5.4. Differential Equations and e^{At}.

- For differential equations, $\frac{dy}{dt} = A \cdot u(t)$. $\xrightarrow{\text{solution}}$ $u(t) = u(0)\, e^{At}$.
  $\downarrow$ exponential of A.

$\Rightarrow$ how to define the exponential of matrix A ?

@ Relationship between eigenvalues and special solutions ($e^{\lambda t}$) of differential eqn.

$$\frac{du(t)}{dt} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} u(t) \end{bmatrix} \rightarrow \begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{matrix} a\,x(t) + b\,y(t) & -① \\ c\,x(t) + d\,y(t) & -② \end{matrix}$$

$$②: \quad x(t) = \frac{1}{c}\left(y'(t) - d\,y(t)\right). \quad -②'$$

$②' \rightarrow ①$

$$\frac{1}{c}\left(y''(t) - d \cdot y'(t)\right) = \frac{a}{c}\left(y'(t) - d\,y(t)\right) + b\,y(t).$$

$$y''(t) - (a+d)\,y'(t) + (ad-bc)\,y(t) = 0. \qquad y(t) = e^{\lambda t}.$$

$\Rightarrow$ characteristic equation, $\lambda^2 - (a+d)\lambda + (ad-bc) = 0$

$\hookrightarrow \quad y(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$

- by eigenvalue problem of A

same !

$$\det \begin{bmatrix} a-\lambda & b \\ c & d-\lambda \end{bmatrix} = (a-\lambda)(d-\lambda) - bc = 0.$$

$$\Downarrow$$

$$\lambda^2 - (a+d)\lambda + (ad-bc) = 0$$

$\Rightarrow \quad \frac{d}{dt}u(t) = A\,u(t) \rightarrow$ find eigenvalue of A to get $e^{\lambda t}$

$\Rightarrow \quad u(t) = c_1 e^{\lambda_1 t} x_1 + c_2 e^{\lambda_2 t} x_2$

Examples. $\dfrac{du}{dt} = Au = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} u.$

대입 → 성립.

$\rightarrow \quad \lambda_1 = -1, \quad x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad \lambda_2 = -3, \quad x_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix},$

$\Rightarrow$ solutions, $u(t) = c_1 e^{\lambda_1 t} x_1 + c_2 e^{\lambda_2 t} x_2$

$$\begin{cases} Ax_1 = \lambda_1 x_1 \\ Ax_2 = \lambda_2 x_2 \end{cases}$$

$$= \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} c_1 e^{\lambda_1 t} \\ c_2 e^{\lambda_2 t} \end{bmatrix}.$$

$$= \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \text{⑧}.$$

Initial condition $u(0) = c_1 x_1 + c_2 x_2 = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = Sc. \quad , c = S^{-1} u(0).$

$$u(t) = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix} \begin{bmatrix} x_1 & x_2 \end{bmatrix}^{-1} u(0).$$

$$= S e^{\Lambda t} S^{-1} u(0) \quad , \quad e^{\Lambda t} = \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix}.$$

↳ solution of $\dfrac{du(t)}{dt} = Au(t)$

exponential of matrix.

∘ definition of exponential of matrix.

$$\rightarrow e^{\Lambda t} = \begin{bmatrix} e^{\lambda_1 t} & & \\ & \ddots & \\ & & e^{\lambda_n t} \end{bmatrix}.$$

∘ for a general matrix A,

→ use, $e^x = 1 + x + \frac{x^2}{2!} + \cdots$

replace $\begin{cases} x \rightarrow At \\ 1 \rightarrow I. \end{cases}$

$$e^{At} = I + At + \frac{(At)^2}{2!} + \cdots\cdots$$ → series converges!

$$\begin{cases} (e^{As})(e^{At}) = e^{A(s+t)}, \\ (e^{At})(e^{-At}) = I. \end{cases}$$

$$\frac{d}{dt}(e^{At}) = Ae^{At}.$$

$$e^{At} = I + At + \frac{(At)^2}{2!} + \cdots$$

$$= I + S\Lambda S^{-1}t + \frac{S\Lambda^2 S^{-1}t^2}{2!} + \cdots$$

$$= S\left(I + \Lambda t + \frac{(\Lambda t)^2}{2!} + \cdots\right)S^{-1}$$

$$= Se^{\Lambda t}S^{-1}$$

∘ $\frac{du}{dt} = Au(t)$ → $u(t) = u(0)e^{At} = u(0) \cdot Se^{\Lambda t}S^{-1}$

(if A is diagonalized.)

$$e^{\Lambda t} = \begin{bmatrix} e^{\lambda_1 t} & & \\ & \ddots & \\ & & e^{\lambda_n t} \end{bmatrix}$$

$$= c_1 e^{\lambda_1 t}x_1 + \cdots + c_n e^{\lambda_n t}x_n$$

where, $c = S^{-1}u(0)$.

→ when A is not diagonalized, there is another form (Jordan form) → $te^{\lambda t}$.