1.4. Matrix Notation and multiplication.

$$
\begin{aligned}
2u + v + w &= 5 \\
4u - 6v \quad &= -2 \\
-2u + 7v + 2w &= 9
\end{aligned}
\Rightarrow
\begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix}
\begin{bmatrix} u \\ v \\ w \end{bmatrix}
=
\begin{bmatrix} 5 \\ -2 \\ 9 \end{bmatrix}
$$

$\Downarrow$ coefficient matrix.

$$
\Rightarrow u\begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix} + v\begin{bmatrix} 1 \\ -6 \\ 7 \end{bmatrix} + w\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ 9 \end{bmatrix}. \Rightarrow Ax = b.
$$

↳ Ax is a combination of the columns of A.
The coefficients are the components of x.

$$b_i = \sum_{j=1}^{n} a_{ij} \cdot x_j \Rightarrow \text{sigma notation.}$$

Identity matrix. I : $Ix = x$.

multiplication by columns : $AB = A[\,b_1 \ b_2 \ b_3\,] = [\,Ab_1 \ Ab_2 \ Ab_3\,]$.

↳ block multiplication

$A_{m\times n} \cdot B_{n\times l} = (AB)_{m\times l}$

$(AB)_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$ $\quad \begin{cases} i = 1, 2, \cdots, m \\ j = 1, 2, \cdots, l \end{cases}$

$(AB)C = A(BC)$ associative law.

$A(B+C) = AB + AC$ distributive law.

$AB \neq BA$ usually.

HW 1.4. : 19, 20, 21, 27, 45, 55, 56.

Elementary matrix in elimination steps.

$$E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -l_{21} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$l_{21} \rightarrow l_{21}$ times of row 1 and subtraction of row 2.

$$E_{21}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -l_{31} & 0 & 1 \end{bmatrix}$$

$$E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -l_{32} & 1 \end{bmatrix}$$

$$E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & l_{32} & 1 \end{bmatrix}$$

$$E_{21} A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ -2 & 7 & 2 \end{bmatrix}$$

$\rightarrow$ Gaussian elimination process result by l_{21}.

$$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A \end{bmatrix} = \begin{bmatrix} \quad \end{bmatrix}$$

식(1) × (-2) + 식(2)

1.5 Triangular Factors & Row Exchanges.

$$Ax = \begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ 9 \end{bmatrix} = b$$

step1 : $\text{row2} - l_{21} * \text{row1}$ ~~row2~~ $\rightarrow \text{row2}'$

step2 : $\text{row3} - l_{31} \cdot \text{row1} \rightarrow \text{row3}'$

step3 : $\text{row3}' - l_{32} \cdot \text{row2}'$

The result is a Upper triangular matrix U.

$$Ux = \begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 5 \\ -12 \\ 2 \end{bmatrix} = c$$

x is solved by back-substitution from $Ux = c$.

For each elementary matrix, $E = E_{21}$, $F = E_{31}$, $G = E_{32}$,

$$E_{32} E_{31} E_{21} A = U$$

$$\begin{bmatrix} 1 & & \\ & 1 & \\ & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & \\ +1 & & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ -2 & 1 & \\ & & 1 \end{bmatrix} = \begin{bmatrix} 1 & & \\ -2 & 1 & \\ -1 & 1 & 1 \end{bmatrix}.$$

from A to U :

By inverses of matrix.

→ general case 형인.

$$A = E_{21}^{-1}E_{31}^{-1}E_{32}^{-1}U. \qquad \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix} = L.$$

$$A = LU.$$

⇒ the entries below the diagonal are the multipliers, l_{21}, l_{31}, l_{32}

→ complete record of elimination

◎ Triangular factorization ; $A = LU$. with no exchanges of rows.

⇒ the diagonal entries of U are the pivots.

Ex 1. $A = \begin{bmatrix} 1 & 2 \\ 3 & 8 \end{bmatrix} \Rightarrow U = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}, \quad L = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$

$LU = A.$

Ex. 4. $A = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}$ Lower triangular case.

$\Rightarrow U = I \qquad A = L.$

$Ax = b$, first $Lc = b$, then $Ux = c$.

1) Factor : from A, find L and U.

2) Solve : from L and U, find the solution x.

→ two triangular systems in $\frac{n^2}{2}$ steps. each, right-side : n^2.

→ $O(n^2)$.

→ l_{ij} 위치만 결정.

Remark 1.

$$U = \begin{bmatrix} d_1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_n \end{bmatrix} \begin{bmatrix} 1 & u_{12}/d_1 & u_{13}/d_1 & \cdots \\ & 1 & u_{23}/d_2 & \cdots \\ & & \ddots & \vdots \\ & & & 1 \end{bmatrix}$$

$A = LU \rightarrow LDU$: D is a diagonal matrix of pivots.

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ & -2 \end{bmatrix} = \begin{bmatrix} 1 & \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & \\ & -2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ & 1 \end{bmatrix} = LDU.$$

◎ Remark 2.

LDU factorization and LU factorization are uniquely determined by A.

◎ Row Exchanges and Permutation Matrices.

· zero in the pivot position $\begin{bmatrix} 0 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \Rightarrow$ exchange rows.

$$3u + 4v = b_2$$
$$2v = b_1$$

· permutation $P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $PA = \begin{bmatrix} 3 & 4 \\ 0 & 2 \end{bmatrix} = Pb = \begin{bmatrix} b_2 \\ b_1 \end{bmatrix}$.

- A permutation matrix $\mathbb{P}$ has the same rows as the $\mathbb{I}$.
  - $\Rightarrow$ There is a single "1" in every row and column.
  - $\Rightarrow$ total $n!$ permutations.

$$\mathbb{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \mathbb{P}_{21} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbb{P}_{31} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\mathbb{P}_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad \mathbb{P}_{32}\mathbb{P}_{21} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \neq \mathbb{P}_{21}\mathbb{P}_{32} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},$$

$$\mathbb{P}^{-1} = \mathbb{P}^{T}$$

Ex. $$A = \begin{bmatrix} 0 & a & b \\ 0 & 0 & c \\ d & e & f \end{bmatrix} \Rightarrow \mathbb{P}_{23}\mathbb{P}_{13}A = \begin{bmatrix} d & e & f \\ 0 & a & b \\ 0 & 0 & c \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} A$$

$$= \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} A.$$

◎1J : · In the nonsingular cases, $\left\{ \begin{array}{l} \text{there exists a permutation matrix } \mathbb{P}. \\ \text{and } PA = LU. \end{array} \right.$

· In the singular cases, there is no $\mathbb{P}$, and elimination fails.

Practically, near-zero pivots are exchanged to reduce the round-off errors.

HW 1.5 5, 7, 9, 18, 19, 25, 33