5. 3 Difference Equations and Powers A^K.

Example) Compound interest ; 0.06, / year. → 5 years, P_0

- Yearly $P_5 = (1.06)^5 \cdot P_0 = 1.338 P_0$.
- Monthly $P_{60} = \left(1 + \frac{0.06}{12}\right)^{60} \cdot P_0 = 1.349 \cdot P_0$
- Daily $P_{5\cdot 365} = \left(1 + \frac{0.06}{365}\right)^{5\cdot 365} P_0 = 1.34983 P_0$.
- continuously $\lim_{N\to\infty} \left(1 + \frac{0.06}{N}\right)^{5\cdot N} P_0 = e^{0.30} P_0 = 1.34987 P_0$

⇒ Discrete to continuous , $\frac{P_{k+1} - P_k}{\Delta t} = 0.06 P_k \rightarrow \frac{dp}{dt} = 0.06 P$.

$p(t) = e^{0.06t} P_0$; $t = 5$, $p(5) = 1.34987 P_0$.

◎ Fibonacci Numbers.

- Fibonacci sequences : 0, 1, 1, 2, 3, 5, 8, 13, ...
- Fibonacci equation : $F_{K+2} = F_{K+1} + F_K \rightarrow F_{1000} = ?$

⇒ $U_{K+1} = AU_k$, $U_k = \begin{bmatrix} F_{K+1} \\ F_K \end{bmatrix}$

$$\begin{bmatrix} F_{K+2} \\ F_{K+1} \end{bmatrix} = \begin{matrix} F_{K+1} + F_K \\ F_{K+1} \end{matrix} = \underbrace{\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}}_{A} \begin{bmatrix} F_{K+1} \\ F_K \end{bmatrix}$$

Solution : $u_{k+1} = A^{k+1}u_0 \Leftarrow Au_k$

If $A = S\Lambda S^{-1}$

then $u_k = A^k u_0 = S\Lambda^k S^{-1}u_0$, $(S^{-1}u_0 = C)$

$$= \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix} \begin{bmatrix} \lambda_1^k & & & \\ & \lambda_2^k & & \\ & & \ddots & \\ & & & \lambda_n^k \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

$$= c_1\lambda_1^k x_1 + c_2\lambda_2^k x_2 + \cdots + c_n\lambda_n^k x_n$$

↳ linear combination of $\{\lambda_i^k x_i\text{'s}\}$

- find the eigenvalues of A

$A - \lambda I = \begin{bmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{bmatrix}$, $\det(A-\lambda I) = \lambda(\lambda-1) - 1 = 0$.

$\lambda_1 = \dfrac{1+\sqrt{5}}{2}$, $\lambda_2 = \dfrac{1-\sqrt{5}}{2}$

$S = \begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix}$, $S^{-1} = \dfrac{1}{\lambda_1 - \lambda_2}\begin{bmatrix} 1 & -\lambda_2 \\ -1 & \lambda_1 \end{bmatrix}$.

$S^{-1}u_0 = S^{-1}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \dfrac{1}{\lambda_1 - \lambda_2}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \dfrac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}. = C\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}.$

$\begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix} = u_k = c_1\lambda_1^k x_1 + c_2\lambda_2^k x_2 . = c_1\lambda_1^k\begin{bmatrix} \lambda_1 \\ 1 \end{bmatrix} + c_2\lambda_2^k\begin{bmatrix} \lambda_2 \\ 1 \end{bmatrix}$