· for 2 ~~of~~ samples, of 1 unknown,

$$E^2 = w_1^2(x-b_1)^2 + w_2^2(x-b_2)^2$$

$$\hat{x}_w = \frac{w_1^2 b_1 + w_2^2 b_2}{w_1^2 + w_2^2}$$ //

$\Rightarrow$ $WAx = Wb$

$\Rightarrow$ $A^T W^T W A x = A^T W^T W b.$

HW 3.3 2, 6, 9, 18, 41

3.4. ORThogonal Basis & Gram-Schmidt

$q_1, \cdots q_n$ are orthonormal

∘ $q_i^T q_j = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$ $\rightarrow$ $Q = [q_1 \; q_2 \cdots q_n].$

∘ $Q^T Q = I$ $\Rightarrow$ left-inverse.

$\Rightarrow$ $Q^{-1} = Q^T$

ex) · rotation matrix $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ $\rightarrow$ axes rotation.

· Permutation matrix $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$ $\Rightarrow$ $P^{-1} = P^T.$

- Geometrically, an orthogonal Q is the product of a rotation and a reflection.

- projection reduces the length of a vector, but orthonormal matrix preserves angles and lengths.

$$\|Qx\|^2 = x^T \underbrace{Q^T Q}_{I} x = \|x\|^2 \rightarrow \text{length conservation.}$$

$$(Qx)^T (Qy) = x^T \underbrace{Q^T Q}_{I} y = x^T y \rightarrow \text{inner product or angle conservation.}$$

- for any vector b,

$$b = x_1 q_1 + x_2 q_2 + \cdots + x_n q_n \leftarrow Qx = b$$

$$q_i^T b = x_i \quad \text{since} \begin{cases} q_i^T q_j = 0, & i \neq j \\ 1, & i = j \end{cases}$$

$$\Rightarrow b = (q_1^T b) q_1 + \cdots + (q_n^T b) q_n$$

$$b = Qx \Leftrightarrow x = Q^T b = Q^{-1} b$$

Remark 1 > 1-D projection onto a line, a. $= \frac{a^T b}{a^T a} a$.

$\rightarrow$ for q_i, $\frac{q_i^T b}{q_i^T q_i} q_i = (q_i^T b) q_i = x_i q_i$, ($x_i$: projection of q_i of b.)

Remark 2. $Q^T = Q^{-1} \Rightarrow QQ^T = I$, for square cases

$\rightarrow$ The rows are also orthonormal

◎ Rectangular Matrix with Orthonormal Columns.

- for $m>n$; least square cases,

$$Qx = b$$

$\rightarrow Q^TQ\hat{x} = Q^Tb$: normal equation for least squares,

$\hookrightarrow \hat{x} = Q^Tb = \cancel{Q^{-1}b}$ since $Q^TQ = I$

$$p = Q\hat{x}$$

$$\Rightarrow p = Q(\underbrace{Q^TQ}_{I})^{-1}Q^Tb$$

$$= QQ^Tb$$

$Q^T \rightarrow$ left-inverse.

$(m>n)$
$$QQ^T = \begin{array}{c c} & \overset{n}{} \\ n\left\{\right. & \begin{bmatrix} 1 & 0 & \cdots & 0 & \\ & & & \vdots & 0 \\ \vdots & 1 & & & \\ 0 & 0 & 1 & & \\ & 0 & & & 0 \end{bmatrix} \end{array} = \begin{bmatrix} I_{n\times n} & 0 \\ 0 & 0 \end{bmatrix}$$

$0_{m-n \times m-n}$.

Ex3) $b = (x, y, z) \longrightarrow$ project onto $(x-y)$ plane.

$$q_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad q_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

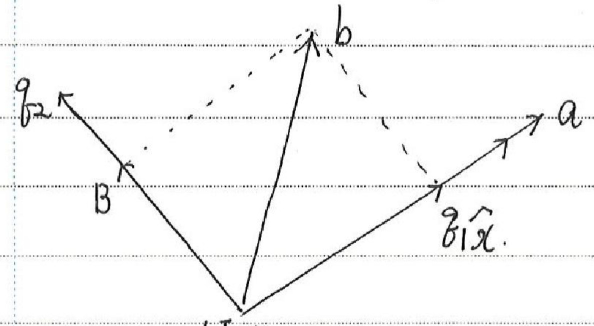
$$q_1^Tb = x, \quad q_2^Tb = y,$$

$$P = QQ^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$Pb = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix}.$$ $\rightarrow x-y$ plane.

◎ Gram - Schmidt Orthogonalization.

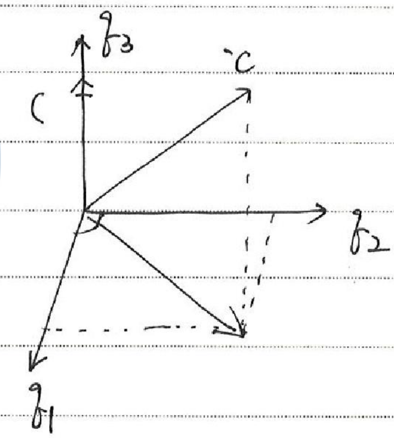
→ find the orthonormal basis given independent vectors.



$$q_1 = \frac{a}{\|a\|}$$

$$B = b - \frac{b^Ta}{a^Ta}a = b - (q_1^Tb)q_1 = (q_2^Tb)q_2$$

$$b = (q_1^Tb)q_1 + (q_2^Tb)q_2$$



$$C = c - [(q_1^Tc)q_1 + (q_2^Tc)q_2]$$
$$= (q_3^Tc)q_3$$

$$c = \sum_{i=1}^{3}(q_i^Tc)q_i$$

⇒ For given independent vectors, $\{a_1, \cdots, a_n\}$

$$\begin{cases} A_j = a_j - \sum_{i=1}^{j-1}(q_i^Ta_j)q_i = (q_j^Ta_j)q_j \\ a_j = \sum_{i=1}^{j}(q_i^Ta_j)q_i \end{cases}$$

$$q_j = \frac{A_j}{\|A_j\|}$$ → normalization.

Ex 5)

$$a = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad c = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

$$q_1 = \frac{a}{\|a\|} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$B = b - (q_1^T b) q_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - \frac{1}{\sqrt{2}} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ 0 \\ -\frac{1}{2} \end{bmatrix}$$

$$q_2 = \frac{\sqrt{2}}{2} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$C = c - (q_1^T c) q_1 - (q_2^T c) q_2$$

$$= \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} - \sqrt{2} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix} - \sqrt{2} \begin{bmatrix} \frac{\sqrt{2}}{2} \\ 0 \\ -\frac{\sqrt{2}}{2} \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} //$$

$$= q_3.$$

◎ Factorization $A = QR$

$$\begin{bmatrix} a_1 & a_2 & a_3 & \cdots & a_n \end{bmatrix} = \begin{bmatrix} (q_1^T a_1) q_1 & (q_1^T a_2) q_1 + (q_2^T a_2) q_2 & (q_1^T a_3) q_1 + (q_2^T a_3) q_2 + (q_3^T a_3) q_3 & \sum_{i=1}^{n} (q_i^T a_n) q_i \\ & & & \ddots \end{bmatrix}$$

→ linear combination of each colum vector.

$$= \begin{bmatrix} q_1 & q_2 & q_3 & \cdots & q_n \end{bmatrix} \begin{bmatrix} (q_1^T a_1) & (q_1^T a_2) & q_1^T a_3 & \cdots & q_1^T a_n \\ 0 & (q_2^T a_2) & q_2^T a_3 & & \vdots \\ 0 & 0 & q_3^T a_3 & \ddots & \vdots \\ \vdots & 0 & 0 & & \\ 0 & 0 & 0 & & q_n^T a_n \end{bmatrix}.$$

$= Q \cdot R.$

$\downarrow$

upper triangular matrix.

◎ Function Spaces and Fourier Series (optional topic).

→ extend vector space to ~~finite~~ function space.

→ apply Gram - Schmidt orthogonalization to function space.

1. Hilbert Space.

→ consider $\mathbb{R}^\infty$ space , $v = (v_1, v_2, \cdots\cdots,)$

→ include a finite length of $\mathbb{R}^\infty$ vector , $\|v\| < \infty$.

↳ function is defined in a finite interval.

∘ vector space of Hilbert space.

$\|v_1 + v_2\| \leq \|v_1\| + \|v_2\| < \infty$ → addition $\in \mathbb{R}^\infty$

$\|cv_1\| = |c|\|v_1\| < \infty$ → scalar multiplication $\in \mathbb{R}^\infty$.

∘ Hilbert space is a vector space in $\mathbb{R}^\infty$ where vectors have finite lengths.