Chapter 4. Determinants

∘ Some important properties of determinant.

1. A^{-1} exist when $\det A \neq 0$.

2. $\det A$ equals the volume of a box in $\mathbb{R}^n$ space.

ex) Jacobian.

$$J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{vmatrix} = r, \quad \left(\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ z &= z \end{aligned} \right. \quad \text{Cylindrical coord.}$$

$$dx\,dy\,dz \doteq r\,dr\,d\theta\,dz.$$

3. $\det A = \pm$ (product of the pivots)

4. Cramer's rule for $Ax = b$.

$$x_1 = \frac{\det A_1}{\det A} \qquad A_1 = \begin{bmatrix} b & a_2 & a_3 \cdots & a_n \end{bmatrix}.$$

4.2 Properties of Determinant

→ demonstrated by 2x2 matrix. $\det\begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc.$

◎ Basic Properties.

1. $\det I = 1.$

2. The determinant changes sign when two rows are exchanged.

3. The determinant depends linearly on the first row.

ex) $\begin{vmatrix} a+a' & b+b' \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a' & b' \\ c & d \end{vmatrix}$

$$\begin{vmatrix} ta & tb \\ c & d \end{vmatrix} = t\begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

· $\det(A+B) \neq \det(A) + \det(B)$

· $\det(tA) \neq t\det(A)$

$\hookrightarrow t^n \det(A).$

◎ Additional Properties by 1~3.

4. If two rows are equal, then $\det(A) = 0.$ → by prop. 2

5. Subtracting a multiple of one row from another row leaves the same determinant.

row operation $\begin{vmatrix} a-lc & b-ld \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \cancel{\begin{vmatrix} -lc & -ld \\ c & d \end{vmatrix}} \to 0$

6. If A has a row of zeros, then $\det(A) = 0$.

7. If A is triangular, $\det(A) =$ product of diagonal entries
$= a_{11}\, a_{22} \cdots a_{nn}$.

pf) Gauss elimination can remove all the off-diagonal elements without changing the determinant (by rule 5).

$\det(D) = a_{11} \cdots a_{nn}$ for diagonal matrix

8. (If A is singular, $\det(A) = 0$.
A is non-singular (invertible), $\det(A) \neq 0$. $\to$ $\pm$product of pivots.

9. $\det(AB) = \det(A) \cdot \det(B)$.

$\det(A^{-1}) = \dfrac{1}{\det(A)}$

$d(A) = \dfrac{\det(AB)}{d(B)} = \det(A)$ 증명

by property 1~3.
(1) $B = I$
(2) row exchange of A.
(3).

10. $\det(A^T) = \det(A)$.

$PA = LDU \to \det(PA) = \det(P) \det A$
$= \det(L) \det(D) \det(U)$

$\det(PA)^T = \det(A^T) \det(P^T)$
$= \det(U^T) \det(D^T) \det(L^T)$

det $\Rightarrow$ $L, D, U \to$ diagonal product.