Since $\det(L)$, $\det(D)$, $\det(U)$ are the products of diagonal elements, $\det(D^T) = \det(D)$, $\underbrace{\det(L^T) = \det(L)}_{=1}$, $\underbrace{\det(U^T) = \det(U)}_{=1}$.

HW. 4.2 16, 18, 27,

4.3. Formulas for the Determinant.

- If A is invertible, then $PA = LDU$ and $\det P = \pm 1$.

$$\det A = \pm \det L \det D \det U = \pm(\text{product of pivots})$$

↳ depends on the # of row exchanges. ↳ MATLAB.

Ex 1)

$$A_{n\times n} = \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & -1 & \\ & & \ddots & \ddots & -1 \\ & & & -1 & 2 \end{bmatrix} = LDU = L \begin{bmatrix} 2 & & & & \\ & \frac{3}{2} & & & \\ & & \frac{4}{3} & & \\ & & & \ddots & \\ & & & & \frac{n+1}{n} \end{bmatrix} U$$

$$\det A = 2 \cdot \left(\tfrac{3}{2}\right)\left(\tfrac{4}{3}\right) \cdots \left(\tfrac{n+1}{n}\right) = n+1.$$

◎ From the properties 1–3, define the determinant generally for $n \times n$.

→ each row is broken down into vectors in the coordinate directions:

$[a \ \ b]^T \to [a \ \ 0]^T + [0 \ \ b]^T$.

$[a \ \ b \ \ c]^T \to [a \ \ 0 \ \ 0]^T + [0 \ \ b \ \ 0]^T + [0 \ \ 0 \ \ c]^T$

By the first property 1. for 2x2 matrix,

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & 0 \\ c & d \end{vmatrix} + \begin{vmatrix} 0 & b \\ c & d \end{vmatrix}$$

$$= \cancel{\begin{vmatrix} a & 0 \\ c & 0 \end{vmatrix}}^{\,0} + \begin{vmatrix} a & 0 \\ 0 & d \end{vmatrix} + \begin{vmatrix} 0 & b \\ c & 0 \end{vmatrix} + \cancel{\begin{vmatrix} 0 & b \\ 0 & d \end{vmatrix}}^{\,0}$$

↳ first row exchange 한 후 property 1 적용 하고, 다시 row exchange.

$$= ad - bc.$$

(when a column is zero vector → determinant is zero.
There are $n!$ permutations for non-zero determinants in decomposition.

◦ for 3x3 matrix,

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & & \\ & a_{22} & \\ & & a_{33} \end{vmatrix} + \begin{vmatrix} & a_{12} & \\ a_{21} & & \\ & & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & & \\ & & a_{23} \\ & a_{32} & \end{vmatrix}$$

$$+ \begin{vmatrix} & a_{12} & \\ & & a_{23} \\ a_{31} & & \end{vmatrix} + \begin{vmatrix} & & a_{13} \\ a_{21} & & \\ & a_{32} & \end{vmatrix} + \begin{vmatrix} & & a_{13} \\ & a_{22} & \\ a_{31} & & \end{vmatrix}.$$

→ by row exchanges to make diagonal matrix,

$$\det A = a_{11}a_{22}a_{33} \underbrace{\begin{vmatrix} 1 & & \\ & 1 & \\ & & 1 \end{vmatrix}}_{P} + \cdots \quad a_{13}a_{22}a_{31} \underbrace{\begin{vmatrix} & & 1 \\ & 1 & \\ 1 & & \end{vmatrix}}_{P}$$

⇒ ◦ Big Formula

$$\det A = \sum_{\text{all } P\text{'s}} (a_{1\alpha} a_{2\beta} \cdots a_{n\nu}) \det P.$$

$(\alpha, \beta, \cdots, \nu) \rightarrow n!$ permutations $\leftarrow (1, 2, \cdots, n)$

ex) : $(1, 3, 2) \rightarrow a_{11} a_{23} a_{32} \begin{vmatrix} 1 & & \\ & & 1 \\ & 1 & \end{vmatrix}$,

When we consider all the terms $a_{1\alpha} a_{2\beta} \cdots a_{n\nu}$ involving a_{11},
$(\beta, \cdots, \nu) \rightarrow (2, \cdots, n) \rightarrow$ sub-permutation.

$a_{11} \underline{C_{11}} \rightarrow$ cofactor of a_{11} : C_{11}

$$C_{11} = \Sigma (a_{2\beta} \cdots a_{n\nu}) \det P = \det(\text{submatrix of } A)$$

along row 1 $\Rightarrow$ $\det A = a_{11} C_{11} + a_{12} C_{12} + \cdots + a_{1n} C_{1n}$.

ex) 3×3 cases

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & & \\ & a_{22} & a_{23} \\ & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} & a_{12} & \\ a_{21} & & a_{23} \\ a_{31} & & a_{33} \end{vmatrix} + \begin{vmatrix} & & a_{13} \\ a_{21} & a_{22} & \\ a_{31} & a_{32} & \end{vmatrix}$$

C_{11}

minors. matrix.

◎ det A by cofactors.

$$\det A = \sum_{j=1}^{n} a_{ij} C_{ij} \quad , \quad C_{ij} = (-1)^{i+j} \det M_{ij}$$

M_{ij} : submatrix that deleted row i and column j

Ex 3) $A_4 = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} \rightarrow \det A_4 = (2) \begin{vmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{vmatrix} + (1) \begin{vmatrix} -1 & -1 & 0 \\ 0 & 2 & -1 \\ 0 & -1 & 2 \end{vmatrix}$

$= 2 \det A_3 - \det A_2$

$= 2\big(2 \times (3) - (-1)(-2)\big) + \big(-1 \cdot (3) - \cancel{(1) \cdot 0}\big) = 5.$