◎ Singular Value Decomposition

- generally Gaussian - Elimination is not proper in the finite-precis[ion] due to rounding-off process.

- SVD : $A = U \Sigma V^T$ for $m \times n$ matrix.

⇒ first find the eigenvalues of A^TA, $(n \times n)$

$$A^TAx = \lambda x$$
$$x^TA^TAx = \lambda \|x\|^2 \Rightarrow \lambda = \frac{\|Ax\|^2}{\|x\|^2} \geq 0 \quad \begin{pmatrix}\text{non-zero}\\ \text{positive}\end{pmatrix}$$

assume, there are r non-zero (positive) eigenvalues

$\lambda_1 \geq \lambda_2 \geq \cdots \lambda_r > 0$, and $\lambda_{r+1} = \lambda_{r+2} = \cdots = \lambda_n = 0$

($A^TA \rightarrow n \times n$ matrix)

⇒ singular values : $\sigma_i = \sqrt{\lambda_i}$

Since A^TA is symmetric → the eigenvectors are orthonormal. in R^n.

$V_1 = \begin{bmatrix} v_1, v_2 & \cdots & v_r \end{bmatrix}$ → orthonormal eigenvectors of A^TA for $\lambda_1, \lambda_2, \cdots, \lambda_r$.

$$\Sigma_1 = \begin{bmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_r \end{bmatrix} \rightarrow \underset{m \times n}{\Sigma} = \begin{bmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{bmatrix}$$

$v_1, v_2 \cdots v_r$ → basis of Row Space $C(A^T)$

for eigenvalues, $\lambda_{r+1} = \lambda_{r+2} = \cdots = \lambda_n = 0$,

the corresponding eigenvectors ; $v_{r+1}, v_{r+2}, \cdots, v_n \rightarrow$ orthonormal,

$$A^T A v_j = \underbrace{\lambda_j}_{0} v_j = 0.$$

($A^T A$ is symmetric)

$$A^T A v_j = 0 \qquad j = r+1, r+2, \cdots, n.$$

v_j is the basis of $N(A^T A) = N(A)$ in $\mathbb{R}^n$

$$V_2 = \begin{bmatrix} v_{r+1} & v_{r+2} & \cdots & v_n \end{bmatrix}$$

$\Rightarrow \; A V_2 = 0. \rightarrow$ null space of A.

$$V = [V_1 \; V_2] \Rightarrow V_1 V_1^T = I$$

Now, we show that $AV = U\Sigma$ (U: orthogonal matrix)

$$A \begin{bmatrix} v_1 & v_2 & \cdots & v_r & v_{r+1} & \cdots & v_n \end{bmatrix} = \begin{bmatrix} u_1 & u_2 & \cdots & u_m \end{bmatrix} \underbrace{\begin{bmatrix} \sigma_1 & & & & \\ & \sigma_2 & & & \\ & & \ddots & & \\ & & & \sigma_r & \\ & & & & 0 \end{bmatrix}}_{m \times n}$$

$$= \begin{bmatrix} \sigma_1 u_1 & \sigma_2 u_2 & \cdots & \sigma_r u_r, & 0 & 0 & \cdots & 0 \end{bmatrix}$$

$$A v_i = \sigma_i u_i \qquad i = 1, 2, \cdots r.$$

$$u_i \stackrel{d}{=} \frac{1}{\sigma_i} A v_i \qquad i = 1, 2, \cdots, r.$$

$$U_1 = \begin{bmatrix} u_1 & u_2 & \cdots & u_r \end{bmatrix}$$

orthonormal matrix.

$$A V_1 = U_1 \Sigma_1 \Rightarrow A = U_1 \Sigma_1 V_1^T \qquad (V_1^{-1} = V_1^T)$$

$$u_i^T u_j = \left(\frac{1}{\sigma_i} A v_i\right)^T \left(\frac{1}{\sigma_j} A v_j\right) = \frac{1}{\sigma_i \sigma_j} v_i^T \underbrace{A^T A v_j} = \frac{1}{\sigma_i \sigma_j} v_i^T \lambda_j v_j^T$$

$$= \delta_{ij} \quad \begin{cases} i=j = 1 \\ i \neq j = 0 \end{cases}$$

(since v_i are orthonormal eigenvectors of $A^T A$).

⇒ thus u_i's $(i = 1, 2, \cdots r)$ form an orthonormal basis in $\mathbb{R}^m$ ($C(A')$) → column space.

⇒ rank $= r$

$$U_2 = \left[u_{r+1} \quad u_{r+2} \quad \cdots \quad u_m \right] \rightarrow m-r \text{ dimension.}$$

→ left null space!

⇒ we usually construct an orthonormal basis for $N(A^T)$, $[u_{r+1}, u_{r+2}, \cdots u_m]$ by G.S.O or inspection.

$$U = [U_1 \quad U_2]$$

U_2 → left null space (A).

finally

$$\Rightarrow \quad U \Sigma V^T = \underset{m \times m}{[U_1 \quad U_2]} \underset{m \times n}{\begin{bmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{bmatrix}} \underset{n \times n}{\begin{bmatrix} V_1^T \\ V_2^T \end{bmatrix}}$$

$$= U_1 \Sigma_1 V_1^T$$

$$= A$$

⇒ U and V are not unique, but $\sigma_1, \sigma_2 \cdots \sigma_r$ are unique.

Example> $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 0 \end{bmatrix}$, $\Rightarrow$ $A^TA = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$, $\Rightarrow \lambda_1 = 4,\ \lambda_2 = 0$
$\Rightarrow \sigma_1 = 2,\ \sigma_2 = 0.$

$r=1.$

$v_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\quad v_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ $\Rightarrow$ $V = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

o $u_1 = \frac{1}{\sigma_1} A v_1 = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 0 \end{bmatrix} \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

$u_2, u_3 \Rightarrow$ the orthonormal basis of $N(A^T)$

eg $u_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$, $u_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$$A = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 0 \end{bmatrix}$$