generally, $2 \det A_{n-1} + \begin{vmatrix} -1 & -1 & 0 & 0 & \cdots \\ 0 & 2 & -1 & 0 & \cdots \\ 0 & -1 & 2 & -1 & \\ \vdots & & & \vdots & \end{vmatrix} = 2 \det A_{n-1} - A_{n-2} + \begin{vmatrix} 0 & 1 & 0 & \cdots \\ 0 & 2 & & \\ \vdots & \vdots & & \\ 0 & & & \end{vmatrix}$

$$\det A_n = 2 \det A_{n-1} - \det A_{n-2}.$$

(→ 등차수열 (등차중항)
↳ Arithmetic mean)

$\det A_1 = 2$, $\det A_2 = 3$, $\det A_3 = 4$, $\cdots$ $\det A_n = n+1$.

HW 4.3 2, 3, 11, 14

4.4 Applications of Determinants

1. Computation of A^{-1}.

C: cofactor matrix. $A^{-1} = \dfrac{C^T}{\det A}$, $(A^{-1})_{ij} = \dfrac{C_{ji}}{\det A}$.

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{n1} & & a_{nn} \end{bmatrix} \begin{bmatrix} C_{11} & & C_{n1} \\ \vdots & & \\ C_{1n} & & C_{nn} \end{bmatrix} = \begin{bmatrix} \det A & & 0 \\ & \ddots & \\ 0 & & \det A \end{bmatrix}$$

ex) → row 1 of A, row 2 of C

$$a_{11}C_{21} + a_{12}C_{22} + \cdots + a_{1n}C_{2n} = 0$$

cofactor는 특정 열과 행을 제외한 행렬에서 얻어지는데, $C_{21} \sim C_{2n}$ 의 경우 row 1 을 포함한 것임. 따라서 row 1 이 두 번 중복되어 위의 식은 zero 가 됨 (마치 row1 을 row2 로 넣고 det A 구하는 것과 같기 때문임.)

2. The solution $Ax = b$.

$$x = A^{-1}b = \frac{C^T b}{\det A} \Rightarrow x_i = \frac{\det B_i}{\det A}$$

$$B_i = \begin{array}{c} \left[\begin{array}{ccccc} a_{11} & a_{12} & \cdots & b_1 & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & b_n & & a_{nn} \end{array}\right] \\ \qquad\qquad \uparrow \\ \qquad\qquad i\text{ th column.} \end{array}$$

$$\det B_i = b_1 C_{1i} + b_2 C_{2i} + \cdots + b_n C_{ni}$$

$\underline{\det(A^T) = \det A}$

b가 column vector 이므로 A^T 형태로 determinant 구해짐.

3. The volume of a Box.

- right-angled box $\rightarrow$ Orthogonal rows.

$$AA^T = \begin{bmatrix} \text{row 1} \\ \text{row 2} \\ \vdots \\ \text{row } n \end{bmatrix} \begin{bmatrix} \text{r} & & \text{r} \\ \text{o} & \cdots & \text{o} \\ \text{w} & & \text{w} \\ 1 & & n \end{bmatrix} = \begin{bmatrix} l_1^2 & & \\ & \ddots & \\ & & l_n^2 \end{bmatrix}.$$

$$\det(AA^T) = \det(A)\det A^T = (\det A)^2 = l_1^2 \cdot l_2^2 \cdots l_n^2.$$

$\det A = l_1 l_2 \cdots l_n$ for right-angled boxes.

$\rightarrow$ The determinant equals the volume

- For parallelogram,
  - $\rightarrow$ find the projection of ~~each~~ each row, to make a rectangular space.
  - $\rightarrow$ using Gram-Schmidt orthogonalization