Deriving and Conceptualizing an Item Characteristic Curve

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Contents

Introduction

This document provides a conceptual derivation of the Item Characteristic Curve (ICC) using the normal ogive. The presentation follows the narrative found in Lord & Novick (1968, pp. 370-371) and Thissen & Orlando (2001, pp. 84-87).

As Lord and Novick wrote:

> "[The equation] may be taken simply as a basic assumption, the utility of which can be investigated for a given set of data (albeit with considerable difficulty). Alternatively [this equation] can be inferred from other, possibly more plausible assumptions. We shall outline one way of doing this, a way that some theorists find interesting and others do not." (Lord & Novick, 1968, p. 370)

Part 1: Statistical Background

Before deriving the ICC, we need to review some foundational concepts about random variables and probability distributions.

Random Variables and Probability Functions

A **random variable** can be loosely defined as a quantity that can have more than one realized value such that the possible values can be assigned to a probability function.

Notation conventions:

- Random variables are denoted using upper case italicized letters: X, Y, Z
- Realized/observed values are denoted using lower case italicized letters: x, y, z
- If we write $P(X = x) = .5$, this says "the probability that random variable X equals the value x is 0.5"

There are two kinds of random variables:

- **Discrete**: Can only take on specific, countable values
- **Continuous**: Can take on any value within a range

Probability Distribution Functions (pdf)

A probability function provides information about the distribution of a random variable.

For Discrete Random Variables

$$p(x) = P(X = x)$$

where:

1. $p(x) \geq 0$ (probabilities must be non-negative)
2. $\sum_x p(x) = 1$ (probabilities must sum to 1)

For Continuous Random Variables

We represent the pdf as a function $f(x)$ where:

$$P(a \leq X \leq b) = \int_a^b f(x)\,dx$$

Properties:

1. $f(x) \geq 0$ for $-\infty < x < +\infty$
2. $\int_{-\infty}^{+\infty} f(x)\,dx = 1$
3. $P(X = c) = 0$ (probability of any exact value is 0)

Cumulative Distribution Functions (cdf)

The cumulative distribution function tells us $P(X \leq x)$.

For Discrete Random Variables

$$F(x) = P(X \leq x) = \sum_{t|t\leq x} p(t)$$

For Continuous Random Variables

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(t)\, dt$$

Key relationship: To go from a cdf to a pdf, take the first derivative:

$$f(x) = \frac{dF(x)}{dx}$$

The Normal (Gaussian) Distribution

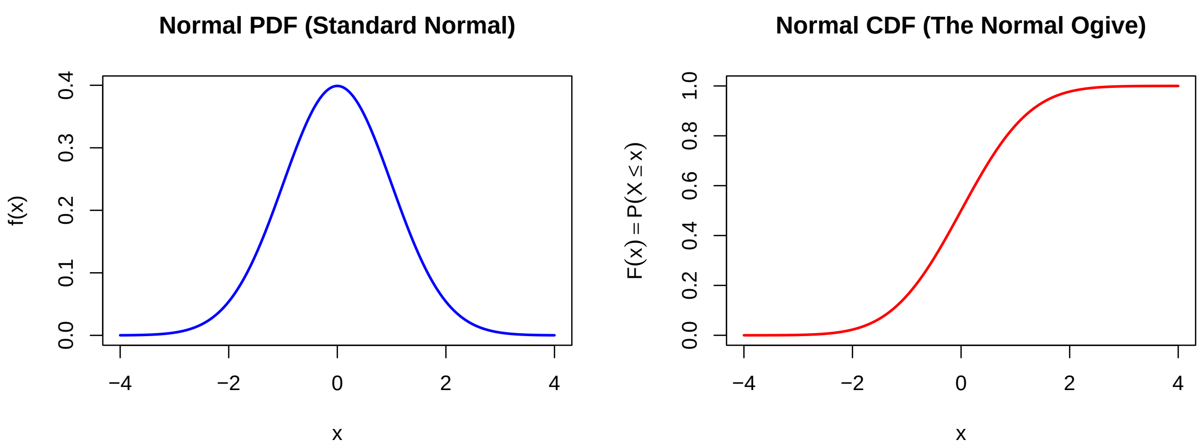
The pdf of the normal distribution is:

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(\frac{-(x-\mu)^2}{2\sigma^2}\right)$$

The cdf (also called the **normal ogive**) is:

$$F(x) = P(X \leq x) = \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{x} \exp\left(\frac{-(t-\mu)^2}{2\sigma^2}\right) dt$$

```r
par(mfrow = c(1, 2))

# Normal PDF
x <- seq(-4, 4, 0.01)
plot(x, dnorm(x), type = "l", lwd = 2, col = "blue",
     main = "Normal PDF (Standard Normal)",
     xlab = "x", ylab = "f(x)")

# Normal CDF (the "ogive")
plot(x, pnorm(x), type = "l", lwd = 2, col = "red",
     main = "Normal CDF (The Normal Ogive)",
     xlab = "x", ylab = expression(F(x) == P(X <= x)))
```



```r
par(mfrow = c(1, 1))
```

Part 2: Building the Model

It Starts with a Test and an Item

Consider the first item on a math test for 6th graders:

> A penny is tossed 20 times. Which of the following is most likely to be the number of times heads came up?
>
> A. 0 B. 2 C. 5 D. 8 E. 15

This item is trying to find out whether the student has a basic understanding of probability.

The Response Process Continuum

We assume there is a "response process" continuum that governs whether any individual will answer this item correctly (see Thissen & Orlando, 2001, p. 85).

- This process is represented by the continuous quantity V_i for item i
- Each item on the test is associated with a distinct response process quantity V_i
- Let θ represent the general unidimensional construct of "mathematical ability"
- We assume both V_i and θ have been standardized (mean = 0, SD = 1)

A Regression Equation

The latent variable underlying any single item can be related to θ with the linear regression equation:

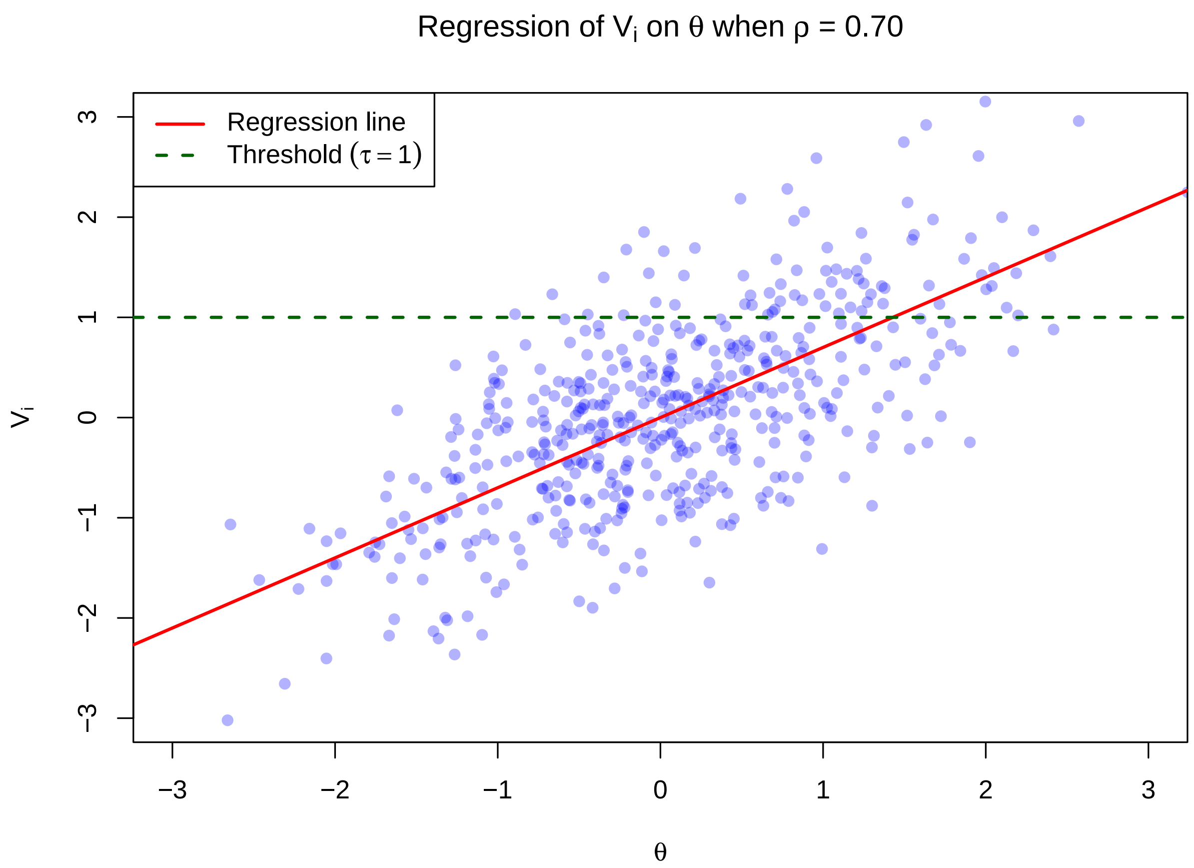
$$V_i = \rho_i \theta + \varepsilon_i$$

where:

- ρ_i represents the correlation between V_i and θ (this is a **biserial correlation**)
- ε_i represents a random error term, where $\varepsilon_i \sim N(0, 1)$
- By construction, ε_i and θ are independent

```r
set.seed(123)
n <- 500
rho <- 0.70

# Generate data
theta <- rnorm(n, 0, 1)
epsilon <- rnorm(n, 0, sqrt(1 - rho^2))
V <- rho * theta + epsilon

# Plot
```

```r
plot(theta, V, pch = 16, col = rgb(0, 0, 1, 0.3),
     xlab = expression(theta), ylab = expression(V[i]),
     main = expression(paste("Regression of ", V[i], " on ", theta, " when ", rho, " = 0.70")),
     xlim = c(-3, 3), ylim = c(-3, 3))
abline(a = 0, b = rho, col = "red", lwd = 2)
abline(h = 1, col = "darkgreen", lwd = 2, lty = 2)
legend("topleft", legend = c("Regression line", expression(Threshold ~ (tau == 1))),
       col = c("red", "darkgreen"), lwd = 2, lty = c(1, 2))
```



Key Properties

The regression line predicts the conditional mean of V_i given θ:

$$E(V_i|\theta) = \rho_i\theta$$

The conditional standard deviation (RMSE) is:

$$\sigma_{\varepsilon|\theta} = \sqrt{1 - \rho_i^2}$$

The distribution of $V_i|\theta$ is assumed to be normal.

Part 3: Deriving the ICC

Using the Normal CDF

Given a threshold τ_i for item i, we want to find:

$$P(V_i > \tau_i | \theta)$$

This is calculated using the conditional normal cdf:

$$P(V_i \leq \tau_i | \theta) = \frac{1}{\sqrt{2\pi}\sigma_{\varepsilon|\theta}} \int_{-\infty}^{\tau_i} \exp\left(\frac{-(t - E(V_i|\theta))^2}{2\sigma_{\varepsilon|\theta}^2}\right) dt$$

Example Calculations

Let $\rho_i = 0.70$ and $\tau_i = 1$.

Then $\sigma_{\varepsilon|\theta} = \sqrt{1 - 0.70^2} = \sqrt{0.51} \approx 0.71$

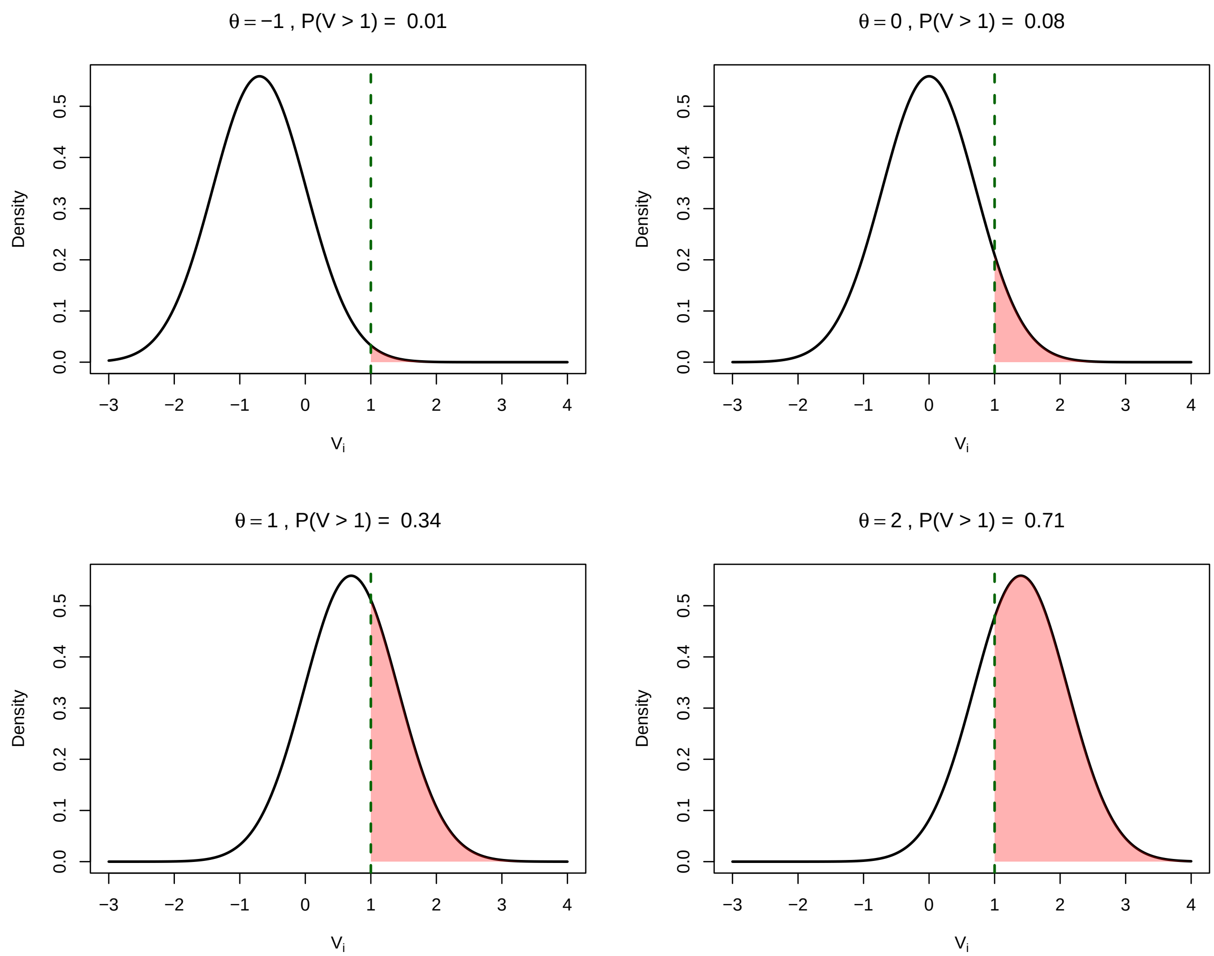
```
rho <- 0.70
tau <- 1
sigma_eps <- sqrt(1 - rho^2)

# For different values of theta
theta_vals <- c(-1, 0, 1, 2)

for (th in theta_vals) {
  E_V <- rho * th
  prob <- 1 - pnorm(tau, mean = E_V, sd = sigma_eps)
  cat(sprintf("When   = %2d: E(V| ) = %5.2f, P(V >  | ) = %.3f\n", th, E_V, prob))
}
```

```
## When   = -1: E(V| ) = -0.70, P(V >  | ) = 0.009
## When   =  0: E(V| ) =  0.00, P(V >  | ) = 0.081
## When   =  1: E(V| ) =  0.70, P(V >  | ) = 0.337
## When   =  2: E(V| ) =  1.40, P(V >  | ) = 0.712
```

Visualizing the Probability Calculation

```
par(mfrow = c(2, 2))

theta_vals <- c(-1, 0, 1, 2)
v_range <- seq(-3, 4, 0.01)

for (th in theta_vals) {
  E_V <- rho * th
  prob <- 1 - pnorm(tau, mean = E_V, sd = sigma_eps)

  # Plot the conditional distribution
  plot(v_range, dnorm(v_range, mean = E_V, sd = sigma_eps), type = "l", lwd = 2,
```

```r
      xlab = expression(V[i]), ylab = "Density",
      main = bquote(theta == .(th) ~ ", P(V > 1) = " ~ .(round(prob, 2))))

  # Shade the area above threshold
  v_above <- v_range[v_range >= tau]
  polygon(c(tau, v_above, max(v_above)),
          c(0, dnorm(v_above, mean = E_V, sd = sigma_eps), 0),
          col = rgb(1, 0, 0, 0.3), border = NA)

  abline(v = tau, col = "darkgreen", lwd = 2, lty = 2)
}
```




```r
par(mfrow = c(1, 1))
```

From Latent to Observed

Both V_i and θ are latent variables. We link the observed item response to θ through a **dichotomization rule**:

Let X_i be a discrete random variable representing the observed response to item i:

- If $V_i > \tau_i$ then $X_i = 1$ (correct)
- If $V_i \leq \tau_i$ then $X_i = 0$ (incorrect)

This implies:

$$P(X_i = 1|\theta) = P(V_i > \tau_i|\theta)$$

Part 4: The Two-Parameter Normal Ogive Model

Putting this all together, we can write:

$$P(X_i = 1|\theta) = \Phi(a_i(\theta - b_i))$$

where Φ is the standard normal cdf, and:

$$a_i = \frac{\rho_i}{\sqrt{1-\rho_i^2}} \quad \text{(discrimination)}$$

$$b_i = \frac{\tau_i}{\rho_i} \quad \text{(difficulty)}$$

Example

Given $\rho_i = 0.70$ and $\tau_i = 1$:

```r
rho <- 0.70
tau <- 1

a <- rho / sqrt(1 - rho^2)
b <- tau / rho

cat(sprintf("Discrimination (a) = %.2f\n", a))
```

```
## Discrimination (a) = 0.98
```

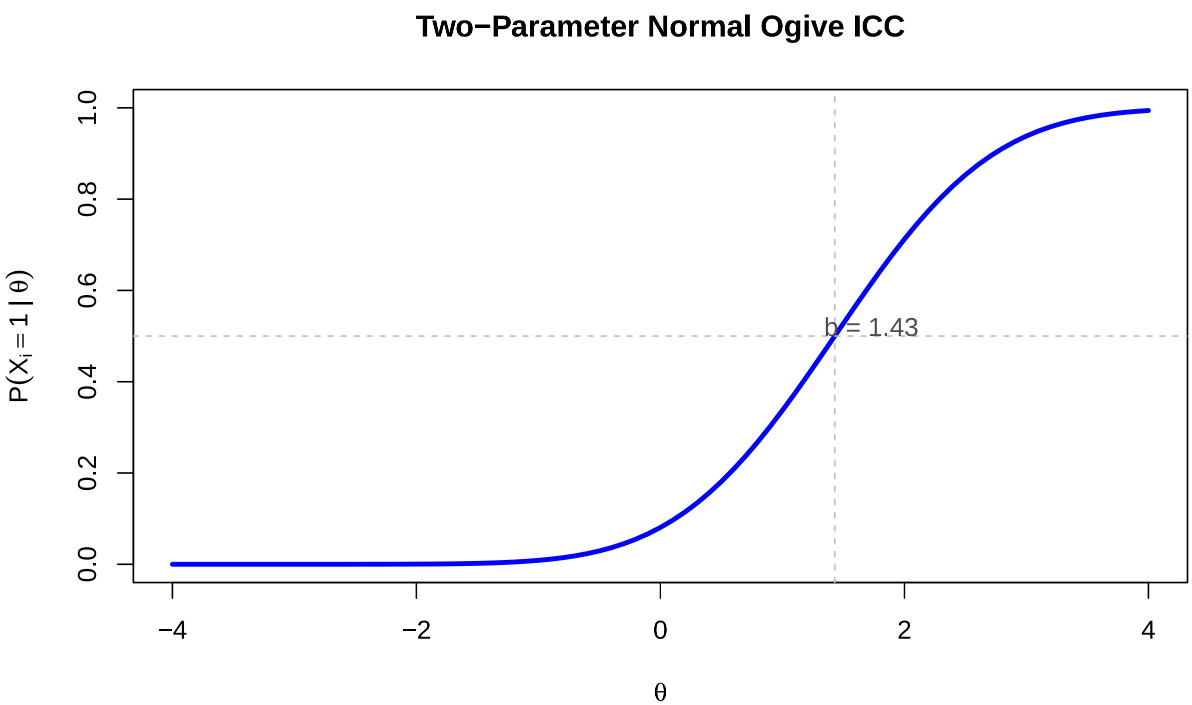
```r
cat(sprintf("Difficulty (b) = %.2f\n", b))
```

```
## Difficulty (b) = 1.43
```

Plotting the ICC

```r
theta <- seq(-4, 4, 0.1)
prob <- pnorm(a * (theta - b))

plot(theta, prob, type = "l", lwd = 3, col = "blue",
     xlab = expression(theta), ylab = expression(P(X[i] == 1 ~ "|" ~ theta)),
```

```r
     main = "Two-Parameter Normal Ogive ICC",
     ylim = c(0, 1))
abline(h = 0.5, lty = 2, col = "gray")
abline(v = b, lty = 2, col = "gray")
text(b + 0.3, 0.52, paste("b =", round(b, 2)), col = "gray30")
```



Interpretation of Parameters

This derivation explains why:

1. The **discrimination parameter** (a_i) in the 2PL IRT model is analogous to the correlation between the item and the construct of measurement
2. The **difficulty parameter** (b_i) is analogous to the threshold between a correct and incorrect response (inversely related to the proportion answering correctly)

Converting Between Parameterizations

To go from IRT parameters back to factor analytic parameters:

$$\rho_i = \frac{a_i}{\sqrt{1 + a_i^2}} \quad \text{(biserial correlation / factor loading)}$$

$$\tau_i = b_i \cdot \rho_i \quad \text{(threshold)}$$

Note: These relationships only hold if θ is normally distributed and there is no guessing on items.

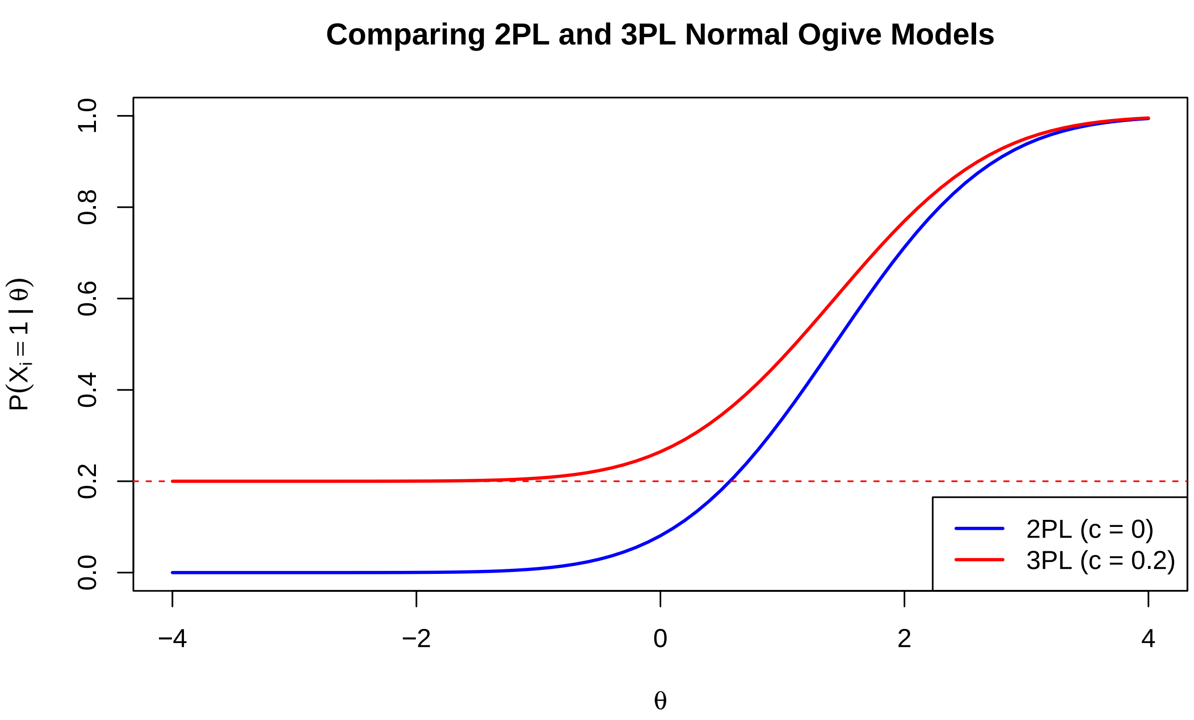
Part 5: The Three-Parameter Normal Ogive Model

Adding a lower asymptote (guessing parameter):

$$P(X_i = 1|\theta, b_i, a_i, c_i) = c_i + (1 - c_i)\Phi(a_i(\theta - b_i))$$

where c_i represents the probability of a correct response by guessing.

```r
theta <- seq(-4, 4, 0.1)
c_param <- 0.20  # guessing parameter

prob_2pl <- pnorm(a * (theta - b))
prob_3pl <- c_param + (1 - c_param) * pnorm(a * (theta - b))

plot(theta, prob_2pl, type = "l", lwd = 2, col = "blue",
     xlab = expression(theta), ylab = expression(P(X[i] == 1 ~ "|" ~ theta)),
     main = "Comparing 2PL and 3PL Normal Ogive Models",
     ylim = c(0, 1))
lines(theta, prob_3pl, lwd = 2, col = "red")
abline(h = c_param, lty = 2, col = "red")
legend("bottomright", legend = c("2PL (c = 0)", paste0("3PL (c = ", c_param, ")")),
       col = c("blue", "red"), lwd = 2)
```



Part 6: The Logistic Approximation

The logistic distribution provides a close approximation to the normal distribution and is computationally simpler.

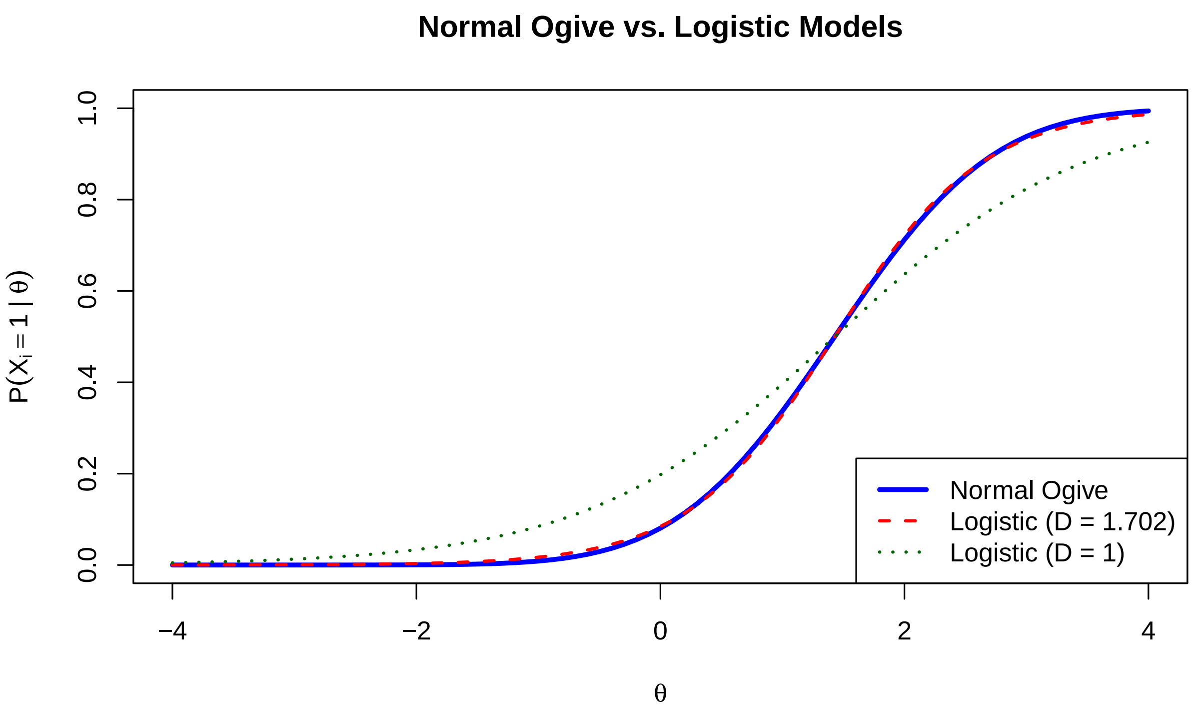
Logistic vs. Normal Ogive

The logistic IRT model can be written as:

$$P(X_i = 1|\theta) = c_i + (1 - c_i)\frac{\exp(Da_i(\theta - b_i))}{1 + \exp(Da_i(\theta - b_i))}$$

where $D = 1.702$ makes the logistic model approximately equal to the normal ogive model.

```r
theta <- seq(-4, 4, 0.1)
D <- 1.702

# Normal ogive
prob_normal <- pnorm(a * (theta - b))

# Logistic with D scaling
prob_logistic_D <- plogis(D * a * (theta - b))

# Logistic without D scaling
prob_logistic <- plogis(a * (theta - b))

plot(theta, prob_normal, type = "l", lwd = 3, col = "blue",
     xlab = expression(theta), ylab = expression(P(X[i] == 1 ~ "|" ~ theta)),
     main = "Normal Ogive vs. Logistic Models",
     ylim = c(0, 1))
lines(theta, prob_logistic_D, lwd = 2, col = "red", lty = 2)
lines(theta, prob_logistic, lwd = 2, col = "darkgreen", lty = 3)
legend("bottomright",
       legend = c("Normal Ogive", "Logistic (D = 1.702)", "Logistic (D = 1)"),
       col = c("blue", "red", "darkgreen"), lwd = c(3, 2, 2), lty = c(1, 2, 3))
```



Modern Practice

Historically, researchers used the $D = 1.702$ scaling to make logistic parameters match normal ogive parameters. Today, most practitioners simply use the logistic version without D because:

1. The logistic is easier to estimate
2. There isn't much practical need to match normal ogive parameters
3. Model fit is essentially the same

Summary

1. The ICC can be derived by conceptualizing a latent response process continuum V_i that underlies observed item responses
2. The regression of V_i on θ with a threshold dichotomization rule leads to the normal ogive model
3. The discrimination parameter a_i reflects how strongly the item relates to the construct
4. The difficulty parameter b_i reflects the threshold for a correct response
5. The logistic model provides a computationally convenient approximation to the normal ogive

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References

Lord, F. M., & Novick, M. R. (1968). *Statistical theories of mental test scores.* Addison-Wesley.

Thissen, D., & Orlando, M. (2001). Item response theory for items scored in two categories. In D. Thissen & H. Wainer (Eds.), *Test scoring* (pp. 73-140). Lawrence Erlbaum Associates.