

Interpreting Item Characteristic Curves

1PL, 2PL, and 3PL Models

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EDUC 8720

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Introduction

This document covers the interpretation of Item Characteristic Curves (ICCs) for the 1PL (Rasch), 2PL, and 3PL IRT models for dichotomously scored items.

Scores in CTT vs. IRT

Classical Test Theory (CTT)	Item Response Theory (IRT)
Assumes $X = T + E$	Uses response patterns to infer θ
Score = sum/average of responses	Score = estimated latent trait level
Assumes observed score is sufficient	Models probability of each response

In IRT, we use the pattern of observed item responses to make an inference about the underlying latent trait level, θ (“theta”).

Terms often used synonymously for θ : construct, ability, proficiency, capability, attribute.

Key property: θ does not theoretically depend on the specific set of items written for any given test (but we use a specific set of item responses to measure it).

Notation

Symbol	Meaning
X_{ip}	Response of person p to item i (0 or 1)
θ_p	Trait or ability level for person p
b_i	“Difficulty” or location of item i
a_i	Discrimination of item i
c_i	Lower asymptote (“guessing”) for item i
$P_i(\theta)$	Probability of answering item i correctly
$Q_i(\theta)$	Probability of NOT answering correctly; $Q_i = 1 - P_i$

The IRT Models

The 1PL/Rasch Model

$$P(X_{pi} = 1 | \theta_p, b_i) = \frac{\exp(\theta_p - b_i)}{1 + \exp(\theta_p - b_i)}$$

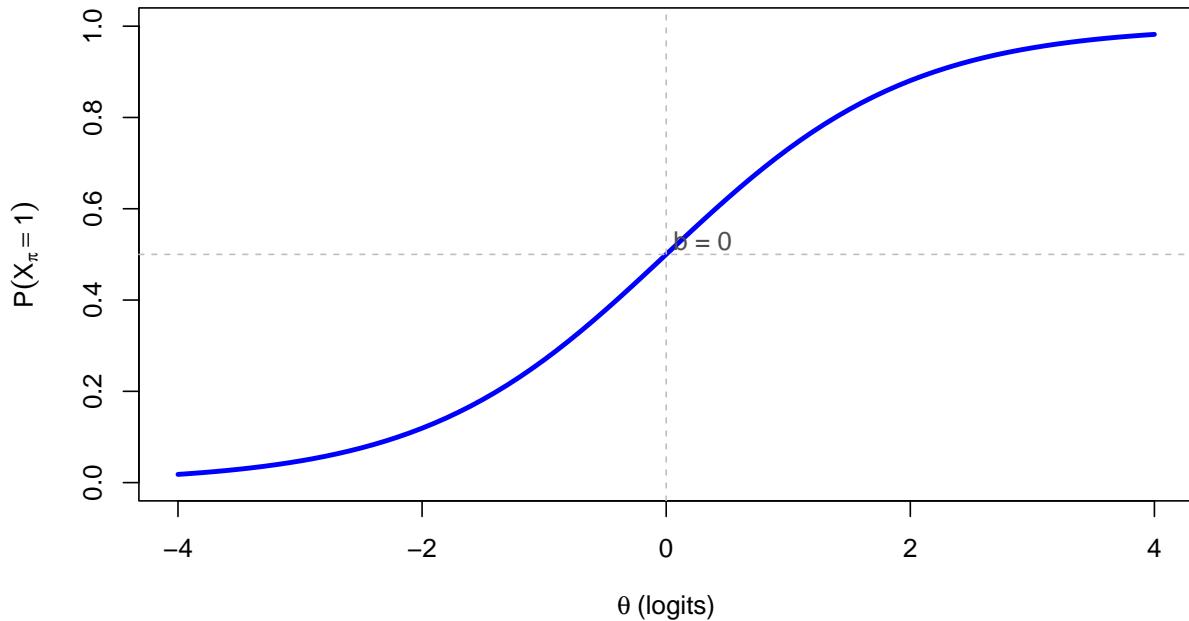
- **One item parameter:** difficulty (b_i)
- b_i is the point where $P(X_{pi} = 1) = 0.5$
- Assumes equal discrimination for all items

```
# Function to calculate probability
calc_prob <- function(theta, a = 1, b = 0, c = 0) {
  c + (1 - c) * exp(a * (theta - b)) / (1 + exp(a * (theta - b)))
}

theta <- seq(-4, 4, 0.1)

# Plot Rasch ICC
plot(theta, calc_prob(theta, a = 1, b = 0), type = "l", lwd = 3, col = "blue",
      xlab = expression(theta ~ "(logits)"), ylab = expression(P(X[pi] == 1)),
      main = "Rasch/1PL Item Characteristic Curve (b = 0)",
      ylim = c(0, 1))
abline(h = 0.5, lty = 2, col = "gray")
abline(v = 0, lty = 2, col = "gray")
text(0.3, 0.53, "b = 0", col = "gray30")
```

Rasch/1PL Item Characteristic Curve ($b = 0$)



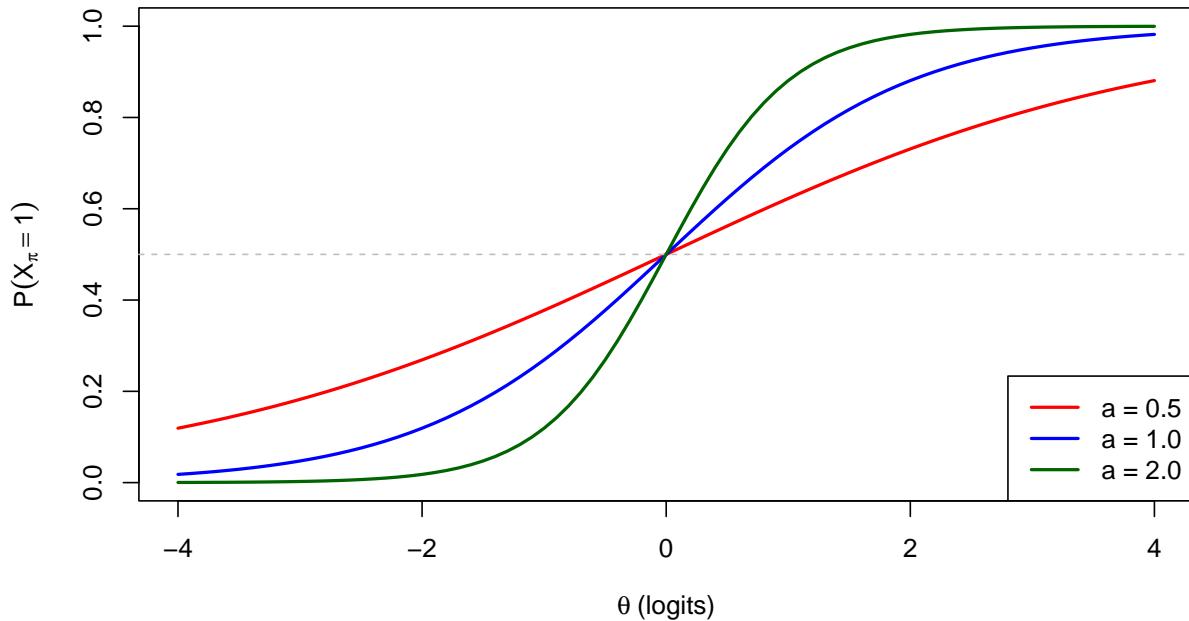
The 2PL Model

$$P(X_{pi} = 1 | \theta_p, a_i, b_i) = \frac{\exp(a_i(\theta_p - b_i))}{1 + \exp(a_i(\theta_p - b_i))}$$

- **Two item parameters:** discrimination (a_i) and difficulty (b_i)
- a_i controls the slope (steepness) of the ICC at b_i
- Higher a_i = steeper slope = better discriminating item

```
# Compare different discrimination values
plot(theta, calc_prob(theta, a = 0.5, b = 0), type = "l", lwd = 2, col = "red",
      xlab = expression(theta ~ "(logits)"), ylab = expression(P(X[pi] == 1)),
      main = "2PL ICCs: Effect of Discrimination Parameter",
      ylim = c(0, 1))
lines(theta, calc_prob(theta, a = 1, b = 0), lwd = 2, col = "blue")
lines(theta, calc_prob(theta, a = 2, b = 0), lwd = 2, col = "darkgreen")
abline(h = 0.5, lty = 2, col = "gray")
legend("bottomright", legend = c("a = 0.5", "a = 1.0", "a = 2.0"),
       col = c("red", "blue", "darkgreen"), lwd = 2)
```

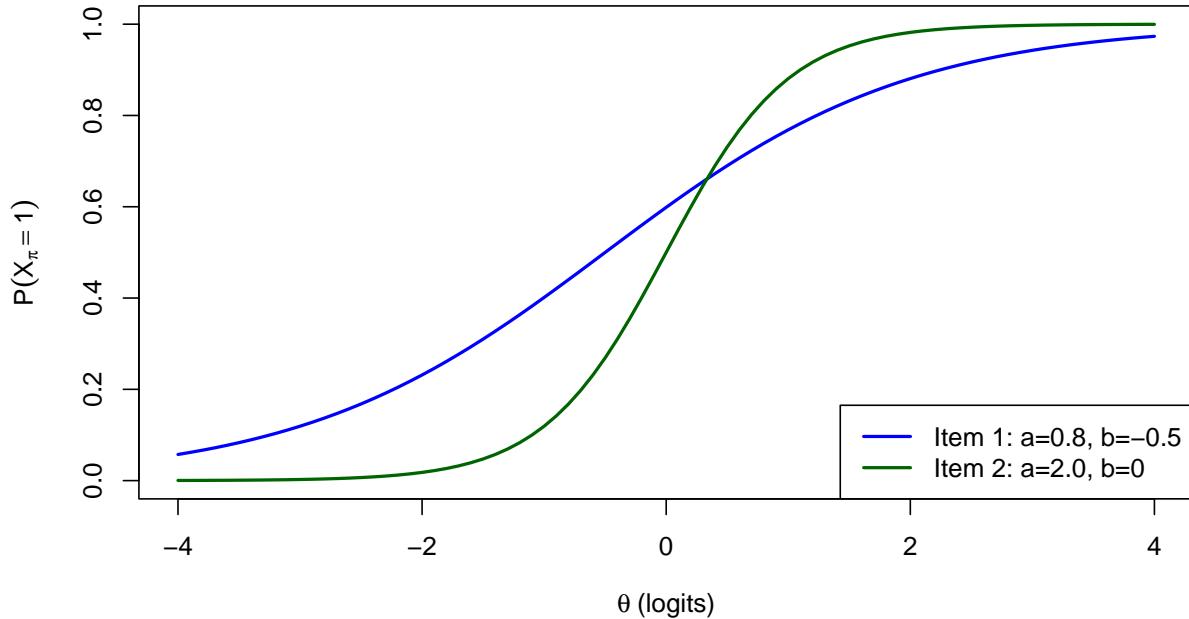
2PL ICCs: Effect of Discrimination Parameter



Important: In the 2PL model, different ICCs can cross. This means relative item difficulty can depend on the ability level of the test-taker.

```
plot(theta, calc_prob(theta, a = 0.8, b = -0.5), type = "l", lwd = 2, col = "blue",
      xlab = expression(theta ~ "(logits)"), ylab = expression(P(X[pi] == 1)),
      main = "2PL: Crossing ICCs",
      ylim = c(0, 1))
lines(theta, calc_prob(theta, a = 2, b = 0), lwd = 2, col = "darkgreen")
legend("bottomright", legend = c("Item 1: a=0.8, b=-0.5", "Item 2: a=2.0, b=0"),
       col = c("blue", "darkgreen"), lwd = 2)
```

2PL: Crossing ICCs



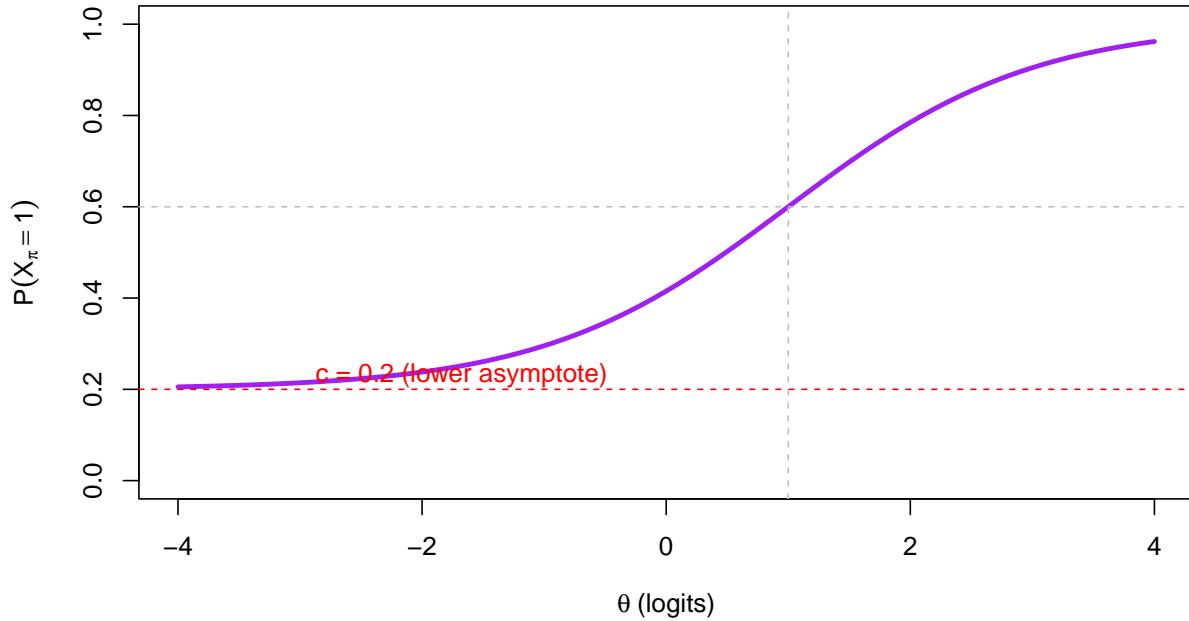
The 3PL Model

$$P(X_{pi} = 1 | \theta_p, a_i, b_i, c_i) = c_i + (1 - c_i) \frac{\exp(a_i(\theta_p - b_i))}{1 + \exp(a_i(\theta_p - b_i))}$$

- **Three item parameters:** discrimination (a_i), difficulty (b_i), pseudo-guessing (c_i)
- c_i is the lower asymptote (probability of correct response by guessing)
- **Note:** In the 3PL, b_i is no longer the 50% point; it corresponds to $P = 0.5 + 0.5c_i$

```
# 3PL ICC
plot(theta, calc_prob(theta, a = 1, b = 1, c = 0.2), type = "l", lwd = 3, col = "purple",
      xlab = expression(theta ~ "(logits)"), ylab = expression(P(X[pi] == 1)),
      main = "3PL ICC (a = 1, b = 1, c = 0.2)",
      ylim = c(0, 1))
abline(h = 0.2, lty = 2, col = "red")
abline(h = 0.6, lty = 2, col = "gray") # 0.5 + 0.5*0.2 = 0.6
abline(v = 1, lty = 2, col = "gray")
text(-3, 0.23, "c = 0.2 (lower asymptote)", col = "red", pos = 4)
```

3PL ICC ($a = 1$, $b = 1$, $c = 0.2$)



Comparing All Three Models

```

par(mfrow = c(1, 3))

# 1PL
b_vals <- c(-1, 0, 1)
plot(theta, calc_prob(theta, a = 1, b = b_vals[1]), type = "l", lwd = 2,
      col = "blue", xlab = expression(theta), ylab = "P(X = 1)",
      main = "1PL/Rasch Model", ylim = c(0, 1))
lines(theta, calc_prob(theta, a = 1, b = b_vals[2]), lwd = 2, col = "red")
lines(theta, calc_prob(theta, a = 1, b = b_vals[3]), lwd = 2, col = "darkgreen")
legend("bottomright", legend = paste("b =", b_vals), col = c("blue", "red", "darkgreen"), lwd = 2, cex = 0.8)

# 2PL
plot(theta, calc_prob(theta, a = 0.5, b = 0), type = "l", lwd = 2,
      col = "blue", xlab = expression(theta), ylab = "P(X = 1)",
      main = "2PL Model", ylim = c(0, 1))
lines(theta, calc_prob(theta, a = 1, b = 0), lwd = 2, col = "red")
lines(theta, calc_prob(theta, a = 2, b = 0), lwd = 2, col = "darkgreen")
legend("bottomright", legend = c("a=0.5", "a=1", "a=2"), col = c("blue", "red", "darkgreen"), lwd = 2, cex = 0.8)

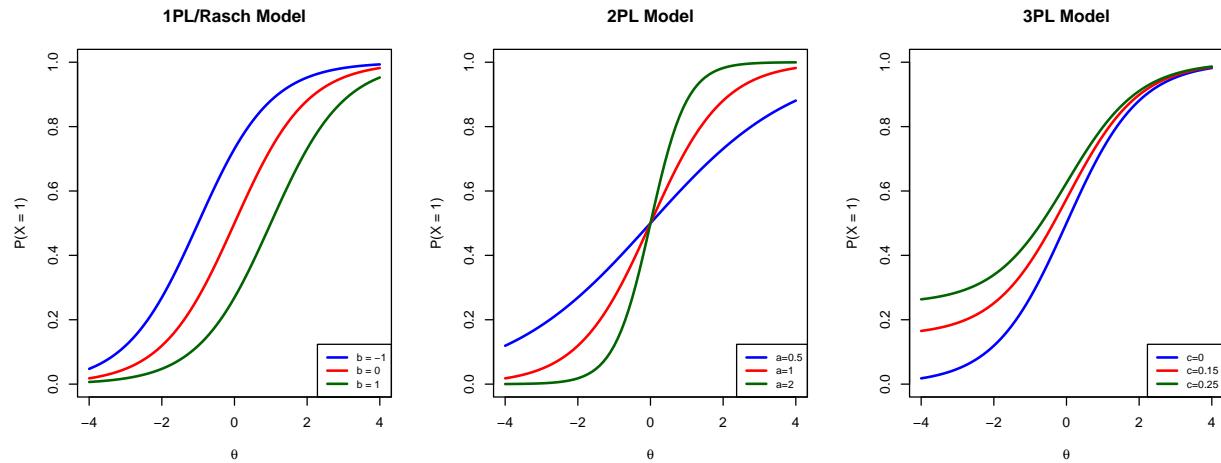
# 3PL
plot(theta, calc_prob(theta, a = 1, b = 0, c = 0), type = "l", lwd = 2,
      col = "blue", xlab = expression(theta), ylab = "P(X = 1)",
      main = "3PL Model", ylim = c(0, 1))
lines(theta, calc_prob(theta, a = 1, b = 0, c = 0.15), lwd = 2, col = "red")

```

```

lines(theta, calc_prob(theta, a = 1, b = 0, c = 0.25), lwd = 2, col = "darkgreen")
legend("bottomright", legend = c("c=0", "c=0.15", "c=0.25"), col = c("blue", "red", "darkgreen"), lwd =

```



```

par(mfrow = c(1, 1))

```

Logits and Alternative Forms

The units of θ , b , and a are **logit** values. The c parameter is on the probability scale.

For the 1PL and 2PL, we can show that:

$$\ln \left(\frac{P(X_{pi} = 1)}{1 - P(X_{pi} = 1)} \right) = a_i(\theta_p - b_i)$$

Slope-Intercept Form

Sometimes you'll see the models written in slope-intercept form:

$$P(X_{pi} = 1) = c_i + (1 - c_i) \frac{\exp(a_i \theta_p + d_i)}{1 + \exp(a_i \theta_p + d_i)}$$

where $d_i = -a_i \cdot b_i$ (or equivalently, $b_i = -d_i/a_i$).

Many IRT programs estimate d_i and then convert to b_i .

Activity: Three-Item Test

Consider a test with three items with the following parameters:

```

items <- data.frame(
  Item = 1:3,
  a = c(1, 1, 2),
  b = c(-1.5, 0, 0.2),
  c = c(0.2, 0.2, 0.3)
)
knitr::kable(items, caption = "Item Parameters for Three-Item Test")

```

Table 3: Item Parameters for Three-Item Test

Item	a	b	c
1	1	-1.5	0.2
2	1	0.0	0.2
3	2	0.2	0.3

Visualizing the ICCs

```

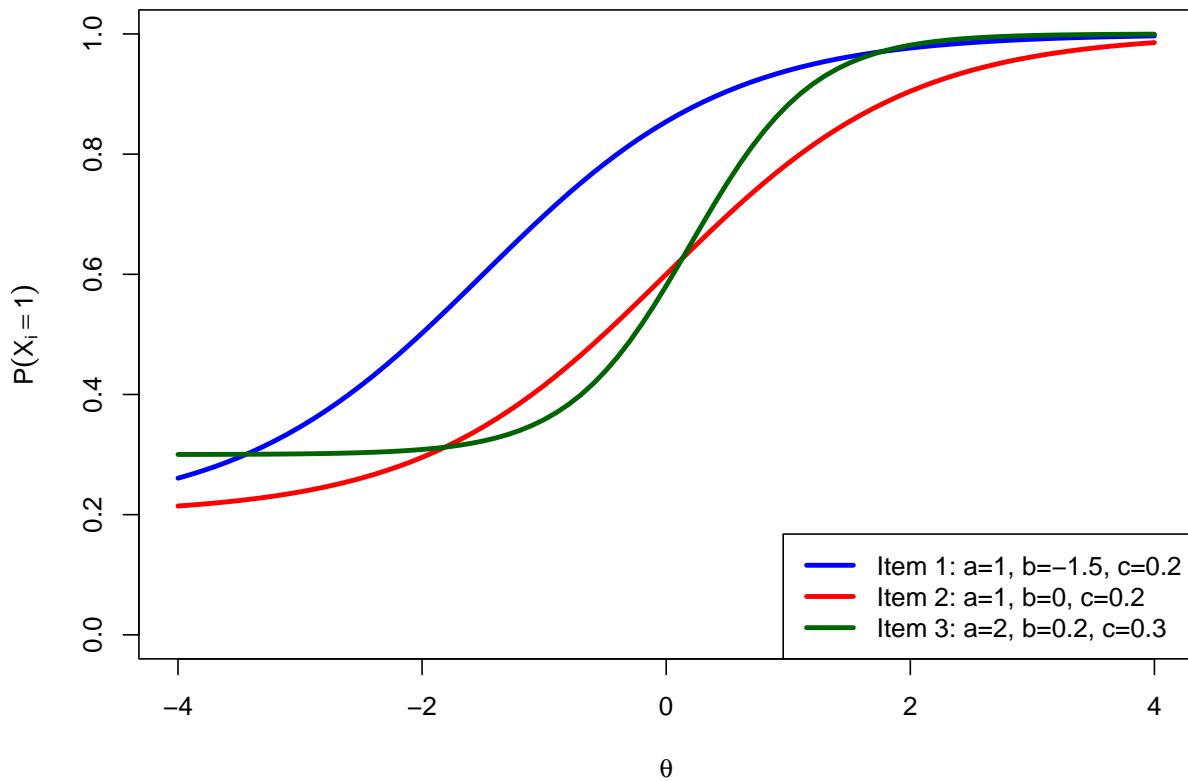
theta <- seq(-4, 4, 0.1)

# Calculate probabilities for each item
p1 <- calc_prob(theta, a = 1, b = -1.5, c = 0.2)
p2 <- calc_prob(theta, a = 1, b = 0, c = 0.2)
p3 <- calc_prob(theta, a = 2, b = 0.2, c = 0.3)

plot(theta, p1, type = "l", lwd = 3, col = "blue",
      xlab = expression(theta), ylab = expression(P(X[i] == 1)),
      main = "ICCs for Three-Item Test",
      ylim = c(0, 1))
lines(theta, p2, lwd = 3, col = "red")
lines(theta, p3, lwd = 3, col = "darkgreen")
legend("bottomright",
       legend = c("Item 1: a=1, b=-1.5, c=0.2",
                 "Item 2: a=1, b=0, c=0.2",
                 "Item 3: a=2, b=0.2, c=0.3"),
       col = c("blue", "red", "darkgreen"), lwd = 3)

```

ICCs for Three-Item Test



Practice Questions

1. Compare the difficulty of Item 1 vs. Item 2
 - Items 1 and 2 have the same discrimination ($a = 1$) and guessing ($c = 0.2$)
 - Item 1 has $b = -1.5$, Item 2 has $b = 0$
 - Item 1 is **easier** (lower b = easier)
2. Compare the difficulty of Item 2 vs. Item 3
 - Different discrimination and guessing parameters make this more complex
 - The ICCs cross, so relative difficulty depends on θ
3. Calculate response vector probabilities

```
# Function to calculate probability of response vector
calc_vector_prob <- function(theta, responses, a, b, c) {
    prob <- 1
    for (i in 1:length(responses)) {
        p_i <- calc_prob(theta, a[i], b[i], c[i])
        if (responses[i] == 1) {
            prob <- prob * p_i
        } else {
            prob <- prob * (1 - p_i)
        }
    }
}
```

```

    }
    return(prob)
}

# Item parameters
a <- c(1, 1, 2)
b <- c(-1.5, 0, 0.2)
c <- c(0.2, 0.2, 0.3)

# Theta values
theta_vals <- c(-4.0, -1.1, 0, 1.6)

# Response patterns
patterns <- list(c(0, 0, 0), c(0, 1, 0), c(1, 1, 1))
pattern_names <- c("000", "010", "111")

# Calculate probabilities
results <- matrix(NA, nrow = length(theta_vals), ncol = length(patterns))
for (i in 1:length(theta_vals)) {
  for (j in 1:length(patterns)) {
    results[i, j] <- calc_vector_prob(theta_vals[i], patterns[[j]], a, b, c)
  }
}

results_df <- data.frame(
  Theta = theta_vals,
  `000` = round(results[, 1], 3),
  `010` = round(results[, 2], 3),
  `111` = round(results[, 3], 3),
  check.names = FALSE
)

knitr::kable(results_df, caption = "Probability of Each Response Vector by Theta")

```

Table 4: Probability of Each Response Vector by Theta

Theta	000	010	111
-4.0	0.406	0.111	0.017
-1.1	0.126	0.084	0.095
0.0	0.024	0.037	0.298
1.6	0.000	0.001	0.802

Interpretation:

- At $\theta = -4.0$: The “000” pattern (all wrong) is most likely
- At $\theta = 1.6$: The “111” pattern (all correct) is most likely
- The “010” pattern (only item 2 correct) has relatively low probability at all theta levels

Fitting IRT Models with mirt

Now let's use the `mirt` package in R to fit IRT models to real data.

```
library(mirt)
```

Load and Prepare Data

```
forma <- read.csv("../Data/pset1_formA.csv")
forma <- forma[, 1:15] # Use first 15 items
```

Fit 1PL, 2PL, and 3PL Models

```
# Fit the models
mirt_1pl <- mirt(forma, model = 1, itemtype = "Rasch", method = "EM")
mirt_2pl <- mirt(forma, model = 1, itemtype = "2PL", method = "EM")
mirt_3pl <- mirt(forma, model = 1, itemtype = "3PL", method = "EM")
```

Extract and Compare Parameters

```
# Extract coefficients in IRT parameterization
coef_1pl <- coef(mirt_1pl, simplify = TRUE, IRTpars = TRUE)
coef_2pl <- coef(mirt_2pl, simplify = TRUE, IRTpars = TRUE)
coef_3pl <- coef(mirt_3pl, simplify = TRUE, IRTpars = TRUE)

# Create comparison table
comparison <- data.frame(
  Item = rownames(coef_2pl$items),
  a_1PL = round(coef_1pl$items[, 1], 3),
  b_1PL = round(coef_1pl$items[, 2], 3),
  a_2PL = round(coef_2pl$items[, 1], 3),
  b_2PL = round(coef_2pl$items[, 2], 3),
  a_3PL = round(coef_3pl$items[, 1], 3),
  b_3PL = round(coef_3pl$items[, 2], 3),
  c_3PL = round(coef_3pl$items[, 3], 3)
)

knitr::kable(comparison, caption = "Parameter Estimates Across Models")
```

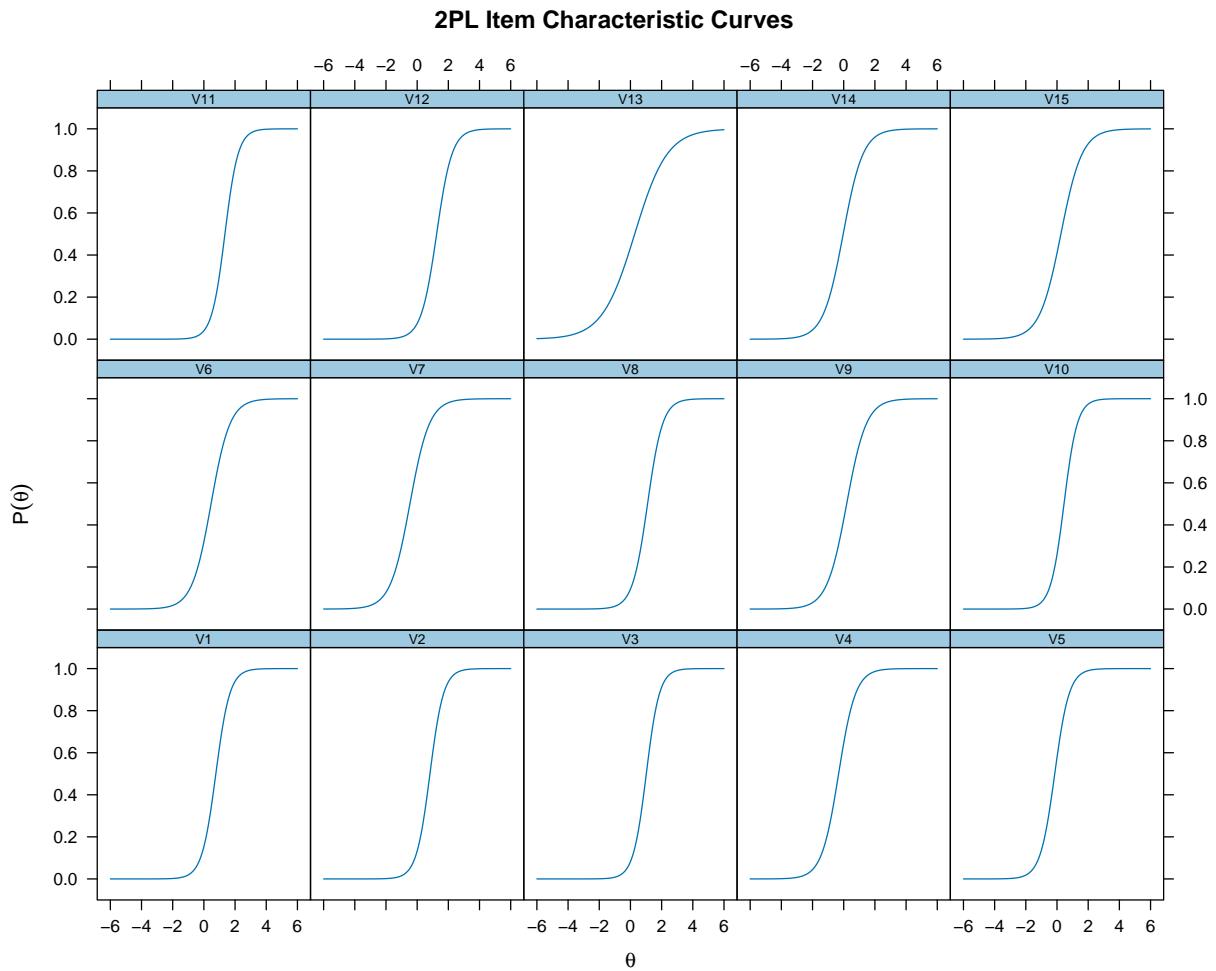
Table 5: Parameter Estimates Across Models

	Item	a_1PL	b_1PL	a_2PL	b_2PL	a_3PL	b_3PL	c_3PL
V1	V1		1	1.469	2.223	0.768	60.370	0.949
V2	V2		1	1.590	2.285	0.821	59.418	0.960
V3	V3		1	2.013	2.371	1.018	11.383	1.007
V4	V4		1	-0.604	1.833	-0.327	1.747	-0.290
								0.044

	Item	a_1PL	b_1PL	a_2PL	b_2PL	a_3PL	b_3PL	c_3PL
V5	V5	1	-0.306	2.152	-0.144	2.048	-0.095	0.000
V6	V6	1	0.774	1.641	0.461	1.545	0.532	0.000
V7	V7	1	-0.791	1.595	-0.465	1.537	-0.433	0.000
V8	V8	1	2.066	2.057	1.098	1.903	1.186	0.000
V9	V9	1	0.306	1.580	0.189	1.531	0.248	0.000
V10	V10	1	0.853	2.338	0.450	2.139	0.532	0.000
V11	V11	1	2.697	2.353	1.350	2.027	1.470	0.000
V12	V12	1	2.342	2.018	1.249	1.838	1.347	0.000
V13	V13	1	0.351	0.956	0.274	0.932	0.324	0.000
V14	V14	1	-0.065	1.566	-0.033	1.476	0.015	0.000
V15	V15	1	0.398	1.468	0.251	1.412	0.310	0.000

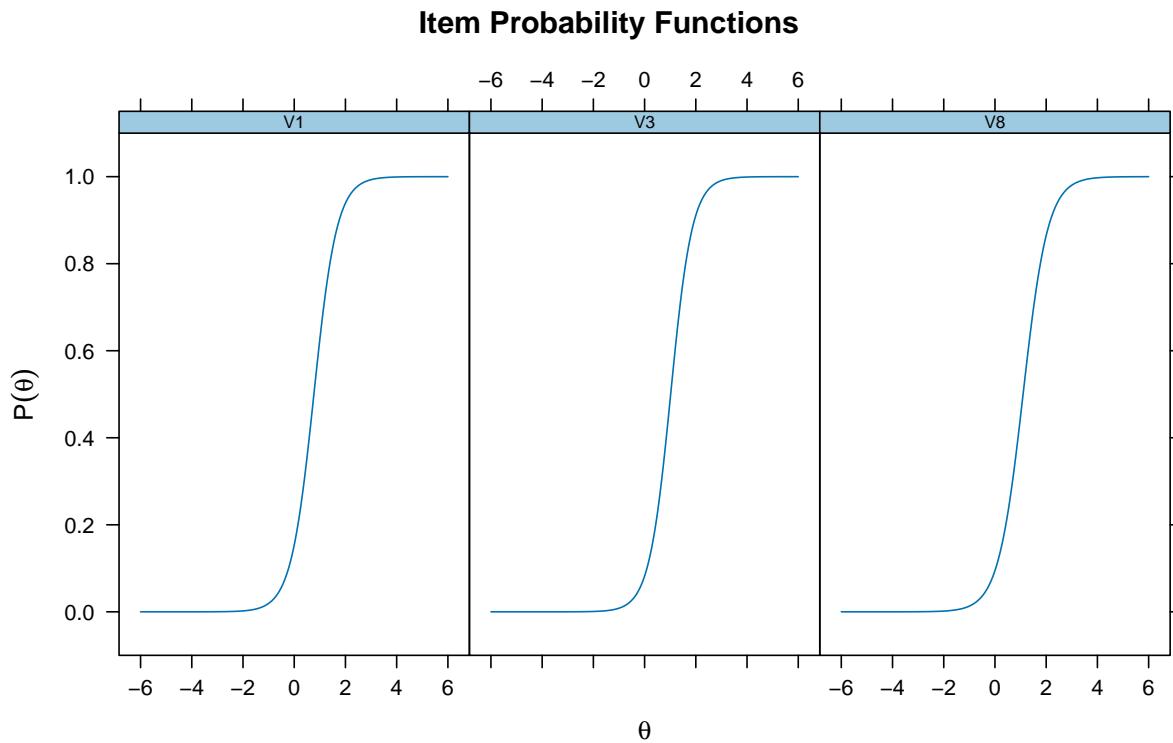
Plot ICCs

```
# Plot ICCs for 2PL model
plot(mirt_2pl, type = 'trace', main = "2PL Item Characteristic Curves")
```



Plot Specific Items

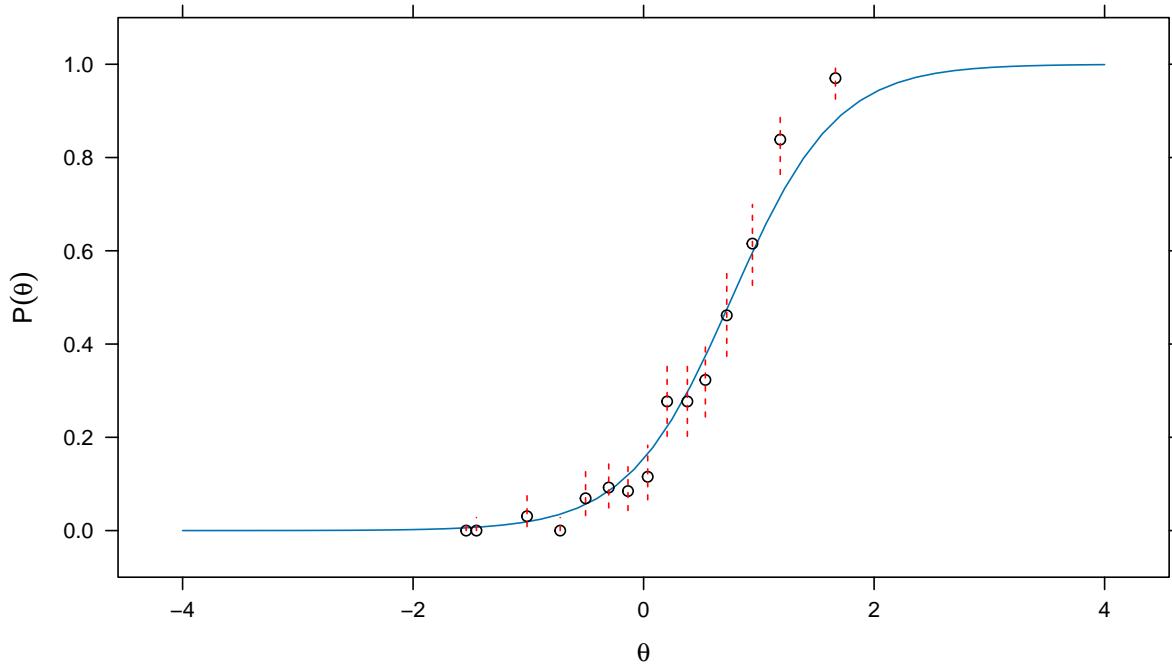
```
# Compare specific items
plot(mirt_2pl, which.items = c(1, 3, 8), type = 'trace')
```



Check Item Fit

```
# Observed vs. predicted for Item 1
itemfit(mirt_2pl, group.bins = 15, empirical.plot = 1)
```

Empirical plot for item 1



Summary

1. The **1PL/Rasch model** has one item parameter (difficulty) and assumes equal discrimination
 2. The **2PL model** adds a discrimination parameter, allowing ICCs to have different slopes
 3. The **3PL model** adds a lower asymptote (guessing) parameter
 4. In the 2PL and 3PL, ICCs can cross, meaning relative item difficulty may depend on ability level
 5. The `mirt` package provides a flexible framework for fitting these models in R
-

Interactive Practice

For an interactive version of the three-item test activity, see the **Shiny app** in the **Shiny Apps** folder.