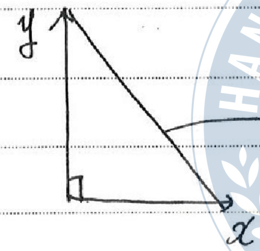
Chapter 3. Orthogonality

3.1 Orthogonal Vectors and Subspaces

- orthogonality ⇒ independent basis, easy calculation .....
- coordinate axes are orthogonal.
- the fundamental subspaces meet at right angles.

- length of vector : $\|x\|^2 = \sum_{i=1}^{n} x_i^2 = x^T x$.
- for a right angle, by Pythagoras



$$\|x\|^2 + \|y\|^2 = \|x - y\|^2$$

$$x^T x + y^T y = (x-y)^T (x-y)$$

$$= x^T x - y^T x - x^T y + y^T y$$

$x^T y = y^T x = 0$ for perpendicular

⇒ $x^T y$: inner product.

◎ $x^T y = 0$ for orthogonal (right angle)

$x^T y < 0$ for angle $> 90°$

$x^T y > 0$ for angle $< 90°$

- If nonzero vectors $v_1, \cdots, v_k$ are mutually orthogonal,

  ⇒ then the vectors are independent.

pf) $c_1 v_1 + c_2 v_2 + \cdots + c_k v_k = 0$

for any v_i, $v_i^T (c_1 v_1 + \cdots + c_k v_k) = c_i \|v_i\|^2 = 0$.

⇒ $c_i = 0$ for $\forall i$

→ any linear combinations of $v_1 \sim v_k$ can't have nonzero coefficients for $\sum c_i v_i = 0$.

◎ Orthogonal Subspaces.

⇒ Every vector in one subspace must be orthogonal to every vector in the other subspace.

$v \in V, \quad w \in W \qquad v^T w = 0$ for $\forall$ v and w.

◎ $\left\langle \begin{array}{l} \text{row space is orthogonal to the nullspace in } \mathbb{R}^n \\ \text{column space is } \quad '' \quad \text{the left nullspace in } \mathbb{R}^m \end{array} \right.$

pf) ∘ $Ax = 0 \rightarrow x$: nullspace.

$$Ax = \begin{bmatrix} \text{row 1} \\ \text{row 2} \\ \vdots \\ \vdots \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

→ each row vector and x are perpendicular. $(\text{row})^T \cdot x = 0$. for all row.

- thus, x is orthogonal to the row space in $\mathbb{R}^n$

∘ $A^T y = 0 \rightarrow y$: left nullspace.

$$\begin{bmatrix} \text{column 1} \\ \vdots \\ \text{column } n \end{bmatrix} \begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix} = 0 \quad \text{for } \forall \quad (\text{column of } A)^T \cdot y = 0$$

→ the left nullspace is orthogonal to the column space.

2nd pf) $Ax = 0$, x: nullspace.

let $v = A^T z$ → linear combination of row vectors.
→ row space.

$x^T v = x^T A^T z = (Ax)^T z = 0.$ for any z.
↳ $x \perp v$.

Ex 3) $A = \begin{bmatrix} 1 & 3 \\ 2 & 6 \\ 3 & 9 \end{bmatrix}$.

- row space : $[1, 3]^T$ and multiple → line.
- null space : $Ax = 0 \to x = [-3, 1]^T$

  ⇒ $(row)^T \cdot x = 0$ → orthogonal.

} in R^2

- column space : $[1, 2, 3]^T$ multiple → line.
- left nullspace : $A^T y = 0 \to y ⇒ y_1 + 2y_2 + 3y_3 = 0$ plane.

⇒ in R^3

$[1, 2, 3]^T$ is the normal vector of plane $y_1 + 2y_2 + 3y_3 = 0$.

(row space) + (nullspace) $= r + (n-r) = n$
(column space) + (left nullspace) $= r + (m-r) = m$.

Definition : $V \in R^n$, the space of all vectors orthogonal to V is called the orthogonal complement of V.

$V^{\perp} = V$ perp.

$N(A) = (C(A^T))^{\perp} \Rightarrow C(A^T) = (N(A))^{\perp}$ rowspace ⊥ nullspace
$C(A) = (N(A^T))^{\perp}$ column space ⊥ left nullspace

dim (row space) + dim (null space) = # of columns
dim (column space) + dim (left nullspace) = # of rows.

⇒ row space contains every orthogonal vectors to the nullspace.
⇒ column space 〃 〃 〃 left null space.

◎ $Ax = b$. → b to be in the column space (linear comb. of c.s)
or → b to be perpendicular to the left nullspace.

⇒ $Ax = b$ is solvable iff $y^T b = 0$ for $y^T A = A^T y = 0$
↳ left nullspace.

when we know one or two vectors, y such that $y^T A = 0$,
it's easy to check the solution by $y^T b = 0$.

◎ orthogonal complement ≠ orthogonal.

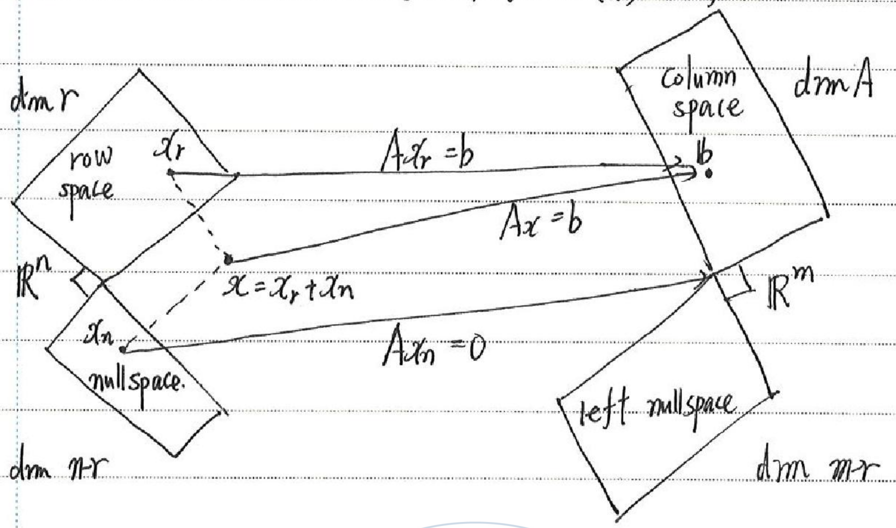
ex) two lines spanned by $V(0,1,0)$, $(0,0,1)$ are not orthogonal complement,
V W

W should be a plane to be V^T.

If $V = W^T$ and $W = V^T$ → $\dim V + \dim W = n$.
$V^{TT} = V$

⇒ $\mathbb{R}^n$ is decomposed into two perpendicular parts.

Fig 3.4 The true action $Ax = A(x_r + x_n)$ of $m \times n$ matrix.



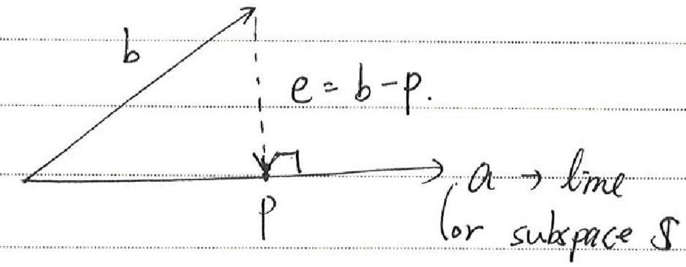
} action between row space and column space.

} → nothing happening

$\left\{ \begin{array}{l} x_r \text{ is a projection of } x \text{ onto row space.} \\ x_n \quad '' \quad ' \quad \text{nullspace.} \end{array} \right.$

HW 3.1 9, 48, 27.

3.2 Cosines and Projections onto lines.

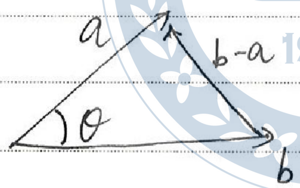


p : projection
(closest point on line a to b.)

Generally → p is the projection of b onto the subspace.

↳ practically, this is the problem of least-squares solution to an overdetermined system, where there is no solution of $Ax = b$. p is an best selection.

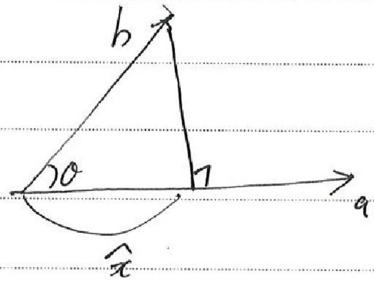
- Inner product and Cosines.



$$\cos\theta = \frac{a^T b}{\|a\|\ \|b\|}$$

Law of Cosines $\quad \|b - a\|^2 = \|b\|^2 + \|a\|^2 - 2\|a\|\ \|b\| \cos\theta.$

- projection onto a line



$(b - \hat{x}a) \perp a.$

$a^T b - \hat{x} a^T a = 0.$

$\hat{x} = \dfrac{a^T b}{a^T a} \qquad p = \dfrac{a^T b}{a^T a} a.$

- Schwarz inequality

$|a^T b| \leq \|a\|\ \|b\|. \quad \rightarrow \quad |\cos\theta| \leq 1.$

equality iff b is a multiple of a

ex 1) $a = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $b = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

$$\hat{x} = \frac{a^Tb}{a^Ta} = \frac{6}{3} = 2. \Rightarrow p = 2a = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}.$$

$$\cos\theta = \frac{6}{\sqrt{3}\sqrt{14}} \Rightarrow 6 \leq \sqrt{3}\cdot\sqrt{14}.$$

◎ Projection Matrix of Rank 1.

→ by a trivial change ($\hat{x} = \frac{a^Tb}{a^Ta} \rightarrow p = \hat{x}a.$)

→ the projection is performed by a matrix. $\underline{Pb \rightarrow p}$

$$p = a\frac{a^Tb}{a^Ta} = \frac{aa^Tb}{a^Ta} \quad \underset{\text{matrix}}{P} = \frac{aa^T}{a^Ta}$$

ex) $a = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ $\quad P = \frac{aa^T}{a^Ta} = \frac{1}{3}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$

1) P : symmetric.

2) $P^2 = P$, → $P^2b = Pp = p$. already on the line.

↳ projection. definition.

P → column space consists of the line through a.

→ null space consists of the plane perpendicular to a

→ rank = 1.

$Pb = \hat{x}a$ → every column of P is the multiple of a, so Pb is the multiple of a ($\mapsto \hat{x}a$).

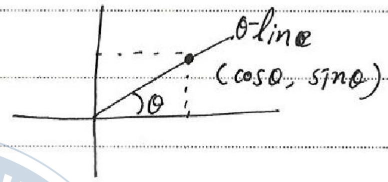
when. $Pb = 0$, then $b \perp a$.

∘ Remark on scaling – no influence on the scaling.

$$a = \begin{bmatrix}2\\2\\2\end{bmatrix} \rightarrow P = \frac{1}{12}\begin{bmatrix}2\\2\\2\end{bmatrix}[2\ .\ 2\ \ 2] = \frac{1}{3}\begin{bmatrix}1&1&1\\1&1&1\\1&1&1\end{bmatrix}$$

ex 3). Project onto θ-line.

$$a = \begin{bmatrix}\cos\theta\\ \sin\theta\end{bmatrix}. \qquad \|a\| = 1.$$



$$P = \frac{a a^T}{a^T a} = \begin{bmatrix}c\\s\end{bmatrix}[c, s] = \begin{bmatrix}c^2 & cs\\ cs & s^2\end{bmatrix}.$$

∘ Transposes from inner Products.

$$(Ax)^T y = x^T(A^T y).$$

HW 3.2 17, 19, 21