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A Closed-Form Execution Strategy to Target Volume Weighted Average Price*

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Abstract. We provide two explicit closed-form optimal execution strategies to target volume weighted average price (VWAP). We do this under very general assumptions about the stochastic process followed by the volume traded in the market, and, unlike earlier studies, we account for permanent price impact stemming from order-flow of the agent and all other traders. One of the strategies consists of time weighted average price adjusted upward by a fraction of instantaneous order-flow and adjusted downward by the average order-flow that is expected over the remaining life of the strategy. The other strategy consists of the Almgren–Chriss execution strategy adjusted by the expected volume and net order-flow during the remaining life of the strategy. We calibrate model parameters to five stocks traded in Nasdaq (FARO, SMH, NTAP, ORCL, INTC) and use simulations to show that the strategies target VWAP very closely and on average outperform the target by between 0.10 and 8 basis points.

Key words. VWAP, POV, TWAP, algorithmic trading, high-frequency trading, acquisition, liquidation

AMS subject classifications. 91G80, 49L20, 60G55, 60G99

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1. Introduction.

How to optimally execute a large order has been, and continues to be, a very active field of research. Set against the backdrop of electronic markets and the rise of computerized trading, algorithmic trading has become an industry where brokerage firms compete to offer the best execution strategies. Typically, algorithms break large orders into smaller child orders and these are sequentially executed over a trading window. Everything else equal, the longer the trading window, the less market impact the child orders will have but the more uncertain the execution prices become; see [3], [2], [10], and [14]. Choosing the length of the execution window to complete the transaction is an important decision, but once the execution program is over, how can the performance of these algorithms be measured?

One of the most widely used benchmarks in algorithmic trading in general, and optimal execution in particular, is the volume weighted average price (VWAP) (see section 3.3.1 in [19] for other benchmarks). This benchmark is straightforward to calculate (ex-post) and is attractive to investors because execution prices close to VWAP indicate that they obtained prices similar to what the market bore over the execution window. Although this transparency and ease of ex-post computation make it a desirable benchmark to target with an execution

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algorithm, its implementation is challenging because volume of trades along the trading horizon is not known in advance. Total daily volume is indeed random, and although it is well known that the intraday pattern is U-shaped (more activity at the start and the end of the trading day), there is much variability from one day to the next and large fluctuations intraday.

In this paper we provide execution algorithms designed to target VWAP, which is done by continuously tracking and trading a percentage of the volume traded (POV) in the market. We assume a general stochastic process for volume and provide a closed-form expression for the optimal execution strategy. To the best of our knowledge, this is the first time in the literature that an execution algorithm targeting VWAP is provided under such general assumptions for the volume process and, more importantly, the strategy is found in closed-form and the result of dynamic optimization. Furthermore, the optimal strategy accounts for the impact that the agent's trading has through both temporary and permanent price impact. When the agent trades, she receives worse prices than the midprice (temporary price impact), and the midprice is pushed in the direction of her trades (permanent price impact), up if she buys and down if she sells. In addition, unlike many earlier works, we treat the order-flow from other traders in a similar fashion to the agent's own trades. More specifically, we account for the permanent impact that other traders have when submitting orders to the market, so that if total order-flow (from the agent and other traders) is buy (sell) heavy, then prices are pushed upward (downward).

The first academic paper that derived a static execution strategy to target VWAP is [17]. This was followed by [5], where they propose a methodology for intraday volume which is then used to target VWAP. The paper [16] proposes a way to incorporate intraday noise into a VWAP trading rule and later [20] formulate a mean-variance optimal VWAP strategy, where it is assumed that final (total) volume is known ahead of time.

Another recent strand in the literature uses optimal control techniques to devise trading algorithms to target VWAP. Frei and Westray [12] consider the optimal liquidation of a large position where the investor's trades have temporary market impact. They develop an algorithm to minimize the mean and variance of the order slippage with respect to VWAP where it is assumed that the total volume (not the trajectory) traded during the execution is known in advance. In a similar vein, [15] develops an algorithm for a broker who guarantees VWAP to an investor and the rate at which shares are executed have permanent market impact. The authors use a performance criteria based on exponential utility, where the utility of the deviation of the investor's execution cost and that of a VWAP strategy are optimized. Although the authors focus on the case when the volume trajectory is deterministic, they provide a brief discussion on how their approach would change if volume is assumed stochastic. Finally, [21] obtain a closed-form discrete-time strategy where the investor tracks the market VWAP using market orders (MOs). Their model allows codependence between stock price innovations and volume dynamics and they perform an extensive data analysis. However, the investor's MOs do not impact the midprice and the intraday midprice is a martingale conditional on the realized stock volume. In all of these prior works, the effects of other agents' trading on prices are ignored.

The remainder of the paper is structured as follows. In section 2 we use Nasdaq data to show volume patterns, estimate temporary and permanent impact of trading activity, and

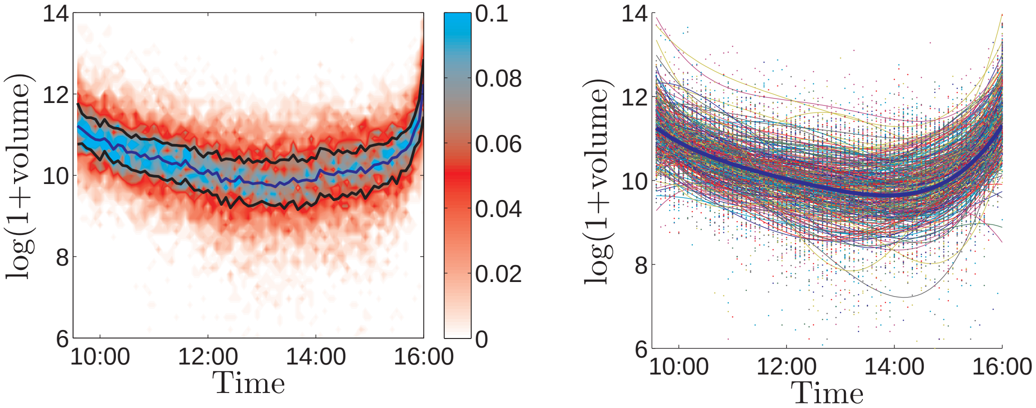

Figure 1. *ORCL traded volume for orders sent to Nasdaq in all of* 2013 *using* 5*-minute buckets: (left) a heat-map of the data and* 25%*,* 50%*, and* 75% *quantiles; (right) functional data regression (using Legendre polynomials) on the daily curves and the estimated mean curve.*

discuss the link between arrival of trades and VWAP. In section 3 we present the general framework and solve the control problem explicitly. In section 4 we use data for five stocks traded in Nasdaq to calibrate a model of volume dynamics and use simulations to show the performance of the VWAP strategy. Finally, section 5 concludes, and we collect proofs in the appendix.

2. Order-flow, price impact, and targeting VWAP.

2.1. Empirical facts: Order-flow and price impact. Traded volume generally follows a U-shaped pattern where at the beginning and the end of the trading day there is more trading than throughout the rest of the day. However, as Figure 1 shows, there is a considerable amount of deviation from this pattern.[1] The figure uses millisecond-stamped messages from Nasdaq ITCH to show the estimate of the volume rate of trading (per 5 minutes) for ORCL for all days of 2013. Here the variable volume amalgamates, buys, and sell orders and we use $\log(1 + \text{volume})$ because there are intervals with no trades. The left panel shows a heat-map of the daily traded volume rates, while the right panel shows the raw data (fine dots) together with regressions of a given day onto a Legendre polynomial basis up to order 4. The thick blue line on the right panel is the mean of every regression curve.[2] In both panels we observe the well-known U-shape of trading volume, but it is also clear that volume is a volatile quantity which must be modeled as a key ingredient of a VWAP algorithm.

Traded volume patterns and the arrival of MOs in particular are important in the design of algorithms that target volume-based benchmarks. In addition, paramount to these algorithms is the inclusion of the effect that trading has on midprices. As new information arrives in the market this is impounded in the midprice of the assets by traders who execute MOs and/or traders who reposition their limit orders (LOs). MOs can have a temporary or permanent

[1] See [19] for volume curves across different markets and locations.

[2] That is, we use a functional data analysis perspective and view the traded volume each day as being drawn from a probability space on the set of functions.

Table 1

Permanent and temporary price impact parameters for Nasdaq stocks, average volume of MOs, average midprice, σ volatility (hourly) of arithmetic price changes, mean arrival (hourly) of MOs $\lambda^{\pm}$, and average volume of MOs $\mathbb{E}[\eta^{\pm}]$. Data are from Nasdaq 2013.

	FARO		SMH		NTAP		ORCL		INTC	
	Mean	stdev	Mean	stdev	Mean	stdev	Mean	stdev	Mean	stdev
$\hat{b}$	1.41E-04	9.61E-05	5.45E-06	4.20E-06	5.93E-06	2.31E-06	1.82E-06	7.19E-07	6.15E-07	2.16E-07
$\hat{k}$	1.86E-04	2.56E-04	8.49E-07	8.22E-07	3.09E-06	1.75E-06	8.23E-07	3.78E-07	2.50E-07	1.25E-07
ADV	23,914	14,954	233,609	148,580	1,209,628	642,376	3,387,954	1,889,322	5,947,502	2,271,967
Midprice	40.55	6.71	37.90	2.44	38.33	3.20	33.67	1.43	23.04	1.34
σ	0.151	0.077	0.067	0.039	0.078	0.045	0.054	0.012	0.039	0.010
λ^+	16.81	9.45	47.29	28.13	300.52	144.48	345.94	151.10	333.23	127.67
$\mathbb{E}[\eta^+]$	103.56	21.16	377.05	118.05	308.45	53.09	746.70	188.31	1,443.60	317.15
λ^-	17.62	10.69	46.37	27.62	293.83	136.13	332.49	158.57	322.09	134.30
$\mathbb{E}[\eta^-]$	104.00	21.79	381.70	126.74	312.81	49.86	788.35	187.78	1,461.86	312.63

impact on the midprice. MOs that walk the limit order book (LOB) have price impact because they obtain average execution prices worse than the best quote and are considered temporary because it is assumed that the LOB will replenish very quickly and, after restocking, there is no effect on the midprice of the asset. On the other hand, investors' MO activity has a permanent effect on midprices because traders' order-flow conveys information that affects the midprice of the asset. In general, when there is positive net order-flow (more buy than sell MOs) midprices tend to drift up, and when there is negative net order-flow (more sell than buy MOs) the midprice tends to drift down.

In Table 1 we show parameter estimates for permanent and temporary price impact for five stocks using data from Nasdaq for the year 2013. In the first row we show the parameter estimate for permanent price impact. We assume a linear relationship between net order-flow and changes in the midprice; thus for every trading day we perform the regression

$$\Delta S_n = b\,\mu_n + \varepsilon_n\,, \tag{1}$$

where $\Delta S_n = S_{n\tau} - S_{(n-1)\tau}$ is the change in the midprice, μ_n is net order-flow defined as the difference between the volume of buy and sell MOs during the time interval $[(n-1)\tau,\, n\tau]$, and ε_n is the error term (assumed normal). In the empirical analysis we choose $\tau = 5\,\text{min}$.[3] The table shows the mean and standard deviation of the daily estimate for b by first Winsorizing the data to exclude the upper and lower 0.5% tails and then carrying out a robust linear regression on the model (1). Furthermore, Table 2 shows summary statistics for the statistical significance of b and R^2's of model (1).

In the second row of the Table 1 we show the parameter estimate for temporary impact. To estimate this parameter, which we denote by k, we assume that temporary price impact is linear in the rate of trading so the difference between the execution price that the investor

[3]See [4] for a discussion on linear market impact using proprietary execution data.

Table 2
Linear permanent model fit diagnostics.

	FARO	SMH	NTAP	ORCL	INTC
Median p-value	4.30×10^{-5}	1.76×10^{-2}	4.75×10^{-7}	2.39×10^{-9}	6.76×10^{-13}
% days p-value < 0.1	91.6%	71.7%	98.8%	99.2%	99.6%
% days p-value < 0.05	89.6%	60.6%	98.0%	98.4%	99.6%
Median R^2	0.40	0.15	0.45	0.52	0.66

receives and the midprice is $k\,\nu$, where ν is the speed of trading. To do this we take a snapshot of the LOB each second, determine the price per share for various volumes (by walking through the LOB), compute the difference between the price per share and the best quote at that time, and perform a linear regression. The slope of the linear regression is an estimate of the temporary price impact per share at that time. We do this for every second of every trading day and in the table we report the mean and standard deviation of these daily estimates when we exclude the first and last half-hour of the trading day and Winsorize the data.

The table also reports the average daily volume (ADV) of shares for each stock, the average midprice, and the volatility of price changes. In addition, we report the average number of buy and sell MOs, λ^+ and λ^-, respectively, and the mean volume of MOs, $\mathbb{E}[\eta^+]$ and $\mathbb{E}[\eta^-]$, respectively. For example, in Nasdaq, ORCL receives on an hourly basis 346 market buy orders, with an average of 747 shares per order.

2.2. Targeting VWAP. It is common for investors to seek average execution prices which are close to what the market has borne during the execution horizon. This widely pursued objective is achieved by targeting VWAP, which, as mentioned above, is very simple to compute ex-post and has the added benefit that it can also be employed by the investor to measure the performance of the broker to whom she delegated the execution program. When a broker outperforms VWAP, and the order was small enough not to drive VWAP, it can be ascribed to the broker's ability and not to good fortune as a result of favorable trends in prices.

In addition to targeting VWAP, investors reduce the market impact of their own orders by breaking the parent order into smaller child orders which are executed over a trading window. These two objectives explain why at the core of any algorithm that targets VWAP is the choice of how many shares are executed at each point of the trading horizon. Intuitively, if the market trades at a constant volume rate, then smoothing the parent order's volume evenly would result in a constant rate of execution and the average execution price would, in the absence of temporary impact, match VWAP. Such a simple strategy would, however, not necessarily match the total volume the agent seeks. Further, if volume is stochastic, it seems reasonable to spread the volume of the parent order so that the execution rate mimics that of the overall market—when the market volume rate is high (low), the algorithm should trade more (less) so the volume of the child orders goes in tandem with the market. Thus, VWAP strategies are deeply connected to simpler algorithmic strategies which are designed to track POV in the market but differ in that under VWAP the investor specifies a number of shares to be executed and also specifies the trading horizon.

A related problem is addressed by [13], which discusses execution and block trade with optimal constant rate of participation. In [8] we study how order-flow affects execution strategies. See also [18], which discusses optimal trading using stochastic approximation methods, and [9], which studies optimal trading when the agent learns from order-flow and price movements.

Algorithms that track VWAP and those that target POV are closely interlinked because the investor also requires minimizing market impact. Reducing price impact forces the algorithm to slice the parent order over a time window, and doing it in a volume-based fashion will help to target VWAP. Our formulation is to set up a performance criteria where the investor seeks to execute a large order over a trading horizon T and the speed of trading targets POV or targets a percentage of cumulative volume (POCV). Once we derive the optimal speed of trading for these two general cases, we show how these strategies include optimal execution strategies that target and achieve VWAP.

If the investor were tracking precisely the POV, then her average execution price per share would equal VWAP. However, if she did track the POV, and hence attain VWAP, she would not liquidate all of her shares, or she would have liquidated more than her initial amount of shares—unless her initial inventory happens to equal a fixed percentage (equal to the POV target) of the total shares traded in the market.

3. The model.

3.1. Targeting POV. In this section we assume that the investor's execution strategy targets a percentage of the speed at which other market participants are trading and focus on the liquidation strategy with MOs only. The setup for optimal acquisition is very similar. In the liquidation problem, the investor searches for an optimal speed to liquidate $\mathfrak{N}$ shares over a trading horizon T. To achieve this she must choose a liquidation strategy, which is denoted by $\nu = \{\nu_t\}_{0\le t\le T}$, to draw down her controlled inventory $Q^\nu = \{Q^\nu_t\}_{0\le t\le T}$ according to

$$dQ^\nu_t = -\nu_t\,dt\,, \qquad Q^\nu_0 = \mathfrak{N}\,. \tag{2}$$

The probability space and, more specifically, the filtration on which the strategies ν are adapted to, and the set of admissible strategies will be provided below.

The investor also requires her liquidation strategy to target a fraction $\tilde\rho \in [0,1]$ of the speed at which the overall market is trading, i.e., the investor targets order-flow. Here we assume that other market participants are also sending buy and sell MOs at overall speeds of $\mu^+ = \{\mu^+_t\}_{0\le t\le T}$ and $\mu^- = \{\mu^-_t\}_{0\le t\le T}$, respectively. At each instant in time the investor therefore targets a percentage of the total, including her own, trading rate (also referred to as total order-flow, or simply order-flow) given by

$$\chi^\nu_t := \tilde\rho \times \left(\mu^+_t + \mu^-_t + \nu_t\right)\,. \tag{3}$$

We could just as easily target instead the one-sided order-flow, $\tilde\rho \times (\mu^- + \nu_t)$; however, most market participants target total order-flow. This target will show up as a penalty term in the investor's performance criteria shown below in (7).

The investor's MOs have a temporary impact on the asset's price because there might not be enough liquidity posted at the best quote, hence her orders may walk the LOB, and she receives an execution price which is worse than the midprice. Even if she breaks up her trades into small buckets that do not take the full liquidity at-the-touch, she will still receive worse than the best quoted prices since high-frequency traders will shuffle their LOs accordingly. To simplify the modeling, we assume that when the investor sells shares these are executed at a price $\hat{S}^\nu = \{\hat{S}^\nu_t\}_{0\le t\le T}$, where

$$\hat{S}^\nu_t = S^\nu_t - k\,\nu_t\,, \tag{4}$$

$k > 0$ is the temporary impact parameter, and $S^\nu = \{S^\nu_t\}_{0\le t\le T}$ is the asset's midprice.

In principle nonlinear temporary impact functions could be applied (see, e.g., [1] and [22]); however, the model then fails to be analytically tractable. Moreover, as [12] points out, given the extremely low predictive accuracy of market impact models (typically $< 5\%\ R^2$), the cost of increased complexity arising from moving away from a linear model would outweigh any gains from better describing market impact. Although numerical methods can still be applied in the nonlinear case, we opt to leave it out of the discussion in this article and focus instead on the effect of order-flow and how we can target VWAP.

We also assume that all MOs (including other traders' and the agent's) have a permanent effect on the midprice: sell MOs induce a downward trend in the midprice, and buy MOs induce an upward trend in the midprice so that the (controlled) midprice satisfies the stochastic differential equation (SDE)

$$dS^\nu_t = b\left(\mu^+_t - (\nu_t + \mu^-_t)\right)\,dt + dM_t\,, \qquad S^\nu_0 = S\,, \tag{5}$$

where $b \ge 0$ is the permanent impact parameter, and $M = \{M_t\}_{0\le t\le T}$ is a martingale (independent of all other processes[4]) which captures revisions to the midprice that come from changes in the LOB such as additions and cancellations of LOs. Here the permanent price impact is assumed the same across all types of investors, but this is not necessarily always the case. For example, if the agent has superior information and other traders are able to detect these informed trades, the agent's trading activity will have a stronger (i.e., larger than b) effect on the trend of the asset. In our setup it is trivial to include a specific price impact parameter for the agent's trades, but for parsimony we assume the same parameter for all market participants.

We assume that the processes M and $\mu^\pm$ are Markov, but not necessarily Markov in M and $\mu^\pm$ solely—there may be additional processes that one needs to include to make the processes Markov (e.g., order-flow may be driven by a number of factors)—and we suppress the dependence on these additional factors, as they do not play a role in the analysis, other than to alter the explicit computation of expectations, such as $\mathbb{E}[\,\mu^\pm_s\,|\,\mathcal{F}_t\,]$ (which appear in the optimal solution), under particular modeling assumptions.

We work on the completed and filtered probability space $(\Omega, \mathbb{P}, \mathbb{F} = \{\mathcal{F}_t\}_{0\le t\le T})$, where $\mathcal{F}_t$ is the natural filtration generated by (M, μ^+, μ^-), and further assume that μ^+ and μ^- are cadlag with finite second moment (i.e., that $\mathbb{E}[(\mu^\pm_s)^2|\mathcal{F}_t] < +\infty$ for $t \le s \in [0,T]$). For example, they could be independent Ornstein–Uhlenbeck processes reverting to zero driven

[4]There could in fact be dependence between M and $\mu^\pm$, e.g., increased order-flow could induce increases in midprice volatility; however, such dependence structures do not alter the final form of the optimal strategy, other than the filtration which one must use in computing certain expectations. To keep the exposition compact, we opt to leave this out of the discussions.

by a compound Poisson process with finite second moment jump size. In section 4 we present a version of such a model.

Our model for the midprice captures two stylized facts that we observe in the market. First, the drift term $b\left(\mu_t^+ - (\nu_t + \mu_t^-)\right) dt$ shows that when there is one-sided buying or selling pressure, the midprice will trend up or down (see [7], [11], and [6]); see also the estimation results in Table 1. Second, in a balanced market, i.e., when the rate of arrival of buys and sells is the same (including the investor's sell orders), there will be no trend and the midprice innovations come solely from the increments of the martingale component.

Finally, the investor's cash process $X^\nu = \{X_t^\nu\}_{0\le t\le T}$, which represents the proceeds from liquidating shares, satisfies

$$dX_t^\nu = \hat{S}_t^\nu\, \nu_t\, dt\,, \qquad X_0^v = X_0\,. \tag{6}$$

3.2. Performance criteria. The investor aims to maximize wealth from selling shares but penalizes strategies that deviate from her target trading rate (in order for her to achieve VWAP). Hence, we set her performance criteria to be

$$H^\nu(t,x,S,\boldsymbol{\mu},q) = \mathbb{E}_{t,x,S,\boldsymbol{\mu},q}\left[X_T^\nu + Q_T^\nu\,(S_T^\nu - \alpha\, Q_T^\nu) - \tilde{\varphi}\int_t^T (\nu_u - \chi_u^\nu)^2\, du\right], \tag{7}$$

where $\boldsymbol{\mu} = \{\mu^+, \mu^-\}$, and $\mathbb{E}_{t,x,S,\boldsymbol{\mu},q}[\,\cdot\,]$ represents expectation conditioned on (with a slight abuse of notation) $X_t = x$, $S_{t-} = S$, $\mu_{t-}^+ = \mu^+$, $\mu_{t-}^- = \mu^-$, and $Q_t = q$, and her value function

$$H(t,x,S,\boldsymbol{\mu},q) = \sup_{\nu\in\mathcal{A}} H^\nu(t,x,S,\boldsymbol{\mu},q)\,, \tag{8}$$

where $\mathcal{A}$ is the set of admissible strategies consisting of $\mathcal{F}$-predictable processes such that $\mathbb{E}[\int_0^T |\nu_u|\, du] < +\infty$.

On the right-hand side of the performance criteria (7) are three terms. The first is the investor's terminal cash from liquidating the shares throughout the trading horizon. The second is the cost incurred by the investor when liquidating any remaining inventory Q_T^ν at the end of the strategy. The terminal inventory is liquidated at the midprice S_T^ν and the costs associated to crossing the spread, liquidity taking fees, and market impact, all of which are captured by the liquidation penalty parameter $\alpha \ge 0$. Finally, the third term is the running penalty $\tilde{\varphi}\int_t^T (\nu_u - \chi_u^\nu)^2\, du$ that the investor receives if the strategy deviates from the target χ_t^ν , where $\tilde{\varphi} \ge 0$ is the target penalty parameter. This penalization curbs the liquidation strategy—high values of $\tilde{\varphi}$ constrain the strategy to closely track the target χ_t^ν at every instant in time, and low values of $\tilde{\varphi}$ result in liquidation strategies which are more lax about tracking the POV target.

3.3. Solving the dynamic programming equation. Applying the dynamic programming principle suggests that the value function should satisfy the dynamic programming equation (DPE)

$$\begin{aligned} 0 &= \left(\partial_t + \mathcal{L}^M + \mathcal{L}^{\boldsymbol{\mu}}\right) H \\ &\quad + \sup_\nu \left\{ (S - k\,\nu)\,\nu\,\partial_x H - \nu\,\partial_q H + b\left((\mu^+ - \mu^-) - \nu\right)\partial_S H - \varphi\,(\nu - \rho\,\mu)^2 \right\} \end{aligned} \tag{9}$$

with terminal condition

$$H(T,x,S,\boldsymbol{\mu},q) = x + q\,(S - \alpha\, q)\,, \tag{10}$$

where $\mu = \mu^+ + \mu^-$, $\mathcal{L}^{\boldsymbol{\mu}}$ represents the infinitesimal generator of $\boldsymbol{\mu}$, $\mathcal{L}^M$ represents the infinitesimal generator of M, and we have introduced the rescaled parameters

$$\varphi = \tilde{\varphi}(1-\tilde{\rho})^2 \qquad \text{and} \qquad \rho = \tilde{\rho}/(1-\tilde{\rho})\,.$$

Here we see the generators of M and $\boldsymbol{\mu}$ separately because we assumed their independence. If as noted earlier we allowed for dependence, e.g., the order-flow affects the volatility of the martingale component, then we would simply replace the sum of the generators, by the generator of the triplet (M, μ^+, μ^-), but all else remains the same.

Proposition 1. Solving the DPE for POV. *The DPE (9) admits the solution*

$$H(t,x,S,\boldsymbol{\mu},q) = x + q\,S + h_0(t,\boldsymbol{\mu}) + h_1(t,\boldsymbol{\mu})\,q + h_2(t)\,q^2\,, \tag{11}$$

where

$$h_2(t) = -\left(\frac{T-t}{k+\varphi} + \frac{1}{\alpha - \frac{1}{2}b}\right)^{-1} - \frac{1}{2}\,b\,, \tag{12a}$$

$$\begin{aligned} h_1(t,\boldsymbol{\mu}) = {} & \frac{2\varphi\,\rho}{(T-t)+\zeta}\int_t^T \mathbb{E}_{t,\boldsymbol{\mu}}\left[\mu_s^+ + \mu_s^-\right]\,ds \\ & + \frac{b}{(T-t)+\zeta}\int_t^T ((T-s)+\zeta)\,\mathbb{E}_{t,\boldsymbol{\mu}}\left[\mu_s^+ - \mu_s^-\right]ds\,, \end{aligned} \tag{12b}$$

$$h_0(t,\boldsymbol{\mu}) = \int_t^T \mathbb{E}_{t,\boldsymbol{\mu}}\left[\frac{1}{4(k+\varphi)}\left(h_1(t,\boldsymbol{\mu}_s) - 2\varphi\,\rho\,(\mu_s^+ + \mu_s^-)\right)^2 - \varphi\,\rho^2\,(\mu_s^+ + \mu_s^-)^2\right]ds\,, \tag{12c}$$

and the constant

$$\zeta = \frac{k+\varphi}{\alpha - \frac{1}{2}\,b}\,.$$

Proof. See Appendix A.1 for the proof. ∎

The above proposition provides us with a candidate solution to our original control problem (8). Note that it is quadratic in inventory q but potentially nonlinear in the net $(\mu^+ - \mu^-)$ and total $(\mu^+ + \mu^-)$ order-flow of other agents—depending on how one specifies the model. Furthermore, the candidate value function splits into two pieces: (i) $x + q\,S$, which is the cash from the sales so far plus the book value (valued at the midprice) of the inventory the agent still has remaining; and (ii) $h_0(t,\boldsymbol{\mu}) + h_1(t,\boldsymbol{\mu})\,q + h_2(t)\,q^2$, which represents the excess value (and cost) of trading optimally from this point onward and is independent of the midprice. Indeed, this additional term can be interpreted as the agent's marginal reservation price, i.e., the price at which she is willing to sell the outstanding inventory so that her value function remains unchanged.

The following theorem shows that this candidate solution is indeed the one we seek and also provides us with the explicit form of the optimal control.

Theorem 2. Verification for POV. *The candidate value function provided in Proposition 1 is indeed the solution to the optimal control problem (8). Moreover, the optimal trading speed is given by*

$$\nu_t^* = \frac{1}{(T-t)+\zeta}\, Q_t^{\nu^*} \tag{13a}$$

$$+ \frac{\varphi}{k+\varphi}\,\rho \left\{ (\mu_t^+ + \mu_t^-) - \frac{1}{(T-t)+\zeta} \int_t^T \mathbb{E}[\, (\mu_s^+ + \mu_s^-) \mid \mathcal{F}_t^{\boldsymbol{\mu}} \,]\, ds \right\} \tag{13b}$$

$$- \frac{b}{k+\varphi} \int_t^T \frac{(T-s)+\zeta}{(T-t)+\zeta}\, \mathbb{E}\left[\, (\mu_s^+ - \mu_s^-) \mid \mathcal{F}_t^{\boldsymbol{\mu}} \right] ds\,, \tag{13c}$$

where $\mathcal{F}_t^{\boldsymbol{\mu}}$ denotes the natural filtration generated by $\boldsymbol{\mu}$, and is the admissible optimal control we seek.

Proof. Since $x + q\,S + h_0(t,\boldsymbol{\mu}) + h_1(t,\boldsymbol{\mu})\,q + h_2(t)\,q^2$ is clearly a classical solution, standard results imply that it suffices to check that the feedback control is indeed an admissible strategy. To this end, from the feedback form of the optimal control ν in (A.2) and the explicit form of $h_2(t)$ and $h_1(t,\boldsymbol{\mu})$ we find the expression above for ν^*. Next, note that since $\mu^\pm$ have finite second moment, then (13b) and (13c) clearly have finite expectation. Then, by explicitly integrating $dQ_t^{\nu^*} = -\nu_t^*\,dt$ (see (14) in Remark 1 below), we see that $Q_t^{\nu^*}$ is at most linear in $\mu^\pm$, and hence also has finite expectation, and decays like $(T-t)+\zeta$ as t approaches T. Therefore, ν^* is admissible. ∎

The optimal strategy appearing above is general, we have not specified a model for order-flow, and it is fully explicit in terms of simple expectations of the future total and net order-flow. To understand the intuition of the strategy (13) we start by pointing out that the investor's performance criteria (7) include competing objectives. On the one hand the strategy aims at liquidating $\mathfrak{N}$ shares by T, and on the other hand the strategy must track a fraction of other market participants' trading rate. Only when $\mathfrak{N}$ is exactly equal to the desired fraction ρ of the total volume over the execution duration T are these two objectives compatible.

The first component of the strategy (13a) is TWAP-like. We refer to the term $\frac{Q_t^{\nu^*}}{(T-t)+\zeta}$ as a TWAP-like strategy because if $\zeta = 0$ we would obtain the so-called time weighted average price (TWAP) strategy, which consists in liquidating shares by slicing the remaining inventory in equal parts over the remaining time to expiry.

The second component of the strategy (13b) consists of targeting instantaneous total order-flow plus a weighted average of future expected total order-flow. Although the POV target is $\rho\,\mu_t$, the strategy targets a lower amount since $\frac{\varphi\,\rho}{k+\varphi} \le \rho$, where equality is achieved if the costs of missing the target are $\varphi \to \infty$ and k remains finite, or there is no temporary impact $k \downarrow 0$. Recall that in our notation the investor wishes to target $\tilde\rho\,(\mu_t + \nu_t)$, which is equivalent to targeting $\rho\,\mu_t$ when $\rho = \tilde\rho/(1-\tilde\rho)$.

The last component (see (13c)) acts to correct her trading based on her expectations of the net order-flow from that point in time until the end of the trading horizon. When there is a current surplus of buy trades she slows down her trading rate to allow the midprice to appreciate before liquidating the rest of her order. When there is a current surplus of sell orders, she speeds up her trades because the action of other sellers in the market will push prices downward, and therefore degrade her cash proceeds from selling shares if she waits. Thus, she attempts to liquidate a larger portion of her inventory now rather than later.

We point out that we solve the unconstrained problem where the investor may buy back some assets prior to the end of the trading horizon. This is optimal because the strategy can benefit from directional moves in the midprice which are caused by net order-flow. In particular, if net order-flow is positive and large, the strategy anticipates an upward trend in the midprice and to profit from this short-term trend it is optimal to purchase shares. In section 4 we show the performance of the strategy and report the percentage of time the strategy ends up purchasing rather than selling shares.

Our results have counterparts in the result of [12], where order-flow is modeled by a Gamma bridge; however, there are some important differences. Like [12], through their Gamma bridge, our optimal rate of trading is perturbed by future order-flow; however, unlike their work, weighted future expected net and total order-flow are important components in our optimal trading rate. In particular, they do not have a component which adjusts net order-flow since they do not account for permanent price impact in their approach.

Remark 1. We can explicitly solve for the optimal inventory to hold using (13) and (A.2) to find

$$Q_t^{\nu^*} = \frac{T-t+\zeta}{T+\zeta}\,\mathfrak{N} + \frac{1}{2(k+\varphi)}\int_0^t \frac{T-t+\zeta}{T-s+\zeta}\left[h_1(s,\boldsymbol{\mu}_s) - 2\,\rho\,\varphi\,\left(\mu_s^+ + \mu_s^-\right)\right]\,ds\,. \tag{14}$$

Moreover, while the optimal trading speed in Theorem 2 is Markov in the state variables, in particular $Q_t^{\nu^*}$, μ_t^+, and μ_t^-, we see from (14) that the optimal inventory at any point in time is non-Markovian and depends on the entire history of μ^+ and μ^- through the integral term.

We continue our discussion of the optimal strategy by looking at some limiting cases.

3.4. Targeting VWAP with POV. To obtain VWAP the liquidation strategy we must, over the trading horizon, (i) liquidate all shares and (ii) target POV at every instant in time. The first requirement is achieved by taking an arbitrarily large final inventory liquidation penalty $\alpha \to \infty$. Although the initial role of the terminal penalty parameter α was to account for liquidation costs, the investor can also choose this parameter to be large enough so that the strategy ensures full liquidation. In the value function the interpretation is that this is not a financial cost so it does not affect the cash process. In this limiting case $\zeta \xrightarrow{\alpha\to\infty} 0$, and the optimal liquidation speed simplifies to

$$\lim_{\alpha\to\infty} \nu_t^* = \frac{1}{T-t} Q_t^{\nu^*} \tag{15a}$$

$$+\frac{\varphi}{k+\varphi}\,\rho\left\{\mu_t - \frac{1}{T-t}\int_t^T \mathbb{E}[\,\mu_s^+ + \mu_s^- \,|\, \mathcal{F}_t^{\boldsymbol{\mu}}\,]\,ds\right\} \tag{15b}$$

$$-\frac{b}{k+\varphi}\int_t^T \frac{T-s}{T-t}\,\mathbb{E}\left[\mu_s^+ - \mu_s^- \,|\, \mathcal{F}_t^{\boldsymbol{\mu}}\right]\,ds\,, \tag{15c}$$

which guarantees that the strategy liquidates all shares by the end of the trading horizon as required.

The second requirement, that the strategy must closely target the appropriate POV χ_t^ν at every instant in time, is attained by taking the limit $\varphi \to \infty$, resulting in

$$\lim_{\varphi\to\infty}\lim_{\alpha\to\infty} \nu_t^* = \frac{1}{T-t} Q_t^{\nu^*} + \rho \left\{ \left(\mu_t^+ + \mu_t^-\right) - \frac{1}{T-t} \int_t^T \mathbb{E}\left[\, \left(\mu_s^+ + \mu_s^-\right) \,\middle|\, \mathcal{F}_t^{\boldsymbol{\mu}} \,\right] ds \right\} . \tag{16}$$

We observe that only total order-flow shows up in the strategy, while the role of the net order-flow disappears in this limit. Moreover, this strategy designed to track VWAP consists in executing TWAP plus the POV target which is adjusted downward to take into account future trades. More precisely, this adjustment is the average trading rate of other traders that is expected over the remaining life of the strategy.

We note that although strategy (16) remains finite, and makes financial sense, in the double limit $\lim_{\varphi\to\infty}\lim_{\alpha\to\infty}$, the value function explodes (becomes minus infinity) due to the penalty term $\tilde\varphi \int_0^T (\nu_t - \chi_t^\nu)^2$. However, a normalized value function $H \to \frac{1}{\varphi} H := \tilde H$ has a sensible limit and the optimal strategy remains the same—indeed, we could have defined the value function as this normalized version. Let us work out this limit. First we take $\alpha \to \infty$ and obtain

$$\lim_{\alpha\to\infty} \tilde H(t,x,S,\boldsymbol{\mu},q) = \tfrac{1}{\varphi}\,(x + q\,S) + \tilde h_0(t,\boldsymbol{\mu}) + \tilde h_1(t,\boldsymbol{\mu})\,q\,, + \tilde h_2(t)\,q^2, \tag{17}$$

where

$$\tilde h_2(t) = -\,\frac{1+\frac{k}{\varphi}}{T-t} - \tfrac{1}{2}\,\frac{b}{\varphi}\,, \tag{18a}$$

$$\tilde h_1(t,\boldsymbol{\mu}) = \frac{\rho}{T-t}\int_t^T \mathbb{E}_{t,\boldsymbol{\mu}}\left[\mu_s^+ + \mu_s^-\right] ds + \frac{\frac{b}{\varphi}}{T-t}\int_t^T (T-s)\,\mathbb{E}_{t,\boldsymbol{\mu}}\left[\,\left(\mu_s^+ - \mu_s^-\right)\,\right] ds\,, \tag{18b}$$

$$\tilde h_0(t,\boldsymbol{\mu}) = \frac{1}{\varphi}\int_t^T \mathbb{E}_{t,\boldsymbol{\mu}}\left[\frac{1}{4(k+\varphi)}\left(h_1(t,\boldsymbol{\mu}_s) - 2\varphi\,\rho\,(\mu_s^+ + \mu_s^-)\right)^2 - \varphi\,\rho^2\,(\mu_s^+ + \mu_s^-)^2\right] ds\,. \tag{18c}$$

This is followed by $\varphi \to \infty$ and we obtain the result below.

Proposition 3. Limiting value function: Complete liquidation then targeting POV. *We have*

$$\lim_{\varphi\to\infty}\lim_{\alpha\to\infty} \tfrac{1}{\varphi} H(t,x,S,\boldsymbol{\mu},q) = q\left(\frac{\rho}{T-t}\int_t^T \mathbb{E}_{t,\boldsymbol{\mu}}\left[\,\left(\mu_s^+ + \mu_s^-\right)\,\right] ds - \frac{q}{T-t}\right). \tag{19}$$

Moreover, when the strategy is approaching expiry, $T - t \ll 1$, the optimal liquidation strategy (16) behaves like

$$\nu_t^* \sim \frac{1}{T-t} Q_t^{\nu^*}\,,$$

which is TWAP. Clearly, if the investor's strategy aims at full liquidation by the end of the trading horizon (as a result of $\alpha \to \infty$), her other objective, which is to target a fraction of the trading rate, needs to be (optimally) phased out very close to expiry even if the target penalty parameter φ is arbitrarily large. This outcome, where the terminal penalty becomes more important than the target penalty parameter as the strategy approaches the terminal date T, results from the order in which the limits were taken, first $\alpha \to \infty$ and then $\varphi \to \infty$.

The order in which we take the limits is important for they do not commute. If, on the other hand, the investor wishes to closely target a fraction ρ of the trading $(\mu_t^+ + \mu^-)$, then she lets $\varphi \to \infty$ first, and noting that $\zeta \xrightarrow{\varphi\to\infty} \infty$, we obtain

$$\nu_t^* \xrightarrow{\varphi\to\infty} \rho\,(\mu_t^+ + \mu^-)\,. \tag{20}$$

This strategy does not depend on α and is independent of the investor's liquidation target $\mathfrak{N}$.

Proposition 4. Limiting value function: Targeting POV. *We have*

$$\begin{aligned}\lim_{\varphi\to\infty} H(t,x,S,\boldsymbol{\mu},q) &= x + q\,S + \alpha\,q^2\\ &\quad - \rho\int_t^T \mathbb{E}_{t,\boldsymbol{\mu}}\left[h_1(t,\boldsymbol{\mu}_s)\,(\mu_s^+ + \mu_s^-)\right]\,ds\\ &\quad + \left(\int_t^T \left\{\rho\,\mathbb{E}_{t,\boldsymbol{\mu}}\left[(\mu_s^+ + \mu_s^-)\right] + b\,\mathbb{E}_{t,\boldsymbol{\mu}}\left[(\mu_s^+ - \mu_s^-)\right]\right\} ds\right) q\,.\end{aligned} \tag{21}$$

3.4.1. Temporary impact from other traders. Above we assumed that the liquidation price received by the investor is worse than the midprice because the investor's MOs walk the LOB, and we assumed that the LOB is replenished immediately. In our model we can also include the temporary price impact that other traders' MOs have on the LOB. In this case, the investor's execution price (when liquidating the asset) is

$$\hat{S}_t^\nu = S_t^\nu - k\left(\nu_t + \mu_t^-\right)\,, \tag{22}$$

rather than (4). In principle, one could make the impact of other traders different from the trader's own impact (i.e., choose a different temporary impact parameter for other traders' MOs). The investor's optimal liquidation strategy can be obtained analogously to the previous section and results in

$$\nu_t^* = \frac{1}{(T-t)+\zeta}\,Q_t^{\nu^*} \tag{23a}$$

$$\quad + \frac{\varphi}{k+\varphi}\,\rho\left\{(\mu_t^+ + \mu_t^-) - \frac{1}{(T-t)+\zeta}\int_t^T \mathbb{E}[\,(\mu_s^+ + \mu_s^-)\mid \mathcal{F}_t^{\boldsymbol{\mu}}\,]\,ds\right\} \tag{23b}$$

$$\quad - \frac{b}{k+\varphi}\int_t^T \frac{(T-s)+\zeta}{(T-t)+\zeta}\,\mathbb{E}\left[\,(\mu_s^+ - \mu_s^-)\mid \mathcal{F}_t^{\boldsymbol{\mu}}\right]\,ds\,, \tag{23c}$$

$$\quad - \frac{k}{2(k+\varphi)}\left\{\mu_t^- - \frac{1}{(T-t)+\zeta}\int_t^T \mathbb{E}[\,\mu_s^-\mid \mathcal{F}_t^{\boldsymbol{\mu}}\,]\,ds\right\}\,. \tag{23d}$$

The liquidation strategy (23) is as the one shown in (13) with the additional terms that appear in line (23d). These extra terms show how the strategy changes as a result of MOs from other participants that also walk the LOB. For example, if the sign of the quantity inside the braces is positive, the investor's liquidation strategy slows down. It is better to liquidate more later to avoid current temporary costs from walking the LOB because μ_t^- is too high relative to the expected sell order-flow (from other traders) during the remaining life of the strategy.

3.5. Targeting VWAP with POCV. In this subsection we assume that the investor's execution strategy targets POCV and the liquidation strategy relies on MOs only. Here the accumulated volume V of orders, excluding the agent's own trades, is given by

$$V_t = \int_0^t \left(\mu_u^+ + \mu_u^-\right) du\,.$$

The assumptions in the previous section on order-flow, midprice, execution price, and admissible strategies remain in force. The investor's performance criteria is now modified to

$$\begin{aligned} H^\nu(t,x,S,\boldsymbol{\mu},V,q) = \mathbb{E}_{t,x,S,\boldsymbol{\mu},V,q}\Bigg[& X_T^\nu + Q_T^\nu\left(S_T^\nu - \alpha Q_T^\nu\right) \\ & - \tilde\varphi \int_t^T \Big((\mathfrak{N} - Q_u^\nu) - \tilde\rho\,\left(V_u + (\mathfrak{N} - Q_u^\nu)\right) \Big)^2 du \Bigg]. \end{aligned} \tag{24}$$

Observe that $(V_u + (\mathfrak{N} - Q_u^\nu))$ is the total volume traded in the market including the investor's own trades. The investor's value function is

$$H(t,x,S,\boldsymbol{\mu},V,q) = \sup_{\nu\in\mathcal{A}} H^\nu(t,x,S,\boldsymbol{\mu},V,q)\,, \tag{25}$$

and her controlled inventory Q^ν, execution price $\hat S^\nu$, midprice S^ν, and cash process X^ν satisfy the equations in (2), (4), (5), and (6), respectively. Applying the dynamic programming principle suggests that the value function should satisfy the DPE

$$\begin{aligned} 0 = \left(\partial_t + \mathcal{L}^M + \mathcal{L}^{\boldsymbol{\mu},V}\right) H - \varphi\left((\mathfrak{N} - q) - \rho\, V\right)^2 & \\ + \sup_\nu \left\{ b\left((\mu^+ - \mu^-) - \nu\right)\partial_S H + (S - k\,\nu)\,\nu\,\partial_x H - \nu\,\partial_q H \right\} & \end{aligned} \tag{26}$$

subject to the terminal condition $H(T,x,S,y,q) = x + q\,(S - \alpha\,q)$ and we have introduced the rescaled parameters

$$\varphi = \tilde\varphi(1-\tilde\rho)^2 \qquad \text{and} \qquad \rho = \tilde\rho/(1-\tilde\rho)\,.$$

Here, $\mathcal{L}^{\boldsymbol{\mu},V}$ denotes the infinitesimal generator of the joint process $(\boldsymbol{\mu}_t, V_t)_{0\le t\le T}$.

Proposition 5. Solving the DPE for POCV. *The DPE* (26) *admits the solution*

$$H(t,x,S,\boldsymbol{\mu},V,q) = x + q\,S + h_0(t,\boldsymbol{\mu},V) + h_1(t,\boldsymbol{\mu},V)\,q + h_2(t)\,q^2\,, \tag{27}$$

where

$$h_2(t) = -\sqrt{k\,\varphi}\,\frac{\gamma\,e^{\xi\,(T-t)} + e^{-\xi\,(T-t)}}{\gamma\,e^{\xi(T-t)} - e^{-\xi\,(T-t)}} - \tfrac{1}{2}\,b\,, \tag{28a}$$

$$\begin{aligned} h_1(t,\boldsymbol{\mu},V) = {} & 2\,\varphi\int_t^T \ell(u,t)\,\left(\mathfrak{N} - \rho\,\mathbb{E}_{t,\boldsymbol{\mu},V}\left[V_u\right]\right) du \\ & + b\int_t^T \ell(u,t)\,\mathbb{E}_{t,\boldsymbol{\mu},V}\left[(\mu_u^+ - \mu_u^-)\right] du\,, \end{aligned} \tag{28b}$$

$$h_0(t,\boldsymbol{\mu},V) = \int_t^T \mathbb{E}_{t,\boldsymbol{\mu}}\left[\tfrac{1}{4k}\big(h_1(u,\boldsymbol{\mu}_u,V_u)\big)^2 - \varphi\big(\mathfrak{N} - \rho\,V_u\big)^2\right] du\,, \tag{28c}$$

the function

$$\ell(u,t) := \frac{\gamma\, e^{\xi\,(T-u)} - e^{-\xi\,(T-u)}}{\gamma\, e^{\xi\,(T-t)} - e^{-\xi\,(T-t)}}\,, \tag{28d}$$

and the constants

$$\xi = \sqrt{\frac{\varphi}{k}}\,, \quad \text{and} \quad \gamma = \frac{\alpha - \frac{1}{2}b + \sqrt{k\,\varphi}}{\alpha - \frac{1}{2}b - \sqrt{k\,\varphi}}\,.$$

Proof. See Appendix A.2 for the proof. ■

As in the previous subsection, the above proposition provides us with a candidate solution for the value function, and the ansatz $H = x + q\,S + h_0(t,\boldsymbol{\mu},V) + h_1(t,\boldsymbol{\mu},V)\,q + h_2(t)\,q^2$ carries the following interpretations: (i) the first two terms represent the book value of the asset evaluated at the midprice together with the cash from liquidation to this point, and (ii) the remaining terms correspond to the value of optimally liquidating the remaining shares. As before, the additional terms can be interpreted as the investor's reservation price for selling the outstanding inventory all at once.

Theorem 6. Verification for POCV. *The candidate value function provided in Proposition 5 is indeed the solution to the optimal control problem* (25)*. Moreover, the trading speed given by*

$$\nu_t^* = \xi\,\frac{1+\gamma}{\gamma\, e^{\xi(T-t)} - e^{-\xi\,(T-t)}}\, Q_t^{\nu^*} \tag{29a}$$

$$- \xi^2 \int_t^T \ell(u,t)\,\left\{\left(\mathfrak{N} - Q_t^{\nu^*}\right) - \rho\,\mathbb{E}\left[V_u \,\middle|\, \mathcal{F}_t^{\boldsymbol{\mu},V}\right]\right\}\,du \tag{29b}$$

$$- \frac{b}{2\,k}\int_t^T \ell(u,t)\,\mathbb{E}\left[(\mu_u^+ - \mu_u^-)\,\middle|\,\mathcal{F}_t^{\boldsymbol{\mu},V}\right]\,du\,, \tag{29c}$$

where $\mathcal{F}_t^{\boldsymbol{\mu},V}$ denotes the natural filtration generated by $(\boldsymbol{\mu},V)$, and $\ell(u,t)$ is given by (28d)*, and ν_t^* in* (29a) *is the admissible optimal control we seek.*

Proof. See Appendix A.3 for the proof. ■

The various terms in the optimal speed to trade in the theorem above carries some natural interpretations. First, (29a) may be interpreted as an Almgren–Chriss-like strategy which approaches TWAP near maturity. The second term (29b) corrects for the weighted average of the difference between what the investor has liquidated so far $\left(\mathfrak{N} - Q_t^{\nu^*}\right)$ and her future targeted volume $\rho\,\mathbb{E}[V_u \,|\, \mathcal{F}_t^{\boldsymbol{\mu},V}]$. This term will be positive most of the time because (i) total volume is a submartingale and so $\mathbb{E}[V_u \,|\, \mathcal{F}_t^{\boldsymbol{\mu},V}] \geq V_t$, and (ii) since at time t the investor targets $\left(\mathfrak{N} - Q_t^{\nu^*}\right)$ to $\rho\,V_t$, it is likely that

$$\left(\mathfrak{N} - Q_t^{\nu^*}\right) - \rho\,\mathbb{E}\left[V_u \,\middle|\, \mathcal{F}_t^{\boldsymbol{\mu},V}\right] \sim \rho\,V_t - \rho\,\mathbb{E}\left[V_u \,\middle|\, \mathcal{F}_t^{\boldsymbol{\mu},V}\right] \leq 0\,.$$

The third term (29c) contributes a weighted average of future net order-flow and accounts for the permanent impact which trades have on the midprice. If future net order-flow is expected to be buy heavy, then the strategy slows down—since the investor wishes to profit from the positive pressure on midprice from this order-flow before selling her assets. If, on the other hand, future net order-flow is expected to be sell heavy, then the strategy speeds up—since the investor wishes to ensure she gets the better prices that prevail in the market now, as she expects midprice to have negative pressure. Finally, as the trading horizon ends, t approaching T, the second and third terms become negligible and the investor ignores order-flow entirely and instead focuses on completing her trades.

Similarly as in the POV target, here we solve the unconstrained problem where the investor may buy back some assets prior to the end of the trading horizon. This is optimal because the strategy can benefit from directional moves in the midprice which are caused by net order-flow. In particular, if net order-flow is positive and large, the strategy anticipates an upward trend in the midprice, and to profit from this short-term trend it is optimal to purchase shares. In section 4 we show the performance of the strategy and report the percentage of time the strategy ends up purchasing rather than selling shares.

Remark 2. We can explicitly solve for the optimal inventory to hold using (29) and solving $dQ_t^{\nu^*} = -\nu_t^*\,dt$, with $Q_0^{\nu^*} = 0$, to find

$$
\begin{aligned}
Q_t^{\nu^*} &= \ell(t,0)\,\mathfrak{N} \\
&\quad + \int_0^t \int_u^T \ell(u,s)\,\left\{\tfrac{\varphi}{k}\left(\mathfrak{N} - \rho\,\mathbb{E}\left[V_s \,\middle|\, \mathcal{F}_u^{\boldsymbol{\mu},V}\right]\right) + \tfrac{b}{2k}\mathbb{E}\left[(\mu_s^+ - \mu_s^-)\,\middle|\,\mathcal{F}_u^{\boldsymbol{\mu},V}\right]\right\}\,ds\,du\,.
\end{aligned}
\tag{30}
$$

Moreover, while the optimal trading speed in Theorem 6 is Markov in the state variables, in particular $Q_t^{\nu^*}$, $\mu_t^{\pm}$, and V_t, we see from (30) that the optimal inventory at any point in time is non-Markovian and depends on the entire history of $\mu^{\pm}$ and V through the integral term.

Note that in the limit as the terminal penalty becomes arbitrarily large $\alpha \to \infty$, so that the investor ensures that she completely liquidates her position by the terminal time, $\gamma \to 1$; therefore $\ell(u,t) \to \frac{\sinh(\xi\,(T-u))}{\sinh(\xi\,(T-t))}$ and so we have

$$
\begin{aligned}
\lim_{\alpha\to\infty} \nu_t^* = \frac{\xi\,Q_t^{\nu^*}}{\sinh\left(\xi\,(T-t)\right)} &- \xi^2 \int_t^T \frac{\sinh\left(\xi\,(T-u)\right)}{\sinh\left(\xi\,(T-t)\right)} \left\{\left(\mathfrak{N} - Q_t^{\nu^*}\right) - \rho\,\mathbb{E}\left[V_u \,\middle|\, \mathcal{F}_t^{\boldsymbol{\mu},V}\right]\right\} du \\
&- \frac{b}{2\,k}\int_t^T \frac{\sinh\left(\xi\,(T-u)\right)}{\sinh\left(\xi\,(T-t)\right)}\,\mathbb{E}\left[(\mu_u^+ - \mu_u^-)\,\middle|\,\mathcal{F}_t^{\boldsymbol{\mu},V}\right] du
\end{aligned}
\tag{31}
$$

and

$$
\begin{aligned}
\lim_{\alpha\to\infty} Q_t^{\nu^*} = \frac{\sinh(\xi\,(T-t))}{\sinh(\xi\,T)}\,\mathfrak{N} &+ \xi^2 \int_0^t \int_u^T \frac{\sinh(\xi\,(T-u))}{\sinh(\xi\,(T-s))}\,\left(\mathfrak{N} - \rho\,\mathbb{E}\left[V_s \,\middle|\, \mathcal{F}_u^{\boldsymbol{\mu},V}\right]\right)\,ds\,du \\
&+ \frac{b}{2k}\int_0^t \int_u^T \frac{\sinh(\xi\,(T-u))}{\sinh(\xi\,(T-s))}\mathbb{E}\left[(\mu_s^+ - \mu_s^-)\,\middle|\,\mathcal{F}_u^{\boldsymbol{\mu},V}\right]\,ds\,du\,.
\end{aligned}
$$

These expressions show how the strategy is similar to Almgren–Chriss with corrections that depend on the expected future order-flow relative to amount already liquidated. In the limit $\varphi \to \infty$, the strategy is not well defined since we are controlling a rate, while the target is the total volume. Hence, if we are not precisely on target, an infinitely large penalty will cause the investor to trade infinitely fast to get back on target.

3.5.1. Temporary impact from other traders. As in subsection 3.4.1 here we assume MOs from the investor and other market participants have temporary impact, i.e., walk the LOB. Thus, the investor receives prices as in (22) and the investor's optimal liquidation strategy is

$$\nu_t^* = \xi \, \frac{1+\gamma}{\gamma \, e^{\xi (T-t)} - e^{-\xi\,(T-t)}} \, Q_t^{\nu^*} \tag{32a}$$

$$- \, \xi^2 \int_t^T \ell(u,t) \, \left\{ \left(\mathfrak{N} - Q_t^{\nu^*} \right) - \rho \, \mathbb{E}\left[V_u \, | \, \mathcal{F}_t^{\boldsymbol{\mu},V} \right] \right\} \, du \tag{32b}$$

$$- \, \frac{b}{2\,k} \int_t^T \ell(u,t) \, \mathbb{E}\left[(\mu_u^+ - \mu_u^-) \, | \, \mathcal{F}_t^{\boldsymbol{\mu},V} \right] \, du \tag{32c}$$

$$- \, \frac{1}{2} \left\{ \mu_t^- - \xi \int_t^T \frac{\gamma \, e^{\xi\,(T-u)} + e^{-\xi\,(T-u)}}{\gamma \, e^{\xi\,(T-t)} - e^{-\xi\,(T-t)}} \, \mathbb{E}\left[\mu_u^- \, | \, \mathcal{F}_t^{\boldsymbol{\mu},V} \right] \, du \right\} . \tag{32d}$$

The liquidation strategy (32) is like the one shown in (29) with two extra terms that appear in line (32d). These two extra terms show how the strategy changes as a result of MOs from other participants that also walk the LOB. For example, if the sign of the quantity inside the braces is positive, the liquidation strategy slows down because it is better to liquidate more later to avoid current temporary costs from walking the LOB.

4. Targeting VWAP: Performance of POV and POCV strategies . Above we derived the optimal liquidation speed for an investor who wishes to liquidate $\mathfrak{N}$ shares over a time horizon T and requires her selling proceeds per share to target VWAP. To show the performance of the strategy we calibrate the model for volume, midprice, and price impact using the parameter estimates presented in Table 1.

So far we have made general assumptions about the midprice and the volume process, but to carry out simulations we must specify a model these two processes. We assume that the martingale component in the midprice (5) is $M_t = \sigma B_t$, where B_t is a standard Brownian motion and σ is as in Table 1.

Moreover, we motivate a choice of model for the speed at which the overall market is trading. Figure 2 uses millisecond-stamped messages from Nasdaq to show the rate of trading (per minute) for ORCL on November 1, 2013, which is computed by counting the volume traded over 5-minute and 1-minute windows, respectively, and scaling the counts to one minute.

As the figure shows, a reasonable first order model is that traded volume comes in bursts of activity which persists for a while (seconds, for instance) and then decays to zero. Thus, for an agent whose objective is to target VWAP over a time window $[0, T]$ following the strategy (15), we assume that the market participants' buy/sell order-flow $\boldsymbol{\mu}_t^{\pm}$, excluding the investor's own trades, are mean-reverting processes which satisfy the SDEs

$$d\mu_t^+ = -\kappa^+ \, \mu_t^+ \, dt + \eta^+_{1+N^+_{t^-}} \, dN_t^+ \, , \tag{33a}$$

$$d\mu_t^- = -\kappa^- \, \mu_t^- \, dt + \eta^-_{1+N^-_{t^-}} \, dN_t^- \, , \tag{33b}$$

where $\kappa^{\pm} \geq 0$ are the mean-reversion rates, N_t^+ and N_t^- are independent homogeneous Poisson processes with intensities λ^+ and λ^-, respectively, $\{\eta_1^{\pm}, \eta_2^{\pm}, \dots\}$ are nonnegative independent and identically distributed random variables with distribution function F, with finite second

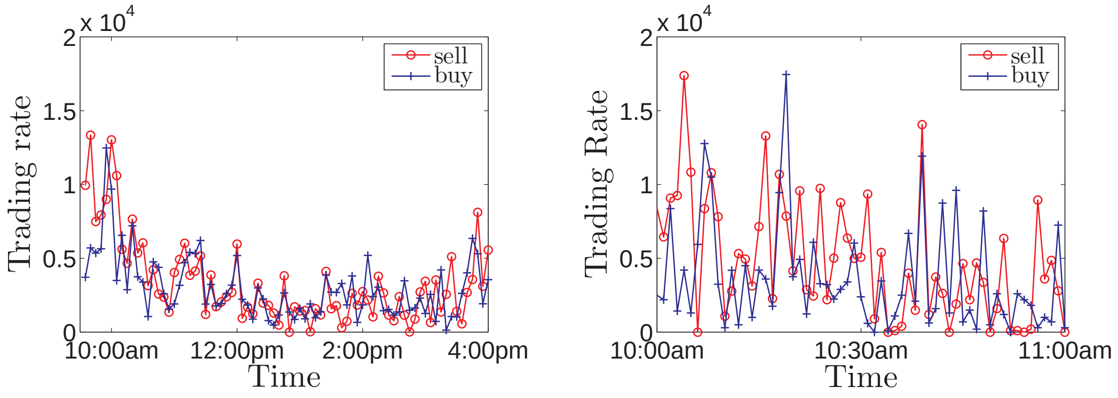

Figure 2. *ORCL per minute traded volume rates for MOs sent to Nasdaq on November* 1, 2013, *using (left)* 5*-minute buckets (right)* 1*-minute buckets.*

moment, independent from all processes. In addition, we require $\kappa^\pm > \lambda^\pm\,\mathbb{E}[\eta_1^\pm]$ to ensure that $\mu^\pm$ remains ergodic.

Model (33) allows buy and sell order-flow to have their own idiosyncratic jumps. Naturally, we could also include a common jump component to reflect times when activity on both sides of the market increases; we could also add diffusive noise terms to both equations, or make the intensity of the jumps time dependent (to mimic the U-shaped activity seen across markets) and/or stochastic. These generalizations, however, do not add much to our intuition and we opt to exclude them.

The solution to (33), for $s > t$, is

$$\mu_s^\pm = e^{-\kappa^\pm(s-t)}\,\mu_t^\pm + \int_t^s e^{-\kappa^\pm(s-u)}\,\eta^\pm_{1+N^\pm_{u-}}\,dN_u^\pm\,,$$

so that

$$\mathbb{E}[\,\mu_s^\pm\,|\,\mathcal{F}_t^\mu\,] = e^{-\kappa^\pm(s-t)}\,\left(\mu_t^\pm - \psi^\pm\right) + \psi^\pm\,,$$

where

$$\psi^\pm = \tfrac{1}{\kappa^\pm}\,\lambda^\pm\,\mathbb{E}[\eta^\pm]\,.$$

The constants $\psi^\pm$ therefore act as the expected long-run activity of buy and sell orders. Moreover, we have

$$\begin{aligned}
\mathbb{E}\left[V_u\;|\,\mathcal{F}_t\right] &= V_t + \int_u^t \mathbb{E}\left[(\mu_s^+ + \mu_s^-)\;|\,\mathcal{F}_t\right]\,ds \\
&= V_t + \int_u^t \left\{\left(e^{-\kappa^+(s-t)}(\mu_t^+ - \psi^+) + \psi^+\right) + \left(e^{-\kappa^-(s-t)}(\mu_t^- - \psi^-) + \psi^-\right)\right\} ds \\
&= V_t + \left(\frac{1-e^{-\kappa^+(u-t)}}{\kappa^+}(\mu_t^+ - \psi^+) + \psi^+(u-t)\right) \\
&\quad + \left(\frac{1-e^{-\kappa^-(u-t)}}{\kappa^-}(\mu_t^- - \psi^-) + \psi^-(u-t)\right)\,.
\end{aligned}$$

As shown earlier in Figure 1, the arrival of trades follows a U-shape which can be incorporated in (33) by introducing a deterministic component in the drift of the process, but here, for simplicity, we assume that the trading rate always mean-reverts to zero (e.g., if trading is taken only over a given 1-hour period of the trading day).

From the above computations, when $\kappa^{\pm} = \kappa$ (so that buy and sell activity have the same level of persistence), the limiting strategy for targeting POV (16) becomes

$$\lim_{\varphi\to\infty}\lim_{\alpha\to\infty} \nu_t^{*POV} = \frac{1}{T-t} Q_t^{\nu^*} + \rho\left[1 - \frac{1-e^{-\kappa(T-t)}}{\kappa(T-t)}\right]\left((\mu_t^+ + \mu_t^-) - (\psi^+ + \psi^-)\right)$$

and for targeting POCV (31) becomes

$$\begin{aligned}\lim_{\alpha\to\infty} \nu_t^{*POCV} = {} & \xi\,\frac{Q_t^{\nu^*}}{\sinh\left(\xi\,(T-t)\right)} \\ & - \xi^2\Bigg\{\left(\left(\mathfrak{N} - Q_t^{\nu^*}\right) - \rho\, V_t\right)\ell_0(t) \\ & \qquad - \rho\left(\frac{\ell_0(t)-\ell_2(t)}{\kappa}\left((\mu_t^+ + \mu_t^-) - (\psi^+ + \psi^-)\right) + (\psi^+ + \psi^-)\ell_1(t)\right)\Bigg\} \\ & - \frac{b}{2\,k}\left(\ell_2(t)\left((\mu_t^+ - \mu_t^-) - (\psi^+ - \psi^-)\right) + (\psi^+ - \psi^-)\ell_0(t)\right),\end{aligned}$$

where

$$\ell_0(t) = \frac{1}{\xi}\frac{\cosh(\xi(T-t)) - 1}{\sinh(\xi(T-t))}, \tag{34a}$$

$$\ell_1(t) = \frac{1}{\xi^2}\left(1 - \frac{\xi\,(T-t)}{\sinh(\xi(T-t))}\right), \tag{34b}$$

$$\ell_2(t) = \frac{\xi}{\xi^2-\kappa^2}\,\frac{\cosh(\xi(T-t)) - e^{-\kappa\,(T-t)}}{\sinh(\xi(T-t))} - \frac{\kappa}{\xi^2-\kappa^2}. \tag{34c}$$

As expected, when $t \nearrow T$, ℓ_0, ℓ_1, and ℓ_2 all approach 0 and both strategies becomes TWAP,

$$\lim_{\varphi\to\infty}\lim_{\alpha\to\infty} \nu^{*POV} \sim \frac{Q_t^{\nu^*}}{T-t}, \qquad \lim_{\alpha\to\infty} \nu^{*POCV} \sim \frac{Q_t^{\nu^*}}{T-t}.$$

4.1. Targeting VWAP: Numerical experiments. For the numerical experiments we use the estimated model parameters in Table 1 and choose $\varphi = 10^5 \times k$ for the POCV strategy. We assume trading occurs over 1 hour. Over that 1 hour, the investor must liquidate an amount of shares equal to 5% of the total expected traded volume, including her own. She is also targeting a trading rate of $\tilde{\rho} = 5\%$. Finally, we assume that when the investor has liquidated the target amount, she stops trading—this could happen before the terminal date of the trading horizon T. This is an ad hoc assumption because the control problem solved by the investor allows share repurchases throughout the life of the strategy and trades up until the terminal date T.

We perform 10,000 simulations of the order-flow, optimal speed to trade, and midprice. Along each midprice path we compute the VWAP (resulting from all traders' order-flow) using

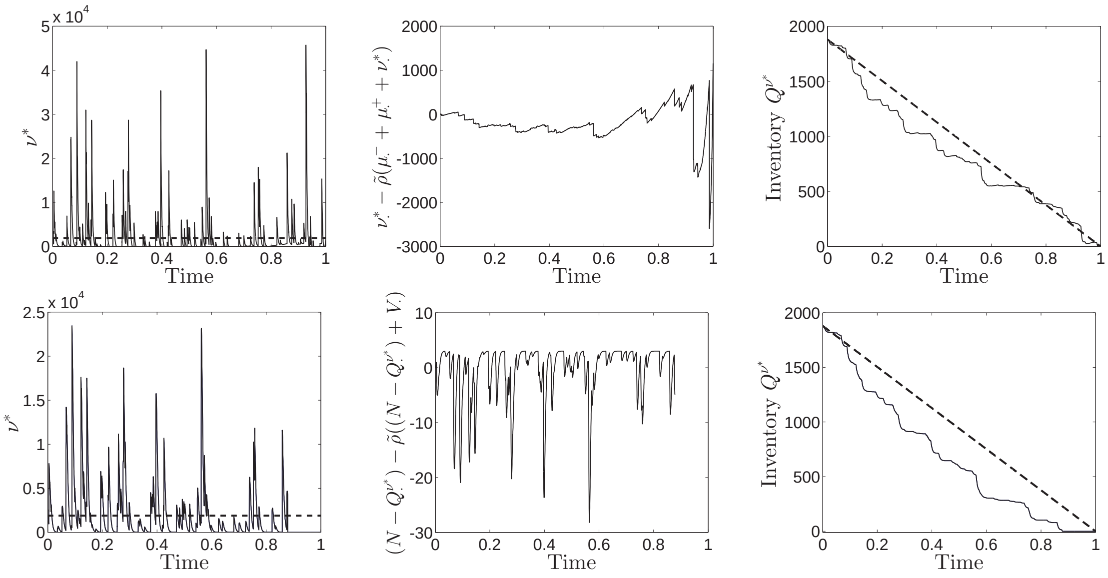

Figure 3. *SMH: Sample path of the investor's optimal trading rate, the difference between it and its target, and the resulting inventory path. The dashed lines in the left and right panels are the trajectory for a TWAP strategy.*

the parameter estimates for FARO, SMH, NTAP, ORCL, and INTC. To illustrate different aspects of the strategies we first discuss the results for SMH and then present a table for all assets.

Figure 3 illustrates one realization of the strategy for SMH where we show (i) market trading rate, (ii) optimal trading speed, (iii) deviations from target (rate for POV and cumulative volume for POCV), and (iv) the resulting inventory path. The dashed line in the first picture of each row represents the constant speed of liquidation of a TWAP strategy, and in the third picture the dashed line shows the investor's TWAP inventory. In the figure the top row is the POV strategy and the bottom row is the POCV strategy.

We see that the difference between the optimal liquidation speed and target rate for POV, second picture in the first row of Figure 3, fluctuates near zero for a large portion of the path, but toward the end of the trading horizon, the optimal strategy ignores the target and instead focuses on liquidating the remaining shares—recall that close to expiry the strategy becomes TWAP and recall also that for POV we are using the limiting strategy where the investor penalizes any remaining inventory $\alpha \to \infty$.

Moreover, the second picture in the second row shows the POCV deviation from the target. The large downward spikes result from the market trading rate spiking upward, but without the strategy fully tracking the increase in the volume traded. This is optimal because the liquidation strategy also takes into account the permanent impact that the market's increase in trading rate has on the midprice. In these cases there is a trade-off between missing the target and taking advantage of the permanent effect on the midprice. Finally, from the same picture we see that for this particular path, the POCV strategy finished liquidation early.

Figure 4 shows the heat-map of the inventory, together with the 5th and 95th percentiles, and the histogram of the difference between execution price/share and VWAP relative to

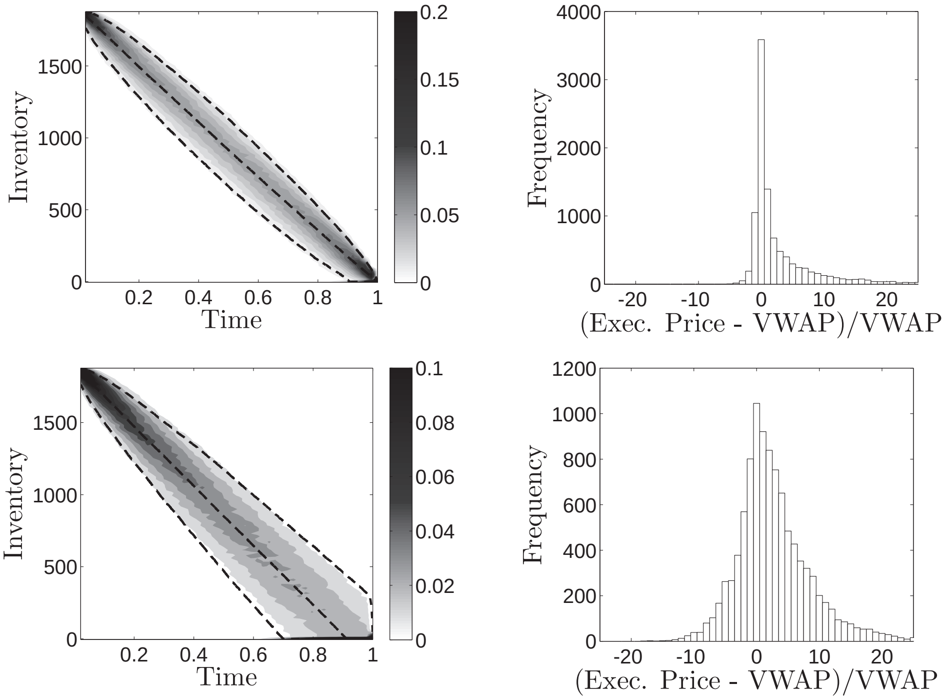

Figure 4. *SMH: Heat-map of the inventory, first and third pictures for POV and POCV, respectively. Histograms of the difference between execution price/share and VWAP relative to VWAP; second and fourth pictures are POV and POCV (in basis points), respectively.*

VWAP for POV and POCV for asset SMH. The computed VWAP does not account for temporary price impact. It does, however, account for the total traded volume and the investor's permanent price impact on the midprice. That is, we compute VWAP as

$$\text{VWAP} = \frac{\int_0^T S_u^\nu \, (\mu_u^+ + \mu_u^- + \nu_u)\, du}{\int_0^T \left(\mu_u^+ + \mu_u^- + \nu_u\right)\, du}, \tag{35a}$$

while the execution price is computed as

$$\text{Exec. Price} = \frac{X_T^\nu}{\mathfrak{N}} = \frac{\int_0^T \hat{S}_u^\nu \, \nu_u \, du}{\mathfrak{N}}. \tag{35b}$$

Table 3 shows summary statistics of the revenue from liquidating the shares when the investor follows POV and POCV to target VWAP. The first panel of the table shows the results for POV and the second panel the results for POCV. We see that the strategy targets VWAP very closely but on average it outperforms the target. For example, if the investor follows a POV strategy in SMH, the strategy outperforms VWAP by 2.98 basis points, or if the investor employs POCV, then it outperforms VWAP 2.75 basis points. From the results we observe that in all cases not only do the two algorithms outperform VWAP by an economically significant amount—brokerage fees for targeting are in the region of 10 to 20 basis points (see [21])—but the distribution of the relative error is skewed to the right.

Table 3

Statistics of the execution price, VWAP, and relative error (computed as $(Exec.Price - VWAP)/VWAP$ for each simulation) and reported in basis points (i.e., $\times 10^4$). For POCV we set $\varphi = 10^5 \times k$.

POV		FARO		SMH		NTAP		ORCL		INTC	
		VWAP	Rel.error	VWAP	Rel.error	VWAP	Rel.error	VWAP	Rel.error	VWAP	Rel.error
	Mean	\$ 40.54	8.9	\$ 37.90	2.98	\$ 38.30	0.19	\$ 33.64	0.13	\$ 23.03	0.16
	stdev	\$ 0.11	16.9	\$ 0.04	6.10	\$ 0.06	0.87	\$ 0.04	0.74	\$ 0.03	0.77
Quantile	5%	\$ 40.35	-4.2	\$ 37.83	-1.03	\$ 38.20	-0.78	\$ 33.57	-0.69	\$ 22.99	-0.65
Quantile	25%	\$ 40.46	-0.4	\$ 37.87	-0.16	\$ 38.26	-0.20	\$ 33.62	-0.18	\$ 23.02	-0.17
Quantile	50%	\$ 40.54	2.5	\$ 37.90	0.53	\$ 38.30	0.03	\$ 33.65	0.02	\$ 23.04	0.03
Quantile	75%	\$ 40.61	12.0	\$ 37.92	3.60	\$ 38.34	0.36	\$ 33.67	0.28	\$ 23.05	0.29
Quantile	95%	\$ 40.72	42.6	\$ 37.96	15.29	\$ 38.40	1.66	\$ 33.71	1.35	\$ 23.08	1.42
$\%t : \nu_t^* < 0$		27.3%		18.4%		0.6%		0.4%		0.5%	

POCV		FARO		SMH		NTAP		ORCL		INTC	
		VWAP	Rel.error	VWAP	Rel.error	VWAP	Rel.error	VWAP	Rel.error	VWAP	Rel.error
	Mean	\$ 40.54	5.0	\$ 37.89	2.75	\$ 38.30	1.88	\$ 33.64	1.81	\$ 23.03	1.81
	stdev	\$ 0.11	14.2	\$ 0.04	6.02	\$ 0.06	2.35	\$ 0.04	2.16	\$ 0.03	2.18
Quantile	5%	\$ 40.35	-13.6	\$ 37.83	-5.43	\$ 38.20	-1.65	\$ 33.57	-1.43	\$ 22.99	-1.45
Quantile	25%	\$ 40.46	-3.0	\$ 37.87	-0.78	\$ 38.26	0.28	\$ 33.61	0.37	\$ 23.01	0.36
Quantile	50%	\$ 40.54	1.9	\$ 37.89	1.79	\$ 38.30	1.72	\$ 33.64	1.68	\$ 23.03	1.66
Quantile	75%	\$ 40.61	11.0	\$ 37.92	5.51	\$ 38.34	3.27	\$ 33.67	3.06	\$ 23.05	3.08
Quantile	95%	\$ 40.73	31.9	\$ 37.96	13.76	\$ 38.40	6.00	\$ 33.72	5.62	\$ 23.08	5.64
$\%t : \nu_t^* < 0$		0.83%		0.01%		0.55%		0%		0%	

The table also shows the percentage of time that the strategy is buying, rather than selling, shares. In general, for these stocks, strategies that target VWAP employing POV spend a percentage of time between 0.5% and 27.3% buying back shares. On the other hand, VWAP strategies based on POCV very seldom purchase shares during the trading horizon.

5. Conclusions.

We provide a closed-form execution strategy for an investor who wishes to liquidate a large position over a trading horizon, her trades have both permanent and temporary impact, and the strategy must track VWAP. The strategy is based on tracking the rate of trading of the market or on tracking the cumulative traded volume in the market. In both cases the strategy is derived assuming a general stochastic process for the market's volume. The strategy based on tracking the market's trading rate consists of TWAP adjusted upward by a fraction of instantaneous order-flow and adjusted downward by the average order-flow that is expected over the remaining life of the strategy. The strategy based on cumulative volume consists of Almgren–Chriss adjusted by expected future volume and a weighted average of expected net order-flow to account for the permanent impact which trades have on the midprice.

We calibrate model parameters to five stocks traded in Nasdaq (FARO, SMH, NTAP, ORCL, INTC) and use simulations to show the performance of the strategies. We show that the strategies target VWAP very closely and on average they outperform the target. For example, if the investor follows a POV strategy in SMH, the strategy outperforms VWAP by 2.98 basis points, or if the investor employs POCV, then it outperforms VWAP by 2.75 basis points. Not only do the two strategies outperform VWAP by economically significant amount—brokerage fees for targeting are in the region of 10 to 20 basis points—but the distribution of the relative error is skewed to the right.

Appendix A. Proofs.

Appendix A.1. Proof of Proposition 1. To solve the DPE (9), first we seek a solution of the form

$$H(t, x, S, \boldsymbol{\mu}, q) = x + q\,S + h(t, \boldsymbol{\mu}, q)\,.$$

Inserting the ansatz into (9), we find that h must satisfy

$$0 = (\partial_t + \mathcal{L}^{\boldsymbol{\mu}})\,h + b\,(\mu^+ - \mu^-)\,q + \sup_{\nu}\Big\{-k\,\nu^2 - (\partial_q h + b\,q)\;\nu - \varphi\,(\nu - \rho\,(\mu^+ + \mu^-))^2\Big\} \tag{A.1}$$

with terminal condition $h(T, \boldsymbol{\mu}, q) = -\alpha\, q^2$. The first order condition provides the optimal speed of trading in feedback form as

$$\nu^* = \frac{-(\partial_q h + b\,q) + 2\,\varphi\,\rho\,(\mu^+ + \mu^-)}{2(k+\varphi)}\,, \tag{A.2}$$

and upon inserting into the DPE we find the nonlinear equation which h should satisfy:

$$\begin{aligned} 0 = {} & (\partial_t + \mathcal{L}^{\boldsymbol{\mu}})\,h + b\,(\mu^+ - \mu^-)\,q \\ & + \frac{1}{4(k+\varphi)}\big(\partial_q h + b\,q - 2\,\varphi\,\rho\,(\mu^+ + \mu^-)\big)^2 - \varphi\,\rho^2\,(\mu^+ + \mu^-)^2\,. \end{aligned} \tag{A.3}$$

Next, to solve (A.3) we observe that the PDE and its terminal conditions are at most quadratic in q, which suggests the further ansatz

$$h(t, \boldsymbol{\mu}, q) = h_0(t, \boldsymbol{\mu}) + h_1(t, \boldsymbol{\mu})\,q + h_2(t, \boldsymbol{\mu})\,q^2\,,$$

subject to the terminal conditions $h_0(T, \boldsymbol{\mu}) = h_1(T, \boldsymbol{\mu}) = 0$ and $h_2(T, \boldsymbol{\mu}) = -\alpha$. On inserting this ansatz, expanding the expression, and collecting terms with like powers in q, setting each power of q to zero separately since it must hold every q, we find that h_i satisfy the following coupled equations:

$$0 = \;(\partial_t + \mathcal{L}^{\boldsymbol{\mu}})\;h_2 + \frac{\big(h_2 + \frac{1}{2}b\big)^2}{k+\varphi}\,, \tag{A.4a}$$

$$0 = \;(\partial_t + \mathcal{L}^{\boldsymbol{\mu}})\;h_1 + \frac{h_1 - 2\,\varphi\,\rho\,(\mu^+ + \mu^-)}{k+\varphi}\,\big(h_2 + \tfrac{1}{2}\,b\big) + b\,(\mu^+ - \mu^-)\,, \tag{A.4b}$$

$$0 = \;(\partial_t + \mathcal{L}^{\boldsymbol{\mu}})\;h_0 + \frac{1}{4(k+\varphi)}\big(h_1 - 2\varphi\,\rho\,(\mu^+ + \mu^-)\big)^2 - \varphi\,\rho^2\,(\mu^+ + \mu^-)^2\,. \tag{A.4c}$$

Now observe that (A.4a) for h_2 contains no source terms dependent on μ and its terminal condition is independent of μ, hence the solution must also be independent from μ, and straightforward computations show that it is given by

$$h_2(t) = -\left(\tfrac{T-t}{k+\varphi} + \tfrac{1}{\alpha - \frac{1}{2}b}\right)^{-1} - \tfrac{1}{2}\,b\,.$$

Next, we observe that (A.4b) is a linear PDE for h_1 with nonlinear source terms. The general solution of such an equation can be represented using the Feynman–Kac theorem as follows:

$$h_1(t,\boldsymbol{\mu}) = \mathbb{E}_{t,\boldsymbol{\mu}}\left[\int_t^T e^{\int_t^s \tilde{h}_2(u,\boldsymbol{\mu}_u)\,du}\left\{-2\,\varphi\,\rho\,\tilde{h}_2(s,\boldsymbol{\mu}_s)\,(\mu_s^+ + \mu_s^-) + b\left(\mu_s^+ - \mu_s^-\right)\right\} ds\right],$$

where $\mathbb{E}_{t,\boldsymbol{\mu}}[\,\cdot\,]$ represents expectation conditional on $\mu_t^\pm = \mu^\pm$ and

$$\tilde{h}_2(t,\boldsymbol{\mu}_t) := \tfrac{1}{k+\varphi}\left(h_2(t,\boldsymbol{\mu}_t) + \tfrac{1}{2}b\right).$$

The above expectation can be simplified by explicitly computing

$$e^{\int_t^s \tilde{h}_2(u,\boldsymbol{\mu}_u)\,du} = \exp\left\{-\tfrac{1}{k+\varphi}\int_t^s \left(\tfrac{T-s}{k+\varphi} + \tfrac{1}{\alpha-\frac{1}{2}b}\right)^{-1} ds\right\} = \frac{(T-s)+\zeta}{(T-t)+\zeta},$$

and recall that $\zeta = \frac{k+\varphi}{\alpha-\frac{1}{2}b}$. Inserting this integral into the expression for h_1 above, and interchanging the expectation and outer integral, we arrive at (12b).

Finally, we can solve for $h_0(t,\boldsymbol{\mu})$ by again noticing it is a linear PDE with nonlinear source term and a straightforward application of Feynman–Kac, and interchanging integration and expectation, we obtain (12c). ■

Appendix A.2. Proof of Proposition 5. To solve the DPE (26), we seek a solution of the form

$$H(t,x,S,\boldsymbol{\mu},V,q) = x + q\,S + h(t,\boldsymbol{\mu},V,q).$$

Inserting this ansatz into (26), we find that h must satisfy

$$\begin{aligned}0 = \left(\partial_t + \mathcal{L}^{\boldsymbol{\mu},V}\right)h - \varphi\left((\mathfrak{N}-q) - \rho\,V\right)^2 + b\,(\mu^+ - \mu^-)\,q \\ + \sup_\nu\left\{-k\,\nu^2 - (\partial_q h + b\,q)\,\nu\right\},\end{aligned} \tag{A.5}$$

subject to the terminal condition $h(T,\boldsymbol{\mu},V,q) = -\alpha\,q^2$. The first order condition provides us with the optimal trading rate in feedback form as

$$\nu^* = -\tfrac{1}{2k}\left(\partial_q h + b\,q\right),$$

and upon inserting this result into (A.5) we find that h satisfies the nonlinear PDE

$$0 = \left(\partial_t + \mathcal{L}^{\boldsymbol{\mu},V}\right)h - \varphi\left((\mathfrak{N}-q) - \rho\,V\right)^2 + b\,(\mu^+ - \mu^-)\,q + \tfrac{1}{4k}\left(\partial_q h + b\,q\right)^2. \tag{A.6}$$

Next, since the terminal conditions and PDE are at most quadratic in q, we use the further ansatz

$$h(t,\boldsymbol{\mu},V,q) = h_0(t,\boldsymbol{\mu},V) + q\,h_1(t,\boldsymbol{\mu},V,q) + q^2\,h_2(t,\boldsymbol{\mu},V,q)$$

with the terminal conditions $h_0(T,\boldsymbol{\mu},V) = h_1(T,\boldsymbol{\mu},V) = 0$ and $h_2(T,\boldsymbol{\mu},V) = -\alpha$. Inserting this ansatz into (A.6), expanding and collecting powers of q, and setting each power of q to zero (since the equation must hold for all q), we find that h_i satisfy the following coupled system of PDEs:

$$0 = \left(\partial_t + \mathcal{L}^{\boldsymbol{\mu},V}\right) h_2 + \tfrac{1}{k}\left(h_2 + \tfrac{1}{2}b\right)^2 - \varphi\,, \tag{A.7a}$$

$$0 = \left(\partial_t + \mathcal{L}^{\boldsymbol{\mu},V}\right) h_1 + \tfrac{1}{k}\left(h_2 + \tfrac{1}{2}b\right) h_1 + 2\,\varphi\,(\mathfrak{N} - \rho\, V) + b\,(\mu^+ - \mu^-)\,, \tag{A.7b}$$

$$0 = \left(\partial_t + \mathcal{L}^{\boldsymbol{\mu},V}\right) h_0 + \tfrac{1}{4k}\,(h_1)^2 - \varphi\,(\mathfrak{N} - \rho\, V)^2\,. \tag{A.7c}$$

Now observe that (A.7a) for h_2 contains no source terms dependent on V or $\boldsymbol{\mu}$ and its terminal condition is also independent of V and $\boldsymbol{\mu}$, hence the solution must also be independent of V and $\boldsymbol{\mu}$ and we write $h_2(t)$ to show this independence. Moreover, h_2 satisfies the Riccati equation

$$0 = \partial_t h_2 + \tfrac{1}{k}\left(h_2 + \tfrac{1}{2}b\right)^2 - \varphi\,,$$

which can be solved explicitly to find (28a). Next, we observe that (A.4b) is a linear PDE for h_1 with nonlinear source terms. The general solution of such an equation can be represented using the Feynman–Kac theorem as follows:

$$h_1(t,\boldsymbol{\mu},V) = \mathbb{E}_{t,\boldsymbol{\mu},V}\left[\int_t^T e^{\frac{1}{k}\int_t^s (h_2(u)+\frac{1}{2}b)\,du}\,\left\{2\varphi(\mathfrak{N} - \rho\, V_s) + b\,(\mu_s^+ - \mu_s^-)\right\} ds\right].$$

By explicitly computing the integral in the expectation we arrive at (28b). Similarly, we can solve for h_0 using Feynman–Kac since it is a linear PDE with nonlinear source terms to arrive at (28c). ■

Appendix A.3. Proof of Theorem 6. Since $x + q\,S + h_0(t,\boldsymbol{\mu},V) + h_1(t,\boldsymbol{\mu},V)\,q + h_2(t)\,q^2$ is clearly a classical solution, standard results imply that it suffices to check that the feedback control is indeed an admissible strategy. To this end, from the feedback form of the optimal control ν in (Appendix A.2), and the explicit form of $h_2(t)$ and $h_1(t,\boldsymbol{\mu})$ we find that

$$\nu_t^* = \xi\,\frac{\gamma\, e^{\xi\,(T-t)} + e^{\xi\,(T-t)}}{\gamma\, e^{\xi(T-t)} - e^{-\xi\,(T-t)}}\,Q_t^{\nu^*} \tag{A.8a}$$

$$- \,\xi^2 \int_t^T \frac{\gamma\, e^{\xi\,(T-u)} - e^{-\xi\,(T-u)}}{\gamma\, e^{\xi\,(T-t)} - e^{-\xi\,(T-t)}}\left(\mathfrak{N} - \rho\,\mathbb{E}\left[V_u \,\middle|\, \mathcal{F}_t^{\boldsymbol{\mu},V}\right]\right) du \tag{A.8b}$$

$$- \,\frac{b}{2\,k}\int_t^T \frac{\gamma\, e^{\xi\,(T-u)} - e^{-\xi\,(T-u)}}{\gamma\, e^{\xi\,(T-t)} - e^{-\xi\,(T-t)}}\,\mathbb{E}\left[(\mu_u^+ - \mu_u^-)\,\middle|\, \mathcal{F}_t^{\boldsymbol{\mu},V}\right] du\,. \tag{A.8c}$$

Next, using the equality

$$\xi\left\{\gamma\, e^{\xi\,(T-t)} + e^{\xi\,(T-t)}\right\} = \xi(\gamma+1) + \xi^2\int_t^T \left(\gamma\, e^{\xi\,(T-u)} - e^{-\xi\,(T-u)}\right) du\,,$$

the first and second lines above can be rewritten as in (29). Finally, note that since $\mu^{\pm}$ are assumed to have finite second moment, clearly (A.8b) and (A.8c) have finite expectation, and by explicitly solving $dQ_t^{\nu^*} = -\nu_t^*\,dt$ (see Remark 2 and explicitly (30)) so does (A.8a). Hence, ν^* is admissible. ■

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