2.6 Linear Transformations

∘ $Ax = b$

↳ b is a linear combination of column vectors of A with coefficients in x.

• x is transformed (mapped) into b by A.

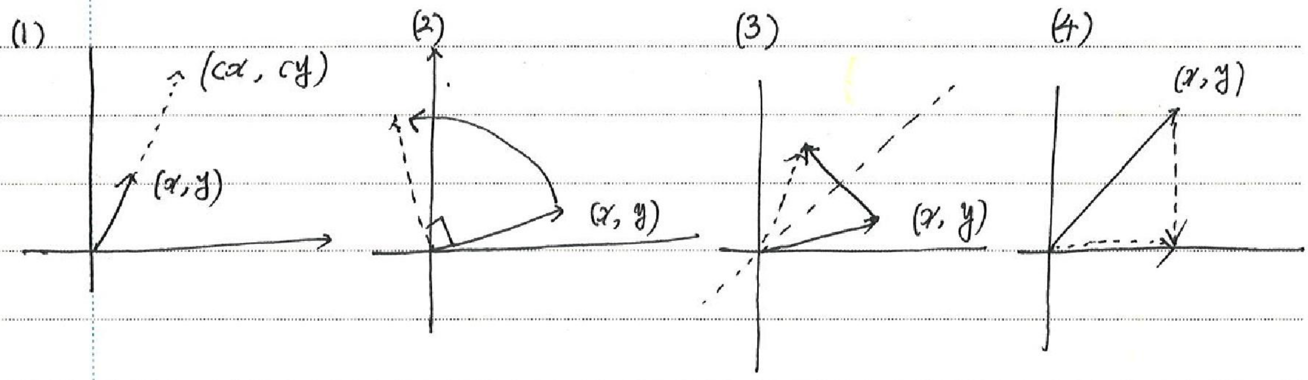
Examples.

1) $A = \begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix}$ → stretching (extending or contracting)

2) $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ → 90° rotation

3) $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ → reflection by ~~$y=x$~~. $x_2 = x_1$

4) $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ → projection onto x.



⊙ Linear Transformation $T(x) = Ax$.

(i) The origin can't be moved, $Ax \Rightarrow A \cdot 0 = 0$ for $\forall A$.

(ii) $A(cx) = c(Ax)$.

(iii) $A(x+y) = Ax + Ay$.

$A(ax + by) = a(Ax) + b(Ay)$.

A matrix $\xrightarrow{\circ}$ linear Transformation
$\xleftarrow[\circ]{}$

x needs not be vectors (function, ⋯)

$R^n \rightarrow R^m$ by $m \times m$ mat.

For polynomial vectors $p(t) = a_0 + a_1 t + a_2 t^2 + \cdots a_n t^n \in P_n(t) \rightarrow \text{rank} = n+1$

a kind of vector.
→ $n+1$ dimension
$p(t) = 0$ for $\forall t \rightarrow a_i = 0$, $i = 0 \sim n$

ex1) differentiation. $A = d/dt$ is linear.

$Ap(t) = a_1 + 2a_2 t + \cdots + n a_n t^{n-1}$

nullspace → 1-D. , rank = n,

→ $Ap(t) = 0$ for $p(t) = a_0$ → 1-dimensional

ex2) integration

$$Ap(t) = \int_0^t (a_0 + \cdots + a_n t^n)\, dt$$

$$= a_0 t + \cdots + \frac{a_n}{n+1} t^{n+1}$$ → n-dimensional column space.

→ no nullspace except zero vector. $Ap(t) = 0$ only if $p(t) = 0$.

→ no constant term → the constant term ∋ left nullspace.

ex3) multiplication by a polynomial

$$Ap(t) = (2+3t)(a_0 + \cdots + a_n t^n) = 2a_0 + \cdots + 3a_n t^{n+1}$$

→ $P_n \rightarrow P_{n+1}$.

→ no nullspace except zero.

◎ Transformations Represented by Matrices.

∘ If we know Ax for each basis vector, then we know Ax for entire vector space.

Linearity : $x = c_1 x_1 + \cdots + c_n x_n \rightarrow Ax = c_1(Ax_1) + \cdots + c_n(Ax_n)$

$= c_1 y_1 + \cdots c_n y_n = [y_1 y_2 \cdots y_n] \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$

ex 4)

가장 기본적으로 A 구하는 경우

basis ⇒ $\begin{bmatrix}1\\0\\0\\ \vdots\end{bmatrix}\begin{bmatrix}0\\1\\ \vdots\end{bmatrix}\cdots\begin{bmatrix}0\\0\\ \vdots\\1\end{bmatrix}$

$$\left.\begin{array}{ll} x_1 = \begin{bmatrix}1\\0\end{bmatrix}, & Ax_1 = \begin{bmatrix}2\\3\\4\end{bmatrix} \\ x_2 = \begin{bmatrix}0\\1\end{bmatrix}, & Ax_2 = \begin{bmatrix}4\\6\\8\end{bmatrix}\end{array}\right\} \Rightarrow T(x) = A = \begin{bmatrix}2 & 4\\3 & 6\\4 & 8\end{bmatrix}$$

계수 $\begin{bmatrix}c_1\\c_2\\ \vdots\\c_n\end{bmatrix}$ = x가 됨

Ax

for a different basis (1, 1), (2, −1)

$$A\begin{bmatrix}1\\1\end{bmatrix} = A\cdot(x_1+x_2) = \begin{bmatrix}4\\6\\8\end{bmatrix} + \begin{bmatrix}2\\3\\4\end{bmatrix} = \begin{bmatrix}6\\9\\12\end{bmatrix}$$

$$A\begin{bmatrix}2\\-1\end{bmatrix} = A(2x_1 - x_2) = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

예) $\begin{bmatrix}2 & 4\\3 & 6\\4 & 8\end{bmatrix}\begin{bmatrix}-1\\2\end{bmatrix} = \begin{bmatrix}6\\9\\12\end{bmatrix}$

다름.

예) $\begin{bmatrix}6 & 0\\9 & 0\\12 & 0\end{bmatrix}\begin{bmatrix}-1\\2\end{bmatrix} = \begin{bmatrix}-6\\-9\\-12\end{bmatrix}$.

- polynomial case : differentiation & integration.
  1) first find ~~an~~ a basis. → most fundamental $\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}, \dots$
  2) then determine transformation (matrix) on the basis.

ex) degree 3 polynomial. $\mathbb{P}_3$

$\begin{bmatrix}1\\t\\t^2\\t^3\end{bmatrix}$ ⇒ 계수로 vector 구성.

⇒ Basis for $\mathbb{P}_3$: $p_1 = 1,\ p_2 = t,\ p_3 = t^2,\ p_4 = t^3$ (not unique)

→ $\frac{d}{dt} = A$: $Ap_1 = 0$, $Ap_2 = 1 = p_1$, $Ap_3 = 2t = 2p_2$, $Ap_4 = 3t^2 = 3p_3$

→ a ~~matrix~~ vector form of each basis polynomial

$p_1 = (1, 0, 0, 0)$, $p_2 = (0, 1, 0, 0)$, $p_3 = (0, 0, 1, 0)$

$p_4 = (0, 0, 0, 1)$

$$Ap_1 = A\begin{bmatrix}1\\0\\0\\0\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\end{bmatrix}, \qquad Ap_2 = A\begin{bmatrix}0\\1\\0\\0\end{bmatrix} = \begin{bmatrix}1\\0\\0\\0\end{bmatrix}$$

$$Ap_3 = A\begin{bmatrix}0\\0\\1\\0\end{bmatrix} = \begin{bmatrix}0\\2\\0\\0\end{bmatrix} = 2p_2, \qquad Ap_4 = 3p_3 = A\begin{bmatrix}0\\0\\0\\1\end{bmatrix} = \begin{bmatrix}0\\0\\3\\0\end{bmatrix}.$$

$$A_{diff} = \begin{bmatrix}0&1&0&0\\0&0&2&0\\0&0&0&3\\0&0&0&0\end{bmatrix}$$ → free var. ⇒ p_1 , pivot. p_2, p_3, p_4

special sol = $\begin{array}{c} \\ p_1\end{array}\begin{bmatrix}1\\0\\0\\0\end{bmatrix}$.

⇒ null space : p_1 ⇒ $Ap = 0$ → 상수항 (p_1) 만 → 0으로

⇒ column space contains : p_1, p_2, p_3. → 미분후 solution space (3차 미분 2차식)

⇒ row space (?) ⇒ 1차, 2차, 3차식 → 적분 결과.

for $p(t) = 2 + t - t^2 - t^3$, by linearity

$$\frac{dp}{dt} = Ap = \begin{bmatrix}0&1&0&0\\0&0&2&0\\0&0&0&3\\0&0&0&0\end{bmatrix}\begin{bmatrix}2\\1\\-1\\-1\end{bmatrix} = \begin{bmatrix}1\\-2\\-3\\0\end{bmatrix} \rightarrow 1 - 2t - 3t^2.$$

If we know, a matrix (transformation) and its corresponding basis

→ transformation of every vector is known.

$x_1, x_2, \cdots, x_n$: a basis for V

$y_1, y_2, \cdots, y_m$: a basis for W.

$T: V \longrightarrow W$

Column j of A : Ax_j.

⑨ Integration. $\mathbb{P}_3 \rightarrow \mathbb{P}_4$

first find basis. $\rightarrow$ $\{1, t, t^2, t^3\} \rightarrow \{1, t, t^2, t^3, t^4\}$

$\int_0^t 1\, dt = t$, or $Ax_1 = y_2$. $\rightarrow (0, 1, 0, 0, 0)^T$

$\int_0^t t\, dt = \frac{1}{2}t^2$ or $Ax_2 = \frac{1}{2}y_3$ $\rightarrow [0, 0, \frac{1}{2}, 0, 0]^T$

$\vdots$ $\qquad Ax_3 = \frac{1}{3}y_4 \qquad \vdots$

$Ax_4 = \frac{1}{4}y_5$

$$A_{int} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & \frac{1}{4} \end{bmatrix}$$

$\Rightarrow$ integration followed by differentiation is invertible,

$$\underset{\substack{\swarrow \\ W \text{ space}}}{A_{diff}}\ \underset{\substack{\downarrow \\ V \text{ space}}}{A_{int}} = \begin{bmatrix} 1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{bmatrix} \Leftarrow \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & \frac{1}{4} \end{bmatrix}$$

$\Rightarrow$ differentiation is a left-inverse of integration

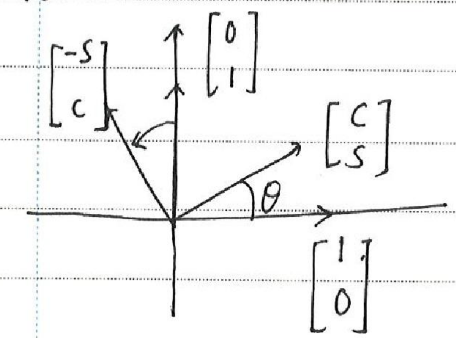
$A_{int} A_{diff} = ?$ $\rightarrow$ not I. (differentiation of constant $\rightarrow$ zero

$\rightarrow$ not unique in integration.

(상수항 term P_1 부분 $\rightarrow 0$)

Q Rotation Q, Projection P, Reflection H.

1. Rotation.



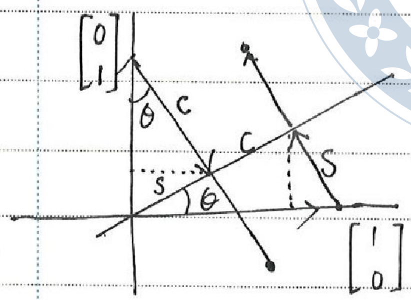
$$Q_\theta \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}, \quad Q_\theta \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}.$$

$$\Downarrow$$

$$\Rightarrow Q_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$\left.\begin{aligned} Q_\theta^{-1} &= Q_{-\theta} \quad \text{rotation backward.} \\ Q_\theta^2 &= Q_{2\theta} \\ Q_\theta \cdot Q_\phi &= Q_{\theta+\phi} \end{aligned}\right\} \text{trigonometric formulation.}$$

2. Projection.



$$P_\theta \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos^2\theta \\ \sin\theta\cos\theta \end{bmatrix}$$

$$P_\theta \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\theta \\ \sin^2\theta \end{bmatrix}$$

$$P_\theta = \begin{bmatrix} c^2 & cs \\ cs & s^2 \end{bmatrix}$$ → projection onto θ-line.

→ no inverse.

$$P_\theta^2 = P_\theta$$