Chapter 5. Eigenvalues & Eigenvectors.

$Ax = \lambda x.$; λ: eigenvalue. , x: eigenvector.

→ row exchanges change the eigenvalue λ. , we use determinan

$\Rightarrow \quad (A - \lambda I)x = 0.$

- vector x : nullspace of $A - \lambda I$.
- λ : eigen value.

⇒ for non-zero vector x, $A - \lambda I$ should be singular.

$\det(A - \lambda I) = 0.$

ex) $A = \begin{bmatrix} 4 & -5 \\ 2 & -3 \end{bmatrix}$. $A - \lambda I = \begin{bmatrix} 4-\lambda & -5 \\ 2 & -3-\lambda \end{bmatrix}$

$\det(A - \lambda I) = (4-\lambda)(-3-\lambda) + 10 = \lambda^2 - \lambda - 2 = 0.$ $\lambda = 2$ or -1.

$\lambda_1 = -1.$ $(A - \lambda I)x = \begin{bmatrix} 5 & -5 \\ 2 & -2 \end{bmatrix} x = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

$\lambda_2 = 2$ $(A - \lambda I)x = \begin{bmatrix} 2 & -5 \\ 2 & -5 \end{bmatrix} x = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $x_2 = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$.

1) Compute $\det(A - \lambda I)$, and find the roots of λ.
2) For each eigenvalue, solve the equation $(A - \lambda I)x = 0$.

Ex 3) $A = \begin{bmatrix} 1 & 4 & 5 \\ 0 & \frac{3}{4} & 6 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$

$$\det(A-\lambda I) = \begin{bmatrix} 1-\lambda & 4 & 5 \\ 0 & \frac{3}{4}-\lambda & 6 \\ 0 & 0 & \frac{1}{2}-\lambda \end{bmatrix} = (1-\lambda)(\tfrac{3}{4}-\lambda)(\tfrac{1}{2}-\lambda)$$

- Eigenvalues are different from pivots in Gauss elimination.

[◎ $\prod \lambda_i = \prod$ pivots $= \det A$.

◎ Trace of $A = \lambda_1 + \lambda_2 + \cdots + \lambda_n = a_{11} + a_{22} + \cdots + a_{nn}$.

⇒ assume $\det(A-\lambda I) = 0$ has n roots, $\lambda_1, \lambda_2, \cdots \lambda_n$, then.

$\det(A-\lambda I) = (\lambda_1-\lambda)(\lambda_2-\lambda)\cdots(\lambda_n-\lambda) = 0.$

↳ $= (-\lambda)^n + (\lambda_1+\lambda_2+\cdots+\lambda_n)(-\lambda)^{n-1} + \cdots\cdots + \lambda_1\lambda_2\cdots\lambda_n$. ①

$$\det(A-\lambda I) = \begin{vmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & \cdots & & a_{nn}-\lambda \end{vmatrix}$$

→ $\lambda=0$ → $\det(A)$

① 에서 $\lambda=0$ → $\lambda_1\lambda_2\cdots\lambda_n = \det A$.

→ to get the terms of $(-\lambda)^n$ and $(-\lambda)^{n-1}$, they are derived from

$(a_{11}-\lambda)C_{11}$ and, consequently, $(-\lambda)^n$ and $(-\lambda)^{n-1}$ come from,

→ $(a_{11}-\lambda)(a_{22}-\lambda)\cdots(a_{n-1\,n-1}-\lambda)(a_{nn}-\lambda)$

$= (-\lambda)^n + (a_{11}+a_{22}+\cdots+a_{nn})(-\lambda)^{n-1} + \cdots\cdots$ ②

by ① = ② for $\forall (-\lambda)^1, (-\lambda)^2, \cdots (-\lambda)^n$,

$\sum_{i=1}^{n} \lambda_i = \sum_{i=1}^{n} a_{ii}$ //

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5.2 Diagonalization of a Matrix

⇒ The eigenvectors diagonalize a matrix.

- for $n \times n$ matrix A, A has n linearly independent eigenvectors. (e_i)

$$S = \begin{bmatrix} e_1 & e_2 & \cdots & e_n \end{bmatrix}, \Rightarrow S^{-1}AS = \Lambda = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$$

↳ eigenvector matrix, eigenvalue matrix.

$Ae_i = \lambda e_i$.

$$AS = A \begin{bmatrix} e_1 & e_2 & \cdots & e_n \end{bmatrix} = \begin{bmatrix} \lambda_1 e_1 & \lambda_2 e_2 & \cdots & \lambda_n e_n \end{bmatrix}$$

$$= \begin{bmatrix} e_1 & e_2 & \cdots & e_n \end{bmatrix} \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$$

$$\begin{cases} S^{-1}AS = \Lambda \\ A = S\Lambda S^{-1} \end{cases}$$

S: invertible, n linearly independent eigenvectors

< Remark 1.>

If $\lambda_1, \lambda_2, \cdots \lambda_n$ are distinct, then n eigenvectors are independent.

pf ⇒ assume $x_1 = c_2x_2 + \cdots + c_nx_n$ → (linearly dependent. x_1 independent $x_2 \sim x_n$

$Ax_1 = A(c_2x_2 + \cdots + c_nx_n)$

$\lambda_1 \underline{x_1} = c_2\lambda_2x_2 + \cdots + c_n\lambda_nx_n$

$\lambda_1(c_2x_2 + \cdots + c_nx_n) = c_2\lambda_2x_2 + \cdots + c_n\lambda_nx_n$

⇒ $c_2(\lambda_1 - \lambda_2)x_2 + c_3(\lambda_1 - \lambda_3)x_3 + \cdots + c_n(\lambda_1 - \lambda_n)x_n = 0.$

→ $c_2 = \cdots = c_n = 0$ //

Remark 2.

⇒ S is not unique, An eigenvector can be multiplied by a constant,

Remark 3.

⇒ The order of eigenvalues and eigenvectors in S and Λ are the same.

Remark 4.

⇒ Not all matrices have n linearly independent eigenvectors, so not all matrices are diagonalizable.

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \rightarrow \lambda^2 = 0. \quad x = \begin{bmatrix} c \\ 0 \end{bmatrix} //$$

↳ double root.

⇒ Diagonalization fails only if there are repeated eigenvalues.

↳ not always. (for $A = I$)

- Invertibility of A depends on non-zero eigenvectors.
  ↳ if $\lambda = 0$ $\det A = 0$

◎ Examples of Diagonalization

ex1) $A = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$ → $\det(A_0 - \lambda I) = 0$ ⇒ $\lambda = 1$ or 0

$S = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$, $\Lambda = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ → $S^{-1}AS = \Lambda$

ex2) $90°$ rotation $K = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ → $\det(K - \lambda I) = 0$, $\lambda = i, -i$
$(\lambda^2 + 1 = 0)$

$S = \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix}$, $\Lambda = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$ → $S^{-1}KS = \Lambda$. in Complex dom

◎ Powers and Products : A^k and AB

⇒ eigenvalues of A^k → λ_i^k $\Lambda^k = S^{-1}A^kS$.
eigenvectors of A^k → S

e.g) $A^2x = A \cdot Ax = A \cdot \lambda x = \lambda Ax = \lambda \cdot \lambda x = \lambda^2 x$.

$A^k x = \lambda^k \cdot x$ → eigenvectors are the same!
↳ λ^k.

$\Lambda^k = S^{-1}A^kS$.

∘ A^{-1} : $Ax = \lambda x$.

$\Rightarrow$ $A^{-1} \cdot x = ?$ $A^{-1}\lambda x$ $\Leftrightarrow$ $A^{-1}x = \frac{1}{\lambda}x$. $\rightarrow$ λ^{-1} //

The eigenvalue of A^{-1} are $\frac{1}{\lambda_i}$.

optimal.

∘ usually AB (product of A, B) does not have same eigenvectors.

$Ax_1 = \lambda_A x_1$. $Bx_2 = \lambda_B x_2$.

$\rightarrow$ $AB = \lambda x$. ??

$\rightarrow$ iff $AB = BA$, $\rightarrow$ diagonalization matrices share the same eigenvectors.

assume.

① $A = S\Lambda_1 S^{-1}$, and $B = S\Lambda_2 S^{-1}$

$AB = S\Lambda_1\Lambda_2 S^{-1}$, $BA = S\Lambda_2\Lambda_1 S^{-1}$.

since $\Lambda_1\Lambda_2 = \Lambda_2\Lambda_1$ $\Rightarrow$ then $AB = BA$

② assume $AB = BA$.

$\rightarrow$ $Ax = \lambda x$.

$\underline{BA}x = B\lambda x = \lambda Bx$. $AB = BA$.

$AB x = \lambda Bx$.

$\rightarrow$ Bx is the eigenvector of A with λ

$\rightarrow$ x and Bx share λ, $\rightarrow$ Bx is multiple of x.

HW 5.2 15, 16, 22, 31