Chapter 1. Matrix & Gaussian Elimination.

1.1 Introduction

How to solve linear equations with n unknowns?

(1) Elimination.

$$x + 2y = 3$$
$$4x + 5y = 6$$

$(4x+5y=6) - 4\times(x+2y=3) \Rightarrow -3y = -6 \Rightarrow y = 2.$

$x = -1.$

(2) Determinants (Cramer's Rule)

$$y = \frac{\begin{vmatrix} 1 & 3 \\ 4 & 6 \end{vmatrix}}{\begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix}} = 2. \qquad x = \frac{\begin{vmatrix} 3 & 2 \\ 6 & 5 \end{vmatrix}}{\begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix}} = -1$$

$\Rightarrow$ usually elimination method is much better than determinants.

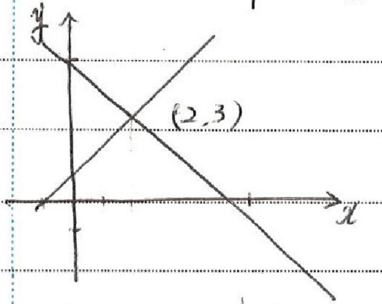
1.2 Geometry of Linear Equations.

→ 도형의 특성을 이용

→ lines (2 unknowns), planes (3 or more unknowns ...)

$2x - y = 1$

$x + y = 5$ } (1) row equations.

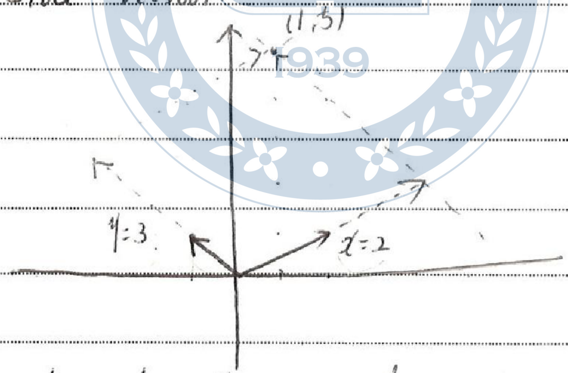


2 lines → solution is the intersection of two lines.

(2) column form.

$$x\begin{bmatrix}2\\1\end{bmatrix} + y\begin{bmatrix}-1\\1\end{bmatrix} = \begin{bmatrix}1\\5\end{bmatrix}.$$

→ to find the combination of column vectors that produces the right-side vector.



solution is to make the geometric parallelogram of vectors.

③ $n=3$ case

$$\left.\begin{aligned} 2u + v + w &= 5 \\ 4u - 6v \quad &= -2 \\ -2u + 7v + 2w &= 9 \end{aligned}\right\} \Rightarrow 3 \text{ planes}$$

(1) row equations.

→ solution is the intersection of 3 planes. → a point.

(2) Column form.

$$u\begin{bmatrix}2\\4\\-2\end{bmatrix}+v\begin{bmatrix}1\\-6\\7\end{bmatrix}+w\begin{bmatrix}1\\0\\2\end{bmatrix}=\begin{bmatrix}5\\-2\\9\end{bmatrix}$$

↳ Linear Combinations of 3 column vectors.

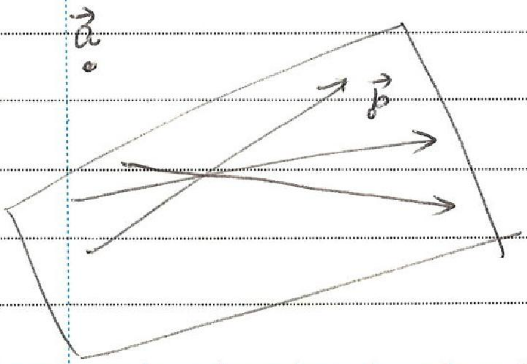
Row picture (equation) ⇒ intersection of planes.
Column 〃 〃 ⇒ Combination of columns

◎ Singular Cases → no solution
↳ infinite solutions.

(1) row form.
→ parallel
→ no intersection of 3 planes.) no solution.
→ line of intersection. → infinite → line (차원↓)

(2) column picture.

⇒ 3 column vectors are in a plane.



$\vec{a}$: not in a plane.
↳ no solution

$\vec{b}$: in a plane.
↳ infinite solutions.

If n planes have no point in common or infinitely many points, then the n columns lie in the same plane.

HW 1.2

3, 5, 11, 12, 21, 22.

1.3 Example of Gaussian Elimination

◎ Forward elimination step.

$$\begin{cases} 2u + v + w = 5 \\ 4u - 6v = -2 \\ -2u + 7v + 2w = 9 \end{cases}$$

↓ to eliminate u from the last two equations.

1st pivot.

$$\begin{cases} 2u + v + w = 5 \\ -8v - 2w = -12 \\ 8v + 3w = 14 \end{cases}$$

2nd pivot

↓ to eliminate v from the last equation

$$\begin{cases} 2u + v + w = 5 \\ -8v - 2w = -12 \\ w = 2 \end{cases} \Rightarrow \text{triangular system.}$$

$w=2 \rightarrow v=1 \rightarrow u=1 \Rightarrow$ back-substitution

by definition, pivots cannot be zero.

$$\begin{bmatrix} 2 & 1 & 1 & 5 \\ 4 & -6 & 0 & -2 \\ -2 & 7 & 2 & 9 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 1 & 1 & 5 \\ 0 & -8 & -2 & -12 \\ 0 & 8 & 3 & 14 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 1 & 1 & 5 \\ 0 & -8 & -2 & -12 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

↳ triangular system.

③ Breakdown of Elimination

when a zero appears in a pivot position, elimination has to stop, and ~~change~~ the order of equations has to be changed.

Ex 1.
$$\begin{aligned} u+v+w &= * \\ 2u+2v+5w &= * \\ 4u+6v+8w &= * \end{aligned} \rightarrow \begin{aligned} u+v+w &= * \\ 3w &= * \\ 2v+4w &= * \end{aligned} \rightarrow \begin{aligned} u+v+w &= * \\ 2v+4w &= * \\ 3w &= * \end{aligned}$$

→ non-singular.

Ex 2.
$$\left.\begin{aligned} u+v+w &= a \\ 2u+2v+5w &= b \\ 4u+4v+8w &= c \end{aligned}\right) \rightarrow \left.\begin{aligned} u+v+w &= a \\ 3w &= b-2a \\ 4w &= c-4a \end{aligned}\right) \rightarrow \text{singular}$$

↳ No exchange. for non-zero pivots.

$\begin{cases} \text{if } \dfrac{b-2a}{3} = \dfrac{c-4a}{4} & \rightarrow \text{infinite.} \\ \text{else} & \rightarrow \text{no solution.} \end{cases}$

⇒ by exchanging the order of equations for non-zero full pivots, we can find a solution by elimination.