

# Item Parameter Estimation via MMLE/EM

## From Intuition to Implementation

Derek Briggs

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## Introduction

In the previous module on person parameter estimation, we saw how to estimate a person's ability ( $\theta$ ) given known item parameters. Now we tackle the other side of the coin: **estimating item parameters when person abilities are unknown.**

Joint Maximum Likelihood Estimation (JMLE) attempts to estimate both simultaneously, but suffers from well-known problems (the Neyman-Scott inconsistency, inflated discrimination estimates). **Marginal Maximum Likelihood Estimation (MMLE)** offers an elegant alternative: instead of estimating each person's  $\theta$ , we *integrate over* a population distribution of  $\theta$ , effectively averaging out the person parameters. This yields consistent item parameter estimates regardless of sample size.

The key insight: **we don't need to know where each individual person falls on the ability scale; we only need to know how persons are distributed across the scale.**

MMLE is paired with the **EM (Expectation-Maximization) algorithm**, an iterative procedure that alternates between:

- **E-step:** Using current item parameter guesses to figure out how examinees are distributed across the ability scale (“artificial data”)
- **M-step:** Using that distribution to re-estimate item parameters, one item at a time, using the same Newton-Raphson approach from JMLE

This document walks through the entire MMLE/EM process with actual data and actual numbers, building from concepts you already know. It builds from the more technical presentation of MMLE found in Chapter 6 of Baker and Kim (2004).

```
library(ggplot2)
library(dplyr)
library(tidyr)
library(knitr)
library(mirt)
```

## From Thor to Marginal Likelihood

### Recall: Thor's Posterior Distribution

In the person parameter estimation module, we met Thor, who takes a two-item test (Item 1:  $a = 1, b = 0$ ; Item 2:  $a = 2, b = 2$ ) and gets the pattern  $\{1, 0\}$  (first item correct, second incorrect).

We computed Thor's posterior distribution by combining a **prior** (our assumption about the population distribution of  $\theta$ , typically standard normal) with the **likelihood** of his response pattern:

$$\text{Posterior}(\theta) \propto \underbrace{L(\theta | \mathbf{u})}_{\text{likelihood}} \times \underbrace{g(\theta)}_{\text{prior}}$$

```
# 2PL model
p_2pl <- function(theta, a, b) {
  1 / (1 + exp(-a * (theta - b)))
}

# Item parameters
a1 <- 1; b1 <- 0
a2 <- 2; b2 <- 2

# Fine grid of theta values
theta <- seq(-4, 4, by = 0.01)

# Prior: standard normal
prior <- dnorm(theta, 0, 1)

# Thor's likelihood: P1(theta) * Q2(theta)
likelihood_thor <- p_2pl(theta, a1, b1) * (1 - p_2pl(theta, a2, b2))

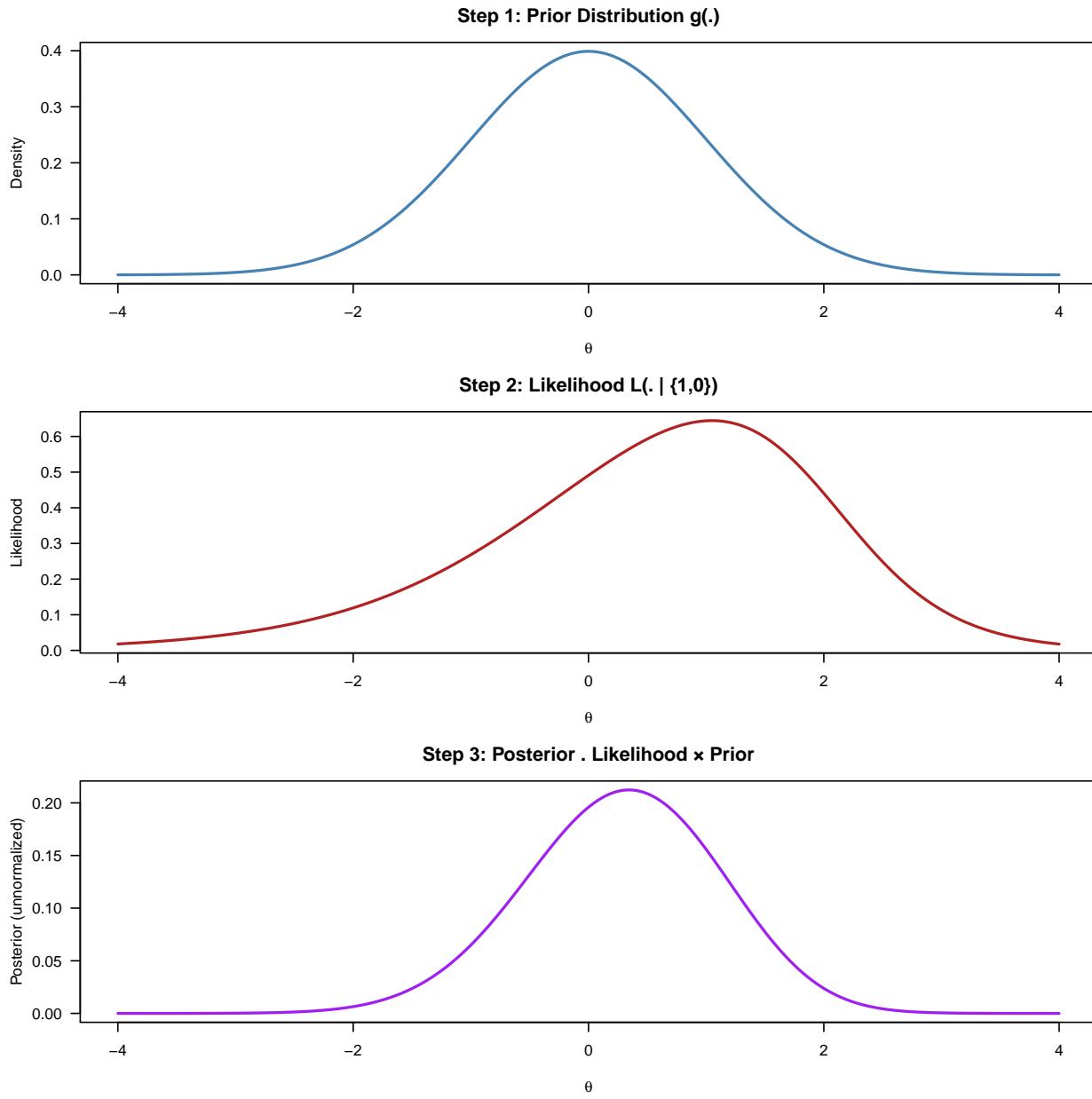
# Posterior (unnormalized)
posterior_unnorm <- likelihood_thor * prior
```

```
par(mfrow = c(3, 1), mar = c(4, 4, 3, 1))

plot(theta, prior, type = "l", col = "steelblue", lwd = 2,
      main = "Step 1: Prior Distribution g(\u03b8)",
      xlab = expression(theta), ylab = "Density", las = 1)

plot(theta, likelihood_thor, type = "l", col = "firebrick", lwd = 2,
      main = "Step 2: Likelihood L(\u03b8 | {1,0})",
      xlab = expression(theta), ylab = "Likelihood", las = 1)

plot(theta, posterior_unnorm, type = "l", col = "purple", lwd = 2,
      main = "Step 3: Posterior \u221d Likelihood \u00d7 Prior",
      xlab = expression(theta), ylab = "Posterior (unnormalized)", las = 1)
```



## The Denominator You May Have Ignored

When we wrote Bayes' theorem for the posterior, the full expression was:

$$P(\theta | \mathbf{u}, \xi) = \frac{L(\mathbf{u} | \theta, \xi) \cdot g(\theta)}{\int L(\mathbf{u} | \theta, \xi) \cdot g(\theta) d\theta}$$

For person estimation, we could ignore the denominator — it's just a normalizing constant that doesn't change where the peak is. But that denominator has a name: it is the **marginal likelihood** of the response pattern  $\mathbf{u}$ . It tells us: *how probable is this response pattern, averaged across all possible ability levels?*

How do we compute this integral? One brute-force approach is to evaluate the function at a very large number of closely spaced  $\theta$  values, multiply each by the spacing between points, and sum up the results.

This is called the **trapezoidal rule** — it's the numerical integration equivalent of adding up lots of very thin rectangles:

```
# The marginal likelihood is the area under the unnormalized posterior.
# Here we approximate the integral using a "brute force" approach:
# evaluate at 801 closely spaced points (theta from -4 to 4 by 0.01)
# and sum up the tiny rectangles (trapezoidal rule).
dx <- theta[2] - theta[1] # spacing = 0.01
marginal_lik_thor <- sum(likelihood_thor * prior) * dx

cat("Number of evaluation points:", length(theta), "\n")

## Number of evaluation points: 801

cat("Spacing between points (dx):", dx, "\n")

## Spacing between points (dx): 0.01

cat("Marginal likelihood of Thor's pattern {1,0}:",
    round(marginal_lik_thor, 6), "\n")

## Marginal likelihood of Thor's pattern {1,0}: 0.447543

# Now we can compute the actual normalized posterior
posterior_norm <- (likelihood_thor * prior) / marginal_lik_thor
cat("Area under normalized posterior:",
    round(sum(posterior_norm) * dx, 4), "(should be ~1)\n")

## Area under normalized posterior: 1 (should be ~1)
```

This works well when we can afford to evaluate the function at hundreds of points. But in MMLE, we need to compute this integral for *every response pattern*, at *every EM cycle*, for *every set of candidate item parameters*. Evaluating at 801 points each time would be extremely slow. We need a much more efficient approach — and that's where **quadrature** comes in (next section).

## The Key Insight: From Person Estimation to Item Estimation

Here is the conceptual leap:

- **Person estimation:** We knew the item parameters and used the posterior to find where *one person* likely falls on the  $\theta$  scale.
- **Item estimation (MMLE):** We *don't* know the item parameters. But if we compute the marginal likelihood for *every response pattern* across *all* examinees, we get a function of the **item parameters alone** (the  $\theta$ 's have been integrated out!). We then find the item parameters that maximize this marginal likelihood.

The marginal likelihood for the full sample is:

$$L(\xi) = \prod_{j=1}^N \int L(\mathbf{u}_j | \theta, \xi) \cdot g(\theta | \tau) d\theta$$

Each examinee's contribution is the area under their own likelihood  $\times$  prior curve — the same denominator we computed for Thor above. The product is over all  $N$  examinees (or equivalently, over all unique response patterns weighted by their frequencies).

**The challenge:** that integral has no closed-form solution. We need a numerical approximation.

## Quadrature: Making Integration Concrete

### The Problem

We need to evaluate integrals of the form:

$$\int_{-\infty}^{\infty} f(\theta) \cdot g(\theta) d\theta$$

where  $f(\theta)$  is a likelihood function and  $g(\theta)$  is the prior density. There's no formula that gives us the answer directly — we need to approximate.

We saw one way to do this above: the **trapezoidal rule**, which evaluates the function at hundreds of closely spaced points (we used 801). This gives a very accurate answer but is computationally expensive. In the MMLE context, this integral must be evaluated repeatedly — for every response pattern, at every EM iteration — so efficiency matters enormously.

### The Solution: Gaussian Quadrature

Quadrature takes a fundamentally different approach: instead of brute-forcing with many points, it uses a **small number of strategically chosen points** with carefully computed weights that are designed to capture the shape of the distribution efficiently.

The core idea: **replace the integral with a weighted sum over a small number of points.**

$$\int f(\theta) \cdot g(\theta) d\theta \approx \sum_{k=1}^q f(X_k) \cdot A(X_k)$$

Here  $X_k$  are the **nodes** (specific  $\theta$  values),  $A(X_k)$  are the **weights** (each reflecting how much probability mass that node represents), and  $q$  is the number of nodes — typically just 10 to 40. The weights  $A(X_k)$  already incorporate the prior density  $g(\theta)$ , so we only need to evaluate the likelihood  $f(X_k)$  at each node.

Think of it as replacing the smooth curve with a bar chart — but a very efficient bar chart where the bars are placed and sized to capture the essential shape of the distribution with as few bars as possible.

Method	Points	What you evaluate	Weights
<b>Trapezoidal rule</b>	Many (e.g., 801)	$f(\theta_j) \cdot g(\theta_j)$ at each point	All equal ( $dx = \text{spacing}$ )
<b>Gaussian quadrature</b>	Few (e.g., 10)	$f(X_k)$ at each node	Pre-computed $A(X_k)$ that absorb $g(\theta)$

```

# Show quadrature with different numbers of nodes
par(mfrow = c(2, 2), mar = c(4, 4, 3, 1))

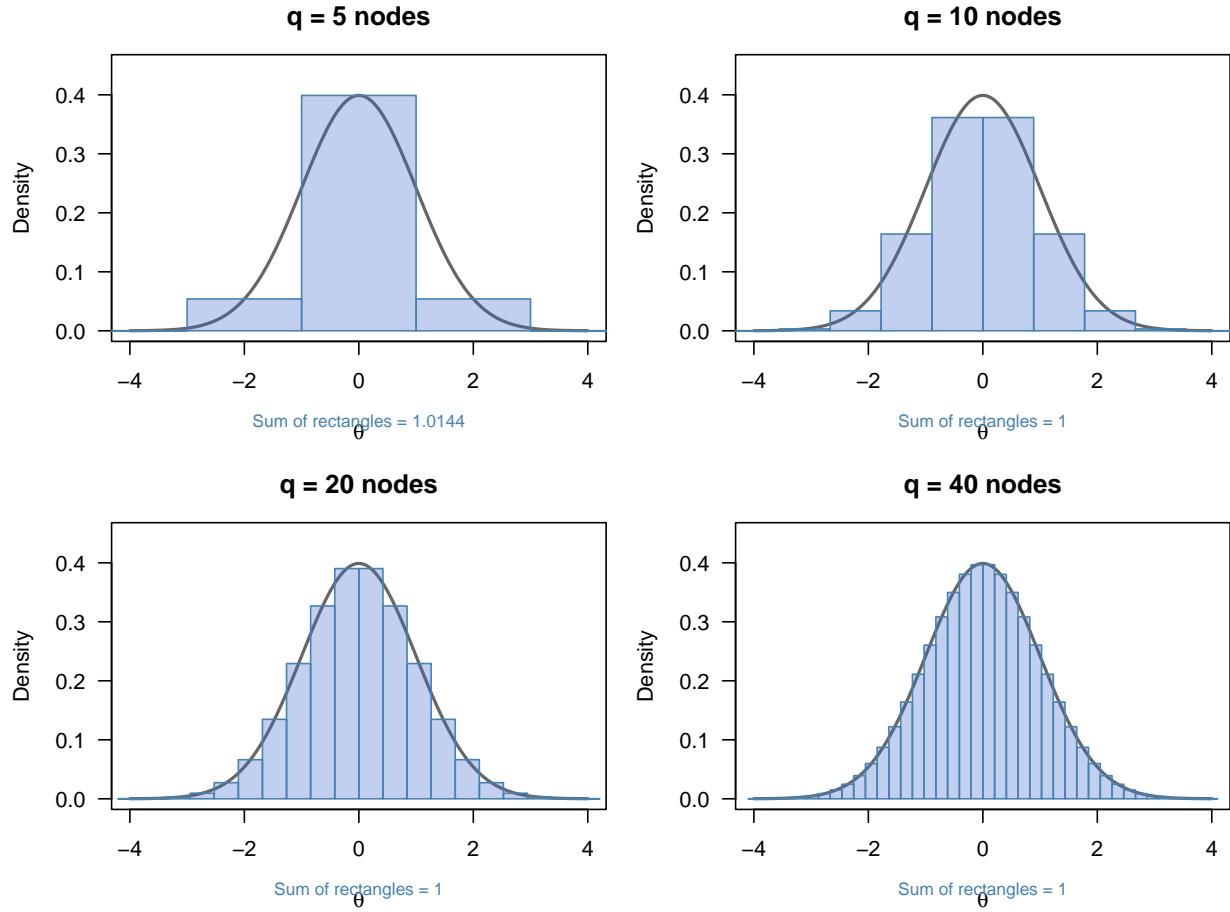
for (q in c(5, 10, 20, 40)) {
  # Equally spaced nodes from -4 to 4
  nodes <- seq(-4, 4, length.out = q)
  width <- nodes[2] - nodes[1]
  weights <- dnorm(nodes) * width
  weights <- weights / sum(weights) # normalize

  # Plot the continuous curve
  plot(theta, dnorm(theta), type = "l", lwd = 2, col = "gray40",
    main = paste0("q = ", q, " nodes"),
    xlab = expression(theta), ylab = "Density", las = 1,
    ylim = c(0, 0.45))

  # Overlay the rectangles
  for (i in seq_along(nodes)) {
    rect(nodes[i] - width/2, 0, nodes[i] + width/2,
      dnorm(nodes[i]), col = rgb(0.2, 0.4, 0.8, 0.3),
      border = "steelblue")
  }

  # Report approximation quality
  approx_area <- sum(dnorm(nodes) * width)
  mtext(paste0("Sum of rectangles = ", round(approx_area, 4)),
    side = 1, line = 2.5, cex = 0.7, col = "steelblue")
}

```



More nodes = better approximation. In practice, 10-40 nodes are typically used.

## The Specific Nodes and Weights Used in Practice

The classic BILOG program (Baker & Kim, Appendix D) uses **10 equally spaced nodes** from  $-4$  to  $+4$  with weights derived from the standard normal density:

```
# 10-point quadrature (from Baker & Kim / BILOG)
q <- 10
X_k <- c(-4.0, -3.1111, -2.2222, -1.3333, -0.4444,
          0.4444, 1.3333, 2.2222, 3.1111, 4.0)
A_k <- c(0.000119, 0.002805, 0.03002, 0.1458, 0.3213,
          0.3213, 0.1458, 0.03002, 0.002805, 0.000119)

# Verify the weights sum to 1 and are centered at 0
cat("Sum of weights:", sum(A_k), "\n")
```

```
## Sum of weights: 1.000088
```

```
cat("Weighted mean:", sum(X_k * A_k), "\n")
```

```
## Weighted mean: 1.029992e-18
```

```

cat("Weighted variance:", sum(X_k^2 * A_k), "\n")

## Weighted variance: 0.9998768

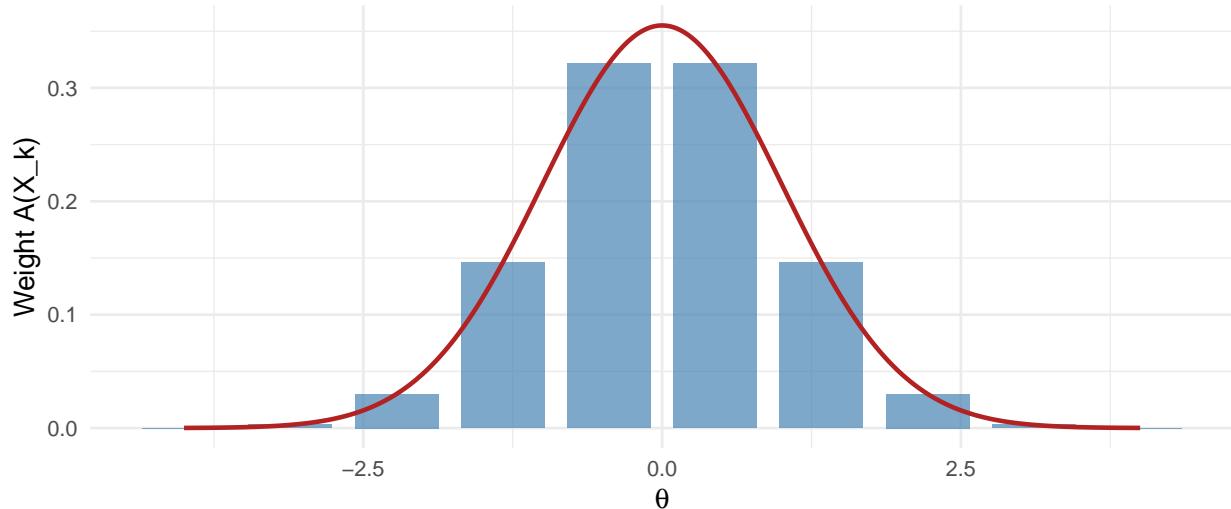
df_quad <- data.frame(node = X_k, weight = A_k)

ggplot(df_quad, aes(x = node, y = weight)) +
  geom_col(fill = "steelblue", alpha = 0.7, width = 0.7) +
  geom_line(data = data.frame(x = theta, y = dnorm(theta)),
            aes(x = x, y = y * 0.89), color = "firebrick", linewidth = 1) +
  labs(title = "10-Point Quadrature Approximation to N(0,1)",
       subtitle = "Blue bars = quadrature weights; Red curve = scaled normal density",
       x = expression(theta), y = "Weight A(X_k)") +
  theme_minimal(base_size = 13)

```

## 10–Point Quadrature Approximation to $N(0,1)$

Blue bars = quadrature weights; Red curve = scaled normal density



## Testing Quadrature: Thor's Marginal Likelihood

Now the payoff: let's verify that 10-point quadrature can match what we got earlier from the 801-point trapezoidal rule for Thor's marginal likelihood:

```

# Compute Thor's likelihood at each of the 10 quadrature nodes
lik_at_nodes <- p_2pl(X_k, a1, b1) * (1 - p_2pl(X_k, a2, b2))

# Marginal likelihood via quadrature: just 10 multiplications and a sum!
marginal_lik_quad <- sum(lik_at_nodes * A_k)

cat("Thor's marginal likelihood:\n")

## Thor's marginal likelihood:

```

```

cat(" Trapezoidal rule (801 points):", round(marginal_lik_thor, 6), "\n")

## Trapezoidal rule (801 points): 0.447543

cat(" Quadrature (10 points):      ", round(marginal_lik_quad, 6), "\n")

## Quadrature (10 points):      0.447608

cat(" Relative error:           ",
    round(abs(marginal_lik_quad - marginal_lik_thor) / marginal_lik_thor * 100, 2), "%\n")

## Relative error:           0.01 %

```

With just 10 evaluation points instead of 801, we get nearly the same answer. That's an 80-fold reduction in computation for a trivial loss in accuracy — and this savings compounds across every response pattern and every EM cycle.

## The LSAT-6 Data: A Classic Example

We'll now walk through the full MMLE/EM algorithm using the **LSAT-6 dataset** from Bock and Lieberman (1970), also used in Baker and Kim's textbook. This dataset has:

- **1000 examinees**
- **5 dichotomous items**
- **32 unique response patterns** (all  $2^5$  possible combinations of 0s and 1s)

```

# All 32 possible response patterns for 5 items
patterns <- as.matrix(expand.grid(rep(list(0:1), 5)[, 5:1])
colnames(patterns) <- paste0("Item", 1:5)

# Frequency of each pattern in the LSAT-6 sample (Baker & Kim, Appendix D)
freq <- c(3, 6, 2, 11, 1, 1, 3, 4, 1, 8, 0, 16, 0, 3, 2, 15,
        10, 29, 14, 81, 3, 28, 15, 80, 16, 56, 21, 173, 11, 61, 28, 298)

N <- sum(freq)  # Total examinees
n_items <- 5
n_patterns <- nrow(patterns)

cat("Total examinees:", N, "\n")

## Total examinees: 1000

cat("Number of items:", n_items, "\n")

## Number of items: 5

```

```
cat("Unique response patterns:", n_patterns, "\n")
```

```
## Unique response patterns: 32
```

Let's look at the most common patterns:

```
# Show top patterns by frequency
df_patterns <- data.frame(patterns, Freq = freq)
df_patterns$Pattern <- apply(patterns, 1, paste, collapse = "")
df_patterns <- df_patterns[order(-df_patterns$Freq), ]

kable(head(df_patterns[, c("Pattern", "Freq")], 10),
      row.names = FALSE,
      caption = "10 Most Common Response Patterns")
```

Table 2: 10 Most Common Response Patterns

Pattern	Freq
11111	298
11011	173
10011	81
10111	80
11101	61
11001	56
10001	29
10101	28
11110	28
11010	21

The most common pattern is {1,1,1,1,1} (all correct) with 298 examinees, suggesting these are relatively easy items.

## The EM Algorithm, Step by Step

### Overview

The EM algorithm alternates between two steps:

Step	What it does	Analogy
<b>E-step</b>	Uses current item parameter guesses to figure out how examinees are probably distributed across the ability scale	“If these were the true item parameters, where would each examinee likely fall?”
<b>M-step</b>	Uses that ability distribution to re-estimate item parameters, treating the E-step output as known data	“Given where we think examinees fall, what item parameters best fit the responses?”

We repeat until the estimates stop changing.

## Initial Setup

We use the **2PL model** with an intercept/slope parameterization:

$$P_i(\theta) = \frac{1}{1 + \exp(-(d_i + a_i\theta))}$$

where  $d_i$  is the intercept and  $a_i$  is the slope (discrimination). The usual difficulty parameter is  $b_i = -d_i/a_i$ .

We start with initial guesses: all intercepts  $d_i = 0$  and all slopes  $a_i = 1$ .

```
# 2PL probability using intercept/slope parameterization
P_icc <- function(theta, d, a) {
  1 / (1 + exp(-(d + a * theta)))
}

# Initial item parameter estimates: d = 0, a = 1 for all items
d_est <- rep(0, n_items)
a_est <- rep(1, n_items)

cat("Initial estimates:\n")

## Initial estimates:

cat("Intercepts (d):", d_est, "\n")

## Intercepts (d): 0 0 0 0 0

cat("Slopes (a):      ", a_est, "\n")

## Slopes (a):      1 1 1 1 1
```

## The E-Step: Creating “Artificial Data”

The E-step is the heart of MMLE. It creates two quantities for each item  $i$  at each quadrature node  $k$ :

- $\bar{n}_{ik}$  = the **expected number of examinees** at ability level  $X_k$  (same for all items)
- $\bar{r}_{ik}$  = the **expected number of correct responses** to item  $i$  at ability level  $X_k$

These are the “artificial data” — they play the same role as the observed frequencies in the JMLE estimation equations.

### Step-by-step walkthrough

#### Step E1: Compute the likelihood of each response pattern at each quadrature node.

For response pattern  $l$  at node  $X_k$ :

$$L_l(X_k) = \prod_{i=1}^n P_i(X_k)^{u_{li}} Q_i(X_k)^{1-u_{li}}$$

This is just the probability of seeing that particular pattern if the examinee’s ability were exactly  $X_k$ .

```

# Compute L_l(X_k) for all patterns and all nodes
# This is a matrix: rows = patterns, columns = nodes
compute_L_matrix <- function(patterns, X_k, d, a) {
  n_pat <- nrow(patterns)
  q <- length(X_k)
  L <- matrix(1, nrow = n_pat, ncol = q)

  for (k in 1:q) {
    for (i in 1:ncol(patterns)) {
      p <- P_icc(X_k[k], d[i], a[i])
      # Multiply in each item's contribution
      L[, k] <- L[, k] * ifelse(patterns[, i] == 1, p, 1 - p)
    }
  }
  return(L)
}

L_mat <- compute_L_matrix(patterns, X_k, d_est, a_est)

```

Let's look at a specific example. Pattern 22 is {1,0,1,0,1} (items 1, 3, 5 correct):

```

l <- 22 # Pattern index
cat("Pattern", l, ":", patterns[l, ], "\n")

```

```
## Pattern 22 : 1 0 1 0 1
```

```
cat("Frequency:", freq[l], "examinees\n\n")
```

```
## Frequency: 28 examinees
```

```
cat("Likelihood at each quadrature node:\n")
```

```
## Likelihood at each quadrature node:
```

```

df_lik <- data.frame(
  Node = 1:q,
  X_k = X_k,
  `L_l(X_k)` = L_mat[l, ],
  check.names = FALSE
)
kable(df_lik, digits = 8, align = "c")

```

Node	X_k	L_l(X_k)
1	-4.0000	0.00000561
2	-3.1111	0.00007111
3	-2.2222	0.00076087
4	-1.3333	0.00568600
5	-0.4444	0.02214009
6	0.4444	0.03452866

Node	X_k	L_l(X_k)
7	1.3333	0.02157010
8	2.2222	0.00702097
9	3.1111	0.00159618
10	4.0000	0.00030636

Notice: the likelihood is tiny at extreme  $\theta$  values (it's very unlikely an examinee at  $\theta = -4$  would get 3 out of 5 correct with all items having  $d = 0, a = 1$ ).

**Step E2: Compute the marginal probability of each response pattern.**

$$P_l = \sum_{k=1}^q L_l(X_k) \cdot A(X_k)$$

This is the marginal likelihood of pattern  $l$  — the same type of integral we computed for Thor, but now using quadrature.

```
# P_l for each pattern: sum of L_l(X_k) * A(X_k) over nodes
P_l <- L_mat %*% A_k # matrix multiplication gives the weighted sum

# Show marginal probabilities for the 10 most common response patterns
top10_idx <- order(-freq)[1:10]
df_marg <- data.frame(
  Pattern = apply(patterns[top10_idx, ], 1, paste, collapse = ""),
  Freq = freq[top10_idx],
  P_l = round(as.numeric(P_l[top10_idx]), 8)
)
kable(df_marg, row.names = FALSE,
      col.names = c("Pattern", "Frequency", "$P_l$"),
      caption = "Marginal Probabilities for the 10 Most Common Response Patterns")
```

Table 5: Marginal Probabilities for the 10 Most Common Response Patterns

Pattern	Frequency	$P_l$
11111	298	0.0955352
11011	173	0.0360619
10011	81	0.0224199
10111	80	0.0360619
11101	61	0.0360619
11001	56	0.0224199
10001	29	0.0224199
10101	28	0.0224199
11110	28	0.0360619
11010	21	0.0224199

**Step E3: Compute posterior probabilities at each node for each pattern.**

$$P(X_k | \mathbf{u}_l, \boldsymbol{\xi}, \boldsymbol{\tau}) = \frac{L_l(X_k) \cdot A(X_k)}{P_l}$$

This is exactly Bayes' theorem — the same posterior we computed for Thor, but now discretized at the quadrature nodes.

```
# Posterior probability matrix: rows = patterns, columns = nodes
posterior <- L_mat * matrix(A_k, nrow = n_patterns, ncol = q, byrow = TRUE)
posterior <- posterior / as.vector(P_1)

# Show the full calculation for pattern 22
cat("Posterior probabilities for pattern", 1, "{1,0,1,0,1}:\n")

## Posterior probabilities for pattern 22 {1,0,1,0,1}:

cat("P_1 =", round(P_1[1], 8), "\n\n")

## P_1 = 0.02241993

df_post <- data.frame(
  Node = 1:q,
  X_k = X_k,
  `L_l(X_k)` = L_mat[l, ],
  `A(X_k)` = A_k,
  `L * A` = L_mat[l, ] * A_k,
  `P_1` = rep(as.numeric(P_1[1]), q),
  Posterior = posterior[l, ],
  check.names = FALSE
)
kable(df_post, digits = 8, align = "c",
      col.names = c("Node", "$X_k$", "$L_l(X_k)$", "$A(X_k)$",
                    "$L_l(X_k) \times A(X_k)$", "$P_1$",
                    "$\\frac{L_l \\times A}{P_1} = Posterior$"))

```

Node	$X_k$	$L_l(X_k)$	$A(X_k)$	$L_l(X_k) \times A(X_k)$	$P_l$	$\frac{L_l \times A}{P_l} = \text{Posterior}$
1	-4.0000	0.00000561	0.000119	0.00000000	0.02241993	0.00000003
2	-3.1111	0.00007111	0.002805	0.00000020	0.02241993	0.000000890
3	-2.2222	0.00076087	0.030020	0.00002284	0.02241993	0.00101879
4	-1.3333	0.00568600	0.145800	0.00082902	0.02241993	0.03697690
5	-0.4444	0.02214009	0.321300	0.00711361	0.02241993	0.31728951
6	0.4444	0.03452866	0.321300	0.01109406	0.02241993	0.49483018
7	1.3333	0.02157010	0.145800	0.00314492	0.02241993	0.14027339
8	2.2222	0.00702097	0.030020	0.00021077	0.02241993	0.00940098
9	3.1111	0.00159618	0.002805	0.00000448	0.02241993	0.00019970
10	4.0000	0.00030636	0.000119	0.00000004	0.02241993	0.00000163

Verify that the posterior probabilities sum to 1 (as they must — they represent a proper probability distribution over the quadrature nodes):

```
cat("Sum of posterior probabilities:", round(sum(posterior[1, ]), 6), "\n")

## Sum of posterior probabilities: 1
```

```

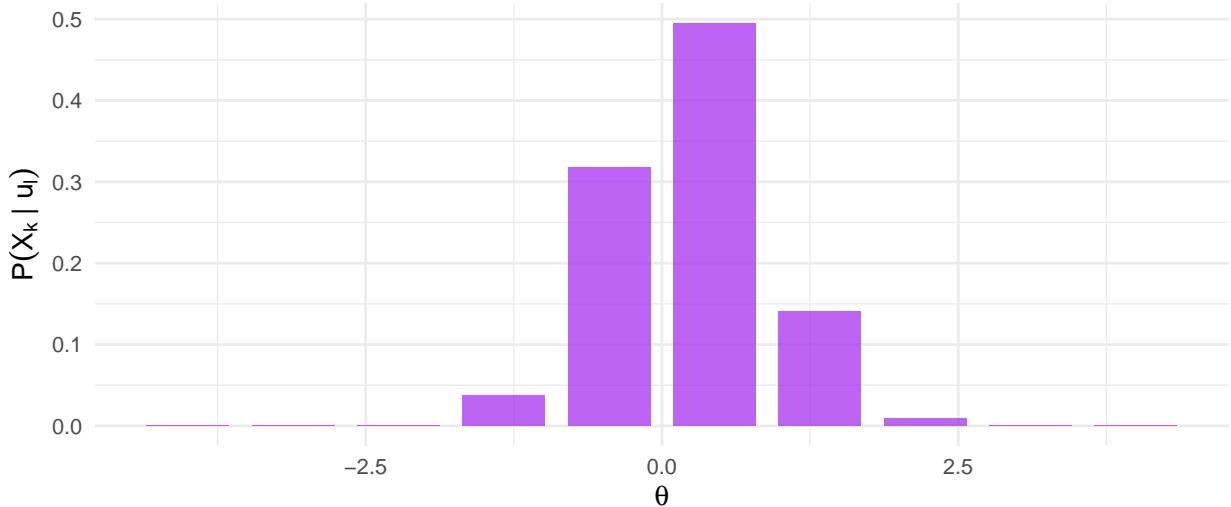
df_post_plot <- data.frame(X_k = X_k, Posterior = posterior[1, ])

ggplot(df_post_plot, aes(x = X_k, y = Posterior)) +
  geom_col(fill = "purple", alpha = 0.7, width = 0.7) +
  labs(title = paste0("Posterior Distribution for Pattern ", l, " {1,0,1,0,1}"),
       subtitle = "Where is this examinee most likely located on the ability scale?",
       x = expression(theta), y = expression(P(X[k] *| *u[l]))) +
  theme_minimal(base_size = 13)

```

## Posterior Distribution for Pattern 22 {1,0,1,0,1}

Where is this examinee most likely located on the ability scale?



The posterior tells us: given the response pattern  $\{1,0,1,0,1\}$  and our current item parameter guesses, this examinee is most likely around  $\theta \approx 0.4$ .

**Step E4: Compute the artificial data ( $\bar{n}_k$  and  $\bar{r}_{ik}$ ).**

Now we aggregate across all examinees. Each examinee's posterior probabilities tell us how to distribute them across the ability scale. We weight by pattern frequency  $f_l$ :

$$\bar{n}_k = \sum_{l=1}^s f_l \cdot P(X_k | \mathbf{u}_l, \boldsymbol{\xi}, \boldsymbol{\tau})$$

$$\bar{r}_{ik} = \sum_{l=1}^s u_{li} \cdot f_l \cdot P(X_k | \mathbf{u}_l, \boldsymbol{\xi}, \boldsymbol{\tau})$$

$\bar{n}_k$  is the expected number of examinees at node  $k$  (same for all items).  $\bar{r}_{ik}$  is the expected number of correct responses to item  $i$  at node  $k$ .

```

compute_artificial_data <- function(patterns, freq, posterior, n_items, q) {
  # n_bar_k: expected number of examinees at each node
  n_bar <- colSums(freq * posterior)

  # r_bar_ik: expected correct responses for each item at each node
  r_bar <- matrix(0, nrow = n_items, ncol = q)
  for (i in 1:n_items) {

```

```

        r_bar[i, ] <- colSums(patterns[, i] * freq * posterior)
    }

    list(n_bar = n_bar, r_bar = r_bar)
}

art_data <- compute_artificial_data(patterns, freq, posterior, n_items, q)

cat("Expected number of examinees at each node (n-bar):\n")

## Expected number of examinees at each node (n-bar):

cat("(Should sum to", N, ")\n\n")

## (Should sum to 1000 )

df_art <- data.frame(
  Node = 1:q,
  X_k = X_k,
  n_bar = round(art_data$n_bar, 2)
)

# Add r_bar for each item
for (i in 1:n_items) {
  df_art[[paste0("r_bar_", i)]] <- round(art_data$r_bar[i, ], 2)
}

kable(df_art, align = "c",
      col.names = c("Node", "X_k", "n-bar",
                    paste0("r-bar (Item ", 1:5, ")")),
      caption = "Artificial Data from E-Step (Cycle 1)")

```

Table 7: Artificial Data from E-Step (Cycle 1)

Node	X_k	n-bar	r-bar (Item 1)	r-bar (Item 2)	r-bar (Item 3)	r-bar (Item 4)	r-bar (Item 5)
1	-4.0000	0.00	0.00	0.00	0.00	0.00	0.00
2	-3.1111	0.15	0.04	0.01	0.00	0.01	0.03
3	-2.2222	2.71	1.36	0.41	0.20	0.53	1.00
4	-1.3333	31.07	22.10	9.59	4.94	11.70	17.98
5	-0.4444	181.98	155.12	91.10	56.33	105.48	137.51
6	0.4444	402.31	374.09	278.74	205.97	304.10	351.35
7	1.3333	295.79	286.41	249.00	211.15	259.64	278.24
8	2.2222	77.55	76.48	71.97	66.50	73.28	75.54
9	3.1111	8.08	8.03	7.83	7.56	7.89	7.99
10	4.0000	0.36	0.36	0.35	0.35	0.36	0.36

```
cat("\nSum of n-bar:", round(sum(art_data$n_bar), 2), "\n")
```

```

##  

## Sum of n-bar: 1000

df_art_long <- data.frame(  

  X_k = rep(X_k, n_items + 1),  

  Count = c(art_data$n_bar, as.vector(t(art_data$r_bar))),  

  Type = c(rep("Expected Examinees (n-bar)", q),  

    rep(paste0("Correct on Item ", 1:n_items), each = q))
)

ggplot(df_art_long, aes(x = X_k, y = Count, fill = Type)) +  

  geom_col(position = "identity", alpha = 0.5, width = 0.6) +  

  facet_wrap(~ Type, scales = "free_y", ncol = 2) +  

  labs(title = "E-Step Artificial Data (Cycle 1)",  

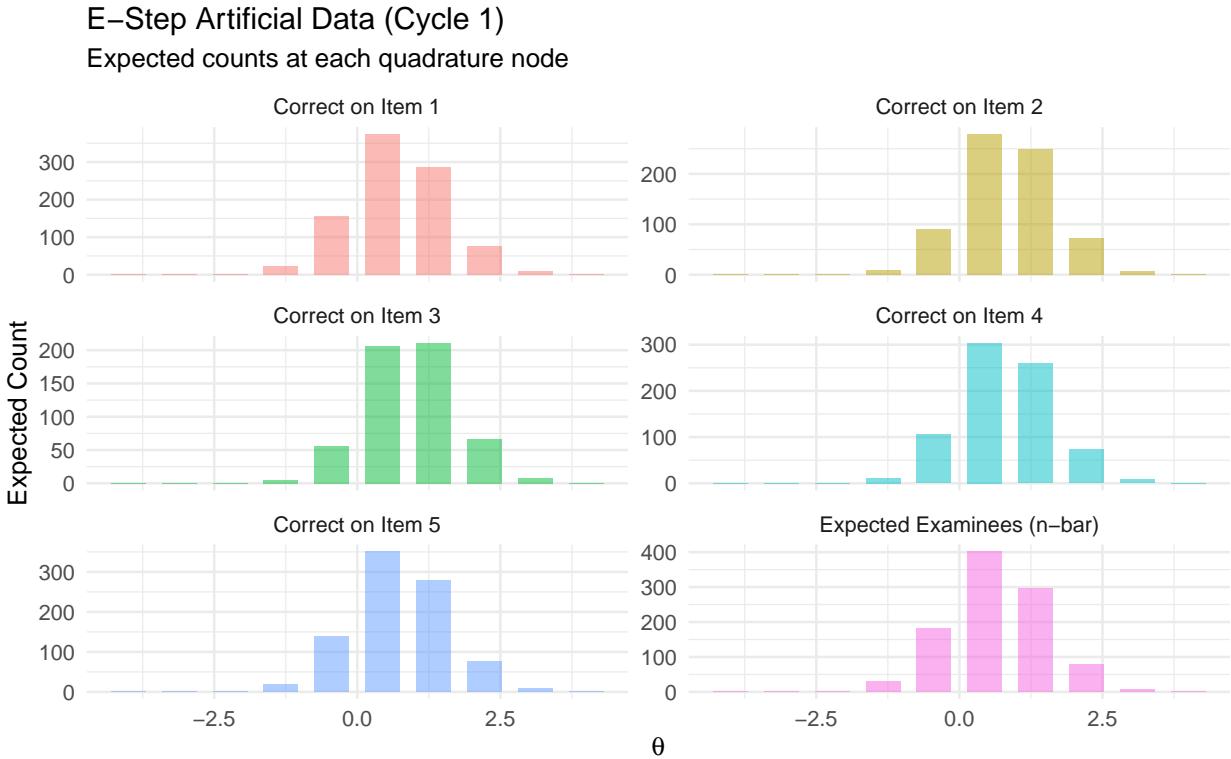
    subtitle = "Expected counts at each quadrature node",  

    x = expression(theta), y = "Expected Count") +  

  theme_minimal(base_size = 12) +  

  theme(legend.position = "none")

```



**This is the key result of the E-step.** We now have, for each item, a set of “data” that mirrors what we had in the JMLE estimation equations: at each ability level  $X_k$ , we know the expected number of examinees ( $\bar{n}_k$ ) and the expected number of correct responses ( $\bar{r}_{ik}$ ). The observed proportion correct at each node is simply  $\bar{r}_{ik}/\bar{n}_k$ . The difference is that in JMLE these counts came directly from the data (using provisional  $\theta$  estimates for each person), whereas here they are *expected* counts derived from the posterior distribution.

**Why do the  $\bar{r}_{ik}$  values differ across items when we started with identical parameters?** You might notice that  $\bar{n}_k$  is the same for all items, yet the  $\bar{r}_{ik}$  columns are different. This is because  $\bar{n}_k$  depends only on the posterior probabilities (which are driven by the full response pattern and the item parameters), while  $\bar{r}_{ik}$  also depends on  $u_{li}$  — the actual observed response to item  $i$ . Look back at the

formula:  $\bar{r}_{ik} = \sum_l u_{li} \cdot f_l \cdot P(X_k | \mathbf{u}_l)$ . The  $u_{li}$  term means that only examinees who answered item  $i$  correctly contribute to  $\bar{r}_{ik}$ . So even though our initial guesses treat all items identically, the *data* break the symmetry: items that more examinees answered correctly will have larger  $\bar{r}_{ik}$  values, and this is what drives the M-step to produce different parameter estimates for each item.

```
# Confirm: proportion correct differs across items in the raw data
p_correct <- colSums(patterns * freq) / N
cat("Proportion correct (from raw data):\n")
```

```
## Proportion correct (from raw data):
```

```
for (i in 1:n_items) {
```

```
  cat("  Item", i, ":", round(p_correct[i], 3), "\n")
```

```
}
```

```
##  Item 1 : 0.924
##  Item 2 : 0.709
##  Item 3 : 0.553
##  Item 4 : 0.763
##  Item 5 : 0.87
```

These different proportions correct are what the artificial data capture, even from the very first cycle.

## The M-Step: Estimating Item Parameters

The M-step treats the artificial data as if it were real and estimates item parameters using the same Newton-Raphson procedure from the JMML approach. The beauty of the Bock-Aitkin reformulation is that **each item is estimated separately**.

For the 2PL model, we solve two equations for each item — one for the intercept  $d_i$  and one for the slope  $a_i$ :

$$\sum_{k=1}^q \bar{n}_k \left[ \frac{\bar{r}_{ik}}{\bar{n}_k} - P_i(X_k) \right] = 0$$

$$\sum_{k=1}^q \bar{n}_k \left[ \frac{\bar{r}_{ik}}{\bar{n}_k} - P_i(X_k) \right] X_k = 0$$

The logic is the same as in JMML: the term in brackets is the difference between the **observed proportion correct** ( $\bar{r}_{ik}/\bar{n}_k$ ) and the **model-predicted probability** ( $P_i(X_k)$ ) at each node. We seek the values of  $d_i$  and  $a_i$  that make these weighted residuals sum to zero. The second equation includes  $X_k$  as a multiplier, which is what allows us to solve for the slope in addition to the intercept.

These are the same estimation equations you've seen in the JMML context — the difference is that the “data” ( $\bar{r}_{ik}, \bar{n}_k$ ) are artificial rather than observed.

```
# Newton-Raphson for one item using 2PL intercept/slope parameterization
# Returns updated d and a for that item
m_step_item <- function(X_k, n_bar, r_bar_i, d_init, a_init, max_iter = 6,
                        tol = 0.05) {
  d <- d_init
  a <- a_init
```

```

for (iter in 1:max_iter) {
  # Compute P_hat at each node using current estimates
  P_hat <- P_icc(X_k, d, a)
  W <- P_hat * (1 - P_hat)

  # Observed proportion at each node
  p_obs <- r_bar_i / n_bar
  p_obs[n_bar == 0] <- 0 # Handle zero counts

  # Residuals
  V <- (p_obs - P_hat) / W
  V[W < 1e-7] <- 0 # Avoid division by near-zero

  # Newton-Raphson sums
  nW <- n_bar * W
  nWV <- n_bar * W * V
  nWX <- n_bar * W * X_k
  nWXV <- n_bar * W * V * X_k
  nWX2 <- n_bar * W * X_k^2

  SNW <- sum(nW)
  SNWV <- sum(nWV)
  SNWX <- sum(nWX)
  SNWXV <- sum(nWXV)
  SNWX2 <- sum(nWX2)

  # Denominator of Newton-Raphson equations
  denom <- SNW * SNWX2 - SNWX^2
  if (abs(denom) < 1e-4) break

  # Increments
  delta_d <- (SNWV * SNWX2 - SNWXV * SNWX) / denom
  delta_a <- (SNW * SNWXV - SNWX * SNWV) / denom

  # Update
  d <- d + delta_d
  a <- a + delta_a

  # Check convergence
  if (abs(delta_d) <= tol & abs(delta_a) <= tol) break

  # Bounds check
  if (abs(d) > 30 | abs(a) > 20) break
}

return(c(d = d, a = a))
}

```

Now let's run the M-step for all 5 items:

```

# M-step: estimate parameters for each item
d_new <- numeric(n_items)
a_new <- numeric(n_items)

```

```

for (i in 1:n_items) {
  result <- m_step_item(X_k, art_data$n_bar, art_data$r_bar[i, ],
                        d_est[i], a_est[i])
  d_new[i] <- result["d"]
  a_new[i] <- result["a"]
}

# Convert to difficulty parameterization
b_new <- -d_new / a_new

cat("After Cycle 1:\n")

## After Cycle 1:

kable(data.frame(
  Item = 1:5,
  Intercept_d = round(d_new, 4),
  Slope_a = round(a_new, 4),
  Difficulty_b = round(b_new, 4)
), row.names = FALSE, caption = "Item Parameter Estimates After 1 EM Cycle")

```

Table 8: Item Parameter Estimates After 1 EM Cycle

Item	Intercept_d	Slope_a	Difficulty_b
1	2.1659	0.9438	-2.2949
2	0.4102	0.9410	-0.4360
3	-0.3787	0.9697	0.3905
4	0.7263	0.9322	-0.7791
5	1.5321	0.9198	-1.6658

Compare these to Baker & Kim's Table (Appendix D, p. 338), which reports after cycle 1: Item 1:  $d = 2.166, a = 0.944$ ; Item 2:  $d = 0.410, a = 0.941$ ; Item 3:  $d = -0.379, a = 0.970$ ; Item 4:  $d = 0.726, a = 0.932$ ; Item 5:  $d = 1.532, a = 0.920$ .

## Putting It All Together: The Full EM Loop

Now we iterate: use the new parameter estimates from the M-step as input to the next E-step, and repeat.

```

# Full EM algorithm
run_em <- function(patterns, freq, X_k, A_k, n_items,
                     max_cycles = 100, tol = 1e-4) {

  N <- sum(freq)
  q <- length(X_k)
  n_patterns <- nrow(patterns)

  # Initialize
  d <- rep(0, n_items)
  a <- rep(1, n_items)

```

```

# Storage for tracking convergence
history <- data.frame()

for (cycle in 1:max_cycles) {
  # --- E-step ---
  # Compute likelihood matrix
  L_mat <- compute_L_matrix(patterns, X_k, d, a)

  # Marginal probabilities
  P_l <- L_mat %*% A_k

  # Posterior probabilities
  posterior <- L_mat * matrix(A_k, nrow = n_patterns, ncol = q, byrow = TRUE)
  posterior <- posterior / as.vector(P_l)

  # Artificial data
  art <- compute_artificial_data(patterns, freq, posterior, n_items, q)

  # Compute log-likelihood
  loglik <- sum(freq * log(pmax(P_l, 1e-300)))

  # --- M-step ---
  d_old <- d
  a_old <- a

  for (i in 1:n_items) {
    result <- m_step_item(X_k, art$n_bar, art$r_bar[i, ], d[i], a[i])
    d[i] <- result["d"]
    a[i] <- result["a"]
  }

  # Record history
  for (i in 1:n_items) {
    history <- rbind(history, data.frame(
      Cycle = cycle, Item = i,
      d = d[i], a = a[i], b = -d[i]/a[i],
      LogLik = loglik
    ))
  }

  # Check convergence
  max_change <- max(abs(c(d - d_old, a - a_old)))
  if (max_change < tol & cycle > 5) break
}

list(d = d, a = a, b = -d/a, history = history, n_cycles = cycle)
}

# Run the EM algorithm
em_result <- run_em(patterns, freq, X_k, A_k, n_items, max_cycles = 100)

cat("Converged after", em_result$n_cycles, "cycles\n\n")

```

```

## Converged after 44 cycles

cat("Final estimates:\n")

## Final estimates:

kable(data.frame(
  Item = 1:5,
  Intercept_d = round(em_result$d, 4),
  Slope_a = round(em_result$a, 4),
  Difficulty_b = round(em_result$b, 4)
), row.names = FALSE, caption = "Final MMLE/EM Item Parameter Estimates")

```

Table 9: Final MMLE/EM Item Parameter Estimates

Item	Intercept_d	Slope_a	Difficulty_b
1	2.7732	0.8256	-3.3588
2	0.9903	0.7231	-1.3695
3	0.2491	0.8901	-0.2798
4	1.2848	0.6887	-1.8657
5	2.0535	0.6573	-3.1239

## Watching EM Converge

One of the informative aspects of the EM algorithm is watching the parameter estimates evolve across cycles:

```

hist_df <- em_result$history
hist_df$Item <- factor(hist_df$Item, labels = paste("Item", 1:5))

p1 <- ggplot(hist_df, aes(x = Cycle, y = d, color = Item)) +
  geom_line(lineWidth = 1) +
  labs(title = "Intercept (d) Estimates Across EM Cycles",
       y = "Intercept (d)") +
  theme_minimal(base_size = 12) +
  theme(legend.position = "right")

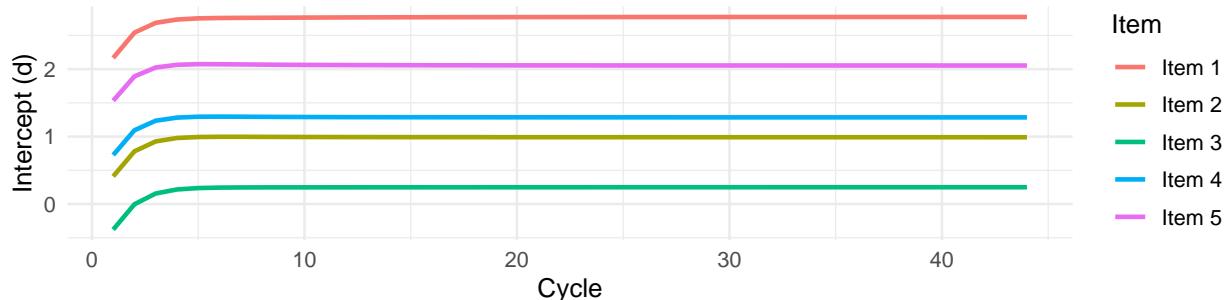
p2 <- ggplot(hist_df, aes(x = Cycle, y = a, color = Item)) +
  geom_line(lineWidth = 1) +
  labs(title = "Slope (a) Estimates Across EM Cycles",
       y = "Slope (a)") +
  theme_minimal(base_size = 12) +
  theme(legend.position = "right")

p3 <- ggplot(hist_df[hist_df$Item == "Item 1", ],
              aes(x = Cycle, y = LogLik)) +
  geom_line(lineWidth = 1, color = "black") +
  labs(title = "Log-Likelihood Across EM Cycles",
       y = "Log-Likelihood") +
  theme_minimal(base_size = 12)

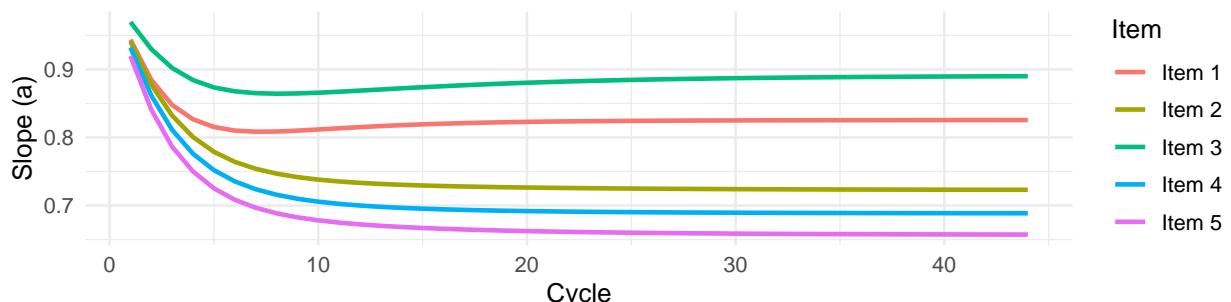
gridExtra::grid.arrange(p1, p2, p3, ncol = 1)

```

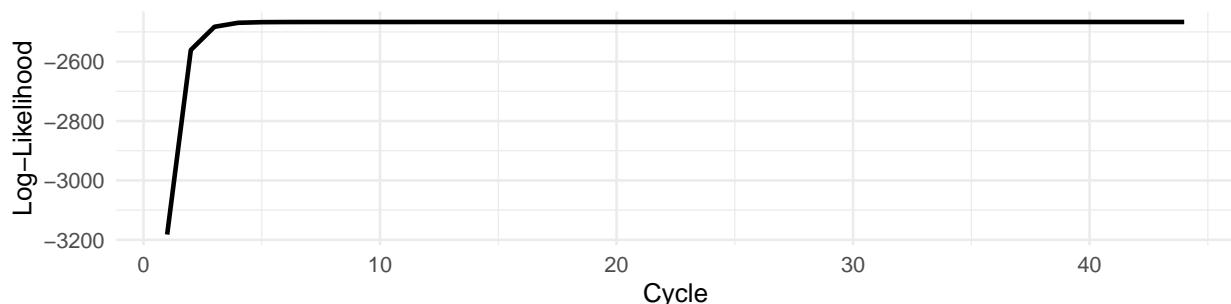
### Intercept (d) Estimates Across EM Cycles



### Slope (a) Estimates Across EM Cycles



### Log-Likelihood Across EM Cycles



Notice that the EM algorithm converges **slowly** — this is a well-known characteristic. The parameters change rapidly in the first few cycles but then creep toward their final values. Production software (like BILOG or mirt) uses acceleration techniques to speed this up.

## Comparison with mirt

Now let's verify our hand-coded results against the `mirt` package, which uses the same MMLE/EM approach but with production-grade optimizations.

```
# Reconstruct the full response matrix from patterns and frequencies
resp_matrix <- patterns[rep(1:n_patterns, freq), ]
colnames(resp_matrix) <- paste0("Item", 1:5)

# Fit 2PL model with mirt
# Use EM algorithm to match our approach
mod_2pl <- mirt(data.frame(resp_matrix), model = 1, itemtype = "2PL",
                  method = "EM", verbose = FALSE)
```

```

# Extract parameters in slope/intercept form
mirt_pars <- coef(mod_2pl, IRTpars = FALSE, simplify = TRUE)$items
mirt_irt <- coef(mod_2pl, IRTpars = TRUE, simplify = TRUE)$items

cat("Comparison of estimates:\n\n")

## Comparison of estimates:

comp_df <- data.frame(
  Item = 1:5,
  Our_a = round(em_result$a, 4),
  mirt_a = round(mirt_pars[, "a1"], 4),
  Our_d = round(em_result$d, 4),
  mirt_d = round(mirt_pars[, "d"], 4),
  Our_b = round(em_result$b, 4),
  mirt_b = round(mirt_irt[, "b"], 4)
)

kable(comp_df,
      col.names = c("Item", "Our a", "mirt a", "Our d", "mirt d",
                    "Our b", "mirt b"),
      caption = "Our MMLE/EM vs. mirt Package Estimates")

```

Table 10: Our MMLE/EM vs. mirt Package Estimates

	Item	Our a	mirt a	Our d	mirt d	Our b	mirt b
Item1	1	0.8256	0.8251	2.7732	2.7728	-3.3588	-3.3607
Item2	2	0.7231	0.7231	0.9903	0.9903	-1.3695	-1.3696
Item3	3	0.8901	0.8900	0.2491	0.2491	-0.2798	-0.2799
Item4	4	0.6887	0.6887	1.2848	1.2849	-1.8657	-1.8657
Item5	5	0.6573	0.6576	2.0535	2.0536	-3.1239	-3.1230

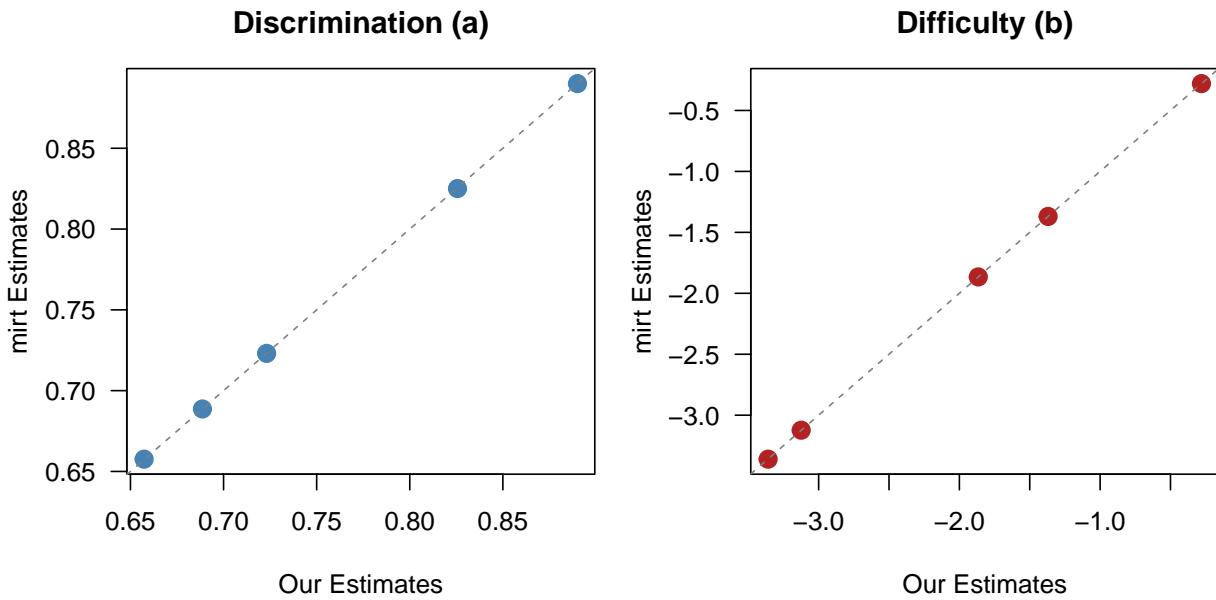
```

par(mfrow = c(1, 2), mar = c(4, 4, 3, 1))

# Discrimination comparison
plot(em_result$a, mirt_pars[, "a1"], pch = 19, col = "steelblue", cex = 1.5,
      xlab = "Our Estimates", ylab = "mirt Estimates",
      main = "Discrimination (a)", las = 1)
abline(0, 1, lty = 2, col = "gray50")

# Difficulty comparison
plot(em_result$b, mirt_irt[, "b"], pch = 19, col = "firebrick", cex = 1.5,
      xlab = "Our Estimates", ylab = "mirt Estimates",
      main = "Difficulty (b)", las = 1)
abline(0, 1, lty = 2, col = "gray50")

```



Any differences reflect implementation details: `mirt` uses more quadrature points by default (61 vs. our 10), updates the quadrature weights between cycles, and uses the Bock-Aitkin acceleration method. The estimates should be in the same ballpark.

## Applying MMLE to Real Data

Let's apply MMLE to the same CDE math data used in the person parameter estimation module:

```
cde <- read.csv("cde_subsample_math.csv")
cat("CDE data:", nrow(cde), "students,", ncol(cde), "items\n")

## CDE data: 1500 students, 31 items

# Fit 2PL model
mod_cde <- mirt(cde, model = 1, itemtype = "2PL", method = "EM", verbose = FALSE)

# Extract IRT parameters
cde_pars <- coef(mod_cde, IRTpars = TRUE, simplify = TRUE)$items

# Display
kable(round(cde_pars[, c("a", "b")], 3),
      caption = "2PL Item Parameter Estimates for CDE Math Data (MMLE/EM)")
```

Table 11: 2PL Item Parameter Estimates for CDE Math Data (MMLE/EM)

	a	b
V1	1.021	-1.060
V2	0.783	-1.311
V3	1.424	-0.585

	a	b
V4	1.204	-1.020
V5	0.934	-0.382
V6	0.867	0.034
V7	0.714	-0.656
V8	0.926	-0.051
V9	1.966	-0.847
V10	1.184	-0.611
V11	1.182	-1.480
V12	1.935	-1.026
V13	1.126	-1.492
V14	0.761	-0.233
V15	1.285	-2.251
V16	2.092	-1.497
V17	1.032	-1.001
V18	0.784	-0.614
V19	1.514	-1.372
V20	1.218	1.316
V21	1.265	-0.582
V22	1.122	-0.584
V23	0.347	-0.120
V24	0.941	-0.094
V25	0.978	-1.151
V26	1.504	0.640
V27	1.003	0.209
V28	0.636	0.544
V29	0.998	1.623
V30	0.901	0.855
V31	0.507	1.901

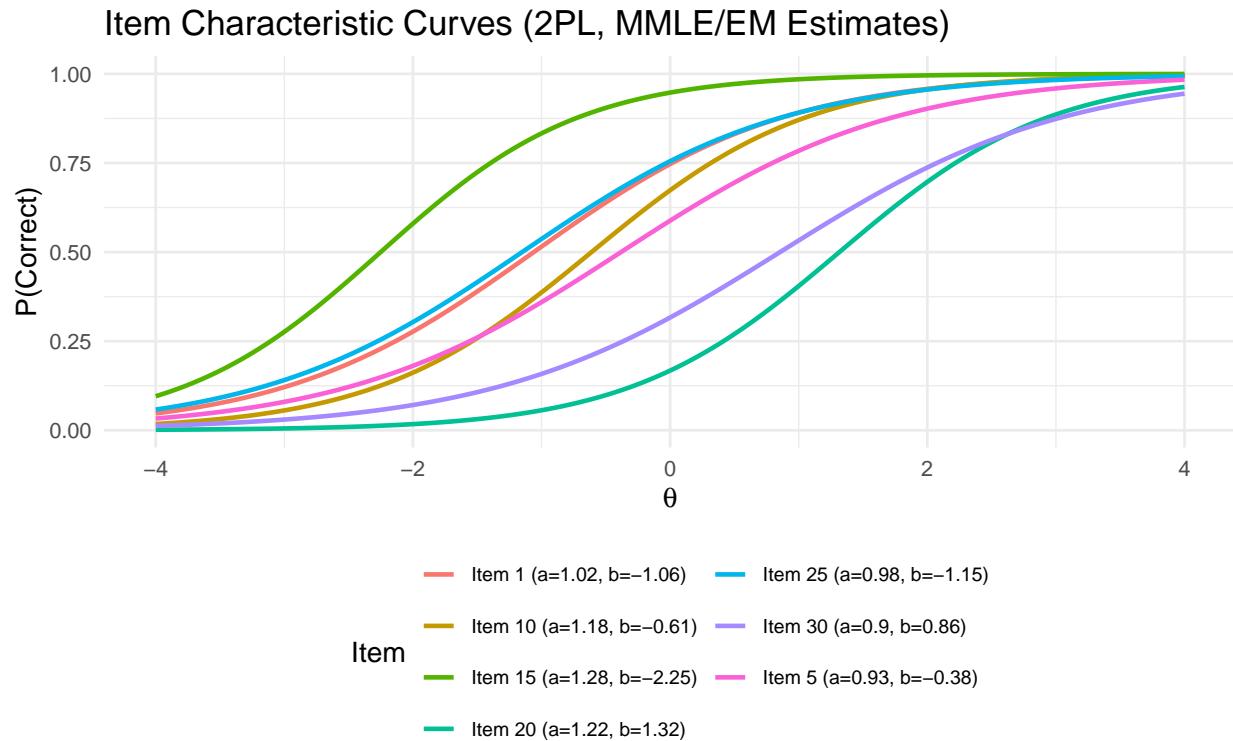
```
# Plot ICCs for a selection of items
theta_plot <- seq(-4, 4, by = 0.01)

# Pick items with varied characteristics
items_to_plot <- c(1, 5, 10, 15, 20, 25, 30)
items_to_plot <- items_to_plot[items_to_plot <= nrow(cde_pars)]

icc_data <- do.call(rbind, lapply(items_to_plot, function(j) {
  a_j <- cde_pars[j, "a"]
  b_j <- cde_pars[j, "b"]
  data.frame(
    theta = theta_plot,
    P = p_2pl(theta_plot, a_j, b_j),
    Item = paste0("Item ", j, " (a=", round(a_j, 2), ", b=", round(b_j, 2), ")")
  )
}))

ggplot(icc_data, aes(x = theta, y = P, color = Item)) +
  geom_line(linewidth = 1) +
  labs(title = "Item Characteristic Curves (2PL, MMLE/EM Estimates)",
       x = expression(theta), y = "P(Correct)") +
  theme_minimal(base_size = 13) +
  theme(legend.position = "bottom", legend.text = element_text(size = 9)) +
```

```
guides(color = guide_legend(ncol = 2))
```



## Summary: The MMLE/EM Big Picture

```
##  
## MMLE/EM Algorithm Summary  
## =====  
##  
## 1. INITIALIZE: Set starting values for all item parameters (d, a)  
##  
## 2. E-STEP ('Expectation'):  
##   - For each response pattern, compute the likelihood at each quadrature node  
##   - Use Bayes' theorem to get the posterior distribution at each node  
##   - Aggregate across all examinees to get the 'artificial data':  
##     * n-bar_k = expected number of examinees at node k  
##     * r-bar_ik = expected number correct on item i at node k  
##  
## 3. M-STEP ('Maximization'):  
##   - For each item separately:  
##     * Treat (n-bar_k, r-bar_ik) as known data (just like observed counts in JMLE)  
##     * Use Newton-Raphson to find item parameters that best fit this data  
##  
## 4. CONVERGED? If parameters have stopped changing, stop. Otherwise, go to step 2.
```

## Key Takeaways

### What MMLE does differently from JMLE:

- JMLE tries to estimate every person's  $\theta$  along with all item parameters simultaneously. This creates the Neyman-Scott problem: as you add more examinees, you add more parameters, and the item parameter estimates never become consistent.
- MMLE *integrates out* the person parameters by assuming they come from a population distribution. This reduces the problem to estimating only item parameters, which produces consistent estimates.

### What the EM algorithm does:

- Since we can't observe where examinees actually fall on the ability scale, the E-step uses the current best guess of item parameters to *infer* the ability distribution ("artificial data").
- The M-step then estimates item parameters as if this inferred distribution were real data.
- Each cycle refines both the inferred ability distribution and the item parameter estimates until they stabilize.

### Practical notes:

Feature	Detail
<b>Pros</b>	Works for 1PL, 2PL, 3PL, and multidimensional models; consistent estimates; better SEs than JMLE
<b>Cons</b>	Requires pre-specifying the $\theta$ distribution (though robust to mild violations); computationally intensive; can converge slowly
<b>Default in mirt</b>	Uses EM with 61 quadrature points; $N(0, 1)$ prior; Bock-Aitkin acceleration
<b>No person estimates</b>	MMLE only produces item parameters. Person $\theta$ estimates require a separate step (MLE, MAP, or EAP — covered in the person estimation module)

```
## R version: R version 4.5.2 (2025-10-31)
```

```
## mirt version: 1.45.1
```

## Reference

Baker, F. B., & Kim, S.-H. (2004). *Item response theory: Parameter estimation techniques* (2nd ed.). Marcel Dekker.