◎ Dimension of a Vector Space.

- infinitely many different bases.
- number of basis vectors are same for the vector space.
  - ↳ degrees of freedom = dimension.

- Any linearly independent set in V can be extended to a basis by adding more vectors.

- Any spanning set in V can be reduced to a basis, by discarding vectors if necessary.

※ basis is a maximal independent set.
minimal spanning set.

ex) 4-dimensional vector → (a, b, c, d)
4-dimensional subspace in $\mathbb{R}^6$ → (0, a, b, c, d, 0)

HW 2.3 4, 10, 19, 31, 34

2.4 4 fundamental Subspaces.

- How to find an explicit basis? → a systematic procedure.

A: $m \times n$ matrix.

1. The column space, $\mathbb{C}(A)$: Its dimension is the rank r.
2. The null space, $N(A)$: Its dimension is $n-r$
3. The row space, $\mathbb{C}(A^T)$: Its dimension is also r.
4. The left null space, $N(A^T) \Rightarrow A^T y = 0$. Its dimension is $m-r$.

$N(A)$ and $\mathbb{C}(A^T)$ are subspaces of $\mathbb{R}^n$.
$N(A^T)$ and $\mathbb{C}(A)$ are subspaces of $\mathbb{R}^m$.

ex) $A = U = R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ 2×3

- $\mathbb{C}(A)$: $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ line.
- $\mathbb{C}(A^T)$: line through $[1, 0, 0]^T \in \mathbb{R}^3$
- $N(A)$: a plane contains $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ $u=0$ 인 plane (u, v, w).
- $N(A^T)$: line contains $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

ex) $A = \begin{bmatrix} 1 & 3 & 3 & 2 \\ 2 & 6 & 9 & 7 \\ -1 & -3 & 3 & 4 \end{bmatrix} \rightarrow U = \begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

- row space of A ⇒ non zero rows of U are a basis, and the # of pivots (or non-zero rows) is the dimension of row space of A or U.

dim A = dim U ⇒ No change of dimension by G.E.

◎ nullspace of A ;→ by elimination $Ax=0 \rightarrow Ux=0$.
nullspaces of A and U are the same.
→ its dimension is $n-r$. (# of free variables)
→ The special solutions are a basis.

special solutions. $\begin{cases} v=1 \\ y=0 \end{cases}$ $x_1 = \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, $\begin{cases} v=0 \\ y=1 \end{cases}$ $x_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$

$c_1x_1 + c_2x_2 = 0 \rightarrow c_1 = c_2 = 0$. for free variables.
$n-r = 4-2$ vectors are a basis.

Kernel : = nullspace. $\dim(N) = \text{nullity} = n-r$.

◎ Column Space of A
→ The pivot columns of A (or U) are a basis.
→ Dim (A) and Dim (C(A)) equals the rank r, → row space dim.
→ # of indep. columns equals # of indep. rows.

◎ left nullspace of A (A^T nullspace) → $y^TA = 0$ left-sided vector.

$$y^TA = [y_1 \ \cdots \ y_m] \begin{bmatrix} A \end{bmatrix} = [0 \ \cdots \ 0]$$

$$\begin{cases} \dim C(A) + \dim N(A) = \# \text{ of columns} = n. \\ r + \dim N(A) = n \end{cases}$$

$$\dim C(A^T) + \dim N(A^T) = \# \text{ of rows} = m.$$
$$r + \dim N(A^T) = m \Rightarrow N(A^T) = m - r$$

Ex 1. $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$, $m = n = 2$, $r = 1$ $\xrightarrow{G \cdot E}$ $\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$

1) column space : all multiples of $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$

orthogonal

2) nullspace : all multiples of $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$ $\leftarrow$ $Ax = 0$

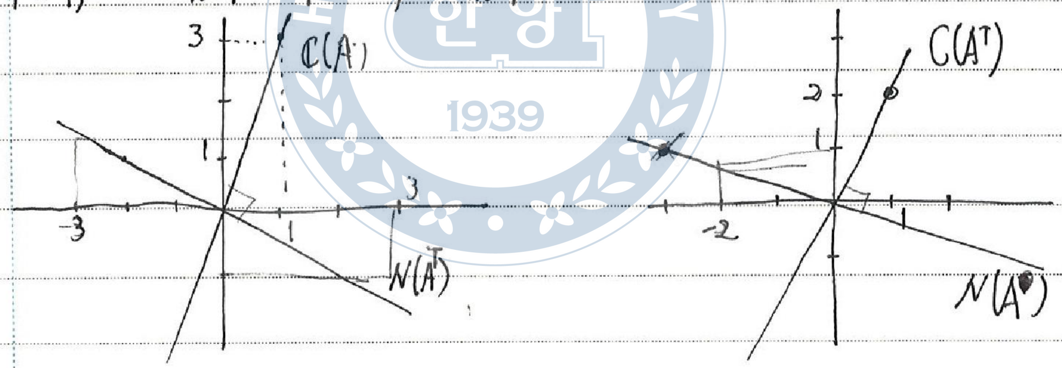
3) row space : all multiples of $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

4) left nullspace : all multiples of $\begin{bmatrix} -3 \\ 1 \end{bmatrix}$ $\leftarrow$ $A^T y = 0$.

$\nwarrow$ $A^T = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \xrightarrow{G \cdot E} \begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix}$

$\Rightarrow$ all four subspaces are lines.

$r = 1$, $n - r = 1$, $m - r = 1$.



◎ Existence of Inverses. (A : $m \times n$ matrix)

$\rightarrow$ an inverse exists only when the rank is as large as possible.

$\rightarrow$ left inverse $A^{-1}A = I$ $\Rightarrow$ $r = n$ ($m \geq n$)

$\rightarrow$ right inverse $AA^{-1} = I$. $\Rightarrow$ $r = m$ $\rightarrow$ ($n \geq m$)

$\rightarrow$ two-sided inverse $AA^{-1} = A^{-1}A = I$ $r = m = n$ for square mat.

∘ for $r = m$ $\rightarrow$ there is a right-inverse $\rightarrow$ $Ax = b$ always has a solution

$AC = I$ $\rightarrow$ $Ax = b$ $\rightarrow$ $A\underline{C}b = b$ $\rightarrow$ for ∞ solutions

$\boxed{A}\boxed{A^{-1}} = I_{m \times m}$ $\rightarrow$ infinitely many sol. $\rightarrow$ infinite right-inverse.

- for $r = n$, → there is a left-inverse, → Solution is <u>unique</u>.

→ $BA = I$ $\quad Ax = b \rightarrow BAx = Bb \rightarrow x = Bb$

↳ unique.

◎ one-sided inverses.

→ $BA = I \Leftrightarrow (A^TA)^{-1}A^T = A^{-1} = B.$

→ $AC = I \Rightarrow A^T(AA^T)^{-1} = A^{-1} = C.$

in chap. 3 → (A^TA have an inverse if the rank is n,
AA^T have an inverse if the rank is m,

Ex. 2. $A = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 5 & 0 \end{bmatrix} \rightarrow r = m = 2.$

(예) $A = \begin{bmatrix} 5 & 0 & 1 \\ 0 & 4 & 2 \end{bmatrix} \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$Ax = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $Ay = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$) 풀면

$x = x_3 \begin{bmatrix} -\frac{1}{5} \\ -\frac{1}{2} \\ 1 \end{bmatrix} + \begin{bmatrix} \frac{1}{5} \\ 0 \\ 0 \end{bmatrix}$, $y = x_3 \begin{bmatrix} -\frac{1}{5} \\ -\frac{1}{2} \\ 1 \end{bmatrix} + \begin{bmatrix} \cdots \\ \cdots \\ 0 \end{bmatrix}$

$AC = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 5 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{5} \\ c_{31} & c_{32} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

↳ arbitrary many right-inverses (c_{31}, c_{32})

no left-inverse. ∵ $n = 3 > r = 2$. not as large as possible.

Best right-inverse : $A^T(AA^T)^{-1} = \begin{bmatrix} 4 & 0 \\ 0 & 5 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{16} & 0 \\ 0 & \frac{1}{25} \end{bmatrix}$

$= \begin{bmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{5} \\ 0 & 0 \end{bmatrix} = C.$

→ pseudo inverse – a way of choosing the best inverse in 6.3

let $A^T = \begin{bmatrix} 4 & 0 \\ 0 & 5 \\ 0 & 0 \end{bmatrix}$.

$$BA^T = \begin{bmatrix} \frac{1}{4} & 0 & b_{13} \\ 0 & \frac{1}{5} & b_{23} \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 5 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

↳ infinitely many left-inverses. for any b_{13} and b_{23}.

$$A^T x = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \Rightarrow \begin{bmatrix} 4 & 0 \\ 0 & 5 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

To have a solution, $b_3 = 0$, if not ($b_3 \neq 0$), no sol.

Then,

$$x = BA^T x = Bb = \begin{bmatrix} \frac{1}{4} & 0 & b_{13} \\ 0 & \frac{1}{5} & b_{23} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{4} b_1 \\ \frac{1}{5} b_2 \end{bmatrix}.$$ ⇒ b_{13}, b_{23}에 무관.

↳ unique x.

- A rectangular matrix can't have both existence and uniqueness.
- A square matrix can't have one property without the other.

$B = C = A^{-1}$ (only one inverse), $r = m = n$.

- Invertible conditions that are necessary & sufficient for $m = n = r$.
  1) The columns span $\mathbb{R}^n$, so $Ax = b$ has one solution for every b. (linear combination)
  2) The columns are independent, $Ax = 0$ has only the sol. $x = 0$.
  3) The rows span $\mathbb{R}^n$
  4) The rows are linearly independent.
  5) Elimination can be complete : $PA = LDU$ with all n pivots

6) $\det(A) \neq 0$

7) The eigenvalue of A is not zero. ($Ax = \lambda x$)

8) $A^T A$ is positive definite.

◎ Matrix of Rank 1.

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 4 & 2 & 2 \\ 8 & 4 & 4 \\ -2 & -1 & -1 \end{bmatrix} \rightarrow (\text{column vector}) \cdot (\text{row vector})$$

$$= \begin{bmatrix} 1 \\ 2 \\ 4 \\ -1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \end{bmatrix}$$

HW 2.4 2, 3, 25, 30, 28.