Math 104 Applied Matrix Theory
Mock Final
Solution

Problem 1. (10pts) **Markov Process**
Trump's moods tend to be variable. If he is in a good mood today, there is a 80% chance he will still be in a good mood tomorrow; but if he is grumpy today, there is only 60% chance that his mood will be good tomorrow.

(a) (5 pts.) Write down the Markov matrix M for Trump's moods.

$$M = \begin{bmatrix} 0.8 & 0.6 \\ 0.2 & 0.4 \end{bmatrix}$$

(b) (5 pts.) Over the long term, what percentage of the time is Trump in a good mood?
M is a positive matrix, i.e. all its entries are positive. Hence by Perron-Frobenius, the equilibrium vector of M is the unique stationary vector of M. $\mathcal{N}(M-I)$ is spanned by $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$, so with probability $\frac{3}{3+1} = 75\%$, Trump will be in a good mood in the long term.

Problem 2. (30pts) **Skew-Symmetric Matrices**
Let A be an $n \times n$ real skew-symmetric matrix, i.e. $A^T = -A$, and all entries of A are real. Suppose $\det(A) \neq 0$ and n is even. Let

$$B = A^T A = -A^2.$$

(a) (10 pts.) Show that the eigenvalues of B are all positive.
$\det(A) \neq 0$, A is invertible. Suppose $B\mathbf{v} = \mu\mathbf{v}$, where $\mathbf{v} \neq 0$. $A\mathbf{v} \neq \mathbf{0}$, since A is invertible.

$$0 < \|A\mathbf{v}\|^2 = \mathbf{v}^T A^T A\mathbf{v} = \mathbf{v}^T B\mathbf{v} = \mu\mathbf{v}^T\mathbf{v} = \mu\|\mathbf{v}\|^2, \qquad \mu > 0$$

(b) (10 pts.) Show that the eigenvalues of A are purely imaginary.
Since $\det(A) \neq 0$, A is invertible and its eigenvalues are all nonzero. A has eigenvalues $\lambda_1, \cdots, \lambda_n$, some may be complex and/or repeated. $B = A^T A = -A^2$ has eigenvalues $-\lambda_k^2$. By the previous part, $\mu = -\lambda_k^2 > 0$, each λ_k must be purely imaginary.

(c) (10 pts.) Show that $\det(A) > 0$.
The complex eigenvalues of real A occur in complex conjugate pairs. So the eigenvalues are $\pm a_k i$ for some real $a \neq 0$. $\det(A)$, the product of all the eigenvalues are $\prod_k a_k^2 > 0$.

Problem 3. (35pts) **Symmetric Matrices**
Fix $a \in \mathbb{R}$, consider the symmetric matrix

$$A = \begin{bmatrix} 1 & 1 & a+1 \\ 1 & a+1 & 1 \\ a+1 & 1 & 1 \end{bmatrix} = J + aP$$

where J is the 3×3 matrix all of whose entries are 1 and P is the reverse-order identity matrix. Note that each row of A sums to $a + 3$.

(a) (10 pts.) Compute $A^T A$ using the fact that $A = J + aP$. Express your result in the form $A^T A = \alpha I + \beta J$, where I is the 3×3 identity matrix.

$$A^T A = (J + aP)^T (J + aP) = (J + aP)(J + aP)$$
$$= J^2 + aPJ + aJP + a^2 P^2 = 3J + 2aJ + a^2 I = a^2 I + (2a + 3)J$$

(b) (15 pts.) Find the singular values of A.
Hint: You may use the fact that $A^T A = \alpha I + \beta J$ has eigenvector $\mathbf{1}$. Compute $Tr(\alpha I + \beta J)$ and $\det(A^T A - \alpha I)$ to deduce the eigenvalues of $A^T A$,

$$A^T A \mathbf{1} = (\alpha I + \beta J)\mathbf{1} = (\alpha + 3\beta)\mathbf{1} = (a^2 + 3(2a + 3))\mathbf{1} = (a + 3)^2 \mathbf{1}$$
$$Tr(\alpha I + \beta J) = 3\alpha + 3\beta = \alpha + 3\beta + \lambda_2 + \lambda_3$$
$$\det(A^T A - \alpha I) = \det(\beta J) = 0$$

So α is an eigenvalue of $A^T A$. Since $\lambda_2 + \lambda_3 = 2\alpha$, we have $\lambda_2 = \lambda_3 = \alpha$. The singular values of A are the positive square roots of the eigenvalues of M:

$$|a + 3|, \qquad |a|, \qquad |a|$$

(c) (10 pts.) Find

$$\|A\|_F, \qquad \|A\|_1, \qquad \|A\|_2, \qquad \|A\|_\infty$$

$$\|A\|_F = \sqrt{(a + 3)^2 + 2a^2} = \sqrt{3}\sqrt{a^2 + 2a + 3}$$
$$\|A\|_1 = \max_{j=1}^{n} \sum_{i=1}^{n} |a_{ij}| = 2 + |a + 1|$$
$$\|A\|_\infty = \max_{i=1}^{n} \sum_{j=1}^{n} |a_{ij}| = 2 + |a + 1|$$
$$\|A\|_2 = \max\{|a + 3|, |a|\}$$

Problem 4. (40pts.) **SVD**
Let

$$A = \begin{bmatrix} 1 & -1 & -1 \\ 1 & -1 & 1 \\ 1 & -1 & 0 \\ 2\sqrt{2} & 0 & 0 \end{bmatrix}$$

(a) (20 pts.) Find the Reduced Singular Value Decomposition (RSVD) of A.

$$A^T A = \begin{bmatrix} 11 & -3 & 0 \\ -3 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

which has characteristic polynomial

$$(\lambda - 2)(\lambda^2 - 14\lambda + 24) = 0$$

$$\lambda_1 = 12, \qquad \mathbf{v_1} = \frac{1}{\sqrt{10}} \begin{bmatrix} 3 & -1 & 0 \end{bmatrix}^T$$

$$\lambda_2 = 2, \qquad \mathbf{v_2} = \frac{1}{\sqrt{10}} \begin{bmatrix} 1 & 3 & 0 \end{bmatrix}^T$$

$$\lambda_3 = 2, \qquad \mathbf{v_3} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T$$

$$A^T A = VEV^T, \qquad V = \frac{1}{\sqrt{10}} \begin{bmatrix} 3 & 1 & 0 \\ -1 & 3 & 0 \\ 0 & 0 & \sqrt{10} \end{bmatrix}, \qquad E = \begin{bmatrix} 12 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$A = UDV^T, \qquad D = \begin{bmatrix} 2\sqrt{3} & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \end{bmatrix}$$

$$\mathbf{u_1} = \frac{A\mathbf{v_1}}{\sqrt{\lambda_1}} = \frac{1}{\sqrt{30}} \begin{bmatrix} 2 \\ 2 \\ 2 \\ 3\sqrt{2} \end{bmatrix}$$

$$\mathbf{u_2} = \frac{A\mathbf{v_2}}{\sqrt{\lambda_2}} = \frac{1}{\sqrt{5}} \begin{bmatrix} -1 \\ -1 \\ -1 \\ \sqrt{2} \end{bmatrix}$$

$$\mathbf{u_3} = \frac{A\mathbf{v_3}}{\sqrt{\lambda_3}} = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$U = \begin{bmatrix} \mathbf{u_1} & \mathbf{u_2} & \mathbf{u_3} \end{bmatrix} = \frac{1}{\sqrt{30}} \begin{bmatrix} 2 & -\sqrt{6} & -\sqrt{15} \\ 2 & -\sqrt{6} & \sqrt{15} \\ 2 & -\sqrt{6} & 0 \\ 3\sqrt{2} & 2\sqrt{3} & 0 \end{bmatrix}$$

(b) (10 pts.) Find the least squares solution to $A\mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ \sqrt{2} \end{bmatrix}$.

$$A^T A\vec{x} = A^T \begin{bmatrix} 1 \\ 1 \\ 1 \\ \sqrt{2} \end{bmatrix}, \qquad \mathbf{x} = (A^T A)^{-1} A^T \begin{bmatrix} 1 \\ 1 \\ 1 \\ \sqrt{2} \end{bmatrix}$$

$$(A^T A)^{-1} = \frac{1}{24} \begin{bmatrix} 3 & 3 & 0 \\ 3 & 11 & 0 \\ 0 & 0 & 12 \end{bmatrix}, \qquad (A^T A)^{-1} A^T = \frac{1}{24} \begin{bmatrix} 0 & 0 & 0 & 6\sqrt{2} \\ -8 & -8 & -8 & 6\sqrt{2} \\ -12 & 12 & 0 & 0 \end{bmatrix}$$

Alternatively,

$$A^+ = VD^{-1}U^T = (A^T A)^{-1} A^T = \frac{1}{24} \begin{bmatrix} 0 & 0 & 0 & 6\sqrt{2} \\ -8 & -8 & -8 & 6\sqrt{2} \\ -12 & 12 & 0 & 0 \end{bmatrix}$$

$$\mathbf{x} = A^+ \begin{bmatrix} 1 \\ 1 \\ 1 \\ \sqrt{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ 0 \end{bmatrix}$$

(c) (10 pts.) Find the least squares solution of minimal norm to $A^T\vec{y} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

The pseudoinverse to A^T is

$$(A^T)^+ = (A^+)^T = A(A^T A)^{-1}.$$

$$\mathbf{y} = (A^T)^+ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = (A^+)^T \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{24} \begin{bmatrix} 0 & -8 & -12 \\ 0 & -8 & 12 \\ 0 & -8 & 0 \\ 6\sqrt{2} & 6\sqrt{2} & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -\frac{5}{6} \\ \frac{1}{6} \\ -\frac{1}{3} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

Alternatively,

$$\mathbf{y} = A(A^T A)^{-1} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{24} \begin{bmatrix} -20 \\ 4 \\ -8 \\ 12\sqrt{2} \end{bmatrix}$$

Problem 5. (30 pts.) **System of Differential Equations**
Consider the following system of differential equations

$$\begin{aligned} x'(t) &= y(t) \\ y'(t) &= -2x(t) + 3y(t) + z(t) \\ z'(t) &= x(t) - y(t) \end{aligned}$$

Given the initial conditions

$$x(0) = 1, \qquad y(0) = 0, \qquad z(0) = 1,$$

solve for $x(t), y(t)$ and $z(t)$.

$$\frac{d}{dt}\begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -2 & 3 & 1 \\ 1 & -1 & 0 \end{bmatrix}\begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} = A\begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & 1 & 0 \\ -2 & 3 & 1 \\ 1 & -1 & 0 \end{bmatrix}$$

has characteristic polynomial

$$\pi_A(\lambda) = \det(A - \lambda I) = (\lambda - 1)^3 = 0.$$

$n(A - I) = 1$, so the Jordan canonical form of A has one Jordan block. A has eigenvector $\mathbf{v_1} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$,

$$(A - I)\mathbf{v_2} = \mathbf{v_1}, \qquad \mathbf{v_2} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

$$(A - I)\mathbf{v_3} = \mathbf{v_2}, \qquad \mathbf{v_3} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

A has 1-chain basis $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$.

$$A\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \qquad AX = XJ$$

$$\exp(Jt) = \begin{bmatrix} e^t & te^t & \frac{t^2}{2}e^t \\ 0 & e^t & te^t \\ 0 & 0 & e^t \end{bmatrix}$$

$$\mathbf{x}(t) = e^{At}\mathbf{x}(0) = Xe^{Jt}X^{-1}\mathbf{x}(0)$$

$$= \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} e^t & te^t & \frac{t^2}{2}e^t \\ 0 & e^t & te^t \\ 0 & 0 & e^t \end{bmatrix}\begin{bmatrix} 2 & -1 & -1 \\ -1 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} e^t - te^t \\ -te^t \\ e^t \end{bmatrix}$$

You can use Cramer's Rule to find X^{-1} for a 3×3 matrix, or note that

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

implies

$$a+g=1, b=-h, c=-i, a+d+g=0, b+e+h=1, c+f+i=0, d=g, e=h, i-f=1$$

$$a=-2d, a+d=1, d=g=-1, a=2$$

$$b=-h=-e, b+2e=1, e=h=1, b=-1$$

$$c=-i, f=0, i=1=-c$$

For more practice with finding Jordan Normal Form and Jordan chain basis, you can try

$$A=\begin{bmatrix} 2 & 2 & 3 \\ 1 & 3 & 3 \\ -1 & -2 & -2 \end{bmatrix}$$

$$A\begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & -1 \\ -1 & 0 & 0 \end{bmatrix}=\begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & -1 \\ -1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & -1 \\ -1 & 0 & 0 \end{bmatrix}^{-1}=\begin{bmatrix} 0 & 0 & -1 \\ 1 & 2 & 3 \\ 0 & -1 & -1 \end{bmatrix}$$

To come up with more numerical examples, you can use reverse engineering. Come up with say 3 linearly independent vectors $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3} \in \mathbb{R}^3$. Let X be a 3×3 matrix with $\mathbf{v_i}$'s as its columns. Suppose

$$(A-3I)\mathbf{v_1}=\mathbf{0}, \qquad (A-3I)\mathbf{v_2}=\mathbf{v_1}, \qquad (A-3I)\mathbf{v_3}=\mathbf{v_2}.$$

then compute

$$A=X\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}X^{-1}.$$

Now start with A, cover up the $\mathbf{v_i}$'s, see if you can recover the Jordan chain basis and the Jordan normal form.
You can be more ambitious and try with 4×4 matrix examples etc.

Problem 6. (35pts.) **Magic Squares**
A magic square is an $n\times n$ array of numbers such that each row, each column, and the two main diagonals, all have the same sum. Call this sum the magic sum. For example, $S_1=\begin{array}{|c|c|c|}\hline 1&1&1\\\hline 1&1&1\\\hline 1&1&1\\\hline\end{array}$ has magic sum 3; $S_2=\begin{array}{|c|c|c|}\hline 2&7&6\\\hline 9&5&1\\\hline 4&3&8\\\hline\end{array}$ has magic sum 15.

(a) (4 pts.) Let $\mathcal{S}$ be the set of all $n\times n$ magic squares. Show that $\mathcal{S}$ is a vector subspace of $\mathbb{R}^{n^2}$.
Just check that if A and B are magic squares with magic sums a and b respectively, then $A+B$ is a magic square with magic sum $a+b$; and kA is a magic square with magic sum ka.

(b) (8 pts.) Find the dimension of the subspace of all 2×2 magic squares.

a	b
c	d

has magic sum

$$a + b = c + d = a + c = b + d = a + d = b + c = s$$

which implies

$$a = b = c = d$$

So the dimension of the subspace of all 2×2 magic square is 1.

(c) (10 pts.) Show that the magic sum of a 3×3 magic square is three times its central entry.
Let the central entry of a magic square be x, and the magic sum be s. Summing over all the entries in the square, we have $3s$, $3\times$ row sum s. Summing the middle row, middle column, and the two diagonals, we have

$$4s = 3s + 3x \Longrightarrow s = 3x$$

(d) (13 pts.) Find the dimension of the subspace of all 3×3 magic squares.
Hint: Denote the central entry by x, the top left entry by $x + y$, and the top right entry by $x + z$, and work out the rest of the entries in the magic square.
Denote the central entry by x, the top left entry by $x+y$, and the top right entry by $x+z$, $s = 3x$, we can fill out the magic square accordingly

$$\begin{array}{|c|c|c|}\hline x+y & x-y-z & x+z \\ \hline x-y+z & x & x+y-z \\ \hline x-z & x+y+z & x-y \\ \hline\end{array} = x \cdot \begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline 1 & 1 & 1 \\ \hline 1 & 1 & 1 \\ \hline\end{array} + y \cdot \begin{array}{|c|c|c|}\hline 1 & -1 & 0 \\ \hline -1 & 0 & 1 \\ \hline 0 & 1 & -1 \\ \hline\end{array} + z \cdot \begin{array}{|c|c|c|}\hline 0 & -1 & 1 \\ \hline 1 & 0 & -1 \\ \hline -1 & 1 & 0 \\ \hline\end{array}$$

Hence the space of 3×3 magic squares is 3-dimensional with basis

$$\left\{ \begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline 1 & 1 & 1 \\ \hline 1 & 1 & 1 \\ \hline\end{array}, \begin{array}{|c|c|c|}\hline 1 & -1 & 0 \\ \hline -1 & 0 & 1 \\ \hline 0 & 1 & -1 \\ \hline\end{array}, \begin{array}{|c|c|c|}\hline 0 & -1 & 1 \\ \hline 1 & 0 & -1 \\ \hline -1 & 1 & 0 \\ \hline\end{array} \right\}$$