci $\sum_{n=0}^{\infty} a_n \cdot u_n = \text{CONV}$

ex $\sum_{n=1}^{\infty} \frac{\sin(nx)}{n}$, $x \in \mathbb{R}$

Notăm: $a_n = \frac{1}{n} \xrightarrow{n \to \infty} 0$ (i) ✓

$$u_n = \sin(nx)$$

$$t_n = u_1 + u_2 + \_\_\_ + u_n = \sin(x) + \sin(2x) + \_\_\_ + \sin(nx) =$$

$$= \boxed{\frac{\sin\left(\frac{nx}{2}\right) \cdot \sin\frac{(n+1)x}{2}}{\sin\left(\frac{x}{2}\right)}}, \quad x \neq 2k\pi, \ k \in \mathbb{Z}$$

$$|t_n| = \frac{\left|\sin\frac{nx}{2}\right| \cdot \left|\sin\frac{(n+1)x}{2}\right|}{\left|\sin\frac{x}{2}\right|} \leq \frac{1}{\left|\sin\frac{x}{2}\right|} \overset{not}{=} M \quad \text{(ii) ✓}$$

$$\xRightarrow[\text{DIRICHLET}]{\text{CRIT}} \sum_{n=1}^{\infty} \frac{1}{n} \cdot \sin(nx), \quad x \neq 2k\pi \text{ este convergentă.}$$

$$\cos x + \cos(2x) + \_\_\_ + \cos(nx) = \frac{\sin\left(\frac{nx}{2}\right) \cdot \cos\frac{(n+1)}{2}}{\sin\frac{x}{2}}, \quad x \neq 2k\pi, \ k \in \mathbb{Z}$$

$$= \frac{\sin\left(\frac{nx}{2}\right) \cdot \cos\frac{(n+1) \cdot x}{2}}{\sin\frac{x}{2}}$$

CRITERIUL lui LEIBNIZ

Dacă $(a_n)_{n \geq 0}$ = șir de nr. reale pozitive

și $a_n \xrightarrow{n \to \infty} 0$, atunci $\sum_{n=0}^{\infty} (-1)^n \cdot a_n$ = CONV ex 2 $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ = CONV. pt. că $a_n = \frac{1}{n} \searrow 0$

Definiție! • $\sum x_n$ s.n absolut conv. dacă

$\sum_{n=0}^{\infty} |x_n|$ = CONV.

• $\sum_{n=0}^{\infty} x_n$ s.n semiconv. dacă este

CONV și nu e absolut conv.

ex: $\sum_{n=1}^{\infty} \underbrace{\frac{(-1)^n}{n^2}}_{= x_n}$ = CONV. (folosind Crit. lui Leibniz)

$\sum_{n=1}^{\infty} |x_n| = \sum_{n=1}^{\infty} \frac{1}{n^2}$ = CONV

$\Rightarrow$ Absolut convergentă

$\rightarrow$ Însă $\sum |x_n| = \sum \frac{1}{n} \Longrightarrow$ DIV