◎ Null space of A.

$Ax = 0$ for more unknowns than equations.

↳ There are infinite solutions. →

↳ They form a vector space – null space of A

∘ The null space of a matrix A consists of all vectors x such that $Ax = 0$. , $N(A)$, → subspace.

(i) $Ax = 0$, and $Ax' = 0$ → $A(x+x') = 0$, closed und add.

(ii) $Ax = 0$, → $Acx = c0 = 0$, closed under scalar multiplication.

ex). (1) $A = \begin{bmatrix} 1 & 0 \\ 5 & 4 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ → homogeneous eqn. $Ax = 0$.

$N(A) \Rightarrow (0,0)$.

(2) $B = \begin{bmatrix} 1 & 0 & 1 \\ 5 & 4 & 9 \\ 2 & 4 & 6 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ → same $C(A) = C(B)$

↳ $N(B)$ is in the line. $(c, c, -c)$. $(1, 1, -1)$.

→ We will find $N(A)$, $C(A)$.

HW 2.1 5, 6, 24, 25.

2.2 Solving $Ax = 0$ and $Ax = b$.

∘ For an invertible matrix A, the nullspace $N(A)$ contains only $x=0$.
The column space is the whole space.

∘ When the nullspace contains more than the zero vector column ~~vector~~ space contains less than all vectors.

b complete solution $Ax_p = b$ and $Ax_n = 0$ → $A(x_p + x_n) = b$.

ex). $\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$.

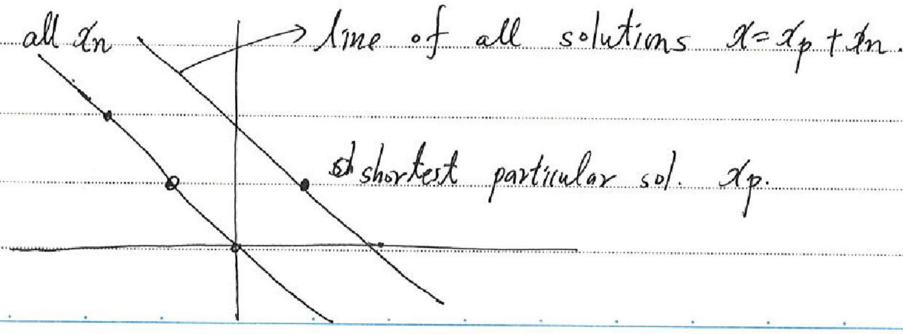
(i) $b_2 \neq 2b_1$.
→ no solution

(ii) $b_2 = 2b_1$
→ infinitely many solutions.

$x_p = (1, 1)$. ← $\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$.

$x_n = (1, 1)$ or $(-c, c)$.

$$x_p + x_n = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1-c \\ 1+c \end{bmatrix}.$$



⑨ Echelon Form U and Row Reduced Form R. for rectangular mat.

↳ the simplest matrix that elimination can give.

ex).

$$A = \begin{bmatrix} 1 & 3 & 3 & 2 \\ 2 & 6 & 9 & 7 \\ -1 & -3 & 3 & 4 \end{bmatrix}$$

$$\downarrow$$

$$A \rightarrow \begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 6 & 6 \end{bmatrix}$$ No pivot in column 2.

$$\downarrow$$

$$A \rightarrow \begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$ → Echelon matrix U.

↳ staircase pattern of pivots.

→ should be zero for last entry.

$$\downarrow$$

$$\begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$ → divide by pivot 3 in 3rd column.

$$\downarrow$$

$$\begin{bmatrix} 1 & 3 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$ → Reduced row Echelon form R.

↳ all pivots are "1".

↳ use the unit pivot row to produce zero above the pivot.

ex)

$$R = \begin{bmatrix} 1 & 0 & * & 0 & * & * & * & 0 \\ 0 & 1 & * & 0 & * & * & * & 0 \\ 0 & 0 & 0 & 1 & * & * & * & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$Rx = 0 \rightarrow Ux = 0$ and $Ax = 0$. (same solution).

Lower Triangular $L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 2 & 1 \end{bmatrix}$

$\hookrightarrow$ pivot multipliers

if necessary
$(P)A = LU$ (for rectangular matrix.)

◎ Pivot Variables and Free Variables.

for solving $Rx = 0$, the pivots are important.

$$Rx = \begin{bmatrix} 1 & 3 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\rightarrow$ pivot columns in 1st, 3rd.

special solution (reverse sign).

- pivot variables with pivot columns : u, w
- free variable without pivot columns : v, y.

solve $\Rightarrow$ $u + 3v - y = 0$, $w + y = 0$ $\Rightarrow$ $u = -3v + y$, $w = -y$

special solutions

$$x = \begin{bmatrix} u \\ v \\ w \\ y \end{bmatrix} = \begin{bmatrix} -3v + y \\ v \\ -y \\ y \end{bmatrix} = v \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + y \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

special solution

$\hookrightarrow$ $(-3, 1, 0, 0)$, $(1, 0, -1, 1)$ $\Leftarrow$ $(v=1, y=0)$ and $(v=0, y=1)$.

$\rightarrow$ Null space contains all combinations of special solutions using free variables.