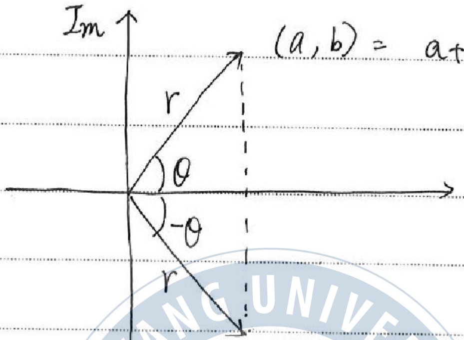
5.5 Complex Matrix

° Complex number : $a+ib$, ($i^2=-1$, a and b are real)

Complex conjugate : $\overline{a+ib} = a-ib$ → mirror image of real axis.



$(a,b) = a+ib = r\cos\theta + i\,r\sin\theta = re^{j\theta}$

$(a,-b) = a-ib = r\cos\theta - i\,r\sin\theta = re^{-j\theta}$

· Absolute value : $(a+ib)\overline{(a+ib)} = (a+ib)(a-ib) = a^2+b^2 = r^2$

$|a+ib| = r = \sqrt{a^2+b^2}$

polar form : $a+ib = re^{j\theta}$ $\quad \theta = \tan^{-1}\left(\frac{b}{a}\right)$

∘ Complex vector.

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad x_j = a_j + ib_j$$

length, $\|x\|^2 = |x_1|^2 + |x_2|^2 + \cdots + |x_n|^2$

$= \bar{x}_1 \cdot x_1 + \bar{x}_2 x_2 + \cdots + \bar{x}_n x_n$

$= \bar{x}^T \cdot x$

· Inner product : $\bar{x}^T y$ $(\neq \bar{y}^T x)$ → the order should be important.

$= x^H y$ $\quad x^H = \bar{x}^T$ (Hermitian)

(when $x^H y$ is real $x^H y = y^H x$.) ↳ conjugate transpose

$$\begin{bmatrix} 2+i & 3i \\ 4-i & 5 \\ 0 & 0 \end{bmatrix}^H = \begin{bmatrix} 2-i & 4+i & 0 \\ -3i & 5 & 0 \end{bmatrix}.$$

1) Orthogonal x and y. $\rightarrow$ $x^H y = 0$ (the order should be kept)
($\Rightarrow y^H x = 0$ real)

2) $\|x\|^2 = x^H x = |x_1|^2 + |x_2|^2 + \cdots + |x_n|^2$

3) $(AB)^H = B^H A^H$

4) $(x^H y)^H = y^H x$. $\quad x^H y \overset{\text{conjugate}}{\longleftrightarrow} y^H x$

◎ Hermitian Matrices.

- Symmetric matrix in real components : $A^T = A$.
- Hermitian matrix : $A^H = A$. $\Rightarrow$ $A_{ij} = \bar{A}_{ji}$
  $\rightarrow$ The diagonal entries must be real.

Symmetric matrices $\subset$ Hermitian matrices.

property 1) If $A = A^H$, $x^H A x$ is real. $A \rightarrow n\times n$ square mat.

$\Rightarrow$ $\underset{\substack{\downarrow \\ \text{a number (not vector)}}}{(x^H A x)^H} = x^H A^H x = x^H A x$. $\Rightarrow$ ($\bar{y} = y \rightarrow y$ is real)

Property 2) If $A^H = A$, every eigenvalue is real.

$\rightarrow$ $Ax = \lambda x$. for complex vector x.

$$\underbrace{x^H A x}_{\text{real.}} = x^H \lambda x = \lambda x^H x = \underbrace{\lambda \| x \|^2}_{\text{real.}}$$

$$\lambda = \frac{x^H A x}{\| x \|^2} \rightarrow \text{real.}$$

(if A = correlation mat $\Rightarrow R^T R$)

$$\frac{\| Rx \|^2}{\| x \|^2} = \lambda > 0 \rightarrow \text{positive.}$$

Property 3) two eigenvectors of Hermitian matrix, if they come from different eigenvalues, are orthogonal to one another.

$\rightarrow$ $Ax_1 = \lambda_1 x_1$, $Ax_2 = \lambda_2 x_2$, $\lambda_1 \neq \lambda_2$. and $A^H = A$.

$$(Ax_1)^H \cdot x_2 = x_1^H A^H x_2 = x_1^H A x_2 \quad ①$$

$$② = (\lambda_1 x_1)^H x_2 = x_1^H \underbrace{\lambda_1^H}_{\text{real}} x_2 = \lambda_1 x_1^H x_2.$$

$$x_1^H A x_2 = x_1^H \lambda_2 x_2 = \lambda_2 x_1^H x_2 = \lambda_1 x_1^H x_2.$$

$$(\lambda_2 - \lambda_1) x_1^H x_2 = 0 \rightarrow \lambda_1 \neq \lambda_2 \text{ thus, } \begin{cases} x_1^H x_2 = 0 \\ x_2^H x_1 = 0. \end{cases}$$

∘ when we choose ~~of~~ the unit eigenvectors, $\| x_j \| = 1$,

for $A^T = A$ (real symmetric matrix)

$\left(\dfrac{x_1 x_1^T}{x_1^T x_1} \Rightarrow \text{Projection} \right)$

$$A = S \Lambda S^{-1} = Q \Lambda Q^{-1} = Q \Lambda Q^T = \lambda_1 \underbrace{x_1 x_1^T}_{\substack{\text{projection onto } x_1. \\ \| x_1 \|^2 = 1}} + \cdots + \lambda_n \underbrace{x_n x_n^T}_{''}$$

(Q: orthonormal matrix.

$Q^T Q = I \rightarrow Q^{-1} = Q^T$)

$A = \Sigma$ projection 이 λ_i 만큼 가중치로 표현

(A는 orthogonal (& independent) eigenvector 에 대해서 λ_i 만큼 projection 성분을 가짐 (A는 주로 correlation matrix : $A^H = A$.)

⇒ All symmetric matrices are combinations of one-dimensional projections
~~which a~~ onto eigenvectors which are orthogonal to one another.

spectral theorem.

$$A = S\Lambda S^{-1} = Q\Lambda Q^{-1} = Q\Lambda Q^{T} \qquad (Q^{-1} = Q^{T} : \text{orthogonal, normal})$$

$$= \begin{bmatrix} | & | & & | \\ x_1 & x_2 & \cdots & x_n \\ | & | & & | \end{bmatrix} \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix} \begin{bmatrix} - & x_1 & - \\ - & x_2 & - \\ & \vdots & \\ - & x_n & - \end{bmatrix}$$

→ for n distinct eigen

$$= \lambda_1 x_1 x_1^T + \lambda_2 x_2 x_2^T + \cdots + \lambda_n x_n x_n^T.$$

↳ 1-D projection.

Remark : If ~~A^T~~ A is real, and its eigenvalues happen to be real,
then eigenvectors are also real.
→ $(A-\lambda I)x = 0$ can be computed by elimination.
↳ not using $\det(A-\lambda I)=0$.

If A is real, all complex eigenvalues come in conjugate pairs.
⇒ $Ax = \lambda x$, $A\bar{x} = \bar{\lambda}\bar{x}$

◎ Unitary matrix.

for real orthogonal , $Q^T Q = I$.
for complex " , $U^H U = I$ → unitary matrix.
↳ a number on the unit circle →

∘ Unitary matrix : $U^H U = U U^H = I$, $U^H = U^{-1}$.

property 1. angle and length are preserved.

$$(Ux)^H (Uy) = x^H U^H U y = x^H y$$ → inner products are preserved after Unitary transform.

$$\|Ux\|^2 = (Ux)^H (Ux) = x^H U^H U x = x^H x = \|x\|^2$$

property 2. Every eigenvalue of U has absolute value $|\lambda| = 1$.

→ $Ux = \lambda x$ → $\|Ux\| = \|\lambda x\| = |\lambda|\,\|x\|$

↓ length is presered.

$\|Ux\| = \|x\| = |\lambda|\,\|x\|$, $|\lambda| = 1$.

property 3. Each eigenvector from different eigenvalue is orthonormal

$Ux_1 = \lambda_1 x_1$, $Ux_2 = \lambda_2 x_2$

→ $x_1^H x_2 = (Ux_1)^H (Ux_2) = x_1^H U^H U x_2 = x_1^H x_2$

↑ angle preserved,

$$= (\lambda_1 x_1)^H (\lambda_2 x_2)$$
$$= \bar{\lambda}_1 \lambda_2 x_1^H x_2$$

$$x_1^H x_2 (1 - \bar{\lambda}_1 \lambda_2) = 0.$$

since $\bar{\lambda}_1 \lambda_1 = 1$, $\bar{\lambda}_2 \lambda_2 = 1$. $1 \neq \bar{\lambda}_1 \lambda_2$. → $x_1^H x_2 = 0$.

- Skew-Hermitian : $K^H = -K$

If A is Hermitian, then $K = iA$ is skew-Hermitian.

⇒ $K^H = (iA)^H = -iA^H = -iA = -K$.

page 288. table : Real versus Complex.

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