→ linear combination of each column vector.

$$= \begin{bmatrix} q_1 & q_2 & q_3 & \cdots & q_n \end{bmatrix} \begin{bmatrix} (q_1^T a_1) & (q_1^T a_2) & q_1^T a_3 & \cdots & q_1^T a_n \\ 0 & (q_2^T a_2) & q_2^T a_3 & & \vdots \\ 0 & 0 & q_3^T a_3 & & \vdots \\ \vdots & 0 & 0 & & \\ 0 & 0 & 0 & & q_n^T a_n \end{bmatrix}.$$

$= Q \cdot R.$

↓

upper triangular matrix.

◎ Function Spaces and Fourier Series (optional topic).

→ extend vector space to ~~finite~~ function space.

→ apply Gram - Schmidt orthogonalization to function space.

1. Hilbert Space.

→ consider $\mathbb{R}^\infty$ space, $v = (v_1, v_2, \cdots\cdots)$

→ include a finite length of $\mathbb{R}^\infty$ vector, $\|v\| < \infty$.

↳ function is defined in a finite interval.

∘ vector space of Hilbert space.

$\|v_1 + v_2\| \leq \|v_1\| + \|v_2\| < \infty$ → addition $\in \mathbb{R}^\infty$

$\|cv_1\| = |c|\|v_1\| < \infty$ → scalar multiplication $\in \mathbb{R}^\infty$.

∘ Hilbert space is a vector space in $\mathbb{R}^\infty$ where vectors have finite lengths.

2. Length and Inner Product.

→ For continuous functions, the summation for length $\|f\|$ is replaced with integration in an interval.

ex) $f(x) = \sin x, \quad \in \quad \mathbb{R}^\infty$

$$\|f(x)\|^2 = \int_0^{2\pi} \sin x \cdot \sin x \; dx = \pi < \infty.$$

∘ inner product of functions. $(f, g) = \int_{Int.} f(x) \cdot g(x) \, dx$

$|(f,g)| \leq \|f\| \cdot \|g\|$ → Schwarz Inequality.

$\|f+g\| \leq \|f\| + \|g\| < \infty.$

3. Fourier Series

⇒ Orthogonal basis functions ; $\{\sin nx, \; \cos mx\}$ n, m : integer.

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{m=1}^{\infty} b_m \sin mx.$$

$$a_1 = \frac{\int f(x) \cdot \cos x \; dx}{(\cos x, \cos x)}$$ ⇒ Fourier Series is a projection onto $\cos nx$ and $\sin mx$.

→ every function is expressed by the linear combination (Fourier Series) of orthogonal basis functions ($\sin mx$, $\cos nx \cdots$) in an interval

→ each function is considered as $\mathbb{R}^\infty$ vector by the Fourier Series.

4. Gram - Schmidt for Functions.

- For polynomial functions, $1, x, x^2, \cdots$ are independent, but not orthogonal, for $0 \le x \le 1$.

$\rightarrow$ change the interval $-1 \le x \le 1$.

$b_1(x) = 1$, $\quad b_2(x) = x$,

$(b_1(x), b_2(x)) = (1, x) = \int_{-1}^{1} x \, dx = 0 \rightarrow$ orthogonal.

$$b_3(x) = x^2 - \frac{(1, x^2)\cdot 1}{(1,1)} - \frac{\overset{\nearrow 0}{(x, x^2)}}{(x,x)} \cdot x$$

$$= x^2 - \frac{1}{3}$$

$\Rightarrow (1, x^2 - \frac{1}{3}) = 0$, $\quad (x, x^2 - \frac{1}{3}) = 0 \rightarrow$ orthogonal.

$$f(x) = \frac{(f(x), 1)}{(1, 1)} \cdot 1 + \frac{(f(x), x)}{(x, x)} x + \frac{(f(x), x^2 - \frac{1}{3})}{(x^2 - \frac{1}{3}, x^2 - \frac{1}{3})} (x^2 - \frac{1}{3}) + \cdots$$

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