1.6 Inverse and Transpose.

- $Ax = b \rightarrow A^{-1}b = x. \Leftrightarrow AA^{-1} = A^{-1}A = I$
- not all A has its inverse matrix.

Note 1. The inverse exists iff elimination produces n pivots.
(row exchanges are allowed.)

Note 2. The inverse matrix is unique.
when $A^{-1} = B$ or C.
$(BA)C = B(AC) \Leftrightarrow C = B$.

Note 3. If A is invertible,

$Ax = b \quad A^{-1}Ax = A^{-1}b \Leftrightarrow x = A^{-1}b.$

Note 4. assume there is a nonzero vector x such that $Ax = 0$
Then, A can't have an Inverse.
$Ax = 0 \rightarrow x = 0.$ for invertible A.

Note 5. for $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \rightarrow A^{-1} = \dfrac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Det $A = ad - bc. \neq 0 \rightarrow$ invertible.
In Matlab, finding n non-zero pivots for invertibility

Note 6. Diagonal matrix.

$$A = \begin{bmatrix} d_1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_n \end{bmatrix} \rightarrow A^{-1} = \begin{bmatrix} 1/d_1 & & & \\ & 1/d_2 & & \\ & & \ddots & \\ & & & 1/d_n \end{bmatrix}$$

◎ The inverses come in reverse order.

$$(AB)^{-1} = B^{-1}A^{-1}$$
$$(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$$

◎ Calculation of A^{-1} : Gauss - Jordan Method.

$$AA^{-1} = I. \quad A^{-1} = [x_1 \; x_2 \; \cdots \; x_n].$$

ex)
$$\begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix} \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ ⇒ 3 system equations for x_1, x_2, x_3. with the same A.

- elimination process at the same time for 3 system equations.
  ⇒ Gauss - Jordan method. : $A = LU \Leftrightarrow A^{-1} = U^{-1}L^{-1}$

$$[A \; e_1 \; e_2 \; e_3] = \begin{bmatrix} 2 & 1 & 1 & 1 & 0 & 0 \\ 4 & -6 & 0 & 0 & 1 & 0 \\ -2 & 7 & 2 & 0 & 0 & 1 \end{bmatrix}$$

- pivot 2 → $\begin{bmatrix} 2 & 1 & 1 & 1 & 0 & 0 \\ 0 & -8 & -2 & -2 & 1 & 0 \\ 0 & 8 & 3 & 1 & 0 & 1 \end{bmatrix}$

- pivot -8 → $\begin{bmatrix} 2 & 1 & 1 & 1 & 0 & 0 \\ 0 & -8 & -2 & -2 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{bmatrix} = [U \; L^{-1}]$ → 1st-half elimination

↳ $E_{32}E_{31}E_{21} = L^{-1}$ G.E이 L^{-1} 임.

by multiplying U^{-1} → $U^{-1}[U \;\; L^{-1}] = [I \;\; U^{-1}L^{-1}]$

$= [I \;\; A^{-1}]$

→ 좌측 3×3 부분을 I로 변환. → 이 과정이 U^{-1} 곱하기. (마치 $E_{32}E_{31}E_{21}$ 있듯)

(row1 − row3, row2 + 2 × row3) → $\left[\begin{array}{ccc|ccc} 2 & 1 & 0 & 2 & -1 & -1 \\ 0 & -8 & 0 & -4 & 3 & 2 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right]$

row 1 + $\frac{1}{8}$ × row2 → $\left[\begin{array}{ccc|ccc} 2 & 0 & 0 & \frac{3}{2} & -\frac{5}{8} & -\frac{3}{4} \\ 0 & -8 & 0 & -4 & 3 & 2 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right]$

divide by pivots. → $\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{3}{4} & -\frac{5}{16} & -\frac{3}{8} \\ 0 & 1 & 0 & \frac{1}{2} & -\frac{3}{8} & -\frac{1}{4} \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right] = [I \;\; U^{-1}L^{-1}] = [I \;\; A^{-1}]$

※ det A = the product of the pivots $(2)\cdot(-8)\cdot(1) = -16$

Remark 1. $Lc = b$ and $Ux = c$ are better than $A^{-1}b = x$.

⊙ Invertible = non-singular. (n pivots)

⊙ Transpose Matrix.

$$(A^T)_{ij} = A_{ji}$$

$$(A+B)^T = A^T + B^T$$

$$(AB)^T = B^T \cdot A^T \rightarrow$$

$(A^{-1})^T = (A^T)^{-1}$ $\qquad A^{-1}A = I \Leftrightarrow (A^{-1}A)^T = I^T = I$

$\Leftrightarrow A^T \cdot (A^{-1})^T = I.$

$\Leftrightarrow$ $(A^T)^{-1}$.

- Symmetric Matrix.

  $A^T = A \Rightarrow$ square matrix.

  $a_{ij} = a_{ji}$

- if A is symmetric and invertible, A^{-1} is too.

  pf) $AA^{-1} = I \Rightarrow (AA^{-1})^T = I^T = I.$

  $(A^{-1})^T A^T = (A^{-1})^T \cdot A = I.$

  $A^{-1} = (A^{-1})^T \rightarrow$ symmetric.

- Symmetric Products R^TR, $R \cdot R^T$, and LDL^T.

  $\Rightarrow R^TR \neq R \cdot R^T$

- suppose $A = A^T$, $A = LDU$. $\rightarrow A^T = A = U^T D^T \cdot L^T$

  $= LDL^T$

  $(L^T = U)$

by symmetricity, we reduce elimination processes.

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HW 1.6 10, 12, 15, 17, 39, 48

HW 1.5 5, 7, 9, 18, 19, 25.